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A generalization of quasi-ideals

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KVAASI-IDEAALIDE ÜLDISTUS

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Lühikokkuvõte

Käesoleva bakalaureusetöö eesmärgiks on üldistada Otto Steinfeldi esitatud kvaasi-ideaali mõistet ning sellega seonduvaid tulemusi. Aastal 1978 ilmunud monograafias "Quasi-ideals in Rings and Semigroups" uurib Steinfeld nii poolrühmade kui ka ringide kvaasi-ideaale, mida võib mõista kui vasakpoolse ja parempoolse ideaali ühisosa. Üldistusprotsessis piirdume poolrühma kvaasi-ideaali üldistamisega. Töö raames kirjeldame kvaasi-ideaalile analoogilist objekti üle bipolügooni ning uurime, millised tulemused, mainitud monograafiast on niisuguse üldistuse järel tõestatavad.

CERCS teaduseriala: P120 Arvuteooria, väljateooria, algebraline geomeetria, algebra, rühmateooria

Märksõnad: Algebralised struktuurid, poolrühmad, ideaalid, ekvivalentsiseosed.

A GENERALIZATION OF QUASI-IDEALS

Bachelor's thesis

Olari Kikas

Abstract

The purpose of this bachelor's thesis is to present a generalization of quasi-ideals, which have been introduced by Otto Steinfeld. In the year 1978 Steinfeld published the monograph "Quasi-ideals in Rings and Semigroups", in which he studies quasi-ideals for both semigroups and rings, which can be considered to be intersections of a left ideal and a right ideal. In this work, we

limit ourselves to generalizing quasi-ideals of a semigroup. For this purpose, we describe an object, similar to the quasi-ideal, over a biact and see which results from the mentioned monograph are possible to prove after making such a generalization.
CERCS research specialisation: P120 Number theory, field theory, algebraic geometry, algebra, group theory
Key Words: Algebraic structures, semigroups, ideals, equivalence relations.

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Introduction

In the case of any algebraic structure, one is likely to study its subsets with certain properties. One of the most common examples of this is studying subalgebras of a chosen algebra. However, in the case of semigroups, we are also interested in the notion of ideal. Left ideals, right ideals and ideals of a semigroup have been studied extensively, although, in 1978, a Hungarian mathematician Otto Steinfeld published a monograph about quasi-ideals, which turns out to be a more general case of all of the the previously mentioned types of ideals.

In this thesis we propose a generalization of the quasi-ideal, which could be regarded as replacing a semigroup, which the quasi-ideal lies in, with a biact, as the biact over two semigroups is in fact a generalization of a semigroup considered as a biact over itself. The structure of a biact is interesting because in some cases it ties together two different semigroups. This is a result of not demanding algebraic closure in the usual sense, while still demanding some compatibility of these semigroup actions. We will make no additional assumptions about the biact, such as the biact being defined over only a single semigroup or the left action and the right action being defined alike. It could be interesting to see if such assumptions would let us generalize a bigger part of Steinfeld's results.

In this work, we will follow the example of [3] and discuss which part of the results in this monograph can be proved in the context of biacts. In the first section we will provide sufficient backround for the reader. In the second section we will prove some elementary results, which will provide useful groundwork for the following sections.

In the third section we show certain similarities between minors (our generalization of the quasi-ideals) and Green's $\mathcal{H}$-relation. Furthermore, this section will contain one of this work's most noteworthy results, which further generalizes O. Steinfeld's generalization of Green's Lemma. Lastly, in the fourth section, we will prove some results regarding minimal minors.

1 Preliminaries

In this thesis, we have two key concepts, which are fundamental before we start with the generalization process. Firstly, we must have a clear understanding of the object we are going to generalize – the quasi-ideal of a semigroup. Secondly, it is of great significance that we describe the algebraic structure on which our work will be done on. By this we refer to the definition of biact. However, we should note that in the process of generalizing quasi-ideals the most notable difference is given by the loss of structure. By this we mean that the following process should be regarded as defining an object resembling a quasi-ideal after replacing the semigroup, which a quasi-ideal was defined on, by a more general structure. Let us now describe all that is necessary for the process.

1.1 Definitions

In this subsection we will briefly give a few fundamental definitions and notations, which the reader should be familiar with. These definitions can also be found, for example, in [1] or [2].

Definition 1.1. Let S be a set and $\cdot : S \times S \to S$ a mapping. S is called a **semigroup** with multiplication $\cdot$ if, for any $a, b, c \in S$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Here we used the infix notation for $\cdot$. We will omit $\cdot$ from this notation and denote the multiplication by juxtaposition of respective elements, whenever the context is clear. For example the above equation would be written as

$$(ab)c = a(bc).$$

Definition 1.2. Let S be a semigroup. An element $e \in S$ is called an **idempotent** if

$$e^2 = e,$$

where e^2 denotes the product ee. We will denote the set of all idempotent elements of S by $E(S)$.

Definition 1.3. Let L be a partially ordered set. If for every two elements $a, b \in L$ the set L contains the least upper bound and the greatest lower bound of a and b, then L is called a **lattice**. If every subset S of a lattice L has a least upper bound and a greatest lower bound, then the lattice L is called **complete**.

1.2 Quasi-ideals of a semigroup

In this subsection we will describe what is necessary to define a quasi-ideal of a semigroup and we will discuss how this object is a generalization of its better-known counterpart – a left/right ideal of a semigroup.

Definition 1.4. Let S be a semigroup and $L \subseteq S$. The set L is called a **left ideal of** S if, for any $l \in L$ and $s \in S$, we have

$$sl \in L.$$

Let us define $SL := \{sl \mid s \in S, l \in L\}$. It is clear, that the last condition is equivalent to $SL \subseteq L$. The **right ideal of a semigroup** S is defined dually.

Definition 1.5. Let S be a semigroup and $I \subseteq S$. If I is both a left and a right ideal of S, then I is called **an ideal of** S.

It is most common, that the following definition is presented as an example, as this is rather easily seen to truly satisfy the conditions of the previous definitions. However we give it as a definition, due to its significance in Otto Steinfeld's work of quasi-ideals.

Definition 1.6. Let S be a semigroup and $a \in S$. The ideals

$$S^1a := \{sa \mid s \in S\} \cup \{a\}; \quad aS^1 := \{as \mid s \in S\} \cup \{a\};$$
$$S^1aS^1 := \{sat \mid s, t \in S\} \cup S^1a \cup aS^1$$

are called the **principal left ideal, principal right ideal** and **principal ideal generated by the element** a, respectively.

Let us now provide the definition of a quasi-ideal of a semigroup, after which we will give some context to the motivation behind studying such an object.

Definition 1.7 ([3])**.** Let S be a semigroup and $Q \subseteq S$. The set Q is called a **quasi-ideal** of S if

$$SQ \cap QS \subseteq Q$$

This definition is equivalent to Q being such that, for any element $a \in S$ for which there exist $s_1, s_2 \in S$ and $q_1, q_2 \in Q$ such that $a = s_1q_1 = q_2s_2$, we have $a \in Q$.

