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Spline approximations

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Chapter 1

Introduction

At approximation of functions or discrete data a well known tool is the interpolation. Historically, the interpolation with polynomials was one of the first means. There are several disadvantages like nonconvergence and instability with respect to round-off errors. From the middle of 20th century an intensive study of splines as piecewise polynomials is started, let us mention here the work by Schoenberg [68] from the year 1946. Piecewise polynomials are free of above mentioned disadvantages, the interpolation process with them is, in general, converging and stable. The use of piecewise polynomial functions as approximants showed success mostly in engineering but also in economics, physics, statistics. They are easy to manage in calculations and storing. An important application is construction of curves and surfaces in industry like car or plane manufacturing.

Besides the interpolation, the histopolation is in many cases more practical as, e.g., the statistical information is rather given in form of histograms. The histopolation problem is equivalent to an interpolation problem, see [69], we will explain this phenomena below. The dissertation will be devoted mainly to the histopolation problems. However, instead of solving an histopolation problem, it is possible to solve the corresponding interpolation problem. The derivative of the interpolant is then the solution of histopolation problem. This additional step in practice accompanies with additional (at least round-off) errors at calculations. Because of that it is preferable to have direct algorithms for finding histopolant.

In the case of polynomial spline interpolants and histopolants it is well known [44] that they do not keep the geometrical properties of data, we mean here positivity, monotonicity and convexity. Many researchers attempted to study the shape preserving polynomial spline interpolants, about the history see, e.g., [41], and also [5, 20]. The initial development of nonlinear spline spaces with rational functions and their generalizations was accomplished by Schaback [62] and similar idea was carried out by Werner [78]. Later on a well generalized space of rational splines was introduced by Schumaker [70]. Carnier [2] generalized the work of [62]. Since then, we see several classes of rational splines to be investigated. Delbourgo and Gregory in their works [9, 10, 11] considered piecewise C^1 and C^2 smooth rational quadratic and cubic splines to preserve the shape of function to interpolate, in particular, to keep the monotonicity. Wang and Wu in their work [77] established a general idea of rational splines to avoid solving nonlinear systems of

equations. A general idea in these works is that rational functions as spline pieces keep the sign of certain derivative like, e.g., linear/linear rational function keeps the sign of its first derivative, quadratic/linear rational function keeps the sign of its second derivative. Polynomial spline pieces do not have such properties.

The convergence problem at spline interpolation is well studied. For the polynomial cases this implies the results at histopolation. The case of rational spline interpolation and histopolation requires special attention, let us mention here the works [18, 19, 24, 52, 53].

To guarantee the shape of data an idea is to add additional spline knots creating in such a way free parameters (see, e.g., [26]). It is also possible to require less smoothness like in [42, 43].

A wide class at solving differential and integral equations is projection methods. The collocation method is also referred to as the interpolation projection method and the subdomain method is to the histopolating projection method. The application of these methods requires detailed description of interpolation and histopolation processes, respectively. Probably the first comparison of quadratic and cubic splines for collocation method at solving boundary value problems for ordinary differential equations was indicated in Khalifa and Eilbeck's work [32]. It was found out that quadratic splines may be better than cubic splines, both having the rate of convergence $\mathcal{O}(h^2)$. Here, the adequate characterization of accuracy was given in the papers [50, 51, 55]. In [55] and later in [54] for more general case was established that the subdomain method with cubic splines has the rate of convergence $\mathcal{O}(h^4)$. This showed an advantage in convergence for subdomain method or histopolation projection method mentioned above. Another advantage is that the subdomain method is very natural if, e.g., on subintervals, the free term function in differential equation is given approximately by mean values. This idea works well in case of Volterra integral equations [12]. There are several works about shape preservation in the case of two variables. Let us mention here the works [4, 63].

The comparison of polynomial and rational splines in collocation methods at solving boundary value problems for second order linear differential equations was systematically carried out in [27, 30, 31]. These reasonings base on comparison of special interpolants which was given [28, 29]. We do not know appropriate studies about subdomain method (or nonlinear histopolation projection method) for this kind of boundary value problems. Thus, there cannot be any comparison of subdomain and collocation methods using rational splines. By our knowledge, there is no studies about the use of rational splines at solving Volterra or Fredholm integral equations by collocation or subdomain methods.

The histopolation of data is named in literature under several different phrases like interpolation in the mean [6], area matching interpolation [7, 13], area true approximation of histogram [61], histopolating splines [73], and also as integral splines.

An important particular case at histopolation is the study of cubic splines [33]. As we have seen above, this is naturally related to the study of interpolating quartic splines, in this area we mention the works [35, 36, 37] with quite systematic approach.

A particular case is also the study of quadratic splines at interpolation, see [40, 45, 46], and at histopolation [49].

A review about shape preserving approximation and interpolation is given in [39].

There are several books about the spline theory where interpolation or histopolation is treated, like, e.g., [1, 5, 41, 48, 72, 74]. The histopolation with splines in two-dimensional case is developed in [4].

Gregory [22] worked out a rational cubic function which has shape preserving interpolation properties, like, monotonic or convex sets of data. There are many works where the histopolation or interpolation by rational spline is considered, especially, focused on monotonicity [17, 64, 65, 66, 67, 71]. Clements [3] studied a twice continuously differentiable piecewise rational interpolant. Also a necessary and sufficient condition are employed to ensure that the interpolating function preserves the local curvature. For given monotone data an histopolating linear/linear rational spline of class C^1 is studied in [18]. In [19] algorithms are introduced to get comonotone histopolating splines consisting of linear/linear rational or quadratic polynomial pieces. Interpolation and histopolation with periodic polynomial splines is studied in several papers, see, e.g., [8, 14, 15, 47, 56, 59, 76, 79].

Special attention to the monotone interpolation and its convergence is given already in [57]. The existence of monotone polynomial spline interpolants is established in [58] where some deficiency of the spline is assumed.

The convexity of histograms and the existence of convex polynomial spline histopolants is studied in [80, 81].

Let us explain the connection between interpolants and histopolants. This is done in several sources in literature, see, e.g., [21, 69]. For the convenience of the reader, we reproduce this here.

Suppose we have a given mesh $a = x_0 < x_1 < \ldots < x_n = b$ and function values (real numbers) $f_i, \quad i = 0, \ldots, n$, corresponding to points x_i. In the interpolation problem it is needed to find a function $I : [a, b] \to \mathbb{R}$ such that $I(x_i) = f_i, \quad i = 0, \ldots, n$, which are called interpolation conditions. In the histopolation problem we have given mesh as above and real numbers z_i, $i = 1, \ldots, n$, corresponding to subintervals $[x_{i-1}, \ x_i], \quad i = 1, \ldots, n$. It is needed to construct a function $H : [a, b] \to \mathbb{R}$ such that

$$\int_{x_{i-1}}^{x_i} H(x)dx = z_i(x_i - x_{i-1}), \quad i = 1, \ldots, n.$$

Let us mention that

$$\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} H(x)dx, \quad i = 1, \ldots, n,$$

are the mean values of H on subintervals and, actually, z_i are the given mean values of H on subintervals.

For a given histopolation problem we may fix arbitrarily $f_0 \in \mathbb{R}$ and calculate $f_i = f_{i-1} + z_i(x_i - x_{i-1}), \ i = 1, \ldots, n$. In such way we obtain the corresponding

interpolation problem. Finding an interpolant I as a solution of this interpolation problem we get I' as a solution of the primary given histopolation problem.

For a given interpolation problem we calculate $z_i = (f_i - f_{i-1})/(x_i - x_{i-1})$, $i = 1, \ldots, n$, and get the corresponding histopolation problem. If H is a solution of this histopolation problem then it is immediate to check that the function

$$I(x) = f_0 + \int_a^x H(s)ds$$

is a solution of the initially given interpolation problem.

We see that if one of the interpolation or histopolation problems is studied then there are also results about the other one. Such an approach works well in case of polynomial splines but not in case of rational splines. For example, the derivative of quadratic/linear rational spline is not linear/linear rational spline, or, the integral of linear/linear spline is not quadratic/linear spline. Thus, even if we have an appropriate interpolation theory for rational splines, the histopolation theory should be created independently. Due to this phenomena the rational spline histopolation problem is studied in present dissertation.

In the following we briefly summarize the main results of the dissertation by chapters. This dissertation consists of six chapters.

In Chapter 1 we have presented a brief summary of interpolation and histopolation problems and also describe connection between interpolation and histopolation problems. A review of main books and publications on spline theory is given.

Chapter 2 consists of some preliminary notions and presents some propositions and corollaries about general features in interpolantion and histopolation problems.

In Chapter 3 we treat the histopolation problem with cubic splines and develop an explicit theory about that. An appropriate representation of the histopolant on interval between spline knots occurs to be the second moments of spline and integrals of spline over parts of the interval. We consider the most common boundary values conditions like given values of the spline and its first and second derivatives in endpoints of given interval. We solve the problem of existence and uniqueness of the solution for such histopolation problem. Another representation of the histopolant via the values of spline and its second derivatives in spline knots is also studied. The results about cubic spline histopolation in Chapter 3 are published in [33].

In Chapter 4 we will discuss about the periodic polynomial spline histopolation. We studied periodic polynomial spline histopolation with arbitrary placement of histogram knots. Spline knots are considered coinciding with histogram knots. The histopolation problem with polynomial splines is known to be equivalent to an interpolation problem with polynomial splines of degree one higher. The existence and uniqueness problem for the corresponding homogeneous problems preserves this equivalence in periodical case. The main problem of this chapter is

the study of existence of solution. This is done for several numbers of grid points and spline degree in dependence whether they are even or odd. Results of Chapter 4 are published in [56].

In Chapter 5 we study the rational spline histopolation of convex data. The convenient tool here is the use of quadratic/linear rational splines of class C^2. Given histogram knots may be placed arbitrarily and the spline knots between them also arbitrarily. A quite special representation of the histopolant on subintervals is considered. The key moment at the study of existence of solution is the existence of solution for a nonlinear system of basic equations to determine the values of second derivatives in spline knots. The other parameters in the representation of spline are determined from a linear system with regular matrix. We arrive to the result that there is a strictly convex histogram without the solution of histopolation problem for any choice of spline knots. The results of Chapter 5 are intended to publish in [34].

Chapter 6 includes the numerical tests and figures which support our theoretical results. These results are in complete accordance with theory.

Chapter 2

Preliminary results about histopolation

In this section we will give some general results about the shape of the histopolant. Suppose we have in an interval $[a, b]$ the points x_i such that $a = x_0 < x_1 < \ldots < x_n = b$ and let $z_i,\ i = 1, \ldots, n$, be given real numbers which will be considered as the histogram heights. We are looking for a function S defined on $[a, b]$ which satisfies the following histopolation conditions

$$\int_{x_{i-1}}^{x_i} S(x)dx = z_i(x_i - x_{i-1}), \quad i = 1, \ldots, n. \tag{2.1}$$

Conditions (2.1) are called histopolation conditions. Denote $h_i = x_i - x_{i-1}$, $i = 1, \ldots, n$.

A function $S : [a, b] \to \mathbb{R}$ is called linear/linear rational spline if S has the form

$$S(x) = \frac{a_i + b_i x}{1 + d_i x}, \quad x \in [x_{i-1}, x_i],\ i = 1, \ldots, n,$$

with $1 + d_i x \neq 0$ for all $x \in [x_{i-1}, x_i]$ and $S \in C^1[a, b]$.

A function $S : [a, b] \to \mathbb{R}$ is called quadratic/linear rational spline if S has the form

$$S(x) = \frac{a_i + b_i x + c_i x^2}{1 + d_i x}, \quad x \in [x_{i-1}, x_i],\ i = 1, \ldots, n,$$

with $1 + d_i x \neq 0$ for all $x \in [x_{i-1}, x_i]$ and $S \in C^2[a, b]$.

Next we will prove a couple of propositions.

Proposition 1. *If $f \in C^1[a, b]$ is such that $f'(x) > 0$ for all $x \in [a, b]$ and $z_i = {h_i}^{-1} \int_{x_{i-1}}^{x_i} f(x)dx$, $i = 1, \ldots, n$, then $z_1 < z_2 < \ldots < z_n$ (shortly, $z_i < z_{i+1}$ for each i).*

Proof. We calculate

$$z_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} f(x)dx$$

$$= \frac{1}{h_i}\int_{x_{i-1}}^{x_i} \left(f(x_i) + \int_{x_i}^{x} f'(s)ds\right) dx$$

$$= f(x_i) + \frac{1}{h_i}\int_{x_{i-1}}^{x_i} \left(\int_{x_i}^{x} f'(s)ds\right) dx.$$

As here $f'(s) > 0$, we have $\int_{x_i}^{x} f'(s)ds < 0$ for $x \in (x_{i-1}, x_i)$ and thus, $z_i < f(x_i)$. Similarly,

$$z_{i+1} = \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f(x)dx$$

$$= \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} \left(f(x_i) + \int_{x_i}^{x} f'(s)ds\right) dx$$

$$= f(x_i) + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} \left(\int_{x_i}^{x} f'(s)ds\right) dx,$$

which gives $f(x_i) < z_{i+1}$ because now $\int_{x_i}^{x} f'(s)ds > 0$ for $x \in (x_i, x_{i+1})$. □

Remark 1. The condition $z_i < z_{i+1}$ for each i does not imply that $f'(x) > 0$ for all x, an appropriate example is the function $f(x) = x^3$ with $[a, b]$ around the value 0.

Corollary 1. *If a linear/linear rational spline S is such that $S'(x) > 0$ for all $x \in [a, b]$ and satisfies histopolation conditions* (2.1) *then necessarily $z_i < z_{i+i}$ for all i.*

Proposition 2. *If $f \in C^2[a, b]$ is such that $f''(x) > 0$ for all $x \in [a, b]$ and $z_i = {h_i}^{-1}\int_{x_{i-1}}^{x_i} f(x)dx$, $i = 1, \ldots, n$, then*

$$D_i = (h_i + h_{i+1})z_{i-1} - (h_{i-1} + 2h_i + h_{i+1})z_i + (h_{i-1} + h_i)z_{i+1} > 0, \quad i = 2, \ldots, n-1.$$

Proof. Let us apply the Taylor expansion with the remainder in integral form

$$f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(l-1)}(a)}{(l-1)!}(x-a)^{l-1}$$
$$+ \frac{1}{(l-1)!}\int_a^x (x-s)^{l-1} f^{(l)}(s)ds. \tag{2.2}$$

By (2.2) we have

$$f(x) = f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + \int_{x_{i-1}}^{x} (x - s) f''(s) ds \tag{2.3}$$

and

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \int_{x_i}^{x} (x - s) f''(s) ds. \tag{2.4}$$

First, using (2.3) we find

$$\begin{aligned} z_{i-1} =& \frac{1}{h_{i-1}} \int_{x_{i-2}}^{x_{i-1}} f(x) dx \\ =& f(x_{i-1}) + \frac{1}{h_{i-1}} f'(x_{i-1}) \left. \frac{(x - x_{i-1})^2}{2} \right|_{x=x_{i-2}}^{x=x_{i-1}} \\ &+ \frac{1}{h_{i-1}} \int_{x_{i-2}}^{x_{i-1}} \left(\int_{x_{i-1}}^{x} (x - s) f''(s) ds \right) dx \\ =& f(x_{i-1}) - \frac{h_{i-1}}{2} f'(x_{i-1}) \\ &+ \frac{1}{h_{i-1}} \int_{x_{i-2}}^{x_{i-1}} \left(\int_{x_{i-1}}^{x} (x - s) f''(s) ds \right) dx. \end{aligned} \tag{2.5}$$

Next we calculate with the help of (2.3) and (2.4)

$$\begin{aligned} z_i &= \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x) dx \\ &= f(x_{i-1}) + \frac{h_i}{2} f'(x_{i-1}) + \frac{1}{h_i} \int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x} (x - s) f''(s) ds \right) dx, \end{aligned} \tag{2.6}$$

$$\begin{aligned} z_i &= \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x) dx \\ &= f(x_i) - \frac{h_i}{2} f'(x_i) + \frac{1}{h_i} \int_{x_{i-1}}^{x_i} \left(\int_{x_i}^{x} (x - s) f''(s) ds \right) dx. \end{aligned} \tag{2.7}$$

Finally, calculate z_{i+1} using (2.4)

$$\begin{aligned} z_{i+1} &= \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f(x) dx \\ &= f(x_i) + \frac{h_{i+1}}{2} f'(x_i) + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} \left(\int_{x_i}^{x} (x - s) f''(s) ds \right) dx. \end{aligned} \tag{2.8}$$

Substitute all the values of z_{i-1}, z_i, z_{i+1} from (2.5) – (2.8) into the expression of D_i. We get

$$
\begin{aligned}
D_i =&(h_{i-1}+h_i)(z_{i+1}-z_i)-(h_i+h_{i+1})(z_i-z_{i-1}) \qquad (2.9)\\
=&(h_{i-1}+h_i)\bigg(f(x_i)+\frac{h_{i+1}}{2}f'(x_i)+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}\Big(\int_{x_i}^{x}(x-s)f''(s)ds\Big)dx\\
&-\Big(f(x_i)-\frac{h_i}{2}f'(x_i)+\frac{1}{h_i}\int_{x_{i-1}}^{x_i}\Big(\int_{x_i}^{x}(x-s)f''(s)ds\Big)dx\Big)\bigg)\\
&-(h_i+h_{i+1})\bigg(f(x_{i-1})+\frac{h_i}{2}f'(x_{i-1})+\frac{1}{h_i}\int_{x_{i-1}}^{x_i}\Big(\int_{x_{i-1}}^{x}(x-s)f''(s)ds\Big)dx\\
&-\Big(f(x_{i-1})-\frac{h_{i-1}}{2}f'(x_{i-1})+\frac{1}{h_{i-1}}\int_{x_{i-2}}^{x_{i-1}}\Big(\int_{x_{i-1}}^{x}(x-s)f''(s)ds)dx\Big)\bigg)\\
\\
=&(h_{i-1}+h_i)\bigg(\frac{f'(x_i)}{2}(h_i+h_{i+1})+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}\Big(\int_{x_i}^{x}(x-s)f''(s)ds\Big)dx\\
&-\frac{1}{h_i}\int_{x_{i-1}}^{x_i}\Big(\int_{x_i}^{x}(x-s)f''(s)ds\Big)dx\bigg)\\
&-(h_i+h_{i+1})\bigg(\frac{f'(x_{i-1})}{2}(h_{i-1}+h_i)+\frac{1}{h_i}\int_{x_{i-1}}^{x_i}\Big(\int_{x_{i-1}}^{x}(x-s)f''(s)ds\Big)dx\\
&-\frac{1}{h_{i-1}}\int_{x_{i-2}}^{x_{i-1}}\Big(\int_{x_{i-1}}^{x}(x-s)f''(s)ds\Big)dx\bigg). \qquad (2.10)
\end{aligned}
$$

The terms without integrals give here

$$
\begin{aligned}
&\frac{1}{2}(h_{i-1}+h_i)(h_i+h_{i+1})\Big(f'(x_i)-f'(x_{i-1})\Big)\\
&\quad=\frac{1}{2}(h_{i-1}+h_i)(h_i+h_{i+1})\int_{x_{i-1}}^{x_i}f''(s)ds.
\end{aligned}
$$

To transform the integral terms in (2.10) we use the formulae in general case like

$$
\int_{x_{i-2}}^{x_{i-1}}\int_{x_{i-1}}^{x}g(x,s)dsdx=-\int_{x_{i-2}}^{x_{i-1}}\int_{x_{i-2}}^{s}g(x,s)dxds, \qquad (2.11)
$$

$$
\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x}g(x,s)dsdx=\int_{x_{i-1}}^{x_i}\int_{s}^{x_i}g(x,s)dxds, \qquad (2.12)
$$

$$
\int_{x_{i-1}}^{x_i}\int_{x_i}^{x}g(x,s)dsdx=-\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{s}g(x,s)dxds, \qquad (2.13)
$$

$$\int_{x_i}^{x_{i+1}} \int_{x_i}^{x} g(x,s)dsdx = \int_{x_i}^{x_{i+1}} \int_{s}^{x_{i+1}} g(x,s)dxds. \tag{2.14}$$

Then we get by (2.11) for the term in (2.10)

$$\begin{aligned}\int_{x_{i-2}}^{x_{i-1}} \Big(\int_{x_{i-1}}^{x} (x-s)f''(s)ds \Big) dx &= -\int_{x_{i-2}}^{x_{i-1}} \int_{x_{i-2}}^{s} (x-s)f''(s)dxds, \\ &= -\int_{x_{i-2}}^{x_{i-1}} f''(s) \Big(\int_{x_{i-2}}^{s} (x-s)dx \Big) ds \\ &= \int_{x_{i-2}}^{x_{i-1}} f''(s) \frac{(s-x_{i-2})^2}{2} ds.\end{aligned}$$

Similarly, by (2.12) we get

$$\int_{x_{i-1}}^{x_i} \Big(\int_{x_{i-1}}^{x} (x-s)f''(s)ds \Big) dx = \int_{x_{i-1}}^{x_i} f''(s) \frac{(x_i-s)^2}{2} ds,$$

by (2.13)

$$\int_{x_{i-1}}^{x_i} \Big(\int_{x_i}^{x} (x-s)f''(s)ds \Big) dx = \int_{x_{i-1}}^{x_i} f''(s) \frac{(s-x_{i-1})^2}{2} ds,$$

and finally, by (2.14)

$$\int_{x_i}^{x_{i+1}} \Big(\int_{x_i}^{x} (x-s)f''(s)ds \Big) dx = \int_{x_i}^{x_{i+1}} f''(s) \frac{(x_{i+1}-s)^2}{2} ds.$$

Collecting all transformed terms of (2.10) we obtain

$$\begin{aligned}D_i =& \frac{1}{2}(h_{i-1}+h_i)(h_i+h_{i+1}) \int_{x_{i-1}}^{x_i} f''(s)ds \\ &+ \frac{h_i+h_{i+1}}{h_{i-1}} \int_{x_{i-2}}^{x_{i-1}} f''(s) \frac{(s-x_{i-2})^2}{2} ds \\ &- \frac{h_i+h_{i+1}}{h_i} \int_{x_{i-1}}^{x_i} f''(s) \frac{(x_i-s)^2}{2} ds \\ &- \frac{h_{i-1}+h_i}{h_i} \int_{x_{i-1}}^{x_i} f''(s) \frac{(s-x_{i-1})^2}{2} ds \\ &+ \frac{h_{i-1}+h_i}{h_i} \int_{x_i}^{x_{i+1}} f''(s) \frac{(x_{i+1}-s)^2}{2} ds. \end{aligned} \tag{2.15}$$

As we supposed that $f''(s) > 0$ for all s, it is clear that the second and last addend in (2.15) are strictly positive.

