DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
37

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37

APPROXIMATION METHODS FOR WEAKLY SINGULAR INTEGRAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Contents

Chapter 1

Introduction

We will be concerned with the numerical solution of linear weakly singular integral equations of the second kind of the form

$$u(t) - \int_a^b K(t,s)u(s)ds = f(t)\,, \quad t \in [a,b]\,. \tag{1.1}$$

Here K is called the kernel and assumed to be known, and the function f, called the free term (or the forcing function), is also assumed known. The unknown function is denoted by u. We denote the limits of integration by a and b here, but we shall usually take $a = 0$ later.

If the kernel $K(t,s)$ of equation (1.1) is the product of a smooth function $g(t,s)$ and an unbounded (but integrable) function $|t-s|^{-\alpha}$,

$$K(t,s) = g(t,s)|t-s|^{-\alpha}\,, \quad 0 < \alpha < 1\,, \tag{1.2}$$

then we speak of a weakly singular integral equation (see, for example, [8, 10]).

However, we shall later attach the description "weakly singular" also to integral equations with kernels in more wide settings. So, we speak that in [69] (cf. also [35, 56]) is studied a class of weakly singular integral equations with kernels of the form

$$K(t,s) = g(t,s)\kappa(t-s)\,, \tag{1.3}$$

where[1]

$$g \in C^m([a,b] \times [a,b])\,, \quad \kappa \in C^{m-1}([a-b, b-a] \setminus \{0\})\,, \quad m \geq 1\,, \tag{1.4}$$

and

$$\left|\kappa^{(m-1)}(\tau)\right| \leq c|\tau|^{-\beta}\,, \quad 0 < \beta < m\,. \tag{1.5}$$

Here c is a positive constant and $\tau \in [a-b, b-a] \setminus \{0\}$. If $m-1 < \beta < m$, then it follows from (1.3)–(1.5) that the kernel $K(t,s)$ may have a weak singularity at $t = s$:

$$|K(t,s)| \leq c|t-s|^{-\beta+m-1}\,.$$

Moreover, a differentiation of a weakly singular kernel (1.2) increases the order of the singularity. For example, if $g \in C^1([a,b] \times [a,b])$, then

$$\frac{\partial}{\partial t}K(t,s) = \frac{\partial g(t,s)}{\partial t}|t-s|^{-\alpha} - \alpha g(t,s)|t-s|^{-\alpha-1}$$

behaves like $|t-s|^{-\alpha-1}$ for $t \to s$. On the other hand,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)K(t,s) = \left[\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)g(t,s)\right]|t-s|^{-\alpha}\,.$$

These observations motivated G. Vainikko in 1993 (see [63]) to introduce the smoothness assumption about the weakly singular kernel

$$K(t,s) = \kappa(t,s) \tag{1.6}$$

in the following form: $\kappa(t,s)$ is m times ($m \geq 1$) continuously differentiable on $(a,b) \times (a,b) \setminus \{t = s\}$ and there exists a real number $\nu \in (-\infty, 1)$ such that the estimate

$$\left|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \kappa(t,s)\right| \leq c \begin{cases} 1\,, & \nu + i < 0\,, \\ 1 + |\ln|t-s||\,, & \nu + i = 0\,, \\ |t-s|^{-\nu-i}\,, & \nu + i > 0 \end{cases} \tag{1.7}$$

holds with a positive constant $c = c(\kappa)$ for all $t, s \in (a,b)$, $t \neq s$, and for all non-negative integers i and j, such that $i + j \leq m$. Putting $i = j = 0$, (1.7) yields

$$|\kappa(t,s)| \leq c \begin{cases} 1\,, & \nu < 0\,, \\ 1 + |\ln|t-s||\,, & \nu = 0\,, \\ |t-s|^{-\nu}\,, & \nu > 0\,. \end{cases} \tag{1.8}$$

[1]By $C^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and p is a nonnegative integer, is denoted the set of p times continuously differentiable functions on Ω (later abbreviate $C^0(\Omega)$ by $C(\Omega)$).

It follows from (1.8) that in case $0 \le \nu < 1$ the function $\kappa(t,s)$ may have a weak singularity at $t = s$. In particular, $\kappa(t,s)$ may have the form

$$\kappa(t,s) = \kappa_1(t,s) \ln|t-s| + \kappa_2(t,s)\,, \tag{1.9}$$

or

$$\kappa(t,s) = \kappa_1(t,s)|t-s|^{-\alpha} + \kappa_2(t,s)\,, \quad 0 < \alpha < 1\,, \tag{1.10}$$

or, more generally,

$$\kappa(t,s) = \kappa_1(t,s) \frac{|\ln|t-s||^{\beta}}{|t-s|^{\alpha}} + \kappa_2(t,s)\,,$$
$$0 \le \beta < \infty\,,\ 0 \le \alpha < 1\,,\ \alpha + \beta \ne 0\,, \tag{1.11}$$

where κ_1 and κ_2 are some smooth functions on $[a,b] \times [a,b]$ without singularities. For such functions the condition (1.7) holds: in case (1.9) we can set $\nu = 0$; in cases (1.10) and (1.11) we can take correspondingly $\nu = \alpha$ and $\nu = \alpha + \varepsilon$ with some small $\varepsilon > 0$ ($\alpha + \varepsilon < 1$).

In this thesis we assume that the kernel $K(t,s)$ of equation (1.1) has the form

$$K(t,s) = g(t,s)\kappa(t,s)\,. \tag{1.12}$$

We assume that 1) $g(t,s)$ is m times ($m \ge 1$) continuously differentiable with respect to t and s for $t \in [a,b]$, $s \in [a,b] \setminus \{d\}$, $a < d < b$, and its derivatives are bounded in the regions $[a,b] \times [a,d)$ and $[a,b] \times (d,b]$; 2) $\kappa(t,s)$ is m times continuously differentiable with respect to t and s for $t, s \in [a,b]$, $t \ne s$ and satisfies the condition (1.7) with a real number $\nu < 1$ for all $t, s \in [a,b]$, $t \ne s$, and for all nonnegative integers i and j such that $i + j \le m$.

Integral equations of this kind arise from potential problems [24, 33], Dirichlet problems [33], the description of hydrodynamic interaction between elements of a polymer chain in solution [49], nuclear physics [9] and atmosphere physics [7, 11, 22, 61].

Most integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. The integral equations (1.1) and methods for their approximate solution have been studied by many authors. First of all we refer here to the monographs [6, 8, 15, 31, 32, 63, 69] and the literature given therein, see also [1, 4, 12, 30]. Our aim is a generalization of some results obtained in [35, 56, 63, 69] to the case of equations (1.1) with kernels of the form (1.12).

When analyzing the convergence of a numerical method for a given integral equation one needs information about the smoothness of the exact solution. This becomes more significant when one wants to find the maximum order of convergence of a method. For equations (1.1) with smooth kernels, the smoothness of the kernel K and the forcing function f determines the smoothness of the solution u on the closed interval $[a, b]$. If we allow weakly singular kernels of type (1.12), with smooth coefficient function g: $[a, b] \times [a, b] \to \mathbb{R}$, then the resulting solutions are typically nonsmooth at the endpoints of the interval of integration $[a, b]$, where their derivatives become unbounded [42, 63, 65, 69], see also [25, 27, 45, 47, 50]. If g is proposed to be smooth only on $[a, b] \times ([a, b] \setminus \{d\})$, where $a < d < b$, then the derivatives of the solution $u(t)$ of equation (1.1) may have singularities at $t = d$, also [44, 57, 63]. This complicates the construction of high order methods for the numerical solution of such equations.

This thesis is concerned with the construction of effective methods for the numerical solution of equations (1.1) with kernels of type (1.12). In particular, numerical methods based on collocation techniques will be considered. Global convergence estimates are derived and the attainable order of local superconvergence is analyzed. In order to calculate the approximate solution by piecewise polynomial collocation method it is necessary to solve large linear systems. We present a two-grid iteration scheme for the solution of such systems and show fast convergence of this method. We also study a quadrature formula method and its adaptation to the case of a weakly singular kernel of type (1.12). Finally we give numerical examples to illustrate the theoretical estimates.

In the following we give a review of this thesis by chapters. Chapters 1–3 have introductory character. In Chapter 2 we introduce some notation and basic results which we will use in this work.

In Chapter 3 we introduce the basic assumptions which we will use during this work and present a basic result characterizing the regularity properties of solutions of equation (1.1). Following [61] we also present an auxiliary result about the conditions under which a weakly singular integral operator is compact as an operator from $L^p(0, b)$ to $C[0, b]$.

In Chapter 4 we consider the numerical solution of equation (1.1) by quadrature methods. Quadrature methods involve the replacement of the integral in equation (1.1) by a quadrature approximation, the resulting equation being then satisfied at selected points. A general discussion of this approach can be found in [1, 4, 6, 15, 31, 59]. The direct application of the standard quadrature method to a weakly singular equation (1.1) may not work as we are now dealing with an unbounded integrand. One of the techniques for treating integral equations with badly behaved integrands is singularity subtraction: equation (1.1) can be written in the form

$$u(t) - \int_0^b K(t,s)[u(s) - u(t)]ds - u(t) \int_0^b K(t,s)ds = f(t)\,,$$

and quadrature-type methods can then be applied to this new equation. This technique was suggested by Kantorovich and Krylov [29]. We study the convergence of the singularity subtraction method based on the mid-point rule of numerical quadrature. The main result of this chapter is formulated in Theorem 4.2. This result is not published yet.

Chapter 5 is devoted to the background material which will be used in Chapters 6 and 7. First we introduce the spline space and a graded grid. Then we derive the estimates for interpolation error on quasi-uniform and graded-grids, which we will use later for convergence analysis. These results are partly published in [16, 18, 19]. We note in passing that Rice (see [46]) seems to be first who introduced and used graded grids for approximation of functions with singularities.

Chapters 6–9 are concerned with high order methods for the numerical solution of equation (1.1) with kernels of type (1.12). In Chapter 6 we study a piecewise polynomial collocation method. A common numerical technique for solving integral equations of the second kind is to look for an approximate solution in the form of a linear combination of certain functions, and then to select a particular linear combination by forcing the approximate solution to satisfy the integral equation at a selected set of points in the integration region. This defines the collocation method, and the points referred to in the definition are the collocation points. This method is one of the most efficient numerical approach for the solution of equation (1.1). By taking the local polynomial degree m sufficiently large, we can obtain an order of convergence as high as we like. In this chapter we extend the results of [16, 18, 19]. We carry out an analysis of the convergence properties of collocation approximations in $S_{m-1}^{(-1)}(\Delta_N)$ to the solution of the integral equations {(1.1), (1.12)}, both for quasi-uniform sequences of meshes and for graded meshes. Note that piecewise polynomial collocation methods for integral equations are studied, for example, in [6, 23, 26, 34, 35, 36, 38, 40, 54, 55, 58, 63, 66, 68, 69].

In Chapter 7 we examine the attainable order of convergence of collocation approximationes at the collocation points. This chapter is an extended expansion of the corresponding results of [18, 19, 20]. Using special collocation points, error estimates at the collocation points are given, showing a more rapid convergence as the global uniform convergence in $[a, b]$ available by piecewise polynomials. The main result of this chapter is formulated in Theorem 7.1. We refer also to the works [41, 43, 63, 69], where the local superconvergence of piecewise polynomial collocation approximations for integral equations with weakly singular kernels is studied. Note that there are quite many other works about superconvergence of the collocation method, but mostly they concern equations with a smooth kernel without singularities, or, at least, without singularities of the exact solution. We refer to [63] for a more detailed discussion in this connection.

In Chapter 8 a two-grid method is constructed for solving large systems arising from application of a collocation method. This chapter is based on our papers [17, 21]. Two-grid iteration methods similar to our treatment are considered in [51, 52, 53, 62, 69]. Although two-grid methods had already been described in the early 1960's, it was not until the mid-seventies that they were realized to be very efficient methods of solution with a broad area of application (see [5, 6, 63]). A systematic treatment of two-grid methods can be found in [14, 15].

In Chapter 9 we consider the numerical solution of equation (1.1) by Galerkin method. In this chapter we extend the results of [16, 20]. The convergence analysis of Galerkin method is based on the results of Chapter 6. We refer also to the works [6, 13, 15, 30, 69], where similar results in some special cases of weakly singular equations are obtained.

In Chapter 10 a series of numerical tests is given to support theoretical results obtained.

Chapter 2

Notation and Basic Results

In this chapter we collect some notation and basic known results, which we will use in this work. We refer to [6, 15, 28, 31, 48, 59, 60].

2.1 Some Notation

Throughout this work we denote by $c, c_0, c_1, \ldots$ real constants, which may be different at different places. By $\mathbb{N} = \{1, 2, \ldots\}$ we denote the set of all positive integers and by $\mathbb{R}^n$ $(n \in \mathbb{N})$ the set of vectors with n real-valued components,

$$\mathbb{R}^1 = \mathbb{R} = (-\infty; \infty)\,, \quad \mathbb{N}_0 = \{0\} \cup \mathbb{N}\,.$$

For $\Omega \subset \mathbb{R}^n$, by $C^m(\Omega)$ we denote the set of m times continuously differentiable functions $x : \Omega \to \mathbb{R}$, $C^0(\Omega) = C(\Omega)$. The set $C[a, b]$ of continuous functions $x : [a, b] \to \mathbb{R}$ is a Banach space with respect to the norm

$$\|x\|_{C[a,b]} = \max_{a \le t \le b} |x(t)|\,, \quad x \in C[a, b]\,.$$

For $p \in [1, \infty)$, by $L^p(a, b)$ we denote the linear space of measurable functions $x : [a, b] \to \mathbb{R}$, such that

$$\int_a^b |x(t)|^p dt < \infty\,.$$

It is a Banach space with respect to the norm

$$\|x\|_{L^p(a,b)} = \Big(\int\limits_a^b |x(t)|^p dt \Big)^{1/p}, \quad x \in L^p(a,b).$$

By $L^\infty(a,b)$ we represent the set of measurable functions $x : [a,b] \to \mathbb{R}$, such that

$$\inf_{\Omega \subset [a,b] : \mu(\Omega)=0} \ \sup_{t \in [a,b] \setminus \Omega} |x(t)| < \infty$$

(here $\mu(\Omega)$ is the Lebesgue measure of the set Ω). The set $L^\infty(a,b)$ is a Banach space with the norm

$$\|x\|_{L^\infty(0,b)} = \inf_{\Omega \subset [a,b] : \mu(\Omega)=0} \ \sup_{t \in [a,b] \setminus \Omega} |x(t)|, \quad x \in L^\infty(a,b).$$

2.2 Some Results for Linear Operators

Let E and F be Banach spaces. By $\mathcal{L}(E,F)$ we denote the Banach space of all linear continuous (=linear bounded) operators $A : E \to F$ with the norm

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{x \in E, \|x\| \le 1} \|Ax\|.$$

For an operator $A : E \to F$ let $A(E) = \{Ax : x \in E\} \subset F$ be the range of A. If for each $y \in A(E)$ there is only one element $x \in E$ with $Ax = y$, then A is said to be injective. If $A(E) = F$, then A is said to be surjective. If A is injective and surjective then A has an inverse operator $A^{-1} : F \to E$.

Theorem 2.1 [6] *Let E be a Banach space, and let $A \in \mathcal{L}(E,E)$ be a bounded linear operator from E into E with $\|A\|_{\mathcal{L}(E,E)} < 1$. Then there exists $(I-A)^{-1} \in \mathcal{L}(E,E)$, and*

$$\big\|(I-A)^{-1}\big\|_{\mathcal{L}(E,E)} \le \frac{1}{1 - \|A\|_{\mathcal{L}(E,E)}},$$

where I is the identity mapping in E.

Theorem 2.2 [6] *Let E and F be Banach spaces, and let $A, A_N \in \mathcal{L}(E,F)$ ($N \in \mathbb{N}$) be bounded linear operators. Let D be a dense subset of E. Then in order that $A_N x \to Ax$, $N \to \infty$ for all $x \in E$, it is necessary and sufficient that*

1) $\exists\, c \in \mathbb{R}$, *so that* $\|A_N\|_{\mathcal{L}(E,F)} \le c \quad \forall\, N \in \mathbb{N}$*;*

2) $A_N x \to Ax$, $N \to \infty \;\forall\, x \in D$*.*

A linear operator $A : E \to F$ is called compact (or completely continuous) if A transforms every bounded set of E into a relatively compact set of F. A linear compact operator $A : E \to F$ is continuous. A subset $M \subset E$ is called relatively compact, if all sequences $\{x_k\} \subset M$ contain a subsequence converging in E.

By $\mathcal{L}_c(E,F)$ we shall denote the set of all compact linear operators from E to F.

Theorem 2.3 [6, 48] *For a closed interval* $[a,b]$*, a set* $M \subset C[a,b]$ *is relatively compact in* $C[a,b]$ *if and only if the functions* $u \in M$ *are (a) uniformly bounded and (b) equicontinuous, i.e.,*

(a) there is a constant c *such that* $|u(x)| \le c$ *(*$x \in [a,b]$*) for all* $u \in M$*;*

(b) for any $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $x_1, x_2 \in [a,b]$ *with* $|x_1 - x_2| < \delta$ *imply* $|u(x_1) - u(x_2)| < \varepsilon$ *for all* $u \in M$*.*

Theorem 2.4 [6] *Let* E *be a Banach space, and let* $T \in \mathcal{L}(E,E)$ *be compact. Then the equation* $u = Tu + f$ *has a unique solution* $u \in E$ *if and only if the homogeneous equation* $u = Tu$ *has only the trivial solution* $u = 0$*. In such a case, the operator* $I - T$ *has a bounded inverse* $(I-T)^{-1} \in \mathcal{L}(E,E)$*.*

2.3 Outlines of Discrete Convergence Theory

Let E and E_N ($N \in \mathbb{N}$) be Banach spaces and let $\mathcal{P} = \{p_N\}$ be a system of operators $p_N \in \mathcal{L}(E, E_N)$ ($N \in \mathbb{N}$) such that

$$\left\|p_N u\right\|_{E_N} \to \|u\|_E \quad \text{as} \quad N \to \infty \tag{2.1}$$

for every $u \in E$; in this case operators p_N are called connection operators.

We say that a sequence $\{u_N\}_{N \in \mathbb{N}'}$ with $u_N \in E_N$ ($N \in \mathbb{N}'$, $\mathbb{N}' \subset \mathbb{N}$) $\mathcal{P}$-converges (or converges discretely) to $u \in E$ if

$$\|u_N - p_N u\|_{E_N} \to 0 \quad \text{as} \quad N \to \infty\,; \tag{2.2}$$

in this case we write $u_N \xrightarrow{\mathcal{P}} u$ $(N \in \mathbb{N}')$. A sequence $\{u_N\}_{N\in\mathbb{N}}$ of elements $u_N \in E_N$ is called $\mathcal{P}$-compact (or discretely compact) if for any $\mathbb{N}' \subset \mathbb{N}$ there exists $\mathbb{N}'' \subset \mathbb{N}'$ such that the sequence $\{u_N\}_{N\in\mathbb{N}''}$ $\mathcal{P}$-converges. Note that if a sequence $\{u^N\}$, $u^N \in E$, $N \in \mathbb{N}$ is relatively compact in the space E then the sequence $\{p_N u^N\}$ is $\mathcal{P}$-compact.

A sequence $\{T_N\}_{N\in\mathbb{N}}$ of linear bounded operators $T_N \in \mathcal{L}(E_N, E_N)$ converges discretely (or $\mathcal{P}$-converges) to an operator $T \in \mathcal{L}(E, E)$, if for any $\mathcal{P}$-convergent sequence $\{u_N\}$ the following implication holds:

$$u_N \xrightarrow{\mathcal{P}} u \quad \Rightarrow \quad T_N u_N \xrightarrow{\mathcal{P}} Tu\,; \tag{2.3}$$

we write $T_N \xrightarrow{\mathcal{P}} T$ $(N \in \mathbb{N})$.

Theorem 2.5 [59, 60] *For $T \in \mathcal{L}(E, E)$, $T_N \in \mathcal{L}(E_N, E_N)$, $N \in \mathbb{N}$, the following conditions (i), (ii) and (iii) are equivalent:*

(i) a sequence of operators $T_N \in \mathcal{L}(E_N, E_N)$, is discretely convergent to operator $T \in \mathcal{L}(E, E)$:

$$T_N \xrightarrow{\mathcal{P}} T \quad (N \in \mathbb{N})\,;$$

(ii) there exists a constant $c > 0$ which does not depend on N so that

$$\|T_N\|_{\mathcal{L}(E_N,E_N)} \le c \quad (N \in \mathbb{N})$$

and

$$\|T_N p_N u - p_N T u\|_{E_N} \to 0 \quad (N \in \mathbb{N})$$

for every $u \in E$;

(iii) $\|T_N\|_{\mathcal{L}(E_N,E_N)} \le c$ $(N \in \mathbb{N})$, and there is a dense set $E' \subset E$ such that

$$\|T_N p_N u - p_N T u\|_{E_N} \to 0 \quad (N \in \mathbb{N})$$

for every $u \in E'$.

Further, we say that a sequence $\{T_N\}_{N\in\mathbb{N}}$ of operators $T_N \in \mathcal{L}(E_N, E_N)$ converges compactly to an operator $T \in \mathcal{L}(E, E)$, if $T_N \xrightarrow{\mathcal{P}} T$ $(N \in \mathbb{N})$ and the following compactness condition holds:

$$\xi_N \in E_N, \ \ \|\xi_N\|_{E_N} \le c \ \ (N \in \mathbb{N}) \Rightarrow \{T_N \xi_N\}_{N\in\mathbb{N}} \ \text{ is } \ \mathcal{P}-\text{compact}\,; \tag{2.4}$$

in this case we simply write $T_N \to T$ compactly.

Consider an equation

$$u = Tu + f\,, \tag{2.5}$$

where $f \in E$ and $T \in \mathcal{L}(E,E)$. We approximate (2.5) by the equations

$$u_N = T_N u_N + p_N f \quad (N \in \mathbb{N})\,, \tag{2.6}$$

with $T_N \in \mathcal{L}(E_N, E_N)$ and $p_N \in \mathcal{L}(E, E_N)$ satisfying the condition (2.1). Assume that equation (2.5) is uniquely solvable. We are interested in unique solvability of (2.6) for a sufficiently large N and in the $\mathcal{P}$-convergence $u_N \xrightarrow{\mathcal{P}} u$, where $u \in E$ and $u_N \in E_N$ are the solutions of equations (2.5) and (2.6), respectively.

Theorem 2.6 [59, 60] *Assume that the following conditions are fulfilled:*

1) E, E_N ($N \in \mathbb{N}$) are Banach spaces;

2) $f \in E$;

3) $T \in \mathcal{L}_c(E,E)$;

4) equation $u = Tu$ has only the trivial solution $u = 0$;

5) $T_N \in \mathcal{L}_c(E_N, E_N)$ ($N \in \mathbb{N}$);

6) $p_N \in \mathcal{L}(E, E_N)$ ($N \in \mathbb{N}$) satisfy the condition (2.1);

7) $T_N \to T$ compactly.

Then equation (2.5) has a unique solution $u^* \in E$ and, for all sufficiently large N, say $N \geq N_0$, equation (2.6) has a unique solution $u_N^* \in E_N$ and $u_N^* \xrightarrow{\mathcal{P}} u^*$ ($N \in \mathbb{N}$), i.e.

$$\|u_N^* - p_N u^*\|_{E_N} \to 0 \quad as \quad N \to \infty\,. \tag{2.7}$$

The following error estimate holds:

$$c_1 \varepsilon_N \leq \|p_N u^* - u_N^*\|_{E_N} \leq c_2 \varepsilon_N\,, \quad N \geq N_0\,, \tag{2.8}$$

where c_1 and c_2 are some positive constants independent of N, and

$$\varepsilon_N = \|T_N p_N u^* - p_N T u^*\|_{E_N}\,. \tag{2.9}$$

Chapter 3

Integral Equation

In this chapter we describe the integral equation which is the subject of this work.

3.1 Compactness of Integral Operators with Weakly Singular Kernels

Consider an integral operator T, defined by the formula

$$(Tu)(t) = \int_0^b K(t,s)u(s)ds\,, \quad t \in [0,b]\,. \tag{3.1}$$

Next we prove a theorem which in special cases have been proved in [28, 61]. In our proof we have followed the approach of [61].

Theorem 3.1 *Assume that the following conditions are fulfilled:*

(i) $K(t,s) = g(t,s)\kappa(t,s)$*;*

(ii) the function $g = g(t,s)$ *is continuous and bounded for* $(t,s) \in [0,b] \times ([0,b] \setminus \{d\})$*, where* $d \in (0,b)$ *is a fixed point in the interval* $(0,b)$*;*

(iii) the function $\kappa = \kappa(t,s)$ *is continuous everywhere on* $[0,b] \times [0,b]$ *except the diagonal* $t = s$ *where a weak singularity is allowed: there*

exist an ν, $0 < \nu < 1$, and a positive constant c such that the estimate

$$|\kappa(t,s)| \leq c\,|t-s|^{-\nu}$$

holds for all $t, s \in [0,b]$, $t \neq s$.

Then T is linear and compact as an operator from $L^p(0,b)$ to $C[0,b]$ for $p > 1/(1-\nu)$.

Proof. It follows from the conditions (i)–(iii) that $Tu \in C[0,b]$ for $u \in L^p(0,b)$, $p > 1/(1-\nu)$. According to the definition of a compact operator we have to show that T maps the unit ball

$$B_p = \{u \in L^p(0,b) : \|u\|_{L^p(0,b)} \leq 1\}$$

of the space $L^p(0,b)$, $p > 1/(1-\nu)$, into the relatively compact set of the space $C[0,b]$. According to Theorem 2.3, we have to show that the functions $v = Tu$ $(u \in B_p)$ are uniformly bounded and equicontinuous.

Let $t \in [0,b]$ and q be such that $1/p + 1/q = 1$. Then $1 - \nu q > 0$ and for every $t \in [0,b]$ we have

$$\int_0^b \big|K(t,s)\big|^q ds \leq c \int_0^b |t-s|^{-\nu q} ds < \infty\,.$$

Using the Hölder's inequality we obtain for $u \in B_p$ that

$$\begin{aligned} |v(t)| &= \Big|\int_0^b K(t,s)u(s)ds\Big| \\ &\leq \Big(\int_0^b |K(t,s)|^q ds\Big)^{1/q} \Big(\int_0^b |u(s)|^p ds\Big)^{1/p} \\ &\leq c\Big(\int_0^b |t-s|^{-\nu q} ds\Big)^{1/q} \leq c_1\,, \quad t \in [0,b]\,. \end{aligned}$$

Thus, functions $v = Tu$ $(u \in B_p)$ are uniformly bounded.

Next we show that the functions $v = Tu$ $(u \in B_p)$ are equicontinuous. Let $t_1, t_2 \in [0,b]$, $u \in B_p$ and $p^{-1} + q^{-1} = 1$. With the help of the Hölder's inequality we have

$$\begin{aligned}\big|v(t_1)-v(t_2)\big| &= \int_0^b \big|K(t_1,s)-K(t_2,s)\big||u(s)|ds\\ &\le \Big\{\int_0^b \big|K(t_1,s)-K(t_2,s)\big|^q ds\Big\}^{1/q}\big\{|u(s)|^p ds\big\}^{1/p}\\ &\le \Big\{\int_0^b \big|K(t_1,s)-K(t_2,s)\big|^q ds\Big\}^{1/q}.\end{aligned}$$

Our purpose is to show that for any $\varepsilon>0$ there is a $\eta=\eta(\varepsilon)>0$ such that

$$\int_0^b \big|K(t_1,s)-K(t_2,s)\big|^q ds \le \varepsilon^q$$

for $t_1,t_2\in[0,b]$, $|t_1-t_2|\le\eta$.

For $t_1,t_2\in[0,b]$, $\delta>0$ we introduce a set

$$U_\delta(t_1,t_2,d)=(t_1-\delta,t_1+\delta)\cup(t_2-\delta,t_2+\delta)\cup(d-\delta,d+\delta)\,.$$

Due to the assumptions (i)–(iii) we can pick $\delta=\delta(\varepsilon)$ so that

$$\int_{[0,b]\cap U_\delta(t_1,t_2,d)} \big|K(t_1,s)-K(t_2,s)\big|^q ds \le \frac{1}{2}\varepsilon^q \tag{3.2}$$

for any $t_1,t_2\in[0,b]$. Indeed, we have

$$\begin{aligned}\int_{[0,b]\cap U_\delta(t_1,t_2,d)} \big|K(t_1,s)-K(t_2,s)\big|^q ds &\le \int_{[0,b]\cap U_\delta(t_1,t_2,d)} \big(\big|K(t_1,s)\big|+\big|K(t_2,s)\big|\big)^q ds\\ &\le c\int_{U_\delta(t_1,t_2,d)} (|t_1-s|^{-\nu}+|t_2-s|^{-\nu})^q ds\\ &\le c_1\int_{t_0-\delta}^{t_0+\delta} |t_0-s|^{-\nu q} ds\,.\end{aligned}$$

Since

$$\int_{t_0-\delta}^{t_0+\delta} |t_0-s|^{-\nu q} ds = \frac{2}{1-\nu q}\delta^{1-\nu q}$$

with $1-\nu q>0$, we can choose δ so small that the estimate (3.2) holds.

In the integral

$$\int\limits_{[0,b]\setminus U_\delta(t_1,t_2,d)} \big|K(t_1,s)-K(t_2,s)\big|^q ds$$

we have $|s-t_1|\geq\delta$ and $|s-t_2|\geq\delta$. Since $K(t,s)$ is uniformly continuous on $\{t,s\in[0,b]: |t-s|\geq\delta,\ |s-d|\geq\delta\}$ for a small $\delta>0$, we get

$$\int\limits_{[0,b]\setminus U_\delta(t_1,t_2,d)} \big|K(t_1,s)-K(t_2,s)\big|^q ds \leq \frac{1}{2}\varepsilon^q,$$

for $t_1,t_2\in[0,b]$, $|t_1-t_2|\leq\eta$, $\eta=\eta(\delta,\varepsilon)>0$.

Since $\delta=\delta(\varepsilon)$, we obtain that η depends only on ε and for every $\varepsilon>0$ there is a $\eta>0$ such that

$$\big|v(t_1)-v(t_2)\big|\leq\varepsilon$$

for $t_1,t_2\in[0,b]$, $|t_1-t_2|\leq\eta$, $u\in B_p$. In other words, the functions $v=Tu$ ($u\in B_p$) are equicontinuous. □

3.2 Integral Equation

Throughout this work we consider a linear integral equation of the second kind of the form

$$u(t)-\int\limits_0^b K(t,s)u(s)ds=f(t)\,,\quad 0\leq t\leq b\,,\ b>0\,, \tag{3.3}$$

where K and f are given functions. The main results obtained concern with kernels of the form

$$K(t,s)=g(t,s)\kappa(t,s) \tag{3.4}$$

with g and κ satisfying the assumptions (A1) and (A2), respectively.

(A1) The function $g=g(t,s)$ is m times ($m\geq1$) continuously differentiable with respect to t and s for $t\in[0,b]$, $s\in[0,b]\setminus\{d\}$, $0<d<b$, and its derivatives are bounded in the regions $[0,b]\times[0,d)$ and $[0,b]\times(d,b]$. Let p ($0\leq p\leq m$) be an integer defined as follows: $p=0$ if g may have a discontinuity across the line $s=d$; $p\geq1$ if $g\in C^{p-1}[0,b]$.

(A2) The function $\kappa = \kappa(t,s)$ is m times ($m \in \mathbb{N}$ is fixed in the assumption (A1)) continuously differentiable with respect to t and s for $t, s \in [0,b]$, $t \neq s$, and there exists a real number ν, $-\infty < \nu < 1$, such that the estimate

$$\left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j \kappa(t,s) \right| \leq c \begin{cases} 1\,, & \text{if} \quad \nu + i < 0\,, \\ 1 + |\ln|t-s||\,, & \text{if} \quad \nu + i = 0\,, \\ |t-s|^{-\nu-i}, & \text{if} \quad \nu + i > 0\,, \end{cases} \tag{3.5}$$

holds with a positive constant c for all $t, s \in [0,b]$, $t \neq s$ and for all $i, j \in \mathbb{N}_0$, $i + j \leq m$.

About the forcing function f we shall assume that it is at least continuous on $[0,b]$ (we usually assume that $f \in C^{m,\nu}_{d,p}[0,b]$, see Section 3.3).

Remark 3.1 It follows from the assumption (A1) that $g \in C^{p-1}([0,b] \times [0,b])$ for $1 \leq p \leq m$, i.e. g is a continuous function on $[0,b] \times [0,b]$. If $p = 0$ then $g(t,s)$ may be discontinuous at $(t,s) = (t,d)$, but there exist one-sided limits

$$g(t, d-0) = \lim_{s<d, s \to d} g(t,s) \tag{3.6}$$

and

$$g(t, d+0) = \lim_{s>d, s \to d} g(t,s) \tag{3.7}$$

for every $t \in [0,b]$. Thus, after extension to the point $(t,s) = (t,d)$, $t \in [0,b]$,

$$\tilde{g}(t,s) = \begin{cases} g(t,s)\,, & 0 \leq t \leq b\,,\ 0 \leq s < d\,, \\ g(t,d-0)\,, & 0 \leq t \leq b\,,\ s = d\,, \end{cases} \tag{3.8}$$

$$\hat{g}(t,s) = \begin{cases} g(t,s)\,, & 0 \leq t \leq b\,,\ d < s \leq b\,, \\ g(t,d+0)\,, & 0 \leq t \leq b\,,\ s = d\,, \end{cases} \tag{3.9}$$

the functions $\tilde{g}$ and $\hat{g}$ will become continuous on $[0,b] \times [0,d]$ and $[0,b] \times [d,b]$, respectively.

