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# $\lambda$-compact operators as a surjective hull of certain nuclear operators

Mathematics and Statistics Curriculum
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# $\lambda$-kompaktsed operaatorid kui teatud tuumaoperaatorite sürjektiivne kate

Magistritöö

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**Lühikokkuvõte.** Magistritöös esitame jadaruumide kolmiku poolt defineeritud tuumaoperaatorite mõiste, mis üldistab nii Pietschi $(r,p,q)$-tuumaoperaatoreid [Pie80, Chapter 18] kui ka Ramanujani $\lambda$-tuumaoperaatoreid [Ram70]. See võimaldab meil kirjeldada Bhari ja Gupta poolt tutvustatud $\lambda$-kompaktseid operaatoreid [GB13] kui teatud kvaasi-Banachi operaatorideaali sürjektiivset katet (järgides [ALO12] ja [Pie14] ideid). Tänu sellele saame näidata, et $\lambda$-kompaktsed operaatorid moodustavad kvaasi-Banachi operaatorideaali, kui jadaruum $\lambda$ rahuldab teatud tagasihoidlikke eeldusi.

**CERCS teaduseriala:** P140 Jadad, Fourier analüüs, funktsionaalanalüüs.

**Märksõnad:** tuumaoperaatorid, kompaktsed operaatorid, jadaruumid, operaatorideaalid.

# $\lambda$-compact operators as a surjective hull of certain nuclear operators

Master's thesis

Triin Taveter

**Abstract.** In this master's thesis we introduce nuclear operators defined by a triplet of sequence spaces extending both the $(r,p,q)$-nuclear operators of Pietsch [Pie80, Chapter 18] as well as $\lambda$-nuclear operators of Ramanujan [Ram70]. This enables us to represent $\lambda$-compact operators introduced by Bhar and Gupta [GB13] as the surjective hull of a certain quasi-Banach operator ideal (following the ideas of [ALO12] and [Pie14]). This, in particular, yields that $\lambda$-compact operators themselves form a quasi-Banach operator ideal given some modest requirements on the sequence space $\lambda$.

**CERCS research specialisation:** P140 Series, Fourier analysis, functional analysis.

**Key words:** nuclear operators, compact operators, sequence spaces, operator ideals.

# Contents

# Introduction

The research field of this master's thesis is functional analysis. In the thesis, we study sequence spaces and certain nuclear and compact operators defined by them.

The theory of nuclear operators was created by Grothendieck in 1953 when he first described $p$-nuclear operators ($0 < p \leq 1$). In the 1970s, Pietsch presented the definition of $(r, p, q)$-nuclear operators and developed their theory, among other things proving that they form a quasi-Banach operator ideal.

In the year 2002 Sinha and Karn introduced $p$-compact operators [SK02]. This was followed by studies about $p$-compact operators by many authors. In [ALO12], published in 2012, Ain, Lillemets and Oja showed that $p$-compact operators can be studied in the context of previously known theory of nuclear operators, because the class of $p$-compact operators is (in the sense of quasi-Banach operator ideals) equal to the surjective hull of $(p, 1, p)$-nuclear operators, meaning $\mathcal{K}_p = \mathcal{N}^{\mathrm{sur}}_{(p,1,p)}$. This was also shown by Pietsch in 2104 [Pie14].

In 1970, M. S. Ramanujan introduced $\lambda$-nuclear operators (for a sequence space $\lambda \subset \ell_\infty$) as an extension of $p$-nuclear operators [Ram70]. In 2013, Bhar and Gupta, introduced $\lambda$-compact operators [GB13]. In 2017, Bhar and Karn [BK18] proved that the dual of the operator ideal of $\lambda$-compact operators $\mathcal{K}^d_\lambda = \{T : T^* \in \mathcal{K}_\lambda\}$ is (in the sense of quasi-Banach operator ideals) equal to the injective hull of $\lambda$-nuclear operators, meaning $\mathcal{K}^d_\lambda = \mathcal{N}^{\mathrm{inj}}_\lambda$. Therefore, $\mathcal{K}^d_\lambda$ is a quasi-Banachi operator ideal. So far, it has not been known, whether $\mathcal{K}_\lambda$ is a quasi-Banachi operator ideal.

These results give a reason to ask, if there exist sequence spaces $(\lambda, \mu, \nu)$ such that for suitably defined $(\lambda, \mu, \nu)$-nuclear operators, the equality $\mathcal{K}_\lambda = \mathcal{N}^{\mathrm{sur}}_{(\lambda,\mu,\nu)}$ would hold.

This thesis consists of four sections. In the first section we present preliminaries about topological vector spaces, operator ideals and sequence spaces. In the second part, we introduce $(\lambda, \mu, \nu)$-nuclear operators and shown that they form a quasi-Banach operator ideal. In the third section we present the factoring of nuclear operators.

In the last section we introduce $(\lambda, \mu)$-compact operators extending both $(p, r)$-compact operators introduced by Ain, Lillemets and Oja [ALO12] and $\lambda$-compact operators introduced by Bhar and Gupta [GB13]. We will show that $(\lambda, \mu)$-compact operators are (in the sense of quasi-Banach operator ideals) equal to the surjective hull of certain nuclear operators defined by a triplet of sequence spaces $(\lambda, \mu, \ell_\infty)$. This enables us to represent $\lambda$-compact operators as the surjective hull of a certain quasi-Banach operator ideal (following the ideas of [ALO12] and [Pie14]). This, in particular, yields that $\lambda$-compact operators themselves form a quasi-Banach operator ideal given some modest requirements on the sequence space $\lambda$.

We consider vector spaces over a field $\mathbb{K}$, which denotes either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Given vector spaces $X$ and $Y$, we denote by $L(X, Y)$ the collection of all linear operators from $X$ to $Y$. If $X$ and $Y$ are topological vector spaces, then $\mathcal{L}(X, Y)$ denotes the space of all continuous and linear operators from $X$ to $Y$. Symbol $I_X$ denotes the identity operator of a vector space $X$. By $B_X$ we denote the closed unit ball of a normed or quasi-normed space $X$. The dual space of $X$ is denoted by $X^*$.

# 1 Preliminaries

We will assume that the reader is familiar with the main definitions and results from the theories of functional analysis and topological vector spaces. We will, however, recall some particular facts and definitions from them. In the last subsection we will touch on the basics about general sequence spaces.

## 1.1 About topological vector spaces

In this section we present some definitions about topological vector spaces that we will use in the following sections. These and other definitions about topological vector spaces can be found in [Koe69]. By $\mathbb{R}^+$ we denote the set of all non-negative real numbers.

**Definition 1.1.** Let $X$ be a vector space. A mapping $q : X \to \mathbb{R}^+$ is a *quasi-norm*, if

1. $q(x) = 0 \Rightarrow x = 0$ for each $x \in X$,
2. $q(kx) = |k|q(x)$ for each $k \in \mathbb{K}, x \in X$ (absolute homogeneity),
3. there exists a constant $\varkappa \geq 1$ such that $q(x + y) \leq \varkappa(q(x) + q(y))$ for each $x, y \in X$.

A *quasi-normed space* is a vector space equipped with a quasi-norm. An extensive outline about quasi-normed spaces can be found in [KPR84].

**Definition 1.2.** Let $X$ be a topological vector space and $x^* \in X^*$. Then a functional $p_{x^*} : X \to \mathbb{R}$, defined as

$$p_{x^*}(x) = |x^*(x)|, \ x \in X,$$

is a seminorm. The *weak topology* (denoted by $w$) on $X$ is the locally convex topology defined by all the seminorms $p_{x^*}, x^* \in X^*$. This means that the base of neighbourhoods of zero is

$$\mathcal{B} = \{U_{\varepsilon,x_1^*,\ldots,x_n^*} | \varepsilon > 0, x_1^*, \ldots, x_n^* \in X^*, n \in \mathbb{N}\},$$

where

$$U_{\varepsilon,x_1^*,\ldots,x_n^*} = \{x \in X | \sup_{1 \leq i \leq n} p_{x_i^*}(x) < \varepsilon\}.$$

The *weak-* topology* for the dual space $X^*$ (denoted by $w^*$) is defined in a dual way, using seminorms $p_x$, instead of $p_{x^*}$, defined as $p_x(x^*) = |x^*(x)|$, for all $x^* \in X^*$.

**Definition 1.3.** A *Fréchet space* is a complete metrizable locally convex space.

**Proposition 1.4** ([Con85, Chapter IV, Prop 2.1])**.** *A locally convex space is a Fréchet space if and only if its topology is defined by a countable family of seminorms.*

## 1.2 About operator ideals

In this section we present some notions from the theory of operator ideals. We use the definitons and notations from the extensive monograph [Pie80] and article [Ste80]. Let $\mathcal{L}$ denote the class of all linear continuous operators between any two Banach spaces. In the following definitions, $X, Y, W, V$ run through the class of all Banach spaces.

**Definition 1.5.** An *operator ideal* is a subclass $\mathcal{A}$ of class $\mathcal{L}$ such that its *components* $\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$ satisfy the following conditions:

1. $I_{\mathbb{K}} \in \mathcal{A}(\mathbb{K},\mathbb{K})$.

2. If $S_1, S_2 \in \mathcal{A}(X,Y)$, then $S_1 + S_2 \in \mathcal{A}(X,Y)$.

3. If $T \in \mathcal{L}(W,X),\ S \in \mathcal{A}(X,Y)$ and $R \in \mathcal{L}(Y,V)$, then $RST \in \mathcal{A}(W,V)$.

**Definition 1.6.** Let $\mathcal{A}$ be an operator ideal. A mapping $A : \mathcal{A} \to \mathbb{R}^+$ is a *quasi-norm (on an operator ideal)* if:

1. $A(I_{\mathbb{K}}) = 1$.