The other addends in (2.15) together are

$$\begin{aligned}\frac{1}{2}\int_{x_{i-1}}^{x_i} f''(s)\Big((h_{i-1}+h_i)(h_i+h_{i+1}) \\ - \frac{h_i+h_{i+1}}{h_i}(x_i-s)^2 - \frac{h_{i-1}+h_i}{h_i}(s-x_{i-1})^2\Big)ds \\ = \frac{1}{2}\int_{x_{i-1}}^{x_i} f''(s)\varphi(s)ds\end{aligned}$$

where we introduced the quadratic polynomial (with respect to s) φ. We see that

$$\begin{aligned}\varphi(x_{i-1}) &= (h_{i-1}+h_i)(h_i+h_{i+1}) - (h_i+h_{i+1})h_i > 0, \\ \varphi(x_i) &= (h_{i-1}+h_i)(h_i+h_{i+1}) - (h_{i-1}+h_i)h_i > 0.\end{aligned}$$

In addition, the coefficient of s^2 in φ is

$$-\frac{(h_i+h_{i+1})}{h_i} - \frac{(h_{i-1}+h_i)}{h_i} < 0,$$

and thus, $\varphi(s) > 0$ for all $s \in [x_{i-1}, x_i]$. In total, we have $D_i > 0$ which completes the proof. □

The condition $D_i > 0$ for all i will be called strict convexity of the histogram.

Remark 2. The condition $D_i > 0$ for each i does not imply that $f''(x) > 0$ for all x, an example is the function $f(x) = x^4$ on the interval $[a, b]$ containing the value 0.

Corollary 2. *If quadratic/linear rational spline S is such that $S''(x) > 0$ for all $x \in [a, b]$ and satisfies histopolation conditions then necessarily $D_i > 0$, $i = 2, \ldots, n-1$.*

Chapter 3

Cubic spline histopolation

Consider the initial situation of Chapter 2 where in the given interval $[a, b]$ we have the points x_i such that $a = x_0 < x_1 < \ldots < x_n = b$. We say that we have a given grid $\triangle$. A function $S : [a, b] \to \mathbb{R}$ is called cubic spline if it is on each subinterval $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, a cubic polynomial and $S \in C^2[a, b]$. The set of all cubic splines on $\triangle$ is a finite dimensional vector space, denoted as $S^3(\triangle)$. It is well known that $\dim S^3(\triangle) = n + 3$.

3.1 Interpolation problem with cubic splines

Suppose we have for a given grid $\triangle$ the values f_i, $i = 0, \ldots, n$, corresponding to the points x_i, $i = 0, \ldots, n$. We are looking for a function $S \in S^3(\triangle)$ such that

$$S(x_i) = f_i, \quad i = 0, \ldots, n, \tag{3.1}$$

called the interpolation conditions. As $\dim S^3(\triangle) = n + 3$ and the number of conditions (3.1) is $n + 1$, we add two conditions (boundary conditions) from

1) $S'(a) = \alpha, \quad S'(b) = \beta,$

2) $S''(a) = \alpha, \quad S''(b) = \beta$

with given α and β at different endpoints. Such a problem is an interpolation problem with cubic splines.

There are several possibilities to represent the cubic spline. One of them is to use $S_i = S(x_i)$ and $M_i = S''(x_i)$, $i = 0, \ldots, n$. The problem with boundary conditions 2) leads to the linear system

$$\begin{cases} M_0 = \alpha, \\ \\ \mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6 \dfrac{\dfrac{f_{i+1} - f_i}{h_{i+1}} - \dfrac{f_i - f_{i-1}}{h_i}}{h_i + h_{i+1}}, \quad i = 1, \ldots, n-1, \\ \\ M_n = \beta \end{cases} \tag{3.2}$$

with $\mu_i = h_i/(h_i + h_{i+1})$, $\lambda_i = h_{i+1}/(h_i + h_{i+1})$. Solving the system (3.2) we get $M_0, \ldots, M_n$. The values $S_i = f_i$, $i = 0, \ldots, n$, are known from interpolation conditions (3.1). The representation of S by the values f_i, M_i on subintervals could be found almost in any book of spline theory, see, e.g., [5, 41, 69]. It is remarkable that the diagonal of the matrix in (3.2) has diagonal domination in rows which guarantees the stability of calculations.

The use of S_i and $m_i = S'(x_i)$, $i = 0, \ldots, n$, in interpolation problem with boundary conditions 1) leads to the system

$$\begin{cases} m_0 = \alpha, \\ \lambda_i m_{i-1} + 2m_i + \mu_i m_{i+1} = 3\left(\mu_i \dfrac{f_{i+1} - f_i}{h_{i+1}} + \lambda_i \dfrac{f_i - f_{i-1}}{h_i}\right), \\ i = 1, \ldots, n-1, \\ m_n = \beta \end{cases} \tag{3.3}$$

for finding $m_0, \ldots, m_n$. The system (3.3) has also the matrix with diagonal domination in rows. Very popular is the use of B-splines to represent cubic splines. It is important to mention here that the diagonal of the matrix in the system to determine the coefficients of representation may be not dominating in rows in case of nonuniform grid. This causes certain instability in calculations.

3.2 The histopolation problem

In addition to the given grid $\triangle$ suppose that we have given real numbers z_i, $i = 1, \ldots, n$, called histogram heights. Denote the width of histograms by $h_i = x_i - x_{i-1}$, $i = 1, \ldots, n$. We consider the problem of finding a function $S : [a, b] \to \mathbb{R}$ such that

$$\int_{x_{i-1}}^{x_i} S(x)dx = h_i z_i, \ i = 1, \ldots, n. \tag{3.4}$$

The function S is not reasonable to search in $S^3(\triangle)$ as the number of conditions in (3.4) is only n. Because of that we choose S to be a cubic spline having knots

$$\xi_1 = x_0, \quad \xi_i \in (x_{i-1}, x_i), \ i = 2, \ldots, n-1, \quad \xi_n = x_n.$$

Then the cubic spline has $n + 2$ free parameters. To the histopolation conditions (3.4) we add two boundary conditions from

$$\begin{aligned} &S(a) = \alpha, \quad S(b) = \beta, \\ &S'(a) = \alpha, \quad S'(b) = \beta, \\ &S''(a) = \alpha, \quad S''(b) = \beta, \end{aligned} \tag{3.5}$$

at different endpoints a and b.

3.3 Representation of the histopolant

Several representations of cubic spline histopolant could be considered, but the one which uses second moments and particular integrals is appropriate. Thus, on the interval $[\xi_i, \xi_{i+1}]$ we use next four parameters to represent the spline:

$$M_i = S''(\xi_i), \quad M_{i+1} = S''(\xi_{i+1}), \quad \lambda_i = \int_{\xi_i}^{x_i} S(x)dx, \quad \rho_i = \int_{x_i}^{\xi_{i+1}} S(x)dx.$$

Denote $\varepsilon_i = x_i - \xi_i$, $\eta_i = \xi_{i+1} - x_i$, $\delta_i = \varepsilon_i + \eta_i$, $i = 1, \ldots, n-1$ (see Figure 1).

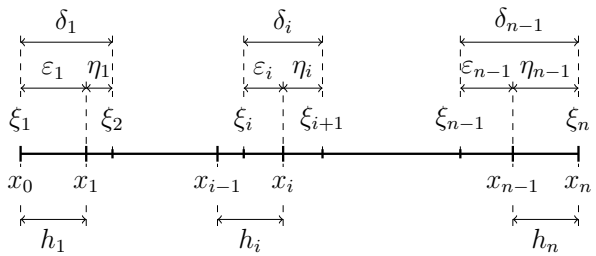

Figure 1: Additional parameters.

The cubic spline as a cubic polynomial in subinterval could be written as

$$S(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, x \in [\xi_i, \xi_{i+1}], i = 1, \ldots, n-1. \tag{3.6}$$

From (3.6) we get

$$S''(x) = 2c_i + 6d_i(x - x_i),$$

which, in turn, gives $M_i = 2c_i - 6d_i\varepsilon_i$ and $M_{i+1} = 2c_i + 6d_i\eta_i$. From these two equations we calculate

$$d_i = \frac{M_{i+1} - M_i}{6\delta_i}, \quad c_i = \frac{M_i\eta_i + M_{i+1}\varepsilon_i}{2\delta_i}.$$

Then we obtain

$$\lambda_i = \int_{\xi_i}^{x_i} S(x)dx = a_i\varepsilon_i - \frac{b_i}{2}\varepsilon_i^2 + \frac{c_i}{3}\varepsilon_i^3 - \frac{d_i}{4}\varepsilon_i^4,$$

$$\rho_i = \int_{x_i}^{\xi_{i+1}} S(x)dx = a_i\eta_i + \frac{b_i}{2}\eta_i^2 + \frac{c_i}{3}\eta_i^3 + \frac{d_i}{4}\eta_i^4$$

and from them

$$a_i = \frac{1}{\delta_i}\left(\lambda_i\frac{\eta_i}{\varepsilon_i} + \rho_i\frac{\varepsilon_i}{\eta_i}\right) - \frac{c_i}{3}\varepsilon_i\eta_i - \frac{d_i}{4}(\eta_i - \varepsilon_i)\varepsilon_i\eta_i,$$

$$b_i = \frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i}\right) - 2\left(\frac{c_i}{3}(\eta_i - \varepsilon_i) + \frac{d_i}{4}(\eta_i^2 - \eta_i\varepsilon_i + \varepsilon_i^2)\right).$$

To determine the parameters $M_i,\ i=1,\dots,n,\ \lambda_i,\ \rho_i,\quad i=1,\dots,n-1$, we use smoothness conditions

$$S(\xi_i-0)=S(\xi_i+0),\quad i=2,\dots,n-1, \tag{3.7}$$

$$S'(\xi_i-0)=S'(\xi_i+0),\quad i=2,\dots,n-1, \tag{3.8}$$

histopolation conditions

$$\rho_{i-1}+\lambda_i=z_ih_i,\quad i=1,\dots,n, \tag{3.9}$$

with $\rho_0=0$, $\lambda_n=0$, and two boundary conditions. Equations (3.7) and (3.8) take the form, respectively,

$$\begin{aligned}
&-\frac{\eta_{i-1}}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1}+\frac{1}{\delta_{i-1}}\left(2+\frac{\varepsilon_{i-1}}{\eta_{i-1}}\right)\rho_{i-1}-\frac{1}{\delta_i}\left(2+\frac{\eta_i}{\varepsilon_i}\right)\lambda_i+\frac{\varepsilon_i}{\delta_i\eta_i}\rho_i\\
=&\frac{1}{24}\Bigg[-\eta_{i-1}(\varepsilon_{i-1}+2\eta_{i-1})M_{i-1}\\
&+(-\eta_{i-1}(3\varepsilon_{i-1}+2\eta_{i-1})+\varepsilon_i(2\varepsilon_i+3\eta_i))M_i\\
&+\varepsilon_i(2\varepsilon_i+\eta_i)M_{i+1}\Bigg],\quad i=2,\dots,n-1,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&-\frac{1}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1}+\frac{1}{\delta_{i-1}\eta_{i-1}}\rho_{i-1}+\frac{1}{\delta_i\varepsilon_i}\lambda_i-\frac{1}{\delta_i\eta_i}\rho_i\\
=&-\frac{1}{24}\Bigg[\frac{\delta_{i-1}^2+\eta_{i-1}\delta_{i-1}+\eta_{i-1}^2}{\delta_{i-1}}M_{i-1}\\
&+\left(\frac{3\delta_{i-1}^2+2\eta_{i-1}\delta_{i-1}+\eta_{i-1}\varepsilon_{i-1}}{\delta_{i-1}}+\frac{3\delta_i^2+2\varepsilon_i\delta_i+\varepsilon_i\eta_i}{\delta_i}\right)M_i\\
&+\frac{\delta_i^2+\varepsilon_i\delta_i+\varepsilon_i^2}{\delta_i}M_{i+1}\Bigg],\quad i=2,\dots,n-1.
\end{aligned} \tag{3.11}$$

Observe that these equations are linear (homogeneous) with respect to the unknowns $\lambda_{i-1},\ \rho_{i-1},\ \lambda_i,\ \rho_i,\ M_{i-1},\ M_i,\ M_{i+1}$.

3.4 Systems defining spline parameters

In total we have to determine $3n-2$ unknowns $M_1,\dots,M_n,\ \lambda_1,\dots,\lambda_{n-1},\ \rho_1,\dots,\rho_{n-1}$ from the system of $3n-2$ equations: (3.9), (3.10), (3.11) and two boundary conditions. This system is of undetermined form to study. We take 9 equations (3.9,$i-1$), (3.10,$i-1$), (3.11,$i-1$), (3.9,i), (3.10,i), (3.11,i), (3.9,$i+1$), (3.10,$i+1$), (3.11,$i+1$) containing eight unknowns $\lambda_{i-2},\ \rho_{i-2},\ \lambda_{i-1},\ \rho_{i-1},\ \lambda_i,\ \rho_i,\ \lambda_{i+1},\ \rho_{i+1}$.

These λ_j, ρ_j could be eliminated by using the linear combination of equations with coefficients indicated below:

$$(3.9, i-1) \qquad -\frac{h_i + h_{i+1}}{h_{i-1}},$$

$$(3.10, i-1) \qquad \frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2},$$

$$(3.11, i-1) \qquad -\frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}^2,$$

$$(3.9, i) \qquad \frac{h_{i-1} + 2h_i + h_{i+1}}{h_i},$$

$$(3.10, i) \qquad \frac{\varepsilon_i(h_i + h_{i+1}) - \eta_{i-1}(h_{i-1} + h_i)}{h_i},$$

$$(3.11, i) \qquad \frac{\varepsilon_i^2(h_i + h_{i+1}) + \eta_{i-1}^2(h_{i-1} + h_i)}{h_i} - (h_{i-1} + h_i)(h_i + h_{i+1}),$$

$$(3.9, i+1) \qquad -\frac{h_{i-1} + h_i}{h_{i+1}},$$

$$(3.10, i+1) \qquad -\frac{h_{i-1} + h_i}{h_{i+1}}\varepsilon_{i+1},$$

$$(3.11, i+1) \qquad -\frac{h_{i-1} + h_i}{h_{i+1}}\varepsilon_{i+1}^2.$$

For $i = 3, \ldots, n-2$ we obtain the equation

$$c_{i,i-2}M_{i-2} + c_{i,i-1}M_{i-1} + c_{ii}M_i + c_{i,i+1}M_{i+1} + c_{i,i+2}M_{i+2} = D_i, \tag{3.12}$$

where

$$D_i = (h_i + h_{i+1})z_{i-1} - (h_{i-1} + 2h_i + h_{i+1})z_i + (h_{i-1} + h_i)z_{i+1}, \tag{3.13}$$

$$c_{i,i-2} = \frac{1}{24}\frac{\eta_{i-2}^4(h_i + h_{i+1})}{\delta_{i-2}h_{i-1}}, \tag{3.14}$$

$$c_{i,i-1} = \frac{1}{24}\Bigg(\Bigg(\eta_{i-2}(3\eta_{i-2}+2\varepsilon_{i-1}+3\eta_{i-1})+(\varepsilon_{i-1}+2\eta_{i-1})(h_{i-1}+\eta_{i-1})$$
$$+\frac{\eta_{i-2}^2}{h_{i-1}}\Big(\frac{\varepsilon_{i-2}\eta_{i-2}}{\delta_{i-2}}+\frac{\varepsilon_{i-1}\eta_{i-1}}{\delta_{i-1}}\Big)\Bigg)(h_i+h_{i+1})$$
$$+\frac{\eta_{i-1}^3\varepsilon_i(h_{i-1}+h_i+h_{i+1})}{\delta_{i-1}h_i}+\frac{\eta_{i-1}^2(h_{i-1}+\eta_{i-1})(\varepsilon_i+h_{i+1})}{\delta_{i-1}}\Bigg), \tag{3.15}$$

$$c_{ii} = \frac{1}{24}\Bigg(\Bigg(\eta_{i-2}(2\varepsilon_{i-1}+\eta_{i-1})+(3\varepsilon_{i-1}+2\eta_{i-1})(h_{i-1}\ +\eta_{i-1})$$
$$+\frac{\eta_{i-2}^2\varepsilon_{i-1}^2}{\delta_{i-1}h_{i-1}}\Bigg)(h_i+h_{i+1})$$
$$+\left(\varepsilon_{i+1}(\varepsilon_i+2\eta_i)+(2\varepsilon_i+3\eta_i)(\varepsilon_i+h_{i+1})+\frac{\eta_i^2\varepsilon_{i+1}^2}{\delta_i h_{i+1}}\right)(h_{i-1}+h_i)$$
$$+\left(\Big(3+\frac{\varepsilon_{i-1}}{\delta_{i-1}}\Big)\eta_{i-1}+\Big(3+\frac{\eta_i}{\delta_i}\Big)\varepsilon_i\right)\Bigg((h_{i-1}+\eta_{i-1})(\varepsilon_i+h_{i+1})$$
$$+\eta_{i-1}\varepsilon_i\frac{h_{i-1}+h_i+h_{i+1}}{h_i}\Bigg)\Bigg), \tag{3.16}$$

$$c_{i,i+1} = \frac{1}{24}\Bigg(\Bigg(\varepsilon_{i+1}(3\varepsilon_i+2\eta_i+3\varepsilon_{i+1})+(2\varepsilon_i+\eta_i)(\varepsilon_i+h_i)$$
$$+\frac{\varepsilon_{i+1}^2}{h_{i+1}}\Big(\frac{\varepsilon_i\eta_i}{\delta_i}+\frac{\varepsilon_{i+1}\eta_{i+1}}{\delta_{i+1}}\Big)\Bigg)(h_{i-1}+h_i)$$
$$+\frac{\eta_{i-1}\varepsilon_i^3(h_{i-1}+h_i+h_{i+1})}{\delta_i h_i}+\frac{\varepsilon_i^2(h_{i-1}+\eta_{i-1})(\varepsilon_i+h_{i+1})}{\delta_i}\Bigg), \tag{3.17}$$

$$c_{i,i+2} = \frac{1}{24}\frac{\varepsilon_{i+1}^4(h_{i-1}+h_i)}{\delta_{i+1}h_{i+1}}. \tag{3.18}$$

Let us notice certain symmetry in equation (3.12). There are symmetric pairs of parameters: $h_{i-1} \leftrightarrow h_{i+1}$, $\delta_{i-1} \leftrightarrow \delta_i$, $\eta_{i-2} \leftrightarrow \varepsilon_{i+1}$, $\varepsilon_{i-1} \leftrightarrow \eta_i$, $\eta_{i-1} \leftrightarrow \varepsilon_i$. Then we see the symmetry between $c_{i,i-2}$ and $c_{i,i+2}$, $c_{i,i-1}$ and $c_{i,i+1}$, inside c_{ii}. However, all coefficients (3.14)–(3.18) are positive.

In case of $i = 2$ we take seven equations (3.9,1), (3.9,2), (3.10,2), (3.11,2), (3.9,3), (3.10,3), (3.11,3) to eliminate six unknowns λ_1, ρ_1, λ_2, ρ_2, λ_3, ρ_3. The

coefficients of the appropriate linear combination are as in general case. This leads to the equation

$$c_{21}M_1 + c_{22}M_2 + c_{23}M_3 + c_{24}M_4 = D_2,$$

where D_2 is determined by (3.13), c_{23} and c_{24} by (3.17) and (3.18), respectively. There is certain difference in c_{21} and c_{22} compared to (3.15) and (3.16), but they could be calculated similarly to the general case taking into account also the configuration of the intervals near the endpoint a.

Similar situation takes place in case of $i = n - 1$.

The simplest boundary equation here is $S''(a) = \alpha$ or $M_1 = \alpha$. The other possible boundary conditions, e.g., $S(a) = \alpha$ and $S'(a) = \alpha$, require the calculation of $S(\xi_1 + 0)$ and $S'(\xi_1 + 0)$ as it was done at transformation of (3.7) and (3.8). This should be followed by the elimination of appearing parameters λ_j, ρ_j. Both cases give us the equation

$$c_{11}M_1 + c_{12}M_2 + c_{13}M_3 = D_1 \tag{3.19}$$

with certain expression D_1 depending on α and given histogram parameters. However, (3.19) includes $M_1 = \alpha$.

The boundary conditions at the endpoint b could be treated similarly.

Thus, the spline parameters $M_1, \ldots, M_n$ are determined by the five-diagonal system

$$\begin{cases} c_{11}M_1 + c_{12}M_2 + c_{13}M_3 = D_1, \\ c_{21}M_1 + c_{22}M_2 + c_{23}M_3 + c_{24}M_4 = D_2, \\ c_{i,i-2}M_{i-2} + c_{i,i-1}M_{i-1} + c_{ii}M_i + c_{i,i+1}M_{i+1} + c_{i,i+2}M_{i+2} = D_i, \\ i = 3, \ldots, n-2, \\ c_{n-1,n-3}M_{n-3} + c_{n-1,n-2}M_{n-2} + c_{n-1,n-1}M_{n-1} + c_{n-1,n}M_n = D_{n-1}, \\ c_{n,n-2}M_{n-2} + c_{n,n-1}M_{n-1} + c_{nn}M_n = D_n. \end{cases} \tag{3.20}$$

Solving this, the system consisting of all equations (3.10), (3.11) allows to determine the parameters λ_j, ρ_j. Its unique solvability is shown in [23]. Note that the values λ_1 and ρ_{n-1} are known due to the histopolation conditions (3.9,1) and (3.9,n). We will discuss the solvability of (3.20) in next section.