Note that, for an arbitrary semigroup S, every one-sided ideal is a quasi ideal. However a quasi-ideal $Q \subseteq S$ need not be either left nor right ideal. This is one of the most considerable remarks Steinfeld made in his work on quasi-ideals, as to why these should even be studied – because they are more general than their one-sided counterparts. This remark can be affirmed by the simple check that in the case of, say, a left ideal L of the semigroup S, we have

$$SL \cap LS \subseteq SL \subseteq L,$$

implying that any left ideal of S must also be a quasi-ideal of S. For right ideals this affirmation is clearly dual.

1.3 Biacts

As we mentioned in the beginning of this section, our work focuses mostly on describing an object similiar to a quasi-ideal of a semigroup, with the key difference of doing it on a more general structure than the semigroup. Let us now describe this structure and reason as to why this could be considered a generalization of semigroups.

Definition 1.8. Let S be a semigroup, A a set and $* : S \times A \to A, (s, a) \mapsto s * a$, a mapping. The set A is called a **left S-act** if for every $s, s' \in S$ and $a \in A$ we have

$$(s's) * a = s' * (s * a).$$

We then denote this as ${}_SA$ and we also say that S acts on A from the left and that the mapping $*$ is a **left S-action**. The **right S-act** is defined dually and denoted by A_S. In the following text we will omit writing out the action $*$, as there will not be any products defined on the set A itself.

Definition 1.9. Let ${}_SA$ be a left S-act and $B \subseteq A$. The set B is called a **subact of ${}_SA$** if for every $s \in S$ and $b \in B$ we have

$$sb \in B.$$

Defining $SB := \{sb \mid s \in S, b \in B\}$ we see that this definition is equivalent to $SB \subseteq B$. The **subact of A_S** is defined dually.

We will now give the definition which is central to our work (see Definition 1.4.24 and Remark 1.4.3 in [2]).

Definition 1.10. Let S and T be semigroups. We say that A is an (S, T)-**biact** if A is a left S-act and right T-act and, for every $a \in A, s \in S$ and $t \in T$, we have

$$(sa)t = s(at).$$

We write ${}_SA_T$ if A is an (S,T)-biact. Note that here the semigroups S and T may be different and the actions of ${}_SA$ and A_T need not be similar in any way.

As to how this is a generalization of semigroups becomes clear from the following remark.

Remark 1.11. Let S be a semigroup. Note that the multiplication of S can inherently be viewed as both a left S action and a right S action on the set S. Considering that the semigroup multiplication is associative, we can describe the biact ${}_SS_S$ with these actions. This implies that we could consider every semigroup as a biact. However it is clear that an arbitrary biact cannot describe a semigroup by similar argumentation.

From a universal algebra point of view, it is clear that an (S,T)-biact A is an algebra of a type, which only has unary operations for every element of S and T. This of course hints, that the most natural way to define a subact of ${}_SA_T$ will be the following.

Definition 1.12. Let A be an (S,T)-biact. We say that $B \subseteq A$ is a **subact of ${}_SA_T$**, if

$$sb \in B, \quad bt \in B,$$

for every $b \in B$, $s \in S$ and $t \in T$, that is B is closed under every operation of the biact ${}_SA_T$.

Definition 1.13. Let A be an (S,T)-biact and let $\rho \subseteq A \times A$ be an equivalence relation. We say that ρ is a **left congruence of the biact ${}_SA_T$** if, for all $a, b \in A$ and $s \in S$, the following implication holds:

$$a\rho b \;\Rightarrow\; sa\rho sb.$$

We define a **right congruence of the biact ${}_SA_T$** dually.

1.4 Minor of a biact

We have given the necessary background to define the main object of this thesis. For the sake of simpler readability, let us define, for any pair of semigroups S, T and any left S-act or right T-act A and $X \subseteq A$, the sets

$$SX := \{sx \mid s \in S, x \in X\}, \quad S^1X := SX \cup X,$$
$$XT := \{xt \mid x \in X, t \in T\}, \quad XT^1 := XT \cup X.$$

Mind that here the juxtapositions sx and xt with $s \in S$, $x \in X$ and $t \in T$, represent the respective products of the actions of ${}_SA$ and A_T – these actions need not be the same nor have any similarities.
Furthermore, here the notation S^1 refers to the use of an external unit. Such a construction could technically be avoided in this thesis, though we will define the following to ensure better clarity of our proofs.

Definition 1.14. Let S be a semigroup. Let us define a semigroup $S^1 := S \sqcup \{1\}$ with multiplication $*$, where

$$s * s' := ss', \quad s * 1 := s, \quad 1 * s := s, \quad 1 * 1 := 1,$$

for all $s, s' \in S$. The semigroup $(S^1, *)$ is called **the semigroup S with an externally adjoined unit** and the element $1 \in S^1$ is called the **external unit of S^1**.

In this thesis we will implicitly assume that for any left S-act ${}_SA$, we can consider the act ${}_{S^1}A$ with the action $*'$ is defined as

$$s *' a := sa, \quad 1 *' a := a,$$

for all $s \in S^1$ and $a \in A$. Note that generally S^1-actions on the set A can be defined differently as well, though we will only use the scenario described. We will make dual assumptions for all right T-acts. In what follows, we will omit the symbols $*$ and $*'$ as this causes no confusion.

Definition 1.15. Let A be an (S,T)-biact and $M \subseteq A$. We say that M **is a minor of** A if

$$SM \cap MT \subseteq M.$$

If we consider the semigroup S as a biact of the form described in Remark 1.11, then it is clear that a minor of this specific biact is precisely a quasi-ideal of the semigroup S, thus a minor of a biact is a valid generalization of the quasi-ideal of a semigroup.

It easily follows from the definition that both A and $\emptyset$ are minors of ${}_SA_T$. We will call these minors **trivial**.

2 Properties of a minor

Now that we have defined the minor of a biact as a generalization of the quasi-ideal of a semigroup, it would be most natural to affirm, which definitive properties of quasi-ideals are preserved in a minor. As the first candidate, Steinfeld proved in [3, Proposition 2.11a] that it is equivalent to define the quasi-ideal as an intersection of a left- and a right ideal. In this section our first goal is to show that any minor M of a biact ${}_SA_T$ is the intersection of a subact of ${}_SA$ and a subact of A_T. After doing so, we will give a small collection of examples exhibiting such minors, which are not trivial in the sense of being a subact of some left- or right act over a semigroup.

2.1 Standard properties

When we introduced the quasi-ideal of a semigroup S, then we mentioned the fact that every left- or right ideal of a semigroup S must be a quasi-ideal of S. It turns out that, for a biact ${}_SA_T$, an analogous fact holds for subacts of ${}_SA$ or A_T and minors of ${}_SA_T$. Let us show this formally

Remark 2.1. Let ${}_SA_T$ be an (S,T)-biact and $L \subseteq A$ be a subact of ${}_SA$. This gives,

$$SL \cap LT \subseteq SL \subseteq L.$$

Thus every subact L of ${}_SA$ is a minor of ${}_SA_T$. For a subact of A_T this is shown dually. We will later provide that generally this inclusion of classes is proper.