2. There exists a constant $\varkappa \geq 1$ such that

$$A(S_1 + S_2) \leq \varkappa[A(S_1) + A(S_2)] \quad \forall S_1, S_2 \in \mathcal{A}(X,Y).$$

3. If $T \in \mathcal{L}(W,X),\ S \in \mathcal{A}(X,Y)$ and $R \in \mathcal{L}(Y,V)$, then

$$A(RST) \leq \|R\| A(S) \|T\|.$$

In that case $(\mathcal{A}, A)$ is a *quasi-normed operator ideal.*

**Definition 1.7.** A *quasi-Banach operator ideal* $(\mathcal{A}, A)$ is an operator ideal $\mathcal{A}$ equipped with a quasi-norm $A$, such that all quasi-normed spaces $(\mathcal{A}(X,Y), A)$ are complete.

**Definition 1.8.** The natural surjection $Q_X : \ell_1(B_X) \to X$ is defined as

$$Q_X(a_x)_{x\in B_X} = \sum_{x\in B_X} a_x x, \quad (a_x)_{x\in B_X} \in \ell_1(B_X).$$

**Definition 1.9.** The *surjective hull* of an operator ideal $\mathcal{A}$ is $\mathcal{A}^{\text{sur}}$ and its components are $\mathcal{A}^{\text{sur}}(X,Y) = \{T \in \mathcal{L}(X,Y) : TQ_X \in \mathcal{A}(\ell_1(B_X), Y)\}$.

**Definition 1.10.** An operator ideal $\mathcal{A}$ is *surjective* if $\mathcal{A} = \mathcal{A}^{\text{sur}}$.

We also recall the description of the surjective hull from [Ste80].

**Proposition 1.11** ([Ste80])**.** *The surjective hull $\mathcal{A}^{\text{sur}}$ of an operator ideal $\mathcal{A}$ is the class of operators $T$ which are subject to an inclusion $T(B_X) \subset S(B_Z)$, operator $S$ being an operator of type $\mathcal{A}$, say $S \in \mathcal{A}(Z,Y), T \in \mathcal{L}(X,Y)$.*

**Definition 1.12.** Let $(\mathcal{A}, A)$ be a quasi-normed operator ideal. Then $A^{\text{sur}}(T) = A(TQ_X)$ for any $T \in \mathcal{A}(X,Y)$ and $(\mathcal{A}^{\text{sur}}, A^{\text{sur}})$ is the *surjective hull* of $(\mathcal{A}, A)$.

**Proposition 1.13** ([Pie80, Prop. 8.5.2])**.** *Let $(\mathcal{A}, A)$ be a quasi-Banach operator ideal. Then $(\mathcal{A}^{\text{sur}}, A^{\text{sur}})$ is a quasi-Banach operator ideal.*

## 1.3 Sequence spaces

In this section we present some definitions and results about sequence spaces. Terminology presented in this section is based on [**Ruckle**].

Given a vector space $X$, we denote by $X^{\mathbb{N}}$ the vector space of all sequences in $X$, that is the vector space of all functions from $\mathbb{N}$ to $X$. We will interchangingly denote the elements of $X^{\mathbb{N}}$ as $\overline{x} = (x_n) = (x_n)_n$, where $x_n \in X$ for every $n \in \mathbb{N}$. Given $\overline{x} \in \mathbb{K}^{\mathbb{N}}$ and $\overline{y} \in X^{\mathbb{N}}$, the pointwise multiplication is denoted by $\overline{xy} := (x_n y_n) \in X^{\mathbb{N}}$. By a sequence space we mean a vector subspace of $\mathbb{K}^{\mathbb{N}}$.

Given $N \subset \mathbb{N}$, we denote by $\chi_N$ the characteristic function of $N$ and put $\chi_m := \chi_{\{m,m+1,\dots\}}$ for $m \in \mathbb{N}$. By $e$ we denote the sequence $e := \chi_{\mathbb{N}} = (1,1,1,\dots)$. By $e_n$ we denote the $n$-th coordinate sequence, namely $e_n := \chi_{\{n\}} = (\underbrace{0,\dots,0,1}_{n},0,\dots)$.

Given two subsets $A, B \subset \mathbb{K}^{\mathbb{N}}$, the product set is denoted by $AB = A \cdot B := \{\overline{xy} : \overline{x} \in A,\ \overline{y} \in B\}$. In the following we would like to relate the product of two quasi-normed sequence spaces to another one. Since the product of two sequence spaces may fail to be a sequence space, it will be convenient, for this relation, to consider a wider class of subsets of $\mathbb{K}^{\mathbb{N}}$, which is closed under taking product: class of sets, which are closed under multiplication by a scalar. Let us denote this class by $\mathcal{S}$. Namely, $\mathcal{S} = \{kA : k \in \mathbb{K}, A \subset \mathbb{K}^{\mathbb{N}}\}$. Given two sets $A, B \in \mathcal{S}$, equipped by absolutely homogeneous functions $p$ and $q$, respectively,

- put $(A,p) \leq (B,q)$, if $A \subset B$ and $p(\overline{x}) \geq q(\overline{x})$ for all $\overline{x} \in A$;
- denote $(A,p) \cdot (B,q) := (AB, pq)$ with $pq(\overline{c}) := \inf p(\overline{x})q(\overline{y})$, where infimum ranges over all representations $\overline{c} = \overline{xy}$ with $\overline{x} \in A$ and $\overline{y} \in B$.

Note that it is easy to verify that $pq$ above is absolutely homogeneous on $AB \in \mathcal{S}$.

Some of the properties usually defined for sequence spaces are also of use in a more general context of vector sequence spaces. Let us present the definitions in this context. In the following, let $X$ denote a Banach space.

**Definition 1.14.** A *vector sequence space* $\alpha(X)$ is a vector subspace of $X^{\mathbb{N}}$ for some Banach space $X$.

In case when $X = \mathbb{K}$, then a vector sequence space $\alpha(\mathbb{K})$ is a sequence space $\alpha$.

**Definition 1.15.** Given a vector sequence $\overline{x} \in X^{\mathbb{N}}$ and $N \subset \mathbb{N}$, the *$N$-section* of $\overline{x}$ is defined as $S_N(\overline{x}) := \chi_N \cdot \overline{x} \in X^{\mathbb{N}}$.

The usual pointwise order $\leq$ on $\mathbb{R}^{\mathbb{N}}$ induces a preorder on $X^{\mathbb{N}}$ via the mapping $j_s : \overline{x} \mapsto \|\cdot\| \circ \overline{x}$.

**Definition 1.16.** A vector sequence space $\alpha(X) \subset X^{\mathbb{N}}$ is *solid* (or *normal*) if $\overline{y} \in \alpha(X)$ whenever $\overline{y} \in X^{\mathbb{N}}$ and there exists $\overline{x} \in \alpha(X)$ such that $\|\cdot\| \circ \overline{y} \leq \|\cdot\| \circ \overline{x}$.

**Definition 1.17.** A vector sequence space $\alpha(X) \subset X^{\mathbb{N}}$ is *monotone* (or *sectionally regular*) if $S_N(\overline{x}) \in \alpha(X)$ for every $\overline{x} \in \alpha(X)$ and $N \subset \mathbb{N}$.

**Definition 1.18.** A vector sequence space $\alpha(X) \subset X^{\mathbb{N}}$ is *symmetric* if $\overline{x} \circ \pi \in \alpha(X)$ for every $\overline{x} \in \alpha(X)$ and every permutation $\pi$ of $\mathbb{N}$.

**Definition 1.19.** A sequence space $\lambda \subset K^{\mathbb{N}}$ with the property that the topology of $\lambda$ is equal or finer than the pointwise topology on $\mathbb{K}^{\mathbb{N}}$ is a *K-space.*

If $\lambda$ is a K-space, then for each $n \in \mathbb{N}$ the coordinate functional $e_n^* : \lambda \to \mathbb{K}$, defined by $e_n^*(\overline{x}) = x_n$, is continuous, meaning $e_n^* \in \lambda^*$.

**Definition 1.20.** A Fréchet space that is also a K-space is an *FK-space.*

The description of Fréchet spaces is in Proposition 1.4 provides the following example.

*Example* 1.21. The sequence space $\mathbb{K}^{\mathbb{N}}$, equipped with the topology of pointwise convergence, is an FK-space.

**Definition 1.22.** A Banach sequence space that is also a K-space is a *BK-space.*

**Definition 1.23.** A K-space $\alpha(X)$ with the property that for all $\overline{x} \in \alpha(X)$ and for all $n \in \mathbb{N}$, we have $\chi_{\{1,2,\dots,n\}} \cdot \overline{x} \in \alpha(X)$ and $\lim_{n\to\infty} \chi_{\{1,2,\dots,n\}} \cdot \overline{x} = \overline{x}$ (which is equivalent to $\lim_{n\to\infty} \chi_n \cdot \overline{x} = 0$) for all $\overline{x} \in \alpha(X)$ is an *AK-space.*

**Definition 1.24.** The *Koethe dual* (or *cross-dual*) of a sequence space $\lambda$ is

$$\lambda^{\times} = \left\{ \overline{x} \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |a_n x_n| < \infty \quad \forall \overline{a} \in \lambda \right\}.$$

**Definition 1.25.** A quasi-norm $q$ on a vector sequence space $\alpha(X) \subset X^{\mathbb{N}}$ is *monotone* if $q(\overline{y}) \leq q(\overline{x})$ whenever $\overline{x}, \overline{y} \in \alpha(X)$ are such that $\|\cdot\| \circ \overline{y} \leq \|\cdot\| \circ \overline{x}$.

**Definition 1.26.** A quasi-norm $q$ on a symmetric vector sequence space $\alpha(X) \subset X^{\mathbb{N}}$ is *K-symmetric* if $q(\overline{x} \circ \pi) = q(\overline{x})$ for any $\overline{x} \in \alpha(X)$ and any permutation $\pi$ of $\mathbb{N}$.