Remark 3.2 It follows from (3.5) (with $i = j = 0$, $0 \leq \nu < 1$) that $\kappa(t,s)$ may have a weak singularity at $t = s$:

$$\left|\kappa(t,s)\right| \leq c(1 + |\ln|t-s||)\,, \quad t, s \in [0,b]\,,\ t \neq s\ (\nu = 0)\,;$$

$$\left|\kappa(t,s)\right| \leq c|t-s|^{-\nu}\,, \quad t, s \in [0,b]\,,\ t \neq s\ (0 < \nu < 1)\,.$$

If $\nu < 0$ then function $\kappa(t,s)$ in (3.4) is bounded for $t, s \in [0,b]$, $t \neq s$, but its derivatives may be singular at $t = s$.

3.3 Smoothness of Solution

As a rule, the singular behaviour of the kernel $K(t,s)$ of equation (3.3) (see the assumptions (A1) and (A2)) causes singularities of the derivatives of the solution $u(t)$ at $t=0$, $t=d$ and $t=b$ (cf. [25, 57, 63]). In order to characterize those singularities we introduce a set of functions $C^{m,\nu}_{d,p}[0,b]$.

Let $m\in\mathbb{N}$, $\nu\in\mathbb{R}$, $\nu<1$, $0<d<b$, $p\in\mathbb{N}_0$, $p\le m$. Define $C^{m,\nu}_{d,p}[0,b]$ as the collection of continuous functions $u:[0,b]\to\mathbb{R}$ which are m times continuously differentiable in $(0,b)\setminus\{d\}$ and such that the estimate

$$\left|u^{(j)}(t)\right| \le c\begin{cases} 1, \quad \text{if} \quad j<1-\nu,\ p\in\{0,1,\dots,m\}, \\ 1+|\ln t|+|\ln(b-t)|, \\ \qquad \text{if} \quad j=1-\nu,\ p\in\{1,\dots,m\}, \\ 1+|\ln t|+|\ln|d-t||+|\ln(b-t)|, \\ \qquad \text{if} \quad j=1-\nu,\ p=0, \\ t^{1-\nu-j}+(b-t)^{1-\nu-j}, \\ \qquad \text{if} \quad 1-\nu<j<1-\nu+p,\ p\in\{1,\dots,m\}, \\ t^{1-\nu-j}+|\ln|d-t||+(b-t)^{1-\nu-j}, \\ \qquad \text{if} \quad j=1-\nu+p,\ p\in\{1,\dots,m-1\}, \\ t^{1-\nu-j}+|d-t|^{1-\nu-j+p}+(b-t)^{1-\nu-j}, \\ \qquad \text{if} \quad j>1-\nu+p,\ p\in\{0,\dots,m-1\}, \end{cases} \tag{3.10}$$

holds with a positive constant $c=c(u)$ for every $t\in(0,b)\setminus\{d\}$ and $j=1,\dots,m$.

The following result characterizes the regularity properties of solutions of equation (3.3) (see [44, 63]).

Theorem 3.2 *Let the conditions (A1) and (A2) about the kernel* (3.4) *be fulfilled. Let* $f\in C^{m,\nu}_{d,p}[0,b]$*, with* m,ν,d,p*, fixed in the assumptions (A1) and (A2). If the integral equation* (3.3) *has an integrable solution* $u\in L^1(0,b)$ *then* $u\in C^{m,\nu}_{d,p}[0,b]$.

Chapter 4

Quadrature Methods

Quadrature methods for solving equation (3.3) are obtained by directly approximating the integral term in (3.3) by some numerical integration rule. Where applicable, these methods appear to be the most efficient way of solving equations (3.3). Some opportunities to solve weakly-singular integral equations by quadrature formula method have been considered, for example, in [2, 29, 35, 37, 39, 64, 67, 69]. We will follow the approach of [2, 29, 69]. The main result of this chapter is formulated in Theorem 4.2.

4.1 Quasi-Uniform Grids

For a given $b \in \mathbb{R}$, $b > 0$, $N \in \mathbb{N}$, $N \geq 2$, let Δ_N be a partition (a grid) for the interval $[0, b]$.

$$\Delta_N = \{t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \ldots < t_N = b\}. \tag{4.1}$$

In the following we usually assume that d $(0 < d < b)$ is an interior knot of Δ_N:

$$d \in \{t_1, \ldots, t_{N-1}\}. \tag{4.2}$$

For the ease of notation we suppress the index N in $t_j = t_j^{(N)}$ $(j = 0, 1, \ldots, N)$, indicating the dependence of the grid points $t_0, t_1, \ldots, t_N$ on N. With Δ_N we associate the subintervals

$$\sigma_1 = [t_0, t_1], \quad \sigma_j = (t_{j-1}, t_j], \quad j = 2, \ldots, N. \tag{4.3}$$

Let h_N be the maximal length and $\underline{h}_N$ be the minimal length of the subintervals σ_j $(j = 1, \ldots, N)$:

$$h_N = \max_{j=1,\ldots,N} (t_j - t_{j-1}), \tag{4.4}$$

$$\underline{h}_N = \min_{j=1,\ldots,N} (t_j - t_{j-1}). \tag{4.5}$$

A sequence (Δ_N) of partitions (4.1) with $N \in \mathbb{N}$ is called quasi-uniform if there exists a positive constant θ, independent of N, so that

$$h_N / \underline{h}_N \le \theta, \quad N \in \mathbb{N}. \tag{4.6}$$

If $\theta = 1$ then the quasi-uniform grid Δ_N is uniform.

It follows from (4.6) that

$$h_N \le b\,\theta N^{-1}. \tag{4.7}$$

Therefore, for the quasi-uniform grid (4.1),

$$h_N = \max_{j=1,\ldots,N} (t_j - t_{j-1}) \to 0 \quad \text{as} \quad N \to \infty. \tag{4.8}$$

4.2 Idea of Quadrature Method

To describe a quadrature method consider equation (3.3) with $K(t,s)$ and $f(t)$ continuous on $[0,b] \times [0,b]$ and $[0,b]$, respectively. Let

$$\int_0^b F(s)ds = \sum_{j=1}^{N} w_j F(s_j) + R_N(F) \tag{4.9}$$

be a quadrature formula (quadrature process) with the node points $s_j = s_j^{(N)} \in [0,b]$ and weights $w_j = w_j^{(N)} \in \mathbb{R}$. Assume that the quadrature process (4.9) converges for all continuous functions F on $[0,b]$, that is the remainder $R_N(F) \to 0$ as $N \to \infty$ for all $F \in C[0,b]$:

$$\lim_{N\to\infty} \sum_{j=1}^{N} w_j F(s_j) = \int_0^b F(s)ds \quad \forall\, F \in C[0,b].$$

Replacing the integral in (3.3) by the rule (4.9) without R_N,

$$\int_0^b K(t,s)u(s)ds \approx \sum_{j=1}^{N} w_j K(t,s_j)u(s_j),$$

we obtain an approximate equation for equation (3.3):

$$u_N(t) - \sum_{j=1}^{N} w_j K(t, s_j) u_N(s_j) = f(t)\,, \quad 0 \le t \le b\,, \tag{4.10}$$

with an unknown $u_N(t) \approx u(t)$, $0 \le t \le b$. By setting $t = s_1, \ldots, s_N$, equation (4.10) reduces to a $N \times N$ linear system

$$z_i - \sum_{j=1}^{N} w_j K(s_i, s_j) z_j = f(s_i)\,, \quad i = 1, \ldots, N\,, \tag{4.11}$$

with respect to $z_i = u_N(s_i) \approx u(s_i)$, $i = 1, \ldots, N$. Notice, that the values z_i $(i = 1, \ldots N)$ can further be used to determine an approximative solution

$$u_N(t) = f(t) + \sum_{j=1}^{N} w_j K(t, s_j) z_j\,, \quad 0 \le t \le b\,. \tag{4.12}$$

The function (4.12) is often called as Nyström approximation of u, the solution of equation (3.3).

The method (4.11) is usually called a quadrature formula method for the numerical solution of equation (3.3). This method has been a point of application for several abstract theories of operator equations (see, for example, [1, 59]). The following result is proved in [59].

Theorem 4.1 *Suppose that $K \in C([0,b] \times [0,b])$, $f \in C[0,b]$ and equation (3.3) has a unique solution $u(t)$. Assume also that the quadrature process (4.9) converges for all continuous functions F on $[0,b]$.*

Then for sufficiently large N, say $N \ge N_0$, the system of equations (4.11) has a unique solution $(z_1, \ldots, z_N)$ and the following estimate holds:

$$c_1 \varepsilon_N \le \max_{1 \le i \le N} \left| z_i - u(s_i) \right| \le c_2 \varepsilon_N\,.$$

Here $c_2 \ge c_1 \ge 0$ are some constants which do not depend on N and

$$\varepsilon_N = \max_{1 \le i \le N} \left| R_N[K(s_i, \cdot) u(\cdot)] \right| \to 0 \quad as \quad N \to \infty\,.$$

If the kernel K in equation (3.3) is weakly singular, the direct application of the quadrature method may not be effective in solving (3.3) numerically. In the next sections we consider this problem more exactly.

4.3 Modified Quadrature Method

Consider now an integral equation

$$u(t) - \int_0^b K(t,s)u(s)ds = f(t)\,, \quad 0 \le t \le b\,, \tag{4.13}$$

with $f \in C[0,b]$ and $K(t,s) = g(t,s)\kappa(t,s)$ where g and κ satisfy the assumptions (A1) and (A2), respectively. In this setting we can not directly apply the quadrature method (4.11). Indeed, it follows from (A2) that in case $0 \le \nu < 1$ the kernel $K(t,s)$ might have a singular factor $\ln|t-s|$ ($\nu = 0$) or $|t-s|^{-\nu}$ ($0 < \nu < 1$), and, therefore,

$$K(t,s) \to \infty \quad \text{as} \quad s \to t\,.$$

Thus, to compose the system (4.11), we have to isolate the singularity of the kernel $K(t,s)$ for $t = s$. For example, instead of (4.11) we can consider the system [64, 69]

$$z_i = h\sum_{j=1}^{N} K\left(ih, \left(j - \frac{1}{2}\right)h\right) z_j = f(ih)\,, \quad i = 1, \dots, N\,. \tag{4.14}$$

In (4.14) z_i ($i = 1, \dots, N$) are the approximate values of the solution u of equation (4.13) in the nodes $\left(i - \frac{1}{2}\right)h$ (or ih), $i = 1, \dots, N$, $h = \frac{b}{N}$. To get (4.14), we first replace the integral in (4.13) by the rule

$$\int_0^b F(s)ds \approx h\sum_{j=1}^{N} F\left(\left(j - \frac{1}{2}\right)h\right), \quad h = \frac{b}{N}\,,$$

and then set $t = ih$, $i = 1, \dots, N$.

But it is more useful to construct approximate solutions with the help of the singularity subtraction technique described by Kantorovich and Krylov in [29] (cf. also [2, 3, 4, 6]). According to this, we rearrange (4.13) in such a way that the singularity of the kernel is at least partially cancelled by the smoothness of the solution:

$$u(t) - \int_0^b K(t,s)[u(s) - u(t)]ds - u(t)\int_0^b K(t,s)ds = f(t)\,. \tag{4.15}$$

The idea is to mitigate the singularity of the kernel by a factor $u(s) - u(t)$. In equation (4.15) the function $K(t, s)[u(s) - u(t)]$ is smoother than the function $K(t, s)u(s)$ in the equation (4.13).

In the following we consider only the use of the rectangular rule

$$\int_0^b F(s)ds = \sum_{j=1}^{N}(t_j - t_{j-1})F(s_j) + R_N(F)\,, \tag{4.16}$$

where t_j $(j = 0, 1, \ldots, N)$ are the grid points introduced by {(4.1),(4.2)} and

$$s_j = \frac{t_{j-1} + t_j}{2}\,, \quad j = 1, \ldots, N\,. \tag{4.17}$$

Using (4.15) we obtain an approximate equation in the form

$$u_N(t) - \sum_{j=1}^{N}(t_j - t_{j-1})K(t, s_j)[u_N(s_j) - u_N(t)] - u_N(t)\int_0^b K(t, s)ds = f(t)\,, \quad 0 \le t \le b\,, \tag{4.18}$$

with an unknown function $u_N(t)$, $0 \le t \le b$. By setting $t = s_i$, $i = 1, \ldots, N$, in a similar way as in Section 4.2, equation (4.18) reduces to a linear system of algebraic equations

$$z_i - \sum_{\substack{j=1\\ j\neq i}}^{N}(t_j - t_{j-1})K(s_i, s_j)[z_j - z_i] - z_i\int_0^b K(s_i, s)ds = f(s_i)\,, \quad i = 1, \ldots, N\,, \tag{4.19}$$

with respect to $z_i = u_N(s_i)$, $i = 1, \ldots, N$.

In the following we will investigate the solvability of the system (4.19). Moreover, we analyse the rate of convergence to zero of the quantity

$$\max_{i=1,\ldots,N} |z_i - u(s_i)|\,.$$

4.4 Convergence of Modified Quadrature Method

In this section we examine the convergence order of method (4.19). In fact, we prove the following result which is not published yet.

Theorem 4.2 *Let the assumptions (A1) and (A2) about the kernel K of equation* (4.13) *hold with $m = 2$ and $p = 0$. Let $f \in C^{2,\nu}_{d,0}[0,b]$, with ν, d, fixed in (A1) and (A2). Assume also that the homogeneous integral equation*

$$u(t) = \int_0^b K(t,s)u(s)ds$$

has only the trivial solution $u = 0$ and the grid $\{(4.1),(4.2)\}$ is quasi-uniform, i.e. it satisfies the condition (4.6).

Then equation (4.13) *has a unique solution $u^* \in C^{2,\nu}_{d,0}[0,b]$. For all sufficiently large N, $N \geq N_0$, the system of equations* (4.19) *has a unique solution $(z_1^*, \dots, z_N^*)$ and the following error estimate holds:*

$$\max_{i=1,\dots,N} \left| z_i^* - u^*(s_i) \right| \leq c \left\{ \begin{array}{ll} N^{-2}, & \nu < 0 \\ N^{-2}(\ln N)^2, & \nu = 0 \\ N^{-2(1-\nu)}, & \nu > 0 \end{array} \right\}, \quad N \geq N_0\,, \qquad (4.20)$$

where c is a constant which is independent of N.

Proof. We intend to make use of Theorem 2.6. We consider equation (4.13) as the equation

$$u = Tu + f \qquad (4.21)$$

in the Banach space $E = C[0,b]$, with T, given by (3.1).

It follows from Theorem 3.1 that $T \in \mathcal{L}_c(C[0,b], C[0,b])$. Further, we consider (4.19) as the equation

$$\overline{z}_N = T_N \overline{z}_N + p_N f \qquad (4.22)$$

in the Banach space $E_N = m_N$ of vectors $\overline{z}_N = (z_1, \dots, z_N)$ with the norm

$$\left\| \overline{z}_N \right\|_{m_N} = \max_{i=1,\dots,N} |z_i|\,. \qquad (4.23)$$

Operators $T_N : m_N \to m_N$ ($N \in \mathbb{N}$) are defined by the formula

$$T_N \overline{z}_N = ((T_N \overline{z}_N)_1, \dots, (T_N \overline{z}_N)_N)\,, \quad \overline{z}_N = (z_1, \dots, z_N) \in m_N\,,$$

where

$$(T_N\overline{z}_N)_i = \sum_{\substack{j=1\\ j\neq i}}^{N} (t_j - t_{j-1}) K(s_i, s_j)[z_j - z_i] + z_i \int_0^b K(s_i, s) ds\,, \quad i = 1, \ldots, N\,. \tag{4.24}$$

It is easy to see that $T_N \in \mathcal{L}_c(m_N, m_N)$. Indeed, $T_N : m_N \to m_N$ is linear and for every $\overline{z}_N \in m_N$ we have

$$\left\|T_N\overline{z}_N\right\|_{m_N} = \max_{i=1,\ldots,N} \left|(T_N\overline{z}_N)_i\right|. \tag{4.25}$$

Using (4.24) and (4.17) we can estimate $(T_N\overline{z}_N)_i$ as follows:

$$\begin{aligned}
\left|(T_N\overline{z}_N)_i\right| &= \left| \sum_{\substack{j=1\\ j\neq i}}^{N} (t_j - t_{j-1}) K(s_i, s_j)[z_j - z_i] + z_i \int_0^b K(s_i, s) ds \right| \\
&\le c\|\overline{z}_N\|_{m_N} \left(\sum_{\substack{j=1\\ j\neq i}}^{N} (t_j - t_{j-1}) \left\{ \begin{array}{ll} 1, & \nu < 0 \\ 1 + |\ln|s_i - s_j||, & \nu = 0 \\ |s_i - s_j|^{-\nu}, & \nu > 0 \end{array} \right\} \right. \\
&\left. + \int_0^b \left\{ \begin{array}{ll} 1, & \nu < 0 \\ 1 + |\ln|s_i - s||, & \nu = 0 \\ |s_i - s|^{-\nu}, & \nu > 0 \end{array} \right\} ds \right) \\
&\le c_1\|\overline{z}_N\|_{m_N} \int_0^b \left\{ \begin{array}{ll} 1, & \nu < 0 \\ 1 + |\ln|s_i - s||, & \nu = 0 \\ |s_i - s|^{-\nu}, & \nu > 0 \end{array} \right\} \\
&= c_2\|\overline{z}_N\|_{m_N}\,, \quad i = 1, \ldots, N\,.
\end{aligned} \tag{4.26}$$

Therefore,

$$\left\|T_N\overline{z}_N\right\|_{m_N} \le c\left\|\overline{z}_N\right\|_{m_N}, \quad \overline{z}_N \in m_N\,,$$

where c is a positive constant which does not depend on N. Thus, $T_N \in \mathcal{L}(m_N, m_N)$. Clearly, $T_N \in \mathcal{L}_c(m_N, m_N)$.

Operators $p_N : C[0, b] \to m_N$ ($N \in \mathbb{N}$) in (4.22) are defined as follows:

$$p_N u = (u(s_1), u(s_2), \ldots, u(s_N))\,, \quad u \in C[0, b]\,. \tag{4.27}$$

It is clear that the operators $p_N : C[0, b] \to m_N$ are linear and bounded. Moreover, for every $u \in C[0, b]$ we have

$$\left\|p_N u\right\|_{m_N} = \max_{i=1,\ldots,N} |u(s_i)| \le \|u\|_{C[0,b]}\,,$$

that is, $p_N \in \mathcal{L}(C[0,b], m_N)$. We also have that

$$\left\|p_N u\right\|_{m_N} \to \|u\|_{C[0,b]} \quad \text{as} \quad N \to \infty$$

for every $u \in C[0,b]$.

In the next section we prove (see Lemma 4.1) that

$$T_N \to T \quad \text{compactly}\,.$$

Thus, we can apply Theorem 2.6. On the basis of this theorem we obtain that equation (4.21) (equation (4.13)) has a unique solution $u^* \in C[0,b]$. By Theorem 3.2, $u^* \in C^{2,\nu}_{d,0}[0,b]$. For all sufficiently large N, $N \geq N_0$, equation (4.22) (system of equations (4.19)) has a unique solution $\overline{z}^*_N = (z^*_1, \ldots, z^*_N) \in m_N$. Moreover, by (2.8) and Lemma 4.3 (see Sec. 4.6) we obtain the estimates (4.20). □

Remark 4.1 Actually it follows from (4.26) that the norms of $T_N \in \mathcal{L}(m_N, m_N)$ are uniformly bounded in N:

$$\begin{aligned} \|T_N\|_{\mathcal{L}(m_N,m_N)} &= \sup_{\overline{z}_N \in m_N : \|\overline{z}_N\|_{m_N} \leq 1} \left\|T_N \overline{z}_N\right\|_{m_N} \\ &= \sup_{\overline{z}_N \in m_N : \|\overline{z}_N\|_{m_N} \leq 1} \max_{i=1,\ldots,N} \left|\left(T_N \overline{z}_N\right)_i\right| \leq c \quad (N \in \mathbb{N})\,. \end{aligned} \tag{4.28}$$

4.5 Compact Convergence of Discretized Operators

The following lemma is a part of the proof of Theorem 4.2.

Lemma 4.1 *Let the conditions of Theorem 4.2 be fulfilled and let the operators T, T_N and p_N be given by* (3.1), (4.24) *and* (4.27), *respectively. Then*

$$T_N \to T \quad \textit{compactly}\,.$$

Proof. First we show that $T_N \xrightarrow{\mathcal{P}} T$ $(N \in \mathbb{N})$. By (4.28) we have that

$$\left\|T_N\right\|_{\mathcal{L}(m_N,m_N)} \leq c \quad (N \in \mathbb{N})\,.$$

According to Theorem 2.5 it remains to prove that

$$\left\|T_N p_N u - p_N T u\right\|_{m_N} \to 0\,, \quad \text{as} \quad N \to \infty\,, \tag{4.29}$$

for $u \in E' \subset E$, where $E' = C^1[0,b]$ is dense in $E = C[0,b]$. Let $u \in C^1[0,b]$. Then for $i = 1, \ldots, N$, we have

$$\begin{aligned}
(p_N T u - T_N p_N u)_i &= \int_0^b K(s_i, s) u(s) ds \\
&\quad - \sum_{\substack{j=1 \\ j \neq i}}^{N} (t_j - t_{j-1}) K(s_i, s_j)[u(s_j) - u(s_i)] \\
&\quad - u(s_i) \int_0^b K(s_i, s) ds \\
&= \int_0^b K(s_i, s)[u(s) - u(s_i)] ds \\
&\quad - \sum_{\substack{j=1 \\ j \neq i}}^{N} (t_j - t_{j-1}) K(s_i, s_j)[u(s_j) - u(s_i)] \\
&= \int_0^b v_i(s) ds - \sum_{\substack{j=1 \\ j \neq i}}^{N} (t_j - t_{j-1}) v_i(s_j)\,, \qquad (4.30)
\end{aligned}$$

where

$$v_i(s) = K(s_i, s)[u(s) - u(s_i)]\,, \quad i = 1, \ldots, N\,. \qquad (4.31)$$

Since $u \in C^1[0,b]$ it follows from the assumptions of Theorem 4.2 that

$$\lim_{s \to s_i} v_i(s) = 0\,, \quad i = 1, \ldots, N\,.$$

Moreover, the functions

$$\tilde{v}_i(s) = \begin{cases} v_i(s)\,, & 0 \le s < d \\ v_i(d-0)\,, & s = d \end{cases} \qquad (i = 1, \ldots, N) \qquad (4.32)$$

and

$$\hat{v}_i(s) = \begin{cases} v_i(s)\,, & d < s \le b \\ v_i(d+0)\,, & s = d \end{cases} \qquad (i = 1, \ldots, N) \qquad (4.33)$$

are continuous on $[0,d]$ and $[d,b]$, respectively.

We can also renounce from the restriction $j \neq i$ in the expression (4.30) for $(p_N Tu - T_N p_N u)_i$. Actually we have

$$(p_N Tu - T_N p_N u)_i = \int_0^d \tilde{v}_i(s)ds - \sum_{j=1}^{N_d} (t_j - t_{j-1})\tilde{v}_i(s_j) + \int_d^b \hat{v}_i(s)ds - \sum_{j=N_d+1}^{N} (t_j - t_{j-1})\hat{v}_i(s_j)\,,$$

where $i = 1, \ldots, N$ and N_d is fixed so that $t_{N_d} = d$ (see (4.2)). Since the grid $\{(4.1), (4.2)\}$ is quasi-uniform, both $N_d \to \infty$ and $N - N_d \to \infty$ for $N \to \infty$. So we have

$$\begin{aligned} \left\| p_N Tu - T_N p_N u \right\|_{m_N} &= \max_{i=1,\ldots,N} \left| \left(p_N Tu - T_N p_N u \right)_i \right| \\ &\leq \max_{i=1,\ldots,N} \left| \int_0^d \tilde{v}_i(s)ds - \sum_{j=1}^{N_d} (t_j - t_{j-1})\tilde{v}_i(s_j) \right| \\ &+ \max_{i=1,\ldots,N} \left| \int_d^b \hat{v}_i(s)ds - \sum_{j=N_d+1}^{N} (t_j - t_{j-1})\hat{v}_i(s_j) \right|, \end{aligned} \tag{4.34}$$

where $s_j = (t_{j-1} + t_j)/2$ $(j = 1, \ldots, N)$ and $\tilde{v}_i$ and $\hat{v}_i$ $(i = 1, \ldots, N)$ are given by the formulae (4.32) and (4.33), respectively. It is easy to see that

$$\sup_{z \in \tilde{M}} \left| \int_0^d z(s)ds - \sum_{j=1}^{N_d} (t_j - t_{j-1}) z(s_j) \right| \to 0\,, \quad N_d \to \infty\,, \tag{4.35}$$

for every relatively compact set $\tilde{M} \subset C[0,d]$. For $t \in [0,b]$ and $u \in C^1[0,d]$ we set

$$\tilde{v}_t(s) = \tilde{g}(t,s)\kappa(t,s)[u(s) - u(t)]\,, \quad s \in [0,d]\,,$$

where $\tilde{g}(t,s)$ is given by the formula (3.8). It follows from the assumptions of lemma that $v_t \in C[0,d]$. Using Theorem 2.3 we obtain that the set

$$\tilde{M} = \{\tilde{v}_t(\cdot)\}_{t \in [0,b]}$$

is relatively compact in $C[0,d]$.

Analogously it can be shown that

$$\sup_{z\in\hat{M}}\left|\int_d^b z(s)ds - \sum_{N_d+1}^{N}(t_j - t_{j-1})z(s_j)\right| \to 0\,, \quad N - N_d \to \infty\,, \qquad (4.36)$$

for every relatively compact set $\hat{M} \subset C[d,b]$,

$$\hat{M} = \{\hat{v}_t(\cdot)\}_{t\in[0,b]}\,,$$

with

$$\hat{v}_t(s) = \hat{g}(t,s)\kappa(t,s)[u(s) - u(t)]\,, \quad u \in C^1[d,b]\,,\ s \in [d,b]\,.$$

With help of (4.35) and (4.36) we now conclude from (4.34) that

$$\left\|p_N Tu - T_N p_N u\right\|_{m_N} \to 0\,, \quad N \to \infty\,,$$

for every $u \in C^1[0,b]$. Thus, we have shown that $T_N \xrightarrow{\mathcal{P}} T$, $N \to \infty$.

Further, let $\{u_N\}_{N\in\mathbb{N}}$ with $\overline{u}_N = (u_{1,N},\ldots,u_{N,N}) \in m_N$ be a given sequence such that $\|\overline{u}_N\|_{m_N} \le c$, $N \in \mathbb{N}$. Let us define

$$\eta_i = \sum_{\substack{j=1\\ j\neq i}}^{N}(t_j - t_{j-1})K(s_i,s_j)(u_{j,N} - u_{i,N}) + u_{i,N}\int_0^b K(s_i,s)ds\,,\ i = 1,\ldots,N.$$

For every $N \in \mathbb{N}$ we construct a broken line $z^N \in C[0,b]$ as follows:

1) $z^N(s_i) = \eta_i$, $i = 1,\ldots,N$;

2) $z^N(t)$ is linear for $t \in [s_{i-1},s_i]$, $i = 2,\ldots,N-1$;

3) $z^N(t) = \eta_1$ for $0 \le t \le s_1$ and $z^N(t) = \eta_N$ for $s_N \le t \le b$.

With help of Theorem 2.3 we obtain that the sequence $\{z^N(t)\}_{N\in\mathbb{N}}$ is relatively compact in $C[0,b]$. Therefore the sequence $\{p_N z^N\}$ is $\mathcal{P}$-compact (see Sec. 2.3). Since $p_N z^N = T_N\overline{u}_N$, we obtain that the sequence $\{T_N\overline{u}_N\}_{N\in\mathbb{N}}$ is $\mathcal{P}$-compact. This together with the convergence $T_N \xrightarrow{\mathcal{P}} T$ $(N \in \mathbb{N})$ yields $T_N \to T$ compactly. $\square$

4.6 Error Estimation

In this section we prove Lemma 4.3 that was used in the proof of Theorem 4.2. In order to estimate the error $\|p_N Tu - T_N p_N u\|_{m_N}$ we need the following result.

Lemma 4.2 [69] *Assume that the grid* $\{(4.1), (4.2)\}$ *is quasi-uniform and* s_j *is given by the formula* (4.17). *Then*

$$\Big| \int_{t_k}^{t_l} F(s)ds - \sum_{j=k+1}^{l} (t_j - t_{j-1})F(s_j) \Big| \le c h_N \int_{t_k}^{t_l} |F'(s)|ds \,, \tag{4.37}$$

$$\Big| \int_{t_k}^{t_l} F(s)ds - \sum_{j=k+1}^{l} (t_j - t_{j-1})F(s_j) \Big| \le c h_N^2 \int_{t_k}^{t_l} |F''(s)|ds \,, \tag{4.38}$$

provided that the function F *is such that all integrals in these inequalities exist. Here* c *is a positive constant which is independent of* N, $0 \le k < l \le N$ *and* h_N *is given by* (4.4).

Lemma 4.3 *Let the conditions of Theorem 4.2 be fulfilled and let the operators* T, T_N *and* p_N *be given by the formulae* (3.1), (4.24) *and* (4.27), *respectively. If* $u \in C^{2,\nu}_{d,0}[0,b]$ *then*

$$\big\|p_N Tu - T_N p_N u\big\|_{m_N} \le c\tau_\nu^2(N) \,, \tag{4.39}$$

where c *is a constant, which is independent of* N *and*

$$\tau_\nu(N) = \begin{cases} N^{-1}, & \text{if} \quad \nu < 0 \,, \\ N^{-1}\ln N, & \text{if} \quad \nu = 0 \,, \\ N^{-(1-\nu)}, & \text{if} \quad \nu > 0 \,. \end{cases} \tag{4.40}$$

Proof. We have

$$\big\|p_N Tu - T_N p_N u\big\|_{m_N} = \max_{i=1,\dots,N} \big|\big(p_N Tu - T_N p_N u\big)_i\big| \,.$$

Since

$$K(s_i, s)[u(s) - u(s_i)] \to 0 \quad \text{as} \quad s \to s_i \ (i = 1, \dots, N)$$

we have to show that (see (4.30) and (4.31))

$$\left| \int_0^b v_i(s)ds - \sum_{j=1}^{N}(t_j - t_{j-1})v_i(s_j) \right| \le c\tau_\nu^2(N)\,, \quad i = 1, \ldots, N\,, \tag{4.41}$$

where $s_j = (t_{j-1} + t_j)/2$, $j = 1, \ldots, N$ and c is a positive constant which does not depend on N. In order to prove (4.41), we devide the interval $[0, b]$ into subintervals and derive the error estimates on the corresponding intervals.

If $i \le N_d - 1$ (recall that N_d is an index such that $t_{N_d} = d$), then we devide the interval $[0, b]$ into subintervals $[0, t_1]$, $[t_1, t_{i-1}]$, $[t_{i-1}, t_i]$, $[t_i, t_{N_d-1}]$, $[t_{N_d-1}, d]$, $[d, t_{N_d+1}]$, $[t_{N_d+1}, t_{N-1}]$, $[t_{N-1}, b]$. If $i \ge N_d + 2$, then we devide the interval $[0, b]$ into subintervals $[0, t_1]$, $[t_1, t_{N_d-1}]$, $[t_{N_d-1}, d]$, $[d, t_{N_d+1}]$, $[t_{N_d+1}, t_{i-1}]$, $[t_{i-1}, t_i]$, $[t_i, t_{N-1}]$, $[t_{N-1}, b]$. On the short intervals $[0, t_1]$, $[t_{i-1}, t_i]$, $[t_{N_d-1}, d]$, $[d, t_{N_d+1}]$, $[t_{N-1}, b]$ we use the estimate (4.37) and on other intervals we use the estimate (4.38). In order to apply Lemma 4.2 we find

$$\begin{aligned} v_i'(s) = &\left\{ \frac{\partial}{\partial s} g(s_i, s)\kappa(s_i, s) + g(s_i, s)\frac{\partial}{\partial s}\kappa(s_i, s) \right\} [u(s) - u(s_i)] \\ &+ g(s_i, s)\kappa(s_i, s)u'(s)\,, \end{aligned} \tag{4.42}$$

$$\begin{aligned} v_i''(s) = &\left\{ \frac{\partial^2}{\partial s^2} g(s_i, s)\kappa(s_i, s) + 2\frac{\partial}{\partial s} g(s_i, s)\frac{\partial}{\partial s}\kappa(s_i, s) \right. \\ &\left. + g(s_i, s)\frac{\partial^2}{\partial s^2}\kappa(s_i, s) \right\} [u(s) - u(s_i)] \\ &+ 2\left\{ \frac{\partial}{\partial s} g(s_i, s)\kappa(s_i, s) + g(s_i, s)\frac{\partial}{\partial s}\kappa(s_i, s) \right\} u'(s) \\ &+ g(s_i, s)\kappa(s_i, s)u''(s)\,, \end{aligned} \tag{4.43}$$

where $i = 1, \ldots, N$.