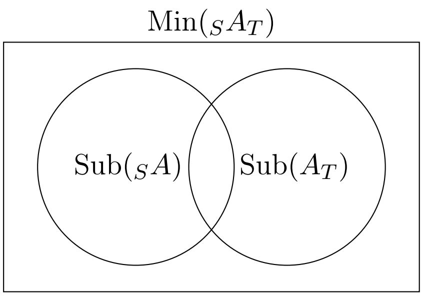

Figure 2.1

Here we illustrate this inclusion of classes with Figure 2.1, where the intersection of classes $\mathrm{Sub}_{_SA}$ and Sub_{A_T} actually represents the class of all subacts of the biact $_SA_T$, which we denote as $\mathrm{Sub}_{_SA_T}$.

Proposition 2.2. *If A is a left S-act and $X \subseteq A$, then S^1X is a subact of $_SA$. Dually if A is a right T-act, then XT^1 is a subact of A_T.*

Proof. Let us prove this statement for the subact of $_SA$. Let $X \subseteq A$, and $y \in S^1X$ and $s \in S$ be arbitrary. If $y \in X$, then, by definition, $sy \in SX \subseteq S^1X$. If $y \in SX$, then there exists such $s' \in S, x \in X$ such that $s'x = y$, then

$$sy = s(s'x) = (ss')x \in SX \subseteq S^1X.$$

Which verifies the fact $S(S^1X) \subseteq S^1X$, meaning that S^1X is a subact of $_SA$. Proof for a right T-act is analogous. $\square$

Corollary 2.3. *If A is an (S,T)-biact, then for any $X \subseteq A$ the sets S^1X and XT^1 are subacts of $_SA$ and A_T respectively.*

We will say that the set S^1X is the **subact of $_SA$ generated by** X and XT^1 is the **subact of A_T generated by** X.

Proposition 2.4 (Cf. [3, Proposition 2.6])**.** *If M is a minor of an (S,T)-biact A, then*

$$M = S^1M \cap MT^1.$$

Proof. Let M be a minor of $_SA_T$. Note that

$$S^1M \cap MT^1 = (SM \cup M) \cap (MT \cup M) = (SM \cap MT) \cup M,$$

from which the inclusion $M \subseteq S^1M \cap MT^1$ becomes clear. For the converse, we see that since M is a minor, we have $SM \cap MT \subseteq M$, implying $S^1M \cap MT^1 \subseteq M$. Thus $M = S^1M \cap MT^1$. $\square$

Note that this proposition already states that every minor is the intersection of a left subact and a right subact.

Proposition 2.5 (Cf. [3, Corollary 2.7])**.** *Let ${}_SA_T$ be a biact. The intersection of a subact of ${}_SA$ and a subact of A_T is a minor of ${}_SA_T$.*

Proof. Let us fix $L, R \subseteq A$ as subacts of ${}_SA$ and A_T, respectively. We see that

$$S(L \cap R) \cap (L \cap R)T \subseteq SL \cap RT \subseteq L \cap R,$$

proving the statement. □

Corollary 2.6. *A subset M of a biact ${}_SA_T$ is a minor if and only if it is the intersection of a subact of ${}_SA$ and a subact of A_T.*

We have now described the equivalence we mentioned at the start of this section. However, we can still obtain a slightly more specific description, if we use the notion of a minor being generated by a subset of a biact.

Definition 2.7. Let A be an (S, T)-biact and $X \subseteq A$. The **minor of A generated by the set** X is the intersection $(X)_m$ of all minors of A containing X. If X consists of a single element $a \in A$, then we say $(a)_m := (X)_m$ is the **principal minor of** a.

One may notice that it is unclear, whether the definition is correct, as we have not shown that the intersection of minors of a biact is in fact such a minor itself. We will prove the following statement, clarifying that this definition is indeed correct.

Proposition 2.8 (Cf. [3, Proposition 2.11a])**.** *The intersection of arbitrarily many minors of a biact ${}_SA_T$ is a minor of ${}_SA_T$.*

Proof. Let $M_i \subseteq A$ be a minor of ${}_SA_T$ for every $i \in \mathcal{I} \neq \emptyset$. Note that, for every

$j \in \mathcal{I}$, we have $\bigcap\limits_{i \in \mathcal{I}} M_i \subseteq M_j$. This fact implies the inclusion

$$S\left(\bigcap_{i\in\mathcal{I}} M_i\right) \cap \left(\bigcap_{i\in\mathcal{I}} M_i\right) T \subseteq SM_j \cap M_jT \subseteq M_j,$$

for every $j \in \mathcal{I}$. By this we get

$$S\left(\bigcap_{i\in\mathcal{I}} M_i\right) \cap \left(\bigcap_{i\in\mathcal{I}} M_i\right) T \subseteq \bigcap_{i\in\mathcal{I}} M_i.$$

Note that this discussion also holds, if the intersection should be empty. □

It is a well known in universal algebra that if a partially ordered set is a complete lower semilattice and contains a biggest element, then it is a complete lattice. Notice that Proposition 2.8 and the fact that A is a minor, lets us pose the following corollary.

Corollary 2.9. *The partially ordered set $Min({}_SA_T)$ of all minors of a biact ${}_SA_T$ ordered by inclusion is a complete lattice, where the meets are intersections.*

We will now prove a statement, which elaborates on Corollary 2.6.

Theorem 2.10 (Cf. [3, Proposition 2.12])**.** *If ${}_SA_T$ is a biact and $X \subseteq A$, then*

$$S^1X \cap XT^1 = (X)_m.$$

Proof. Let A be an (S,T)-biact and $X \subseteq A$. By Proposition 2.5, we know that $S^1X \cap XT^1$ is a minor, furthermore this minor contains X. This gives us

$$(X)_m \subseteq S^1X \cap XT^1,$$

as $(X)_m$ is defined to be the intersection of all minors of ${}_SA_T$ containing X. For the converse, we shall note that $(X)_m$ is a minor, with regards to Proposition 2.8.

By Proposition 2.4, we now get

$$(X)_m = S^1(X)_m \cap (X)_m T^1.$$

Since X is contained in $(X)_m$, we have

$$S^1 X \cap XT^1 \subseteq S^1(X)_m \cap (X)_m T^1 = (X)_m,$$

yielding the inclusion $S^1 X \cap XT^1 \subseteq (X)_m$. $\square$

2.2 Examples

The purpose of this subsection is rather self-explanatory – to give examples. We would however prefer to present such minors of a biact, which are neither left nor right subacts, as latter would not highlight the difference shown on Figure 2.1.

Example 2.11. Let R be a ring, $n, m \in \mathbb{N}$ and $A = \mathrm{Mat}_{m,n}(R)$, $S = \mathrm{Mat}_m(R)$, $T = \mathrm{Mat}_n(R)$, where $\mathrm{Mat}_{n,m}(R)$ denotes the set of all matrices with n rows and m columns and entries from R and $\mathrm{Mat}_n(R)$ denotes the set of all square matrices of size n, with entries from R. Considering actions defined by the usual matrix multiplication, we obtain a biact ${}_SA_T$. If we consider M as the subset of A consisting of matrices that have non-zero elements only on the intersection of some fixed rows and columns, then we see, that M is neither a subact of ${}_SA$ nor A_T, as SM contains matrices with non-zero elements not on our fixed rows and MT will contain matrices with non-zero elements not on our fixed columns. This set however is indeed a minor of ${}_SA_T$, as $SM \cap MT$ consists of only matrices in M.