In the following, by $\phi(X) \subset X^{\mathbb{N}}$ we denote the sequence space of all sequences with finitely many non-zero coordinates. If $X = \mathbb{K}$, then

$$\phi = \operatorname{span}\{e_n \in \mathbb{K}^{\mathbb{N}} : n \in \mathbb{N}\}.$$

**Lemma 1.27.** *If a sequence space $\lambda$ is monotone, symmetric and non-zero, then $\phi \subset \lambda$.*

*Proof.* Let $\lambda$ be monotone, symmetric and non-zero. To see that $\phi \subset \lambda$, it is enough to show that for all $n \in \mathbb{N}$ we have $e_n \in \lambda$. Let $n \in \mathbb{N}$ and let $\overline{x} \in \lambda$ be such that $\overline{x} \neq 0$. Thus exists $i \in \mathbb{N}$ such that $x_i \neq 0$. Therefore $(1/x_i)\overline{x} \in \lambda$ is a sequence with element 1 at the $i$-th position. Since $\lambda$ is monotone, we can "make all other elements zero" in this sequence, so we have $e_i \in \lambda$. Since $\lambda$ is symmetric, we can "arrange" this element 1 in the sequence $e_i$ to any position, by some permutation. Therefore for all $n \in \mathbb{N}$ we have $e_n \in \lambda$. $\square$

We need the following lemma about a property of monotone and symmetric vector sequence space. For any vector sequence $\overline{x} = (x_n)$ we denote $j_1\overline{x} := (x_1, 0, x_2, 0, x_3, 0, \dots)$ and $j_2\overline{x} := (0, x_1, 0, x_2, 0, x_3, 0, \dots)$.

**Lemma 1.28.** *If a vector sequence space $\lambda(X)$ is monotone and symmetric, then $j_1\overline{x} + j_2\overline{y} \in \lambda(X)$ for any $\overline{x}, \overline{y} \in \lambda(X)$.*

*Proof.* Let a vector sequence space $\lambda(X)$ be monotone and symmetric and let $\overline{x}$, $\overline{y} \in \lambda(X)$. Since $\lambda(X)$ is monotone, we have

$$\overline{x}^1 = (x_1, 0, x_3, 0, \dots), \overline{x}^2 = (0, x_2, 0, x_4, \dots) \in \lambda(X).$$

We define $\pi_1, \pi_2$, permutations of $\mathbb{N}$, as follows:

$$\pi_1(n) = \begin{cases} 2n-1, & \text{if } n = 2k+1, k \in \mathbb{N} \cup \{0\}, \\ 2, & \text{if } n = 2, \\ \pi_1(n-2)+1, & \text{if } n = 2k, k \in \mathbb{N} \setminus \{1\}, n+1 \not\equiv 0 \pmod 3, \\ \pi_1(n-2)+2, & \text{if } n = 2k, k \in \mathbb{N} \setminus \{1\}, n+1 \equiv 0 \pmod 3, \end{cases}$$

$$\pi_2(n) = \begin{cases} 2n-1, & \text{if } n = 2k, k \in \mathbb{N}, \\ 1, & \text{if } n = 1, \\ \pi_2(n-2)+1, & \text{if } n = 2k+1, k \in \mathbb{N}, n+1 \not\equiv 0 \pmod 3, \\ \pi_2(n-2)+2, & \text{if } n = 2k+1, k \in \mathbb{N}, n+1 \equiv 0 \pmod 3. \end{cases}$$

Since the vector sequence space $\lambda(X)$ is symmetric, we have

$$\overline{x}^1 \circ \pi_1 = (x_1, 0, 0, 0, x_3, 0, 0, 0, x_5, \dots) \in \lambda(X),$$
$$\overline{x}^2 \circ \pi_2 = (0, 0, x_2, 0, 0, 0, x_4, 0, 0, 0, x_6, \dots) \in \lambda(X)$$

and therefore $\overline{x}^1 \circ \pi_1 + \overline{x}^1 \circ \pi_2 = j_1\overline{x} \in \lambda(X)$. Analogously $j_2\overline{y} \in \lambda(X)$ and in conclusion $j_1\overline{x} + j_2\overline{y} \in \lambda(X)$. □

**Definition 1.29.** Let $X$ be a Banach space and let $q$ be a quasi-norm on a vector sequence space $\alpha(X)$. We shall call quasi-norm $q$ *normalized* if $\phi(X) \subset \alpha(X)$ and $q((x, 0, 0, \dots)) = \|x\|$ for any $x \in X$.

### 1.3.1 Weak sequence spaces

Let $X$ be a Banach space. There is a natural injection $j_w$ from $X^{\mathbb{N}}$ into the space $L(X^*, \mathbb{K}^{\mathbb{N}})$ of linear operators between $X^*$ and $\mathbb{K}^{\mathbb{N}}$: given $\overline{x} = (x_n) \in X^{\mathbb{N}}$ put

$$(j_w\overline{x})(x^*) = x^* \circ \overline{x} = (x^*(x_n)) \text{ for all } x^* \in X^*.$$

Note that $j_w$ is an injection because $X^*$ separates points of $X$, which means that for each $x \in X \setminus \{0\}$ there is a linear functional in $X^*$ that is non-zero on $x$. It is straightforward to verify that the following lemma holds.

**Lemma 1.30.** *Every operator in $j_w(X^{\mathbb{N}})$ is continuous as an operator between locally convex spaces $(X^*, w^*)$ and $\mathbb{K}^{\mathbb{N}}$ (the latter with the usual pointwise topology).*

**Corollary 1.31.** *Every operator in $j_w(X^{\mathbb{N}})$ has the closed graph.*

This is connected to the application of the closed graph theorem below.

**Definition 1.32.** Let $\lambda \subset \mathbb{K}^{\mathbb{N}}$ be a sequence space equipped with a linear topology. The *weak sequence space* $\hat{\lambda}^w(X)$ is defined as $j_w^{-1}[\mathcal{L}(X^*, \lambda)] \subset X^{\mathbb{N}}$.

Therefore, $\overline{x} \in \hat{\lambda}^w(X)$ means that $j_w\overline{x} \in \mathcal{L}(X^*, \lambda)$. If $\lambda$ is a quasi-normed sequence space, then the inclusion $j_w(\hat{\lambda}^w(X)) \subset \mathcal{L}(X^*, \lambda)$ defines a natural quasi-norm on $\hat{\lambda}^w(X)$ by $\|\overline{x}\|_w := \|j_w(\overline{x})\|$ for all $\overline{x} \in \hat{\lambda}^w(X)$. If $\lambda$ is a BK-space, then this inclusion defines a natural complete norm on $\hat{\lambda}^w(X)$.

**Definition 1.33.** Let $\lambda \subset \mathbb{K}^{\mathbb{N}}$ be a sequence space. The *weak sequence space* $\lambda^w(X)$ is defined as $j_w^{-1}[L(X^*, \lambda)] \subset X^{\mathbb{N}}$.

Therefore, $\overline{x} \in \lambda^w(X)$ means that for all $x^* \in X^*$ we have $(x^*(x_n)) \in \lambda$.

If $\lambda$ is a topological vector space, when can we assume that every element of $j_w(\lambda^w(X))$ is continuous? We will use a version of the closed graph theorem.

**Theorem 1.34** ([Trè95, p 173]). *A linear operator from a barrelled space to a Fréchet space is continuous whenever it has the closed graph.*

Definition of a barrelled space and other notions about topological vector spaces not presented in this paper can be found from [[Koe69, chapter 27]]. A Banach space is also a barrelled space.

So in order to ensure that every element of $j_w(\lambda^w(X))$ is continuous we need that

- $\lambda$ is a K-space (then the graph of every element of $j_w(\lambda^w(X))$ is closed)
- $\lambda$ is a Fréchet space.

Therefore, if $\lambda$ is an FK-space, then by Theorem 1.34 we have $j_w(\lambda^w(X)) \subset \mathcal{L}(X^*, \lambda)$.

The arguments above say that $\hat{\lambda}^w(X) = \lambda^w(X)$ whenever $\lambda$ is an FK-space.

On the other hand, if $\lambda$ does not have a topology, then $\lambda^w(X)$ is still always a special case of $\hat{\lambda}^w(X)$ if we equip $\lambda$ with the subspace topology of $\mathbb{K}^{\mathbb{N}}$.

Similarly to the above, one can define an injection $j_{w^*} : (X^*)^{\mathbb{N}} \to L(X, \mathbb{K}^{\mathbb{N}})$ by

$$(j_{w^*}\overline{x}^*)(x) = (x_n^*(x)) = j_X(x) \circ \overline{x}^*$$

for $\overline{x}^* = (x_n^*) \in (X^*)^{\mathbb{N}}$ and $x \in X$, because $X$ separates points of $X^*$. Here, $j_X : X \to X^{**}$ is the natural embedding defined as

$$(j_X(x))(x^*) = x^*(x) \quad \forall x^* \in X^*$$

for each $x \in X$. Again, elements of $j_{w^*}((X^*)^{\mathbb{N}})$ are continuous, when considered as operators from $(X, w)$ to $\mathbb{K}^{\mathbb{N}}$.

**Definition 1.35.** Let $\lambda \subset \mathbb{K}^{\mathbb{N}}$ be a sequence space. The *weak-* sequence space* $\lambda^{w^*}(X^*)$ is defined as $j_{w^*}^{-1}[L(X, \lambda)] \subset (X^*)^{\mathbb{N}}$.

Therefore, $\overline{x}^* \in \lambda^{w^*}(X^*)$ means that for all $x \in X$ we have $(x_n^*(x)) = j_X(x) \circ \overline{x}^* \in \lambda$.

**Definition 1.36.** Let $\lambda \subset \mathbb{K}^{\mathbb{N}}$ be a sequence space equipped with a linear topology. The *weak-* sequence space* $\hat{\lambda}^{w^*}(X^*)$ is defined as $j_{w^*}^{-1}[\mathcal{L}(X, \lambda)]$

The inclusion $j_{w^*}\hat{\lambda}^{w^*}(X^*) := \mathcal{L}(X, \lambda)$ defines a quasi-norm on $\hat{\lambda}^{w^*}(X^*)$ if $\lambda$ is quasi-normed, defined as $\|\overline{x}\|_{w^*} := \|j_{w^*}(\overline{x})\|$ for all $\overline{x} \in \hat{\lambda}^{w^*}(X)$. If $\lambda$ is an FK-space, then $\hat{\lambda}^{w^*}(X^*) = \lambda^{w^*}(X^*)$.