Let us consider the subinterval $[0, t_1]$. On the basis of (4.37) and (4.42) we have

$$\left| \int_0^{t_1} v_i(s)ds - t_1 v_i(s_1) \right| \le ch_N \int_0^{t_1} |v_i'(s)|ds$$

$$= ch_N \int_0^{t_1} \left| \left\{ \frac{\partial}{\partial s} g(s_i, s)\kappa(s_i, s) + g(s_i, s) \frac{\partial}{\partial s}\kappa(s_i, s) \right\} [u(s) - u(s_i)] \right.$$

$$\left. + g(s_i, s)\kappa(s_i, s)u'(s) \right| ds \le I_1^{(i)} + I_2^{(i)} ,$$

with

$$I_1^{(i)} = c_1 h_N \int_0^{t_1} \left[\left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|s_i - s||\,, & \nu = 0 \\ |s_i - s|^{-\nu}, & \nu > 0 \end{array} \right\} \right.$$

$$\left. + \left\{ \begin{array}{ll} 1\,, & \nu + 1 < 0 \\ 1 + |\ln|s_i - s||\,, & \nu + 1 = 0 \\ |s_i - s|^{-\nu - 1}, & \nu + 1 > 0 \end{array} \right\} \right] |u(s) - u(s_i)| ds\,, \quad (4.44)$$

$$I_2^{(i)} = c_2 h_N \int_0^{t_1} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|s_i - s||\,, & \nu = 0 \\ |s_i - s|^{-\nu}, & \nu > 0 \end{array} \right\}$$

$$\times \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln s| + |\ln|d - s|| + |\ln(b - s)|\,, & \nu = 0 \\ s^{-\nu} + |d - s|^{-\nu} + (b - s)^{-\nu}, & \nu > 0 \end{array} \right\} ds\,, \quad (4.45)$$

where $i = 1, \ldots, N$.

Let us estimate $I_1^{(i)}$, $i = 1, \ldots, N$, given by (4.44). We present only the detailed proof for the case $0 < \nu < 1$. For $\nu \le 0$ the proof is analogous.

We find

$$
\begin{aligned}
I_1^{(i)} &= c_1 h_N \int_0^{t_1} (|s_i - s|^{-\nu} + |s_i - s|^{-\nu-1}) |u(s) - u(s_i)| ds \\
&\le c_2 h_N \int_0^{t_1} |s_i - s|^{-\nu-1} \left| \int_{s_i}^{s} |u'(t)| dt \right| ds \\
&\le c_3 h_N \int_0^{t_1} |s_i - s|^{-\nu-1} \left| \int_{s_i}^{s} (t^{-\nu} + |d - t|^{-\nu} + (b - t)^{-\nu}) dt \right| ds \\
&\le c_4 h_N \int_0^{t_1} |s_i - s|^{-\nu} \Big[\max\big\{ s^{-\nu}, s_i^{-\nu} \big\} + \max\big\{ (d - s)^{-\nu}, (d - s_i)^{-\nu} \big\} \\
&\quad + \max\{ (b - s)^{-\nu}, (b - s_i)^{-\nu} \} \Big] ds \,,
\end{aligned}
$$

where $i = 1, \dots, N$. So we have

$$
I_1^{(i)} \le c_4 h_N \big(I_{1,1}^{(i)} + I_{1,2}^{(i)} + I_{1,3}^{(i)} \big) \,, \quad i = 1, \dots, N \,, \tag{4.46}
$$

where

$$
\begin{aligned}
I_{1,1}^{(i)} &= \int_0^{t_1} |s_i - s|^{-\nu} \max\big\{ s^{-\nu}, s_i^{-\nu} \big\} ds \,, \\
I_{1,2}^{(i)} &= \int_0^{t_1} |s_i - s|^{-\nu} \max\big\{ (d - s)^{-\nu}, (d - s_i)^{-\nu} \big\} ds \,, \\
I_{1,3}^{(i)} &= \int_0^{t_1} |s_i - s|^{-\nu} \max\big\{ (b - s)^{-\nu}, (b - s_i)^{-\nu} \big\} ds \,.
\end{aligned}
$$

Let us consider $I_{1,1}^{(i)}$, $i = 1, \ldots, N$. Using the property (4.6) we obtain for $i = 1, \ldots, N$ that

$$I_{1,1}^{(i)} \le \int_0^{t_1} |s_1 - s|^{-\nu} \max\left\{s^{-\nu}, s_1^{-\nu}\right\} ds$$

$$\le \int_0^{t_1/2} (s_1 - s)^{-\nu} s^{-\nu} ds + s_1^{-\nu} \int_{t_1/2}^{t_1} (s - s_1)^{-\nu} ds$$

$$\le c\underline{h}_N^{-\nu} \Bigg(\int_0^{t_1/4} s^{-\nu} ds + \int_{t_1/4}^{t_1/2} (s_1 - s)^{-\nu} ds \Bigg) + \underline{h}_N^{-\nu} \int_{t_1/2}^{t_1} (s - s_1)^{-\nu} ds$$

$$\le c\underline{h}_N^{-\nu} h_N^{1-\nu} \le c_1 h_N^{1-2\nu} \,.$$

Let us consider $I_{1,2}^{(i)}$, $i = 1, \ldots, N$. We have

$$I_{1,2}^{(i)} \le \begin{cases} \displaystyle\int_0^{t_1} |s_i - s|^{-\nu} |d - s_i|^{-\nu} ds\,, & \text{if} \quad d \ne t_1 \\ \displaystyle\int_0^{t_1} |s_i - s|^{-\nu} (d - s)^{-\nu} ds\,, & \text{if} \quad d = t_1 \end{cases}$$

$$\le c\underline{h}_N^{-\nu} \begin{cases} \displaystyle\int_0^{t_1} |s_1 - s|^{-\nu} ds\,, & \text{if } d \ne t_1 \\ \displaystyle\int_0^{t_1/2} (s_1 - s)^{-\nu} ds + \int_{t_1/2}^{3t_1/4} (s - s_1)^{-\nu} ds + \int_{3t_1/4}^{t_1} (t_1 - s)^{-\nu} ds\,, & \\ & \text{if } d = t_1 \end{cases}$$

$$\le c_1 \underline{h}_N^{-\nu} h_N^{1-\nu} \le c_2 h_N^{1-2\nu} \,,$$

where $i = 1, \ldots, N$. In a similar way we find that

$$I_{1,3}^{(i)} \leq \int_0^{t_1} |s_1 - s|^{-\nu}(b - s_N)^{-\nu} ds$$

$$\leq \underline{h}_N^{-\nu} \int_0^{t_1} |s_1 - s|^{-\nu} ds \leq c\underline{h}_N^{-\nu} h_N^{1-\nu} \leq c_1 h_N^{1-2\nu},$$

where $i = 1, \ldots, N$. On the basis (4.46) we now obtain that

$$I_1^{(i)} \leq c h_N^{2(1-\nu)}, \quad 0 < \nu < 1\,,\; i = 1, \ldots, N\,, \tag{4.47}$$

where c is a positive constant which does not depend on N.

Next we estimate $I_2^{(i)}$, $i = 1, \ldots, N$, given by (4.45) for $0 < \nu < 1$. We have

$$I_2^{(i)} = c_2 h_N \int_0^{t_1} |s_i - s|^{-\nu}\left[s^{-\nu} + |d - s|^{-\nu} + (b - s)^{-\nu}\right] ds$$

$$\leq c_3 h_N \left(I_{2,1}^{(i)} + I_{2,2}^{(i)} + I_{2,3}^{(i)}\right), \quad i = 1, \ldots, N\,,$$

where

$$I_{2,1}^{(i)} = \int_0^{t_1} |s_i - s|^{-\nu} s^{-\nu} ds\,,$$

$$I_{2,2}^{(i)} = \int_0^{t_1} |s_i - s|^{-\nu}(d - s)^{-\nu} ds\,,$$

$$I_{2,3}^{(i)} = \int_0^{t_1} |s_i - s|^{-\nu}(b - s)^{-\nu} ds\,.$$

Consider $I_{2,1}^{(i)}$ $(i = 1, \ldots, N)$:

$$I_{2,1}^{(i)} \leq \int_0^{t_1/4} (s_1 - s)^{-\nu} s^{-\nu} ds + \int_{t_1/4}^{t_1/2} (s_1 - s)^{-\nu} s^{-\nu} ds + \int_{t_1/2}^{t_1} (s - s_1)^{-\nu} s^{-\nu} ds$$

$$\leq c\underline{h}_N^{-\nu} h_N^{1-\nu} \leq c_1 h_N^{1-2\nu}.$$

Analogously,

$$I_{2,2}^{(i)} \le \underline{h}_N^{-\nu} \int\limits_0^{t_1/2} (s_1 - s)^{-\nu} ds + \begin{cases} \underline{h}_N^{-\nu} \int\limits_{t_1/2}^{t_1} (s - s_1)^{-\nu} ds\,, & \text{if} \quad d \neq t_1 \\ \int\limits_{t_1/2}^{t_1} (s - s_1)^{-\nu} (t_1 - s)^{-\nu} ds\,, & \text{if} \quad d = t_1 \end{cases}$$

$$\le c \underline{h}_N h_N^{1-\nu} \le c_1 h_N^{1-2\nu}\,,$$

where $i = 1, \ldots, N$. Finally,

$$I_{2,3}^{(i)} \le h_N^{-\nu} \int\limits_0^{t_1} |s_1 - s|^{-\nu} ds \le c \underline{h}_N^{-\nu} h_N^{1-\nu} \le c_1 h_N^{1-2\nu},$$

where $i = 1, \ldots, N$. Thus,

$$I_2^{(i)} \le c h_N^{2(1-\nu)}, \quad 0 < \nu < 1\,,\ i = 1, \ldots, N\,, \tag{4.48}$$

with a positive constant c not depending on N. Combining (4.44), (4.45), (4.47), (4.48) and (4.7) we conclude that

$$\left| \int\limits_0^{t_1} v_i(s) ds - t_1 v_i(s_1) \right| \le c N^{-2(1-\nu)}\,, \quad i = 1, \ldots, N\,.$$

In case $\nu \le 0$ the estimation of $\int\limits_0^{t_1} v_i(s) ds - t_1 v_i(s_1)$ is analogous. In summary, we have

$$\left| \int\limits_0^{t_1} v_i(s) ds - t_1 v_i(s_1) \right| \le c \tau_\nu^2(N)\,, \tag{4.49}$$

where $i = 1, \ldots, N$ and c is a positive constant not depending on N.

In a similar way we obtain that

$$\left| \int\limits_{t_{i-1}}^{t_i} v_i(s) ds - (t_i - t_{i-1}) v_i(s_i) \right| \le c \tau_\nu^2(N)\,, \quad i = 1, \ldots, N\,;$$

$$\Big|\int_{t_{N_d-1}}^{d} v_i(s)ds-(d-t_{N_d-1})v_i(s_{N_d})\Big|\le c\tau_\nu^2(N)\,,\quad i=1,\dots,N\,;$$

$$\Big|\int_{d}^{t_{N_d+1}} v_i(s)ds-(t_{N_d+1}-d)v_i(s_{N_d+1})\Big|\le c\tau_\nu^2(N)\,,\quad i=1,\dots,N\,;$$

$$\Big|\int_{t_{N-1}}^{b} v_i(s)ds-(b-t_{N-1})v_i(s_N)\Big|\le c\tau_\nu^2(N)\,,\quad i=1,\dots,N\,.$$

Here c is a positive constant which does not depend on N.

Consider now the subinterval $[t_1,t_{i-1}]$, $i\ge 3$. We give a detailed proof only for the case $i\le N_d-1$. In case $i\ge N_d+2$ the argument is analogous. Indeed, in this case we have

$$\int_{t_1}^{t_{i-1}} v_i(s)-\sum_{j=2}^{i-1}(t_j-t_{j-1})v_i(s_j)$$

$$=\int_{t_1}^{t_{N_d-1}} v_i(s)-\sum_{j=2}^{N_d-1}(t_j-t_{j-1})v_i(s_j)+\int_{t_{N_d-1}}^{d} v_i(s)-(d-t_{N_d-1})v_i(s_{N_d})$$

$$+\int_{d}^{t_{N_d+1}} v_i(s)-(t_{N_d+1}-d)v_i(s_{N_d+1})+\int_{t_{N_d+1}}^{t_{i-1}} v_i(s)-\sum_{j=N_d+2}^{i-1}(t_j-t_{j-1})v_i(s_j)\,.$$

Now we can estimate every term separately.

On the basis of (4.38) and (4.43) we find that

$$\Big|\int_{t_1}^{t_{i-1}} v_i(s)ds-\sum_{j=2}^{i-1}(t_j-t_{j-1})v_i(s_j)\Big|\le ch_N^2\int_{t_1}^{t_{i-1}}\big|v_i''(s)\big|ds$$

$$\le ch_N^2\big(I_1^{(i)}+I_2^{(i)}+I_3^{(i)}\big)\,,\qquad (4.50)$$

where

$$I_1^{(i)} = \int_{t_1}^{t_{i-1}} \Big| \Big\{ \frac{\partial^2}{\partial s^2} g(s_i, s)\kappa(s_i, s) + 2\frac{\partial}{\partial s} g(s_i, s)\frac{\partial}{\partial s}\kappa(s_i, s) + g(s_i, s)\frac{\partial^2}{\partial s^2}\kappa(s_i, s) \Big\} [u(s) - u(s_i)] \Big| ds\,,$$

$$I_2^{(i)} = \int_{t_1}^{t_{i-1}} \Big| \Big\{ 2\frac{\partial}{\partial s} g(s_i, s)\kappa(s_i, s) + 2g(s_i, s)\frac{\partial}{\partial s}\kappa(s_i, s) \Big\} u'(s) \Big| ds\,,$$

$$I_3^{(i)} = \int_{t_1}^{t_{i-1}} \big| g(s_i, s)\kappa(s_i, s)u''(s) \big| ds\,.$$

Using the assumptions about g and κ we get for $i = 3, \ldots, N_d - 1$ that

$$I_1^{(i)} \le c \int_{t_1}^{t_{i-1}} \left\{ \begin{array}{ll} 1\,, & \nu + 2 < 0 \\ 1 + |\ln|s_i - s||\,, & \nu + 2 = 0 \\ |s_i - s|^{-\nu-2}, & \nu + 2 > 0 \end{array} \right\}$$

$$\times \left| \int_{s_i}^{s} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln t| + |\ln|d - t|| + |\ln(b - t)|\,, & \nu = 0 \\ t^{-\nu} + |d - t|^{-\nu} + (b - t)^{-\nu}, & \nu > 0 \end{array} \right\} dt \right| ds\,; \quad (4.51)$$

$$I_2^{(i)} \le c \int_{t_1}^{t_{i-1}} \left[\left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|s_i - s||\,, & \nu = 0 \\ |s_i - s|^{-\nu}, & \nu > 0 \end{array} \right\} + \left\{ \begin{array}{ll} 1\,, & \nu + 1 < 0 \\ 1 + |\ln|s_i - s||\,, & \nu + 1 = 0 \\ |s_i - s|^{-\nu-1}, & \nu + 1 > 0 \end{array} \right\} \right]$$

$$\times \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln s| + |\ln|d - s|| + |\ln(b - s)|\,, & \nu = 0 \\ s^{-\nu} + |d - s|^{-\nu} + (b - s)^{-\nu}, & \nu > 0 \end{array} \right\} ds\,; \quad (4.52)$$

$$I_3^{(i)} \le c \int_{t_1}^{t_{i-1}} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln |s_i - s||\,, & \nu = 0 \\ |s_i - s|^{-\nu}, & \nu > 0 \end{array} \right\}$$

$$\times \left\{ \begin{array}{ll} 1\,, & \nu + 1 < 0 \\ 1 + |\ln s| + |\ln |d-s|| + |\ln(b-s)|\,, & \nu + 1 = 0 \\ s^{-\nu-1} + |d-s|^{-\nu-1} + (b-s)^{-\nu-1}, & \nu + 1 > 0 \end{array} \right\} ds\,. \quad (4.53)$$

In the following we consider only the case $0 < \nu < 1$. For $\nu \le 0$ the proof is analogous. We find

$$\begin{aligned} I_1^{(i)} &\le c \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-2} \left| \int_{s_i}^{s} \big(t^{-\nu} + |d-t|^{-\nu} + (b-t)^{-\nu}\big) dt \right| ds \\ &\le c_1 \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-2} \left| \int_{s_i}^{s} \big(t^{-\nu} + (d-t)^{-\nu}\big) dt \right| ds \\ &\le c_1 \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-1} \big[\max\big\{s^{-\nu}, s_i^{-\nu}\big\} + \max\big\{(d-s)^{-\nu}, (d-s_i)^{-\nu}\big\}\big] ds \\ &= c_1 \big(I_{1,1}^{(i)} + I_{1,2}^{(i)}\big)\,, \quad i = 3, \dots, N_d - 1\,, \end{aligned} \quad (4.54)$$

where

$$I_{1,1}^{(i)} = \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-1} \max\big\{s^{-\nu}, s_i^{-\nu}\big\} ds\,,$$

$$I_{1,2}^{(i)} = \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-1} \max\big\{(d-s)^{-\nu}, (d-s_i)^{-\nu}\big\} ds\,.$$

Consider $I_{1,1}^{(i)}$, $i = 3, \dots, N_d - 1$. Using the property (4.6), we obtain that

$$I_{1,1}^{(i)} \le c \underline{h}_N^{-\nu} \int_{t_1}^{t_{i-1}} (s_i - s)^{-\nu-1} ds \le c_1 \underline{h}_N^{-2\nu} \le c_2 h_N^{-2\nu}\,, \quad i = 3, \dots, N_d - 1\,.$$

Similarly,

$$I_{1,2}^{(i)} \le c \int_{t_1}^{t_{i-1}} (s_i - s)^{-\nu-1}(d - s_i)^{-\nu} ds$$

$$\le c_1 \underline{h}_N^{-\nu} \int_{t_1}^{t_{i-1}} (s_i - s)^{-\nu-1} ds \le c_2 \underline{h}_N^{-2\nu} \le c_3 h_N^{-2\nu},$$

where $i = 3, \dots, N_d - 1$. On the basis of (4.50), (4.51) and (4.54) we conclude that

$$I_1^{(i)} \le c h_N^{2(1-\nu)}, \quad 0 < \nu < 1\,,\ i = 3, \dots, N_d - 1\,, \tag{4.55}$$

where c is a positive constant not depending on N.

Further, it follows from (4.52) for $i = 3, \dots, N_d - 1$ that

$$I_2^{(i)} \le c \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-1} \big\{ s^{-\nu} + |d - s|^{-\nu} + (b - s)^{-\nu} \big\} ds$$

$$\le c_1 \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu-1} \big\{ s^{-\nu} + (d - s)^{-\nu} \big\} ds$$

$$\le c_2 \underline{h}_N^{-\nu} \int_{t_1}^{t_{i-1}} (s_i - s)^{-\nu-1} ds$$

$$\le c_3 \underline{h}_N^{-2\nu} \le c_4 h_N^{-2\nu}\,.$$

Thus,

$$I_2^{(i)} \le c h_N^{2(1-\nu)}, \quad 0 < \nu < 1\,,\ i = 3, \dots, N_d - 1\,, \tag{4.56}$$

with a positive constant c which does not depend on N.

Finally, by (4.53),

$$I_3^{(i)} \le c \int_{t_1}^{t_{i-1}} |s_i - s|^{-\nu} \big(s^{-\nu-1} + |d - s|^{-\nu-1} \big) ds$$

$$\le c_1 \underline{h}_N^{-\nu} \int_{t_1}^{t_{i-1}} \big(s^{-\nu-1} + (d - s)^{-\nu-1} \big) ds$$

$$\le c_2 \underline{h}_N^{-2\nu} \le c_3 h_N^{-2\nu}\,, \quad i = 3, \dots, N_d - 1\,.$$

Thus,

$$I_3^{(i)} \leq c h_N^{2(1-\nu)}\,, \quad 0 < \nu < 1\,,\ i = 3, \dots, N_d - 1\,, \tag{4.57}$$

where c is a positive constant not depending on N. Combining (4.51)–(4.57) and (4.7) we conclude that

$$\Big| \int_{t_1}^{t_{i-1}} v_i(s) - \sum_{j=2}^{i-1} (t_j - t_{j-1}) v_i(s_j) \Big| \leq c\tau_\nu^2(N)\,, \tag{4.58}$$

where $i = 3, \dots, N_d - 1$ and c is a positive constant not depending on N. Analogously we find that

$$\Big| \int_{t_i}^{t_{N_d}-1} v_i(s) - \sum_{j=i+1}^{N_d-1} (t_j - t_{j-1}) v_i(s_j) \Big| \leq c\tau_\nu^2(N)\,, \quad i = 3, \dots, N_d - 2\,;$$

$$\Big| \int_{t_{N_d+1}}^{t_{N-1}} v_i(s)ds - \sum_{j=N_d+2}^{N-1} (t_j - t_{j-1}) v_i(s_j) \Big| \leq c\tau_\nu^2(N)\,, \quad i = 3, \dots, N_d - 1\,.$$

□

Chapter 5

Piecewise Polynomial Interpolation

This chapter is devoted to the background material which we will use in Chapters 6 and 7. These results are partly published in [16, 18, 19].

5.1 Spline Space $S_m^{(k)}(\Delta_N)$

For $N \in \mathbb{N}$, $N \geq 2$ let Δ_N be a grid (4.1) for the interval $[0, b]$. For given integers $m \geq 0$ and $-1 \leq k \leq m-1$, let $S_m^{(k)}(\Delta_N)$ be the set of splines of degree m and of continuity class k of piecewise polynomial functions on the grid Δ_N:

$$\begin{aligned} S_m^{(k)}(\Delta_N) &= \left\{ u \in C^k[0,b] \, : u\big|_{\sigma_j} \in \pi_m, \, j = 1, \dots, N \right\}, \, 0 \leq k \leq m-1\,; \\ S_m^{(-1)}(\Delta_N) &= \left\{ u : u\big|_{\sigma_j} \in \pi_m\,, \; j = 1, \dots, N \right\}, \quad k = -1\,. \end{aligned} \tag{5.1}$$

In (5.1) π_m denotes the set of polynomials of degree not exceeding m and $u\big|_{\sigma_j}$ is the restriction of u to the subinterval σ_j (see (4.3)).

Note that the elements of $S_m^{(-1)}(\Delta_N)$ may have jump discontinuities at the interior grid points $t_1, \dots, t_{N-1}$. The space $S_m^{(-1)}(\Delta_N)$ is thus the least smooth of the polynomial spline spaces on Δ_N, while $S_m^{(m-1)}(\Delta_N)$ (the classical spline space) is the smoothest of these spaces.

It is easy to see that $S_m^{(k)}(\Delta_N)$ is a linear space which has finite dimension:

$$\dim S_m^{(k)}(\Delta_N) = N(m-k) + k + 1\,, \quad -1 \leq k \leq m-1\,. \tag{5.2}$$

5.2 Interpolation Operator

For $N \in \mathbb{N}$, $N \geq 2$, let Δ_N be a grid for the interval $[0, b]$ (see (4.1)). In every subinterval $[t_{j-1}, t_j]$, $j = 1, \ldots, N$ we define $m \in \mathbb{N}$ interpolation points

$$\xi_{j,q} = t_{j-1} + \frac{\eta_q + 1}{2}(t_j - t_{j-1}), \quad q = 1, \ldots, m; \; j = 1, \ldots, N, \tag{5.3}$$

where

$$-1 \leq \eta_1 < \ldots < \eta_m \leq 1 \tag{5.4}$$

is some fixed system of m parameters on the interval $[-1, 1]$, which is the same for every j and N.

To a continuous function $u : [0, b] \to \mathbb{R}$ we assign a piecewise polynomial interpolation function $P_N u = P_{N,m-1} u \in S_{m-1}^{(-1)}(\Delta_N)$ which interpolates u at the nodes (5.3). Let $P_N = P_{N,m-1} : C[0, b] \to S_{m-1}^{(-1)}(\Delta_N)$ be an interpolation operator which assigns to every continuous function $u : [0, b] \to \mathbb{R}$ its piecewise interpolation function $P_N u$:

$$\begin{gathered} P_N u \in S_{m-1}^{(-1)}(\Delta_N), \quad u \in C[0, b], \\ (P_N u)(\xi_{j,q}) = u(\xi_{j,q}), \quad q = 1, \ldots, m; \; j = 1, \ldots, N. \end{gathered} \tag{5.5}$$

Thus, $(P_N u)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$ ($j = 1, \ldots, N$) and may be discontinuous at $t = t_j$, $j = 1, \ldots, N-1$; we can treat $P_N u$ as a two-valued function at these points. If $\eta_1 = -1$, $\eta_m = 1$ then $P_N u$ is a continuous function on the interval $[0, b]$.

We can redefine P_N in a more explicit form, using a Lagrange basis for $S_{m-1}^{(-1)}(\Delta_N)$. Using the node points (5.3) with grid points (4.1) and parameters (5.4) we define

$$\varphi_{j,q}(t) = \frac{(t - \xi_{j,1}) \ldots (t - \xi_{j,q-1})(t - \xi_{j,q+1}) \ldots (t - \xi_{j,m})}{(\xi_{j,q} - \xi_{j,1}) \ldots (\xi_{j,q} - \xi_{j,q-1})(\xi_{j,q} - \xi_{j,q+1}) \ldots (\xi_{j,q} - \xi_{j,m})}, \quad t_{j-1} \leq t \leq t_j, \tag{5.6}$$

$$\psi_{j,q}(t) = \begin{cases} \varphi_{j,q}(t), & t \in [t_{j-1}, t_j], \\ 0, & t \in [0, b] \setminus [t_{j-1}, t_j], \end{cases} \tag{5.7}$$

for $q = 1, \ldots, m$; $j = 1, \ldots, N$. Using this, we can write the interpolation of $u \in C[0,b]$ by an element of $S_{m-1}^{(-1)}(\Delta_N)$ as follows:

$$(P_N u)(t) = \sum_{j=1}^{N} \sum_{q=1}^{m} u(\xi_{j,q}) \psi_{j,q}(t) \,, \quad t \in [0,b] \,. \tag{5.8}$$

Moreover, in every subinterval $[t_{j-1}, t_j]$ ($j = 1, \ldots, N$) we may use the representation

$$(P_N u)(t) = \sum_{q=1}^{m} u(\xi_{j,q}) \varphi_{j,q}(t) \,, \quad t \in [t_{j-1}, t_j] \,, \; j = 1, \ldots, N \,. \tag{5.9}$$

It follows from [63], p. 115, the following result.

Lemma 5.1 *Assume that the node points* (5.3) *with grid points* (4.1) *and parameters* (5.4) *are used. Let* $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$ *be determined by the conditions* (5.5).

Then $P_N \in \mathcal{L}(C[t_{j-1}, t_j], C[t_{j-1}, t_j])$ *($j = 1, \ldots, N$) and* $P_N \in \mathcal{L}(C[0,b], L^\infty(0,b))$. *Moreover, the norms of these operators are uniformly bounded in N:*

$$\max_{j=1,\ldots,N} \left\| P_N \right\|_{\mathcal{L}(C[t_{j-1},t_j], C[t_{j-1},t_j])} \le c \,, \quad N \in \mathbb{N} \,, \tag{5.10}$$

$$\left\| P_N \right\|_{\mathcal{L}(C[0,b], L^\infty(0,b))} \le c \,, \quad N \in \mathbb{N} \,. \tag{5.11}$$

Here c is a positive constant which is independent of j and N.

5.3 Error Estimates for Interpolation

On the basis of Lemma 5.1 we obtain the following result (cf. [63, 69]).

Lemma 5.2 *Let* $u \in C[0,b]$. *Let the node points* (5.3) *with grid points* (4.1) *and parameters* (5.4) *be used and let* $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$ *be determined by the conditions* (5.5).

Then

$$\left\| u - P_N u \right\|_{L^\infty(0,b)} \le c \max_{j=1,\ldots,N} \inf_{v \in S_{m-1}^{(-1)}(\Delta_N)} \max_{t_{j-1} \le t \le t_j} \left| u(t) - v(t) \right| . \tag{5.12}$$

where c is a positive constant not depending on N.

Proof. Taking a $v \in S^{(-1)}_{m-1}(\Delta_N)$, we obtain on the basis of (5.10) that

$$\begin{aligned}
\|u - P_N u\|_{L^\infty(0,b)} &= \max_{j=1,\dots,N} \|u - v + v - P_N u\|_{L^\infty(t_{j-1},t_j)} \\
&= \max_{j=1,\dots,N} \|u - v + P_N(v-u)\|_{C[t_{j-1},t_j]} \\
&\le \max_{j=1,\dots,N} \big(1 + \|P_N\|_{\mathcal{L}(C[t_{j-1},t_j],C[t_{j-1},t_j])}\big)\|u - v\|_{C[t_{j-1},t_j]} \\
&\le c \max_{j=1,\dots,N} \max_{t_{j-1}\le t\le t_j} |u(t) - v(t)|\,,
\end{aligned}$$

where c is a positive constant not depending on N. This together with the arbitrariness of $v \in S^{(-1)}_{m-1}(\Delta_N)$ yields (5.12). □

Next we present a result about the rate of the error $\|u - P_N u\|_{L^\infty(0,b)}$ (Lemma 5.3) in a situation which has not been studied before.

Lemma 5.3 *Assume that* $u \in C^{m,\nu}_{d,0}[0,b]$*,* $m \in \mathbb{N}$*,* $-\infty < \nu < 1$*. Let the node points* (5.3) *with grid points* $\{(4.1),(4.2)\}$ *and parameters* (5.4) *be used and let* $P_N : C[0,b] \to S^{(-1)}_{m-1}(\Delta_N)$ *be determined by the conditions* (5.5)*.*

Then

$$\big\|u - P_N u\big\|_{L^\infty(0,b)} \le c \begin{cases} h_N^m\,, & m < 1-\nu\,, \\ h_N^m(1 + |\ln h_N|)\,, & m = 1-\nu\,, \\ h_N^{1-\nu}\,, & m > 1-\nu\,, \end{cases} \tag{5.13}$$

where c *is a positive constant not depending on* N *and* h_N *is given by* (4.4)*.*

Proof. Let $t_{j/2} = (t_{j-1}+t_j)/2$, $j = 1,\dots,N$. Taking the Taylor polynomial

$$\begin{aligned}
v(t) = u(t_{j/2}) + u'(t_{j/2})(t - t_{j/2}) + \frac{1}{2!}u''(t_{j/2})(t - t_{j/2})^2 + \dots \\
+ \frac{1}{(m-1)!}u^{(m-1)}(t_{j/2})(t - t_{j/2})^{m-1}\,, \quad t \in [t_{j-1},t_j]\,,
\end{aligned} \tag{5.14}$$

we have, due to Lemma 5.2,

$$\big\|u - P_N u\big\|_{L^\infty(0,b)} \le c \max_{j=1,\dots,N} \max_{t_{j-1}\le t\le t_j} |u(t) - v(t)|\,, \tag{5.15}$$

where c is a positive constant which is independent of N.

Let us estimate

$$u(t) - v(t) = \frac{1}{(m-1)!} \int_{t_{j/2}}^{t} (t-s)^{m-1} u^{(m)}(s) ds \,, \tag{5.16}$$

where $t_{j-1} \le t \le t_j$. For every $t \in [t_{j-1}, t_j]$ $(j = 1, \dots, N)$ we obtain that

$$\bigl|u(t) - v(t)\bigr| \le c \times$$
$$\left| \int_{t_{j/2}}^{t} |t-s|^{m-1} \left\{ \begin{array}{ll} 1\,, & m < 1-\nu \\ 1 + |\ln s| + |\ln|s-d|| + |\ln(b-s)|\,, & m = 1-\nu \\ s^{1-\nu-m} + |d-s|^{1-\nu-m} + (b-s)^{1-\nu-m}, & m > 1-\nu \end{array} \right\} ds \right| \tag{5.17}$$

Let us temporarily consider the case $m > 1-\nu$, $t_{j-1} \le t < s \le t_{j/2}$. Then

$$\max\left\{s^{1-\nu-m}, |d-s|^{1-\nu-m}, (b-s)^{1-\nu-m}\right\} \le (s-t)^{1-\nu-m} \,.$$

Analogously,

$$\max\left\{s^{1-\nu-m}, |d-s|^{1-\nu-m}, (b-s)^{1-\nu-m}\right\} \le (t-s)^{1-\nu-m}$$

for $m > 1-\nu$, $t_{j/2} \le s < t \le t_j$. In a similar way we can estimate the second row in (5.17) for $m = 1-\nu$:

$$1 + |\ln s| + |\ln|s-d|| + |\ln(b-s)| \le c(|\ln|s-t|| + 1) \,,$$

with $t_{j-1} \le t < s \le t_{j/2}$ or $t_{j/2} \le s < t \le t_j$. Thus, for $t \in [t_{j-1}, t_{j/2}]$ $(j = 1, \dots, N)$, it follows from (5.17) that

$$\bigl|u(t) - v(t)\bigr| \le c_1 \int_{t}^{t_{j/2}} (s-t)^{m-1} \left\{ \begin{array}{ll} 1\,, & m < 1-\nu \\ 1 + |\ln(s-t)|\,, & m = 1-\nu \\ (s-t)^{1-\nu-m}, & m > 1-\nu \end{array} \right\} ds$$

$$\le c_2 \left\{ \begin{array}{ll} (t_{j/2} - t)^m, & m < 1-\nu\,, \\ (t_{j/2} - t)^m (1 + |\ln(t_{j/2} - t)|)\,, & m = 1-\nu\,, \\ (t_{j/2} - t)^{1-\nu}, & m > 1-\nu\,, \end{array} \right.$$

$$\le c_3 \left\{ \begin{array}{ll} h_N^m, & m < 1-\nu\,, \\ h_N^m (1 + |\ln h_N|)\,, & m = 1-\nu\,, \\ h_N^{1-\nu}, & m > 1-\nu\,. \end{array} \right.$$

In a similar way we obtain that

$$|u(t)-v(t)| \le c \begin{cases} h_N^m\,, & m<1-\nu\,,\\ h_N^m(1+|\ln h_N|)\,, & m=1-\nu\,,\\ h_N^{1-\nu}\,, & m>1-\nu\,, \end{cases}$$

where $t_{j/2} \le t \le t_j$.