3.5 Existence and uniqueness of the solution

It is clear that the unique solvability of system (3.20) is equivalent to the existence of unique solution to the histopolation problem. Let us start with particular cases.

Consider the case of spline knots as $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, n-1$. Then $\eta_{i-1} = \varepsilon_i = h_i/2$, $i = 2, \ldots, n-1$, $\varepsilon_1 = h_1$, $\eta_{n-1} = h_n$. The coefficients in (3.12)

are (we write them also keeping symmetrical structure)

$$
\begin{aligned}
c_{i,i-2} =& \frac{1}{192}(h_i + h_{i+1})\frac{h_{i-1}^3}{h_{i-2}+h_{i-1}},\\
c_{i,i-1} =& \frac{1}{192}\left((h_i + h_{i+1})\left(14h_{i-1}^2 + 17h_{i-1}h_i + 6h_i^2 + \frac{h_{i-2}h_{i-1}^2}{h_{i-2}+h_{i-1}}\right) + h_i^2 h_{i+1}\right),\\
c_{ii} =& \frac{1}{192}\Big((h_i + h_{i+1})(17h_{i-1}^2 + 30h_{i-1}h_i + 10h_i^2)\\
&+ (h_{i-1} + h_i)(10h_i^2 + 30h_i h_{i+1} + 17h_{i+1}^2) + 2h_{i-1}h_i h_{i+1}\Big),\\
c_{i,i+1} =& \frac{1}{192}\left((h_{i-1} + h_i)\left(6h_i^2 + 17h_i h_{i+1} + 14h_{i+1}^2 + \frac{h_{i+1}^2 h_{i+2}}{h_{i+1}+h_{i+2}}\right) + h_{i-1}h_i^2\right),\\
c_{i,i+2} =& \frac{1}{192}(h_{i-1} + h_i)\frac{h_{i+1}^3}{h_{i+1}+h_{i+2}}.
\end{aligned}
$$

We see here the diagonal dominance in rows as

$$
\begin{aligned}
&c_{ii} - (c_{i,i-2} + c_{i,i-1} + c_{i,i+1} + c_{i,i+2})\\
&\quad =\frac{1}{192}((h_i + h_{i+1})(2h_{i-1}^2 + 13h_{i-1}h_i + 3h_i^2)\\
&\qquad + (h_{i-1} + h_i)(3h_i^2 + 13h_i h_{i+1} + 2h_{i+1}^2) + 2h_i^3 + 2h_{i-1}h_i h_{i+1}).
\end{aligned}
$$

Similar calculations give the diagonal dominance in near-boundary equations which yields the unique solvability of (3.20) in this case.

In case of uniform mesh with $h_i = h$, $i = 1, \dots, n$, and $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \dots, n-1$, the interior equations of (3.20) are

$$
\begin{cases}
\dfrac{h^3}{192}(52M_1 + 255M_2 + 76M_3 + M_4) = D_2,\\
\dfrac{h^3}{576}(2M_1 + 229M_2 + 690M_3 + 228M_4 + 3M_5) = D_3,\\
\dfrac{h^3}{192}(M_{i-2} + 76M_{i-1} + 230M_i + 76M_{i+1} + M_{i+2}) = D_i, \quad i = 4, \dots, n-3,\\
\dfrac{h^3}{576}(3M_{n-4} + 228M_{n-3} + 690M_{n-2} + 229M_{n-1} + 2M_n) = D_{n-2},\\
\dfrac{h^3}{192}(M_{n-3} + 76M_{n-2} + 255M_{n-1} + 52M_n) = D_{n-1}.
\end{cases}
$$

The boundary condition $S(a) = \alpha$ gives the equation

$$
\frac{1}{1152}(386M_1 + 379M_2 + 3M_3) = \frac{1}{h^2}(2\alpha - 3z_1 + z_2),
$$

$S(b) = \beta$ gives

$$\frac{1}{1152}(3M_{n-2} + 379M_{n-1} + 386M_n) = \frac{1}{h^2}(z_{n-1} - 3z_n + 2\beta),$$

$S'(a) = \alpha$ leads to

$$\frac{1}{1152}(706M_1 + 443M_2 + 3M_3) = \frac{1}{h^2}(z_2 - z_1 - \alpha h),$$

$S'(b) = \beta$ to

$$\frac{1}{1152}(3M_{n-2} + 443M_{n-1} + 706M_n) = \frac{1}{h^2}(h\beta + z_{n-1} - z_n).$$

In general case, there may be no diagonal dominance in equations (3.12). Let us prove that. Consider in coefficients (3.14) – (3.18) the situation where $\eta_{i-2} = \text{const} > 0$ and other used parameters η_j, ε_j are equal to $\varepsilon \to 0$. Then $c_{i,i-2}$ is of order $\eta_{i-2}^2\varepsilon$ but c_{ii} has the order $\eta_{i-2}\varepsilon^2$.

The unique solvability of system (3.20) follows from the next result.

Proposition 3. *The histopolation problem posed in Section 3.2 has the unique solution.*

Proof. It is sufficient to prove that the corresponding homogeneous problem has only trivial solution. Suppose a cubic spline S satisfies

$$\int_{x_{i-1}}^{x_i} S(x)dx = 0, \quad i = 1, \ldots, n, \tag{3.21}$$

and two of the boundary conditions $S(a) = 0$, $S(b) = 0$, $S'(a) = 0$, $S'(b) = 0$, $S''(a) = 0$, $S''(b) = 0$ at different endpoints a and b. By (3.21) it exists $\eta_i \in (x_{i-1}, x_i)$ such that $S(\eta_i) = 0$, $i = 1, \ldots, n$.

If $S(a) = S(b) = 0$ then there are $\overline{\eta}_i \in (\eta_{i-1}, \eta_i)$, $i = 2, \ldots, n$, $\overline{\eta}_1 \in (a, \eta_1)$, $\overline{\eta}_{n+1} \in (\eta_n, b)$ such that $S'(\overline{\eta}_i) = 0$, $i = 1, \ldots, n+1$. Therefore, there are $\overline{\overline{\eta}}_i \in (\overline{\eta}_i, \overline{\eta}_{i+1})$, $i = 1, \ldots, n$, such that $S''(\overline{\overline{\eta}}_i) = 0$. Consequently, an interval $[\xi_k, \xi_{k+1}]$ contains two (distinct) zeros of S'' which means that $S''(x) = 0$, $x \in [\xi_k, \xi_{k+1}]$.

If $S'(a) = S'(b) = 0$ then again there are $n + 1$ zeros of S' in $[x_0, x_n]$ and n zeros of S'' in (x_0, x_n). If $S''(a) = S''(b) = 0$ then S'' has n zeros in $[x_0, x_n]$. Using different kind boundary conditions at different endpoints we also arrive at the situation with $S''(x) = 0$, $x \in [\xi_k, \xi_{k+1}]$.

Let us make some observations about the situation of $S''(x) = 0$, $x \in [\xi_k, \xi_{k+1}]$. Then S is at most first degree polynomial on $[\xi_k, \xi_{k+1}]$. If S keeps the sign in $[\xi_k, x_k]$ then due to $\int_{x_{k-1}}^{x_k} S(x)dx = 0$ we have $\eta_k \in (x_{k-1}, \xi_k)$ with $S(\eta_k) = 0$. We call this case suitable for the left. If S keeps the sign in $[x_k, \xi_{k+1}]$ then S has a zero in (ξ_{k+1}, x_{k+1}) and this case is called suitable for the right. If, e.g., $k = 1$, then S has a zero in (x_0, x_1), S keeps the sign in $[x_1, \xi_2]$, due to $\int_{x_1}^{x_2} S(x)dx = 0$ there is a zero of S in $[\xi_2, x_2)$ and this case is suitable for the right. Similarly, $k = n - 1$ (i.e., $k + 1 = n$) is a case suitable for the left.

Consider now the case of $[\xi_k, \xi_{k+1}]$ suitable for the left. The interval $[a, \xi_k]$ contains $k-1$ subintervals $[\xi_1, \xi_2], \ldots, [\xi_{k-1}, \xi_k]$. We know that $S(\eta_i) = 0$, $\eta_i \in (x_{i-1}, x_i)$, $i = 1, \ldots, k-1$, and $S(\eta_k) = 0$, $\eta_k \in (x_{k-1}, \xi_k]$. We have:

1) case $S(a) = 0$, then S has $k+1$ zeros $a, \eta_1, \ldots, \eta_k$, S' has k zeros, S'' has $k-1$ zeros in (a, ξ_k) and $S''(\xi_k) = 0$;

2) case $S'(a) = 0$, then S has k zeros $\eta_1, \ldots, \eta_k$, S' has $k-1$ zeros in (a, ξ_k) and $S'(a) = 0$, S'' has $k-1$ zeros in (a, ξ_k) and $S''(\xi_k) = 0$;

3) case $S''(a) = 0$, then S has k zeros in $(a, \xi_k]$, S' has $k-1$ zeros, S'' has $k-2$ zeros in (a, ξ_k) and $S''(a) = 0$, $S''(\xi_k) = 0$.

Anyway, S'' has k zeros in $k-1$ subintervals and, thus, S'' is again equal to zero in some of them.

Observe that receiving $S''(x) = 0$, $x \in [\xi_{k-1}, \xi_{k+1}]$, first degree polynomial S on $[\xi_{k-1}, \xi_{k+1}]$, due to $\int_{x_{k-1}}^{x_k} S(x)dx = 0$, has a zero in (x_{k-1}, x_k) and keeps the sign in $[\xi_{k-1}, x_{k-1}]$, consequently, $[\xi_{k-1}, \xi_k]$ is suitable for the left. At the same time, S keeps the sign in $[x_k, \xi_{k+1}]$ and $[\xi_k, \xi_{k+1}]$ is suitable for the right.

Presented reasonings allow to assert that during the process there are always adjacent subintervals $[\xi_j, \xi_{j+1}], \ldots, [\xi_{k-1}, \xi_k]$ where the nullity of S'' is not yet established, but $[\xi_{j-1}, \xi_j]$ is suitable for the right and $[\xi_k, \xi_{k+1}]$ is suitable for the left which yields that S'' is equal to zero on one of them. However, it may be as well $j = 1$ or $k = n$. The process ends at $S''(x) = 0$, $x \in [a, b]$, and then $S(x) = 0$, $x \in [a, b]$, by histopolation and boundary conditions. Naturally, suppose that $n \geq 2$ if we use $S''(a) = \alpha$ and $S''(b) = \beta$. □

3.6 Another representation

Consider the histopolation problem posed in Section 3.2. A classical representation of cubic spline is the use of $S_i = S(\xi_i)$, $M_i = S''(\xi_i)$, $i = 1, \ldots, n$.

Any cubic spline satisfies the internal equations (continuity of S' at knots ξ_i)

$$\frac{\delta_{i-1}}{\delta_{i-1}+\delta_i} M_{i-1} + 2M_i + \frac{\delta_i}{\delta_{i-1}+\delta_i} M_{i+1} = 6 \frac{\dfrac{S_{i+1}-S_i}{\delta_i} - \dfrac{S_i - S_{i-1}}{\delta_{i-1}}}{\delta_{i-1}+\delta_i}, \quad i = 2, \ldots, n-1. \tag{3.22}$$

For definiteness, add boundary conditions $M_1 = \alpha$ (first equation) and $M_n = \beta$ (last equation). We obtain the system

$$AM = BS + d \tag{3.23}$$

where $M = (M_1, \ldots, M_n)$, $S = (S_1, \ldots, S_n)$, first and last rows of B are zero rows, $d = (\alpha, 0, \ldots, 0, \beta)$. The matrix A has diagonal dominance in rows which gives its invertibility. Note that the diagonal dominance of A in rows takes place also in case of other boundary conditions.

Basing on (3.6) we have

$$S_i = a_i - b_i\varepsilon_i + c_i\varepsilon_i^2 - d_i\varepsilon_i^3,$$

$$S_{i+1} = a_i + b_i\eta_i + c_i\eta_i^2 + d_i\eta_i^3.$$

From them we obtain

$$a_i = \frac{\eta_i S_i + \varepsilon_i S_{i+1}}{\delta_i} - c_i\varepsilon_i\eta_i + d_i(\varepsilon_i - \eta_i)\varepsilon_i\eta_i,$$

$$b_i = \frac{S_{i+1} - S_i}{\delta_i} + c_i(\varepsilon_i - \eta_i) - d_i(\eta_i^2 - \varepsilon_i\eta_i + \varepsilon_i^2).$$

The coefficients c_i and d_i were expressed via M_i and M_{i+1} in Section 3.3. Using (3.6) the histopolation conditions could be written

$$a_{i-1}\eta_{i-1} + \frac{b_{i-1}}{2}\eta_{i-1}^2 + \frac{c_{i-1}}{3}\eta_{i-1}^3 + \frac{d_{i-1}}{4}\eta_{i-1}^4 + a_i\varepsilon_i - \frac{b_i}{2}\varepsilon_i^2 + \frac{c_i}{3}\varepsilon_i^3 - \frac{d_i}{4}\varepsilon_i^4 = z_i h_i$$

or

$$\begin{aligned}
&\frac{\eta_{i-1}^2}{2\delta_{i-1}}S_{i-1} + \left(\frac{\varepsilon_{i-1}\eta_{i-1}}{\delta_{i-1}} + \frac{\eta_{i-1}^2}{2\delta_{i-1}} + \frac{\varepsilon_i\eta_i}{\delta_i} + \frac{\varepsilon_i^2}{2\delta_i}\right)S_i + \frac{\varepsilon_i^2}{2\delta_i}S_{i+1} \\
&- \frac{\eta_{i-1}^2}{24\delta_{i-1}}(2\varepsilon_{i-1}^2 + 4\varepsilon_{i-1}\eta_{i-1} + \eta_{i-1}^2)M_{i-1} \\
&- \left(\frac{\eta_{i-1}^2}{24\delta_{i-1}}(4\varepsilon_{i-1}^2 + 4\varepsilon_{i-1}\eta_{i-1} + \eta_{i-1}^2) + \frac{\varepsilon_i^2}{24\delta_i}(\varepsilon_i^2 + 4\varepsilon_i\eta_i + 4\eta_i^2)\right)M_i \\
&- \frac{\varepsilon_i^2}{24\delta_i}(\varepsilon_i^2 + 4\varepsilon_i\eta_i + 2\eta_i^2)M_{i+1} = z_i h_i, \quad i = 2, \ldots, n-1.
\end{aligned} \tag{3.24}$$

Near the boundary we get

$$\begin{aligned}
&\left(\frac{\eta_1 h_1}{\delta_1} + \frac{h_1^2}{2\delta_1}\right)S_1 + \frac{h_1^2}{2\delta_1}S_2 \\
&- \frac{h_1^2}{24\delta_1}(4\eta_1^2 + 4\eta_1 h_1 + h_1^2)M_1 - \frac{h_1^2}{24\delta_1}(2\eta_1^2 + 4\eta_1 h_1 + h_1^2)M_2 = z_1 h_1
\end{aligned} \tag{3.25}$$

with the counterpart containing $z_n h_n$. These equations together form the system

$$CS = DM + Ez. \tag{3.26}$$

Note that in matrices C and D the diagonal dominates in rows, E is diagonal matrix with entries h_i and $z = (z_1, \ldots, z_n)$. Clearly, to construct the cubic spline histopolant it is necessary and sufficient to solve the system (3.23), (3.26). An opportunity to solve it is the following. Take, e.g., a guess value $M^0 = (M_1, M_2^0, \ldots, M_{n-1}^0, M_n)$, $M_i^0 = D_i/2h_i^3$, $i = 2, \ldots, n-1$, (note that, in uniform grid case, $D_i/2$ is close to $h^3 f''(x_i)$ if the values z_i are determined as

in Chapter 2, Proposition 2) then find S^0 from $CS^0 = DM^0 + Ez$, M^1 from $AM^1 = BS^0 + d$, S^1 from $CS^1 = DM^1 + Ez$, in general, the iteration process is $AM^k = BS^{k-1} + d$, $CS^k = DM^k + Ez$, $k = 1, 2, \ldots$. It may be deduced here also the process

$$M^k = A^{-1}BC^{-1}DM^{k-1} + A^{-1}BC^{-1}Ez + A^{-1}d$$

and the convergence is defined by the spectrum of $A^{-1}BC^{-1}D$. Another opportunity is to take a guess value $S^0 = (S_1^0, \ldots, S_n^0)$, e.g., $S_i^0 = z_i$, $i = 1, \ldots, n$, then find M^0 from $AM^0 = BS^0 + d$, in general, $AM^{k-1} = BS^{k-1} + d$, $CS^k = DM^{k-1} + Ez$, $k = 1, 2, \ldots$. This process could be described as

$$S^k = C^{-1}DA^{-1}BS^{k-1} + C^{-1}DA^{-1}d + C^{-1}Ez.$$

To compare the spectra of the matrices $C^{-1}DA^{-1}B$ and $A^{-1}BC^{-1}D$ let us apply a more general result as follows: if one of the matrices P and Q is invertible then the eigenvalues of PQ and QP coincide. Indeed, assume that P^{-1} exists and $x \neq 0$, λ are such that $PQx = \lambda x$. Then, denoting $y = P^{-1}x$, from $PQx = \lambda PP^{-1}x$ we get $Qx = \lambda P^{-1}x$ or $QPy = \lambda y$ with $y \neq 0$. Thus, it is sufficient to consider $P = C^{-1}D$ and $Q = A^{-1}B$ where the invertibility of P takes place as C and D have diagonal domination in rows.

We will give information about spectra of these matrices in Chapter 6.

Let us consider now uniform mesh with central spline knots, i.e., $h_i = h$, $i = 1, \ldots, n$, $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, n-1$. Equation (3.22) is well known in treatments about cubic splines, it should be taken into account that $\delta_1 = \delta_{n-1} = 3h/2$, $\delta_i = h$, $i = 2, \ldots, n-2$. Equation (3.24) is

$$S_{i-1} + 6S_i + S_{i+1} = \frac{h^2}{48}(7M_{i-1} + 18M_i + 7M_{i+1}) + 8z_i, \quad i = 3, \ldots, n-2,$$

$$2S_1 + 19S_2 + 3S_3 = \frac{h^2}{48}(34M_1 + 77M_2 + 21M_3) + 24z_2$$

and (3.25) is now

$$2S_1 + S_2 = \frac{h^2}{24}(8M_1 + 7M_2) + 3z_1.$$

Chapter 4

Periodic polynomial spline histopolation

We have seen in Chapter 1 that the given histopolation problem, in general, may be reduced to an equivalent interpolation problem and derivative of the interpolant is the histopolant. We have also seen that, on the contrary, certain integral of the histopolant is the solution of a corresponding interpolation problem. This correspondence keeps the periodicity only in one direction, namely, the derivative of a periodic interpolant is periodic but not vice versa. This means that, at periodic histopolation, some problems like, e.g., convergence or error estimates cannot be reduced to similar problems at periodic interpolation. Fortunately, asking about the existence and uniqueness of the solution in spline spaces we are successful because the uniqueness problem could be solved for corresponding homogeneous problems in finite dimensional spaces and the periodicity is preserved in both directions. The existence and uniqueness of the solution at periodic polynomial spline histopolation is the main problem in this chapter. Several cases are treated and the reader can see that different tools are needed in the proofs of assertions.

4.1 The histopolation problem for periodicity

For a given grid Δ_n of points $a = x_0 < x_1 < \ldots < x_n = b$ define the spline space

$$X_m(\Delta_n) = \{S | S : [x_{i-1}, x_i] \to \mathbb{R} \textit{ is in } \mathcal{P}_m(\textit{the set of all polynomials of degree at most } m) \textit{ for } i = 1, \ldots, n,\ S \in C^{m-1}[a, b]\}.$$

It is known that $\dim X_m(\Delta_n) = n + m$. The space $X_{p,m}(\Delta_n)$ of periodic splines is

$$X_{p,m}(\Delta_n) = \left\{ S \in X_m(\Delta_n) | S^{(j)}(a) = S^{(j)}(b), \quad j = 0, 1, \ldots, m-1 \right\}.$$

Then $\dim X_{p,m}(\Delta_n) = n$ and this could be shown, e.g., in following way. From linear algebra it is known the next assertion.

Lemma 1. *Let X be a vector space with $\dim X = n, \phi_i, i = 1, \ldots, k$, be linear functionals defined on X which are linearly independent. Then $\dim(\cap_{i=1}^{k} \ker \phi_i) = n - k$.*

To use this result we take functionals $\phi_i(S) = S^{(i)}(b) - S^{(i)}(a),\ i = 0, \dots, m-1$, defined on $X_m(\Delta_n)$. Then $\varphi_i(S) = 0,\ i = 0, \dots, m-1$, is equivalent to $S^{(i)}(a) = S^{(i)}(b),\ i = 0, \dots, m-1$, which means that, for $S \in X_m(\triangle_n)$ we get $S \in X_{p,m}(\triangle_n)$ if and only if $S \in \cap_{i=0}^{m-1} \ker \varphi_i$. Let us show that the functionals $\varphi_i,\ i = 0, \dots, m-1$, are linearly independent. Suppose $\sum_{i=0}^{m-1} \alpha_i \varphi_i = 0$. Consider the polynomials $p_i(x) = x^{i+1}/(i+1)!,\ i = 0, \dots, m-1$, which belong to the space $X_m(\triangle_n)$. For $p_0(x) = x$ we calculate $\varphi_0(p_0) = b - a \neq 0$ and $\varphi_i(p_0) = p_0^{(i)}(b) - p_0^{(i)}(a) = 0,\ i = 1, \dots, m-1$, hence $\alpha_0 = 0$. For $p_1(x) = x^2/2$ we have $p_1'(x) = x$ and $\varphi_1(p_1) = b - a \neq 0,\ \varphi_i(p_1) = 0,\ i = 2, \dots, m-1$, which implies $\alpha_1 = 0$. Continuing this calculation we establish that $\alpha_i = 0,\ i = 0, \dots, m-1$.

Denote the sizes of the intervals $h_i = x_i - x_{i-1},\ i = 1, \dots, n$. In the periodic histopolation problem we have to find $S \in X_{p,m}(\Delta_n)$ such that

$$\frac{1}{h_i} \int_{x_{i-1}}^{x_i} S(x)dx = z_i, \quad i = 1, \dots, n, \tag{4.1}$$

for given numbers z_i.