**Lemma 1.37.** *If $X$ is reflexive, then the spaces $\lambda^w(X^*)$ and $\lambda^{w^*}(X^*)$ coincide.*

*Proof.* Space $X$ is reflexive when $j_X(X) = X^{**}$. Let $\overline{x}^* \in \lambda^w(X^*)$. This means that $j_w\overline{x}^* \in L(X^{**}, \lambda)$, meaning that for all $x^{**} \in X^{**}$ we have $(x^{**}(x_n^*))_n \in \lambda$.

To get $\overline{x}^* \in \lambda^{w^*}(X^*)$, we need to have that $j_{w^*}\overline{x}^* \in L(X, \lambda)$, meaning that $j_X(x) \circ \overline{x}^* \in \lambda$ would hold for all $x \in X$.

Since $j_X(X) = X^{**}$, then for every $x \in X$ exists $x^{**} \in X^{**}$ such that $j_X(x) = x^{**}$. Thus we can see that $\overline{x}^* \in \lambda^{w^*}(X^*)$ and $\lambda^w(X^*) = \lambda^{w^*}(X^*)$. □

We will present a proof for a well known fact that $\ell_p^w(X^*) = \ell_p^{w^*}(X^*)$ for any $1 \leq p \leq \infty$. It is easy to see using Goldstine theorem. By $\mathrm{Cl}_{w^*} A$ we denote the closure of $A$ in weak-* topology.

**Theorem 1.38** (Goldstine). *Let $X$ be a Banach space. Then the image of $B_X$ under the natural embedding is weak-* dense in $B_{X^{**}}$, meaning $B_{X^{**}} \subset \mathrm{Cl}_{w^*} j_X(B_X)$.*

**Lemma 1.39.** *Let $1 \le p \le \infty$. Then $\ell_p^w(X^*) = \ell_p^{w^*}(X^*)$ for any dual space $X^*$.*

*Proof.* Let us show it for $1 \le p < \infty$. For the case $p = \infty$ some small modifications need to be made.

Assume $\overline{x}^* \in \ell_p^{w^*}(X^*)$. It means that for any $x \in X$ one has $(x_n^*(x))_n \in \ell_p$ and moreover $\|(x_n^*(x))_n\|_{\ell_p} \le \|\overline{x}^*\|_{w^*}\|x\|$. Here, $\|\overline{x}^*\|_{w^*} = \|j_{w^*}\overline{x}^*\|$.

For us, it is enough to show that for any $x^{**} \in B_{X^{**}}$ one has $(x^{**}(x_n^*))_n \in \ell_p$, that is, there exists $M \ge 0$ such that for all $N \in \mathbb{N}$,

$$\sum_{i=1}^{N} |x^{**}(x_i^*)|^p < M.$$

By Goldstine's theorem $B_{X^{**}} \subset \mathrm{Cl}_{w^*}\, j_X(B_X)$. Thus for every $N \in \mathbb{N}$ and for every $\varepsilon > 0$ we can find $x \in B_X$ such that $|x^{**}(x_i^*) - x_i^*(x)| < \varepsilon/N^2$ for all $i \le N$. By using inequality

$$|a+b|^p \le 2^{p-1}(|a|^p + |b|^p)$$

that holds for $p \ge 1$, we can arrange so that

$$|x^{**}(x_i^*)|^p < 2^{p-1}(|x_i^*(x)|^p + (\varepsilon/N^2)^p).$$

Thus, we can see that

$$\sum_{i=1}^{N} |x^{**}(x_i^*)|^p \le 2^{p-1}\left(\sum_{i=1}^{N} |x_i^*(x)|^p + (\varepsilon/N)^p\right) \le 2^{p-1}(\|\overline{x}^*\|_{w^*}^p + (\varepsilon/N)^p)$$

which is enough. $\square$

However, if we look at coordinate functionals $e_n^*$, then $\overline{e}^* \in c_0^{w^*}(c_0^*)\backslash c_0^w(c_0^*)$. It is easy to see that $\overline{e}^* \in c_0^{w^*}(c_0^*)$, meaning $j_{w^*}\overline{e}^* \in L(c_0, c_0)$, because $j_{w^*}\overline{e}^* = I_{c_0}$. To see that $\overline{e}^* \notin c_0^w(c_0^*)$, meaning $j_w\overline{e}^* \notin L(\ell_\infty, c_0)$, it is enough to notice that for $e \in \ell_\infty$ we have $(j_w\overline{e}^*)(e) = e \notin c_0$.

If $\lambda$ is a K-space, then it always holds that $\overline{e}^* \in \lambda^{w^*}(\lambda^*)$, meaning that $j_{w^*}\overline{e}^* = I_\lambda \in L(\lambda, \lambda)$.

### 1.3.2 Strong sequence spaces

**Definition 1.40.** Let $\lambda$ be a sequence space. The *strong sequence space* is defined as

$$\lambda^s(X) := \{\overline{x} = (x_n) \in X^{\mathbb{N}} \colon (\|x_n\|) \in \lambda\}.$$

This may fail to be a vector space unless $\lambda$ is solid. If a solid sequence space $\lambda$ is equipped with the norm $\|\cdot\|$, then $\|\overline{x}\|_s = \|(\|x_n\|)\|$ defines a norm on $\lambda^s(X)$ whenever the original norm $\|\cdot\|$ on $\lambda$ is monotone.

It is well known and easy to verify that $\ell_\infty^s(X)$ and $\ell_\infty^w(X)$ coincide. If $X = Y^*$ is a dual space, then these spaces also coincide with $\ell_\infty^{w^*}(Y^*)$, as discussed in the previous section.

### 1.3.3 Sequence systems

In the following we will need similar properties of two slightly different types of nuclear operators defined by a triplet of sequence spaces. For this, it is convenient to introduce the following general definition.

**Definition 1.41.** By a *sequence system* $\alpha$ we mean a rule that for every Banach space $X$ fixes a linear subspace $\alpha(X)$ of $X^{\mathbb{N}}$ such that for every $\overline{x} = (x_n) \in \alpha(X)$, for every Banach space $Y$ and for every $T \in \mathcal{L}(X, Y)$ one has $T \circ \overline{x} = (Tx_n) \in \alpha(Y)$.

**Definition 1.42.** By a $w^*$*-sequence system* $\alpha$ we mean a rule that for every dual Banach space $X^*$ fixes a linear subspace $\alpha(X^*)$ of $(X^*)^{\mathbb{N}}$ such that for every $\overline{x}^* = (x_n^*) \in \alpha(X^*)$, for every Banach space $Y$ and for every $T \in \mathcal{L}(Y, X)$ one has $T^* \circ \overline{x}^* = (T^* x_n^*) \in \alpha(Y^*)$.

Note that every sequence system is, in particular, also a $w^*$-sequence system.

*Example* 1.43. If $\lambda$ is equipped with a linear topology, then also $\hat{\lambda}^w$ is a sequence system and $\hat{\lambda}^{w^*}$ is a $w^*$-sequence system. We explain why the first claim holds. Indeed, from $\overline{x} \in \lambda(X)$ and $T \in \mathcal{L}(X, Y)$ follows that $(Tx_n) \in \lambda(Y)$. Thus for every $y^* \in Y^*$ we have $(y^*(Tx_n))_n \in \lambda$. Therefore $j_w[(Tx_n)] \in \mathcal{L}(Y^*, \lambda)$, meaning $(Tx_n) \in \hat{\lambda}^w(Y)$.

*Example* 1.44. Given a sequence space $\lambda$, both $\lambda^w : X \mapsto \lambda^w(X)$ and $\lambda^s : X \mapsto \lambda^s(X)$ (assuming that $\lambda$ is solid) are sequence systems and $\lambda^{w^*} : X^* \mapsto \lambda^{w^*}(X^*)$ is a $w^*$-sequence system. We explain why the claim for $\lambda^s$ holds. Indeed, from $\overline{x} \in \lambda(X)$ we have $(\|x_n\|)_n \in \lambda$. Since $\lambda$ is solid, it follows from $\|Tx_n\| \leq \|T\|\|x_n\|$ that $(\|Tx_n\|) \in \lambda$. Thus, $(Tx_n) \in \lambda^s(Y)$.

Let $X$ be a Banach space and let $q$ be a quasi-norm on a vector sequence space $\alpha(X)$.

**Definition 1.45.** We say that a sequence system ($w^*$-sequence system) $\alpha$ is *quasi-normed*, if for all Banach spaces $X$ the vector sequence space $\alpha(X)$ ($\alpha(X^*)$) is equipped with a quasi-norm $q$ such that for every Banach space $Y$, every operator $T \in \mathcal{L}(X,Y)$ ($R \in \mathcal{L}(Y,X)$) and every $\overline{x} \in \alpha(X)$ ($\overline{x}^* \in \alpha(X^*)$), one has $q(T \circ \overline{x}) \leq \|T\| q(\overline{x})$ ($q(R^* \circ \overline{x}^*) \leq \|R\| q(\overline{x}^*)$).

Given any property P introduced for (quasi-normed) vector sequence spaces, we will say that a (quasi-normed) ($w^*$-)sequence system $\alpha$ enjoys property P whenever every component of $\alpha$ enjoys P.

We say that a sequence system $\alpha$ is *non-zero*, when every component of $\alpha$ is non-zero.

*Example* 1.46. If $\lambda$ is monotone and $q$ is monotone, then both $\hat{\lambda}^w$ and $\hat{\lambda}^{w^*}$ are monotone with monotone quasi-norms. We explain why the claim for $\hat{\lambda}^w$ holds. Indeed, let $\lambda$ be monotone, let $X$ be a Banach space, let $\overline{x} \in \hat{\lambda}^w(X)$, $N \subset \mathbb{N}$ and $x^* \in X^*$. Then $x^* \circ S_N(\overline{x}) = S_N[(x^*(x_n))] \in \lambda$. Since $q$ is monotone, it follows that the the norm of operator $j_w S_N(\overline{x})$ is bounded. Therefore $\hat{\lambda}^w$ is monotone. Let $q$ be a monotone quasi-norm on $\lambda$. Let $\overline{x}, \overline{y} \in X$ be such that $\| \cdot \| \circ \overline{y} \leq \| \cdot \| \circ \overline{x}$. Since

$$\|\overline{x}\|_w = \|j_w \overline{x}\| = \sup_{\|x^*\| \leq 1} q((j_w \overline{x})x^*) = \sup_{\|x^*\| \leq 1} q((x^*(x_n))_n)$$

and $q$ is monotone, we have that $\|\overline{y}\|_w \leq \|\overline{x}\|_w$ and therefore $\| \cdot \|_w$ is also monotone.