For example, if $A = \mathrm{Mat}_{4,5}(\mathbb{Z})$, then

$$Q = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & a_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| a_{ij} \in \mathbb{Z} \right\}$$

is a minor. Here we have fixed rows 1 and 3 and columns $2, 3$ and 5.

Example 2.12. We may consider the set $A := \mathbb{C}$ as a biact ${}_SA_T$, where we choose $S := \mathbb{R} \subseteq \mathbb{C}$ and $T := \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 0\} \leq \mathbb{C}$ to be groups with respect to the addition of complex numbers. The left S-action and right T-action are also defined by the addition of complex numbers. Here we could suggest quite a few proper minors of A, but let us only look at the following, for now.
Say we fix $M := \mathbb{C} \setminus (S \cup T)$. It is clear, that M is neither a left nor a right subact of ${}_SA$ or A_T, respectively. Considering that we could fix an arbitrary $z = a + ib \in M$ for which we have $-a \in S$ and $-ib \in T$ with the properties

$$(-a) + z \notin M, \qquad z + (-ib) \notin M.$$

However, if we fix $m \in SM \cap MT$, then we have $m_1, m_2 \in M$ and $s \in S, it \in T$, where $s, t \in \mathbb{R}$ and $m_1 = a_1 + ib_1$, $m_2 = a_2 + ib_2$, with

$$m = s + m_1 = m_2 + it \;\Rightarrow\; m = (s + a_1) + ib_1 = a_2 + i(b_2 + t).$$

This suggests, that $\mathrm{Im}(m) = b_1$ and $\mathrm{Re}(m) = a_2$, which cannot be zero, as this would contradict $m_1, m_2 \in M$, due to M containing only the complex numbers z satisfying $\mathrm{Im}(z) \neq 0$ and $\mathrm{Re}(z) \neq 0$. This implies that $m \in M$, i.e. $SM \cap MT \subseteq M$. On a further note, we see that the principal minors of this biact consist of a single element. If we fix an arbitrary $z = x + iy \in \mathbb{C}$, then

$$S^1 z = \{u + iy \mid u \in \mathbb{R}\}, \quad zT^1 = \{x + ik \mid k \in \mathbb{R}\}.$$

It is rather evident that the intersection $S^1 z \cap zT^1$ is precisely the set $\{z\}$. Considering Theorem 2.10, we get $(z)_m = \{z\}$.

Example 2.13. Let X and Y be arbitrary non-empty sets of the same cardinality and $S := \{f : Y \to Y \mid f \text{ is injective}\}$ and $T := \{g : X \to X \mid g \text{ is surjective}\}$. Here we can take A as the set of all mappings from X to Y. It is clear, that S and T form semigroups under the composition operation of mappings. Note that we also obtain a biact ${}_SA_T$, if we define the actions as composing functions from the respective side, i.e.

$$\begin{aligned} * : S \times A \to A, &\quad (s, a) \mapsto s \circ a, \\ *' : A \times T \to A, &\quad (a, t) \mapsto a \circ t. \end{aligned}$$

Associativity of "$\circ$" confirms, that ${}_SA_T$ satisfies the "associativity-like" properties of a biact. Closure of $*$ and $*'$ is clear. We will omit the notation of $*, *'$ and denote the products of this biact simply by their juxtaposition, as it is usual for mapping compositions. Let us choose the proper subset M of A to be

$$M := \{u : X \to Y \mid u \text{ is a bijection}\} \subset A.$$

Now we have that, for arbitrary $s \in S, u \in M$ and $t \in T$, we always obtain, that su is an injection and ut is a surjection, yet neither of these generally need be a bijection, thus M describes neither a left nor a right subact. Yet it turns out that M is a minor. Indeed, if we choose $u \in SM \cap MT$, then there exist $s \in S, m_1, m_2 \in M$ and $t \in T$ such that

$$u = sm_1 = m_2 t.$$

Here it is clear that sm_1 is an injection and m_2t is a surjection, thus u is a bijection. This indeed implies $SM \cap MT \subseteq M$.

The previous example is actually quite a specific case of a far more general scenario. This following example will expect the reader to be familiar with some standard category-theoretical constructions.

Example 2.14. Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ (this notation refers to X and Y being objects of $\mathcal{C}$ without necessarily implying, that the collection of objects of $\mathcal{C}$ forms a set). We can now consider the sets

$$S := \{s \in \mathcal{C}(Y, Y) \mid s \text{ is a monomorphism}\},$$
$$T := \{t \in \mathcal{C}(X, X) \mid t \text{ is an epimorphism}\}$$

as semigroups (even monoids with the respective unit morphism) by the composition operation of $\mathcal{C}$, as the composition must be associative. Now we can choose $A := \mathcal{C}(X, Y)$, the set of all morphisms from X to Y in $\mathcal{C}$, and construct the biact ${}_SA_T$, similiarly to the previous example, as composing a morphism in A from the respective side. It is easy to confirm, that this describes a biact. Now we choose $M \subset A$ to be the subset of bimorphisms of $\mathcal{C}(X, Y)$. Fixing an arbitrary $u \in SM \cap MT$ will describe $s \in S, m_1, m_2 \in M$ and $t \in T$, which satisfy

$$u = sm_1 = m_2t.$$

Here it is clear that sm_1 is a composition of monomorphisms – therefore u is a monomorphism. Similiarly, since m_2t is a composition of epimorphisms, we have that u is an epimorphism. Thus $u \in M$. Note that M is a minor, but generally not a left nor a right subact.

Considering that, in the category of sets, epimorphisms and monomorphisms are precisely surjections and injections, respectively, it is clear that this is an accurate generalization of Example 2.13. Although we should note, that the category of sets is balanced (bimorphisms and isomorphisms coincide in this category), whereas this generally is not the case. Taking the difference of isomorphisms and bimorphisms into consideration, we can give one more example.

Example 2.15. Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. Furthermore let us define

$$S := \{s \in \mathcal{C}(Y, Y) \mid s \text{ left-invertible}\},$$
$$T := \{t \in \mathcal{C}(X, X) \mid t \text{ is right-invertible}\},$$

Now we describe $A := \mathcal{C}(X, Y)$ and the actions of ${}_SA$ and A_T identically to Example 2.14. This describes the biact ${}_SA_T$. Here we can define the $M \subset A$ to be the set of all isomorphisms in $\mathcal{C}(X, Y)$. Similiarly to the previous examples, M is neither a left, nor a right subact of A, due to the composites in SM not necessarily being left invertible and the composites in MT not necessarily being right invertible. Although a morphism being in $SM \cap MT$ would imply, that this morphism is both left and right invertible, i.e. an isomorphism. This means that M is a minor.

3 Green's relations

In semigroup theory, we are often interested in the principal ideals, as these can be somewhat considered to be the "building blocks" of the very semigroup they lie in. One of the most used tools for studying these are Green's relations (see [1]), named after James Alexander Green. Green's relations for biacts over monoids have been introduced in [2, page 61]. We adapt these definitions for biacts over semigroups.