In summary,

$$\big|u(t)-v(t)\big| \le c \left\{ \begin{array}{ll} h_N^m\,, & m<1-\nu \\ h_N^m(1+|\ln h_N|\,, & m=1-\nu \\ h_N^{1-\nu}, & m>1-\nu \end{array} \right\}, \qquad t\in[t_{j-1},t_j],\ j=1,\ldots,N\,, \quad (5.18)$$

where c is a positive constant not depending on N. Now (5.13) follows from (5.15) and (5.18). □

5.4 Error Estimates on Graded Grids

Let $n \in \mathbb{N}$, $N=4n$, $0<d<b$, $r, r_d \in [1,\infty)$. Assume that the grid points $t_0, t_1, \ldots, t_N$ of the grid Δ_N (see (4.1)) are given by the following formulae:

$$\begin{aligned} t_j &= \frac{d}{2}\left(\frac{j}{n}\right)^r, \quad j=0,1,\ldots,n\,;\\ t_{n+j} &= d-\frac{d}{2}\left(\frac{n-j}{n}\right)^{r_d}, \quad j=1,\ldots,n\,;\\ t_{2n+j} &= d+\frac{b-d}{2}\left(\frac{j}{n}\right)^{r_d}, \quad j=1,\ldots,n\,;\\ t_{3n+j} &= b-\frac{b-d}{2}\left(\frac{n-j}{n}\right)^{r}, \quad j=1,\ldots,n\,. \end{aligned} \qquad (5.19)$$

Then Δ_N is called a graded grid for $[0,b]$. In the present context the so-called grading exponents $r, r_d \in \mathbb{R}$ will always satisfy $r \ge 1$ and $r_d \ge 1$. These parameters characterize the accumulation of nodes $t_0, t_1, \ldots t_N$ near the points of possible unboundedness of the derivatives of the solution u of equation (3.1) (see Theorem 3.2). For larger r and r_d the grid $\Delta_N = \Delta_N^{r,r_d}$ is thicker near 0, d and b. We use two different parameters r and r_d because the order of singularity of the solution u can be different at points 0, b and d. If $r = r_d = 1$ then the grid points (5.19) are uniformly located in the intervals $[0, d]$ and $[d, b]$. For $d = \frac{b}{2}$ and $r = r_d = 1$ we obtain the uniform grid for $[0, b]$ with $N+1$ nodes $t_j = b\frac{j}{N}$, $j = 0, 1, \ldots, N$. Note that if $r > 1$ or $r_d > 1$ then the graded grid is not quasi-uniform.

Notice also that for graded grids Δ_N with the grid points (5.19) an estimate

$$h_N \le cN^{-1} \tag{5.20}$$

holds with a positive constant c which is independent of N. Indeed, let us consider the case $t \in [t_{j-1}, t_j] \subset [0, \frac{d}{2}]$ $(j = 1, \ldots, n)$. Due to (5.19),

$$t_j - t_{j-1} = \frac{d}{2} n^{-r} \big[j^r - (j-1)^r \big] \le \frac{d}{2} r n^{-1} = 2drN^{-1}, \quad j = 1, \ldots, n\,.$$

For other subintervals the argument is analogous. Thus, it follows from (5.20) that for the grid (5.19) the convergence (4.8) holds.

In the following we present a new result about the rate of the error $\|u - P_N u\|_{L^\infty(0,b)}$ on graded grids which is partly published in [16, 18, 19].

Lemma 5.4 *Let* $u \in C^{m,\nu}_{d,p}[0, b]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $p \in \{0, 1, \ldots, m\}$. *Let the node points* (5.3) *with grid points* (5.19) *and parameters* (5.4) *be used. Let* $P_N : C[0, b] \to S^{(-1)}_{m-1}(\Delta_N)$ *be determined by the conditions* (5.5).

Then

$$\big\| u - P_N u \big\|_{L^\infty(0,b)} \le c\varepsilon_N\,, \tag{5.21}$$

where c is a positive constant not depending on N *and* $\varepsilon_N = \varepsilon_N(m, \nu, p, r, r_d)$

is given by the following formula:

$$\varepsilon_N = \begin{cases} N^{-m}, & \text{for} \quad m < 1-\nu,\ p \geq 0,\ r \geq 1,\ r_d \geq 1; \\ N^{-m}, & \text{for} \quad m = 1-\nu,\ p = 0,\ r > 1,\ r_d > 1; \\ N^{-m}\ln N, & \text{for} \quad m = 1-\nu,\ p = 0,\ r = 1,\ r_d \geq 1; \\ N^{-m}\ln N, & \text{for} \quad m = 1-\nu,\ p = 0,\ r \geq 1,\ r_d = 1; \\ N^{-m}, & \text{for} \quad m = 1-\nu,\ p > 0,\ r > 1,\ r_d \geq 1; \\ N^{-m}\ln N, & \text{for} \quad m = 1-\nu,\ p > 0,\ r = 1,\ r_d \geq 1; \\ N^{-r(1-\nu)}, & \text{for} \quad 1-\nu < m < 1-\nu+p,\ p > 0, \\ & \qquad 1 \leq r < \dfrac{m}{1-\nu},\ r_d \geq 1; \\ N^{-m}, & \text{for} \quad 1-\nu < m < 1-\nu+p,\ p > 0, \\ & \qquad r \geq \dfrac{m}{1-\nu},\ r_d \geq 1; \\ N^{-m}\ln N, & \text{for} \quad m = 1-\nu+p,\ p > 0, \\ & \qquad r \geq \dfrac{m}{1-\nu},\ r_d = 1; \\ N^{-m}, & \text{for} \quad m = 1-\nu+p,\ p > 0, \\ & \qquad r \geq \dfrac{m}{1-\nu},\ r_d > 1; \\ N^{-\min\{r(1-\nu),r_d(1-\nu+p)\}}, & \\ & \text{for} \quad m > 1-\nu+p,\ p \geq 0, \\ & \qquad 1 \leq r < \dfrac{m}{1-\nu},\ 1 \leq r_d < \dfrac{m}{1-\nu+p}; \\ N^{-r_d(1-\nu+p)}, & \text{for} \quad m > 1-\nu+p,\ p \geq 0, \\ & \qquad r \geq \dfrac{m}{1-\nu},\ 1 \leq r_d < \dfrac{m}{1-\nu+p}; \\ N^{-r(1-\nu)}, & \text{for} \quad m > 1-\nu+p,\ p \geq 0, \\ & \qquad 1 \leq r < \dfrac{m}{1-\nu},\ r_d \geq \dfrac{m}{1-\nu+p}; \\ N^{-m}, & \text{for} \quad m > 1-\nu+p,\ p \geq 0, \\ & \qquad r \geq \dfrac{m}{1-\nu},\ r_d \geq \dfrac{m}{1-\nu+p}. \end{cases} \tag{5.22}$$

Proof. In order to study the approximation properties of $P_N u$ we follow the approach of [63]. It follows from Lemma 5.2 that

$$\left\|u - P_N u\right\|_{L^\infty(0,b)} \leq c \max_{j=1,\ldots,N} \max_{t_{j-1} \leq t \leq t_j} |u(t) - v(t)|\,, \tag{5.23}$$

where c is a positive constant not depending on N and v is an arbitrary element of the space $S_{m-1}^{(-1)}(\Delta_N)$. We shall estimate $|u(t)-v(t)|$ for a suitable $v(t)$ on every subinterval $[t_{j-1}, t_j]$, $j = 1, \ldots, N$.

Actually, we derive the estimates for $|u(t)-v(t)|$ on any subinterval $[t_{j-1}, t_j] \subset [0, d]$ with $j = 1, \ldots, 2n$ (see (5.19)). For $[t_{j-1}, t_j] \subset [d, b]$ with $j = 2n+1, \ldots, 4n$ the argument is analogous.

First we study the case $[t_{j-1}, t_j] \subset \left[0, \frac{d}{2}\right]$, $j = 1, \ldots, n$ (CASE A). After that we consider the case $[t_{j-1}, t_j] \subset \left[\frac{d}{2}, d\right]$, $j = n+1, \ldots, 2n$ (CASE B).

CASE A.

Denote

$$v(t) = u(t_j) + u'(t_j)(t-t_j) + \frac{1}{2!}u''(t_j)(t-t_j)^2 + \ldots \\ + \frac{1}{(m-1)!}u^{(m-1)}(t_j)(t-t_j)^{m-1}, \quad t \in [t_{j-1}, t_j]. \quad (5.24)$$

Then v is a polynomial of degree $m-1$ and

$$u(t) - v(t) = \frac{1}{(m-1)!}\int_{t_j}^{t}(t-s)^{m-1}u^{(m)}(s)ds, \quad t \in [t_{j-1}, t_j], \quad (5.25)$$

or

$$u(t) - v(t) = \frac{1}{m!}(t-t_j)^m u^{(m)}(\xi_j), \quad \xi_j \in (t_{j-1}, t_j),\ t \in [t_{j-1}, t_j]. \quad (5.26)$$

Let $j = 1$, $t \in [0, t_1]$. Making use of the estimates for the derivatives of $u \in C^{m,\nu}_{d,p}[0, b]$ (see (3.10)) and (5.25), we obtain that

$$|u(t) - v(t)| = \left|\frac{1}{(m-1)!}\int_{t_1}^{t}(t-s)^{m-1}u^{(m)}(s)ds\right|$$

$$\leq c_1 \int_{t}^{t_1}(s-t)^{m-1}\left\{\begin{array}{ll} 1, & m < 1-\nu \\ 1 + |\ln s| + |\ln|d-s|| + |\ln(b-s)|, & m = 1-\nu \\ s^{1-\nu-m} + |d-s|^{1-\nu-m} + (b-s)^{1-\nu-m}, & m > 1-\nu \end{array}\right\} ds$$

$$\leq c_2 \int_{t}^{t_1}(s-t)^{m-1}\left\{\begin{array}{ll} 1, & m < 1-\nu \\ 1 + |\ln s|, & m = 1-\nu \\ s^{1-\nu-m}, & m > 1-\nu \end{array}\right\} ds \quad (5.27)$$

Clearly, for $m < 1-\nu$,

$$c_2 \int_t^{t_1} (s-t)^{m-1} ds = \frac{c_2}{m}(t_1-t)^m \le \frac{c_2}{m} t_1^m \le c_3 n^{-rm}, \quad t \in [0, t_1]. \tag{5.28}$$

Consider the case $m = 1-\nu$. Then (5.19) and (5.27) yield

$$\begin{aligned} |u(t)-v(t)| &\le c_4 t_1^{m-1} \int_t^{t_1} (1+|\ln s|) ds \\ &\le c_5 t_1^m (1+|\ln t_1|) \\ &= c_5 \left[\frac{d}{2}\left(\frac{1}{n}\right)^r\right]^m \left[1+\left|\ln\left(\frac{d}{2}\left(\frac{1}{n}\right)^r\right)\right|\right] \end{aligned} \tag{5.29}$$

Thus, if $m = 1-\nu$, $r = 1$ then

$$|u(t)-v(t)| \le c_6 n^{-m}(1+\ln n), \quad t \in [0, t_1]. \tag{5.30}$$

If $m = 1-\nu$, $r > 1$, then

$$\begin{aligned} |u(t)-v(t)| &\le c_7 \left(\frac{1}{n}\right)^{rm} \left[1+\left|\ln\left(\frac{1}{n}\right)^r\right|\right] \\ &\le c_8 \frac{1}{n^m}\left[1+\frac{\ln n}{n^{(r-1)m}}\right] \\ &\le c_9 n^{-m}, \quad t \in [0, t_1], \end{aligned} \tag{5.31}$$

for all sufficiently large n, say $n \ge n_0$.

Consider now the case $m > 1-\nu$ in (5.27). Set $s = t_1\sigma$, $t = t_1\tau$. Then the estimate (5.27) takes the form

$$\begin{aligned} |u(t)-v(t)| &\le c_2 \int_t^{t_1} (s-t)^{m-1} s^{1-\nu-m} ds \\ &= c_2 \int_\tau^1 (t_1\sigma - t_1\tau)^{m-1} (t_1\sigma)^{1-\nu-m} t_1 d\sigma \\ &= c_2 t_1^{1-\nu} \int_\tau^1 (\sigma-\tau)^{m-1} \sigma^{1-\nu-m} d\sigma \\ &\le c_2 \gamma t_1^{1-\nu}, \end{aligned}$$

where

$$\gamma = \sup_{0\le\tau\le 1} \int_\tau^1 (\sigma-\tau)^{m-1}\sigma^{-\nu-m+1}d\sigma < \infty \quad (m\in\mathbb{N},\ \nu<1)\,.$$

By the formula (5.19),

$$t_1^{1-\nu} = \left[\frac{d}{2}\left(\frac{1}{n}\right)^r\right]^{1-\nu} \le c_{10}n^{-r(1-\nu)}.$$

Therefore for $m>1-\nu$, $1\le r<\dfrac{m}{1-\nu}$ we obtain the estimate

$$|u(t)-v(t)| \le c_{11}n^{-r(1-\nu)}, \quad t\in[0,t_1]\,. \tag{5.32}$$

For $m>1-\nu$, $r\ge\dfrac{m}{1-\nu}$ we can estimate $u(t)-v(t)$ as follows:

$$|u(t)-v(t)| \le c_{12}n^{-m}, \quad t\in[0,t_1]\,. \tag{5.33}$$

Let now $t\in[t_{j-1},t_j]$, $j=2,\dots,n$, $n\ge 2$. Using the estimates for the derivatives of $u\in C^{m,\nu}_{d,p}[0,b]$ (see (3.10)) and (5.26), we have

$$\begin{aligned} |u(t)-v(t)| &= \left|\frac{1}{m!}(t-t_j)^m u^{(m)}(\xi_j)\right| \\ &\le c_1(t_j-t)^m \begin{cases} 1\,, & m<1-\nu\,,\\ 1+|\ln t_{j-1}|\,, & m=1-\nu\,,\\ t_{j-1}^{1-\nu-m}, & m>1-\nu\,,\end{cases} \end{aligned} \tag{5.34}$$

where $t\in[t_{j-1},t_j]$, $j=2,\dots,n$, $n\ge 2$. Therefore, for $m<1-\nu$,

$$|u(t)-v(t)| \le c_2n^{-rm}, \quad t\in[t_{j-1},t_j],\ j=2,\dots,n\,. \tag{5.35}$$

For $m=1-\nu$, $r=1$ we obtain that

$$\begin{aligned} |u(t)-v(t)| &\le c_1(t_j-t)^m(1+|\ln t_{j-1}|) \\ &\le c_3n^{-m}(1+\ln n), \quad t\in[t_{j-1},t_j],\ j=2,\dots,n\,. \end{aligned} \tag{5.36}$$

Consider the case $m=1-\nu$, $r>1$. We have

$$\begin{aligned} |u(t)-v(t)| &\le c_1(t_j-t)^m(1+|\ln t_{j-1}|) \\ &\le c_4\left[\left(\frac{j}{n}\right)^r-\left(\frac{j-1}{n}\right)^r\right]^m\left(1+\left|\ln\left(\frac{j-1}{n}\right)^r\right|\right) \\ &= c_4n^{-rm}\big[j^r-(j-1)^r\big]^m\left(1+\left|\ln\left(\frac{j-1}{n}\right)^r\right|\right). \end{aligned}$$

Since

$$\left|j^r-(j-1)^r\right|\le rj^{r-1}\quad (j=1,\dots,n;\ r\ge 1)\,, \tag{5.37}$$

$$\left|\ln\left(\frac{j-1}{n}\right)\right|\le c\left(\frac{j-1}{n}\right)^{-\varepsilon}\quad (j=2,\dots,n;\ \varepsilon>0) \tag{5.38}$$

then for $j=2,\dots,n$ we obtain that

$$\begin{aligned}
&n^{-rm}\big[j^r-(j-1)^r\big]^m\left(1+\left|\ln\left(\frac{j-1}{n}\right)^r\right|\right)\\
&\le cn^{-m}n^{-m(r-1)}j^{(r-1)m}\big(1+(j-1)^{-\varepsilon}n^{\varepsilon}\big)\\
&\le cn^{-m}n^{-m(r-1)}n^{(r-1)m}\big(1+(n-1)^{-\varepsilon}n^{\varepsilon}\big)\\
&\le c_1 n^{-m}\,.
\end{aligned}$$

Therefore, for $m=1-\nu$, $r>1$ we obtain

$$|u(t)-v(t)|\le c_6 n^{-m}\,,\quad t\in[t_{j-1},t_j]\,,\ j=2,\dots,n\,. \tag{5.39}$$

Consider the case $m>1-\nu$ in (5.34). Then (5.19) and (5.34) yield

$$\begin{aligned}
|u(t)-v(t)| &= c_1|t-t_j|^m t_{j-1}^{-\nu-(m-1)}\\
&\le c_1|t_j-t_{j-1}|^m t_{j-1}^{-\nu-(m-1)}\\
&\le c_7\left\{\left[\left(\frac{j}{n}\right)^r-\left(\frac{j-1}{n}\right)^r\right]^m\left(\frac{j-1}{n}\right)^{-r(\nu+(m-1))}\right\}\\
&= c_7\left\{\left[j^r-(j-1)^r\right]n^{-rm}(j-1)^{-r(\nu+(m-1))}n^{r(\nu+(m-1))}\right\}.
\end{aligned}$$

Using the inequality (5.37) we obtain

$$\begin{aligned}
|u(t)-v(t)| &\le c_7\big(rj^{r-1}\big)^m n^{r(\nu-1)}(j-1)^{-r(\nu+(m-1))}\\
&\le c_8 n^{r(\nu-1)}(j-1)^{r(1-\nu)-m}.
\end{aligned} \tag{5.40}$$

Let us study the term $n^{r(\nu-1)}(j-1)^{r(1-\nu)-m}$. For $m>1-\nu$, $1\le r<\dfrac{m}{1-\nu}$, $j=2,\dots,n$, $n\ge 2$ we get

$$n^{r(\nu-1)}(j-1)^{r(1-\nu)-m}\le n^{r(\nu-1)}. \tag{5.41}$$

For $m>1-\nu$, $r\ge\dfrac{m}{1-\nu}$, $j=2,\dots,n$, $n\ge 2$ we obtain that

$$n^{r(\nu-1)}(j-1)^{r(1-\nu)-m}\le n^{r(\nu-1)}(n-1)^{r(1-\nu)-m}\le n^{-m}. \tag{5.42}$$

From the estimates (5.40)–(5.41) it now follows that if $m > 1-\nu$, $1 \le r < \dfrac{m}{1-\nu}$ then

$$|u(t) - v(t)| \le c_8 n^{-r(1-\nu)}, \quad t \in [t_{j-1}, t_j];\ j = 2, \dots, n;\ n \ge 2\,. \tag{5.43}$$

If $m > 1-\nu$, $r \ge \dfrac{m}{1-\nu}$ then we obtain that

$$|u(t) - v(t)| \le c_8 n^{-m}, \quad t \in [t_{j-1}, t_j];\ j = 2, \dots, n;\ n \ge 2\,. \tag{5.44}$$

CASE B.

Let now $[t_{j-1}, t_j] \subset \left[\dfrac{d}{2}, d\right]$, $j = n+1, \dots, 2n$. Denote

$$v(t) = u(t_{j-1}) + u'(t_{j-1})(t - t_{j-1}) + \frac{1}{2!}u''(t_{j-1})(t - t_{j-1})^2 + \dots$$
$$+ \frac{1}{(m-1)!}u^{(m-1)}(t_{j-1})(t - t_{j-1})^{m-1}, \quad t \in [t_{j-1}, t_j]\,. \tag{5.45}$$

Then v is a polynomial of degree $m-1$ and

$$u(t) - v(t) = \frac{1}{(m-1)!}\int\limits_{t_{j-1}}^{t} (t-s)^{m-1} u^{(m)}(s)ds\,, \quad t \in [t_{j-1}, t_j]\,, \tag{5.46}$$

or

$$u(t) - v(t) = \frac{1}{m!}(t - t_{j-1})^m u^{(m)}(\xi_j)\,, \quad \xi_j \in (t_{j-1}, t_j)\,,\ t \in [t_{j-1}, t_j]\,. \tag{5.47}$$

We apply similar proof as in the previous part. Let $j = 2n$, $t \in [t_{2n-1}, d]$. Using the estimates for the derivatives of the solution $u \in C^{m,\nu}_{d,p}[0,b]$ (see (3.10)) and (5.46), we obtain

$$|u(t) - v(t) = \left|\frac{1}{(m-1)!}\int\limits_{t_{2n-1}}^{t} (t-s)^{m-1} u^{(m)}(s)ds\right|$$
$$\le c_1 \int\limits_{t_{2n-1}}^{t} (t-s)^{m-1} \left\{\begin{array}{ll} 1\,, & m < 1-\nu+p \\ 1 + |\ln(d-s)|\,, & m = 1-\nu+p \\ (d-s)^{1-\nu-m+p}, & m > 1-\nu+p \end{array}\right\} ds \tag{5.48}$$

Clearly, for $m < 1 - \nu + p$,

$$|u(t) - v(t)| \le c_1 \int\limits_{t_{2n-1}}^{t} (t-s)^{m-1} ds \le c_2 n^{-r_d m}, \quad t \in [t_{2n-1}, d]. \qquad (5.49)$$

Consider the case $m = 1 - \nu + p$. Then (5.19) and (5.48) yield

$$\begin{aligned} |u(t) - v(t)| &\le c_3 (d - t_{2n-1})^{m-1} \int\limits_{t_{2n-1}}^{t} (1 + |\ln(d-s)|) ds \\ &\le c_4 \left[\frac{d}{2} \left(\frac{1}{n} \right)^{r_d} \right]^m \left[1 + \left| \ln \left(\frac{d}{2} \left(\frac{1}{n} \right)^{r_d} \right) \right| \right]. \end{aligned} \qquad (5.50)$$

Thus, if $m = 1 - \nu + p$, $r_d = 1$, then

$$|u(t) - v(t)| \le c_5 n^{-m} (1 + \ln n), \quad t \in [t_{2n-1}, d], \qquad (5.51)$$

if $m = 1 - \nu + p$, $r_d > 1$, then

$$|u(t) - v(t)| \le c_6 n^{-m}, \quad t \in [t_{2n-1}, d], \qquad (5.52)$$

for all sufficiently large n, say $n \ge n_0$.

Consider now the case $m > 1 - \nu + p$ in (5.48). Set $d - s = (d - t_{2n-1})\sigma$, $d - t = (d - t_{2n-1})\tau$. Then the estimate (5.48) for $m > 1 - \nu + p$ takes the form

$$|u(t) - v(t)| \le c_1 \int\limits_{t_{2n-1}}^{t} (t-s)^{m-1} (d-s)^{1-\nu-m+p} ds \le c_7 (d - t_{2n-1})^{1-\nu+p}.$$

By the formulae (5.19)

$$(d - t_{2n-1})^{1-\nu+p} = \left[\frac{d}{2} \left(\frac{1}{n} \right)^{r_d} \right]^{1-\nu+p} \le c_8 r^{-r_d(1-\nu+p)}.$$

Therefore for $m > 1 - \nu + p$, $1 \le r_d < \dfrac{m}{1 - \nu + p}$ we obtain the estimate

$$|u(t) - v(t)| \le c_9 n^{-r_d(1-\nu+p)}, \quad t \in [t_{2n-1}, d]. \qquad (5.53)$$

For $m > 1 - \nu + p$, $r_d > \dfrac{m}{1 - \nu + p}$, we have

$$|u(t) - v(t)| \le c_{10} n^{-m}, \quad t \in [t_{2n-1}, d], \qquad (5.54)$$

Let now $t \in [t_{j-1}, t_j]$, $j = n+1, \dots, 2n-1$. Making use of the estimates for the derivatives of the solution $u \in C^{m,\nu}_{d,p}[0,b]$ (see (3.10)) and (5.47), we obtain

$$|u(t) - v(t)| = \left| \frac{1}{m!}(t - t_{j-1})^m u^{(m)}(\xi_j) \right|$$
$$\leq c_1 |t - t_{j-1}|^m \begin{cases} 1\,, & m < 1 - \nu + p\,, \\ 1 + |\ln(d - t_j)|\,, & m = 1 - \nu + p\,, \\ (d - t_j)^{1-\nu-m+p}, & m > 1 - \nu + p\,, \end{cases} \tag{5.55}$$

where $t \in [t_{j-1}, t_j]$, $j = n+1, \dots, 2n-1$. Therefore, for $m < 1 - \nu + p$

$$|u(t) - v(t)| \leq c_2 n^{-r_d m}, \quad t \in [t_{j-1}, t_j]\,,\ j = n+1, \dots, 2n-1\,. \tag{5.56}$$

For $m = 1 - \nu + p$, $r_d = 1$ we have

$$|u(t) - v(t)| \leq c_1 |t - t_{j-1}|^m \big(1 + \big|\ln(d - t_j)\big|\big)$$
$$\leq c_3 n^{-m}(1 + \ln n)\,, \quad t \in [t_{j-1}, t_j]\,;\ j = n+1, \dots, 2n-1\,. \tag{5.57}$$

In case $m = 1 - \nu + p$, $r_d > 1$, we obtain that

$$|u(t) - v(t)| \leq c_4 n^{-m}, \quad t \in [t_{j-1}, t_j],\ j = n+1, \dots, 2n-1\,. \tag{5.58}$$

Consider the case $m > 1 - \nu + p$ in (5.55). Using (5.19) and (5.55) we obtain for $m > 1 - \nu + p$, $1 \leq r_d < \dfrac{m}{1 - \nu + p}$ that

$$|u(t) - v(t)| \leq c_5 n^{-r_d(1-\nu+p)}, \quad t \in [t_{j-1}, t_j];\ j = n+1, \dots, 2n-1\,. \tag{5.59}$$

In a similar way we get from (5.19) and (5.55) for $m > 1 - \nu + p$, $r_d \geq \dfrac{m}{1 - \nu + p}$ that

$$|u(t) - v(t)| \leq c_6 n^{-m}, \quad t \in [t_{j-1}, t_j];\ j = n+1, \dots, 2n-1\,. \tag{5.60}$$

Further, in order to combine the estimates for the error $|u(t) - v(t)|$, $t \in [0, d]$, set

$$w_n = \max_{j=1,\dots,2n} \max_{t_{j-1} \leq t \leq t_j} |u(t) - v(t)|\,, \tag{5.61}$$

where $u \in C^{m,\nu}_{d,p}[0,b]$ and $v \in S^{(-1)}_{m-1}(\Delta_N)$ is given by (5.24) or (5.45).

If $m < 1-\nu$ then $m < 1-\nu+p$, $p \geq 0$. Therefore, using (5.28), (5.35), (5.49) and (5.56) we obtain for $m < 1-\nu$, $p \geq 0$, $r \geq 1$ and $r_d \geq 1$ that $w_n \leq c_1(n^{-rm} + n^{-r_d m}) \leq c_1(n^{-m} + n^{-m}) \leq c_2 n^{-m}$, i.e.

$$w_n \leq cN^{-m}, \quad m < 1-\nu\,, \quad p \geq 0\,, \quad r \geq 1\,, \quad r_d \geq 1\,, \tag{5.62}$$

with a positive constant c not depending on $N = 4n$.

For $m = 1-\nu$, $p = 0$, $r > 1$ and $r_d > 1$ the estimates (5.31), (5.39), (5.52) and (5.58) yield

$$w_n \leq cN^{-m}, \quad m = 1-\nu\,, \quad p = 0\,, \quad r > 1\,, \quad r_d > 1\,. \tag{5.63}$$

For $m = 1-\nu$, $p = 0$, $r = 1$ and $r_d = 1$ on the basis of the estimates (5.30), (5.36), (5.51) and (5.57) we get $w_n \leq cn^{-m}\ln n$ and for $m = 1-\nu$, $p = 0$, $r = 1$ and $r_d > 1$ on the basis of (5.30), (5.36), (5.52) and (5.58) we have $w_n \leq c_1(n^{-m}(1+\ln n) + n^{-m}) \leq c_2 n^{-m}(1+\ln n)$, i.e.

$$w_n \leq cN^{-m}\ln N\,, \quad m = 1-\nu,\; p = 0,\; r = 1,\; r_d \geq 1\,. \tag{5.64}$$

Analogously, for $m = 1-\nu$, $p = 0$, $r \geq 1$, $r_d = 1$ the estimates (5.30), (5.31), (5.36), (5.39), (5.51) and (5.57) yield

$$w_n \leq cN^{-m}\ln N\,, \quad m = 1-\nu,\; p = 0,\; r \geq 1,\; r_d = 1\,. \tag{5.65}$$

If $m = 1-\nu$, $p > 0$, $r > 1$ and $r_d \geq 1$ then with the help of (5.31), (5.39), (5.49) and (5.56) we have $w_n \leq c_1(n^{-m} + n^{-r_d m}) \leq c_7(n^{-m} + n^{-m}) \leq c_2 n^{-m}$, i.e.

$$w_n \leq cN^{-m}, \quad m = 1-\nu,\; p > 0,\; r > 1,\; r_d \geq 1\,. \tag{5.66}$$

In case $m = 1-\nu$, $p > 0$, $r = 1$, $r_d \geq 1$ we get from (5.30), (5.36), (5.49) and (5.56) that $w_n \leq c_1(n^{-m}(1+\ln n) + n^{-r_d m}) \leq c_2 n^{-m}(1+\ln n)$ i.e.

$$w_n \leq cN^{-m}\ln N, \quad m = 1-\nu,\; p > 0,\; r = 1,\; r_d \geq 1\,. \tag{5.67}$$

For $1-\nu < m < 1-\nu+p$, $p > 0$, $1 \leq r < \dfrac{m}{1-\nu}$ and $r_d \geq 1$ on the basis of the estimates (5.32), (5.43), (5.49) and (5.56) we obtain $w_n \leq c_1(n^{-r(1-\nu)} + n^{-r_d m}) \leq c_2 n^{-r(1-\nu)}$, i.e.

$$w_n \leq cN^{-r(1-\nu)}, \quad 1-\nu < m < 1-\nu+p,\; p > 0,\; 1 \leq r < \frac{m}{1-\nu},\; r_d \geq 1\,. \tag{5.68}$$

Let $1-\nu < m < 1-\nu+p$, $p>0$, $r \geq \dfrac{m}{1-\nu}$ and $r_d \geq 1$ then by (5.33), (5.44), (5.49) and (5.56) we obtain $w_n \leq c_1(n^{-m} + n^{-r_d m}) \leq c_2 n^{-m}$ i.e.

$$w_n \leq cN^{-m}, \quad 1-\nu < m < 1-\nu+p,\ p>0,\ r \geq \frac{m}{1-\nu},\ r_d \geq 1\,. \quad (5.69)$$

In case $m = 1-\nu+p$, $p>0$, $r \geq \dfrac{m}{1-\nu}$, $r_d = 1$ we get with the help of (5.33), (5.44), (5.51) and (5.57) that $w_n \leq c_1(n^{-m} + n^{-m}(1+\ln n)) \leq c_2 n^{-m}(1+\ln n)$ i.e.

$$w_n \leq cN^{-m}\ln N\,, \quad m = 1-\nu+p,\ p>0,\ r \geq \frac{m}{1-\nu}, r_d = 1\,. \quad (5.70)$$

For $m = 1-\nu+p$, $p>0$, $r \geq \dfrac{m}{1-\nu}$ and $r_d > 1$ the estimates (5.33), (5.44), (5.52) and (5.58) yield

$$w_n \leq cN^{-m}, \quad m = 1-\nu+p,\ p>0,\ r \geq \frac{m}{1-\nu},\ r_d > 1\,. \quad (5.71)$$

Consider the case $m > 1-\nu+p$, $p \geq 0$, $1 \leq r < \dfrac{m}{1-\nu}$ and $1 \leq r_d < \dfrac{m}{1-\nu+p}$. Then using (5.32), (5.43), (5.53) and (5.59) we have $w_n \leq c_1(n^{-r(1-\nu)} + n^{-r_d(1-\nu+p)}) \leq c_2 n^{-\min\{r(1-\nu), r_d(1-\nu+p)\}}$ i.e.

$$w_n \leq cN^{-\min\{r(1-\nu), r_d(1-\nu+p)\}}, \quad m > 1-\nu+p,\ p \geq 0,$$
$$1 \leq r < \frac{m}{1-\nu},\ 1 \leq r_d < \frac{m}{1-\nu+p}\,. \quad (5.72)$$

Let $m > 1-\nu+p$, $p \geq 0$, $r \geq \dfrac{m}{1-\nu}$ and $1 \leq r_d < \dfrac{m}{1-\nu+p}$. Making use of the estimates (5.33), (5.44), (5.53) and (5.59) we obtain $w_n \leq c_1(n^{-m} + n^{-r_d(1-\nu+p)}) \leq c_2 n^{-r_d(1-\nu+p)}$ i.e.

$$w_n \leq cN^{-r_d(1-\nu+p)}, \quad m > 1-\nu+p,\ p \geq 0,\ r \geq \frac{m}{1-\nu},\ 1 \leq r_d < \frac{m}{1-\nu+p}. \quad (5.73)$$

If $m > 1-\nu+p$, $p \geq 0$, $1 \leq r < \dfrac{m}{1-\nu}$ and $r_d \geq \frac{m}{1-\nu+p}$ then the estimates (5.32), (5.43), (5.54) and (5.60) yield $w_n \leq c_1(n^{-r(1-\nu)} + n^{-m}) \leq c_2 n^{-r(1-\nu)}$ i.e.

$$w_n \leq cN^{-r(1-\nu)}, \quad m > 1-\nu+p,\ p \geq 0,\ 1 \leq r < \frac{m}{1-\nu},\ r_d \geq \frac{m}{1-\nu+p}. \quad (5.74)$$

For $m > 1-\nu+p$, $p \geq 0$, $r \geq \dfrac{m}{1-\nu}$ and $r_d \geq \dfrac{m}{1-\nu+p}$ on the basis of the estimates (5.33), (5.44), (5.54) and (5.60) we find

$$w_n \leq cN^{-m}, \quad m > 1-\nu+p,\ p \geq 0,\ r \geq \frac{m}{1-\nu},\ r_d \geq \frac{m}{1-\nu+p}. \tag{5.75}$$

Denote

$$w'_n = \max_{j=2n+1,\ldots,4n} \max_{t_{j-1}\leq t\leq t_j} |u(t) - v(t)|\,, \tag{5.76}$$

where $u \in C^{m,\nu}_{d,p}[0,b]$ and $v \in S^{(-1)}_{m-1}(\Delta_N)$ is given by the formula (5.24) for $t \in [t_{j-1}, t_j] \subset [d,(d+b)/2]$, $j = 2n+1,\ldots,3n$, and by the formula (5.45) for $t \in [t_{j-1}, t_j] \subset [(d+b)/2, b]$, $j = 3n+1,\ldots,4n = N$. If w'_n is used instead of w_n then due to the symmetry, the analogous estimates (5.62)–(5.75) hold for w'_n. Now the assertion of lemma follows from (5.23). □

We present also a result about the error $\|u - P_N u\|_{L^1(0,b)}$ in a situation which has not been considered before.