*Example* 1.47. Let $\lambda$ be a sequence space equipped with a quasi-norm $q$. If $\lambda$ is symmetric and $q$ is K-symmetric, then both $\hat{\lambda}^w$ and $\hat{\lambda}^{w^*}$ are symmetric with K-symmetric quasi-norms.

# 2 $(\lambda, \mu, \nu)$-nuclear operators

In this section we introduce the notion of $(\lambda, \alpha, \beta)$-nuclear operators. In the following, we assume that sequence spaces and sequence systems are non-zero.

**Definition 2.1.** Let $\lambda$ be a sequence space, let $\alpha$ be $w^*$-sequence system and let $\beta$ be a sequence system. Let $X$ and $Y$ be Banach spaces. We say that an operator $T \in \mathcal{L}(X, Y)$ is $(\lambda, \alpha, \beta)$-*nuclear* if

$$T = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n,$$

where $(\sigma_n) \in \lambda$, $(x_n^*) \in \alpha(X^*)$ and $(y_n) \in \beta(Y)$ (the series converges in the usual operator norm). Here, given $x^* \in X^*$ and $y \in Y$, the one-rank operator $x^* \otimes y$ is defined by $(x^* \otimes y)(x) = x^*(x) \cdot y$ for all $x \in X$. We denote the collection of of all $(\lambda, \alpha, \beta)$-nuclear operators by $\mathcal{N}_{(\lambda,\alpha,\beta)}$. In the following, we will also use notation $T = \sum \overline{\sigma x}^* \otimes \overline{y} = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n$ for a given nuclear representation of operator $T$.

Given a triplet of sequence spaces $\lambda$, $\mu$, $\nu$, we denote $\mathcal{N}_{(\lambda,\mu,\nu)} := \mathcal{N}_{(\lambda,\mu^{w^*},\nu^w)}$. The operators belonging to the latter collection will be called $(\lambda, \mu, \nu)$-*nuclear*. Often, we will also consider a related class $\mathcal{N}_{(\lambda,\mu^w,\nu^w)}$ without giving it a special notation.

*Example* 2.2. Let $1 \le r, p, q \le \infty$ and $1 + 1/r \ge 1/p + 1/q$. The $(r, p, q)$-nuclear operators (see [[Pie80, Chapter 18]]) by definition are exactly the $(\ell_r, \ell_{q^*}^w, \ell_{p^*}^w)$-nuclear operators, which in this special case coincide with the $(\ell_r, \ell_{q^*}, \ell_{p^*})$-nuclear operators, due to Lemma 1.39.

*Example* 2.3. The $\lambda$-nuclear operators (see [Ram70]) are exactly the $(\lambda, \ell_\infty, \lambda^\times)$-nuclear operators. Note that Ramanujan assumed that $\|x_n^*\| \le 1$ and $\|y_n\| \le 1$ for all $n \in \mathbb{N}$. These conditions can be attained by normalizing the nuclear-representation of the operator, considering $\sigma_n x_n^* \otimes y_n = \|(x_n^*)\| \|(y_n)\| \sigma_n \|(x_n^*)\|^{-1} \|(y_n)\|^{-1} x_n^* \otimes y_n$.

We continue with showing that under some modest assumptions, $\mathcal{N}_{(\lambda,\alpha,\beta)}$ is an operator ideal.

**Theorem 2.4.** *Let $\lambda$ be a sequence space, let $\alpha$ be $w^*$-sequence system and let $\beta$ be a sequence system. Let $\lambda$, $\alpha$ and $\beta$ be symmetric and monotone. Then $\mathcal{N}_{(\lambda,\alpha,\beta)}$ is an operator ideal.*

*Proof.* Denote $\mathcal{N} := \mathcal{N}_{(\lambda,\alpha,\beta)}$ for brevity.

1. Since $e_1 \in \lambda, (I_{\mathbb{K}}, 0, 0, \dots) \in \alpha(\mathbb{K}^*), e_1 \in \beta(\mathbb{K})$ and $I_{\mathbb{K}} = 1 \cdot I_{\mathbb{K}} \otimes 1$, we have $I_{\mathbb{K}} \in \mathcal{N}$.

2. Given Banach spaces $X$ and $Y$, let us show that $\mathcal{N}(X, Y)$ is closed under addition. Let $S_1, S_2 \in \mathcal{N}(X, Y)$ and we pick representations

$$S_1 = \sum \overline{\sigma}_1 \overline{x}^*{}_1 \otimes \overline{y}_1, \quad S_2 = \sum \overline{\sigma}_2 \overline{x}^*{}_2 \otimes \overline{y}_2.$$

$$S_1 + S_2 = \sum (j_1\overline{\sigma}_1 + j_2\overline{\sigma}_2)(j_1\overline{x}^*{}_1 + j_2\overline{x}^*{}_2) \otimes (j_1\overline{y}_1 + j_2\overline{y}_2).$$

   Since $\lambda$, $\alpha$ and $\beta$ are symmetric and monotone, it follows from Lemma 1.28 that

$$j_1\overline{\sigma}_1 + j_2\overline{\sigma}_2 \in \lambda,\ j_1\overline{x}^*{}_1 + j_2\overline{x}^*{}_2 \in \alpha,\ \text{and } j_1\overline{y}_1 + j_2\overline{y}_2 \in \beta.$$

   In conclusion, $S_1 + S_2 \in \mathcal{N}$.

3. Let $X$, $Y$, $V$ and $W$ be Banach spaces. Consider operators $S \in \mathcal{N}(X, Y)$, $R \in \mathcal{L}(Y, V)$ and $T \in \mathcal{L}(W, X)$. We need to show that $RST \in \mathcal{N}(W, V)$. Pick a representation $S = \sum \overline{\sigma x}^* \otimes \overline{y}$ with $\overline{\sigma} \in \lambda$, $\overline{x}^* \in \alpha(X^*)$, $\overline{y} \in \beta(Y)$. Then

$$RST = \sum_{n=1}^{\infty} \sigma_n T^* x_n^* \otimes Ry_n.$$

   Since $(T^* x_n^*) \in \alpha(W^*)$ and $(Ry_n) \in \beta(V)$, we have $RST \in \mathcal{N}$.

□

**Corollary 2.5.** *Let $\lambda$, $\mu$ and $\nu$ be symmetric and monotone sequence spaces. Then both $\mathcal{N}_{(\lambda,\mu,\nu)}$ and $\mathcal{N}_{(\lambda,\mu^w,\nu^w)}$ are operator ideals.*

## 2.1 Nuclear quasi-norm

Let $X$ be a Banach space and let $q$ be a quasi-norm on a vector sequence space $\alpha(X)$.

**Corollary 2.6.** *Let $\lambda$, $\mu$ and $\nu$ be symmetric and monotone sequence spaces. Let $\mu$ and $\nu$ be equipped with K-symmetric and monotone quasi-norms. Then both $\mathcal{N}_{(\lambda,\hat{\mu}^{w^*},\hat{\nu}^w)}$ and $\mathcal{N}_{(\lambda,\hat{\mu}^w,\hat{\nu}^w)}$ are operator ideals.*

**Definition 2.7.** Let $(\lambda, q_\lambda)$ be a quasi-normed sequence space, let $(\alpha, q_\alpha)$ be a quasi-normed $w^*$-sequence system and let $(\beta, q_\beta)$ be a quasi-normed sequence system. Let $X$ and $Y$ be Banach spaces. Given an operator $T \in \mathcal{N}_{(\lambda,\alpha,\beta)}(X, Y)$, we define the *nuclear quasi-norm* as

$$N_{(\lambda,\alpha,\beta)}(T) := \inf q_\lambda(\overline{\sigma}) q_\alpha(\overline{x}^*) q_\beta(\overline{y}),$$

where the infimum ranges over all nuclear representations of $T = \Sigma\overline{\sigma}\overline{x}^* \otimes \overline{y}$ with $\overline{\sigma} \in \lambda$, $\overline{x}^* \in \alpha(X^*)$ and $\overline{y} \in \beta(Y)$.

**Theorem 2.8.** *Let $(\lambda, q_\lambda)$ be a quasi-normed sequence space, let $(\alpha, q_\alpha)$ be a quasi-normed $w^*$-sequence system and let $(\beta, q_\beta)$ be a quasi-normed sequence system. Let $\lambda$, $\alpha$ and $\beta$ be symmetric and monotone, all equipped with normalized K-symmetric quasi-norms. Assume also that $(\lambda, q_\lambda) \cdot (\alpha(\mathbb{K}), q_\alpha) \cdot (\beta(\mathbb{K}), q_\beta) \leq (\ell_1, \|\cdot\|_1)$. Then $(\mathcal{N}_{(\lambda,\alpha,\beta)}, N_{(\lambda,\alpha,\beta)})$ is a quasi-normed operator ideal.*

*Proof.* Denote $\mathcal{N} := \mathcal{N}_{(\lambda,\alpha,\beta)}$ and $N := N_{(\lambda,\alpha,\beta)}$ for brevity.

1. Let us show that $N(I_\mathbb{K}) = 1$. Clearly,

   $$N(I_\mathbb{K}) \leq q_\lambda(e_1) q_\alpha((I_\mathbb{K}, 0, 0, \dots)) q_\beta(e_1) = 1,$$

   because the quasi-norms are normalized.