Our main task in this chapter is to answer the question: when for any given values $z_i,\ i = 1, \dots, n$, the formulated periodic histopolation problem has a unique solution?

As our problem is linear, this question could be reformulated equivalently as follows: when the corresponding homogeneous problem has only trivial solution, i.e., when

$$S \in X_{p,m}(\Delta_n), \quad \int_{x_{i-1}}^{x_i} S(x)dx = 0, \quad i = 1, \dots, n, \quad \text{implies } S = 0\ ?$$

4.2 Existence and uniqueness

In this section we first indicate the cases where the solution exists and is unique.

Proposition 4. *For m even the periodic histopolation problem has a unique solution.*

Proof. Let $m = 2k$. Consider in $X_m(\Delta_n)$ the seminorm $\|S\| = \left(\int_a^b \left(S^{(k)}(x) \right)^2 dx \right)^{1/2}$. Suppose that $S \in X_{p,m}(\Delta_n)$. Then we get using integration by parts and periodicity properties of the spline S

$$\|S\|^2 = \int_a^b S^{(k)}(x)S^{(k)}(x)dx$$

$$= S^{(k)}(x)S^{(k-1)}(x)\Big|_a^b - \int_a^b S^{(k+1)}(x)S^{(k-1)}(x)dx$$

$$= \ldots = (-1)^{k-1}S^{(2k-1)}(x)S(x)\Big|_a^b + (-1)^k \int_a^b S^{(2k)}(x)S(x)dx$$

$$= (-1)^k \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S^{(2k)}(x)S(x)dx$$

$$= (-1)^k \sum_{i=1}^{n} S^{(2k)}\left(\frac{x_{i-1}+x_i}{2}\right) \int_{x_{i-1}}^{x_i} S(x)dx.$$

Let now, in addition, $\int_{x_{i-1}}^{x_i} S(x)dx = 0, i = 1, \ldots, n$. Then $S^{(k)}(x) = 0$ for all $x \in [a, b]$ or $S \in \mathcal{P}_{k-1}$. In fact, S being a periodic polynomial, S is constant. Indeed, for $S(x) = c_0 + c_1 x + \ldots + c_l x^l$ with $1 < l \leq k-1$ and $c_l \neq 0$ we have $S^{(l-1)}(x) = (l-1)!\, c_{l-1} + l!\, c_l x$. Then $S^{(l-1)}(a) = S^{(l-1)}(b)$ implies $c_l = 0$ which is a contradiction. The homogeneous histopolation conditions then yield $S = 0$ which completes the proof. □

Recall that sign change zero of a function f is a number z such that $f(z) = 0$ and there exists $\varepsilon_0 > 0$ such that $f(z-\varepsilon)f(z+\varepsilon) < 0$ for all $\varepsilon \in (0, \varepsilon_0)$. If $S \in X_m(\Delta_n)$ then let $Z(S)$ be the number of sign change zeros of S in the interval $[x_0, x_n)$. In the case $m = 0$ we talk here about sign change point z requiring only $f(z-\varepsilon)f(z+\varepsilon) < 0$ for all $\varepsilon \in (0, \varepsilon_0)$.

Lemma 2. (*see, e.g.,*[72]). *For $S \in X_{p,m}(\Delta_n)$ it holds*

$$Z(S) \leq \begin{cases} n-1, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

This holds for all $m \in \mathbb{N} \cup \{0\}$.

Lemma 3. *If $S \in X_m(\Delta_n)$, $\int_{x_{j-1}}^{x_j} S(x)dx = 0, \quad j = 1, \ldots, n,$ and $S(x) = 0,$ where $x \in [x_{i-1}, x_i]$ for some i, then $S(x) = 0, x \in [a, b]$.*

Proof. If $S(x) = 0, x \in [x_{i-1}, x_i]$, then for $x \in [x_i, x_{i+1}]$ use Taylor expansion

$$S(x) = S(x_i) + S'(x_i)(x - x_i) + \ldots + \frac{S^{(m-1)}(x_i)}{(m-1)!}(x - x_i)^{m-1}$$

$$+ \frac{S^{(m)}(x_i + 0)}{m!}(x - x_i)^m = \frac{S^{(m)}(x_i + 0)}{m!}(x - x_i)^m.$$

As $\int_{x_i}^{x_{i+1}} S(x)dx = 0$, it holds $S^{(m)}(x_i + 0) = 0$ and $S(x) = 0,\ x \in [x_i, x_{i+1}]$.

We may continue going from x_{i+1} to the right or similarly from x_{i-1} to the left and establish $S(x) = 0,\ x \in [a, b]$. □

Proposition 5. *For m odd and n odd the periodic histopolation problem has a unique solution.*

Proof. Let $S \in X_{p,m}(\Delta_n)$ and $\int_{x_{i-1}}^{x_i} S(x)dx = 0, i = 1, \ldots, n$. Let $S \neq 0$. If $S(x) = 0,\ \ x \in [x_{i-1}, x_i]$, then by Lemma 3 it holds $S = 0$ which is already a contradiction. The condition $S(x) \geq 0$ for all $x \in [x_{i-1}, x_i]$ and $S(\xi) > 0$ for some $\xi \in [x_{i-1}, x_i]$ gives $\int_{x_{i-1}}^{x_i} S(x)dx > 0$ which is not the case. Similarly, $S(x) \leq 0$ for all $x \in [x_{i-1}, x_i]$ and $S(\xi) < 0$ for some $\xi \in [x_{i-1}, x_i]$ does not take place. Thus, there are sign change zeros $\eta_i \in (x_{i-1}, x_i),\ i = 1, \ldots, n$, of S and $Z(S) \geq n$. But by Lemma 2 it holds $Z(S) \leq n - 1$, which is a contradiction. This means that the homogeneous problem has only trivial solution. □

Let us remark that the proof of Proposition 5 is valid for arbitrary $m \geq 1$ and n odd.

Proposition 6. *For $m = 1$ and n even the homogeneous periodic histopolation problem has a non-trivial solution.*

Proof. Take $\eta_i = (x_{i-1} + x_i)/2,\ i = 1, \ldots, n$. Let $c \neq 0$. Consider the function $S(x) = c_i(x - \eta_i),\ x \in [x_{i-1}, x_i],\ i = 1, \ldots, n$. It holds $\int_{x_{i-1}}^{x_i} S(x)dx = 0,\ i = 1, \ldots, n$, for any choice of numbers c_i. The choice of $c_i = (-2c)/h_i$ for $i = 1, 3, \ldots$ and $c_i = (2c)/h_i$ for $i = 2, 4, \ldots$ ensures that $S \in X_{p,1}(\Delta_n),\ S \neq 0$, with

$$S(x_0) = S(x_2) = \ldots = S(x_n) = c$$

and

$$S(x_1) = S(x_3) = \ldots = S(x_{n-1}) = -c.$$ □

Proposition 7. *For m odd and $n = 2$ the homogeneous periodic histopolation problem has a non-trivial solution.*

Proof. For $m = 1$ the assertion is already proved by Proposition 6. We prove the general case by induction.

Denote $\eta_i = (x_{i-1} + x_i)/2,\ i = 1, 2$. Let $m = 2k - 1$ and $S \in X_{p,m}(\Delta_2)$ be such that $S \neq 0$ and

$$S(x) = c_{1,i}(x - \eta_i) + c_{3,i}(x - \eta_i)^3 + \ldots + c_{2k-1,i}(x - \eta_i)^{2k-1}, x \in [x_{i-1}, x_i],\ i = 1, 2. \tag{4.2}$$

Clearly, this holds for the spline S from the proof of Proposition 6 in the case $m = 1$. Define

$$S_1(x) = c_{0,i} + \int_{\eta_i}^{x} S(s)ds$$

or

$$S_1(x) = c_{0,i} + \frac{c_{1,i}}{2}(x-\eta_i)^2 + \frac{c_{3,i}}{4}(x-\eta_i)^4 + \ldots + \frac{c_{2k-1,i}}{2k}(x-\eta_i)^{2k}, x \in [x_{i-1}, x_i]. \tag{4.3}$$

Then $S_1' = S$ and (4.3) implies that, for any numbers $c_{0,i}$,

$$S_1(x_{i-1}+0) = S_1(x_i - 0), \quad i = 1, 2. \tag{4.4}$$

If $c_{0,1}$ and $c_{0,2}$ are such that

$$S_1(x_1 - 0) = S_1(x_1 + 0) \tag{4.5}$$

then $S_1 \in X_{p,m+1}(\Delta_2)$. Next, define $\bar{S}$ by

$$\bar{S}(x) = \int_{\eta_i}^{x} S_1(s)ds$$

or

$$\bar{S}(x) = c_{0,i}(x-\eta_i) + \frac{c_{1,i}}{2\cdot 3}(x-\eta_i)^3 + \ldots + \frac{c_{2k-1,i}}{2k(2k+1)}(x-\eta_i)^{2k+1}, x \in [x_{i-1}, x_i].$$

We see that $\bar{S}$ has the form (4.2), $\bar{S}' = S_1$ and

$$\bar{S}(x_{i-1}+0) = -\bar{S}(x_i - 0), \quad i = 1, 2. \tag{4.6}$$

If, in addition to (4.5), we have

$$\bar{S}(x_1 - 0) = \bar{S}(x_1 + 0) \tag{4.7}$$

then $\bar{S} \in X_{p,m+2}(\Delta_2)$ due to (4.4) – (4.7).

It remains to show that by (4.5) and (4.7) we can determine suitable numbers $c_{0,1}$ and $c_{0,2}$. The equation (4.5) is, in fact,

$$\begin{aligned} c_{0,1} - c_{0,2} &= \frac{c_{1,2}}{2}\left(\frac{-h_2}{2}\right)^2 + \ldots + \frac{c_{2k-1,2}}{2k}\left(\frac{-h_2}{2}\right)^{2k} \\ &\quad - \left(\frac{c_{1,1}}{2}\left(\frac{h_1}{2}\right)^2 + \ldots + \frac{c_{2k-1,1}}{2k}\left(\frac{h_1}{2}\right)^{2k}\right) \end{aligned}$$

and (4.7) is

$$\begin{aligned} c_{0,1}\frac{h_1}{2} + c_{0,2}\frac{h_2}{2} &= \frac{c_{1,2}}{2\cdot 3}\left(\frac{-h_2}{2}\right)^3 + \ldots + \frac{c_{2k-1,2}}{2k(2k+1)}\left(\frac{-h_2}{2}\right)^{2k+1} \\ &\quad - \left(\frac{c_{1,1}}{2\cdot 3}\left(\frac{h_1}{2}\right)^3 + \ldots + \frac{c_{2k-1,1}}{2k(2k+1)}\left(\frac{h_1}{2}\right)^{2k+1}\right). \end{aligned}$$

But this system has non-zero determinant $(h_1 + h_2)/2$. However, as $S \neq 0$ then $\bar{S} \neq 0$. □

We say that the grid $x_0 < x_1 < \ldots < x_n$ is pairwise uniform if n is even and for any i even it holds $x_{i+1} - x_i = h_1$ and $x_{i+2} - x_{i+1} = h_2$.

Corollary 3. *The homogeneous periodic histopolation problem has a non-trivial solution for m odd and pairwise uniform grid.*

In particular, the case of uniform grid for m odd and n even is included in Corollary 3. This result could be found in [72, 76].

In general, we state as an open problem the following.

Conjecture. *For m odd and n even the homogeneous periodic histopolation problem has a non-trivial solution.*

Define the subspace of $X_{p,m}(\Delta_n)$ as

$$X_{0,p,m}(\Delta_n) = \left\{ S \in X_{p,m}(\Delta_n) \mid \int_{x_{i-1}}^{x_i} S(x)dx = 0, i = 1, \ldots, n \right\}.$$

For m odd and n even it may be that $X_{0,p,m}(\Delta_n) \neq \{0\}$ (if the Conjecture is true then always). It is natural to ask what is in this case $\dim X_{0,p,m}(\Delta_n)$?

Remove from the grid $\Delta_n : a = x_0 < x_1 < \ldots < x_n = b$ a knot x_i. We get the grid $\Delta'_{n-1} : a = x_0 < x_1 < \ldots < x_{i-1} < x_{i+1} < \ldots < x_n = b$ with the number of subintervals $n - 1$ which is odd. By Proposition 5 it holds $X_{0,p,m}(\Delta'_{n-1}) = \{0\}$. Clearly, $X_{p,m}(\Delta'_{n-1}) + X_{0,p,m}(\Delta_n) \subset X_{p,m}(\Delta_n)$. The sum $X_{p,m}(\Delta'_{n-1}) + X_{0,p,m}(\Delta_n)$ is a direct sum which follows from the relation

$$X_{p,m}(\Delta'_{n-1}) \cap X_{0,p,m}(\Delta_n) = \{0\}.$$

Thus, the equality

$$\dim X_{p,m}(\Delta_n) = \dim X_{p,m}(\Delta'_{n-1}) + \dim X_{0,p,m}(\Delta_n)$$

implies $\dim X_{0,p,m}(\Delta_n) = 1$.

The obtained results about the existence of non-trivial solutions for homogeneous problem yield the following.

Theorem 1. *For m even or m and n odd the periodic histopolation problem has for each z_i, $i = 1, \ldots, n$, the unique solution. For m odd and n even there may exist (if the Conjecture is true then exist always) z_i, $i = 1, \ldots, n$, such that the periodic histopolation problem does not have solution.*

4.3 Bibliographical notes

In this section we acquaint the reader with a subjective list of works on periodic spline interpolation and histopolation. The results about existence and uniqueness of solution for periodic polynomial spline interpolation could be found in [1]. A short overview of existence results by several authors are presented in [76], this work contains also convergence estimates for problems on uniform grid with interpolation knots not necessarily in grid points. The paper [47] contains results about

properties of periodic interpolating polynomial splines on subintervals. The existence and uniqueness results of periodic solutions for uniform grid case in several papers are based on the theory of circulant matrices, see, e.g., [14, 15]. General non-uniform grid is considered in [16] for low degree periodic splines with convergence estimates. The work [60] gives error estimates for periodic quadratic spline interpolation problem arising from the histopolation problem with these splines. In [38] the existence and uniqueness problem of solution in periodic quartic polynomial spline histopolation ($m = 4$) is stated generally but solved only for uniform grid. Unlike the other studies the spline representation via moments is used. Our Proposition 4 gives here the answer for general grid case. Periodic interpolation problem on uniform grid with certain non-polynomial functions is studied in [8], and histopolation in [7]. Interpolation with periodic polynomial splines of defect greater than minimal is studied in [59, 75, 79]. Cubic spline histopolation on general grid is treated in [33] in several aspects, including methods of practical construction of the histopolant.

Chapter 5

Quadratic/linear rational spline histopolation

Histopolation problem is important to study as a lot of practical information is given in the form of histograms. Besides, it is preferable to keep geometrical properties of given data like positivity, monotonicity, convexity, in general, see [41, 65]. It occurs that, concerning the convexity, an appropriate tool is interpolation or histopolation with quadratic/linear rational splines [25, 29]. While the monotonicity is preserved at interpolation and histopolation with linear/linear rational splines where main problems like, e.g., existence of solution, are solved positively [18, 19, 52, 53], for quadratic/linear rational spline histopolation the situation is completely different [25]. Despite of the freedom at choosing spline knots which seems to create a large flexibility, the solution of the histopolation problem may not exist for any choice of spline knots. The proof of this result is main task in this chapter.

A decisive point at working with splines is the choice of representation. Our study of histopolation with quadratic/linear rational splines shows that the representation via second derivatives of the spline at spline knots and integral values on parts of particular intervals achieves desirable results. We are almost convinced that the use of other parameters is not appropriate which is quite different compared to the case of polynomial splines where several representations are successful at working with them.

5.1 Histopolation problem

In this section we pose the histopolation problem with quadratic/linear rational splines.

As in Chapter 3, let it be given a mesh $a = x_0 < x_1 < \ldots < x_n = b$ and real numbers $z_i,\ i = 1, \ldots, n$, with $n \geq 2$, corresponding to the subintervals $[x_{i-1}, x_i]$ as histopolation data in the form of a given histogram. Denote the lengths of subintervals by $h_i = x_i - x_{i-1},\ i = 1, \ldots, n$. We look for a function S which

satisfies the histopolation conditions

$$\int_{x_{i-1}}^{x_i} S(x)dx = h_i z_i, \quad i = 1, \ldots, n. \tag{5.1}$$

Now we consider the histopolant S to be a quadratic/linear rational function on each subinterval between its knots and require the smoothness condition $S \in C^2[a, b]$.

Choose the spline knots ξ_i $i = 1, \ldots, n$, as follows: $\xi_1 = x_0$, $x_{i-1} < \xi_i < x_i$, $i = 2, \ldots, n-1$, $\xi_n = x_n$. The function S is supposed to have the form

$$S(x) = a_i + b_i(x - x_i) + \frac{c_i}{1 + d_i(x - x_i)}, \quad x \in [\xi_i, \xi_{i+1}], \tag{5.2}$$

with $1 + d_i(x - x_i) > 0$ for all $x \in [\xi_i, \xi_{i+1}]$. We assume the spline S to satisfy the histopolation conditions (5.1) and, in addition, two boundary conditions of the form

$$S(a) = \alpha, \quad S(b) = \beta, \tag{5.3}$$

$$S'(a) = \alpha, \quad S'(b) = \beta, \tag{5.4}$$

$$S''(a) = \alpha, \quad S''(b) = \beta, \tag{5.5}$$

for given α and β. However, two conditions of different kind from (5.3) $-$ (5.5) could be posed at endpoints a and b.

It is known that the solution of such an histopolation problem is unique provided it exists [25]. From (5.2) it follows

$$S''(x) = \frac{2c_i d_i^2}{(1 + d_i(x - x_i))^3}, \quad x \in [\xi_i, \xi_{i+1}], \tag{5.6}$$

which implies that S'' preserves the sign on each particular interval $[\xi_i, \xi_{i+1}]$ and, consequently, on the whole interval $[a, b]$. This means that a quadratic/linear rational spline S of class C^2 is strictly convex or $-S$ is strictly convex or S is a linear function.

5.2 Representation of the histopolant

Besides of the spline representation (5.2) we will use another one for the histopolant via the second derivatives $M_i = S''(\xi_i)$, $\quad i = 1, \ldots, n$, and the values

$$\lambda_i = \int_{\xi_i}^{x_i} S(x)dx, \quad \rho_i = \int_{x_i}^{\xi_{i+1}} S(x)dx.$$

The histopolation conditions (5.1) are $\rho_{i-1} + \lambda_i = h_i z_i$, $i = 1, \ldots, n$, particularly, $\lambda_1 = h_1 z_1$ and $\rho_{n-1} = h_n z_n$ with $\rho_0 = 0$, $\lambda_n = 0$ if needed. Our next task is to represent the histopolant S on $[\xi_i, \xi_{i+1}]$ by the values M_i, M_{i+1}, λ_i, ρ_i.

We mentioned that S'' preserves the sign on the whole interval $[a, b]$. Thus, we consider the case $M_i > 0$ and $d_i \neq 0, \quad i = 0, \ldots, n$. A remark about the treatment of the case $M_i = 0$ and, more generally, about the case of quadratic polynomial pieces is given in [25].

As in Chapter 3, let us introduce on each interval $[\xi_i, \xi_{i+1}]$ the values $\varepsilon_i = x_i - \xi_i$, $\eta_i = \xi_{i+1} - x_i$, $\delta_i = \varepsilon_i + \eta_i = \xi_{i+1} - \xi_i$, then $h_i = \eta_{i-1} + \varepsilon_i$, particularly, $\varepsilon_1 = h_1$, $\delta_1 = h_1 + \eta_1$, $\eta_{n-1} = h_n$, $\delta_{n-1} = \varepsilon_{n-1} + h_n$.

From (5.6) we get

$$M_i = \frac{2c_i d_i^2}{(1 - d_i\varepsilon_i)^3}, \quad M_{i+1} = \frac{2c_i d_i^2}{(1 + d_i\eta_i)^3}. \tag{5.7}$$

Then

$$\left(\frac{M_{i+1}}{M_i}\right)^{1/3} = \frac{(1 - d_i\varepsilon_i)}{(1 + d_i\eta_i)}$$

which gives

$$(1 + d_i\eta_i)M_{i+1}^{1/3} = (1 - d_i\varepsilon_i)M_i^{1/3}$$

and from that

$$d_i = \frac{M_i^{1/3} - M_{i+1}^{1/3}}{\varepsilon_i M_i^{1/3} + \eta_i M_{i+1}^{1/3}}. \tag{5.8}$$

For brevity, use the values $\mu_i = (M_{i+1}/M_i)^{1/3}$, then by (5.8) we have the representation

$$d_i = \frac{1 - \left(\dfrac{M_{i+1}}{M_i}\right)^{1/3}}{\varepsilon_i + \eta_i \left(\dfrac{M_{i+1}}{M_i}\right)^{1/3}} = \frac{1 - \mu_i}{\varepsilon_i + \eta_i\mu_i}.$$

Denote also $\alpha_i = 1 - d_i\varepsilon_i$ and $\beta_i = 1 + d_i\eta_i$. We see that $\alpha_i = \mu_i\delta_i/(\varepsilon_i + \eta_i\mu_i) > 0$ and $\beta_i = \delta_i/(\varepsilon_i + \eta_i\mu_i) > 0$ because $\mu_i > 0$. By symmetry consideration we use also $\nu_i = (M_{i-1}/M_i)^{1/3}$, then, e.g., $d_i = (\nu_{i+1} - 1)/(\varepsilon_i\nu_{i+1} + \eta_i)$.