3.1 Definitions

Definition 3.1. Let ${}_SA_T$ be a biact over semigroups S and T. **Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on the biact ${}_SA_T$** are defined as follows: for $a, b \in A$,

$$\begin{aligned}
a \,\mathcal{L}\, b &\iff S^1a = S^1b,\\
a \,\mathcal{R}\, b &\iff aT^1 = bT^1,\\
a \,\mathcal{J}\, b &\iff S^1aT^1 = S^1bT^1,\\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R},\\
\mathcal{D} &= \mathcal{L} \vee \mathcal{R}.
\end{aligned}$$

Here $S^1a = \{a\} \cup \{sa \mid s \in S\} =: \{a\} \cup Sa$ etc.

For the case of monoids, it has been shown, in [2, Proposition 1.4.48], that for a biact ${}_SA_T$, where S and T are monoids, we have $\mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R}$. This equality will hold in the case of A being an (S, T)-biact, as this is a specific case of the mentioned proposition. We will proceed with the alternative definition

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

Note that all of these relations are clearly equivalence relations. Furthermore, we can show that $\mathcal{L}$ is a right congruence of ${}_SA_T$ and $\mathcal{R}$ is a left congruence of ${}_SA_T$.

Lemma 3.2. *Let ${}_SA_T$ be a biact. The relation $\mathcal{L}$ is a right congruence of ${}_SA_T$ and the relation $\mathcal{R}$ is a left congruence of ${}_SA_T$.*

Proof. Let ${}_SA_T$ be a biact. We will only prove that $\mathcal{L}$ is a right congruence of ${}_SA_T$, as the rest of the proof is symmetric. Suppose $a\,\mathcal{L}\,b$ and $t \in T$ is arbitrary. By the definition of $\mathcal{L}$, we have $S^1a = S^1b$. This implies that there exist $s, s' \in S^1$ with

$$sa = b, \qquad s'b = a.$$

Acting on both sides of both equalities with t yields

$$sat = bt, \qquad s'bt = at.$$

This implies $S^1bt \subseteq S^1at$ and $S^1at \subseteq S^1bt$, i.e. $S^1at = S^1bt$, meaning precisely that $at\,\mathcal{L}\,bt$. This concludes that $\mathcal{L}$ is a right congruence of ${}_SA_T$. □

3.2 Green's $\mathcal{H}$ relation

We have shown in Theorem 2.10 that for any subset X of an (S,T)-biact A, the minor generated by X is of form $S^1X \cap XT^1$. This also implies that $(a)_m = S^1a \cap aT^1$. With this we will show that, in fact, Green's $\mathcal{H}$-relation precisely consists of pairs having the same principal minor. The next proposition generalizes a result by B. Kolibiarová.

Proposition 3.3 (Cf. [3, Proposition 4.1])**.** *Let A be an (S,T)-biact and $a, b \in A$. Then*

$$a\,\mathcal{H}\,b \iff (a)_m = (b)_m.$$

Proof. If $a\,\mathcal{H}\,b$, then we have $S^1a = S^1b$ and $aT^1 = bT^1$, implying

$$(a)_m = S^1a \cap aT^1 = S^1b \cap bT^1 = (b)_m.$$

For the converse, suppose $(a)_m = (b)_m$. With this equality, we get the implications

$$a \in (b)_m \Rightarrow a \in S^1 b \Rightarrow S^1 a \subseteq S^1 b,$$
$$a \in (b)_m \Rightarrow a \in bT^1 \Rightarrow aT^1 \subseteq bT^1.$$

Dually we obtain inclusions $S^1 b \subseteq S^1 a$ and $bT^1 \subseteq aT^1$, from which we get $S^1 a = S^1 b$ and $aT^1 = bT^1$, implying $a \,\mathcal{L}\, b$ and $a \,\mathcal{R}\, b$. This is, by definition, $a \,\mathcal{H}\, b$. $\square$

For the following propositions we shall use the natural notations for certain equivalence classes in the case of a biact ${}_S A_T$ and an element $a \in A$:

$$H_a := \{b \in A \mid a \,\mathcal{H}\, b\}, \quad R_a := \{b \in A \mid a \,\mathcal{R}\, b\}, \quad L_a := \{b \in A \mid a \,\mathcal{L}\, b\}.$$

Lemma 3.4. *If M is a minor of a biact ${}_S A_T$, then $H_a \subseteq M$ for any $a \in M$.*

Proof. Let A be an (S,T)-biact, M be a minor of ${}_S A_T$ and let $a \in M$ and $b \in H_a$ be arbitrary. Considering $a \,\mathcal{H}\, b$, by Proposition 3.3 and Theorem 2.10, we see that

$$b \in (b)_m = (a)_m = S^1 a \cap aT^1 \subseteq S^1 M \cap MT^1 = M,$$

thus $H_a \subseteq M$ for all $a \in M$. $\square$

Corollary 3.5. *Every minor of a biact is the union of the $\mathcal{H}$-classes of its elements.*

Corollary 3.6. *Let ${}_S A_T$ be a biact. For any $a \in A$ we have $H_a \subseteq (a)_m$.*

For the following propositions, let us define, for a biact ${}_S A_T$ and elements $s \in S$, $a \in A$, $t \in T$, the following subsets of A:

$$sH_a := \{sa' \mid a' \in H_a\}, \quad H_a t := \{a't \mid a' \in H_a\}.$$

Proposition 3.7 (Cf. [3, Lemma 4.2])**.** *Let A be an (S,T)-biact and $a \in A$, $s \in$*

S, $t \in T$. *The following implications hold:*

$$sa \in H_a \;\Rightarrow\; sH_a = H_a,$$
$$at \in H_a \;\Rightarrow\; H_a t = H_a.$$

Proof. Let us only prove the first implication – the second one follows by a symmetric argument. Suppose $sa \in H_a$. We first show $sH_a \subseteq H_a$. To this end, let $h \in sH_a$ be arbitrary. We must show $h \,\mathcal{H}\, a$, for which we first show $h \,\mathcal{R}\, a$. Since $h \in sH_a$, we have $a' \in H_a$ such that $h = sa'$. Due to $\mathcal{R}$ being a left congruence of A and $a' \,\mathcal{R}\, a$, we get

$$sa' \,\mathcal{R}\, sa \,\mathcal{R}\, a,$$

implying $h \,\mathcal{R}\, a$. Next, we will prove that $h \,\mathcal{L}\, a$. Due to $a' \,\mathcal{R}\, a$, we have $t \in T^1$ such that $a' = at$. With $\mathcal{L}$ being a right congruence of A, we obtain, from our assumption, that $sat \,\mathcal{L}\, at$, i.e. $S^1 at = S^1 sat$. Now we can conclude

$$S^1 a = S^1 a' = S^1 at = S^1 sat = S^1 sa' = S^1 h,$$

thus $h \,\mathcal{L}\, a$. With this we have proved the inclusion $sH_a \subseteq H_a$.
For the converse, let us fix $h \in H_a$ arbitrarily and show that there is $x \in H_a$ such that $h = sx$. By having already proven $sH_a \subseteq H_a$, we have $sh \,\mathcal{H}\, a \,\mathcal{H}\, h$. By this we have $s' \in S^1$ and $t' \in T^1$ such that $s'sh = h$ and $sht' = h$. Now we get

$$h = sht' = ss'sht' = ss'h,$$

by which it would suffice to show $s'h \in H_a$. For this we, use the fact that $s'H_a = s'H_{sh} \subseteq H_{sh} = H_a$ (this holds due to the proof of the previous inclusion and $sh \,\mathcal{H}\, a$). This implies $s'h \in H_a$, allowing us to choose $x := s'h$ with the properties $x \in H_a$ and $h = sx$. With this, we have shown $H_a \subseteq sH_a$, finishing the proof. $\square$

We will now state a theorem which Otto Steinfeld proved in [3] for Green's relations of a semigroup. In this theorem we use the notations, regarding Green's relations, in the same way.