   On the other hand, if $I_\mathbb{K} = \sum \overline{\sigma}\overline{x}^* \otimes \overline{y}$ for some $\overline{\sigma} \in \lambda$, $\overline{x}^* \in \alpha(\mathbb{K}^*)$ and $\overline{y} \in \beta(\mathbb{K})$, then

   $$\begin{aligned} 1 = I_\mathbb{K}(1) = \sum_{i=1}^{\infty} \sigma_i x_i^*(1) y_i \leq \|\overline{\sigma} \cdot (j_\mathbb{K}(1) \circ \overline{x}^*) \cdot \overline{y}\|_1 \leq \\ \leq \inf q_\lambda(\overline{\sigma}) q_\alpha(\overline{x}^*) q_\beta(\overline{y}) = N(I_\mathbb{K}), \end{aligned}$$

   In conclusion, $N(I_\mathbb{K}) = 1$.

2. Let $X$ and $Y$ be Banach spaces and let $\kappa_\lambda$, $\kappa_\alpha$, $\kappa_\beta$ be the quasi-norm constants of $q_\lambda$, $q_\alpha$ on $\alpha(X^*)$ and $q_\beta$ on $\beta(Y)$, respectively. Let $S_1, S_2 \in \mathcal{N}(X,Y)$. We shall show that $N(S_1+S_2) \leq 4\kappa_\lambda\kappa_\alpha\kappa_\beta(N(S_1)+N(S_2))$. As in the proof of Theorem 2.4 pick representations $S_1 = \sum \overline{\sigma}_1 \overline{x}^*{}_1 \otimes \overline{y}_1$ and $S_2 = \sum \overline{\sigma}_2 \overline{x}^*{}_2 \otimes \overline{y}_2$ and note that

$$S := S_1 + S_2 = \sum (j_1\overline{\sigma}_1 + j_2\overline{\sigma}_2)(j_1\overline{x}^*{}_1 + j_2\overline{x}^*{}_2) \otimes (j_1\overline{y}_1 + j_2\overline{y}_2),$$

where $j_1$ and $j_2$ are isometries due to K-symmetricity of the quasi-norms. We can assume that $q_\alpha(\overline{x}^*{}_k) = q_\beta(\overline{y}_k) = 1$ for $k = 1, 2$. (If this does not hold for initial nuclear representations, we can normalize them as in Example 2.3.) Then

$$\begin{aligned} N(S) &\leq q_\lambda(j_1\overline{\sigma}_1 + j_2\overline{\sigma}_2) \cdot q_\alpha(j_1\overline{x}^*{}_1 + j_2\overline{x}^*{}_2) \cdot q_\beta(j_1\overline{y}_1 + j_2\overline{y}_2) \leq \\ &\leq 4\kappa_\lambda\kappa_\alpha\kappa_\beta(q_\lambda(\overline{\sigma}_1) + q_\lambda(\overline{\sigma}_2)). \end{aligned}$$

This implies the claim.

3. Let $X$, $Y$, $V$ and $W$ be Banach spaces. Consider operators $S \in \mathcal{N}(X,Y)$, $R \in \mathcal{L}(Y,V)$ and $T \in \mathcal{L}(W,X)$. We need to show that $N(RST) \leq \|R\| N(S) \|T\|$. This is immediate from the definition of a quasi-normed ($w^*$-)sequence system and the representation $RST = \sum \overline{\sigma}(T^* \circ \overline{x}^*) \otimes (R \circ \overline{y})$ (see the proof of Theorem 2.4) for $S = \sum \overline{\sigma x}^* \otimes \overline{y}$ with $\overline{\sigma} \in \lambda$, $\overline{x}^* \in \alpha(X^*)$, $\overline{y} \in \beta(Y)$.

□

**Corollary 2.9.** *Let $\lambda$, $\mu$, $\nu$ be symmetric and monotone sequence spaces all equipped with K-symmetric normalized quasi-norms. Let $\mu$ and $\nu$ be FK-spaces or have monotone quasi-norms. Then both $(\mathcal{N}_{(\lambda,\hat{\mu}^{w^*},\hat{\nu}^w}, N_{(\lambda,\hat{\mu}^{w^*})})$ and $(\mathcal{N}_{(\lambda,\hat{\mu}^w,\hat{\nu}^w}, N_{(\lambda,\hat{\mu}^w,\hat{\nu}^w)})$ are quasi-normed operator ideals.*

## 2.2 Completeness of the nuclear quasi-norm

Given a double indexed sequence $a^i_k$, consider its *diagonalization* $\eta_j = a^{i(j)}_{k(j)}$, where

$$(k(j)) = (1,2,1,3,2,1,4,3,2,1,\dots) \text{ and } i(j) = (1,1,2,1,2,3,1,2,3,4,\dots).$$

We need the following lemma about diagonalization of double indexed series in a Banach space.

**Lemma 2.10.** *Let $X$ be a Banach space, let $(a_k^i)$ be a double indexed sequence in $X$ and let $\left(a_{k(j)}^{i(j)}\right)_j$ be its diagonalization. Assume that for every $k \in \mathbb{N}$, the series $\sum_{i=1}^{\infty} a_k^i$ converges. If, for every $k \in \mathbb{N}$ it is possible to fix a function $p_k : \mathbb{N} \to [0, \infty)$ such that:*

1. *$\|\sum_{i=m}^{\infty} a_k^i\| \leq p_k(m)$ for all $m, k \in \mathbb{N}$,*
2. *$\sum_{k=1}^{\infty} p_k(1) < \infty$,*
3. *$p_k(m)$ monotonely goes to $0$ as $m$ goes to $\infty$ for every $k \in \mathbb{N}$,*

*then*

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a_k^i = \sum_{j=1}^{\infty} a_{k(j)}^{i(j)} \in X.$$

*Proof.* Denoting $b_k^i := p_k(i) - p_k(i+1) \geq 0$, we get that

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} b_k^i < \infty.$$

This series converges under any rearrangement. In particular, under diagonalization.

Since $X$ is a Banach space, (1) and (2) implies that $\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a_k^i$ converges to some $x \in X$. Consider $y_n := x - \sum_{j=1}^{n} a_{k(j)}^{i(j)}$. This must be of the form $y_n = \sum_{k=1}^{\infty} \sum_{i=t_k^n}^{\infty} a_k^i$, where $t_k^n$ differs from 1 for only finitely many indeces $k$. We have that

$$\|y_n\| \leq \sum_{k=1}^{\infty} \left\| \sum_{i=t_k^n}^{\infty} a_k^i \right\| \leq \sum_{k=1}^{\infty} \sum_{i=t_k^n}^{\infty} b_k^i = \sum_{j=n+1}^{\infty} b_{k(j)}^{i(j)} \xrightarrow[n\to\infty]{} 0,$$

as required. □

**Definition 2.11.** We say that a quasi-normed vector sequence space $(\alpha(X), q_\alpha)$ is *stable under diagonalization* if given a sequence $((x_k^i)_i)_k \subset \alpha(X)$ such that

$$\sum_{k=1}^{\infty} q_\alpha((x_k^i)_i) < \infty$$

also its diagonalization $\left(x^{i(n)}_{k(n)}\right)_n$ belongs to $\alpha(X)$ and

$$q_\alpha\left(\left(x^{i(n)}_{k(n)}\right)_n\right) \leq \sum_{k=1}^{\infty} q_\alpha((x^i_k)_i).$$

**Theorem 2.12.** *Let $(\lambda, q_\lambda)$ be a quasi-normed sequence space, let $(\alpha, q_\alpha)$ be a quasi-normed $w^*$-sequence system and let $(\beta, q_\beta)$ be a quasi-normed sequence system such that $(\mathcal{N}_{(\lambda,\alpha,\beta)}, N_{(\lambda,\alpha,\beta)})$ is a quasi-normed operator ideal. Assume that $\lambda$, $\alpha$ and $\beta$ are monotone, have monotone quasi-norms and are stable under diagonalization. Assume that the quasi-norm of at least one of $\lambda$, $\alpha$, or $\beta$ is AK. Then $(\mathcal{N}_{(\lambda,\alpha,\beta)}, N_{(\lambda,\alpha,\beta)})$ is a quasi-Banach operator ideal.*

*Proof.* Denote $\mathcal{N} := \mathcal{N}_{(\lambda,\alpha,\beta)}$ and $N := N_{(\lambda,\alpha,\beta)}$ for brevity. The proof follows the scheme in [[Ram70, p. 192]]. Take a Cauchy sequence $(S_n)$ in $\mathcal{N}(X, Y)$. We can assume that $N(S_{n+1} - S_n) < \frac{1}{8^n}$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, fix $\overline{\sigma}_n \in \lambda$, $\overline{x}^*{}_n \in \alpha(X^*)$ and $\overline{y}_n \in \beta(Y)$ such that $S_{n+1} - S_n = \sum \overline{\sigma}_n \overline{x}^*{}_n \otimes \overline{y}_n$ and $q_\lambda(\overline{\sigma}_n), q_\alpha(\overline{x}^*{}_n), q_\beta(\overline{y}_n) < \frac{1}{2^n}$. Then

$$S_{n+p} - S_n = \sum_{k=n}^{n+p-1} \sum \overline{\sigma}_k \overline{x}^*{}_k \otimes \overline{y}_k.$$

Since $\mathcal{N}$ is a quasi-normed operator ideal, its norm dominates the usual operator norm ([Ram70]), so $(S_n)$ converges to some $S \in \mathcal{L}(X, Y)$ in the usual operator norm. Hence

$$S - S_n = \sum_{k=n}^{\infty} \sum \overline{\sigma}_k \overline{x}^*{}_k \otimes \overline{y}_k = \sum_{k=n}^{\infty} \sum_{i=1}^{\infty} \sigma^i_k x^{*i}_k \otimes y^i_k \text{ in } \mathcal{L}(X, Y).$$

Denote $p_k(m) := q_\lambda(\chi_m \cdot \overline{\sigma}_k) q_\alpha(\chi_m \cdot \overline{x}^*{}_k) q_\beta(\chi_m \cdot \overline{y}_k)$. Then $p_k(m)$ monotonely goes to 0 as $m \to \infty$, because the quasi-norms are monotone and one of them is AK. Also $\sum_{k=n}^{\infty} p_k(1) < \infty$ by construction. Therefore Lemma 2.10 assumptions for $a^i_k = \sigma^i_k x^{*i}_k \otimes y^i_k$ are fulfilled, considering that

$$\sum_{i=1}^{\infty} a^i_k = S_{k+1} - S_k \in \mathcal{N} \text{ and} \|\sum_{i=m}^{\infty} a^i_k\| \leq p_k(m)$$

for all $k, m \in \mathbb{N}$. Lemma 2.10 now implies that $S - S_n = (\hat{\sigma}, \hat{x}^*, \hat{y})$, where $\hat{\sigma}, \hat{x}^*, \hat{y}$ are the diagonalizations of $(\overline{\sigma}_k)_{k=n}^{\infty}$, $(\overline{x}^*{}_k)_{k=n}^{\infty}$, $(\overline{y}_k)_{k=n}^{\infty}$, respectively.