The second equality of (5.7) gives for $\gamma_i = c_i/d_i$

$$\gamma_i = \frac{\beta_i^3}{2d_i^3} M_{i+1} = \frac{\delta_i^3}{2(1 - \mu_i)^3} M_{i+1} = \frac{\delta_i^3 M_i M_{i+1}}{2\left(M_i^{1/3} - M_{i+1}^{1/3}\right)^3}. \tag{5.9}$$

From (5.2) we calculate

$$\lambda_i = \int_{\xi_i}^{x_i} S(x)dx = \left(a_i x + b_i \frac{(x - x_i)^2}{2} + \frac{c_i}{d_i}\log(1 + d_i(x - x_i)\right)\Big|_{x=\xi_i}^{x=x_i}$$

$$= a_i\varepsilon_i - \frac{b_i}{2}\varepsilon_i^2 - \frac{c_i}{d_i}\log\alpha_i.$$

Similarly we get

$$\rho_i = a_i\eta_i + \frac{b_i}{2}\eta_i^2 + \frac{c_i}{d_i}\log\beta_i.$$

These two equalities allow to find

$$b_i = \frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\left(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\right)\right) \tag{5.10}$$

and

$$a_i = \frac{1}{\delta_i}\left(\frac{\eta_i}{\varepsilon_i}\lambda_i + \frac{\varepsilon_i}{\eta_i}\rho_i + \gamma_i\left(\frac{\eta_i}{\varepsilon_i}\log\alpha_i - \frac{\varepsilon_i}{\eta_i}\log\beta_i\right)\right). \tag{5.11}$$

The formulae (5.8) – (5.11) express the parameters of (5.2) via $M_i, M_{i+1}, \lambda_i, \rho_i$ and this gives the representation of the histopolant (we keep here d_i and γ_i given in (5.8) and (5.9) for brevity and better stress the structure of S)

$$\begin{aligned} S(x) = &\frac{1}{\delta_i}\left(\frac{\eta_i}{\varepsilon_i}\lambda_i + \frac{\varepsilon_i}{\eta_i}\rho_i + \gamma_i\left(\frac{\eta_i}{\varepsilon_i}\log\alpha_i - \frac{\varepsilon_i}{\eta_i}\log\beta_i\right)\right) \\ &+\frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\left(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\right)\right)(x - x_i) \\ &+\frac{\gamma_i d_i}{1 + d_i(x - x_i)}, \quad x \in [\xi_i, \xi_{i+1}]. \end{aligned} \tag{5.12}$$

5.3 Basic equations

The representation (5.12) allows to calculate the derivatives of S on subintervals $[\xi_i, \xi_{i+1}]$. The continuity of S'' is ensured by the use of values M_i. In this section we write out the continuity conditions of S and S'. From them with histopolation conditions we obtain the equations to determine the spline representation parameters M_i, λ_i, ρ_i.

The representation (5.12) gives

$$\begin{aligned} S(\xi_i + 0) = &\frac{1}{\delta_i}\left(\left(2 + \frac{\eta_i}{\varepsilon_i}\right)\lambda_i - \frac{\varepsilon_i}{\eta_i}\rho_i + \gamma_i\left(\left(2 + \frac{\eta_i}{\varepsilon_i}\right)\log\alpha_i + \frac{\varepsilon_i}{\eta_i}\log\beta_i\right)\right) \\ &+ \frac{\gamma_i d_i}{\alpha_i}, \\ S(\xi_{i+1} - 0) = &\frac{1}{\delta_i}\left(-\frac{\eta_i}{\varepsilon_i}\lambda_i + \left(2 + \frac{\varepsilon_i}{\eta_i}\right)\rho_i - \gamma_i\left(\frac{\eta_i}{\varepsilon_i}\log\alpha_i + \left(2 + \frac{\varepsilon_i}{\eta_i}\right)\log\beta_i\right)\right) \\ &+ \frac{\gamma_i d_i}{\beta_i}. \end{aligned}$$

The continuity requirements $S(\xi_i - 0) = S(\xi_i + 0)$ are

$$
\begin{aligned}
&-\frac{\eta_{i-1}}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1} + \frac{1}{\delta_{i-1}}\left(2 + \frac{\varepsilon_{i-1}}{\eta_{i-1}}\right)\rho_{i-1} - \frac{1}{\delta_i}\left(2 + \frac{\eta_i}{\varepsilon_i}\right)\lambda_i + \frac{\varepsilon_i}{\delta_i\eta_i}\rho_i \\
=&\frac{\gamma_{i-1}}{\delta_{i-1}}\left(\frac{\eta_{i-1}}{\varepsilon_{i-1}}\log\alpha_{i-1} + \left(2 + \frac{\varepsilon_{i-1}}{\eta_{i-1}}\right)\log\beta_{i-1}\right) - \frac{\gamma_{i-1}d_{i-1}}{\beta_{i-1}} \\
&+ \frac{\gamma_i}{\delta_i}\left(\left(2 + \frac{\eta_i}{\varepsilon_i}\right)\log\alpha_i + \frac{\varepsilon_i}{\eta_i}\log\beta_i\right) + \frac{\gamma_i d_i}{\alpha_i},\ i = 2, \ldots, n-1. \qquad (5.13)
\end{aligned}
$$

We calculate from (5.12)

$$
\begin{aligned}
S'(x) =&\frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\left(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\right)\right) \\
&- \frac{\gamma_i d_i^2}{(1 + d_i(x - x_i))^2}, \quad x \in [\xi_i, \xi_{i+1}].
\end{aligned}
$$

From that we find

$$
S'(\xi_i + 0) = \frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\Big)\right) - \frac{\gamma_i d_i^2}{\alpha_i^2}
$$

and

$$
S'(\xi_{i+1} - 0) = \frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\Big)\right) - \frac{\gamma_i d_i^2}{\beta_i^2}.
$$

By them, the conditions $S'(\xi_i - 0) = S'(\xi_i + 0)$ give the equations

$$
\begin{aligned}
&-\frac{1}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1} + \frac{1}{\delta_{i-1}\eta_{i-1}}\rho_{i-1} + \frac{1}{\delta_i\varepsilon_i}\lambda_i - \frac{1}{\delta_i\eta_i}\rho_i \\
= \ &\frac{\gamma_{i-1}}{\delta_{i-1}}\left(\frac{1}{\varepsilon_{i-1}}\log\alpha_{i-1} + \frac{1}{\eta_{i-1}}\log\beta_{i-1}\right) + \frac{\gamma_{i-1}^2 d_{i-1}^2}{2\beta_{i-1}^2} \\
&-\frac{\gamma_i}{\delta_i}\left(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\right) - \frac{\gamma_i d_i^2}{2\alpha_i^2}, \quad i = 2, \ldots, n-1. \qquad (5.14)
\end{aligned}
$$

The histopolation conditions could be written in the form

$$
\rho_{i-1} + \lambda_i - h_i z_i = 0, \quad i = 1, \ldots, n. \qquad (5.15)
$$

Let us refer to the just introduced equations (5.13) − (5.15) also as $(5.13, i) - (5.15, i)$.

Similarly to the procedure from Chapter 3, 8 unknowns $\lambda_{i-2},\ \rho_{i-2},\ \lambda_{i-1},\ \rho_{i-1},\ \lambda_i, \rho_i,\ \lambda_{i+1},\ \rho_{i+1}$ could be eliminated from the 9 equations (5.13,$i-1$), (5.15,$i-1$), (5.14,$i-1$), (5.13,i), (5.15,i), (5.14,i), (5.13,$i+1$), (5.15,$i+1$), (5.14,$i+1$). This should be done with the help of coefficients $(3.9, i-1)$, (3.10, $i-1$), (3.11, $i-1$),

(3.9, i), (3.10, i), (3.11, i), (3.9, $i+1$), (3.10, $i+1$), (3.11, $i+1$) from Section 3.4. The obtained combination results a nonlinear equation

$$\Phi_i(M_{i-2}, M_{i-1}, M_i, M_{i+1}, M_{i+2}) = D_i \tag{5.16}$$

with

$$D_i = (h_i + h_{i+1})z_{i-1} - (h_{i-1} + 2h_i + h_{i+1})z_i + (h_{i-1} + h_i)z_{i+1}, i = 2, \ldots, n-1.$$

We will analyze this elimination process more in details in the next section. The equation (5.16) will be called basic equation. We suppose in the sequel that $D_i > 0$ for all i. As stated in Chapter 2, such condition is called strict convexity of the histogram. Recall also from Chapter 2, Proposition 2 which asserts that if we calculate the values $z_i = h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)dx$ for a given function $f \in C^2[a,b]$ having $f''(x) > 0$ for all $x \in [a,b]$ then $D_i > 0$.

5.4 Structure of basic equations

In this section we consider the elimination process at establishing the equation (5.16). At left hand side of the formed combination the unknowns λ_{i-2}, ρ_{i-2}, λ_{i-1}, ρ_{i-1}, λ_i, ρ_i, λ_{i+1}, ρ_{i+1} will be eliminated. For example, the coefficient of λ_{i-2} is obtained in $(3.10, i-1)$ multiplied by $(5.13, i-1)$ and in $(3.11, i-1)$ multiplied by $(5.14, i-1)$ which gives

$$\frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}\Big(-\frac{\eta_{i-2}}{\delta_{i-2}\varepsilon_{i-2}}\Big) - \frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}^2\Big(-\frac{1}{\delta_{i-2}\varepsilon_{i-2}}\Big)$$

$$= \frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}^2\Big(-\frac{1}{\delta_{i-2}\varepsilon_{i-2}} + \frac{1}{\delta_{i-2}\varepsilon_{i-2}}\Big) = 0.$$

Similarly, the multipliers of $\rho_{i-2}, \ldots, \rho_{i+1}$ appear to be equal to zero. The combination of z_{i-1}, z_i, z_{i+1} collecting from equations (5.15, $i-1$), (5.15, i), (5.15, $i+1$) gives us D_i. On the other side compared to D_i we have γ_{i-2} presented only in (5.13, $i-1$) and (5.14, $i-1$) with the multiplier as follows:

$$\frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}\left(\frac{1}{\delta_{i-2}}\Big(\frac{\eta_{i-2}}{\varepsilon_{i-2}}\log\alpha_{i-2} + \Big(2 + \frac{\varepsilon_{i-2}}{\eta_{i-2}}\Big)\log\beta_{i-2}\Big) - \frac{d_{i-2}}{\beta_{i-2}}\right)$$

$$- \frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}^2\left(\frac{1}{\delta_{i-2}}\Big(\frac{1}{\varepsilon_{i-2}}\log\alpha_{i-2} + \frac{1}{\eta_{i-2}}\log\beta_{i-2}\Big) + \frac{d_{i-2}^2}{2\beta_{i-2}^2}\right)$$

$$= \frac{h_i + h_{i+1}}{h_{i-1}}\left(\log\beta_{i-2} - \frac{\eta_{i-2}d_{i-2}}{\beta_{i-2}} - \frac{1}{2}\left(\frac{\eta_{i-2}d_{i-2}}{\beta_{i-2}}\right)^2\right).$$

Denoting for brevity $\sigma_{i-2} = \varepsilon_{i-2}/\eta_{i-2}$ (recall that $\nu_{i-1} = (M_{i-2}/M_{i-1})^{1/3}$), we calculate

$$\begin{aligned}
\beta_{i-2} &= 1 + d_{i-2}\eta_{i-2} \\
&= 1 + \frac{M_{i-2}^{1/3} - M_{i-1}^{1/3}}{\varepsilon_{i-2}M_{i-2}^{1/3} + \eta_{i-2}M_{i-1}^{1/3}}\eta_{i-2} \\
&= \frac{\delta_{i-2}M_{i-2}^{1/3}}{\varepsilon_{i-2}M_{i-2}^{1/3} + \eta_{i-2}M_{i-1}^{1/3}} \\
&= \frac{\delta_{i-2}\left(\dfrac{M_{i-2}}{M_{i-1}}\right)^{1/3}}{\varepsilon_{i-2}\left(\dfrac{M_{i-2}}{M_{i-1}}\right)^{1/3} + \eta_{i-2}} \\
&= \frac{\dfrac{\varepsilon_{i-2}+\eta_{i-2}}{\eta_{i-2}}\nu_{i-1}}{\dfrac{\varepsilon_{i-2}}{\eta_{i-2}}\nu_{i-1} + 1} \\
&= \frac{(\sigma_{i-2}+1)\nu_{i-1}}{\sigma_{i-2}\nu_{i-1}+1}
\end{aligned}$$

and, by (5.9)

$$\begin{aligned}
\gamma_{i-2} &= \frac{\delta_{i-2}^3 M_{i-2}M_{i-1}}{2\left(M_{i-2}^{1/3} - M_{i-1}^{1/3}\right)^3} \\
&= \frac{\delta_{i-2}^3 M_{i-2}}{2\left(\nu_{i-1}-1\right)^3}.
\end{aligned}$$

We find also

$$\frac{\eta_{i-2}d_{i-2}}{\beta_{i-2}} = \frac{\nu_{i-1}-1}{(\sigma_{i-2}+1)\nu_{i-1}}.$$

In total, we have the term with γ_{i-2} as

$$\frac{h_i + h_{i+1}}{h_{i-1}}\frac{\delta_{i-2}^3}{2}\frac{M_{i-2}}{(\nu_{i-1}-1)^3}\left(\log\frac{(\sigma_{i-2}+1)\nu_{i-1}}{\sigma_{i-2}\nu_{i-1}+1}\right.$$
$$\left. - \frac{\nu_{i-1}-1}{(\sigma_{i-2}+1)\nu_{i-1}} - \frac{1}{2}\left(\frac{\nu_{i-1}-1}{(\sigma_{i-2}+1)\nu_{i-1}}\right)^2\right).$$

Define the function

$$\varphi_A(x,\sigma) = \frac{1}{(x-1)^3}\left(\log\frac{(\sigma+1)x}{\sigma x+1} - \frac{x-1}{(\sigma+1)x} - \frac{1}{2}\left(\frac{x-1}{(\sigma+1)x}\right)^2\right) \qquad (5.17)$$

which we call one of the basic histopolation functions. Then the summand in (5.16) containing γ_{i-2} is the term

$$\frac{h_i+h_{i+1}}{h_{i-1}}\frac{\delta_{i-2}^3}{2}M_{i-2}\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\frac{\varepsilon_{i-2}}{\eta_{i-2}}\right).$$

Similar reasoning with (5.13, $i+1$) and (5.14, $i+1$) containing γ_{i+1} gives the symmetric term

$$\frac{h_{i-1}+h_i}{h_{i+1}}\frac{\delta_{i+1}^3}{2}M_{i+2}\varphi_A\left(\left(\frac{M_{i+2}}{M_{i+1}}\right)^{1/3},\frac{\eta_{i+1}}{\varepsilon_{i+1}}\right).$$

The value γ_{i-1} is present in (5.13, $i-1$), (5.14, $i-1$), (5.13, i), (5.14,i). This part at elimination gives the multiplier of γ_{i-1} as follows:

$$\begin{aligned}
&\frac{h_i+h_{i+1}}{h_{i-1}}\eta_{i-2}\left(\frac{1}{\delta_{i-1}}\left(\Big(2+\frac{\eta_{i-1}}{\varepsilon_{i-1}}\Big)\log\alpha_{i-1}+\frac{\varepsilon_{i-1}}{\eta_{i-1}}\log\beta_{i-1}\right)+\frac{d_{i-1}}{\alpha_{i-1}}\right)\\
&-\frac{h_i+h_{i+1}}{h_{i-1}}\eta_{i-2}^2\left(-\frac{1}{\delta_{i-1}}\left(\frac{1}{\varepsilon_{i-1}}\log\alpha_{i-1}+\frac{1}{\eta_{i-1}}\log\beta_{i-1}\right)-\frac{d_{i-1}^2}{2\alpha_{i-1}^2}\right)\\
&+\frac{(h_i+h_{i+1})\varepsilon_i-\eta_{i-1}(h_{i-1}+h_i)}{h_i}\\
&\times\left(\frac{1}{\delta_{i-1}}\left(\frac{\eta_{i-1}}{\varepsilon_{i-1}}\log\alpha_{i-1}+\Big(2+\frac{\varepsilon_{i-1}}{\eta_{i-1}}\Big)\log\beta_{i-1}\right)-\frac{d_{i-1}}{\beta_{i-1}}\right)\\
&+\frac{\eta_{i-1}^2(h_{i-1}+h_i)-(h_{i-1}h_i+\eta_{i-1}(h_i+\varepsilon_i))(h_i+h_{i+1})}{h_i}\\
&\times\left(\frac{1}{\delta_{i-1}}\left(\frac{1}{\varepsilon_{i-1}}\log\alpha_{i-1}+\frac{1}{\eta_{i-1}}\log\beta_{i-1}\right)+\frac{d_{i-1}^2}{2\beta_{i-1}^2}\right).
\end{aligned}$$

We calculate in this expression the multiplier of $\log\alpha_{i-1}$ as the sum of following terms:

$$\begin{aligned}
&\frac{h_i+h_{i+1}}{h_{i-1}}\eta_{i-2}\ \frac{1}{\delta_{i-1}}\left(2+\frac{\eta_{i-1}}{\varepsilon_{i-1}}+\frac{\eta_{i-2}}{\varepsilon_{i-1}}\right)\\
=&\frac{h_i+h_{i+1}}{h_{i-1}}\eta_{i-2}\ \frac{1}{\delta_{i-1}}\ \frac{\delta_{i-1}+h_{i-1}}{\varepsilon_{i-1}},\\
&\frac{h_i+h_{i+1}}{h_i}\ \frac{1}{\delta_{i-1}\varepsilon_{i-1}}\Big(\varepsilon_i\eta_{i-1}-\Big(h_{i-1}h_i+\eta_{i-1}(h_i+\varepsilon_i)\Big)\Big)\\
=&-(h_i+h_{i+1})\frac{1}{\delta_{i-1}\varepsilon_{i-1}}(h_{i-1}+\eta_{i-1})
\end{aligned}$$

(the term with $(h_{i-1}+h_i)/h_i$ is equal to zero). This sum gives us

$$\begin{aligned}
&(h_i+h_{i+1})\ \frac{1}{\delta_{i-1}\varepsilon_{i-1}}\left(\frac{\eta_{i-2}\delta_{i-1}}{h_{i-1}}+\eta_{i-2}-h_{i-1}-\eta_{i-1}\right)\\
&=(h_i+h_{i+1})\ \frac{1}{\varepsilon_{i-1}}\ \frac{\eta_{i-2}-h_{i-1}}{h_{i-1}}\\
&=-\frac{h_i+h_{i+1}}{h_{i-1}}.
\end{aligned}$$

Similar calculations give us the multiplier of $\log\beta_{i-1}$ as

$$-\frac{h_{i-1}+2h_i+h_{i+1}}{h_i}.$$

In total we get at elimination the term containing γ_{i-1} as

$$\begin{aligned}
&\gamma_{i-1}\frac{h_i+h_{i+1}}{h_{i-1}}\left(-\log\alpha_{i-1}+\frac{\eta_{i-2}d_{i-1}}{\alpha_{i-1}}+\frac{1}{2}\left(\frac{\eta_{i-2}d_{i-1}}{\alpha_{i-1}}\right)^2\right)\\
&+\gamma_{i-1}\left(\frac{h_{i-1}+2h_i+h_{i+1}}{h_i}\left(-\log\beta_{i-1}+\frac{\eta_{i-1}d_{i-1}}{\beta_{i-1}}+\frac{1}{2}\left(\frac{\eta_{i-1}d_{i-1}}{\beta_{i-1}}\right)^2\right)\right.\\
&\left.-(h_i+h_{i+1})\frac{d_{i-1}}{\beta_{i-1}}-\frac{1}{2}(h_i+h_{i+1})(2\eta_{i-1}+h_{i-1})\left(\frac{d_{i-1}}{\beta_{i-1}}\right)^2\right).
\end{aligned}\tag{5.18}$$

Taking into account $\gamma_{i-1}=\delta_{i-1}^3M_{i-1}/(2(\nu_i-1)^3)$ with $\nu_i=(M_{i-1}/M_i)^{1/3}$ we obtain (5.18) in the form

$$\begin{aligned}
&\frac{h_i+h_{i+1}}{h_{i-1}}\ \frac{\delta_{i-1}^3}{2}M_{i-1}\varphi_{B1}\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\sigma_{i-1}\right)\\
&+\frac{h_{i-1}+2h_i+h_{i+1}}{h_i}\ \frac{\delta_{i-1}^3}{2}M_{i-1}\left(-\varphi_A\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\sigma_{i-1}\right)\right)\\
&+\frac{h_i+h_{i+1}}{h_{i-1}}\ \frac{\delta_{i-1}^3}{2}M_{i-1}\varphi_{B2}\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\ \sigma_{i-1},\ \frac{h_{i-1}}{\delta_{i-1}}\right)
\end{aligned}\tag{5.19}$$

where

$$\varphi_{B1}(x,\sigma)=\frac{1}{(x-1)^3}\left(\log\frac{\sigma x+1}{\sigma+1}-\frac{\sigma(x-1)}{\sigma+1}+\frac{1}{2}\left(\frac{\sigma(x-1)}{\sigma+1}\right)^2\right),\tag{5.20}$$

$$\varphi_{B2}(x,\sigma,\tau)=\tau\left(\frac{1-\sigma x}{(\sigma+1)x^2}+\frac{1}{2}\tau\ \frac{x+1}{x^2}\right).\tag{5.21}$$

The functions φ_{B1} and φ_{B2} are also called basic histopolation functions.

Similar calculations should be done with the terms containing γ_i. This gives us also three summands symmetrical to (5.19).