Theorem 3.8 ([3, Theorem 4.3])**.** *If a and b are elements of a semigroup S, then $ab \in R_a \cap L_b$ implies*

$$aH_b = H_a b = xH_b = H_a y = H_a H_b = H_{ab} = R_a \cap L_b,$$

where $x \in H_a$ and $y \in H_b$.

It is important to notice that some of these sets use the fact that, in S, we can multiply the elements of H_a and H_b. Now, if we were to attempt generalizing this result to the context of biacts, then we generally cannot multiply the elements of a biact in this manner. We will make adjustments to this result, to obtain the full theorem.

Theorem 3.9. *If ${}_SA_T$ is a biact and $s \in S,\ a \in A,\ t \in T$ are such that $sat \in R_{sa} \cap L_{at}$, then the following equalities hold:*

$$sH_{at} = H_{sa}t = H_{sat} = R_{sa} \cap L_{at}.$$

Proof. Let us assume $sat \in R_{sa} \cap L_{at}$. To prove these equalities, we will show that the inclusions/equality

$$sH_{at} \subseteq R_{sa} \cap L_{at} = H_{sat} \subseteq sH_{at}$$

hold. It is symmetric to show that $H_{sa}t$ is equal to these sets.

Firstly, we will show that $sH_{at} \subseteq R_{sa} \cap L_{at}$. Let $sh \in sH_{at}$, where $h \in H_{at}$, be arbitrary. We must show that $sh\,\mathcal{R}\,sa$ and $sh\,\mathcal{L}\,at$. For the first relation, using the facts $sat\,\mathcal{R}\,sa$ and $\mathcal{R}$ is a left congruence, we conclude that

$$h\,\mathcal{H}\,at \;\Rightarrow\; h\,\mathcal{R}\,at \;\Rightarrow\; sh\,\mathcal{R}\,sat\,\mathcal{R}\,sa \;\Rightarrow\; sh\,\mathcal{R}\,sa.$$

Now, in order to obtain $sh\,\mathcal{L}\,at$ we note that, since $h\,\mathcal{R}\,at$, we have $hT^1 = atT^1$, implying the existence of $t' \in T^1$ such that

$$h = att'.$$

Now with $sat\,\mathcal{L}\,at$ and $\mathcal{L}$ being a right congruence, we obtain

$$sat\,\mathcal{L}\,at \;\Rightarrow\; satt'\,\mathcal{L}\,att' \;\Rightarrow\; sh\,\mathcal{L}\,h\,\mathcal{L}\,at \;\Rightarrow\; sh\,\mathcal{L}\,at.$$

With this we have shown that $sH_{at} \subseteq R_{sa} \cap L_{at}$.

Secondly, we will prove that the equality $R_{sa} \cap L_{at} = H_{sat}$ holds. Since $sat \in R_{sa} \cap L_{at}$, we see that

$$R_{sa} \cap L_{at} = R_{sat} \cap L_{sat} = H_{sat}.$$

Lastly, we will show that $H_{sat} \subseteq sH_{at}$. For this, suppose $h \in H_{sat}$ and we must find $x \in H_{at}$ such that

$$h = sx. \tag{3.1}$$

By our assumption $sat\,\mathcal{L}\,at$ and $h\,\mathcal{R}\,sat$, we have $\tilde{s} \in S^1$ and $\tilde{t} \in T^1$, satisfying

$$at = \tilde{s}sat, \qquad h = sat\tilde{t}.$$

Now if we choose $x_0 := \tilde{s}h$, then we can calculate

$$sx_0 = s\tilde{s}h = s\tilde{s}(sat\tilde{t}) = s(\tilde{s}sat)\tilde{t} = sat\tilde{t} = h.$$

To confirm that x_0 is a solution for (3.1), it is left to show that $x_0 \in H_{at}$. Due to

$x_0 = \tilde{s}h$ and $h = sx_0$, we have $S^1h = S^1x_0$, thus

$$x_0 \,\mathcal{L}\, h \,\mathcal{L}\, sat \,\mathcal{L}\, at \;\Rightarrow\; x_0 \,\mathcal{L}\, at.$$

Furthermore, considering that $\mathcal{R}$ is a left congruence, we get

$$h \,\mathcal{R}\, sat \;\Rightarrow\; \tilde{s}h \,\mathcal{R}\, \tilde{s}sat \;\Rightarrow\; x_0 \,\mathcal{R}\, at.$$

With these relations we have obtained $x_0 \in H_{at}$, implying the inclusion $H_{sat} \subseteq sH_{at}$. We have obtained

$$sH_{at} = R_{sa} \cap L_{at} = H_{sat}.$$

As mentioned previously, by symmetry we get

$$H_{sa}t = R_{sa} \cap L_{at} = H_{sat},$$

in conclusion giving us all of the equalities proposed. □

With Figure 3.2, we illustrate how Theorem 3.9 lets us represent certain intersections of $\mathcal{L}$-classes and $\mathcal{R}$-classes as $\mathcal{H}$-classes, under the assumption $sat \in R_{sa} \cap L_{at}$.

	L_{at}	L_{sa}	L_a
R_{sa}	H_{sat}	H_{sa}	
R_{at}	H_{at}		
R_a			H_a

Figure 2.1

The next theorem says that if two elements of a biact are $\mathcal{D}$-related, then their principal minors have the same cardinality.

Theorem 3.10 (Cf. [3, Theorem 4.8]). *Let ${}_SA_T$ be a biact. If $a, c \in A$ are $\mathcal{D}$-equivalent, then there exists a bijection φ between the principal minors $(a)_m$ and $(c)_m$. Furthermore φ restricts to a bijection between the $\mathcal{H}$-classes H_a and H_c.*

Proof. Let $a \,\mathcal{D}\, c$. Considering that $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, we have $b \in A$ such that $a \,\mathcal{R}\, b$ and $b \,\mathcal{L}\, c$. By these relations, there exist elements $s, s' \in S^1$ and $t, t' \in T^1$ satisfying

$$at = b, \qquad bt' = a, \qquad sb = c, \qquad s'c = b.$$

Let us now define $\varphi : (a)_m \to (c)_m$ by

$$\varphi(h) := sht,$$

for all $h \in (a)_m$. To see that this definition is correct, we must verify that the image of an arbitrary $h \in (a)_m$ lies in $(c)_m$. Due to $h \in (a)_m = S^1 a \cap aT^1$, we have $\tilde{s} \in S^1, \tilde{t} \in T^1$ with

$$h = \tilde{s}a = a\tilde{t}.$$

We can calculate

$$\varphi(h) = sht = s\tilde{s}at = s\tilde{s}b \in S^1 b = S^1 c,$$
$$\varphi(h) = sht = sa\tilde{t}t = sbt'\tilde{t}t = ct'\tilde{t}t \in cT^1.$$