Since all the quasi-normed vector sequence spaces in question are stable under diagonalization, we have that $\hat{\sigma} \in \lambda$ and

$$q_\lambda(\hat{\sigma}) \leq \sum_{k=n}^{\infty} q_\lambda(\overline{\sigma}_k) < \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}.$$

Similarly, $\hat{x}^* \in \alpha(X^*)$, $\hat{y} \in \beta(Y)$ and $q_\alpha(\hat{x}^*), q_\beta(\hat{y}) < \frac{1}{2^{n-1}}$. Therefore, $S - S_n \in \mathcal{N}(X,Y)$, so also, in particular, $S \in \mathcal{N}(X,Y)$ and $N(S - S_n) < \frac{1}{8^{n-1}} \to 0$. □

# 3 Factoring nuclear operators

Similarly to [[Pie80, Theorem 18.1.3]] we obtain a factorization of $(\lambda, \mu^{w^*}, \ell_\infty^s)$-nuclear operators, which we can denote shortly as $(\lambda, \mu, \ell_\infty)$-nuclear operators, because $\ell_\infty^s(X) = \ell_\infty^w(X)$, as discussed before (see Section 1.3.2). Before we start, let us fix some notation for some well-known operators related to sequence spaces.

Let $X$ be a Banach space, then $\Sigma_1 : \overline{x} \mapsto \sum_{n=1}^\infty x_n$ defines a bounded linear surjection $\Sigma_1 \in \mathcal{L}(\ell_1^s(X), X)$ of norm 1.

If sequence spaces $\lambda$, $\mu$, $\nu$ satisfy $\lambda\mu \subset \nu$, then every $\overline{\sigma} \in \lambda$ defines a linear multiplication operator $M_{\overline{\sigma}} \in \mathcal{L}(\mu, \nu)$ by $M_{\overline{\sigma}}(\overline{m}) = \overline{\sigma m} \in \nu$. If these sequence spaces are quasi-normed and $\lambda\mu \leq \nu$, then $M_{\overline{\sigma}}$ is bounded and $\|M_{\overline{\sigma}}\| \leq \|\overline{\sigma}\|_\lambda$. Indeed,

$$\|M_{\overline{\sigma}}(\overline{m})\|_\nu = \|\overline{\sigma m}\|_\nu \leq \|\overline{\sigma}\|_\lambda \|\overline{m}\|_\mu.$$

Similarly, in this case, $M_{\overline{x}} \in \mathcal{L}(\mu, \nu^s(X))$ with $\|M_{\overline{x}}\| \leq \|\overline{x}\|_\lambda$ if $\overline{x} \in \lambda^s(X)$ (assuming $\lambda$ and $\nu$ are solid).

If, in addition, $\nu = \ell_1$, $\lambda$ is AK-space and $\mu$ is a BK-space, then $M_{\overline{\sigma}} = \sum \overline{\sigma e}^* \otimes \overline{e} \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(\mu, \ell_1)$ with $N_{(\lambda,\mu,\ell_\infty)}(M_{\overline{\sigma}}) \leq \|\overline{\sigma}\|_\lambda$.

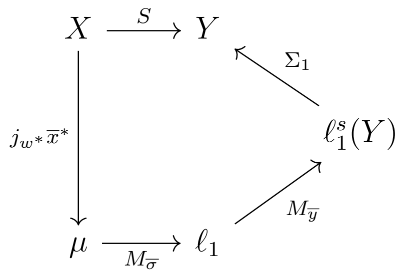
Indeed, $e_n^* \in \mu^*$, because $\mu$ is BK. Then $\overline{e}^* \in \mu^{w^*}(\mu^*)$, because $j_{w^*}\overline{e}^* = I_\mu \in L(\mu, \mu)$. Clearly, $\overline{e} \in \ell_\infty^s(\ell_1)$ with $\|\overline{e}\| = 1$. Finally,

$$\left\| M_{\overline{\sigma}} - \sum_{i=1}^n \sigma_i e_i^* \otimes e_i \right\| \leq \|(0, \ldots, 0, \sigma_{n+1}, \sigma_{n+2}, \ldots)\|_\lambda \xrightarrow[n\to\infty]{} 0,$$

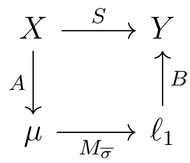
because $\lambda$ is AK.

Let $S \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(X, Y)$ have a representation $S = \sum \overline{\sigma x}^* \otimes \overline{y}$. Then $S$ can be factorized as $S = \Sigma_1 M_{\overline{y}} M_{\overline{\sigma}} j_{w^*}\overline{x}^*$ (see the diagram below), where $j_{w^*}\overline{x}^* \in \mathcal{L}(X, \mu)$ (because $\overline{x}^* \in \mu^{w^*}(X)$), $M_{\overline{\sigma}} \in \mathcal{L}(\mu, \ell_1)$ with $M_{\overline{\sigma}}(\overline{m}) = \overline{\sigma m}$,

$M_{\overline{y}} \in \mathcal{L}(\ell_1, \ell_1^s(Y))$ with $M_{\overline{y}}(\overline{z}) = \overline{yz}$ and $\Sigma_1 \in \mathcal{L}(\ell_1^s(Y), Y)$ with $\Sigma_1(\overline{u}) = \sum \overline{u}$.




In particular, in this case, $S$ can be factorized as $S = BM_{\overline{\sigma}}A$, where $A(= j_{w^*}\overline{x}^*) \in \mathcal{L}(X, \mu)$, $B(= \Sigma_1 M_{\overline{y}}) \in \mathcal{L}(\ell_1, Y)$ and $M_{\overline{\sigma}} \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(\mu, \ell_1)$.




Note that $\|A\| = \|\overline{x}^*\|_{\mu^{w^*}}$ and $\|B\| = \|\Sigma_1 M_{\overline{y}}\| \leq \|\overline{y}\|_{\ell_\infty^s}$. This gives that

$$N_{(\lambda,\mu,\ell_\infty)}(S) = \inf \|\overline{x}^*\|_{\mu^{w^*}} \|\overline{\sigma}\|_\lambda \|\overline{y}\|_{\ell_\infty^s} \geq \inf \|A\| \|\overline{\sigma}\|_\lambda \|B\|,$$

where the infimum ranges over all representations $S = BM_{\overline{\sigma}}A$. The other inequality "$\leq$" holds when $(\mathcal{N}_{(\lambda,\mu,\ell_\infty)}, N_{(\lambda,\mu,\ell_\infty)})$ is a quasi-normed operator ideal, because

$$N_{(\lambda,\mu,\ell_\infty)}(S) = N_{(\lambda,\mu,\ell_\infty)}(BM_{\overline{\sigma}}A) \leq \|B\| N_{(\lambda,\mu,\ell_\infty)}(M_{\overline{\sigma}}) \|A\|$$

and $N_{(\lambda,\mu,\ell_\infty)}(M_{\overline{\sigma}}) \leq \|\overline{\sigma}\|_\lambda$. Thus we can state the following.

**Theorem 3.1.** *Let $\lambda$ and $\mu$ be quasi-normed sequence spaces such that $\lambda$ is AK, $\mu$ is BK, $\lambda\mu \leq \ell_1$ and $(\mathcal{N}_{(\lambda,\mu,\ell_\infty)}, N_{(\lambda,\mu,\ell_\infty)})$ is a quasi-normed operator ideal. Then every operator $S \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(X, Y)$ can be factorized as $S = BM_{\overline{\sigma}}A$, where $A \in \mathcal{L}(X, \mu)$, $B \in \mathcal{L}(\ell_1, Y)$ and $M_{\overline{\sigma}} \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(\mu, \ell_1)$ for some $\overline{\sigma} \in \lambda$. Moreover, $N_{(\lambda,\mu,\ell_\infty)}(S) = \inf \|A\| \|\overline{\sigma}\|_\lambda \|B\|$, where the infimum ranges over all such representations of $S$.*

## 4 $(\lambda, \mu)$-compact operators

In this section we shall mimic the approach in [ALO12] (done for $(p,r)$-compact operators). Let $\mu$ be a BK-space, let $X$ be a Banach space and let $(x_n) \in (\mu^\times)^s(X)$. It is well known that the mapping $\Phi_{(x_n)} : (a_n) \mapsto \sum_{n=1}^\infty a_n x_n$ defines a bounded linear operator from $\mu$ to $X$. Indeed, using the notation of the previous section, $\Phi_{\overline{x}} = \Sigma_1 M_{\overline{x}}$ with $M_{\overline{x}} \in \mathcal{L}(\mu, \ell_1^s(X))$ and $\Sigma_1 \in \mathcal{L}(\ell_1^s(X), X)$.

**Definition 4.1.** Let $X$ and $Y$ be Banach spaces. Let $\mu$ be a BK-space and let $\lambda$ be a solid sequence space such that $\lambda \subset \mu^\times$, which simply means that $\lambda\mu \subset \ell_1$. We say that an operator $T \in \mathcal{L}(Y, X)$ is $(\lambda, \mu)$-*compact* if $T(B_Y) \subset \Phi_{(x_n)}(B_\mu)$ for some $(x_n) \in \lambda^s(X)$. We denote the collection of all $(\lambda, \mu)$-compact operators by $\mathcal{K}_{(\lambda,\mu)}$.