In total, we arrive at the basic equation (5.16) in the form

$$\begin{aligned}
&\frac{h_i+h_{i+1}}{h_{i-1}}\ \frac{\delta_{i-2}^3}{2}M_{i-2}\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\frac{\varepsilon_{i-2}}{\eta_{i-2}}\right)\\
+\ &\frac{h_i+h_{i+1}}{h_{i-1}}\ \frac{\delta_{i-1}^3}{2}M_{i-1}\varphi_{B1}\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\frac{\varepsilon_{i-1}}{\eta_{i-1}}\right)\\
+\ &\frac{h_{i-1}+2h_i+h_{i+1}}{h_i}\ \frac{\delta_{i-1}^3}{2}M_{i-1}\left(-\varphi_A\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\frac{\varepsilon_{i-1}}{\eta_{i-1}}\right)\right)\\
+\ &\frac{h_i+h_{i+1}}{h_{i-1}}\ \frac{\delta_{i-1}^3}{2}M_{i-1}\varphi_{B2}\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\frac{\varepsilon_{i-1}}{\eta_{i-1}},\ \frac{h_{i-1}}{\delta_{i-1}}\right)\\
+\ &\frac{h_{i-1}+h_i}{h_{i+1}}\ \frac{\delta_i^3}{2}M_{i+1}\varphi_{B2}\left(\left(\frac{M_{i+1}}{M_i}\right)^{1/3},\frac{\eta_i}{\varepsilon_i},\ \frac{h_{i+1}}{\delta_i}\right)\\
+\ &\frac{h_{i-1}+2h_i+h_{i+1}}{h_i}\ \frac{\delta_i^3}{2}M_{i+1}\left(-\varphi_A\left(\left(\frac{M_{i+1}}{M_i}\right)^{1/3},\frac{\eta_i}{\varepsilon_i}\right)\right)\\
+\ &\frac{h_{i-1}+h_i}{h_{i+1}}\ \frac{\delta_i^3}{2}M_{i+1}\varphi_{B1}\left(\left(\frac{M_{i+1}}{M_i}\right)^{1/3},\frac{\eta_i}{\varepsilon_i}\right)\\
+\ &\frac{h_{i-1}+h_i}{h_{i+1}}\ \frac{\delta_{i+1}^3}{2}M_{i+2}\varphi_A\left(\left(\frac{M_{i+2}}{M_{i+1}}\right)^{1/3},\frac{\eta_{i+1}}{\varepsilon_{i+1}}\right)=D_i. \qquad (5.22)
\end{aligned}$$

The equations (5.16) or (5.22) could be considered for $i=3,\dots,n-2$ (recall that we have, in general, as unknowns $M_1,\dots,M_n$, the values $M_i=S''(\xi_i)$, $i=1,\dots,n$).

Let us show next shortly how to write basic equations near the boundary. However, we may calculate from initial data D_i, $i=2,\dots,n-1$. At elimination of parameters λ_i, ρ_i we take the equations $(5.15,1)$, $(5.13,2)$, $(5.15,2)$, $(5.14,2)$, $(5.13,3)$, $(5.15,3)$, $(5.14,3)$. The corresponding coefficients are as in Chapter 3, namely

$$\begin{aligned}
&-\frac{h_2+h_3}{h_1},\\
&\frac{h_1+2h_2+h_3}{h_2},\\
&\frac{\varepsilon_2(h_2+h_3)-\eta_1(h_1+h_2)}{h_2},
\end{aligned}$$

$$\frac{\varepsilon_2^2(h_2+h_3)+\eta_1^2(h_1+h_2)}{h_2}-(h_1+h_2)(h_2+h_3),$$

$$-\frac{h_1+h_2}{h_3},$$

$$-\frac{h_1+h_2}{h_3}\varepsilon_3,$$

$$-\frac{h_1+h_2}{h_3}\varepsilon_3^2.$$

The elimination of parameters λ_i, ρ_i results the equation

$$\Phi_2(M_1,M_2,M_3,M_4)=D_2 \tag{5.23}$$

or

$$\begin{aligned}
&\frac{h_1+2h_2+h_3}{h_2}\ \frac{\delta_1^3}{2}M_1\left(-\varphi_A\left(\left(\frac{M_1}{M_2}\right)^{1/3},\frac{\varepsilon_1}{\eta_1}\right)\right)\\
&+\frac{h_2+h_3}{h_1}\ \frac{\delta_1^3}{2}M_1\varphi_{B2}\left(\left(\frac{M_1}{M_2}\right)^{1/3},\ \frac{\varepsilon_1}{\eta_1},\ \frac{h_1}{\delta_1}\right)\\
&+\frac{h_1+h_2}{h_3}\ \frac{\delta_2^3}{2}M_3\varphi_{B2}\left(\left(\frac{M_3}{M_2}\right)^{1/3},\ \frac{\eta_2}{\varepsilon_2},\ \frac{h_3}{\delta_2}\right)\\
&+\frac{h_1+2h_2+h_3}{h_2}\ \frac{\delta_2^3}{2}M_3\left(-\varphi_A\left(\left(\frac{M_3}{M_2}\right)^{1/3},\frac{\eta_2}{\varepsilon_2}\right)\right)\\
&+\frac{h_1+h_2}{h_3}\ \frac{\delta_2^3}{2}M_3\varphi_{B1}\left(\left(\frac{M_3}{M_2}\right)^{1/3},\ \frac{\eta_2}{\varepsilon_2}\right)\\
&+\frac{h_1+h_2}{h_3}\ \frac{\delta_3^3}{2}M_4\varphi_A\left(\left(\frac{M_4}{M_3}\right)^{1/3},\ \frac{\eta_3}{\varepsilon_3}\right)=D_2.
\end{aligned}$$

Near the point b, for $i=n-1$, we obtain the equation

$$\Phi_{n-1}(M_{n-3},M_{n-2},M_{n-1},M_n)=D_{n-1} \tag{5.24}$$

which could be derived symmetrically to the equation (5.23).

The simplest boundary conditions here are (5.5) which give $M_1=\alpha$ and $M_n=\beta$. The condition (5.3) or (5.4) could be implemented also using the elimination of parameters λ_i, ρ_i which is described in [25] and results, e.g., for the point a as

$$\Phi_1(M_1,M_2,M_3)=D_1 \tag{5.25}$$

with appropriate value D_1.

In total, we arrive to the system of basic equations

$$\begin{cases} \text{boundary condition } M_1 = \alpha \text{ or (5.25)}, \\ \Phi_2(M_1, M_2, M_3, M_4) = D_2, \\ \Phi_i(M_{i-2}, M_{i-1}, M_i, M_{i+1}, M_{i+2}) = D_i, \quad i = 3, \dots, n-2, \\ \Phi_{n-1}(M_{n-3}, M_{n-2}, M_{n-1}, M_n) = D_{n-1}, \\ \text{boundary condition } M_n = \beta \text{ or counterpart of (5.25)}. \end{cases} \tag{5.26}$$

5.5 Positivity of histopolation functions

Our solution of the main problem is based on the behaviour of basic functions φ_A, φ_{B1}, φ_{B2}. They are defined for $x > 0$, $\sigma > 0$, $\tau > 0$, however, for $x = 1$ as limit values.

Lemma 4. *It holds*

1) $\lim_{x\to 1} \varphi_A(x,\sigma) = 1/(3(\sigma+1)^3)$,

2) $\lim_{x\to 1} \partial\varphi_A/\partial x(x,\sigma) = -(4\sigma+3)/(4(\sigma+1)^4)$,

3) $\lim_{x\to 1} \partial^2\varphi_A/\partial x^2(x,\sigma) = 2(10\sigma^2+15\sigma+6)/(5(\sigma+1)^5)$,

4) $\varphi_A(x,\sigma) > 0$ *for* $x > 0,\ \sigma > 0$.

Proof. 1) For the function φ_A in (5.17) we consider the decomposition

$$\varphi_A(x,\sigma) = (x-1)^{-3}\widetilde{\varphi}_A(x,\ \sigma). \tag{5.27}$$

The use of L'Hospital's rule gives

$$\lim_{x\to 1} \varphi_A(x,\sigma) = \lim_{x\to 1} \frac{1}{3(x-1)^2}\ \frac{\partial\widetilde{\varphi}_A}{\partial x}(x,\ \sigma).$$

Here

$$\begin{aligned} \frac{\partial\widetilde{\varphi}_A}{\partial x}(x,\sigma) &= \frac{\sigma x+1}{(\sigma+1)x}\ \frac{(\sigma+1)(\sigma x+1)-(\sigma+1)x\sigma}{(\sigma x+1)^2} \\ &\quad - \frac{1}{(\sigma+1)x^2} - \frac{x-1}{(\sigma+1)x}\ \frac{1}{(\sigma+1)x^2} \\ &= \frac{1}{x(\sigma x+1)} - \frac{1}{x^2(\sigma+1)} - \frac{x-1}{(\sigma+1)^2x^3} \\ &= \frac{(x-1)^2}{x^3(\sigma+1)^2(\sigma x+1)}. \end{aligned} \tag{5.28}$$

Now

$$\lim_{x\to 1}\varphi_A(x,\sigma)=\lim_{x\to 1}\frac{1}{3x^3(\sigma+1)^2(\sigma x+1)}=\frac{1}{3(\sigma+1)^3}.$$

2) Basing on (5.27) and (5.28) we find

$$\begin{aligned}\frac{\partial\varphi_A}{\partial x}(x,\sigma)&=-\frac{3}{(x-1)^4}\,\widetilde{\varphi}_A(x,\sigma)+\frac{1}{(x-1)^3}\frac{\partial\widetilde{\varphi}_A}{\partial x}(x,\sigma)\\&=\frac{1}{x-1}\left(\frac{1}{(\sigma+1)^2(\sigma x+1)x^3}-3\varphi_A(x,\sigma)\right).\end{aligned}$$

Then

$$\begin{aligned}\lim_{x\to 1}\frac{\partial\varphi_A}{\partial x}(x,\sigma)&=\lim_{x\to 1}\frac{\partial}{\partial x}\left(\frac{1}{(\sigma+1)^2(\sigma x+1)x^3}-3\varphi_A(x,\sigma)\right)\\&=\lim_{x\to 1}\left(\frac{-(4\sigma x+3)}{(\sigma+1)^2(\sigma x+1)^2x^4}-3\frac{\partial\varphi_A}{\partial x}(x,\sigma)\right)\\&=-\frac{4\sigma+3}{(\sigma+1)^4}-3\lim_{x\to 1}\frac{\partial\varphi_A}{\partial x}(x,\sigma)\end{aligned}$$

and from that

$$\lim_{x\to 1}\frac{\partial\varphi_A}{\partial x}(x,\sigma)=-\frac{4\sigma+3}{4(\sigma+1)^4}.$$

3) Similarly, to the proof of 2) we obtain

$$\frac{\partial^2\varphi_A}{\partial x^2}(x,\sigma)=\frac{1}{(x-1)^2}\left(\frac{-8\sigma x^2+4\sigma x-7x+3}{(\sigma+1)^2(\sigma x+1)^2x^4}+12\varphi_A(x,\sigma)\right).$$

Applying twice L'Hospital's rule we get

$$\lim_{x\to 1}\frac{\partial^2\varphi_A}{\partial x^2}(x,\sigma)=\frac{2(10\sigma^2+15\sigma+6)}{5(\sigma+1)^5}.$$

4) For the proof of positivity of φ_A we consider again the decomposition (5.27) and the formula (5.28). Now, if $x\in(0,1)$ then by Taylor expansion in $x=1$ for some $\xi\in(x,1)$

$$\widetilde{\varphi}_A(x,\sigma)=\widetilde{\varphi}_A(1,\sigma)+\frac{\partial\widetilde{\varphi}_A}{\partial x}(\xi,\sigma)(x-1)<0,$$

which gives $\varphi_A(x,\sigma)>0$. Similarly, if $x>1$ then by the same expansion with $\xi\in(1,x)$ we get $\widetilde{\varphi}_A(x,\sigma)>0$ and also $\varphi_A(x,\sigma)>0$. □

It is important for us later that the summands in basic equation are strictly positive. We will restrict ourselves to the uniform grid, i.e., $h_i=h,\ i=1,\dots,n$. In this case, define the function

$$\varphi_B(x,\sigma,\tau)=\varphi_{B1}(x,\sigma)-2\varphi_A(x,\sigma)+\varphi_{B2}(x,\sigma,\tau).$$

Then equation (5.22) takes the form

$$\delta_{i-2}^3\ M_{i-2}\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right)$$

$$+\ \delta_{i-1}^3\ M_{i-1}\varphi_B\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\ \frac{\varepsilon_{i-1}}{\eta_{i-1}},\ \frac{h}{\delta_{i-1}}\right)$$

$$+\ \delta_i^3\ \ M_i\varphi_B\left(\left(\frac{M_{i+1}}{M_i}\right)^{1/3},\ \frac{\eta_i}{\varepsilon_i},\ \frac{h}{\delta_i}\right)$$

$$+\ \delta_{i+1}^3\ M_{i+1}\varphi_A\left(\left(\frac{M_{i+2}}{M_{i+1}}\right)^{1/3},\ \frac{\eta_{i+1}}{\varepsilon_{i+1}}\right) = D_i. \tag{5.29}$$

It occurs that in this equation all summands are strictly positive for any choice of spline knots. We will show that φ_B is positive in certain feasible domain of arguments.

Lemma 5. *It holds*

1) $\lim_{x\to 1}\varphi_{B1}(x,\sigma) = \sigma^3/(3(\sigma+1)^3)$,

2) $\lim_{x\to 1}\varphi_{B2}(x,\sigma,\tau) = \tau\,((1-\sigma)/(1+\sigma)+\tau)$,

3) $\varphi_B(x,\sigma,\tau) > 0$ *in* $\Omega = \{(x,\ \sigma,\ \tau)\mid x>0,\ \sigma=\varepsilon/\eta,\ \tau=h/(\varepsilon+\eta),\ \varepsilon,\ \eta\in(0,h)\}$.

Proof. Likewise in the proof of Lemma 4, 1) and 2) are the results of standard calculations and we prove the assertion 3).

By Lemma $4,1)$ and Lemma $5,1),2)$ we have

$$\varphi_B(1,\sigma,\tau) = \tau\left(\frac{1-\sigma}{1+\sigma}+\tau\right)+\frac{\sigma^3-2}{3(\sigma+1)^3} = \varphi(\sigma,\tau)$$

(we introduced here the function φ). Actually, $\tau = h/\delta_i > 1/2$ as $\delta_i < 2h$. It holds

$$\frac{\partial\varphi}{\partial\tau}(\sigma,\tau) = \frac{1-\sigma}{1+\sigma}+2\tau = 0$$

only if $\tau = (\sigma-1)/2(\sigma+1) < 1/2$. Thus, for $\tau > 1/2$, $\partial\varphi/\partial\tau(\sigma,\tau)\neq 0$. Because $\partial\varphi/\partial\tau(\sigma,\tau) > 0$ for great τ, we have $\partial\varphi/\partial\tau(\sigma,\tau) > 0$ for each $\tau > 1/2$ which means that the function $\tau\to\varphi(\sigma,\tau)$ is increasing. Now, for $\tau > 1/2$

$$\varphi(\sigma,\tau) > \varphi\left(\sigma,\frac{1}{2}\right) = \frac{\sigma^3+3\sigma^2+15\sigma+1}{12(\sigma+1)^3} > 0$$

or $\varphi_B(1,\sigma,\tau) > 0$ for $\sigma > 0,\ \tau > 1/2$.

Write $\varphi_B(x,\sigma,\tau) = (x-1)^{-3}\widetilde{\varphi}_B(x,\sigma,\tau)$, then $\widetilde{\varphi}_B(1,\sigma,\tau) = 0$. Using the Taylor expansion of the function $\widetilde{\varphi}_B$ in $x = 1$ it is sufficient to prove that $\partial\widetilde{\varphi}_B/\partial x(x,\sigma,\tau) > 0$, $(x,\sigma,\tau) \in \Omega$. We calculate

$$\frac{\partial\widetilde{\varphi}_B}{\partial x}(x,\sigma,\tau) = (x-1)^2\overline{\varphi}_B(x,\sigma,\tau),$$

where

$$\begin{aligned}\overline{\varphi}_B(x,\sigma,\tau) &= 3\tau\left(\frac{1-\sigma x}{(\sigma+1)x^2} + \frac{1}{2}\tau\,\frac{x+1}{x^2}\right)\\ &\quad +(x-1)\tau\left(\frac{\sigma x-2}{(\sigma+1)x^3} - \tau(\frac{1}{2}x+1)\frac{1}{x^3}\right)\\ &\quad +\frac{\sigma^3}{(\sigma+1)^2(\sigma x+1)} - \frac{2}{(\sigma+1)^2(\sigma x+1)x^3}. \qquad (5.30)\end{aligned}$$

Let us write expression (5.30) with the common denominator $(\sigma+1)^2(\sigma x+1)x^3$ and show that the numerator is positive. Consider $\sigma = \varepsilon/\eta$, $\tau = h/(\varepsilon+\eta)$, $\varepsilon,\eta \in (0,h)$. The numerator of (5.30) could be written as

$$\begin{aligned}&\frac{1}{\eta^3}(\varepsilon(h-\varepsilon)^2x^3 + (h\varepsilon(h-\varepsilon) + h\eta(h-\varepsilon))x^2\\ &+(h\eta^2 + h\eta^2 + h\varepsilon\eta + h^2\varepsilon)x + 2\eta^2(h-\eta) + h^2\eta)\end{aligned}$$

which is strictly positive for $x > 0$. This ends of the proof of assertion 3). □

Let us remark that if we assume only $x > 0$, $\sigma > 0$, $\tau > 0$ then $\varphi_B(x,\sigma,\tau)$ may be negative.

5.6 Asymptotic behaviour of histopolation functions

In this section we establish the behaviour of functions φ_A and φ_B if one of the arguments is going to 0 or ∞ with other arguments fixed. We do not describe all asymptotics in full details but only those of them which seem more essential. In the following assertions the sign $\sim$ means that the quotient of two expressions has limit 1 in the limit process under consideration.

Lemma 6. *It holds*

1) $\varphi_A(x,\sigma) \sim 1/(2(\sigma+1)^2x^2)$ *as* $x \to 0$, $\sigma > 0$ *fixed,*

2) $\varphi_A(x,\sigma) \sim \varphi(\sigma)/x^3$ *as* $x \to \infty$, $\sigma > 0$ *fixed with* $\varphi(\sigma) > 0$ *for all* $\sigma > 0$, $\varphi(\sigma) \sim -\log\sigma$ *as* $\sigma \to 0$ *and* $\varphi(\sigma) \sim 1/(3\sigma^3)$ *as* $\sigma \to \infty$,

3) $\varphi_A(x,\sigma) \sim \varphi(x)$ *as* $\sigma \to 0$, x *fixed with* $\varphi(x) \sim 1/(2x^2)$ *as* $x \to 0$ *and* $\varphi(x) \sim (\log x)/x^3$ *as* $x \to \infty$,

4) $\varphi_A(x,\sigma) \sim 1/(3\sigma^3 x^3)$ *as* $\sigma \to \infty$, $x > 0$ *fixed.*

Proof. 1) Let us treat the function φ_A given in (5.17) as $x \to 0$ with fixed $\sigma > 0$. Then

$$\lim_{x\to 0}(x-1)^3 = -1,$$

$$\lim_{x\to 0} \frac{\log \dfrac{(\sigma+1)x}{\sigma x+1}}{\dfrac{1}{2(\sigma+1)^2x^2}} = 2(\sigma+1)^2 \lim_{x\to 0} x^2 \log(\sigma+1)x = 0,$$

$$\lim_{x\to 0} \frac{\dfrac{x-1}{(\sigma+1)x}}{\dfrac{1}{2(\sigma+1)^2x^2}} = 2(\sigma+1)\lim_{x\to 0}(x-1)x = 0,$$

and the main term is obtained from the summand $-((x-1)/((\sigma+1)x))^2/2$.

2) It is immediate to obtain the indicated behaviour $\varphi_A(x,\sigma) \sim \varphi(\sigma)/x^3$ but let us study here the multiplying function $\varphi(\sigma) = \log((\sigma+1)/\sigma) - 1/(\sigma+1) - 1/(2(\sigma+1)^2)$. Clearly, $\varphi_A(x,\sigma) > 0$ for all $x > 0$, $\sigma > 0$ implies $\varphi(\sigma) > 0$ for all $\sigma > 0$. The asymptotics of $\varphi(\sigma)$ as $\sigma \to 0$ is immediate. Taylor expansion

$$\log(1+\frac{1}{\sigma}) = \frac{1}{\sigma} - \frac{1}{2\sigma^2} + \frac{1}{3\sigma^3} - \frac{1}{4(1+\xi)^4}\frac{1}{\sigma^4}, \quad \xi \in \left(0,\ \frac{1}{\sigma}\right),$$

gives

$$\varphi(\sigma) = \frac{1}{3\sigma^3} - \frac{1}{2\sigma^2(\sigma+1)^2} - \frac{1}{4(1+\xi)^4\sigma^4} \tag{5.31}$$

and the asymptotics of $\varphi(\sigma)$ as $\sigma \to \infty$.

From (5.31) we get also that $\varphi(\sigma) \sim 1/(3\sigma^3)$ as $\sigma \to \infty$. It holds also $\varphi(\sigma) \sim -\log\sigma \to \infty$ as $\sigma \to 0$.

3) Here we obtain immediately the asymptotical function

$$\varphi(x) = \frac{1}{(x-1)^3}\left(\log x - \frac{x-1}{x} - \frac{1}{2}\left(\frac{x-1}{x}\right)^2\right)$$

as $\sigma \to 0$, $x > 0$ fixed.

For $x \to 0$ the main term in $\varphi(x)$ is obtained from the last summand and for $x \to \infty$ from the first summand.