Lemma 5.5 *Let $u \in C^{m,\nu}_{d,p}[0,b]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $p \in \{0,1,\ldots,m\}$. Let the node points* (5.3) *with grid points* (5.19) *and parameters* (5.4) *be used. Let $P_N : C[0,b] \to S^{(-1)}_{m-1}(\Delta_N)$ be determined by the conditions* (5.5).

Then

$$\|u - P_N u\|_{L^1(0,b)} \leq cN^{-m} \tag{5.77}$$

for

$$\begin{cases} m < 2-\nu,\ p \geq 0,\ r \geq 1,\ r_d \geq 1, \\ m = 2-\nu,\ p = 0,\ r > 1,\ r_d > 1, \\ m = 2-\nu,\ p > 0,\ r > 1,\ r_d \geq 1, \\ 2-\nu < m < 2-\nu+p,\ p > 0,\ r > \dfrac{m}{2-\nu},\ r_d \geq 1, \\ m = 2-\nu+p,\ p > 0,\ r > \dfrac{m}{2-\nu},\ r_d > 1, \\ m > 2-\nu+p,\ p \geq 0,\ r > \dfrac{m}{2-\nu},\ r_d > \dfrac{m}{2-\nu+p}. \end{cases} \tag{5.78}$$

Here c is a positive constant not depending on N.

Proof. Let $u \in C^{m,\nu}_{d,p}[0,b]$ and let r and r_d be chosen so that the conditions (5.78) are fulfilled. Since

$$\int_0^b |u(t) - (P_N u)(t)|dt = \int_0^d |u(t) - (P_N u)(t)|dt + \int_d^b |u(t) - (P_N u)(t)|dt\,,$$

the statement (5.77) follows from the estimates

$$\int_0^d |u(t) - (P_N u)(t)| dt \le c N^{-m} \tag{5.79}$$

and

$$\int_d^b |u(t) - (P_N u)(t)| dt \le c_1 N^{-m}, \tag{5.80}$$

where c and c_1 are some positive constants not depending on N. We prove only the estimate (5.79). For (5.80) the argument is analogous.

We have (see (5.19))

$$\begin{aligned}\int_0^d |u(t) - (P_N u)(t)| dt &= \int_0^{d/2} |u(t) - (P_N u)(t)| dt + \int_{d/2}^d |u(t) - (P_N u)(t)| dt \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |u(t) - (P_N u)(t)| dt \\ &+ \sum_{j=n+1}^{2n} \int_{t_{j-1}}^{t_j} |u(t) - (P_N u)(t)| dt \,. \end{aligned} \tag{5.81}$$

Let us estimate the first sum on the right of the equality (5.81). Since the norms of $P_N \in \mathcal{L}(C[t_{j-1}, t_j], C[t_{j-1}, t_j])$ are uniformly bounded in N (see (5.10)) then

$$\max_{t_{j-1} \le t \le t_j} |u(t) - (P_N u)(t)| \le c \max_{t_{j-1} \le t \le t_j} |u(t) - v(t)| \,, \ j = 1, \dots, N \,, \tag{5.82}$$

where c is a positive constant not depending on N and $v \in \pi_{m-1}$ is a polynomial of degree $m-1$. For $[t_{j-1}, t_j] \subset [0, \frac{d}{2}]$, $j = 1, \dots, n$, let v be the Taylor polynomial (5.24). For $[t_{j-1}, t_j] \subset [\frac{d}{2}, d]$, $j = n+1, \dots, 2n$, let v be the Taylor polynomial (5.45).

Consider the case $m < 1 - \nu$. On the basis of (5.25)–(5.27), (5.34) and (5.82) we obtain that

$$|u(t) - (P_N u)(t)| \le c (t_j - t_{j-1})^m, \quad t \in [t_{j-1}, t_j], \quad j = 1, \dots, n \,. \tag{5.83}$$

Recall that (see (5.19))

$$t_j - t_{j-1} = \frac{d}{2}\Big[\Big(\frac{j}{n}\Big)^r - \Big(\frac{j-1}{n}\Big)^r\Big] \le \frac{d}{2} r j^{r-1} n^{-r}, \quad j = 1, \dots, n\,. \tag{5.84}$$

Notice also that

$$t_{j-1} \ge \Big(\frac{1}{2}\Big)^r t_j\,, \quad j = 2, \dots, n\,. \tag{5.85}$$

Thus, using (5.81), (5.83) and (5.84) we get

$$\int_0^{d/2} |u(t) - (P_N u)(t)| dt \le c n^{-r(m+1)} \sum_{j=1}^{n} j^{(r-1)(m+1)}. \tag{5.86}$$

It follows from [69], p. 42, that

$$\sum_{j=1}^{n} j^{\alpha} \le c \begin{cases} 1\,, & \alpha < -1\,, \\ 1 + \ln n\,, & \alpha = -1\,, \\ n^{\alpha+1}, & \alpha > -1\,, \end{cases} \tag{5.87}$$

where c is a positive constant not depending on n. On the basis (5.86) and (5.87) we obtain for $m < 1 - \nu$ that

$$\int_0^{d/2} |u(t) - (P_N u)(t)| dt \le c\, n^{-m}\,. \tag{5.88}$$

Consider the case $m = 1 - \nu$. Then it follows from (5.25)–(5.27), (5.34), (5.82) and (5.85) that

$$|u(t) - (P_N u)(t)| \le c(t_j - t_{j-1})^m (1 + |\ln t_j|), \;\; t \in [t_{j-1}, t_j], \;\; j = 1, \dots, n. \tag{5.89}$$

With help of (5.79), (5.81) and (5.84) we get

$$\int_0^{d/2} |u(t) - (P_N u)(t)| dt \le c\, n^{-r(m+1)} \sum_{j=1}^{n} j^{(r-1)(m+1)} \Big(1 + \Big|\ln \frac{j}{n}\Big|\Big)\,.$$

Since $|\ln j/n| \le c(j/n)^{-\varepsilon}$, $j = 1, \dots, n$, with a small $\varepsilon > 0$, and by (5.87),

$$n^{r(m+1)+\varepsilon} \sum_{j=1}^{n} j^{(r-1)(m+1)-\varepsilon} \le c\, n^{-m},$$

then we obtain for $m = 1-\nu$ that

$$\int\limits_0^{d/2} |u(t) - (P_N u)(t)| dt \le c\, n^{-m}. \tag{5.90}$$

Consider tha case $m > 1-\nu$. On the basis of (5.25)–(5.27), (5.34), (5.82) and (5.85) we have

$$|u(t) - (P_N u)(t)| \le c(t_j - t_{j-1})^m t_j^{1-\nu-m},\ t \in [t_{j-1}, t_j],\ j = 1, \dots, n\,. \tag{5.91}$$

Thus, using (5.79), (5.81) and (5.84) we get

$$\int\limits_0^{d/2} |u(t) - (P_N u)(t)| dt \le c\, n^{-r(2-\nu)} \sum_{j=1}^{n} j^{r(2-\nu)-m-1}. \tag{5.92}$$

It follows from (5.87) and (5.92) for $m > 1-\nu$ that

$$\int\limits_0^{d/2} |u(t) - (P_N u)(t)| dt \le c \begin{cases} n^{-r(2-\nu)}, & 1 \le r < \dfrac{m}{2-\nu}, \\ & m > 2-\nu\,, \\ n^{-r(2-\nu)}(1+\ln n), & r = \dfrac{m}{2-\nu},\ m \ge 2-\nu, \\ n^{-m}, & r > \dfrac{m}{2-\nu},\ m > 2-\nu. \end{cases} \tag{5.93}$$

In summary, with help of (5.88), (5.90) and (5.93) we get

$$\int\limits_0^{d/2} |u(t) - (P_N u)(t)| dt \le c \begin{cases} n^{-m}, & m < 2-\nu,\ r \ge 1, \\ n^{-m}(1+\ln n), & m = 2-\nu,\ r = 1, \\ n^{-m}, & m = 2-\nu,\ r > 1, \\ n^{-r(2-\nu)}, & m > 2-\nu,\ 1 \le r < \dfrac{m}{2-\nu}, \\ n^{-m}(1+\ln n), & m > 2-\nu,\ r = \dfrac{m}{2-\nu}, \\ n^{-m}, & m > 2-\nu,\ r > \dfrac{m}{2-\nu}, \end{cases}$$

where c is a positive constant not depending on n.

It remains to estimate the second sum on the right of the equality (5.81).

Consider the case $m < 1-\nu+p$. In this case on the basis of (5.46)–(5.48), (5.55) and (5.82) we obtain that

$$|u(t) - (P_N u)(t)| \le c(t_j - t_{j-1})^m,\ t \in [t_{j-1}, t_j],\ j = n+1, \dots, 2n\,. \tag{5.94}$$

Recall that (see (5.19))

$$t_{n+i} - t_{n+i-1} = \frac{d}{2}\Big[\Big(\frac{n-i+1}{n}\Big)^{r_d} - \Big(\frac{n-i}{n}\Big)^{r_d}\Big]$$
$$\le \frac{d}{2} r_d (n-i+1)^{r_d-1} n^{-r_d}, \ i = 1, \ldots, n. \qquad (5.95)$$

Thus, using (5.81), (5.94) and (5.95) we get

$$\int_{d/2}^{d} |u(t) - (P_N u)(t)| dt \le c\, n^{-r_d(m+1)} \sum_{i=1}^{n} (n-i+1)^{(r_d-1)(m+1)}. \qquad (5.96)$$

On the basis (5.87) and (5.96) we obtain for $m < 1 - \nu + p$, $p \ge 0$, that

$$\int_{d/2}^{d} |u(t) - (P_N u)(t)| dt \le c\, n^{-m}. \qquad (5.97)$$

Consider the case $m = 1 - \nu + p$. Then it follows from (5.46)–(5.48), (5.55) and (5.82) that

$$|u(t) - (P_N u)(t)| \le c(t_j - t_{j-1})^m (1 + |\ln(d - t_{j-1})|),$$
$$t \in [t_{j-1}, t_j], \ j = n+1, \ldots, 2n. \qquad (5.98)$$

With help of (5.79), (5.81), (5.87) and (5.95) we get

$$\int_{d/2}^{d} |u(t) - (P_N u)(t)| dt$$
$$\le c\, n^{-r_d(m+1)} \sum_{i=1}^{n} (n-i+1)^{(r_d-1)(m+1)} \Big(1 + \Big| \ln \frac{n-i+1}{n} \Big|^{r_d}\Big).$$

Thus,

$$\int_{d/2}^{d} |u(t) - (P_N u)(t)| dt \le c\, n^{-m}, \quad m = 1 - \nu + p, \ p \ge 0\,. \qquad (5.99)$$

Consider the case $m > 1-\nu+p$. On the basis of (5.46)–(5.48), (5.55) and (5.82) we have

$$|u(t)-(P_N u)(t)| \le c(t_j - t_{j-1})^m (d-t_{j-1})^{1-\nu-m+p},$$
$$t \in [t_{j-1}, t_j],\ j = n+1,\dots,2n. \quad (5.100)$$

Thus, using (5.79), (5.81) and (5.95) we get

$$\int\limits_{d/2}^{d} |u(t)-(P_N u)(t)|dt \le c\, n^{-r_d(2-\nu+p)} \sum_{i=1}^{n} (n-i+1)^{r_d(2-\nu+p)-m-1}. \quad (5.101)$$

It follows from (5.87) and (5.101) for $m > 1-\nu+p$, $p \ge 0$, that

$$\int\limits_{d/2}^{d} |u(t)-(P_N u)(t)|dt$$
$$\le c \begin{cases} n^{-r_d(2-\nu+p)}, & 1 \le r_d < \dfrac{m}{2-\nu+p},\ m > 2-\nu+p, \\ n^{-r_d(2-\nu+p)}(1+\ln n), & r_d = \dfrac{m}{2-\nu+p},\ m \ge 2-\nu+p, \\ n^{-m}, & r_d > \dfrac{m}{2-\nu+p},\ m > 2-\nu+p. \end{cases} \quad (5.102)$$

In summary it follows from (5.97), (5.99) and (5.102) that

$$\int\limits_{d/2}^{d} |u(t)-(P_N u)(t)|dt$$
$$\le c \begin{cases} n^{-m}, & m < 2-\nu+p,\ r_d \ge 1, \\ n^{-m}(1+\ln n), & m = 2-\nu+p,\ r_d = 1, \\ n^{-m}, & m = 2-\nu+p,\ r_d > 1, \\ n^{-r_d(1-\nu+p)}, & m > 2-\nu+p,\ 1 \le r_d < \dfrac{m}{2-\nu+p}, \\ n^{-m}(1+\ln n), & m > 2-\nu+p,\ r_d = \dfrac{m}{2-\nu+p}, \\ n^{-m}, & m > 2-\nu+p,\ r_d > \dfrac{m}{2-\nu+p}, \end{cases}$$

where $p \ge 0$ and c is a positive constant not depending on n.

Arquing in a similar way as in the proof of Lemma 5.4 we obtain (5.77). □

Chapter 6

Collocation Method

A common numerical technique for solving integral equations of the second kind is to look for an approximate solution in the form of a linear combination of certain functions, and then to select a particular linear combination by forcing the approximate solution to satisfy the integral equation at a selected set of points in the integration region. This defines the collocation method, and the points referred to in the definition are the collocation points. This method is one of the most efficient numerical approach for the solution of integral equations. The following treatment is an extended expansion of the corresponding results of [16, 18, 19].

6.1 Description of Collocation Method

In this section we introduce a piecewise polynomial collocation method for the numerical solution of equation

$$u(t) - \int_0^b K(t,s)u(s)ds = f(t)\,, \quad 0 \le t \le b\,. \tag{6.1}$$

We assume that $f \in C[0,b]$ and the kernel K satisfies the conditions of Theorem 3.1.

We look for an approximation u_N to the solution u of equation (6.1) determing u_N from the following conditions:

$$\begin{gathered}\Big[u_N(t) - \int_0^b K(t,s)u_N(s)ds - f(t)\Big]_{t=\xi_{i,p}} = 0\,, \\ u_N \in S^{(-1)}_{m-1}(\Delta_N)\,, \quad m \geq 1\,, \\ p = 1,\ldots,m\,;\ i = 1,\ldots,N\,,\end{gathered} \tag{6.2}$$

with $\{\xi_{i,p}\}$, given by (5.3). We can represent $u_N \in S^{(-1)}_{m-1}(\Delta_N)$ in the form

$$u_N(t) = \sum_{j=1}^{N}\sum_{q=1}^{m} c_{j,q}\psi_{j,q}(t)\,, \quad t \in [0,b]\,, \tag{6.3}$$

with $\psi_{j,q}(t)$ ($q = 1,\ldots,m$; $j = 1,\ldots,N$), given by (5.7). Now the conditions (6.2) will take the following form of a system of algebraic equations with respect to the coefficients $c_{j,q} = u_N(\xi_{j,q})$, $q = 1,\ldots,m$; $j = 1,\ldots,N$:

$$c_{i,p} = \sum_{j=1}^{N}\sum_{q=1}^{m} a_{i,p,j,q}c_{j,q} + f(\xi_{i,p})\,, \tag{6.4}$$

$$p = 1,\ldots,m\,; i = 1,\ldots,N\,,$$

where

$$a_{i,p,j,q} = \int_{t_{j-1}}^{t_j} K(\xi_{i,p},s)\psi_{j,q}(s)ds\,, \tag{6.5}$$

$$q = 1,\ldots,m\,;\ j = 1,\ldots,N\,;\ p = 1,\ldots,m\,;\ i = 1,\ldots,N\,.$$

If $\eta_1 > -1$ or $\eta_m < 1$ (see (5.4)), then all collocation points $\xi_{j,q}$ ($q = 1,\ldots,m$; $j = 1,\ldots,N$) are different and there are mN collocation points. In this case the system (6.4) (system (6.2)) has $mN = \dim S^{(-1)}_{m-1}(\Delta_N)$ equations and the same number of unknowns. If $\eta_1 = -1$, $\eta_m = 1$, then a part of the collocation points will coincide. The number of different collocation points is $N(m-1)+1 = \dim S^{(0)}_{m-1}(\Delta_N)$ and the system (6.4) (system (6.2)) has the same number equations and unknowns.

We write the system (6.4) in the form

$$\overline{u}_N = T_N\overline{u}_N + \overline{f}_N\,, \tag{6.6}$$

where

$$\overline{u}_N = (c_{1,1},\ldots,c_{1,m},c_{2,1},\ldots,c_{2,m},\ldots,c_{N,1},\ldots,c_{N,m})^T,$$

$$\overline{f}_N = (f(\xi_{1,1}), \ldots, f(\xi_{1,m}), f(\xi_{2,1}), \ldots, f(\xi_{2,m}), \ldots, f(\xi_{N,1}), \ldots, f(\xi_{N,m}))^T,$$

are vectors and

$$T_N =$$

$$\begin{pmatrix}
a_{1,1,1,1} & \cdots & a_{1,1,1,m} & a_{1,1,2,1} & \cdots & a_{1,1,2,m} & \cdots & a_{1,1,N,1} & \cdots & a_{1,1,N,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{1,m,1,1} & \cdots & a_{1,m,1,m} & a_{1,m,2,1} & \cdots & a_{1,m,2,m} & \cdots & a_{1,m,N,1} & \cdots & a_{1,m,N,m} \\
a_{2,1,1,1} & \cdots & a_{2,1,1,m} & a_{2,1,2,1} & \cdots & a_{2,1,2,m} & \cdots & a_{2,1,N,1} & \cdots & a_{2,1,N,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{2,m,1,1} & \cdots & a_{2,m,1,m} & a_{2,m,2,1} & \cdots & a_{2,m,2,m} & \cdots & a_{2,m,N,1} & \cdots & a_{2,m,N,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{N,1,1,1} & \cdots & a_{N,1,1,m} & a_{N,1,2,1} & \cdots & a_{N,1,2,m} & \cdots & a_{N,1,N,1} & \cdots & a_{N,1,N,m} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{N,m,1,1} & \cdots & a_{N,m,1,m} & a_{N,m,2,1} & \cdots & a_{N,m,2,m} & \cdots & a_{N,m,N,1} & \cdots & a_{N,m,N,m}
\end{pmatrix}$$

is a matrix with elements (6.5).

6.2 Convergence of Collocation Approximations

In the present section we shall study the order of global convergence of the collocation method introduced in the previous section. In the next theorem we present a result about the convergence of collocation approximations generalizing our results in [16, 18, 19].

Theorem 6.1 *Let the following conditions be fulfilled:*

1) *K is subject to the conditions given in Theorem 3.1;*

2) *$f \in C[0,b]$;*

3) *the homogeneous integral equation*

$$u(t) = \int_0^b K(t,s)u(s)ds\,, \quad 0 \le t \le b\,, \tag{6.7}$$

has only the trivial solution $u = 0$;

4) the grid $\{(4.1), (4.2)\}$ *satisfies the condition* (4.8) *and the collocation points* (5.3) *are used.*

Then equation (6.1) *has a unique solution* $u^* \in C[0,b]$. *For all sufficiently large* N, *say* $N \geq N_0$, *the collocation conditions* (6.2) *determine for every choice of parameters* $-1 \leq \eta_1 < \ldots < \eta_m \leq 1$ *a unique approximation* $u_N^* \in S_{m-1}^{(-1)}(\Delta_N)$ *to* u^* *and*

$$\sup_{t\in[0,b]} \left|u_N^*(t) - u^*(t)\right| \to 0 \quad \textit{as} \quad N \to \infty\,. \tag{6.8}$$

Proof. We consider equation (6.1) as the equation

$$u = Tu + f \tag{6.9}$$

in the Banach space $E = L^\infty(0,b)$, with the operator T, defined by (3.1). By Theorem 3.1, T is compact as an operator from $L^\infty(0,b)$ to $C[0,b]$ and from $L^\infty(0,b)$ to $L^\infty(0,b)$, also. Since equation $u = Tu$ has only the trivial solution $u = 0$, then it follows from Theorem 2.4 that there exists the inverse operator $(I-T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ and equation (6.9) has a unique solution $u^* = (I-T)^{-1}f \in L^\infty(0,b)$. Since $f \in C[0,b]$ and $T \in \mathcal{L}(L^\infty(0,b), C[0,b])$, then $u^* \in C[0,b]$.

The collocation conditions (6.2) can be written in the form

$$u_N = P_N T u_N + P_N f\,, \tag{6.10}$$

with $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$, defined in Section 5.2. By Lemma 5.1, $P_N \in \mathcal{L}(C[0,b], L^\infty(0,b))$ and the norms of these operators are uniformly bounded in N (see (5.11)). Since $\|P_N v - v\|_{L^\infty(0,b)} \to 0$ as $N \to \infty$ for every $v \in C^m[0,b]$, then, by Theorem 2.2,

$$\left\|P_N v - v\right\|_{L^\infty(0,b)} \to 0\,, \quad N \to \infty\,, \tag{6.11}$$

for every $v \in C[0,b]$. Using this we can show that

$$\left\|T - P_N T\right\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \to 0 \quad \text{for} \quad N \to \infty\,. \tag{6.12}$$

We prove (6.12) by contradiction. Suppose that

$$\left\|T - P_N T\right\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \geq \delta > 0\,,\ N \to \infty\ (N \in \mathbb{N}' \subset \mathbb{N})\,.$$

Then there exist $z_N \in L^\infty(0,b)$ $(N \in \mathbb{N}')$ such that $\|z_N\|_{L^\infty(0,b)} = 1$ and

$$\left\|(T - P_N T) z_N\right\|_{L^\infty(0,b)} \geq \frac{\delta}{2}\,,$$

for $N \in \mathbb{N}' \subset \mathbb{N}$, $N \geq N_0$. Since $T : L^\infty(0,b) \to C[0,b]$ is compact, the sequence Tz_N ($N \in \mathbb{N}'$) is relatively compact in $C[0,b]$. Hence, there exists $w \in C[0,b]$, so that

$$\big\|Tz_N - w\big\|_{L^\infty(0,b)} \to 0\,, \quad N \in \mathbb{N}'' \subset \mathbb{N}' \subset \mathbb{N}\,,\ N \to \infty\,.$$

Further, we have

$$\begin{aligned}\big\|(I-P_N)w\big\|_{L^\infty(0,b)} &= \big\|(I-P_N)Tz_N + (I-P_N)w - (I-P_N)Tz_N\big\|_{L^\infty(0,b)}\\ &\geq \big\|(I-P_N)Tz_N\big\|_{L^\infty(0,b)} - \big\|(I-P_N)(Tz_N - w)\big\|_{L^\infty(0,b)}.\end{aligned}$$

Then, due to the choice of z_N,

$$\big\|(I-P_N)Tz_N\big\|_{L^\infty(0,b)} \geq \frac{\delta}{2} \quad \text{for} \quad N \in \mathbb{N}''\,. \tag{6.13}$$

On the other hand, by (5.11),

$$\big\|(I-P_N)(Tz_N - w)\big\|_{L^\infty(0,b)} \leq (1+c)\big\|Tz_N - w\big\|_{L^\infty(0,b)}, \quad N \in \mathbb{N}''\,.$$

Therefore, $\|(I-P_N)(Tz_N - w)\|_{L^\infty(0,b)} \to 0$ for $N \in \mathbb{N}''$, $N \to \infty$, and we get by (6.13) a contradiction with (6.11).

Using (6.12) we can show that $(I - P_N T)$ is invertible for all sufficiently large N, say for $N \geq N_0$. Indeed, write $I - P_N T$ in the following form

$$I - P_N T = I - T + (T - P_N T) = (I-T)[I + (T - P_N T)(I-T)^{-1}]\,. \tag{6.14}$$

Due to (6.12) we can pick N such that

$$\varepsilon_{N_0} := \sup_{N \geq N_0} \big\|T - P_N T\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} < \frac{1}{\|(I-T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))}}\,. \tag{6.15}$$

Then

$$\big\|(I-T)^{-1}(T - P_N T)\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq$$

$$\big\|(I-T)^{-1}\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))}\big\|T - P_N T\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} < 1\,,\ N \geq N_0\,,$$

and, by Theorem 2.1, the inverse operators

$$\big[I + (I-T)^{-1}(T - P_N T)\big]^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$$

exist for $N \geq N_0$ and are uniformly bounded in N:

$$\big\|[I+(I-T)^{-1}(T-P_NT)]^{-1}\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq \frac{1}{1-\varepsilon_{N_0}\|(I-T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))}}, \quad N \geq N_0 . \quad (6.16)$$

Using (6.14)–(6.16) we obtain that

$$(I-P_NT)^{-1} = [I+(I-T)^{-1}(T-P_NT)]^{-1}(I-T)^{-1},$$

$$\big\|(I-P_NT)^{-1}\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c, \quad N \geq N_0 , \quad (6.17)$$

where c is a positive constant which is independent of N. This shows that for $N \geq N_0$ equations (6.10) have unique solutions $u_N^* = (I-P_NT)^{-1}P_Nf$. We have for u_N^* or u^*, the solution of equation (6.1),

$$\begin{aligned} (I-P_NT)(u^*-u_N^*) &= (I-P_NT)u^* - (I-P_NT)u_N^* \\ &= u^* - P_NTu^* - P_Nf \\ &= u^* - P_Nf - (P_Nu^* - P_Nf) \\ &= u^* - P_Nu^* . \end{aligned}$$

Therefore,

$$u^* - u_N^* = (I-P_NT)^{-1}(u^* - P_Nu^*) .$$

Taking the norms and using (6.17),

$$\big\|u^* - u_N^*\big\|_{L^\infty(0,b)} \leq c\,\big\|u^* - P_Nu^*\big\|_{L^\infty(0,b)}, \quad N \geq N_0 , \quad (6.18)$$

where c is a constant which is independent of N. Since $u^* \in C[0,b]$, the convergence (6.8) follows from (6.11) and (6.18). □

The following results formulated in Theorems 6.2 and 6.3 are original results that are not published yet.

Theorem 6.2 *Let the following conditions be fulfilled:*

1) *$K(t,s) = g(t,s)\kappa(t,s)$ is subject to the conditions stated in the assumptions (A1) and (A2), with $p = 0$ (see Sec. 3.2);*

2) *$f \in C^{m,\nu}_{d,0}[0,b]$, with m, ν, d, fixed in (A1) and (A2);*

3) *equation (6.7) has only the trivial solution $u = 0$;*

4) the underlying grid $\{(4.1),(4.2)\}$ *satisfies the condition* (4.8) *and the collocation points* (5.3) *with the grid points* (4.1) *and the parameters* $-1 \le \eta_1 < \ldots < \eta_m \le 1$ *are used.*

Then for all sufficiently large N*, say* $N \ge N_0$*, the collocation conditions* (6.2) *determine for every choice of parameters* $-1 \le \eta_1 < \ldots < \eta_m \le 1$ *a unique approximation* $u_N^* \in S_{m-1}^{(-1)}(\Delta_N)$ *to* u^**, the exact solution of equation* (6.1)*, and for* $N \ge N_0$ *the following error estimate holds:*

$$\sup_{0\le t\le b} \left|u^*(t) - u_N^*(t)\right| \le c \begin{cases} h_N^m\,, & m < 1-\nu\,,\\ h_N^m(1+|\ln h_N|)\,, & m = 1-\nu\,,\\ h_N^{1-\nu}\,, & m > 1-\nu\,, \end{cases} \tag{6.19}$$

where c *is a positive constant not depending on* N *and* h_N *is given by* (4.4)*.*

Proof. According to Theorem 6.1 we have to prove the estimate (6.19). It follows from Theorem 3.2 that $u^* \in C_{d,0}^{m,\nu}[0,b]$. Now the estimate follows from (6.18) and Lemma 5.3. □

Theorem 6.3 *Let the conditions 1)–3) of Theorem 6.2 be fulfilled. Assume that the underlying grid* $\{(4.1),(4.2)\}$ *is quasi-uniform (i.e. satifies the condition* (4.6)*) and the collocation points* (5.3) *are used.*

Then for all sufficiently large N*, say* $N \ge N_0$*, the collocation conditions* (6.2) *determine for every choice of parameters* $-1 \le \eta_1 < \ldots < \eta_m \le 1$ *a unique approximation* $u_N^* \in S_{m-1}^{(-1)}(\Delta_N)$ *to* u^**, the exact solution of equation* (6.1)*, and for* $N \ge N_0$ *the following error estimate holds:*

$$\sup_{0\le t\le b} \left|u^*(t) - u_N^*(t)\right| \le c \begin{cases} N^{-m}, & m < 1-\nu\,,\\ N^{-m}\ln N\,, & m = 1-\nu\,,\\ N^{-(1-\nu)}, & m > 1-\nu\,, \end{cases} \tag{6.20}$$

where c *is a positive constant not depending on* N*.*

Proof. The statement (6.20) follows immediately from the estimates (4.7) and (6.19). □

In the next theorem we present some results generalizing our results in [16, 18, 19].

Theorem 6.4 *Let the following conditions be fulfilled:*

1) $K(t,s) = g(t,s)\kappa(t,s)$ *is subject to the conditions, stated in the assumptions (A1) and (A2) (see Sec. 3.2);*

2) $f \in C^{m,\nu}_{d,p}[0,b]$*, with* m, ν, d, p*, fixed in (A1) and (A2);*

3) *equation* (6.7) *has only the trivial solution* $u = 0$*;*

4) *the underlying grid* $\{(4.1),(4.2)\}$ *is graded (i.e. given by* (5.19)*) and the collocation points* (5.3) *with the grid points* (5.19) *and the collocation parameters* $-1 \le \eta_1 < \ldots < \eta_m \le 1$ *are used.*

Then for all sufficiently large N*, say* $N \ge N_0$*, the collocation conditions* (6.2) *determine for every choice of parameters* $-1 \le \eta_1 < \ldots < \eta_m \le 1$ *a unique approximation* $u^*_N \in S^{(-1)}_{m-1}(\Delta_N)$ *to* u^**, the exact solution of equation* (6.1)*. The following error estimate holds:*

$$\sup_{0 \le t \le b} \left| u^*(t) - u^*_N(t) \right| \le c\varepsilon_N \, , \quad N \ge N_0 \, , \tag{6.21}$$

where c *is a positive constant not dependending on* N *and* ε_N *is defined by the formula* (5.22)*.*

Proof. Due to Theorem 6.1 we have to prove only the estimate (6.21). By Theorem 3.2, $u^* \in C^{m,\nu}_{d,p}[0,b]$. Now the estimate (6.21) follows from Lemma 5.4 and the inequality (6.18). □

Chapter 7

Superconvergence at Collocation Points

Theorem 6.4 suggests that by using a collocation method based on piecewise polynomials of degree $m-1$ ($m \geq 1$) and graded grids of type (5.19), one can reach a convergence order

$$\sup_{0 \leq t \leq b} \left| u^*(t) - u_N^*(t) \right| \leq cN^{-m} , \quad N \geq N_0 , \tag{7.1}$$

for sufficiently large values of grid parameters r and r_d, see (6.21).

In (7.1) the order m cannot be improved, whereas piecewise polynomials of the order $m-1$ are used for the approximation. Nevertheless, as it will be seen from Theorem 7.1 below, the convergence order at the collocation points will be higher than $O(N^{-m})$ for a special choice of collocation parameters (5.4). Actually, we shall assume that the points (5.4) are the nodes of a quadrature formula

$$\int_{-1}^{1} g(s)ds = \sum_{k=1}^{m} w_k g(\eta_k) + R_m(g) , \quad -1 \leq \eta_1 < \ldots < \eta_m \leq 1 , \tag{7.2}$$

which is exact for all polynomials of degree m.

Note that the weights w_k ($k = 1, \ldots, m$) will not be used in our algorithms. The existence of a quadrature formula (7.2) which is exact for polynomials of degree m is used in the proof of the following theorem which generalizes our previous results in [18, 19, 20].