Thus $\varphi(h) \in S^1 c \cap cT^1 = (c)_m$. Let us also define $\psi : (c)_m \to (a)_m$ with

$$\psi(u) = s'ut',$$

for all $u \in (c)_m$. By symmetry, we note that this definition is also correct. Now we will show that φ and ψ are inverses of one another, that is $(\psi \circ \varphi)(h) = h$ and $(\varphi \circ \psi)(u) = u$, for all $u \in (c)_m$ and $h \in (a)_m$. We will only show the first equality, for an arbitrary $h \in (a)_m$, as the second one follows from symmetry. Let $h, \tilde{s}, \tilde{t}$

satisfy the same properties as earlier ($h = \tilde{s}a = a\tilde{t}$). We can calculate

$$(\psi \circ \varphi)(h) = \psi(\varphi(h)) = \psi(sht) = s'shtt' = s's\tilde{s}bt' = s's\tilde{s}a$$
$$= s'sh = s'sa\tilde{t} = s'sbt'\tilde{t} = s'ct'\tilde{t} = bt'\tilde{t} = a\tilde{t} = h.$$

We have shown that φ is a one-to-one correspondence from $(a)_m$ onto $(c)_m$. In order to show that φ is also a one-to-one correspondence from H_a onto H_c, it suffices to show that $\varphi(h) \in H_c$ and $\psi(u) \in H_a$ for arbitrary $h \in H_a$ and $u \in H_c$. Once again, we will only show the first of these, as the latter follows from symmetry. Let $h \in H_a$ be arbitrary. Due to $h \,\mathcal{L}\, a$, we have $l \in S^1$, such that

$$h = la.$$

With this we obtain

$$htt' = latt' = lbt' = la = h,$$

implying $hT^1 = htT^1$, i.e. $ht \,\mathcal{R}\, h$. Since $\mathcal{R}$ is a left congruence, we obtain

$$ht \,\mathcal{R}\, h \,\mathcal{R}\, a \,\mathcal{R}\, b \;\Rightarrow\; ht \,\mathcal{R}\, b \;\Rightarrow\; sht \,\mathcal{R}\, sb \;\Rightarrow \varphi(h) \,\mathcal{R}\, c.$$

Due to $ht \,\mathcal{R}\, b$, we have $r \in T^1$ satisfying

$$ht = br.$$

We can now calculate

$$s'sht = s'sbr = s'cr = br = ht,$$

which implies $sht \,\mathcal{L}\, ht$. Considering that $\mathcal{L}$ is a right congruence, we get

$$h \,\mathcal{L}\, a \;\Rightarrow\; ht \,\mathcal{L}\, at \;\Rightarrow\; sht \,\mathcal{L}\, b \,\mathcal{L}\, c \;\Rightarrow\; \varphi(h) \,\mathcal{L}\, c.$$

With these relations we have shown $\varphi(h) = sht \in H_c$. Considering that ψ is the inverse function of φ and maps H_c into H_a, we have that φ is also a one-to-one correspondence between H_a and H_c, finishing this proof. $\square$

4 Minimal minors

It is well known that for both semigroups and acts, the question of minimality of a respective subalgebra is a noteworthy topic. In the following section we will define minimal minors, describe some conditions equivalent to minimality and present some examples of these. Let us assume in this section, that the biact ${}_SA_T$ is always non-empty.

4.1 Minimal minors

Definition 4.1. Let $M \neq \emptyset$ be a minor of ${}_SA_T$. We say that M is **minimal**, if for every non-empty minor M' of ${}_SA_T$ the following implication holds.

$$M' \subseteq M \;\Rightarrow\; M = M'.$$

In the first section we saw that a minor is the intersection of a left and a right subact of ${}_SA$ and A_T, respectively. It turns out that in a similiar manner we can describe minimal minors by minimal subacts.

Proposition 4.2 (Cf. [3, Theorem 5.1])**.** *Let $M \subseteq {}_SA_T$ be a non-empty minor. Then M is minimal if and only if it is the intersection of a minimal subact of ${}_SA$ and a minimal subact of A_T.*

Proof. For necessity, let us assume that M is minimal. Since M is non-empty, we have an element $a \in M$. Since $(a)_m$ is a non-empty minor of ${}_SA_T$ and

$$(a)_m = S^1a \cap aT^1 \subseteq S^1M \cap MT^1 = M,$$

the inclusion $(a)_m \subseteq M$ implies, by the minimality of M, that

$$S^1a \cap aT^1 = (a)_m = M.$$

Let us show that S^1a and aT^1 are minimal subacts of ${}_SA$ and A_T, respectively. For this, we assume that $L \subseteq S^1a$ is a non-empty subact of ${}_SA$. By Theorem 2.10 we see that $L \cap aT^1$ is a minor of ${}_SA_T$, thus we obtain

$$L \cap aT^1 \subseteq S^1a \cap aT^1 = M.$$

From this we get $L \cap aT^1 = M$, implying $M \subseteq L$. By this inclusion, we obtain

$$S^1a \subseteq S^1M \subseteq S^1L = L.$$

Thus $S^1a \subseteq L$, which will clearly give $S^1a = L$, meaning that S^1a is a minimal subact of ${}_SA$. Showing that aT^1 is a minimal subact of A_T, is symmetric, thus necessity holds.

For sufficiency, let us assume that $M = L \cap R$, where L and R are minimal subacts of ${}_SA$ and A_T, respectively. Furthermore, let $M' \subseteq M$ be a non-empty minor of ${}_SA_T$. By Corollary 2.3 and the minimality of the subacts L and R, we obtain

$$S^1M' \subseteq S^1L \subseteq L \Rightarrow S^1M' = L,$$
$$M'T^1 \subseteq RT^1 \subseteq R \Rightarrow M'T^1 = R.$$

By Corollary 2.6, we have $S^1M' \cap M'T^1 = M'$, hence it follows that

$$M' = S^1M' \cap M'T^1 = L \cap R = M,$$

having proven this proposition. □

Proposition 4.3 (Cf. [3, Theorem 5.2.]). *A minor M of a biact ${}_SA_T$ is minimal if and only if M is an $\mathcal{H}$-class.*

Proof. Let M be a minor of ${}_SA_T$. If we assume M to be minimal, then it is clear that, for any $u, v \in M$ we have $(u)_m = M = (v)_m$. Thus every element of M belongs to the same $\mathcal{H}$-class, meaning $M \subseteq H_a$ for some $a \in M$. By Lemma 3.4,

we also have $H_a \subseteq M$, thus we have obtained $H_a = M$.
Conversely, if we assume $M = H_a$ for some $a \in A$, then let us fix a non-empty minor $M' \subseteq M$ with $b \in M'$. Considering that $M' \subseteq H_a$, we have that $b \in H_a$, thus $H_a = H_b$. By Lemma 3.4, we get

$$H_b \subseteq M'.$$

This implies $M = H_a = H_b \subseteq M' \subseteq M$, finishing the proof. $\square$

Let us see, what principal minors, equivalence classes of $\mathcal{H}$ and minimal minors look like in the biact of non-square matrices of certain size. This biact is viewed with respect to matrix multiplication.