4) We apply the expansion

$$\log\frac{(\sigma+1)x}{\sigma x+1} = \frac{x-1}{\sigma x+1} - \frac{1}{2}\left(\frac{x-1}{\sigma x+1}\right)^2 + \frac{1}{3}\left(\frac{x-1}{\sigma x+1}\right)^3 - \frac{1}{4(1+\xi)^4}\left(\frac{x-1}{\sigma x+1}\right)^4,$$

$$\xi \in \left(0, \frac{x-1}{\sigma x+1}\right),$$

with $\xi \to 0$ as $\sigma \to \infty$. This gives

$$\varphi_A(x,\sigma) = \frac{1}{3(\sigma x+1)^3} - \frac{x-1}{2(\sigma+1)^2(\sigma x+1)^2 x^2} - \frac{x-1}{4(1+\xi)^4(\sigma x+1)^4}$$

and the asserted asymptotics. □

Lemma 7. *It holds in the feasible domain*

1) $\varphi_B(x,\sigma,\tau) \sim \varphi(\sigma,\tau)/x^2$ *as* $x \to 0$, σ, τ *fixed, with the function* φ *bounded from below by a positive constant,*

2) $\varphi_B(x,\sigma,\tau) \sim \varphi(\sigma,\tau)/x$ *as* $x \to \infty$, σ, τ *fixed,*

3) $\varphi_B(x,\sigma,\tau) \sim \varphi(x,\tau)$ *as* $\sigma \to 0$, x, τ *fixed,*

4) $\varphi_B(x,\sigma,\tau) \sim \varphi(x,\tau)$ *as* $\sigma \to \infty$, x, τ *fixed,*

5) $\varphi_B(x,\sigma,\tau) \sim \varphi(x)\tau^2$ *as* $\tau \to \infty$, x, σ *fixed.*

Proof. 1) In the process $x \to 0$

$$\varphi_{B1}(x,\sigma) \sim -\left(\log\frac{1}{\sigma+1} + \frac{\sigma}{\sigma+1} + \frac{1}{2}\left(\frac{\sigma}{\sigma+1}\right)^2\right),$$

$$\varphi_A(x,\sigma) \sim \frac{1}{2(\sigma+1)^2x^2},$$

$$\varphi_{B2}(x,\sigma,\tau) \sim \tau\left(\frac{1}{\sigma+1} + \frac{1}{2}\tau\right)\frac{1}{x^2},$$

thus

$$\varphi(\sigma,\tau) = \tau\left(\frac{1}{\sigma+1} + \frac{1}{2}\tau\right) - \frac{1}{(\sigma+1)^2}.$$

Taking $\sigma = \varepsilon/\eta$, $\tau = h/(\varepsilon+\eta)$, $\varepsilon, \eta \in (0,h)$, we have

$$\varphi(\sigma,\tau) = \frac{h\eta + \frac{1}{2}h^2 - \eta^2}{(\varepsilon+\eta)^2} \geq \frac{h^2}{2(\varepsilon+\eta)^2} \geq \frac{1}{8}$$

and, in addition, $\varphi(\sigma,\tau) \leq 3h^2/(2(\varepsilon+\eta)^2)$.

2) In this case

$$\varphi(\sigma,\tau) = \tau\left(-\frac{\sigma}{\sigma+1} + \frac{1}{2}\tau\right) + \frac{1}{2}\frac{\sigma^2}{(\sigma+1)^2} = \frac{1}{2}\frac{(h-\varepsilon)^2}{(\varepsilon+\eta)^2} > 0.$$

3) For $\sigma \to 0$, x, τ fixed, we have $\varphi_{B1}(x,\sigma) \sim 0$, the asymptotics of $\varphi_A(x,\sigma)$ is given in Lemma 6, 3) and

$$\varphi_{B2}(x,\sigma,\tau) \sim \tau\left(\frac{1}{x^2} + \frac{1}{2}\tau + \frac{x+1}{x^2}\right)$$

which yields that $\varphi_B(x,\sigma,\tau)$ behaves as a constant depending on x and τ.

4) For $\sigma \to \infty$, x, τ fixed

$$\varphi_{B1}(x,\sigma) \sim \frac{1}{(x-1)^3}\left(\log x - (x-1) + \frac{1}{2}(x-1)^2\right),$$

$\varphi_A(x,\sigma) \sim \ 1/(3\sigma^3 x^3)$, Lemma 6, 4), now $\varphi_A(x,\sigma) \sim 0$,

$$\varphi_{B1}(x,\sigma,\tau) \sim \frac{\tau}{x}\left(\frac{1}{2}\tau\ \frac{x+1}{x} - 1\right) = \varphi(x,\tau)$$

with $\varphi(x,\tau) \sim (x+1)\tau^2/(2x^2)$ as $\tau \to \infty$ and $\varphi(x,\tau) \sim \ \tau^2/(2x^2)$ as $x \to 0$.

5) The main term here comes from φ_{B2} and $\varphi(x) = \ (x+1)/(2x^2)$. □

5.7 Non-existence of the histopolant

The chosen representation (5.12) of the quadratic/linear rational spline histopolant shows that for its existence it is necessary and sufficient that the parameters M_i, λ_i, ρ_i satisfy a nonlinear system consisting of all equations (5.13) – (5.15) together with two boundary conditions from (5.3) – (5.5). The elimination of parameters λ_i, ρ_i leads to the system of basic equations (5.26), therefore the parameters M_i should satisfy (5.26). If (5.26) has a solution then the parameters λ_i, ρ_i are determined from a linear system (see [23, 25]) which has always the unique solution. Thus, the existence of solution for histopolating problem is equivalent to the existence of solution for the system of basic equations.

We will show that it exists a strictly convex histogram based on uniform grid such that the histopolating problem with quadratic/linear rational spline has no solution regardless how the spline knots are chosen.

Let us refer to the equation (5.29) as $(5.29, i)$, we need them for several values of i. Suppose that in the case of fixed uniform grid, for given $D_i > 0$ for all i it is possible to choose ξ_i in such a way that the solution of basic system exists. With the knots ξ_i we have also the numbers ε_i, η_i, δ_i. Consider the case $D_i \to \infty$, $D_{i-1} \to 0$, $D_{i-2} \to 0$, $D_{i+1} \to 0$, $D_{i+2} \to 0$ and suppose that there are always appropriate knots ξ_i for the existence.

From $D_i \to \infty$ basing on $(5.29, i)$ and positivity of φ_A and φ_B we get

$$\delta_{i-2}^3 M_{i-2}\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right) \to \infty \tag{5.32}$$

or

$$\delta_{i-1}^3 M_{i-1}\varphi_B\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\ \frac{\varepsilon_{i-1}}{\eta_{i-1}},\ \frac{h}{\delta_{i-1}}\right) \to \infty \tag{5.33}$$

(the reasoning with third and forth summands in $(5.29, i)$ is symmetrical). However, sometimes there is the need to consider subsequences but we keep simple writings without sequence indices. From $D_{i-1} \to 0$ by $(5.29, i-1)$ we obtain

$$\delta_{i-3}^3 M_{i-3}\varphi_A\left(\left(\frac{M_{i-3}}{M_{i-2}}\right)^{1/3},\ \frac{\varepsilon_{i-3}}{\eta_{i-3}}\right) \to 0 \tag{5.34}$$

and

$$\delta_{i-2}^3 M_{i-2}\varphi_B\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2},},\ \frac{h}{\delta_{i-2}}\right)\to 0 \tag{5.35}$$

and

$$\delta_{i-1}^3 M_{i}\varphi_B\left(\left(\frac{M_{i}}{M_{i-1}}\right)^{1/3},\ \frac{\eta_{i-1},}{\varepsilon_{i-1}},\ \frac{h}{\delta_{i-1}}\right)\to 0 \tag{5.36}$$

and

$$\delta_{i}^3 M_{i+1}\varphi_A\left(\left(\frac{M_{i+1}}{M_{i}}\right)^{1/3},\ \frac{\eta_{i},}{\varepsilon_{i}}\right)\to 0. \tag{5.37}$$

Similar convergences we get from $D_{i-2}\to 0$ and $D_{i+1}\to 0$, $D_{i+2}\to 0$.

1) Assume (5.32). As δ_{i-2} is bounded, it holds

$$M_{i-2}\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right)\to\infty. \tag{5.38}$$

1a) Consider the case $M_{i-2}\le$ const. Then (5.38) gives

$$\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right)\to\infty.$$

By Lemma 6 this yields $M_{i-2}/M_{i-1}\to 0$. Due to that by Lemma 7, 1),

$$\varphi_B\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2}},\ \frac{h}{\delta_{i-2}}\right)\ge\frac{1}{8}\frac{1}{\left(\frac{M_{i-2}}{M_{i-1}}\right)^{2/3}}$$

and (5.35) gives

$$\delta_{i-2}^3\ M_{i-2}^{1/3}\ M_{i-1}^{2/3}\to 0. \tag{5.39}$$

We have also by Lemma 6

$$\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3},\ \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right)\sim\frac{1}{2(\delta_{i-2}+1)^2\left(\frac{M_{i-2}}{M_{i-1}}\right)^{2/3}}\le\frac{1}{2\left(\frac{M_{i-2}}{M_{i-1}}\right)^{2/3}}$$

and the limit of the expression in (5.32) could be estimated by

$$\frac{1}{2}\delta_{i-2}^3\ M_{i-2}^{1/3}\ M_{i-1}^{2/3}\to 0,$$

contradicting to (5.32).

1b) Let $M_{i-2} \to \infty$.

1ba) Consider the case $0 < c_1 \le M_{i-2}/M_{i-1} \le c_2$ with some constants c_1 and c_2. Then by Lemma 6 $\varphi_A\left((M_{i-2}/M_{i-1})^{1/3}, \varepsilon_{i-2}/\eta_{i-2}\right)$ is bounded from above and (5.32) yields

$$\delta_{i-2}^3 \; M_{i-2} \to \infty. \tag{5.40}$$

But by Lemma 7

$$\varphi_B\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3}, \; \frac{\varepsilon_{i-2}}{\eta_{i-2}}, \; \frac{h}{\delta_{i-2}}\right) \sim \text{const}$$

and (5.35) gives $\delta_{i-2}^3 \; M_{i-2} \to 0$ which is contradicting to (5.40).

1bb) The case $M_{i-2}/M_{i-1} \to 0$ could be treated as 1a).

1bc) Assume $M_{i-2}/M_{i-1} \to \infty$. Then $M_{i-1}/M_{i-2} \to 0$. By Lemma 6, 2),

$$\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3}, \; \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right) \sim \varphi(\sigma_{i-2})\frac{M_{i-1}}{M_{i-2}}$$

and by (5.32)

$$\delta_{i-2}^3 \; M_{i-1} \; \varphi(\sigma_{i-2}) \to \infty. \tag{5.41}$$

If now $\delta_{i-2}^3 \; M_{i-1} \ge \text{const} > 0$ then the counterpart of (5.36) obtained from $D_{i-2} \to 0$ is

$$\delta_{i-2}^3 M_{i-1} \varphi_B\left(\left(\frac{M_{i-1}}{M_{i-2}}\right)^{1/3}, \; \frac{\eta_{i-2},}{\varepsilon_{i-2}}, \; \frac{h}{\delta_{i-2}}\right) \to 0 \tag{5.42}$$

with (by Lemma 7, 1))

$$\varphi_B\left(\left(\frac{M_{i-1}}{M_{i-2}}\right)^{1/3}, \; \frac{\eta_{i-2},}{\varepsilon_{i-2}}, \; \frac{h}{\delta_{i-2}}\right) \ge \frac{1}{8} \; \frac{1}{\left(\frac{M_{i-1}}{M_{i-2}}\right)^{2/3}}.$$

The convergence (5.42) gives

$$\delta_{i-2}^3 \; M_{i-1}\left(\frac{M_{i-2}}{M_{i-1}}\right)^{2/3} \to 0$$

which is a contradiction to $\delta_{i-2}^3 \; M_{i-1} \ge \text{const} > 0$ and $M_{i-2}/M_{i-1} \to \infty$.

It remains to consider the case $\delta_{i-2}^3 \; M_{i-1} \to 0$ and $\varphi(\sigma_{i-2}) \to \infty$. Then $\sigma_{i-2} \to 0$, $\varepsilon_{i-2} \to 0$ and $\varphi(\sigma_{i-2}) \sim -\log \sigma_{i-2} = -\log(\varepsilon_{i-2}/\eta_{i-2})$. Thus, by (5.41),

$$\delta_{i-2}^3 \; M_{i-1}\left(-\log \frac{\varepsilon_{i-2}}{\eta_{i-2}}\right) \to \infty. \tag{5.43}$$

From (5.35) with the help of Lemma 7, 2) we obtain

$$\delta_{i-2}\ M_{i-1}^{1/3}\ M_{i-2}^{2/3}(h-\varepsilon_{i-2})^2 \to 0$$

and, by $\varepsilon_{i-2} \to 0$,

$$\delta_{i-2}^3\ M_{i-1}\left(\frac{M_{i-2}}{M_{i-1}}\right)^{2/3} \to 0. \tag{5.44}$$

Now, (5.43) and (5.44) give

$$q = \frac{-\log\sigma_{i-2}}{\left(\dfrac{M_{i-2}}{M_{i-1}}\right)^{2/3}} \to \infty,$$

and, in turn

$$\sigma_{i-2}\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3} = e^{-q\left(\frac{M_{i-2}}{M_{i-1}}\right)^{2/3}}\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3} \to 0.$$

In this case the asymptotics of φ_A could be calculated as

$$\varphi_A\left(\left(\frac{M_{i-2}}{M_{i-1}}\right),\ \sigma_{i-2}\right) \sim \frac{\log\left(\dfrac{M_{i-2}}{M_{i-1}}\right)^{1/3}}{\dfrac{M_{i-2}}{M_{i-1}}}$$

which in (5.32) yields

$$\delta_{i-2}^3 M_{i-1}\log\left(\frac{M_{i-2}}{M_{i-1}}\right)^{1/3} \to \infty,$$

giving contradiction to (5.44).

2) Assume (5.33).

2a) Let $M_{i-1} \le \text{const}$. Then

$$\varphi_B\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\ \frac{\varepsilon_{i-1}}{\eta_{i-1}},\ \frac{h}{\delta_{i-1}}\right) \to \infty$$

which is possible only if $M_{i-1}/M_i \to 0$ and, by Lemma 7, 1), (5.33) gives

$$\delta_{i-1}^3\ M_{i-1}^{1/3}\ M_i^{2/3} \to \infty. \tag{5.45}$$

From (5.36), using Lemma 7, 2), we get

$$\delta_{i-1}^3\ M_{i-1}^{1/3}\ M_i^{2/3}\ \frac{(h-\eta_{i-1})^2}{\delta_{i-1}^2} \to 0.$$

Because of (5.45) this yields $\eta_{i-1} \to h$ and $\delta_{i-1} \geq \text{const} > 0$. The counterpart of (5.34) obtained from $D_{i+1} \to 0$ implies via Lemma 6, 1)

$$\delta_{i-1}\ \eta_{i-1}^2\ M_{i-1}^{1/3}\ M_i^{2/3} \to 0$$

contradicting to (5.45).

2b) Let $M_{i-1} \to \infty$.

2ba) Consider the case $M_i \leq \text{const}$, then $M_{i-1}/M_i \to \infty$. Lemma 7, 2) gives

$$\varphi_B\left(\left(\frac{M_{i-1}}{M_i}\right)^{1/3},\ \frac{\varepsilon_{i-1}}{\eta_{i-1}},\ \frac{h}{\delta_{i-1}}\right) \sim \frac{1}{2}\ \frac{(h-\varepsilon_{i-1})^2}{\delta_{i-1}^2\left(\frac{M_{i-1}}{M_i}\right)^{1/3}}$$

and (5.33) then yields

$$\delta_{i-1}\ M_{i-1}^{2/3}\ M_i^{1/3} \to \infty. \tag{5.46}$$

On the other hand, in (5.36)

$$\varphi_B\left(\left(\frac{M_i}{M_{i-1}}\right)^{1/3},\ \frac{\eta_{i-1}}{\varepsilon_{i-1}},\ \frac{h}{\delta_{i-1}}\right) \sim\ \frac{h\varepsilon_{i-1}+\frac{1}{2}h^2-\varepsilon_{i-1}^2}{\delta_{i-1}^2\left(\frac{M_i}{M_{i-1}}\right)^{2/3}}$$

which gives

$$\delta_{i-1}\ M_{i-1}^{2/3}\ M_i^{1/3} \to 0$$

contradicting to (5.46).

2bb) Suppose $M_i \to \infty$.

2bba) Let $0 < c_1 \leq M_i/M_{i-1} \leq c_2$. Then (5.33) implies $\delta_{i-1}^3\ M_{i-1} \to \infty$ and, consequently, $\delta_{i-1}^3\ M_i \to \infty$. But (5.36) gives $\delta_{i-1}^3\ M_i \to 0$, contradiction.

2bbb) Let $M_{i-1}/M_i \to \infty$. This case leads to a contradiction as 2ba).

2bbc) Let $M_{i-1}/M_i \to 0$. Now, similarly to 2ba), (5.33) leads to $\delta_{i-1}\ M_{i-1}^{1/3}\ M_i^{2/3} \to 0$ and (5.36) to $\delta_{i-1}M_{i-1}^{1/3}\ M_i^{2/3} \to \infty$.

We see that in all cases the existence of appropriate knots ξ_i leads to a contradiction. With this we proved

Theorem 2. *There is a strictly convex histogram on uniform mesh without any possibility to choose spline knots so that quadratic/linear spline histopolant exists.*

Remark 3. In [25] are given sufficient conditions for existence of the solution at quadratic/linear rational spline histopolation. These conditions seem to be quite restrictive to the quotients D_{i-1}/D_i. Heuristically the proof of Theorem 2 suggests that if the solution exists then these quotients cannot be very small or great. This is confirmed to a certain extent by the numerical experiments presented in the following section.

Let us conclude the presentation of theoretical results with the section

Main results

At cubic spline histopolation the existence of solution for the problem is described in Proposition 3.

Results about periodic spline histopolation are given in Propositions $4-7$ with Corollary 3 and summarized in Theorem 1.

Quadratic/linear rational spline histopolation defines histopolation functions and their properties are presented in Lemmas $4-7$. Theorem 2 formulates the main results about non-existence of the histopolant.

Chapter 6

Numerical results

6.1 Tests in Chapter 3

We histopolated the function $f(x) = 1/x^2$, $x \in [-2, -0.1]$, on uniform grid for $n = 8$ and central spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, n-1$. Histogram heights were computed as $z_i = \dfrac{1}{h} \displaystyle\int_{x_{i-1}}^{x_i} f(x)dx$, $i = 1, \ldots, n$. Resulting histopolants S are given in Figures 2–4:

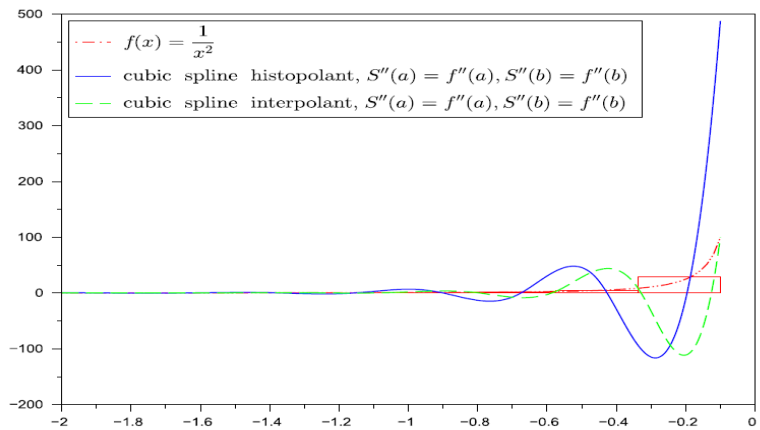

Figure 2: Cubic spline histopolant and interpolant for n = 8

In Figure 2 the histopolant with boundary conditions $S''(a) = f''(a)$, $S''(b) = f''(b)$; in Figure 3 the histopolant with boundary conditions $S'(a) = f'(a)$, $S'(b) = f'(b)$; in Figure 4 the histopolant with boundary conditions $S(a) = f(a)$, $S(b) = f(b)$. In comparison also cubic spline interpolants are given satisfying interpolation conditions $S(x_i) = f(x_i)$, $i = 0, \ldots, n$.

Considering the representation used in Section 3.6 we tested the dependence of eigenvalues of matrix $A^{-1}BC^{-1}D$ on grid points x_i and spline knots ξ_i. Again the case $n = 8$ is analysed for different meshes.

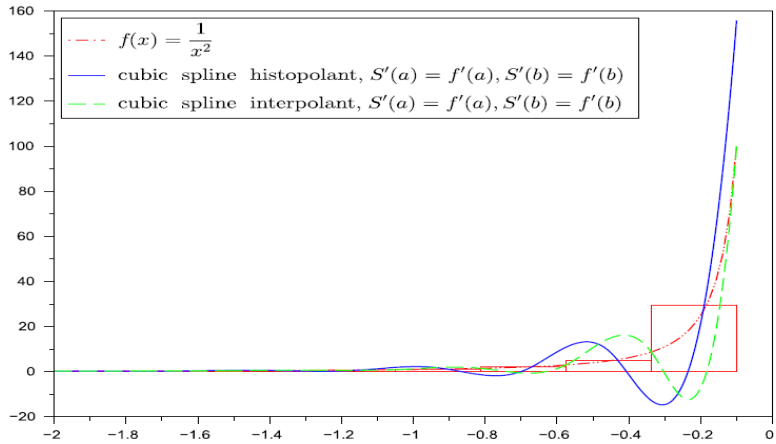

Figure 3: Cubic spline histopolant and interpolant for n = 8

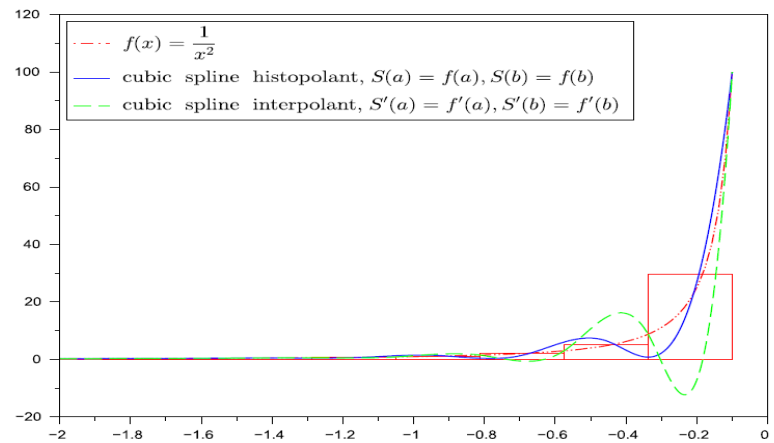

Figure 4: Cubic spline histopolant and interpolant for n = 8

1) Uniform grid $x_i = a + ih$, $i = 0, \ldots, n$, and spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, n-1$, give the maximal by modulus eigenvalue $|\lambda_{\max}| = 0.271$.

2) For uniform histogram grid $x_i = a + ih$, $i = 0, \ldots, n$, and spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, 3, 6, 7$, $\xi_4 = 0.1x_3 + 0.9x_4$, $\xi_5 = 0.9x_4 + 0.1x_5$ (ξ_4 and ξ_5 are close to x_4) it holds $|\lambda_{\max}| = 2.388$.

3) For uniform grid $x_i = a + ih$, $i = 0, \ldots, n$, spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, 3, 6, 7$, $\xi_4 = 0.9x_3 + 0.1x_4$, $\xi_5 = 0.1x_4 + 0.9x_5$ (ξ_4 and ξ_5 are close to x_3 and x_5, respectively) it holds $|\lambda_{\max}| = 0.803$.