*Example* 4.2. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, where $p^*$ is the conjugate index of $p$. Note that $(\ell_p, \ell_r)$-compact operators are exactly the $(p,r)$-compact operators of [ALO12].

*Example* 4.3. Let $\lambda$ be a BK-space with the property that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then $(\lambda, \lambda^\times)$-compact operators are exactly the $\lambda$-compact operators of [GB13].

For an example of a $\lambda$-compact operator which is not $p$-compact, we refer to [GB13].

If, in addition, $\lambda$ is a normed AK-space and $\lambda\mu \leq \ell_1$, then $\Phi_{(x_n)} \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(\mu, X)$ for any $\overline{x} \in \lambda^s(X)$. Indeed,

$$\Phi_{(x_n)} = \sum_{n=1}^\infty \|x_n\| e_n^* \otimes \frac{x_n}{\|x_n\|} = \Sigma_1 M_{\left(\frac{x_n}{\|x_n\|}\right)} M_{(\|x_n\|)}.$$

with $(e_n^*) \in \mu^{w^*}(\mu^*)$.

**Proposition 4.4.** *The class of $(\lambda, \mu)$-compact operators $\mathcal{K}_{(\lambda,\mu)}$ is an operator ideal.*

*Proof.* 1. Let us pick $e_n \in \lambda^s(\mathbb{K})$. Then

$$I_\mathbb{K}(B_\mathbb{K}) = B_\mathbb{K} \subset \Phi_{e_n}(B_\mu) = \left\{ \sum_{i=1}^\infty e_n^i a_i = a_n : (a_n) \in B_\mu \right\} = B_\mathbb{K}.$$

Thus, we have $I_{\mathbb{K}} \in \mathcal{K}_{(\lambda,\mu)}(\mathbb{K},\mathbb{K})$.

2. Let $T, S \in \mathcal{K}_{(\lambda,\mu)}(Y,X)$. Then there exist $(x_n), (z_n) \in \lambda^s(X)$ such that

$$T(B_Y) \subset \Phi_{(x_n)}(B_\mu) \text{ and } S(B_Y) \subset \Phi_{(z_n)}(B_\mu).$$

Therefore,

$$\begin{aligned}(T+S)(B_Y) \subset T(B_Y) + S(B_Y) &\subset \Phi_{(x_n)}(B_\mu) + \Phi_{(z_n)}(B_\mu)\\ &= (\Phi_{(x_n)} + \Phi_{(z_n)})(2B_\mu) = (\Phi_{2(\overline{x}+\overline{z})})(B_\mu)\end{aligned}$$

and thus $T + S \in \mathcal{K}_{(\lambda,\mu)}(Y,X)$.

3. Let $S \in \mathcal{K}_{(\lambda,\mu)}(Y,X)$, $T \in \mathcal{L}(W,Y)$ and $R \in \mathcal{L}(X,V)$. Then $RST \in \mathcal{K}_{(\lambda,\mu)}(Y,X)$, because $RST(B_Y) \subset \Phi_{(a_n)}(B_\mu)$ for $(a_n) = \|T\|(Rx_n) \in \lambda^s(V)$.

□

**Proposition 4.5.** *The operator ideal $\mathcal{K}_{(\lambda,\mu)}$ is surjective.*

*Proof.* This follows directly from the description of the surjective hull in Proposition 1.11 . □

Let $T \in \mathcal{K}_{(\lambda,\mu)}(Y,X)$. Let $(x_n) \in \lambda^s(X)$ be such that $T(B_Y) \subset \Phi_{(x_n)}(B_\mu)$. Since $\Phi_{(x_n)} \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}$ it follows from Proposition 1.11 that $\mathcal{K}_{(\lambda,\mu)} \subset \mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}$.

On the other hand, if

$$T = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(X,Y)$$

with $(\sigma_n) \in \lambda$, $(x_n^*) \in \mu^{w^*}(X^*)$ and $(y_n) \in \ell_\infty(Y)$, then

$$T = \sum_{n=1}^{\infty} x_n^* \otimes (\sigma_n y_n) = \Phi_{(\sigma_n y_n)} \circ j_{w^*}\overline{x}^*,$$

so

$$T(B_X) \subset \Phi_{(\sigma_n y_n)}(\|j_{w^*}\overline{x}^*\| B_\mu) \subset \Phi_{(\|j_{w^*}\overline{x}^*\|(\sigma_n y_n))}(B_\mu).$$

This means that $\mathcal{N}_{(\lambda,\mu,\ell_\infty)} \subset \mathcal{K}_{(\lambda,\mu)}$.

Thus, due to Proposition 4.5 we have $\mathcal{K}_{(\lambda,\mu)} = \mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}$. The last result is summarized in the following theorem.

**Theorem 4.6.** *Let $\mu$ be a BK-space. Let $\lambda$ be a normed AK-space such that $\lambda\mu \leq \ell_1$. Then the operator ideal $\mathcal{K}_{(\lambda,\mu)}$ is equal to $\mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}$.*

**Corollary 4.7.** *Let $\lambda$ be a normed AK-space. Then the operator ideal $\mathcal{K}_\lambda$ is equal to $\mathcal{N}^{\text{sur}}_{(\lambda,\lambda^\times,\ell_\infty)}$.*

Let us show that the norm $N^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}(T)$ of $T \in \mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}(Y,X) = \mathcal{K}_{(\lambda,\mu)}(Y,X)$ can be expressed as $k_{(\lambda,\mu)}(T) := \inf\|\overline{x}\|_\lambda$, where the infimum ranges over all $\overline{x} \in \lambda^s(X)$ such that $T(B_Y) \subset \Phi_{\overline{x}}(B_\mu)$ (which is the norm that was was used in [BK18]). We denote $k_\lambda := k_{(\lambda,\lambda^\times)}$.

On one hand, [[Pie80, Proposition 8.5.4]] gives that

$$N^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}(T) \leq N_{(\lambda,\mu,\ell_\infty)}(\Phi_{\overline{x}}) \leq \|\overline{x}\|_\lambda,$$

so $N^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}(T) \leq k_{(\lambda,\mu)}(T)$. On the other hand, given the factorization $TQ_Y = BM_{\overline{\sigma}}A \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(\ell_1(B_Y),X)$ with $M_{\overline{\sigma}} \in \mathcal{N}_{(\lambda,\mu,\ell_\infty)}(\mu,\ell_1)$ for some $\overline{\sigma} \in \lambda$ (we can assume that $\|A\| = \|B\| = 1$), note that

$$BM_{\overline{\sigma}} = \Sigma_1 M_{(\sigma_n Be_n)} = \sum_{n=1}^{\infty} e_n^* \otimes (\sigma_n Be_n) = \Phi_{(\sigma_n Be_n)}.$$

So,

$$k_{(\lambda,\mu)}(T) \leq \|(\sigma_n Be_n)\|_\lambda \leq \|A\|\|\overline{\sigma}\|_\lambda\|B\|,$$

from which it follows that

$$k_{(\lambda,\mu)}(T) \leq \inf\|A\|\|\overline{\sigma}\|_\lambda\|B\| = N_{(\lambda,\mu,\ell_\infty)}(TQ_Y) = N^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}(T).$$

In particular, this shows that in this case $(\mathcal{K}_{(\lambda,\mu)}, k_{(\lambda,\mu)})$ is a quasi-Banach surjective operator ideal whenever $\mathcal{N}_{(\lambda,\mu,\ell_\infty)}$ is quasi-Banach. Considering Proposition 1.13, $\mathcal{N}_{(\lambda,\mu,\ell_\infty)}$ is a quasi-Banach operator ideal. The last result is summarized in the following theorem.

**Theorem 4.8.** *Let $\mu$ be a BK-space. Let $\lambda$ be a normed AK-space such that $\lambda\mu \leq \ell_1$. If $(\mathcal{N}_{(\lambda,\mu,\ell_\infty)}, N_{(\lambda,\mu,\ell_\infty)})$ is a quasi-Banach operator ideal, then its surjective hull $(\mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}, \mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)})$ is equal to $(\mathcal{K}_{(\lambda,\mu)}, k_{(\lambda,\mu)})$.*

**Corollary 4.9.** *Let $\mu$ be a BK-space. Let $\lambda$ be a normed AK-space such that $\lambda\mu \leq \ell_1$. If $(\mathcal{N}_{(\lambda,\mu,\ell_\infty)}, N_{(\lambda,\mu,\ell_\infty)})$ is a quasi-Banach operator ideal, then $(\mathcal{K}_{(\lambda,\mu)}, k_{(\lambda,\mu)})$ is a surjective quasi-Banach operator ideal.*

**Corollary 4.10.** *Let $(\mu, \|\cdot\|_\mu)$ be a BK-space. Let $(\lambda, \|\cdot\|_\lambda)$ be a normed AK-space, such that $\lambda\mu \leq \ell_1$. Let $\lambda$ and $\mu$ be symmetric and monotone, both equipped with normalized monotone K-symmetric (quasi-)norms. Assume also that $\lambda$ and $\mu$ are are stable under diagonalization. Then quasi-Banach operator ideal $(\mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)}, \mathcal{N}^{\text{sur}}_{(\lambda,\mu,\ell_\infty)})$ is equal to $(\mathcal{K}_{(\lambda,\mu)}, k_{(\lambda,\mu)})$.*

**Corollary 4.11.** *Let $(\lambda, \|\cdot\|_\lambda)$ be a normed AK-space that is symmetric and monotone and equipped with a normalized monotone K-symmetric norm. Assume that $\lambda$ is stable under diagonalization. Then quasi-Banach operator ideal $(\mathcal{N}_{(\lambda,\lambda^\times,\ell_\infty)}, N_{(\lambda,\lambda^\times,\ell_\infty)})$ is equal to $(\mathcal{K}_\lambda, k_\lambda)$*

**Corollary 4.12.** *Let $\lambda$ be a normed AK-space. If $(\mathcal{N}_{(\lambda,\lambda^\times,\ell_\infty)}, N_{(\lambda,\lambda^\times,\ell_\infty)})$ is a quasi-Banach operator ideal, then $(\mathcal{K}_\lambda, k_\lambda)$ is a surjective quasi-Banach operator ideal.*

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