Example 4.4. Suppose we have the biact ${}_S\mathrm{Mat}_{2,3}(K)_T$, where K is a field and $S = \mathrm{Mat}_2(K)$, $T = \mathrm{Mat}_3(K)$ are semigroups with respect to matrix multiplication. The actions are also matrix multiplication. Let us fix the matrix B as

$$B := \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}.$$

It is easy to check that the left subact $S^1 B$ and right subact BT^1 are

$$S^1 B = \left\{ \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \end{pmatrix} \middle| a_1, a_2, b_1, b_2 \in K \right\},$$
$$BT^1 = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \middle| a_i, b_i \in K, i = 1, 2, 3 \right\}.$$

Thus it is clear that the principal minor $(B)_m$ is the set of all matrices with the third column consisting of 0. Now note that for an arbitrary $C \in \mathrm{Mat}_{2,3}(K)$ we have

$$B \,\mathcal{H}\, C \Leftrightarrow (B)_m = (C)_m \Leftrightarrow C = \begin{pmatrix} u & y & 0 \\ x & v & 0 \end{pmatrix},$$

where either u and v are non-zero or x and y are non-zero. This will result in C generating the same principal minor, as B. Thus

$$H_B = \left\{ \begin{pmatrix} x_1 & y_2 & 0 \\ x_2 & y_1 & 0 \end{pmatrix} \middle| x_1, x_2, y_1, y_2 \in K,\ \ i \in \{1, 2\},\ \ x_i, y_i \neq 0 \right\}.$$

Note that every minor, besides the empty minor, of this biact must contain the zero matrix. This implies that H_B is not a minor. With the previous note in mind, it becomes evident that, by definiton, the minimal minor in this biact can only be the principal minor of the zero matrice, which consists of only the zero matrice itself. Though if we would modify the example with limiting S and T to consist of only regular square matrices (matrices with a non-zero determinant) of respective size and remove the zero matrix of size 2×3 from our biact, then one can check, that that all minimal minors in this biact must be generated by a matrix with a single non-zero entry.

4.2 S-unitality

In his work, O. Steinfeld had the benefit of proving a large variety of results about minimal quasi-ideals with the use of idempotents in a semigroup. However it is clear that the use of such elements is far more complicated in the context of biacts, as a biact itself (or even a one-sided act) does not generally have a binary algebraic operation defined on it. This would imply that if we were to attempt to generalize the results from [3], which involve the use of idempotents, then we should find some efficient generalization of an idempotent. In our attempts to solve this problem, we propose the following generalization, which is motivated by s-unital elements for rings, described by H. Tominaga in [4].

Definition 4.5. Let ${}_SA_T$ be a biact. We say that an element $a \in A$ is **s-unital** if there exist $s_0 \in S$ and $t_0 \in T$, such that

$$a = s_0 a = a t_0.$$

If $a = sa = at$ for all $s \in S$ and $t \in T$, then we say a is a **zero element**. If $X \subseteq A$ and every $x \in X$ is s-unital, then we say, that X is s-unital.

Let us note that, for any idempotent e of a semigroup S, if we view S as the biact ${}_S S_S$, described in Remark 1.11, we notice that $e \in {}_S S_S$ is s-unital. This fact asserts that s-unitality generalizes idempotency although in a very weak way.

Proposition 4.6. *If X is an s-unital subset of a biact ${}_S A_T$, then $X \subseteq SX \cap XT$.*

Proof. Suppose X is an s-unital subset of the biact ${}_S A_T$. This implies that, for any $x \in X$, there exist $s_0 \in S$ and $t_0 \in T$ with

$$x = s_0 x \in SX,$$
$$x = x t_0 \in XT.$$

thus $x \in SX \cap XT$. □

Corollary 4.7. *If M is an s-unital minor of a biact ${}_S A_T$, then*

$$M = SM \cap MT.$$

In the following proposition we will show that, for a minor, minimality is a stronger condition than s-unitality. We are comparing these properties in order to have some insight, whether s-unitality is a good enough property to generalize the results in [3], which use idempotents.

Proposition 4.8. *Every minimal minor of a biact is s-unital.*

Proof. Suppose S and T are semigroups and A is an (S,T)-biact. Now let M be a minimal minor of ${}_SA_T$. We can consider two cases.

1. $|M| = 1$. In this case M must only contain a zero element. This zero element is s-unital.

2. $|M| > 1$. In this case, let $u \in M$ be arbitrary and $v \in M \setminus \{u\}$. Considering that M is minimal, then Proposition 4.3 gives us $M = H_u$. Now $v \in H_u$, implies the existence of $s, s' \in S^1$ and $t, t' \in T^1$ such that

   $$u = sv = vt, \qquad v = s'u = ut'.$$

   Although the assumption $u \neq v$ will let us specify $s, s' \in S$ and $t, t' \in T$. With this last note we can conclude

   $$u = s_0 u = u t_0,$$

   where $s_0 := ss' \in S$ and $t_0 := t't \in T$. This means that $u \in M$ is s-unital. Due to $u \in M$ being arbitrary, M itself must be s-unital.

This concludes that, for any cardinality of M, we have that M is s-unital, finishing the proof. $\square$

We will now give an example to show that s-unitality of a minor does not imply its minimality.

Example 4.9. Let the semigroup S be defined as odd natural numbers with standard multiplication. We can now choose our biact as ${}_S\mathbb{N}_S$ with both the left- and right action defined as standard multiplication. This definiton is correct, due to the associativity of integer multiplication. Now note that the element $1 \in S$ is such that it preserves all elements of $\mathbb{N}$ by both actions. This means that $\mathbb{N}$ is s-unital. Furthermore, this is (even though trivial) a minor of ${}_S\mathbb{N}_S$. Now we also have that

$S \subset \mathbb{N}$ is a non-empty minor of ${}_S\mathbb{N}_S$. This shows that $\mathbb{N}$ is such an s-unital minor of ${}_S\mathbb{N}_S$, which is not minimal, as it contains a proper minor S.

This example showed us that even in the case of a unique $s_0 \in S$, which preserves all elements of the act, would an s-unital minor not necessarily be minimal.

Conclusions

In the given thesis, we managed to generalize those results of [3], which do not involve the use of semigroup idempotents. Moreover, we have made necessary adjustments in order to prove Theorem 3.9. The rest of the content in [3] would demand far more precise tools with the likely need to describe ordering of s-unital elements. It is unclear, whether doing this would provide useful.

For the proof of the proposition "A quasi-ideal is minimal if and only if it is an equivalence class of $\mathcal{H}$", Steinfeld used an idempotent of a semigroup. We generalized this result (Proposition 4.3) without having to resort to s-unital elements of a biact. This hints at the possibilities that perhaps there are alternative representations of a semigroups (which do not rely on idempotents), which might have more clear or more efficient generalizations in the context of biacts.

We have also given multiple examples, highlighting the presence of biacts and minors of a biact, in rather natural topics in mathematics. One may suggest that a further study on any example among these, could provide interesting results.

References

[1] J. Howie, Fundamentals of Semigroup Theory, Oxford University Press, New York, 1995.

[2] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, Acts and Categories, Walter de Gruyter & Co., Berlin, 2000.

[3] O. Steinfeld, Quasi-ideals in Rings and Semigroups, Disquisitiones Mathematicae Hungaricae 10, Akadémiai Kiadó, Budapest, 1978.

[4] H. Tominaga, On s-unital rings, Math. J. Okayama Univ. 18 (1976), 117–134.

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