Theorem 7.1 *Let $\nu \in \mathbb{R}$, $\nu < 1$, $m \in \mathbb{N}$, $0 < d < b$, $p \in \{0, 1, \ldots, m+1\}$. Assume that the following conditions are fulfilled.*

(i) *The kernel* $K(t,s) = g(t,s)\kappa(t,s)$ *in equation* (6.1) *satisfies the conditions (A1) and (A2) with* $m+1$ *instead of* m.

(ii) $f \in C^{m+1,\nu}_{d,p}[0,b]$.

(iii) *The integral equation* (6.7) *has only the trivial solution* $u = 0$.

(iv) *The collocation points*

$$\xi_{j,q} = t_{j-1} + \left(\frac{\eta_q + 1}{2}\right)(t_j - t_{j-1})\,, \quad q = 1,\dots,m;\ j = 1,\dots,N\,,$$

with grid points (5.19) *and parameters* (5.4) *are used, where* r *and* r_d *are chosen so that*
if $m < 1-\nu$, $p \geq 0$, *then* $r \geq 1$, $r_d \geq 1$;
if $m = 1-\nu$, $p = 0$, *then* $r > 1$, $r_d > 1$;
if $m = 1-\nu$, $p \geq 1$, *then* $r > 1$, $r_d \geq 1$;
if $1-\nu+p > m > 1-\nu$, $p \geq 1$, *then* $r \geq \dfrac{m}{1-\nu}$, $r_d \geq 1$;
if $m = 1-\nu+p$, $p \geq 1$, *then* $r \geq \dfrac{m}{1-\nu}$, $r_d > 1$;
if $m > 1-\nu+p$, $p \geq 0$, *then* $r \geq \dfrac{m}{1-\nu}$, $r_d \geq \dfrac{m}{1-\nu+p}$.

(v) *The quadrature formula* (7.2) *is exact for all polynomials of degree* m.

Then for all sufficiently large N, *say* $N \geq N_0$, *the collocation conditions* (6.2) *determine a unique approximation* $u^*_N \in S^{(-1)}_{m-1}(\Delta_N)$ *to* $u^* \in C[0,b]$, *the exact solution of equation* (6.1). *The following error estimate holds:*

$$\max_{q=1,\dots,m;j=1,\dots,N} \left| u^*_N(\xi_{j,q}) - u^*(\xi_{j,q}) \right| \leq cN^{-m}\tau_\nu(N)\,, \quad N \geq N_0\,. \qquad (7.3)$$

Here c *is a positive constant independent of* N *and* $\tau_\nu(N)$ *is defined by the formula* (4.40).

Proof. The first statement of Theorem 7.1 follows from Theorem 6.1.

It is clear that

$$\begin{aligned} \left|u^*_N(\xi_{j,q}) - u^*(\xi_{j,q})\right| &= \left|u^*_N(\xi_{j,q}) - (P_N u^*)(\xi_{j,q})\right| \\ &\leq \left\|u^*_N - P_N u^*\right\|_{L^\infty(0,b)}, \quad q = 1,\dots,m\,;\ j = 1,\dots,N\,. \end{aligned} \qquad (7.4)$$

Let us study the error $\|u_N^* - P_N u^*\|_{L^\infty(0,b)}$. We have

$$\begin{aligned} u_N^* - P_N u^* &= P_N T(u_N^* - u^*) \\ &= P_N T(u_N^* - P_N u^*) - P_N T(u^* - P_N u^*) \end{aligned}$$

i.e.

$$(I - P_N T)(u_N^* - P_N u^*) = -P_N T(u^* - P_N u^*)\,. \tag{7.5}$$

As for every $N \geq N_0$ there exists the inverse operator $(I - P_N T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ (see (6.17)), we obtain from (7.5) that

$$u_N^* - P_N u^* = -(I - P_N T)^{-1} P_N T(u^* - P_N u^*)\,, \quad N \geq N_0\,. \tag{7.6}$$

Further, since the norms of operators P_N and $(I - P_N T)^{-1}$ are uniformly bounded in N (see (5.11) and (6.17)), we get from (7.6) that

$$\big\|u_N^* - P_N u^*\big\|_{L^\infty(0,b)} \leq c\big\|T(u^* - P_N u^*)\big\|_{L^\infty(0,b)}\,, \quad N \geq N_0\,, \tag{7.7}$$

where c is a positive constant not depending on N.

Let us estimate $\|T(u^* - P_N u^*)\|_{L^\infty(0,b)}$. Fix $t \in [0,b]$ and let

$$h = \frac{b}{N}\,,$$

$$\theta(t,h) = (t-h, t+h) \cap [0,b]\,, \tag{7.8}$$

$$J_1 = \big\{j \in \{1,\dots,N\} : \, [t_{j-1}, t_j] \cap \theta(t,h) \neq \emptyset\big\}\,, \tag{7.9}$$

$$J_2 = \big\{j \in \{1,\dots,N\} : \, [t_{j-1}, t_j] \cap \theta(t,h) = \emptyset\big\}\,. \tag{7.10}$$

We have

$$\big(T(u^* - P_N u^*)\big)(t) = \int_0^b K(t,s)\big[u^*(s) - P_N u^*(s)\big]\,ds\,, \quad t \in [0,b]\,. \tag{7.11}$$

Thus,

$$\left|\int_0^b K(t,s)\big[u^*(s) - (P_N u^*)(s)\big]\,ds\right| \leq I_1(t) + I_2(t)\,, \quad t \in [0,b]\,, \tag{7.12}$$

where

$$I_1(t) = \left|\sum_{j \in J_1} \int_{t_{j-1}}^{t_j} K(t,s)[u^*(s) - (P_N u^*)(s)]ds\right|, \tag{7.13}$$

$$I_2(t) = \left| \sum_{j\in J_2} \int\limits_{t_{j-1}}^{t_j} K(t,s)[u^*(s) - (P_N u^*)(s)]ds \right| . \tag{7.14}$$

It follows from assumption (i) that

$$I_1(t) \le \sum_{j\in J_1} \int\limits_{t_j-1}^{t_j} |K(t,s)||u^*(s) - (P_N u^*)(s)|ds$$

$$\le c\|u^* - P_N u^*\|_{L^\infty(0,b)} \sum_{j\in J_1} \int\limits_{t_{j-1}}^{t_j} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|t-s||\,, & \nu = 0 \\ |t-s|^{-\nu}, & \nu > 0 \end{array} \right\} ds\,,$$

$$t \in [0,b]\,.$$

Due to (7.8) and (7.9)

$$\sum_{j\in J_1} \int\limits_{t_{j-1}}^{t_j} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|t-s||\,, & \nu = 0 \\ |t-s|^{-\nu}, & \nu > 0 \end{array} \right\} ds$$

$$\le c_1 \int\limits_{t-2h\max\{r,r_d\}}^{t+2h\max\{r,r_d\}} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|t-s||\,, & \nu = 0 \\ |t-s|^{-\nu}, & \nu > 0 \end{array} \right\} ds$$

$$\le c_2 \left\{ \begin{array}{ll} h\,, & \nu < 0 \\ h(1 + |\ln h|)\,, & \nu = 0 \\ h^{1-\nu}, & \nu > 0 \end{array} \right\}$$

$$\le c_3 \tau_\nu(N)\,,$$

where $\tau_\nu(N)$ is given by the formula (4.40).

Using the assumptions about r and r_d and Lemma 5.4, we have

$$\left\| u^* - P_N u^* \right\|_{L^\infty(0,b)} \le cN^{-m}\,, \tag{7.15}$$

where c is a positive constant which does not depend on N. Thus,

$$I_1(t) \le cN^{-m}\tau_\nu(N)\,, \quad t \in [0,b]\,, \tag{7.16}$$

with a constant c which is independent of N.

Consider the term $I_2(t)$, $t \in [0,b]$, given by (7.14). Let

$$s_j = \frac{t_{j-1} + t_j}{2}\,, \quad j = 1, \ldots, N\,.$$

In addition to the points (5.4) we fix in $[-1,1]$ a point η_{m+1} ($\eta_{m+1} \neq \eta_i$, $i = 1,\dots,m$). By an affine transformation we transfer η_{m+1} into the point $\xi_{j,m+1} \in [t_{j-1}, t_j]$ so that $\xi_{j,m+1} \neq \xi_{j,i}$, $i = 1,\dots,m$ ($j = 1,\dots,N$):

$$\xi_{j,m+1} = t_{j-1} + \frac{\eta_{m+1}+1}{2}(t_j - t_{j-1}) .$$

Similarly to the definition of the operator $P_N = P_{N,m-1}$ (see Chapter 5) we define an operator $P_{N,m} : C[0,b] \to S_m^{(-1)}(\Delta_N)$ as follows: if $u \in C[0,b]$ then $P_{N,m}u$ is a polynomial of degree not exceeding m on every subinterval $[t_{j-1}, t_j]$ ($j = 1,\dots,N$) and

$$\big(P_{N,m}u\big)(\xi_{j,q}) = u(\xi_{j,q}) , \quad q = 1,\dots,m+1 ;\ j = 1,\dots,N .$$

Since

$$\begin{aligned} &K(t,s)\big[u^*(s) - \big(P_{N,m-1}u^*\big)(s)\big] \\ &= \big[K(t,s) - K(t,s_j)\big]\big[u^*(s) - \big(P_{N,m-1}u^*\big)(s)\big] + K(t,s_j)\big[u^*(s) - \big(P_{N,m}u^*\big)(s)\big] \\ &\qquad + K(t,s_j)\big[\big(P_{N,m}u^*\big)(s) - \big(P_{N,m-1}u^*\big)(s)\big] , \end{aligned}$$

we can estimate $I_2(t)$ as follows:

$$I_2(t) \le I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t) , \quad t \in [0,b] , \tag{7.17}$$

where

$$I_{2,1}(t) = \sum_{j\in J_2} \int_{t_{j-1}}^{t_j} \big|K(t,s) - K(t,s_j)\big|\big|u^*(s) - \big(P_{N,m-1}u^*\big)(s)\big| ds , \tag{7.18}$$

$$I_{2,2}(t) = \sum_{j\in J_2} \big|K(t,s_j)\big| \int_{t_{j-1}}^{t_j} \big|u^*(s) - \big(P_{N,m}u^*\big)(s)\big| ds , \tag{7.19}$$

$$I_{2,3}(t) = \Big| \sum_{j\in J_2} K(t,s_j) \int_{t_{j-1}}^{t_j} \big[\big(P_{N,m}u^*\big)(s) - \big(P_{N,m-1}u^*\big)(s)\big] ds \Big| . \tag{7.20}$$

Let us estimate $I_{2,1}(t)$, $t \in [0,b]$, given by (7.18). We have

$$I_{2,1}(t) = \sum_{j \in J_2} \int\limits_{t_{j-1}}^{t_j} \big|K(t,s) - K(t,s_j)\big|\big|u^*(s) - \big(P_{N,m-1}u^*\big)(s)\big| ds$$

$$\leq c\big\|u^* - P_N u^*\big\|_{L^\infty(0,b)} \sum_{j \in J_2} \int\limits_{t_{j-1}}^{t_j} \left|\frac{\partial K(t,s)}{\partial s}\right|_{s=\mu_j} |s - s_j| ds\,,$$

where $\mu_j \in (s, s_j)$. Using (i) we have for $s \in [t_{j-1}, t_j]$ that

$$\left|\frac{\partial K(t,s)}{\partial s}\right|_{s=\mu_j} |s - s_j| \leq ch \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1 + |\ln|t-\mu_j||\,, & \nu+1=0 \\ |t-\mu_j|^{-\nu-1}, & \nu+1>0 \end{array} \right\}.$$

Since $[t_{j-1}, t_j] \cap \theta(t,h) = \emptyset$, $s \in [t_{j-1}, t_j]$ and $\mu_j \in (s, s_j)$, then

$$c_1 \leq \frac{|t-\mu_j|}{|t-s|} \leq c_2\,,$$

where c_1 and c_2 are some positive constants. Therefore, using (7.15), we obtain that

$$I_{2,1}(t) \leq c_3 N^{-(m+1)} \sum_{j \in J_2} \int\limits_{t_{j-1}}^{t_j} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1 + |\ln|t-s||\,, & \nu+1=0 \\ |t-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds$$

$$\leq c_4 N^{-(m+1)} \int\limits_{[0,b]\setminus\theta(t,h)} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1 + |\ln|t-s||\,, & \nu+1=0 \\ |t-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds\,. \quad (7.21)$$

For every $t \in [0,b]$ we have

$$\int\limits_{[0,b]\setminus\theta(t,h)} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1 + |\ln|t-s||\,, & \nu+1=0 \\ |t-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds \leq c \left\{ \begin{array}{ll} 1\,, & \nu<0\,, \\ \ln N\,, & \nu=0\,, \\ N^\nu, & \nu>0\,. \end{array} \right. \quad (7.22)$$

Indeed, we consider only the case $(t-h, t+h) \subset [0,b]$, $h = b/N$. Then,

$$\int\limits_{[0,b]\setminus\theta(t,h)} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1 + |\ln|t-s||\,, & \nu+1=0 \\ |t-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds$$

$$= \int_0^{t-h} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1+|\ln(t-s)|\,, & \nu+1=0 \\ (t-s)^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds$$

$$+ \int_{t+h}^{b} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1+|\ln(s-t)|\,, & \nu+1=0 \\ (s-t)^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds$$

$$\leq \begin{cases} c_1\,, & \nu<-1\,, \\ c_2\,, & \nu=-1\,, \\ c_3\,, & -1<\nu<0\,, \\ c_4 \ln N\,, & \nu=0\,, \\ c_5 N^{\nu}\,, & \nu>0\,, \end{cases}$$

where $c_1, \ldots, c_5$ are some positive constants not depending on N.

Thus, (4.40), (7.21) and (7.22) yield

$$I_{2,1}(t) \leq cN^{-m}\tau_\nu(N)\,, \quad t\in[0,b]\,, \tag{7.23}$$

where c is a positive constant which does not depend on N.

Let us turn to $I_{2,2}(t)$, $t\in[0,b]$, given by (7.19). It follows from $[t_{j-1},t_j]\cap\theta(t,h)=\emptyset$, $s_j=(t_{j-1}+t_j)/2$ $(j=1,\ldots,N)$ and (i), that

$$|K(t,s_j)| \leq c \left\{ \begin{array}{ll} 1\,, & \nu<0 \\ 1+|\ln|t-s_j||\,, & \nu=0 \\ |t-s_j|^{-\nu}, & \nu>0 \end{array} \right\} \leq c_1 \left\{ \begin{array}{ll} 1\,, & \nu<0\,, \\ 1+|\ln h|\,, & \nu=0\,, \\ h^{-\nu}, & \nu>0\,. \end{array} \right. \tag{7.24}$$

On the basis of (i) and (ii) we obtain from Theorem 3.2 that $u^*\in C^{m+1,\nu}_{d,p}[0,b]$. Therefore, using the assumptions about r and r_d and Lemma 5.5, with $m+1$ instead of m, we have

$$\left\|u^*-P_{N,m}u^*\right\|_{L^\infty(0,b)} \leq cN^{-(m+1)}, \tag{7.25}$$

where c is a positive constant which does not depend on N. Thus, by (7.24) and (7.25)

$$I_{2,2}(t) = \sum_{j\in J_2} |K(t,s_j)| \int\limits_{t_j-1}^{t_j} |u^*(s) - (P_{N,m}u^*)(s)|ds$$

$$\leq cN^{-(m+1)} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln h|\,, & \nu = 0 \\ h^{-\nu}, & \nu > 0 \end{array} \right\}$$

$$\leq c_1 N^{-m}\tau_\nu(N)\,, \qquad t \in [0,b]\,, \tag{7.26}$$

where $\tau_\nu(N)$ is given by the formula (4.40).

Consider $I_{2,3}(t)$, $t \in [0,b]$. Due to assumption (iv), the quadrature formula

$$\int\limits_{t_{j-1}}^{t_j} g(s)ds = \frac{t_j - t_{j-1}}{2} \sum_{k=1}^{m} w_k g(\xi_{j,k}) + \frac{t_j - t_{j-1}}{2} R_m(g)$$

$$(j = 1,\ldots,N)$$

remains to be exact for all polynomials of degree m. In other words, if g is a polynomial of degree not exceeding m, then

$$\int\limits_{t_{j-1}}^{t_j} g(s)ds = \frac{t_j - t_{j-1}}{2} \sum_{q=1}^{m} w_q g(\xi_{j,q})\,, \quad j = 1,\ldots,N\,.$$

Using this, we have

$$\int\limits_{t_{j-1}}^{t_j} \big[\big(P_{N,m}u^*\big)(s) - \big(P_{N,m-1}u^*\big)(s)\big]ds$$

$$= \frac{t_j - t_{j-1}}{2} \sum_{q=1}^{m} w_q \big[\big(P_{N,m}u^*\big)(\xi_{j,q}) - (P_{N,m-1}u^*)(\xi_{j,q})\big]\,, \quad j = 1,\ldots,N\,.$$

Since

$$\big(P_{N,m}u^*\big)(\xi_{j,q}) = \big(P_{N,m-1}u^*\big)(\xi_{j,q}) = u^*(\xi_{j,q})\,, \quad q = 1,\ldots,m\,;\; j = 1,\ldots,N\,,$$

we obtain that

$$\int\limits_{t_{j-1}}^{t_j} \big[P_{N,m}u^*(s) - P_{N,m-1}u^*(s)\big]ds = 0\,, \quad j = 1\ldots,N\,.$$

Therefore,

$$I_{2,3}(t) = 0\,, \quad t \in [0,b]\,. \tag{7.27}$$

Combining (7.23), (7.26) and (7.27) we obtain

$$I_2(t) \leq cN^{-m}\tau_\nu(N)\,, \quad t \in [0,b]\,, \tag{7.28}$$

with a positive constant c, which does not depend on N.

Now the estimate (7.3) follows from (7.4), (7.7), (7.12), (7.16) and (7.28). □

Chapter 8

Two-Grid Method

To apply the collocation method it is necessary to solve the linear system (6.6). Usually the number of equations in (6.6) is large and, as a result of this, direct solving of (6.6) is rather complicated. An effective method for solving this system is a two-grid iteration method. Note that two-grid iteration methods are studied, for example, in [17, 21, 48, 51, 52, 53, 62, 69]. Our treatment will follow the approach of [17, 21, 52].

8.1 Description of Two-Grid Method

In addition to the original grid $\Delta_N = \Delta_N^{r,r_d}$, $N = 4n$, $n \in \mathbb{N}$, $n \geq 2$, (see (4.1) and (5.19)), corresponding to n, we define an another grid, the coarse grid $\Delta_M = \Delta_M^{r,r_d}$, $M = 4n_M$, $n_M \in \mathbb{N}$, corresponding to n_M, with $n_M < n$:

$$\begin{aligned}
t_i^{(M)} &= \frac{d}{2}\left(\frac{i}{n_M}\right)^r, \quad i = 0, 1, \ldots, n_M\,; \\
t_{n_M+i}^{(M)} &= d - \frac{d}{2}\left(\frac{n_M - i}{n_M}\right)^{r_d}, \quad i = 1, \ldots, n_M\,; \\
t_{2n_M+i}^{(M)} &= d + \frac{b-d}{2}\left(\frac{i}{n_M}\right)^{r_d}, \quad i = 1, \ldots, n_M\,; \\
t_{3n_M+i}^{(M)} &= b - \frac{b-d}{2}\left(\frac{n_M - i}{n_M}\right)^{r}, \quad i = 1, \ldots, n_M\,.
\end{aligned} \tag{8.1}$$

More precisely, we choose n and n_M so that n/n_M is an integer greater than 1:

$$n, n_M, \frac{n}{n_M} \in \mathbb{N}, \quad \frac{n}{n_M} \geq 2\,.$$

Then every subinterval $[t_{j-1}, t_j]$ $(j = 1, \ldots, N)$ of the original grid Δ_N is fully contained in some subinterval $[t_{i-1}^{(M)}, t_i^{(M)}]$ $(i = 1, \ldots, M)$ of the coarse grid Δ_M (cf. [52]).

To solve the system (6.6) the following two-grid iteration method is used:

$$\begin{cases} \overline{v}_N^l = \overline{u}_N^l - T_N \overline{u}_N^l - \overline{f}_N\,, \\ \overline{w}_M^l = (I_M - T_M)^{-1} T_{N,M} \overline{v}_N^l\,, \\ \overline{u}_N^{l+1} = \overline{u}_N^l - \overline{v}_N^l - Q_{M,N} \overline{w}_M^l\,, \quad l = 0, 1, \ldots\,, \end{cases} \tag{8.2}$$

where $\overline{u}_N^0$ is a initial guess of $\overline{u}_N$, I_M is the identity matrix, $T_{N,M} : \mathbb{R}^{d_N} \to \mathbb{R}^{d_M}$ is the restriction operator defined by

$$\Big(T_{N,M}\,\overline{v}_N^l\Big)\big(\xi_{i,p}^{(M)}\big) = \sum_{j=1}^{N} \sum_{q=1}^{m} \left(\int\limits_{t_{j-1}}^{t_j} K\big(\xi_{i,p}^{(M)}, s\big) \varphi_{j,q}(s) ds \right) v_N^l\big(\xi_{j,q}\big)\,,$$

$$p = 1, \ldots, m\,;\ i = 1, \ldots, M\,,$$

and $Q_{M,N} : \mathbb{R}^{d_M} \to \mathbb{R}^{d_N}$ is the prolongation operator defined by

$$\big(Q_{M,N}\,\overline{w}_M^l\big)\big(\xi_{j,q}\big) = \sum_{p=1}^{m} w_M^l\big(\xi_{i,p}^{(M)}\big)\varphi_{i,p}^{(M)}\big(\xi_{j,q}\big) \quad \text{if} \quad \xi_{j,q} \in \big[t_{i-1}^{(M)}, t_i^{(M)}\big]\,,$$

$$i = 1, \ldots, M\,;\ q = 1, \ldots, m\,;\ j = 1, \ldots, N\,.$$

Here the node points $\{\xi_{j,q}\}$ are given by the formula (5.3), the functions $\{\varphi_{j,q}\}$ are defined by the formula (5.6), $\{\xi_{i,p}^{(M)}\}$ and $\{\varphi_{i,p}^{(M)}\}$ are defined in following way:

$$\xi_{i,p}^{(M)} = t_{i-1}^{(M)} + \frac{\eta_p + 1}{2}(t_i^{(M)} - t_{i-1}^{(M)})\,, \quad p = 1, \ldots, m\,;\ i = 1, \ldots, M\,, \tag{8.3}$$

with $\eta_1, \ldots, \eta_m$ satisfying the condition (5.4), and

$$\varphi_{i,p}^{(M)}(t) = \frac{(t - \xi_{i,1}^{(M)}) \ldots (t - \xi_{i,p-1}^{(M)})(t - \xi_{i,p+1}^{(M)}) \ldots (t - \xi_{i,m}^{(M)})}{(\xi_{i,p}^{(M)} - \xi_{i,1}^{(M)}) \ldots (\xi_{i,p}^{(M)} - \xi_{i,p-1}^{(M)})(\xi_{i,p}^{(M)} - \xi_{i,p+1}^{(M)}) \ldots (\xi_{i,p}^{(M)} - \xi_{i,m}^{(M)})}\,,$$

$$t_{i-1}^{(M)} \leq t \leq t_i^{(M)}\,, \tag{8.4}$$

for $p = 1, \ldots, m$; $i = 1, \ldots, M$. Notice also, that $v_N^l(\xi_{j,q})$, $(T_{N,M}\,\overline{v}_N^l)(\xi_{i,p}^{(M)})$, $w_M^l(\xi_{i,p}^{(M)})$ and $(Q_{M,N}\overline{w}_M^l)(\xi_{j,q})$ are the corresponding components of the vectors $\overline{v}_N^l$, $T_{N,M}\overline{v}_N^l$, $\overline{w}_M^l$ and $Q_{M,N}\overline{w}_M^l$, respectively. The ordering of the components of these vectors is the same as the one for the vectors $\overline{u}_N$ and $\overline{f}_N$ in Sec. 6.1 (see (6.6)). For example,

$$\overline{v}_N^l = (v^l(\xi_{1,1}), \ldots, v^l(\xi_{1,m}), v^l(\xi_{2,1}), \ldots, v^l(\xi_{2,m}), \ldots, v^l(\xi_{N,1}), \ldots, v^l(\xi_{N,m}))^T,$$

$$\overline{w}_M^l = (w^l(\xi_{1,1}^{(M)}), \ldots, w^l(\xi_{1,m}^{(M)}), w^l(\xi_{2,1}^{(M)}), \ldots, w^l(\xi_{2,m}^{(M)}), \ldots, w^l(\xi_{M,1}^{(M)}), \ldots, w^l(\xi_{M,m}^{(M)}))^T.$$

Note that

$$d_N = \dim \mathbb{R}^{d_N} = \dim S_{m-1}^{(-1)}(\Delta_N) = mN$$

if $\eta_1 > -1$ or $\eta_m < 1$ (see(5.4)) and

$$d_N = \dim S_{m-1}^{(0)}(\Delta_N) = (m-1)N + 1$$

if $\eta_1 = -1$, $\eta_m = 1$.

Following some ideas of [52, 63] we can prove Lemma 8.1.

Lemma 8.1 *Let the conditions (A1) and (A2) about the kernel* $K(t,s) = g(t,s)\kappa(t,s)$ *hold with* $m = 1$, $p = 0$. *Let the node points* (5.3) *with grid points* (5.19) *and parameters* (5.4) *be used. Let* $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$ *be determined by the conditions* (5.5).

Then

$$\big\|T - P_N T\big\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \le c\varepsilon_{\nu,h_N}\,, \quad N \in \mathbb{N}\,,\ N \ge N_0\,. \tag{8.5}$$

Here c *is a positive constant not depending on* N, *the operator* T *is given by the formula* (3.1) *and*

$$\varepsilon_{\nu,h_N} = \begin{cases} h_N\,, & \nu < 0\,, \\ h_N(1 + |\ln h_N|)\,, & \nu = 0\,, \\ h_N^{1-\nu}, & \nu > 0\,, \end{cases} \tag{8.6}$$

where h_N *is defined in* (4.4).

Proof. Let $u \in L^\infty(0,b)$. Then, due to Theorem 3.1, $Tu \in C[0,b]$. Let $t \in [t_{j-1}, t_j]$ $(j = 1, \dots, N)$. Then

$$\big[(T - P_N T)u\big](t) = \int_0^b K(t,s)u(s)ds - \sum_{q=1}^m \varphi_{j,q}(t) \int_0^b K(\xi_{j,q}, s)u(s)ds$$
$$= \sum_{q=1}^m \varphi_{j,q}(t) \int_0^b \big[K(t,s) - K(\xi_{j,q}, s)\big]u(s)ds\,.$$

At first we estimate $\varphi_{j,q}(t)$, $t \in [t_{j-1}, t_j]$, $j = 1, \dots, N$. The change of the variable

$$t = t_{j-1} + \frac{\eta + 1}{2}(t_j - t_{j-1})$$

gives

$$\varphi_{j,q}(t) = \prod_{\substack{k=1 \\ k \neq q}}^m \frac{t - \xi_{j,k}}{\xi_{j,q} - \xi_{j,k}} = \prod_{\substack{k=1 \\ k \neq q}}^m \frac{\eta - \eta_k}{\eta_q - \eta_k}\,.$$

Therefore,

$$\big|\varphi_{j,q}(t)\big| \leq c\,, \quad t \in [t_{j-1}, t_j]\,,\ q = 1, \dots, m\,,\ j = 1, \dots, N\,, \tag{8.7}$$

where

$$c = \max_{1 \leq q \leq m} \ \max_{-1 \leq \eta \leq 1} \left| \prod_{\substack{k=1 \\ k \neq q}}^m \frac{\eta - \eta_k}{\eta_q - \eta_k} \right|.$$

Thus,

$$\Big|\big[(T - P_N T)u\big](t)\Big| \leq c \sum_{q=1}^m \int_0^b \big|K(t,s) - K(\xi_{j,q}, s)\big| |u(s)| ds$$
$$\leq c\|u\|_{L^\infty(0,b)} \sum_{q=1}^m \int_0^b \big|K(t,s) - K(\xi_{j,q}, s)\big| ds\,, \tag{8.8}$$

where $t \in [t_{j-1}, t_j]$, $j = 1, \dots, N$.

Let us estimate $\int_0^d \big|K(t,s) - K(\xi_{j,q}, s)\big| ds$, with $t, \xi_{j,q} \in [t_{j-1}, t_j]$ $(j = 1, \dots, 2n$, see (5.19)). Denote

$$h = 2\big|\xi_{j,q} - t\big|\,;$$
$$B(x,h) = \big\{s \in \mathbb{R} : |x - s| \leq h\big\}\,, \quad x \in [t_{j-1}, t_j]\,.$$

We have

$$\int_0^d \big|K(t,s)-K(\xi_{j,q},s)\big|ds = \int\limits_{[0,d]\cap\{B(t,h)\cap B(\xi_{j,q},h)\}} \big|K(t,s)-K(\xi_{j,q},s)\big|ds$$

$$+ \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \big|K(t,s)-K(\xi_{j,q},s)\big|ds$$

$$\le I_1 + I_2 + I_3\,, \tag{8.9}$$

where

$$I_1 = \int\limits_{[0,d]\cap B(t,h)} \big|K(t,s)-K(\xi_{j,q},s)\big|ds\,; \tag{8.10}$$

$$I_2 = \int\limits_{[0,d]\cap B(\xi_{j,q},h)} \big|K(t,s)-K(\xi_{j,q},s)\big|ds\,; \tag{8.11}$$

$$I_3 = \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \big|K(t,s)-K(\xi_{j,q},s)\big|ds\,. \tag{8.12}$$

Let us consider I_1, given by (8.10). Using the assumptions (A1), (A2), we find that

$$I_1 = \int\limits_{[0,d]\cap B(t,h)} \big|K(t,s)-K(\xi_{j,q},s)\big|ds \le \int\limits_{[0,d]\cap B(t,h)} \big(|K(t,s)|+|K(\xi_{j,q},s)|\big)ds$$

$$\le c\,(I_{1,1}+I_{1,2})\,,$$

where

$$I_{1,1} = \int\limits_{[0,d]\cap B(t,h)} \left\{\begin{array}{ll} 1\,, & \nu<0 \\ 1+|\ln|t-s||\,, & \nu=0 \\ |t-s|^{-\nu}, & \nu>0 \end{array}\right\} ds\,,$$

$$I_{1,2} = \int\limits_{[0,d]\cap B(t,h)} \left\{\begin{array}{ll} 1\,, & \nu<0 \\ 1+|\ln|\xi_{j,q}-s||\,, & \nu=0 \\ |\xi_{j,q}-s|^{-\nu}, & \nu>0 \end{array}\right\} ds\,,$$

Let us estimate $I_{1,1}$. We have

$$I_{1,1} \le \int_{t-h}^{t+h} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|t-s||\,, & \nu = 0 \\ |t-s|^{-\nu}, & \nu > 0 \end{array} \right\} ds$$

$$\le c \left\{ \begin{array}{ll} h\,, & \nu < 0\,, \\ h(1+|\ln h|)\,, & \nu = 0\,, \\ h^{1-\nu}, & \nu > 0\,. \end{array} \right.$$

Let us estimate $I_{1,2}$. We get

$$I_{1,2} = \int_{[0,d]\cap B(t,h)} \left\{ \begin{array}{ll} 1\,, & \nu < 0 \\ 1 + |\ln|\xi_{j,q}-s||\,, & \nu = 0 \\ |\xi_{j,q}-s|^{-\nu}, & \nu > 0 \end{array} \right\} ds$$

$$\le c \left\{ \begin{array}{ll} h\,, & \nu < 0\,, \\ h(1+|\ln h|)\,, & \nu = 0\,, \\ h^{1-\nu}, & \nu > 0\,. \end{array} \right.$$

In summary,

$$I_1 = \int_{[0,d]\cap B(t,h)} \big|K(t,s) - K(\xi_{j,q},s)\big| ds$$

$$\le c \left\{ \begin{array}{ll} h\,, & \nu < 0\,, \\ h(1+|\ln h|)\,, & \nu = 0\,, \\ h^{1-\nu}, & \nu > 0\,. \end{array} \right. \tag{8.13}$$

In a similar way we find that

$$I_2 = \int_{[0,d]\cap B(\xi_{j,q},h)} \big|K(t,s) - K(\xi_{j,q},s)\big| ds$$

$$\le c \left\{ \begin{array}{ll} h\,, & \nu < 0\,, \\ h(1+|\ln h|)\,, & \nu = 0\,, \\ h^{1-\nu}, & \nu > 0\,. \end{array} \right. \tag{8.14}$$

Finally, let us estimate I_3, given by (8.12). We have

$$I_3 = \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \big|K(t,s) - K(\xi_{j,q},s)\big| ds$$

$$= \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \Big| \int\limits_{\xi_{j,q}}^{t} \frac{\partial K(x,s)}{\partial x} dx \Big| ds$$

$$\leq \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \Big| \int\limits_{\xi_{j,q}}^{t} \Big| \frac{\partial K(x,s)}{\partial x} \Big| dx \Big| ds\,.$$

Since

$$\frac{\partial K(x,s)}{\partial x} = \frac{\partial g(x,s)}{\partial x}\kappa(x,s) + g(x,s)\frac{\partial \kappa(x,s)}{\partial x}\,,$$

then using the assumptions (A1) and (A2) we get

$$\Big| \frac{\partial K(x,s)}{\partial x} \Big| \leq c \begin{cases} 1\,, & \nu+1<0\,, \\ 1 + |\ln|x-s||\,, & \nu+1=0\,, \\ |x-s|^{-\nu-1}, & \nu+1>0\,, \end{cases}$$

where $x \neq s$. Thus,

$$I_3 \leq c \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \Big| \int\limits_{\xi_{j,q}}^{t} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1 + |\ln|x-s||\,, & \nu+1=0 \\ |x-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} dx \Big| ds$$

$$\leq c \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ \max\{1 + |\ln|t-s||\,, & \\ \quad 1 + |\ln|\xi_{j,q}-s||\}\,, & \nu+1=0 \\ \max\{|t-s|^{-\nu-1}, & \\ \quad |\xi_{j,q}-s|^{-\nu-1}\}\,, & \nu+1>0 \end{array} \right\} \Big| \int\limits_{\xi_{j,q}}^{t} dx \Big| ds$$

$$= c\frac{h}{2} \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ \max\{1 + |\ln|t-s||\,, & \\ \quad 1 + |\ln|\xi_{j,q}-s||\}\,, & \nu+1=0 \\ \max\{|t-s|^{-\nu-1}, & \\ \quad |\xi_{j,q}-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds$$

We have for $t, \xi_{j,q} \in [t_{j-1}, t_j]$ and $s \in [0,d] \setminus \{B(t,h) \cap B(\xi_{j,q},h)\}$ that

$$0 < c' \leq \frac{|t-s|}{|\xi_{j,q}-s|} \leq c''\,,$$

where c' and c'' are some positive constants. Therefore,

$$I_3 \leq c_2 h \int\limits_{[0,d]\setminus\{B(t,h)\cap B(\xi_{j,q},h)\}} \left\{ \begin{array}{ll} 1\,, & \nu+1<0 \\ 1+|\ln|t-s||\,, & \nu+1=0 \\ |t-s|^{-\nu-1}, & \nu+1>0 \end{array} \right\} ds$$

$$\leq c_3 \begin{cases} h^2\,, & \nu+1<0\,, \\ h^2(1+|\ln h|)\,, & \nu+1=0\,, \\ h^{1-\nu}, & \nu+1>0\,. \end{cases} \tag{8.15}$$

Combining (8.13), (8.14) and (8.15), we conclude that for $t, \xi_{j,q} \in [t_{j-1}, t_j]$ ($j = 1, \ldots, 2n$) the following estimate holds:

$$\int\limits_0^d \big|K(t,s) - K(\xi_{j,q}, s)\big| ds \leq c \begin{cases} h\,, & \nu<0\,, \\ h(1+|\ln h|)\,, & \nu=0\,, \\ h^{1-\nu}, & \nu>0\,. \end{cases}$$

Here $h = 2|t-\xi_{j,q}|$ and c is a positive constant which is independent of h.