4) Take $h = (b-a)/n$, $h_i = 0.1h$, $i = 1, 3, 5, 7$, $h_i = 1.9h$, $i = 2, 4, 6, 8$, and central spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, n-1$, then $|\lambda_{\max}| = 0.241$.

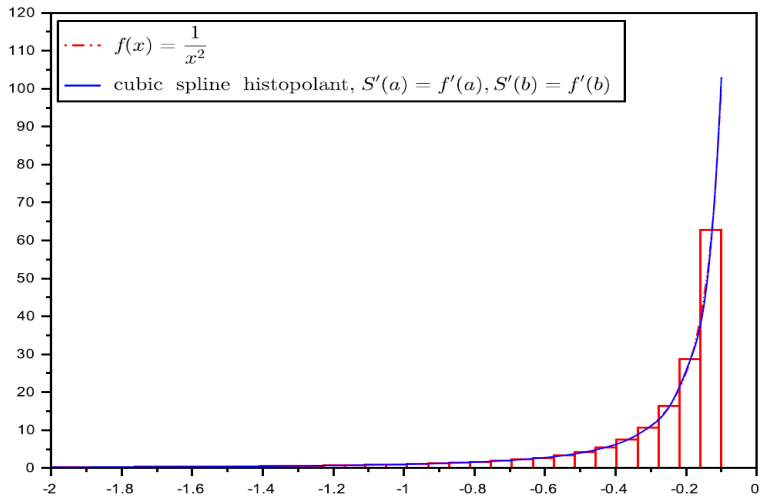

Figure 5: Cubic spline histopolant for n = 32

Construction of histopolating cubic spline could be done using second derivatives M_i and particular integrals λ_i, ρ_i. The crucial moment here is the solution of system (3.20). In case of diagonal dominance in the matrix of (3.20) standard methods (e.g., Gaussian elimination) are stable. In absence of diagonal dominance it may be that other methods should be applied. One way to continue is to solve the system determining parameters λ_i, ρ_i. Another natural way is to solve system (3.26) where the matrix C has diagonal dominance. An opportunity is to use an iteration process described in Section 3.6 to determine either second derivatives or spline values and then the others by (3.23) or (3.26) with a matrix having diagonal dominance. We have seen above in this section that the convergence may be slow or be absent at all. In return, in the presence of convergence, the calculations at interation are stable.

Numerical tests with the function $1/x^2$ confirmed the known fact that polynomial splines (at interpolation or histopolation) do not preserve geometrical properties like positivity, monotonicity, convexity. However, increasing the number n of knots, the cubic spline histopolant occurs to have these properties because of the uniform convergence of values, first and second derivatives (see Figure 5). For the cubic spline histopolant this follows from the uniform convergence (see, e.g., [1]) of first, second and third derivatives of interpolating quartic splines in equivalent problem as described in Chapter 1.

6.2 Tests in Chapter 4

We tested the case $m = 3$ (cubic splines) with $n = 4$ to check the validity of Conjecture. For arbitrary values of $h_i = 1, \ldots, 4$, Taylor representation was used. The symbolic computations showed that, in this case, the Conjecture is true.

6.3 Tests in Chapter 5

In this section we illustrate obtained theoretical results with particular examples. All histograms are based on uniform mesh in the interval $[a, b]$ with $a = -2$, $b = 2$, $h = (b-a)/8$, $x_i = a+ih$, $i = 0, \ldots, 8$. The boundary conditions are given by fixed $M_1 = S''(a) = 1$ and $M_8 = S''(b) = 1$. In addition, we fix $z_1 = z_8 = 1$. Assume that there are given $D_2, \ldots, D_7$ corresponding to the interior knots $\xi_2, \ldots, \xi_7$ of the quadratic/linear rational spline S. In each example we indicated how the knots ξ_i and values D_i are chosen, however, except the common choice of $\xi_0 = a$ and $\xi_8 = b$.

In tests we used Matlab software at solving the system of basic equations and also at solving the linear system to determine the parameters λ_i, ρ_i.

Remark 4. For chosen knots x_i, ξ_i and values D_i, M_1, M_n we may form the system of basic equations. If this has the solution then solving an appropriate linear system (using z_1, z_n), see [25], we determine the parameters λ_i, ρ_i. Now (5.15) allows to calculate the values z_i and get all initial data needed to pose the histopolation problem with quadratic/linear rational splines.

Example 1. We take
$\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, 7$, $D_2 = D_3 = D_5 = D_6 = D_7 = h^3$, $D_4 = ch^3$, with a parameter $c > 0$. The influence of c to the values M_i is reflected by following results.

Table 1 Results for $c = 0.5$ in Example 1.

i	1	2	3	4	5	6	7	8
D_i		1/8	1/8	1/16	1/8	1/8	1/8	
M_i	1	0.373	0.860	0.101	0.807	0.441	0.448	1

Table 2 Results for $c = 2$ in Example 1.

i	1	2	3	4	5	6	7	8
D_i		1/8	1/8	1/4	1/8	1/8	1/8	
M_i	1	0.506	0.262	2.338	0.244	0.668	0.401	1

Table 3 Results for $c = 5.6$ in Example 1.

i	1	2	3	4	5	6	7	8
D_i		1/8	1/8	7/10	1/8	1/8	1/8	
M_i	1	0.918	0.0005	379.97	0.0002	1.887	0.298	1

The used Matlab software could not solve the system of basic equations for $c = 5.7$.

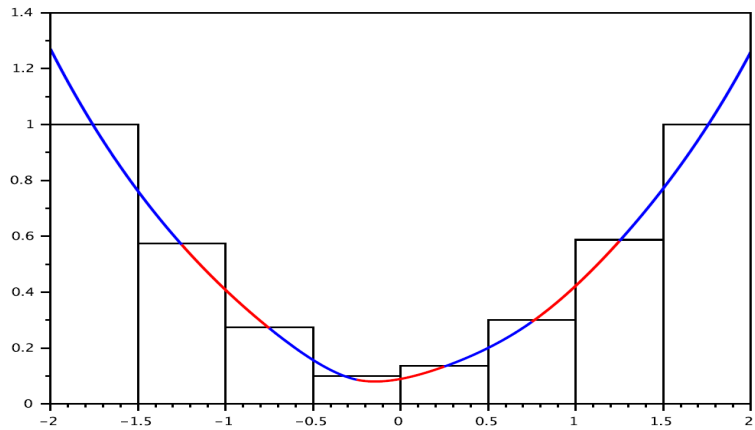

Figure 6: Histogram and histopolant for $c = 1.7$ in Example 1.

Additional tests show that for $c = 3$ we get $z_4,\ z_5 < 0$; for $c = 4$ $z_3,\ z_4,\ z_5 < 0$; for $c = 5$ and $c = 5.6$ $z_2,\ z_3,\ z_4,\ z_5 < 0$.

We see that the influence of c to the histopolant is remarkable.

Remark 5. The system consisting of basic equations and two boundary conditions is homogeneous in the sense that if we multiply all values z_i and data α, β in boundary conditions by a number $\lambda \neq 0$ then the solution of obtained system is the multiple by λ of the solution for initial system. Thus, considering $c \to \infty$ with $D_i = c^{1/2}h^3$, $D_i = h^3/c^{1/2}$, $i = 2, 3, 5, 6, 7$, $z_1 = z_2 = 1/c^{1/2}$, we arrive to the case $D_4 \to \infty$, $D_3 \to 0$, $D_2 \to 0$, $D_5 \to 0$, $D_6 \to 0$, which was the starting assumption in the proof of Theorem 2. So the numerical results are consistent with the theoretical results about the existence of solution.

Example 2. Let $D_2 = h^3$, $D_3 = h^3/c$, $D_4 = ch^3$, $D_5 = h^3/c$, $D_6 = D_7 = h^3$ with $c = 1.7$. Take first $\xi_i = (x_{i-1} + x_i)/2,\ i = 2, \ldots, 7$. The results are in Table 4 and in Figure 7.

Table 4 Results for $\xi_i = (x_{i-1} + x_i)/2$ in Example 2.

i	1	2	3	4	5	6	7	8
ε_i/η_i	2	1	1	1	1	1	1/2	
M_i	1	0.653	0.063	4.169	0.055	0.994	0.358	1

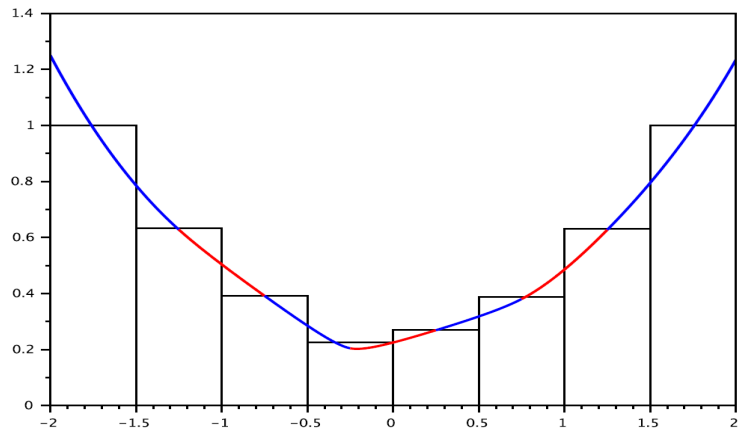

Figure 7: Histogram and histopolant in Example 2 for Table 4.

It is natural to ask how the replacement of knots ξ_i influence the histopolant. Let us shift 1) ξ_3 and ξ_5 closer to ξ_4 ($\eta_2/\varepsilon_3 = 9,\ \eta_4/\varepsilon_5 = 1/9$); 2) ξ_2, ξ_3, ξ_5 and ξ_6 farther from ξ_4 ($\eta_1/\varepsilon_2 = 1/9,\ \eta_2/\varepsilon_3 = 1/99,\ \eta_4/\varepsilon_5 = 99,\ \eta_5/\varepsilon_6 = 9$).

Table 5 Results where ξ_3 and ξ_5 are shifted closer to ξ_4 in Example 2.

i	1	2	3	4	5	6	7	8
ε_i/η_i	2	0.556	0.2	5	1.8	1	0.5	
M_i	1	0.605	0.065	8.309	0.053	0.878	0.371	1

Table 6 Results where ξ_2, ξ_3, ξ_5 and ξ_6 are shifted farther from ξ_4 in Example 2.

i	1	2	3	4	5	6	7	8
ε_i/η_i	10	90	1.98	0.505	0.011	0.2	0.5	
M_i	1	1.396	0.059	2.669	0.057	2.606	0.179	1

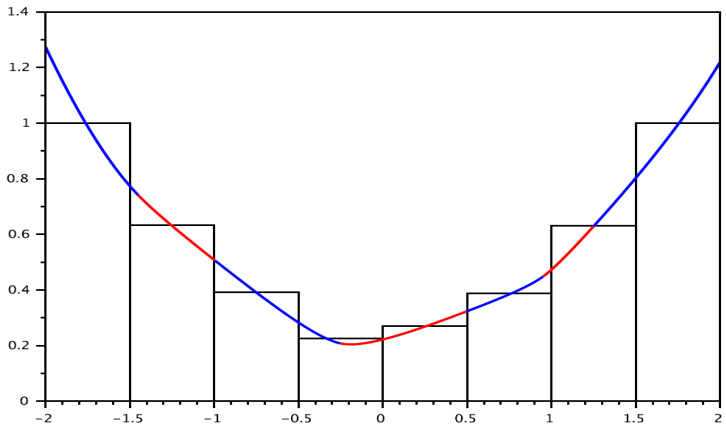

Figure 8: Histogram and histopolant in Example 2 for Table 6.

We see by the presented results that the replacement of knots ξ_i has relatively mild influence to the histopolant. This was confirmed also by our other numerous tests.

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Summary in Estonian

Splainidega lähendamine

Interpoleerimine on üks tähtsamatest vahenditest funktsioonide ligikaudseks taastamiseks diskreetse andmestiku korral. Ajalooliselt üks esimesi lähendavaid funktsioone olid polünoomid. Splainide kui tükiti polünoomide uurimise alguseks loetakse 20. sajandi keskpaika. Esimeseks tähtsamaks tööks loetakse Schönbergi 1946. aastal avaldatud artiklit [68]. Tükiti polünomiaalsete funktsioonide kasutamine lähenditena oli edukas inseneriteadustes, samuti majanduses, füüsikas, statistikas. Nende funktsioonidega on lihtne arvutada, samuti on neid lihtne salvestada arvuti mälus. Üks tähtsamaid rakendusi on siin kõverate ja pindade konstrueerimine näteks auto-ja lennukitööstuses.

Dissertatsioonis käsitletakse põhiliselt histopoleerimisülesandeid. Võrreldes interpoleerimisülesannetega on histopoleerimisülesanded tihti palju praktilisemad, sest näiteks statistiline informatsioon on enamasti antud histogrammidena.

On teada, et antud histopoleerimisülesande korral saab moodustada samal võrgul määratud interpoleerimisülesande, kusjuures interpoleerimisülesande lahendi tuletis on histopoleerimisülesande lahend. Ka vastupidi, antud interpoleerimisülesande korral saab püstitada sama võrguga histopoleerimisülesande ning saadud histopoleerimisülesande lahendist võetud integraal on interpoleerimisülesande lahend. Seega on ühte tüüpi ülesande uurimisel saadud tulemustest kohe leitavad ka teist tüüpi ülesande kohta käivad tulemused. Selline käsitlus on sobiv polünomiaalsete splainide uurimisel, aga mitte ratsionaalsplainide korral. Näiteks ei ole ruut/lineaar ratsionaalsplaini tuletis lineaar/lineaar ratsionaalsplain, samuti ei ole lineaar/lineaar ratsionaalsplaini integraal ruut/lineaar ratsionaalsplain. Niisiis, kui on olemas sobiv interpoleerimise teooria ratsionaalsplainide jaoks, tuleb ikkagi käsitleda eraldi ratsionaalsplainidega histopoleerimist. Eelkõige just sellepärast on käesolevas dissertatsioonis uuritud ratsionaalsplainidega histopoleerimist.

Dissertatsiooni põhimaterjal on esitatud kuues peatükis.

Esimeses peatükis, mis on sissejuhatus, antakse ülevaade antud temaatikaga seotud töödest. Puudutatakse interpoleerimist ja histopoleerimist splainidega, samuti diferentsiaalvõrrandite lahendamist splainide abil. Viimasel juhul on tegemist rajaülesannete korral projektsioonimeetoditega, kus kasutatakse interpolatiooniprojektoreid, mis vastavad kollokatsioonimeetodile, või histopolatsiooniprojektoreid, mis vastavad osapiirkondade meetodile. Üksikasjalikult on käsitletud interpoleerimise ja histopoleerimise vahekorda.

Peatükis 2 tuuakse lineaar/lineaar ja ruut/lineaar ratsionaalsplaini mõisted. Tõestatakse tulemused histogrammi monotoonsuse ja kumeruse kohta sõltuvalt histogrammi tekitava funktsiooni omadustest.

Peatükis 3 vaadeldakse kuupsplainidega histopoleerimist. Seda ei taandata kvartsplainidega interpoleerimise teooriale, seepärast võib sellist käsitlust nimetada ka ilmutatud teooriaks. Splain esitatakse teiste momentide ehk teiste tuletiste abil splaini sõlmedes. Lisaks on veel esitusparameetriteks integraalid splainist osalõikude osadel. Osaintegraalide elimineerimisel võrranditest, mis määravad histopoleerimisülesande lahendi, saadakse teiste momentide suhtes lineaarne süsteem. Olenevalt splaini sõlmede valikust (need asuvad histogrammi sõlmede vahel) võib selle süsteemi maatriksi peadiagonaal alla domineeriv ridade kaupa või mitte. Sellele vaatamata on histopoleerimisülesanne alati üheselt lahenduv. On käsitletud ka teist esitust teiste momentide ja splaini väärtuste kaudu sõlmedes. Vastavate süsteemide lahendamiseks sobib interatsioonimeetod , mille koondumine ja selle kiirus sõltub splaini sõlmede valikust . Peatüki tulemused on publitseeritud artiklis [33].

Peatükis 4 vaadeldakse suvalist järku polünoomiaalsete perioodiliste splainidega histopoleerimist. Splaini ja histogrammi sõlmed langevad kokku, aga need võivad paikneda suvaliselt. Peamine probleem, mida lahendatakse, on histopolatsiooniülesande lahendi olemasolu ja ühesus. Varasemates töödes on sellele antud terviklik vastus ühtlase võrgu korral. Need kõik järelduvad selles peatükis esitatud tulemustest. On tõestatud, et lahend on olemas ja ühene, kui 1) splaini aste on paarisarv ; 2) splaini aste ja osalõikude arv on mõlemad paaritud. On näidatud, et lahend ei ole ühene (kui eksisteerib) või ei eksisteeri, kui 1) splaini aste on 1 ja osalõikude arv on paaris; 2) splaini aste on paaritu arv ja osalõikude arv on 2. Esitatakse hüpotees, et lahend ei ole ühene või ei eksisteeri, kui splaini aste on paaritu arv ja osalõikude arv on paarisarv. Näidatakse, et kui lahend ei ole ühene, siis histopolatsioonioperaatori tuum on ühedimensionaalne. Peatüki tulemused on publitseeritud artiklis [56].

Peatükk 5 on pühendatud ruut/lineaar ratsionaalsplainidega histopoleerimisele. Ruut/ lineaar ratsionaalsplainid klassist C^2 on rangelt kumerad või rangelt nõgusad. Seepärast on loomulik eeldada, et histopoleerimisülesandes antud histogramm on rangelt kumer . Splaini sõlmed valitakse histogrammi sõlmede vahel, selline oli olukord ka kuupsplainidega histopoleerimisel peatükis 3. Ka siin nagu eelmises peatükis on põhiprobleemiks lahendi olemasolu küsimus. Vabadus splaini sõlmede valikus näib suure paindlikkusena, kuid selgus, et on olemas rangelt kumer histogramm, kus histopoleerimisülesandel ei ole lahendit ühegi splaini sõlmede valiku korral. Selle peatüki tulemuste saamisel taandati probleem nn. baasvõrrandite mittelineaarse süsteemi lahendi olemasolule.

Selleks tuli põhjalikult uurida baasvõrrandite süsteemis esinevate histopoleerimisfunktsinoonide käitumist, milles ilmnes mõndagi ootamatut. Peatüki 5 tulemused on vormistatud artikliks [34] ning esitatud publitseerimiseks.

Peatükis 6 on toodud eelnevates peatükkides käsitletud ülesannete näiteid, mille kohta on arvutil tehtud testid. Peatükis 4 toodud hüpoteesi on kontrollitud kuupsplainide korral, kus osalõikude arv on 4. Peatükkide 3 ja 5 kohta tehtud testid on illustreerivad ja kooskõlas teoreetliste tulemustega.

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22^{nd} International Conference on Mathematical Modelling and Analysis, May 30-June 2, 2017, Druskininkai, Lithuania,

23^{rd} International Conference on Mathematical Modelling and Analysis May 29-June 1, 2018, Sigulda, Latvia,

Workshop "Series of lectures on waves and imaging (I)", June 14-15, 2018, ETH Zurich, Switzerland,

4^{th} International conference on Optimization and Analysis of Structure, August 21-23, 2018, Tartu Estonia.

Elulookirjeldus

Gul Wali Shah

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Haridus

1984 – 1997	Põhi -ja keskkool, Jamrud, Khyber
1997 – 1999	Gümnaasium, Degree College, Landi Kotal
2000 – 2002	Kraadiõppe Kolledž, Landi Kotal baccalaureuse kraad matemaatikas ja statistikas
2002 – 2005	Matemaatika osakond, Peshawari ülikool magistrikraad
2012 – 2014	Baasteaduste teaduskond, Peshawar magistrikraad rakendusmatemaatikas
2015 – 2019	Tartu Ülikool, matemaatika ja statistika instituut doktoriõppe üliõpilane matemaatika erialal

Erialane koolitus

2009 – 2012	Juhtimisteaduste instituut, Hayatabad, Peshawar magister ärijuhtimises
2013 – 2014	Allama Iqbali avatud ülikool, Islamabad bakalaureus kasvatusteadustes

Teenistuskäik

2005 – 2011 Kraadiõppe Kolledž, Jamrud
matemaatika õpetaja

2011 – 2012 Kraadiõppe Juhtimisteaduste Kolledž, Jamrud
matemaatika õpetaja

2012 – 2013 arvuti -ja juhtimisteaduste instituut, Hayatabd
matemaatika õpetaja

Feb–Jun 2018 Tartu Ülikooli, arvutiteaduse instituut
matemaatika õpetaja

Teadustegevus

Peamine uurimisvaldkond on splainidega lähendusmeetodid. Tulemused dissertatsiooni teemal on ilmunud kahes teadusartiklis ja üks artikkel on esitatud publitseerimiseks. varem on ilmunud artikkel teisest valdkonnast konverentsikogumikus. Võtnud osa ja esinenud ettekandega järgmistel konverentsidel:

"13th International Conference on Statistical Science" March 16-18, 2015, Peshawar, Pakistan,

"21st International Conference on Mathematical Modelling and Analysis" June 1-4, 2016, Tartu, Estonia,

"22nd International Conference on Mathematical Modelling and Analysis" May 30-June 2, 2017, Druskininkai, Lithuania,

"23rd International Conference on Mathematical Modelling and Analysis" May 29-June 1, 2018, Sigulda, Latvia,

Workshop "Series of lectures on waves and imaging (I)" June 14-15, 2018, ETH Zurich, Switzerland,

"4th International conference on Optimization and Analysis of Structure" August 21-23, 2018, Tartu Estonia.

List of Publications

1) E. Kirsiaed, P. Oja, and G. W. Shah, Cubic spline histopolation. Math. Model. Anal., **22**(2017), 514-527.

2) P. Oja and G. W. Shah, Periodic polynomial spline histopolation, Proc.Est.Acad. Sci., **67**(2018), 246-251.

3) E. Kirsiaed, P. Oja, and G. W. Shah, Rational spline histopolation of convex data (submitted).

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

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