In a similar way we obtain for $t, \xi_{j,q} \in [t_{j-1}, t_j]$ ($j = 2n+1, \ldots, 4n$) that

$$\int\limits_d^b \big|K(t,s) - K(\xi_{j,q}, s)\big| ds \leq c \begin{cases} h\,, & \nu<0\,, \\ h(1+|\ln h|)\,, & \nu=0\,, \\ h^{1-\nu}, & \nu>0\,, \end{cases} \tag{8.16}$$

where $h = 2|t-\xi_{j,q}|$ and c is a positive constant which is independent of h. Due to symmetry the proof of (8.16) is analogous.

Since $h \leq 2h_N$, then on the basis of (8.8) we obtain for every $t \in [t_{j-1}, t_j]$ ($j = 1, \ldots, N$) that

$$\big|\big[(T - P_N T)u\big](t)\big| \leq c\varepsilon_{\nu,h_N} \|u\|_{L^\infty(0,b)}\,,$$

where ε_{ν,h_N} is defined by (8.6) and c is a positive constant not depending on N. We conclude from this that

$$\big\|T - P_N T\big\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} = \sup_{u\in L^\infty(0,b):\|u\|_{L^\infty(0,b)}\leq 1} \big\|(T-P_N T)u\big\|_{L^\infty(0,b)}$$

$$= \max_{j=1,\ldots,N} \sup_{t\in[t_{j-1},t_j]} \big|\big[(T-P_N T)u\big](t)\big| \leq c\,\varepsilon_{\nu,h_N}\,.$$

□

8.2 Convergence of Two-Grid Approximations

In this section we study the convergence of two grid method. The following Theorem 8.1 generalizes our results in [17, 21].

Theorem 8.1 *Let the conditions of Theorem 6.4 be fulfilled.*

Then there exists an integer $M_0 > 0$ *such that, for* $N \geq M_0$, $N = 4n$, *the system* (6.6) *has a unique solution* $\overline{u}_N^*$. *The two-grid iteration method* (8.2) *converges to this solution for* $M \geq M_0$, $M = 4n_M$, $n/n_M = 2, 3, \ldots$, *and for every choice of the initial guess* $\overline{u}_N^0$ *to* $\overline{u}_N^*$:

$$\left\|\overline{u}_N^{l+1} - \overline{u}_N^*\right\|_{d_N} \leq c\varepsilon_{\nu,h_M}\left\|\overline{u}_N^l - \overline{u}_N^*\right\|_{d_N}, \quad l = 0, 1, \ldots, \tag{8.17}$$

where c *is a positive constant which is independent of* N, ε_{ν,h_M} *is given by*

$$\varepsilon_{\nu,h_M} = \begin{cases} h_M\,, & \nu < 0\,, \\ h_M(1 + |\ln h_M|)\,, & \nu = 0\,, \\ h_M^{1-\nu}, & \nu > 0\,, \end{cases} \tag{8.18}$$

for $h_M = \max\limits_{i=1,\ldots,M}(t_i^{(M)} - t_{i-1}^{(M)})$ *and*

$$\left\|\overline{u}_N\right\|_{d_N} = \max_{q=1,\ldots,m;j=1,\ldots,N}\left|u_N(\xi_{j,q})\right|,$$

for

$$\overline{u}_N = \big(u_N(\xi_{1,1}), \ldots, u_N(\xi_{1,m}), u_N(\xi_{2,1}), \ldots, u_N(\xi_{2,m}), \ldots, \\ u_N(\xi_{N,1}), \ldots, u_N(\xi_{N,m})\big) \in \mathbb{R}^{d_N}\,.$$

Proof. In order to give to the method (8.2) a more convenient form for convergence analysis, we introduce the operators $R_{\infty,N} : S_{m-1}^{(-1)}(\Delta_N) \to \mathbb{R}^{d_N}$ and $Q_{N,\infty} : \mathbb{R}^{d_N} \to S_{m-1}^{(-1)}(\Delta_N)$ as follows:

$$R_{\infty,N}u = \big(u(\xi_{1,1}), \ldots, u(\xi_{1,m}), \ldots, u(\xi_{N,1}), \ldots, u(\xi_{N,m})\big)\,, u \in S_{m-1}^{(-1)}(\Delta_N)\,, \tag{8.19}$$

$$\big(Q_{N,\infty}\overline{u}_N\big)(t) = \sum_{q=1}^{m} u_N(\xi_{j,q})\varphi_{j,q}(t)\,, \quad t \in [t_{j-1}, t_j]\,,\ j = 1, \ldots, N\,,$$

Actually, we shall use the definition (8.19) for applying $R_{\infty,N}$ to all functions $u(t)$ which are defined at $t = \xi_{j,q}$, $q = 1, \ldots, m$; $j = 1, \ldots, N$.

For later use, denote

$$u_N^l = Q_{N,\infty}\overline{u}_N^l\,,$$

$$v_N^l = Q_{N,\infty}\overline{v}_N^l\,,$$

$$w_M^l = Q_{M,\infty}\overline{w}_M^l\,,$$

where $\overline{u}_N^l$, $\overline{v}_N^l$, $\overline{w}_M^l$ $(l = 0, 1, \ldots)$ are determined by (8.2) for an initial guess $\overline{u}_N^0 \in R^{d_N}$. Then

$$\begin{aligned}
&\overline{u}_N^l = R_{\infty,N}u_N^l\,,\\
&\overline{v}_N^l = R_{\infty,N}v_N^l\,,\\
&\overline{w}_M^l = R_{\infty,M}w_M^l\,,\\
&R_{\infty,N}Q_{N,\infty} = I_N\,,\\
&Q_{N,\infty}R_{\infty,N} = P_N\,,\\
&R_{\infty,N}TQ_{N,\infty} = T_N\,,\\
&Q_{N,\infty}Q_{M,N} = Q_{M,\infty}\,.
\end{aligned}$$

Using these notations and identities we can rewrite formulae (8.2) as follows:

$$\begin{cases}
v_N^l = u_N^l - P_N T u_N^l - P_N f\,,\\
w_M^l = P_M T w_M^l + P_M T v_N^l\,,\\
u_N^{l+1} = u_N^l - v_N^l - w_M^l\,, \quad l = 0, 1, \ldots\,.
\end{cases} \tag{8.20}$$

Whereas $u_N^0 = Q_{N,\infty}\overline{u}_N^0 \in S_{m-1}^{(-1)}(\Delta_N)$, we also have $v_N^l \in S_{m-1}^{(-1)}(\Delta_N)$, $w_M^l \in S_{m-1}^{(-1)}(\Delta_M) \subset S_{m-1}^{(-1)}(\Delta_N)$ and $u_N^{l+1} \in S_{m-1}^{(-1)}(\Delta_N)$, $l = 0, 1, \ldots$. Therefore the methods (8.2) and (8.20) are equivalent. At the same time the method (8.20) is an iteration method to solve (6.10).

By Theorem 3.1, T is compact as an operator from $L^\infty(0,b)$ to $L^\infty(0,b)$. As the homogeneous equation $u = Tu$ has only the trivial solution $u = 0$, the operator $I - T \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ has a bounded inverse $(I-T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$. By Lemma 8.1,

$$\|T - P_M T\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \to 0 \quad \text{for} \quad M \to \infty\,. \tag{8.21}$$

Therefore there occurs an $M_0 \in \mathbb{N}$ such that for all $M \geq M_0$, $(I - P_M T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ exist and their norms are uniformly bounded (cf. (6.14)-(6.17)):

$$\|(I - P_M T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c\,, \quad M \geq M_0\,, \tag{8.22}$$

where c is a positive constant which is independent of M. This shows that the system (6.6) has a unique solution $\overline{u}_N^* \in \mathbb{R}^{d_N}$ for every $N = 4n \geq M_0$. Consequently, for $M = 4n_M \geq M_0$, $n/n_M = 2, 3, \ldots$ the formulas (8.2) and (8.20) determine unique sequences $\overline{u}_N^l \in \mathbb{R}^{d_N}$ and $u_N^l \in S_{m-1}^{(-1)}(\Delta_N)$, $l = 1, 2, \ldots$.

From (8.20) it follows that

$$
\begin{aligned}
&(I - P_M T)(u_N^{l+1} - u_N^*) \\
&= (I - P_M T)[u_N^l - v_N^l - w_M^l - u_N^*] \\
&= (I - P_M T)[P_N T u_N^l + P_N f - (I - P_M T)^{-1} P_M T v_N^l - u_N^*] \\
&= P_N T u_N^l + P_N f - P_M T P_N T u_N^l - P_M T P_N f \\
&- P_M T v_N^l - u_N^* + P_M T u_N^* \\
&= P_N T(u_N^l - u_N^*) - P_M T(u_N^l - u_N^*) \\
&= (P_N - P_M) T(u_N^l - u_N^*)\,.
\end{aligned}
$$

Therefore

$$
u_N^{l+1} - u_N^* = (I - P_M T)^{-1}(P_N - P_M) T(u_N^l - u_N^*)\,, \quad l = 0, 1, \ldots\,. \tag{8.23}
$$

Using (8.22) for $M \geq M_0$, $n/n_M = 2, 3, \ldots$, we obtain that

$$
\begin{aligned}
\left\| u_N^{l+1} - u_N^* \right\|_{L^\infty(0,b)} &\leq c_1 \left\| (P_N - P_M) T \right\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \\
&\times \left\| u_N^l - u_N^* \right\|_{L^\infty(0,b)}\,, \quad l = 0, 1, \ldots\,.
\end{aligned} \tag{8.24}
$$

Further, using Lemma 8.1 we have,

$$
\begin{aligned}
&\left\| (P_N - P_M) T \right\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \\
&\leq \left\| T - P_N T \right\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} + \left\| T - P_M T \right\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \\
&\leq c_1 \varepsilon_{\nu, h_N} + c_2 \varepsilon_{\nu, h_M} \leq c \varepsilon_{\nu, h_M}\,,
\end{aligned} \tag{8.25}
$$

where c, c_1, c_2 are some positive constants which are independent of M and N. From (8.24) and (8.25) it follows that for $M \geq M_0$, $n/n_M = 2, 3, \ldots$,

$$
\left\| u_N^{l+1} - u_N^* \right\|_{L^\infty(0,b)} \leq c \varepsilon_{\nu, h_M} \left\| u_N^l - u_N^* \right\|_{L^\infty(0,b)}\,, \quad l = 0, 1, \ldots\,, \tag{8.26}
$$

where c is a constant which is independent of N, M and l. Since

$$\begin{aligned}\left\|\overline{u}_N^{l+1} - \overline{u}_N^*\right\|_{d_N} &= \left\|R_{\infty,N}\left(u_N^{l+1} - u_N^*\right)\right\|_{L^\infty(0,b)} \\ &\le \left\|u_N^{l+1} - u_N^*\right\|_{L^\infty(0,b)} \end{aligned} \tag{8.27}$$

and

$$\begin{aligned}\left\|u_N^l - u_N^*\right\|_{L^\infty(0,b)} &= \left\|Q_{N,\infty}\left(\overline{u}_N^l - \overline{u}_N^*\right)\right\|_{d_N} \\ &\le c_3\left\|\overline{u}_N^l - \overline{u}_N^*\right\|_{d_N}, \end{aligned} \tag{8.28}$$

the convergence (8.17) follows from (8.26), (8.27) and (8.28). □

Chapter 9

Galerkin Method

A popular method for the numerical solution of equation (1.1) is also Galerkin method. To a exhaustive study of this method, we refer to [6, 8, 15, 30, 48]. The following treatment is based on the results of [16, 20].

9.1 Description of Galerkin Method

In this section we introduce a Galerkin method for the numerical solution of equation

$$u(t) - \int_0^b K(t,s)u(s)ds = f(t)\,, \quad 0 \le t \le b\,. \tag{9.1}$$

We assume that $f \in C[0,b]$ and the kernel K satisfies the conditions of Theorem 3.1.

Let $E = L_2(0,b)$, by $(\cdot,\cdot)$ we denote the inner product for E. Let $E_N = S_{m-1}^{(-1)}(\Delta_N)$ be a subspace of E. We choose a basis for E_N, in particular we use the same basis functions $\{\psi_{j,q}\}$, $q = 1,\ldots,m$; $j = 1,\ldots,N$ introduced in Chapter 5.

We look for an approximation u_N to the solution u of equation determing u_N from the following conditions:

$$\begin{gathered} u_N \in S_{m-1}^{(-1)}(\Delta_N)\,, \quad m \ge 1\,, \\ (u_N - Tu_N - f, \psi_{j,q}) = 0, \quad q = 1,\ldots,m\,;\ j = 1,\ldots,N\,, \end{gathered} \tag{9.2}$$

with T, given by (3.1). We can represent $u_N \in S_{m-1}^{(-1)}(\Delta_N)$ in the form

$$u_N(t) = \sum_{j=1}^{N} \sum_{q=1}^{m} c_{j,q} \psi_{j,q}(t) \,, \quad t \in [0,b] \,. \tag{9.3}$$

with $\psi_{j,q}(t)$ ($q = 1, \ldots, m$; $j = 1, \ldots, N$), given by (5.7). Now the conditions (9.2) will take the following form of a system of algebraic equations with respect to the coefficients $c_{j,q}$, $q = 1, \ldots, m$; $j = 1, \ldots, N$:

$$\sum_{j=1}^{N} \sum_{q=1}^{m} a_{i,p,j,q} c_{j,q} = b_{i,p} \,, \quad p = 1, \ldots, m; i = 1, \ldots, N \,, \tag{9.4}$$

where

$$a_{i,p,j,q} = \int_0^b \psi_{j,q}(s) \psi_{i,p}(s) ds - \int_0^b \left[\int_0^b K(t,s) \psi_{j,q}(s) ds \right] \psi_{i,p}(t) dt \,,$$

$$q = 1, \ldots, m \,;\; j = 1, \ldots, N \,;\; p = 1, \ldots, m \,;\; i = 1, \ldots, N \,,$$

and

$$b_{i,p} = \int_0^b f(t) \psi_{i,p}(t) dt \,, \quad p = 1, \ldots, m \,;\; i = 1, \ldots, N \,.$$

9.2 Convergence of Galerkin Approximations

In this section we study the convergence of Galerkin method. In Theorems 9.1 and 9.3 we present some results which are partly published in [16, 20]. The results of Theorem 9.2 are not published yet.

Theorem 9.1 *Let the conditions of Theorem 6.1 be fulfilled.*

Then equation (9.1) *has a unique solution* $u^* \in C[0,b]$. *For all sufficiently large* N, *say* $N \geq N_0$, *the conditions* (9.2) *determine for every choice of parameters* (5.4) *an approximation* $u_N^* \in S_{m-1}^{(-1)}(\Delta_N)$ *to* u^* *uniquely and*

$$\sup_{t \in [0,b]} \left| u_N^*(t) - u^*(t) \right| \to 0 \quad as \quad N \to \infty \,. \tag{9.5}$$

Proof. Consider equation (9.1) as the operator equation $u - Tu = f$ in the Banach space $E = L^\infty(0,b)$, where the operator T is defined by (3.1). By Theorem 6.1, equation (9.1) has a unique solution $u^* \in C[0,b]$.

Let $Q_N : L^2(0,b) \to E_N$ be the orthogonal projector of $L^2(0,b)$ onto $E_N = S_{m-1}^{(-1)}(\Delta_N)$, which we will also regard as the projection operator on $L^\infty(0,b)$ to $S_{m-1}^{(-1)}(\Delta_N)$. With Q_N we can rewrite (9.2) as

$$u_N - Q_N T u_N = Q_N f \,. \tag{9.6}$$

We have:

1) if $N \to \infty$, then $\big\|Q_N v - v\big\|_{L^\infty(0,b)} \to 0$ for every $v \in C[0,b]$;

2) $\|Q_N\|_{\mathcal{L}(C[0,b],L^\infty(0,b))} \le c$, $N \ge N_0$ with a constant c, independent of N.

Therefore, $I - Q_N T$ is invertible in $L^\infty(0,b)$ for all sufficiently large N, say $N \ge N_0$, with some constant c, independent of N, and

$$\begin{aligned} \|u_N^* - u^*\|_{L^\infty(0,b)} &= \big\|(I - Q_N T)^{-1}\big\|_{L^\infty(0,b)} \big\|u^* - Q_N u^*\big\|_{L^\infty(0,b)} \\ &\le c\|u^* - Q_N u^*\|_{L^\infty(0,b)} \,. \end{aligned} \tag{9.7}$$

For $u^* \in C[0,b]$ we have $Q_N P_N u^* = P_N u^*$, with P_N defined in Chapter 5. Therefore

$$\begin{aligned} u^* - Q_N u^* &= u^* - Q_N P_N u^* - Q_N u^* + Q_N P_N u^* \\ &= u^* - P_N u^* - Q_N(u^* - P_N u^*) \end{aligned}$$

and

$$\big\|u^* - Q_N u^*\big\|_{L^\infty(0,b)} \le c\big\|u^* - P_N u^*\big\|_{L^\infty(0,b)} \,, N \ge N_0, \tag{9.8}$$

where c is a positive constant not depending on N. Now (9.5) follows since $\|u^* - P_N u^*\|_{L^\infty(0,b)} \to 0$ for $N \to \infty$. $\square$

Theorem 9.2 *Let the conditions of Theorem 6.3 be fulfilled. Then, in notation of Theorem 9.1, for $N \ge N_0$, the following error estimate holds:*

$$\sup_{0 \le t \le b} \big|u^*(t) - u_N^*(t)\big| \le c \begin{cases} N^{-m} \,, & m < 1 - \nu \,, \\ N^{-m} \ln N \,, & m = 1 - \nu \,, \\ N^{-(1-\nu)} \,, & m > 1 - \nu \,, \end{cases} \tag{9.9}$$

where c is a positive constant not depending on N.

Proof. It follows from (9.7) and (9.8) that

$$\left\|u_N^* - u^*\right\|_{L^\infty(0,b)} \le c\left\|u^* - P_N u^*\right\|_{L^\infty(0,b)}, \tag{9.10}$$

where the constant c does not depend on N. By Theorem 3.2 the solution u^* of equation (9.1) belongs to the space $C_{d,0}^{m,\nu}[0,b]$. Applying Lemma 5.3 and using (4.7), we obtain the estimate (9.9). □

Theorem 9.3 *Let the conditions of Theorem 6.4 be fulfilled. Then, in notation of Theorem 9.1, for* $N \ge N_0$*, the following error estimate holds:*

$$\sup_{0\le t\le b} \left|u^*(t) - u_N^*(t)\right| \le c\varepsilon_N, \tag{9.11}$$

where c *is a positive constant independent of* N *and* ε_N *is defined by the formula* (5.22)*.*

Proof. By Theorem 3.2 $u^* \in C_{d,p}^{m,\nu}[0,b]$. Now the statement (9.11) follows from the inequality (9.10) and Lemma 5.4. □

Chapter 10

Numerical Examples

This chapter contains a series of original numerical tests to support theoretical results.

10.1 Test Equations

Since there was not suitable test equation available in literature, we constructed it by ourselves.

We have considered a weakly singular integral equation of the form

$$u(t) = \int_0^b g(t,s)\frac{1}{\sqrt{|t-s|}}u(s)ds + f(t)\,, \quad t \in [0,b]\,, \tag{10.1}$$

where g and f are some fixed functions.

Example 10.1 The first example is described by

$$g(t,s) = \begin{cases} 1\,, & (t,s) \in [0,b] \times [0,d)\,, \\ 2\,, & (t,s) \in [0,b] \times [d,b]\,, \end{cases} \tag{10.2}$$

and

$$f(t) = \begin{cases} f_1(t)\,, & t < d\,, \\ f_2(t)\,, & t \geq d\,, \end{cases} \tag{10.3}$$

where $0 < d < b$ and

$$f_1(t) = t\left[\frac{\pi}{2} - \frac{1}{2}\ln t - \ln\left(\sqrt{d} + \sqrt{d-t}\right) + 2\ln\left(\sqrt{b} + \sqrt{b-t}\right)\right]$$

$$+ (d-t)\left[\frac{\pi}{2} + \ln\left(\sqrt{d-t}\right) + \ln\left(\sqrt{d} + \sqrt{t}\right) - 2\Big(\ln\sqrt{b-t} + \sqrt{b-d}\Big)\right]$$

$$+ (b-t)\left[\frac{\pi}{2} + \arcsin\left(\sqrt{\frac{b-d}{b-t}}\right) + \ln\left(\sqrt{b} + \sqrt{t}\right) - \ln\sqrt{b-t}\right]$$

$$+ \sqrt{t}\left[\sqrt{d} + \sqrt{b}\right]$$

$$- \sqrt{d-t}\left[\sqrt{b-d} + \sqrt{d}\right]$$

$$+ 2\sqrt{b-t}\left[\sqrt{b-d} + \sqrt{b}\right],$$

$$f_2(t) = t\left[\pi - \arcsin\sqrt{\frac{d}{t}} - \ln t + 2\ln\left(\sqrt{b} + \sqrt{b-t}\right)\right]$$

$$+ (t-d)\left[\pi - \ln\sqrt{t-d} - \ln\left(\sqrt{d} + \sqrt{t}\right) + 2\ln\left(\sqrt{b-d} + \sqrt{b-t}\right)\right]$$

$$+ (b-t)\left[\pi - 2\ln\sqrt{b-t} + \ln\left(\sqrt{b} + \sqrt{t}\right) + \ln\left(\sqrt{b-d} + \sqrt{t-d}\right)\right]$$

$$+ \sqrt{t}\left[\sqrt{d} + \sqrt{b}\right]$$

$$+ \sqrt{t-d}\left[\sqrt{b-d} + \sqrt{d}\right]$$

$$+ 2\sqrt{b-t}\left[\sqrt{b-d} + \sqrt{b}\right].$$

We can see that the assumptions (A1)–(A2) about the kernel $K(t,s) = g(t,s)|t-s|^{-1/2}$ hold with $\nu = \frac{1}{2}$, $p = 0$ and an arbitrary m. Note that in this case

$$u^*(t) = t^{1/2} + |d-t|^{1/2} + (b-t)^{1/2}, \quad t \in [0,b],$$

is the exact solution of equation {(10.1)–(10.3)}.

Example 10.2 The second example is described by

$$g(t,s) = \begin{cases} d-s\,, & (t,s) \in [0,b] \times [0,d)\,, \\ s-d\,, & (t,s) \in [0,b] \times [d,b]\,, \end{cases} \tag{10.4}$$

and

$$f(t) = \begin{cases} f_1(t)\,, & t < d\,, \\ f_2(t)\,, & t \geq d\,, \end{cases} \tag{10.5}$$

where $0 < d < b$ and

$$\begin{aligned}
f_1(t) = {} & \frac{5}{8}(d-t)^3\left[\frac{1}{2}\pi + \ln\left(\sqrt{d}+\sqrt{t}\right) - \ln\left(\sqrt{b-d}+\sqrt{b-t}\right)\right] \\
& + \frac{3}{4}(b-t)^2\left[\frac{1}{2}\pi - 2\arcsin\sqrt{\frac{b-d}{b-t}} - \ln\sqrt{b-t} + \ln\left(\sqrt{b}+\sqrt{t}\right)\right] \\
& + \frac{3}{4}t^2\left[-\frac{1}{2}\pi + \frac{1}{2}\ln t - 2\ln\left(\sqrt{d}+\sqrt{d-t}\right) + \ln\left(\sqrt{b}+\sqrt{b-t}\right)\right] \\
& + dt\left[\frac{1}{2}\pi - \frac{1}{2}\ln t + 2\ln\left(\sqrt{d}+\sqrt{d-t}\right) - \ln\left(\sqrt{b}+\sqrt{b-t}\right)\right] \\
& + (b-d)(b-t)\left[-\frac{1}{2}\pi + 2\arcsin\sqrt{\frac{b-d}{b-t}} + \ln\sqrt{b-t} - \ln\left(\sqrt{b}+\sqrt{t}\right)\right] \\
& + \sqrt{t}\left[\frac{1}{3}d^{5/2} + \frac{5}{12}(d-t)d^{3/2} + \frac{5}{8}(d-t)^2\sqrt{d}\right. \\
& \qquad \left. + \frac{1}{2}b^{3/2} + \frac{3}{4}\sqrt{b}(b-t) - \sqrt{b}(b-d)\right] \\
& + \sqrt{d-t}\left[-(b-d)^{3/2} + \frac{3}{2}(b-t)\sqrt{b-d} - \frac{3}{2}t\sqrt{d} + d^{3/2}\right] \\
& + \sqrt{b-t}\left[\frac{1}{3}(b-d)^{5/2} + \frac{5}{12}(t-d)(b-d)^{3/2} + \frac{5}{8}(t-d)^2\sqrt{b-d}\right. \\
& \qquad \left. + \frac{1}{2}b^{3/2} + \frac{3}{4}t\sqrt{b} - d\sqrt{b}\right],
\end{aligned}$$

$$
\begin{aligned}
f_2(t) = {} & \frac{5}{8}(t-d)^3\left[\frac{1}{2}\pi - \ln\left(\sqrt{d}+\sqrt{t}\right) + \ln\left(\sqrt{b-d}+\sqrt{b-t}\right)\right] \\
& + \frac{3}{4}(b-t)^2\left[-\frac{1}{2}\pi + \ln\left(\sqrt{b}+\sqrt{t}\right) - 2\ln\left(\sqrt{b-d}+\sqrt{t-d}\right) + \ln(b-t)\right] \\
& + \frac{3}{4}t^2\left[\frac{1}{2}\pi - 2\arcsin\sqrt{\frac{d}{t}} - \frac{1}{2}\ln t + \ln\left(\sqrt{b}+\sqrt{b-t}\right)\right. \\
& + dt\left[-\frac{1}{2}\pi + 2\arcsin\sqrt{\frac{d}{t}} + \frac{1}{2}\ln t - \ln\left(\sqrt{b}+\sqrt{b-t}\right)\right] \\
& + (b-d)(b-t)\left[\frac{1}{2}\pi - \ln\left(\sqrt{b}+\sqrt{t}\right) - \ln\sqrt{b-t} + 2\ln\left(\sqrt{b-d}+\sqrt{t-d}\right)\right] \\
& + \sqrt{t}\left[\frac{1}{3}d^{5/2} - \frac{5}{12}(t-d)d^{3/2} + \frac{5}{8}(t-d)^2\sqrt{d}\right. \\
& \qquad \left. + \frac{1}{2}b^{3/2} + \frac{3}{4}\sqrt{b}(b-t) - \sqrt{b}(b-d)\right] \\
& + \sqrt{t-d}\left[(b-d)^{3/2} - \frac{3}{2}(b-t)\sqrt{b-d} + \frac{3}{2}t\sqrt{d} - d^{3/2}\right] \\
& + \sqrt{b-t}\left[\frac{1}{3}(b-d)^{5/2} + \frac{5}{12}(t-d)(b-d)^{3/2} + \frac{5}{8}(t-d)^2\sqrt{b-d}\right. \\
& \qquad \left. + \frac{1}{2}b^{3/2} + \frac{3}{4}t\sqrt{b} - d\sqrt{b}\right].
\end{aligned}
$$

We can see that the assumptions (A1)–(A2) about the kernel $K(t,s) = g(t,s)|t-s|^{-1/2}$ hold with $\nu = \frac{1}{2}$, $p = 1$ and an arbitrary m. Notice that in this case

$$
u^*(t) = t^{1/2} + |d-t|^{3/2} + (b-t)^{1/2}, \quad t \in [0,b],
$$

is the exact solution of equation {(10.1),(10.4),(10.5)}.

To test the efficiency of our numerical approaches, we have applied the methods defined in Chapters 4–8 to equations {(10.1)–(10.3)} and {(10.1), (10.4),(10.5)}. The programs for these examples were written using the C++ language.

10.2 Quadrature Method

At first we solved equations {(10.1)–(10.3)} and {(10.1), (10.4),(10.5)} by the quadrature formula method, described in Chapter 4 (see (4.18)). In both of these two cases we used the uniform grid on the interval $[0, b]$ with $d = \frac{1}{2}$. The nodes of the quadrature formula (4.16) we calculated as follows:

$$s_i = \left(i + \frac{1}{2}\right)h\,, \quad h = \frac{b}{N}\,, \quad i = 0, \ldots, N-1\,.$$

The integrals, which we needed for the construction of the system (4.19), we found analytically.

In Table 10.1 the errors

$$\varepsilon_N = \max_{i=1,\ldots,N} \left| z_i^* - u^*(s_i) \right|,$$

where z_i^* are the approximate values of the solution $u^*(t)$ in nodes $s_1, \ldots, s_N$ and the ratios $\varrho_N = \varepsilon_{N/2}/\varepsilon_N$ are presented.

	$b = 1,\ \nu = \frac{1}{2}$			
N	$p = 0$		$p = 1$	
	ε_N	ϱ_N	ε_N	ϱ_N
32	$4.5E-02$		$2.9E-03$	
64	$1.5E-02$	3.0	$1.3E-03$	2.3
128	$6.0E-03$	2.5	$5.4E-04$	2.3
256	$2.5E-03$	2.4	$2.4E-04$	2.3
512	$1.1E-03$	2.3	$1.1E-04$	2.2
1024	$4.7E-04$	2.3	$5.0E-05$	2.2

Table 10.1: Examples {(10.1)–(10.3)} and {(10.1),(10.4),(10.5)}. Quadrature formula method with the rectangular rule.

From Theorem 4.2 it follows that for $\nu = \frac{1}{2}$ the ratio ϱ_N should be approximately $2 (= 2^{2(1-\nu)})$. From Table 10.1 we can see that the numerical results are a little bit better than the theoretical results.

10.3 Collocation Method

Next equations {(10.1)–(10.3)} and {(10.1),(10.4),(10.5)} were solved numerically by the collocation method (6.3)–(6.4) with $b = 1$ and $d = 1/2$. All the integrals needed for the construction of the system (6.4) we found analytically.

In order to estimate the error

$$\sup_{0\le t\le b} \left|u^*(t) - u_N^*(t)\right|, \quad N \ge N_0\,, \tag{10.6}$$

for the solutions $u^*(t)$ and $u_N^*(t)$ of (10.1) and (6.3)–(6.4), respectively, we introduced a partition of the interval $[0, b]$ with the points $\varepsilon_{i,j}$, $i = 0, \dots, N-1$, $j = 0, \dots, 10$ defined by

$$t'_{i,j} = t_i + j\left(\frac{t_{i+1} - t_i}{10}\right). \tag{10.7}$$

Here t_i $(i = 0, 1, \dots, N)$ are the grid points (5.19) with $d = \dfrac{1}{2}$, $b = 1$.

In Tables 10.2, 10.3, 10.5, 10.6, 10.8, 10.9, 10.11, 10.12 we have modelled the error (10.6) by

$$\varepsilon'_N = \max_{i=0,\dots,N-1;j=1,\dots,10} \left|u^*(t'_{i,j}) - u_N^*(t'_{i,j})\right|. \tag{10.8}$$

We have also used the ratios

$$\varrho'_N = \frac{\varepsilon'_{N/2}}{\varepsilon'_N} \tag{10.9}$$

to characterize the rate of convergence of method (6.3)–(6.4). In Tables 10.4, 10.7, 10.10, 10.13 we have presented the errors ε''_N and corresponding ratios ϱ''_N at the collocation points (5.3):

$$\varepsilon''_N = \max_{i=1,\dots,m;j=1,\dots,N} \left|u^*(\xi_{j,i}) - u_N^*(\xi_{j,i})\right|, \tag{10.10}$$

$$\varrho''_N = \frac{\varepsilon''_{N/2}}{\varepsilon''_N}\,, \tag{10.11}$$

where $\{\xi_{j,i}\}$ are the collocation points defined by (5.3).

10.3.1 Test Equation {(10.1)–(10.3)}

First we consider collocation by piecewise constant interpolation at the graded grid (5.19). In this case $m = 1$ and the number of the equations and unknowns in the system (6.6) is $N = 4n$. We choose $\eta_1 = -2/5$ (see Table 10.2) and $\eta_1 = 0$ (see Table 10.3 and 10.4).

From Tables 10.2 and 10.3 we can see that the numerical results are in good agreement with the theoretical estimates of Theorem 6.4.

Indeed, it follows from Theorem 6.4 that for the graded grid (5.19) with $\nu = \frac{1}{2}$, $p = 0$, $m = 1$ the ratio ϱ'_N (see (10.9)) ought to be approximately $1.4 (= \sqrt{2} = 2^{r_d(1-\nu+p)})$ for $r = r_d = 1$. For $r = r_d = 2 \left(\geq \dfrac{m}{1-\nu+p} \right)$ the ratio ϱ'_N should be approximately $2 (= 2^m)$.

The results reported in Table 10.4 show that we get a better convergence order in collocation points.

n	$b = 1,\ d = \frac{1}{2},\ m = 1,\ \nu = \frac{1}{2},\ p = 0,\ \eta_1 = -\frac{2}{5}$			
	$r = r_d = 1$		$r = r_d = 2$	
$(n = \frac{N}{4})$	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$2.9E-01$		$1.8E-01$	
16	$1.3E-01$	2.3	$5.2E-02$	3.4
32	$7.9E-02$	1.6	$2.1E-02$	2.5
64	$5.4E-02$	1.4	$1.1E-02$	2.0
128	$3.8E-02$	1.4	$5.5E-03$	1.9
256	$2.7E-02$	1.4	$2.9E-03$	1.9
512	$1.9E-02$	1.4	$1.5E-03$	2.0
1024	$1.3E-02$	1.4	$7.4E-04$	2.0

Table 10.2: Example {(10.1)–(10.3)}. Collocation with piecewise constant interpolation.

Indeed, from Theorem 7.1 it follows that the ratio ϱ''_N (see (10.11)) ought to be approximately $2.8 (= 2^m 2^{1-\nu})$ for $r = 2 \left(\geq \dfrac{m}{1-\nu} \right)$ and $r_d = 2 \left(\geq \dfrac{m}{1-\nu+p} \right)$. From Table 10.4 we can see that the numerical results are a little better than the theoretical results.

	$b=1,\ d=\frac{1}{2},\ m=1,\ \nu=\frac{1}{2},\ p=0,\ \eta_1=0$			
n	$r=r_d=1$		$r=r_d=2$	
$(n=\frac{N}{4})$	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$1.6E-01$		$5.6E-02$	
16	$1.1E-01$	1.5	$2.7E-02$	2.1
32	$7.0E-02$	1.5	$1.3E-02$	2.1
64	$4.7E-02$	1.5	$6.0E-03$	2.1
128	$3.3E-02$	1.5	$2.9E-03$	2.1
256	$2.3E-02$	1.4	$1.4E-03$	2.0
512	$1.6E-02$	1.4	$7.0E-04$	2.0
1024	$1.1E-02$	1.4	$3.5E-04$	2.0

Table 10.3: Example {(10.1)–(10.3)}. Collocation with piecewise constant interpolation.

	$b=1,\ d=\frac{1}{2},\ m=1,\ \nu=\frac{1}{2},\ p=0,\ \eta_1=0$	
n	$r=r_d=2$	
$(n=\frac{N}{4})$	ε''_N	ϱ''_N
8	$1.8E-02$	
16	$6.4E-03$	2.8
32	$1.9E-03$	3.3
64	$5.2E-04$	3.7
128	$1.4E-04$	3.8
256	$3.5E-05$	3.9
512	$8.9E-06$	3.9
1024	$2.3E-06$	3.9

Table 10.4: Example {(10.1)–(10.3)}. Superconvergence with piecewise constant interpolation.

Next we consider the use of piecewise linear interpolation at the graded grid (5.19). In this case $m=2$ and the number of the equations and unknowns in the system (6.6) is $N=8n$. First we choose $\eta_1=-1/2$ and $\eta_2=1/2$ (see Table 10.5) and then $\eta_1=-1/\sqrt{3}$ and $\eta_2=1/\sqrt{3}$ (see Table 10.6 and 10.7).

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=0,\ \eta_1=-\frac{1}{2},\ \eta_2=\frac{1}{2}$					
n	$r=r_d=1$		$r=r_d=2$		$r=r_d=4$	
$\left(n=\frac{N}{4}\right)$	ε'_N	ϱ'_N	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$7.1E-02$		$2.2E-02$		$5.1E-03$	
16	$4.6E-02$	1.6	$1.0E-02$	2.1	$1.2E-03$	4.4
32	$3.1E-02$	1.5	$5.1E-03$	2.0	$3.3E-04$	3.5
64	$2.1E-02$	1.5	$2.5E-03$	2.0	$8.7E-05$	3.8
128	$1.4E-02$	1.4	$1.2E-03$	2.0	$2.2E-05$	3.9
256	$1.0E-02$	1.4	$6.2E-04$	2.0	$5.6E-06$	3.9
512	$7.1E-03$	1.4	$3.1E-04$	2.0	$1.4E-06$	4.0

Table 10.5: Example {(10.1)–(10.3)}. Collocation with piecewise linear interpolation.

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=0,\ \eta_1=-\frac{1}{\sqrt{3}},\ \eta_2=\frac{1}{\sqrt{3}}$					
n	$r=r_d=1$		$r=r_d=2$		$r=r_d=4$	
$\left(n=\frac{N}{4}\right)$	ε'_N	ϱ'_N	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$6.1E-02$		$2.0E-02$		$4.5E-03$	
16	$4.1E-02$	1.5	$9.6E-03$	2.0	$1.2E-03$	3.6
32	$2.8E-02$	1.5	$4.8E-03$	2.0	$3.3E-04$	3.8
64	$1.9E-02$	1.4	$2.4E-03$	2.0	$8.5E-05$	3.8
128	$1.4E-02$	1.4	$1.2E-03$	2.0	$2.2E-05$	3.9
256	$9.6E-03$	1.4	$5.9E-04$	2.0	$5.5E-06$	4.0
512	$6.7E-03$	1.4	$3.0E-04$	2.0	$1.4E-06$	4.0

Table 10.6: Example {(10.1)–(10.3)}. Collocation with piecewise linear interpolation.

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=0,\ \eta_1=-\frac{1}{\sqrt{3}},\ \eta_2=\frac{1}{\sqrt{3}}$	
n	$r=r_d=4$	
$\left(n=\frac{N}{4}\right)$	ε''_N	ϱ''_N
8	$2.5E-04$	
16	$5.9E-05$	4.2
32	$9.1E-06$	6.5
64	$1.1E-06$	8.4
128	$1.9E-07$	5.6
256	$3.5E-08$	5.5
512	$6.3E-09$	5.6

Table 10.7: Example {(10.1)–(10.3)}. Superconvergence with piecewise linear interpolation.

The results reported in Table 10.5 and 10.6 show that the numerical results are consistent with the theoretical estimates of Theorem 6.4. In detail, it follows from Theorem 6.4 that for the graded grid (5.19) with $\nu=\dfrac{1}{2}$, $p=0$, $m=2$, the ratio ϱ'_N should be approximately $1.4\big(=\sqrt{2}=2^{r_d(1-\nu+p)}\big)$ for $r=r_d=1$. For $r=r_d=2$ the ratio ϱ'_N ought to be approximately $2\big(=2^{r_d(1-\nu+p)}\big)$ and the ratio ϱ'_N ought to be approximately $4(=2^m)$ for $r=r_d=4\left(\geq\dfrac{m}{1-\nu+p}\right)$.

From Theorem 7.1 it follows that the ratio ϱ''_N should be approximately $5.6(=2^m2^{1-\nu})$ for $r=4\left(\geq\dfrac{m}{1-\nu}\right)$, $r_d=4\left(\geq\dfrac{m}{1-\nu+p}\right)$. From Table 10.7 we can see that the numerical results are in good agreement with the theoretical results.

10.3.2 Test Equation {(10.1),(10.4),(10.5)}

First we consider collocation by piecewise constant interpolation at the graded grid (5.19). In this case $m=1$. We choose $\eta_1=-2/5$ (see Table 10.8) and $\eta_1=0$ (see Tables 10.9 and 10.10).

	$b=1,\ d=\frac{1}{2},\ m=1,\ \nu=\frac{1}{2},\ p=1,\ \eta_1=-\frac{2}{5}$			
n	$r=1,\ r_d=1$		$r=2,\ r_d=1$	
$(n=\frac{N}{4})$	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$1.2E-01$		$5.1E-02$	
16	$9.1E-02$	1.3	$2.6E-02$	1.9
32	$6.7E-02$	1.3	$1.3E-02$	2.0
64	$4.9E-02$	1.4	$6.8E-03$	2.0
128	$3.5E-02$	1.4	$3.4E-03$	2.0
256	$2.5E-02$	1.4	$1.7E-03$	2.0
512	$1.8E-02$	1.4	$8.5E-04$	2.0
1024	$1.3E-02$	1.4	$4.3E-04$	2.0

Table 10.8: Example {(10.1),(10.4),(10.5)}. Collocation with piecewise constant interpolation.

	$b=1,\ d=\frac{1}{2},\ m=1,\ \nu=\frac{1}{2},\ p=1,\ \eta_1=0$			
n	$r=r_d=1$		$r=r_d=2$	
$(n=\frac{N}{4})$	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$1.6E-01$		$5.6E-02$	
16	$1.1E-01$	1.5	$2.7E-02$	2.1
32	$7.0E-02$	1.5	$1.3E-02$	2.1
64	$4.7E-02$	1.5	$6.0E-03$	2.1
128	$3.3E-02$	1.5	$2.9E-03$	2.1
256	$2.3E-02$	1.4	$1.4E-03$	2.0
512	$1.6E-02$	1.4	$7.0E-04$	2.0
1024	$1.1E-02$	1.4	$3.5E-04$	2.0

Table 10.9: Example {(10.1),(10.4),(10.5)}. Collocation with piecewise constant interpolation.

	$b=1,\, d=\frac{1}{2},\, m=1,\, \nu=\frac{1}{2},\, p=1,\, \eta_1=0$	
n	$r=r_d=2$	
$\left(n=\frac{N}{4}\right)$	ε''_N	ϱ''_N
8	$1.4E-03$	
16	$4.4E-04$	3.1
32	$1.3E-04$	3.3
64	$3.9E-05$	3.4
128	$1.1E-05$	3.5
256	$3.1E-06$	3.6
512	$8.6E-07$	3.6
1024	$2.4E-07$	3.6

Table 10.10: Example {(10.1),(10.4),(10.5)}. Superconvergence with piecewise constant interpolation.

From Tables 10.8 and 10.9 we can see that the numerical results are consistent with the theoretical estimates of Theorem 6.4. Indeed, it follows from Theorem 6.4 that for the graded grid (5.19) with $\nu = \dfrac{1}{2}$, $p = 1$, $m = 1$ the ratio ϱ'_N should be approximately $1.4(=\sqrt{2} = 2^{r(1-\nu)})$ for $r = 1\left(< \dfrac{m}{1-\nu}\right)$, $r_d = 1(\geq 1)$. For $r = 2\left(\geq \dfrac{m}{1-\nu}\right)$ and $r_d = 1(\geq 1)$ the ratio ϱ'_N ought to be approximately $2(=2^m)$.

From Theorem 7.1 it follows that the ratio ϱ''_N ought to be approximately $2.8(=2^m 2^{1-\nu})$ for $r = 2\left(\geq \dfrac{m}{1-\nu}\right)$ and $r_d = 1(\geq 1)$. Table 10.10 shows than the numerical results are a little better than theoretical results.

Now we consider collocation by piecewise linear interpolation at the graded grid (5.19). In this case $m = 2$. We choose $\eta_1 = -1/2$ and $\eta_2 = 1/2$ (see Table 10.11) and $\eta_1 = -1/\sqrt{3}$ and $\eta_2 = 1/\sqrt{3}$ (see Tables 10.12 and 10.13).

From Table 10.11 and 10.12 we can see that the numerical results are in good agreement with the theoretical estimates of Theorem 6.4.

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=1,\ \eta_1=-\frac{1}{2},\ \eta_2=\frac{1}{2}$					
n	$r=2,\ r_d=1$		$r=3,\ r_d=1$		$r=4,\ r_d=1.4$	
$\left(n=\frac{N}{4}\right)$	ε'_N	ϱ'_N	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$2.0E-02$		$7.4E-03$		$4.5E-03$	
16	$1.0E-02$	2.0	$2.6E-03$	2.9	$1.1E-03$	4.0
32	$5.0E-03$	2.0	$9.0E-04$	2.9	$2.8E-04$	4.0
64	$2.5E-03$	2.0	$3.2E-04$	2.8	$7.1E-05$	4.0
128	$1.2E-03$	2.0	$1.1E-04$	2.8	$1.8E-05$	4.0
256	$6.2E-04$	2.0	$3.9E-05$	2.8	$4.4E-06$	4.0
512	$3.1E-04$	2.0	$1.4E-05$	2.8	$1.1E-06$	4.0

Table 10.11: Example {(10.1),(10.4),(10.5)}. Collocation with piecewise linear interpolation.

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=1,\ \eta_1=-\frac{1}{\sqrt{3}},\ \eta_2=\frac{1}{\sqrt{3}}$					
n	$r=2,\ r_d=1$		$r=3,\ r_d=1$		$r=4,\ r_d=1.4$	
$\left(n=\frac{N}{4}\right)$	ε'_N	ϱ'_N	ε'_N	ϱ'_N	ε'_N	ϱ'_N
8	$1.9E-02$		$6.7E-03$		$3.8E-03$	
16	$9.5E-03$	2.0	$2.4E-03$	2.8	$9.4E-04$	4.0
32	$4.7E-03$	2.0	$8.4E-04$	2.8	$2.3E-04$	4.0
64	$2.4E-03$	2.0	$3.0E-04$	2.8	$5.8E-05$	4.0
128	$1.2E-03$	2.0	$1.0E-04$	2.8	$1.5E-05$	4.0
256	$5.9E-04$	2.0	$3.7E-05$	2.8	$3.7E-06$	4.0
512	$3.0E-04$	2.0	$1.3E-05$	2.8	$9.1E-07$	4.0

Table 10.12: Example {(10.1),(10.4),(10.5)}. Collocation with piecewise linear interpolation.

Indeed, it follows from Theorem 6.4 that for the graded grid (5.19) with $\nu = \dfrac{1}{2}$, $p = 1$, $m = 2$, the ratio ϱ'_N ought to be approximately $2(= 2^{\min\{r(1-\nu), r_d(1-\nu+p)\}})$ for $r = 2\left(< \dfrac{m}{1-\nu}\right)$ and $r_d = 1\left(< \dfrac{m}{1-\nu+p}\right)$. For $r = 4\left(< \dfrac{m}{1-\nu}\right)$ and $r_d = 1.4\left(\geq \dfrac{m}{1-\nu+p}\right)$ the ratio ϱ'_N should be approximately $2.8(= 2^{r(1-\nu)})$ and for $r = 4\left(\geq \dfrac{m}{1-\nu}\right)$ and $r_d = 1.4\left(\geq \dfrac{m}{1-\nu+p}\right)$ the ratio ϱ'_N ought to be approximately $4(= 2^m)$.

From Theorem 7.1 it follows that the ratio ϱ''_N should be approximately $5.6(= 2^m 2^{1-\nu})$ for $r = 4\left(\geq \dfrac{m}{1-\nu}\right)$ and $r_d = 1.4\left(\geq \dfrac{m}{1-\nu+p}\right)$. Table 10.13 shows that the numerical results are consistent with the theoretical results.

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=1,\ \eta_1=-\frac{1}{\sqrt{3}},\ \eta_2=\frac{1}{\sqrt{3}}$	
n	$r=4,\ r_d=1.4$	
$\left(n=\frac{N}{4}\right)$	ε''_N	ϱ''_N
8	$7.7E-05$	
16	$8.0E-06$	9.6
32	$1.1E-06$	7.5
64	$1.7E-07$	6.4
128	$2.8E-08$	6.0
256	$4.8E-09$	5.8
512	$8.5E-10$	5.7

Table 10.13: Example {(10.1),(10.4),(10.5)}. Superconvergence with piecewise linear interpolation.

10.4 Two-Grid Method

In this section we present the results, which have been obtained by the solution of the system (6.6) with the two-grid method (8.2). We consider only the equation {(10.1)–(10.3)} with $b = 1$ and $d = 1/2$, results for the equation {(10.1),(10.4),(10.5)} are analogous.

First we compute the number of iteration steps $l \in \mathbb{N}$ leading to an optimal error $\|\overline{u}^* - \overline{u}_N^l\|_{d_N} \le cN^{-m}$ (cf. (6.21)), where u^* is the exact solution of equation {(10.1)–(10.3)}. Let $n_M = n^{1/2}$, $0 < \nu < 1$, and let the error of an initial guess $\overline{u}_N^0$ of the two-grid method be given by

$$\left\|\overline{u}_N^0 - \overline{u}_N^*\right\|_{d_N} \le cM^{-m} = c_1 N^{-\frac{m}{2}} .$$

The error made on the l-th level of iteration is

$$\begin{aligned}\left\|\overline{u}_N^l - \overline{u}_N^*\right\|_{d_N} &\le ch_M^{1-\nu}\left\|\overline{u}_N^{l-1} - \overline{u}_N^*\right\|_{d_N} \\ &\le c^l h_M^{(1-\nu)l}\left\|\overline{u}_N^0 - \overline{u}_N^*\right\|_{d_N} \\ &\le c^l h_M^{(1-\nu)l} M^{-m} .\end{aligned}$$

Hence after l iterations we have

$$\begin{aligned}\left\|\overline{u}^* - \overline{u}_N^l\right\|_{d_N} &\le \left\|\overline{u}^* - \overline{u}_N^*\right\|_{d_N} + \left\|\overline{u}_N^* - \overline{u}_N^l\right\|_{d_N} \\ &\le c(N^{-m} + N^{-\frac{m}{2}} N^{-\frac{1-\nu}{2}l}).\end{aligned}$$

From which we get

$$\left\|\overline{u}^* - \overline{u}_N^l\right\|_{d_N} \le cN^{-m} , \tag{10.12}$$

if $N^{-\frac{m}{2}} N^{-\frac{1-\nu}{2}l} \le N^{-m}$ or $l \ge \dfrac{m}{1-\nu}$.

In the following we take $\nu = 1/2$ and check out how many iteration steps one needs to obtain the result so that the errors of the two-grid method are within 10% of the error of the corresponding collocation method. The initial guess of the two-grid method has been taken $\overline{u}_N^0 = Q_{MN}\overline{u}_M$.

Let $m = 1$, $n_M = \sqrt{n}$. Then by formula (10.12) the number of iteration steps is $l \ge 2$. The results reported in Figure 10.1 show that for a large n 2 iterations may be sufficient. But a good strategy will be 3 or 4 iteration steps.

Let $m = 2$, $n_M = \sqrt{n}$. Then by formula (10.12) the number of iteration steps is $l \geq 4$. From the numerical example (see Figure 10.2) it follows that for a large n 4 iteration steps may be enough, we recommend to use 8 iterations.

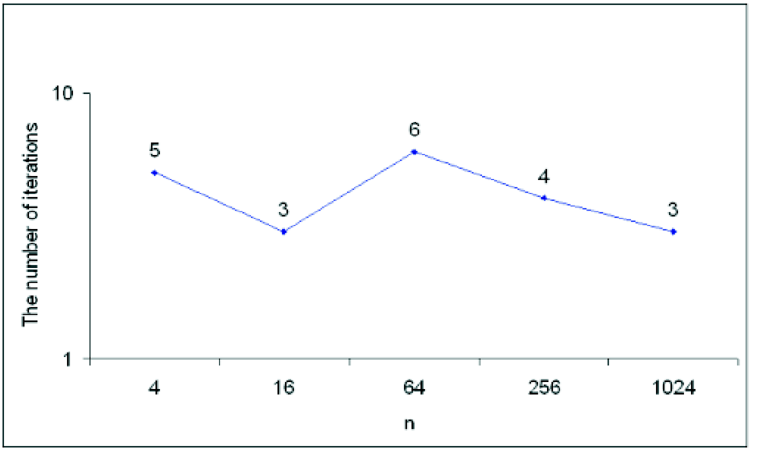

Figure 10.1: Number of iteration steps for $m = 1$.

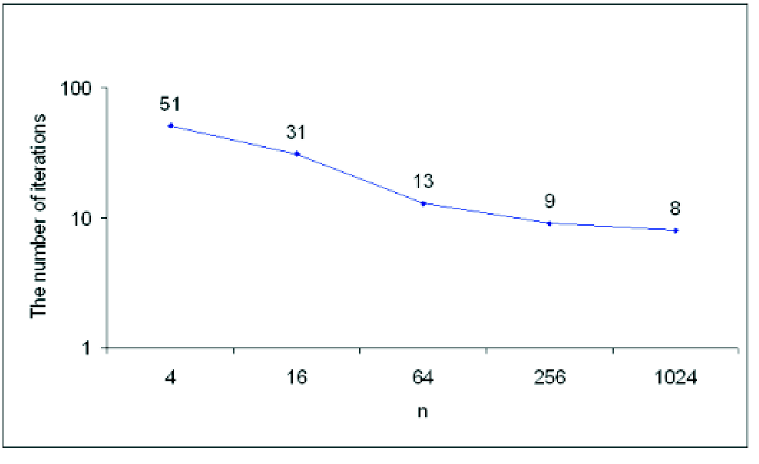

Figure 10.2: Number of iteration steps for $m = 2$.

Next we consider the same test equation {(10.1)–(10.3)} to compare time consumptions that one needs to obtain the result so that the errors of two-grid method are within 10% of the errors of the corresponding collocation method.

	$b=1,\ d=\frac{1}{2},\ m=1,$ $\nu=\frac{1}{2},\ p=0,\ \eta=0,\ r=r_d=2$			
n $\left(n=\frac{N}{4}\right)$	Collocation method		Two-grid method	
	Time consumptions (sec)	ε'_N	Time consumptions (sec)	ε'_N
4	0.0	$1.2E-01$	0.1	$1.2E-01$
16	0.1	$2.7E-02$	0.1	$2.5E-02$
64	0.8	$6.0E-03$	0.2	$6.4E-03$
256	53.1	$1.4E-03$	1.3	$1.5E-03$
1024	3101.8	$3.5E-04$	16.1	$3.6E-04$

Table 10.14: Example {(10.1)–(10.3)}. Convergence speed and time consumption for $m = 1$.

	$b=1,\ d=\frac{1}{2},\ m=2,$ $\nu=\frac{1}{2},\ p=0,\ \eta_1=-\frac{1}{\sqrt{3}},\ \eta_2=\frac{1}{\sqrt{3}},\ r=r_d=4$			
n $\left(n=\frac{N}{4}\right)$	Collocation method		Two-grid method	
	Time consumptions (sec)	ε'_N	Time consumptions (sec)	ε'_N
4	0.0	$2.4E-02$	0.4	$2.6E-02$
16	0.1	$1.2E-03$	0.5	$1.3E-03$
64	6.3	$8.5E-05$	1.2	$9.2E-05$
256	374.8	$5.5E-06$	13.0	$6.0E-06$
1024	-	$-$	2066.4	$3.7E-07$

Table 10.15: Example {(10.1)–(10.3)}. Convergence speed and time consumption for $m = 2$.

We have chosen $m = 1$, $\eta = 0$, $r = r_d = 2$ in the case of Table 10.14 and $m = 2$, $\eta_1 = -1/\sqrt{3}$, $\eta_2 = 1/\sqrt{3}$ in the case of Table 10.15. The error ε'_N of the approximate solution u^*_N is described by (10.8). The numerical results from Table 10.14 and Table 10.15 clearly show, that the time consumption for two-grid method is much smaller than for the collocation method for the same n. We can also see that the two-grid method allows solving an integral equation for a larger n. In the case of Table 10.15 we were not capable of solving the equation with the collocation method for $n = N/4 = 1024$. But we could do it with the two-grid method.

So the advantages of the two-grid method are the smaller time consumption and the opportunity to solve large systems of equations.

In the following we try to find out which is the best choice for n_M. We consider the dependence of the computation time on n_M for $n = N/4 = 512$, $m = 1$ and $m = 2$ in Table 10.16 and Table 10.17, respectively.

	$b = 1,\ d = \frac{1}{2},\ m = 1,\ \nu = \frac{1}{2},\ p = 0,\ \eta = 0,\ r = r_d = 2$	
n_M	Time consumptions (sec)	ε'_N
8	9.5	$7.6E-04$
16	5.2	$7.4E-04$
32	3.8	$7.4E-04$
64	6.7	$7.0E-04$
128	18.8	$7.2E-04$
256	126.6	$7.1E-04$

Table 10.16: Example {(10.1)–(10.3)}. Time depending upon n_M for $n = 512$ and $m = 1$.

	$b = 1,\ d = \frac{1}{2},\ m = 2,\ \nu = \frac{1}{2},\ p = 0,$ $\eta_1 = -\frac{1}{\sqrt{3}},\ \eta_2 = \frac{1}{\sqrt{3}},\ r = r_d = 4$	
n_M	Time consumptions (sec)	ε'_N
8	86	$1.5E-06$
16	53	$1.5E-06$
32	52	$1.5E-06$
64	89	$1.4E-06$
128	290	$1.5E-06$
256	1544	$1.5E-06$

Table 10.17: Example {(10.1)–(10.3)}. Time depending upon n_M for $n = 512$ and $m = 2$.

From Table 10.16 and Table 10.17 it follows that the small time consumptions are achieved for $n_M = n^{1/3} = 8$, $n_M = n^{4/9} = 16$, $n_M = n^{5/9} = 32$ and $n_M = n^{2/3} = 64$.

In Table 10.18 and Table 10.19 we consider the dependence of the computation time on n_M for $n = N/4 = 1024$, $m = 1$ and $m = 2$, respectively.

	$b=1,\ d=\frac{1}{2},\ m=1,\ \nu=\frac{1}{2},\ p=0,\ \eta=0,\ r=r_d=2$	
n_M	Time consumptions (sec)	ε'_N
8	41.4	$3.8E-04$
16	22.2	$3.7E-04$
32	16.1	$3.6E-04$
64	17.0	$3.5E-04$
128	28.0	$3.7E-04$
256	139.1	$3.5E-04$
512	1015.6	$3.5E-04$

Table 10.18: Example $\{(10.1)–(10.3)\}$. Time depending upon n_M for $n = 1024$ and $m = 1$.

	$b=1,\ d=\frac{1}{2},\ m=2,\ \nu=\frac{1}{2},\ p=0,$ $\eta_1=-\frac{1}{\sqrt{3}},\ \eta_2=\frac{1}{\sqrt{3}},\ r=r_d=4$	
n_M	Time consumptions (sec)	ε'_N
8	7769	$3.7E-07$
16	3541	$3.7E-07$
32	2066	$3.7E-07$
64	1648	$3.8E-07$
128	1719	$3.6E-07$
256	3326	$3.6E-07$
512	13301	$3.6E-07$

Table 10.19: Example $\{(10.1)–(10.3)\}$. Time depending upon n_M for $n = 1024$ and $m = 2$.

From Table 10.18 we can see that the time consumption is the smallest for $n_M = n^{2/5} = 16$, $n_M = n^{1/2} = 32$, $n_M = n^{3/5} = 64$ and $n_M = n^{7/10} = 128$. In the case of Table 10.19 the time consumption is the smallest for $n_M = n^{1/2} = 32$, $n_M = n^{3/5} = 64$, $n_M = n^{7/10} = 128$. So from numerical examples it follows that a good choice for n_M is $n_M = n^\tau$, where $1/2 \le \tau \le 2/3$.

To solve the system (6.6) it is necessary to do $O(d_N^3)$ arithmetical operations. Arguing analogously to [62], we see that the approximate solution of the system (6.6) of suitable accuracy can be found by the two-grid method (8.2) applying $O(d_N^2)$ arithmetical operations. For this end, we need to choose n_M so that $d_M = {d_N}^{-\tau}$, where $0 < \tau < 2/3$.

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Aproksimatsioonimeetodid katkeva kordajaga nõrgalt singulaarsete integraalvõrrandite jaoks

Kokkuvõte

Käesolevas töös vaadeldakse lineaarset integraalvõrrandit

$$u(t) - \int_0^b K(t,s)u(s)ds = f(t)\,, \quad t \in [0,b]\,, \tag{1}$$

kus $b > 0$ on reaalarv, vabaliige $f(t)$ on pidev lõigus $[0,b]$ ja tuum $K(t,s)$ on kujul $K(t,s) = g(t,s)\kappa(t,s)$. Eeldatakse, et 1) $g(t,s)$ on m korda ($m \geq 1$) pidevalt diferentseeruv, kui $t \in [0,b]$, $s \in [0,b] \setminus \{d\}$, $0 < d < b$ ja tema tuletised on tõkestatud piirkondades $[0,b] \times [0,d)$ ja $[0,b] \times (d,b]$; 2) $\kappa(t,s)$ on m korda pidevalt diferentseeruv, kui $t,s \in [0,b]$, $t \neq s$, kusjuures leidub reaalarv $\nu \in (-\infty,1)$ nii, et kõigi mittenegatiivsete täisarvude i ja j korral, mille puhul $i + j \leq m$, kehtib võrratus

$$\left|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \kappa(t,s)\right| \leq c \begin{cases} 1\,, & \text{kui} \quad \nu + i < 0\,, \\ 1 + |\ln|t-s||\,, & \text{kui} \quad \nu + i = 0\,, \\ |t-s|^{-\nu-i}, & \text{kui} \quad \nu + i > 0\,, \end{cases}$$

kus $c = c(\kappa)$ on mingi positiivne konstant ja $t,s \in [0,b]$, $t \neq s$. Sellisel juhul võrrandi (1) lahend u (kui ta leidub) on küll pidev lõigus $[0,b]$, kuid tema tuletised võivad olla tõkestamata integreerimislõigu $[0,b]$ otspunktide 0 ja b ning punkti $d \in (0,b)$ ümbruses, isegi juhul, kui f on m korda pidevalt diferentseeruv lõigul $[0,b]$.

Dissertatsioonis on konstrueeritud rida võrrandi tuuma ja lahendi singulaarsusi arvestavaid meetodeid võrrandi (1) ligikaudseks lahendamiseks. Erilist tähelepanu on pööratud kvadratuurvalemite meetodi, kollokatsioonimeetodi, Galjorkini meetodi ning kollokatsioonimeetodil baseeruva kahevõrgu iteratsioonimeetodi koonduvusele ja koonduvuskiirusele mitmesuguste võrkude korral. On üldistatud mitmete uurijate (K. Atkinson, A. Pedas, E. Tamme, P. Uba, G. Vainikko jt.) poolt nõrgalt singulaarsete integraalvõrrandite valdkonnas saadud tulemusi. Dissertatsioonis esitatud tulemused tuginevad dissertandi poolt aastatel 1998–2002 avaldatud kuuele artiklile.

Doktoritöö koosneb kümnest peatükist. Esimesed kolm peatükki on sissejuhatava iseloomuga. Neljandas peatükis käsitletakse võrrandi (1) lahendamist kvadratuurvalemite meetodil. Viiendas peatükis tuletatakse tükiti polünomiaalse interpolandi veahinnangud kvaasiühtlasel ja spetsiaalsel ebaühtlasel võrgul iseärasustega funktsioonide aproksimeerimiseks. Kuuendas peatükis uuritakse lähislahendit määrava kollokatsioonisüsteemi ühest lahenduvust ning tükiti polünomiaalse kollokatsioonimeetodi koonduvust ja koonduvuskiirust. Seitsmendas peatükis on vaadeldud kollokatsioonimeetodiga kaasnevat superkoonduvuse efekti spetsiaalsete kollokatsioonipunktide valiku korral. Integraalvõrrandi (1) lahendamisel tükiti polünomiaalse kollokatsioonimeetodiga tuleb lahendada suuri lineaarseid võrrandisüsteeme. Kaheksandas peatükis on esitatud selliste süsteemide lahendamiseks kahevõrgu iteratsioonimeetod ja selgitatud selle koonduvuskiirus. Üheksandas peatükis vaadeldakse võrrandi (1) lahendamist Galjorkini meetodiga. Meetodi koonduvuse ja kooduvuskiiruse analüüs põhineb kuuenda peatüki tulemustel. Töö kümnendas peatükis on teoreetilisi tulemusi kontrollitud arvukate numbriliste eksperimentide abil.

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List of Publications

1. K. Hakk. *Numerical solution of weakly singular integral equations by piecewise polynomial approximation.* In: Math. Research, **6**, Proceedings of the 3rd International Conference "Differential Equations Applications" (St.-Petersburg, 2000), Izd. St.-Petersburg Gos. Tekh. Univ., St.-Petersburg, 2000, 88–97. Abstracted/Indexed in Mathematical Reviews (MR1826236) and Zentralblatt für Mathematik (Zbl 0973.65128).

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