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Chapter 1

Introduction

1.1 Background

Daugavet and Δ-points were introduced as pointwise versions of the Daugavet property and the diametral local diameter-two property, respectively (see [AHLP20]). The Daugavet property originates from a classical paper by Daugavet, in which he proved that every compact operator T on $C[0,1]$ satisfies the equation

$$\|\mathrm{Id} - T\| = 1 + \|T\|,$$

which is now known as the Daugavet equation (see [Dau63]). Soon after, it was shown that the Daugavet equation holds for every compact operator on $L_1[0,1]$ as well (see [Loz66]). A Banach space in which every rank-one operator satisfies the Daugavet equation is said to have the *Daugavet property* (for further background, see [Wer01]).

The Daugavet property has significant implications for the geometric structure of a Banach space. For instance, every Banach space with the Daugavet property contains an isomorphic copy of ℓ_1 (see [Wer01, Theorem 2.6]) and cannot have an unconditional basis (see [Kad96, Corollary 2.3]). Moreover, the Daugavet property is a diameter-two property, meaning that every slice of the unit ball has diameter exactly 2 – the largest possible diameter. In this sense, it stands in stark contrast to the Radon–Nikodým property, which implies that there exist slices of the unit ball with arbitrarily small diameter.

The Daugavet property admits several equivalent characterizations. In particular, a Banach space X has the Daugavet property if and only

if for every $x \in S_X$ and every $\varepsilon > 0$, we have

$$B_X = \overline{\operatorname{conv}}\,\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\}$$

(see [Wer01, Corollary 2.3]). This characterization motivates a natural pointwise version of the property: an element $x \in S_X$ is called a *Daugavet point* if $B_X = \overline{\operatorname{conv}}\,\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\}$ for every $\varepsilon > 0$. Similarly, an element $x \in S_X$ is called a Δ-*point* if $x \in \overline{\operatorname{conv}}\,\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\}$ for every $\varepsilon > 0$. It is immediate that every Daugavet point is a Δ-point, but the converse fails in general (see, e.g, [AHLP20, Example 4.7]). The study of Daugavet and Δ-points has attracted significant interest in recent years. For example, Daugavet and Δ-points have been investigated in

- $L_1(\mu)$ spaces and their preduals (see [AHLP20]);
- $C(K)$ spaces (see [AHLP20]);
- Absolute sums of Banach spaces (see [AHLP20], [HPV21]);
- Banach spaces with unconditional bases (see [ALMT21]);
- Lipschitz-free spaces (see [JRZ22]);
- Projective tensor products (see [DJRZ22]);
- Asymptotically smooth and convex spaces (see [ALMP22]).

In the course of studying Daugavet and Δ-points, several related notations have been introduced, including super and ccs versions of these points (see [MPRZ24]), relative Daugavet points (see [AAL$^+$24b]), and ∇-points (see [HLPV24]).

An important distinction between the Daugavet property and the existence of Daugavet points is highlighted by the construction in [ALMT21], which provides an example of a Banach space with a 1-unconditional basis that nevertheless contains a Daugavet point. In this thesis, we further explore this distinction by constructing an example of a Banach space with the Radon–Nikodým property that contains Daugavet points. Moreover, we provide examples of reflexive Banach spaces that admit Daugavet points.

1.2 Summary of the thesis

The main aim of the thesis is to study Daugavet and Δ-points in various Banach spaces. In particular, we examine these points in Lipschitz-free spaces and their duals – spaces of Lipschitz functions – providing characterizations for certain subclasses of these spaces, as well as, a complete characterization of Daugavet points in Lipschitz-free spaces. Additionally, we present an example of a metric space, whose Lipschitz-free space has the Radon–Nikodým property and a Daugavet point. Moreover, we show that this Lipschitz-free space is also a dual space isomorphic to ℓ_1. The thesis further investigates renormings of Banach spaces that admit Daugavet or Δ-points, through which we provide examples of reflexive spaces containing Daugavet points. Finally, we introduce several classes of Banach spaces that cannot contain Daugavet or Δ-points.

The thesis is organized into five chapters.

Chapter 1 provides an overview of the main definitions and preliminary results relevant to the thesis.

In Chapter 2, we study Daugavet and Δ-points in Lipschitz-free spaces. We prove that Daugavet points satisfy the distance-to-denting characterization for arbitrary Lipschitz-free spaces. Additionally, based on [JRZ22, Theorem 4.7], we derive a potential characterization of Δ-points, which we will prove for all finitely supported elements, as well as for certain classes of Lipschitz-free spaces. In addition, we provide a fully metric characterization of molecules that are Δ-points. Furthermore, we give an example of a Lipschitz-free space that is a dual space isomorphic to ℓ_1 and contains a Daugavet point. To conclude the chapter, we prove that Daugavet and Δ-points coincide in Lipschitz-free spaces over subsets of $\mathbb{R}$-trees. This chapter is based on [AAL$^+$24a], [AAL$^+$24b], [Vee23], [Vee25a], and [Vee25b].

Chapter 3 is dedicated to the study of Daugavet and Δ-points in the duals of Lipschitz-free spaces – that is, spaces of Lipschitz functions. We prove that Daugavet and Δ-points coincide in spaces of Lipschitz functions over compact metric spaces, and furthermore, that they are precisely the local functions. We also prove that every space of Lipschitz functions contains a Δ-point. A broad class of these spaces – namely, spaces of Lipschitz functions over metric spaces that are unbounded or fail to be uniformly discrete – also contain a Daugavet point. However, we show the existence of a metric space whose space of

Lipschitz functions does not contain any Daugavet points. This proves that, in general, Daugavet and Δ-points do not coincide in spaces of Lipschitz functions. This chapter is based on [Vee25a].

In Chapter 4, we study renormings of Banach spaces that admit Daugavet or Δ-points. The motivation stems from the question of whether there exists a reflexive Banach space that contains Daugavet or Δ-points. We prove that every infinite-dimensional Banach space can be renormed with a Δ-point. Furthermore, we show that every infinite-dimensional Banach space with an unconditional basis can be renormed with a Daugavet point. This shows that the classical spaces ℓ_p, for $p \in [1, \infty)$, admit equivalent norms, under which the spaces contain Daugavet points, thus also providing us with the desired example. This chapter is based on [AAL$^+$24a] and [HLPV24].

The last part of the thesis – Chapter 5 – focuses on finding properties of Banach spaces that prevent the containment of Daugavet or Δ-points. We show that whenever a Banach space X contains a $\mathfrak{D}$-point (a weaker cousin of Δ-points), or its dual X^* contains a weak* Δ-point, then the dual must also contain a weak* super Δ-point. As a corollary, we deduce that a Banach space with a shrinking k-unconditional basis for $k < 2$ cannot contain any $\mathfrak{D}$-points, and thus also does not contain any Δ-points. We further prove that if a Banach space with the Radon–Nikodým property contains a Daugavet point, then it must contain a Daugavet point that is also an extreme point of the unit ball. This implies that a Banach space with the Radon–Nikodým property, in which every extreme point of the unit ball is also denting, cannot contain any Daugavet points. This chapter is based on [AAL$^+$24a] and [HLPV24].

To keep this thesis more focused, we have only included a selection of the results from the papers on which it is based; we refer the reader to these papers for further detail.

1.3 Notation

In this thesis, we consider only real Banach spaces and use common notation. Given a Banach space X, we denote the closed unit ball by B_X, the unit sphere by S_X, and the dual space by X^*. For a subset A of a Banach space X, we denote its closed convex hull by $\overline{\operatorname{conv}} A$ and its diameter by $\operatorname{diam}(A)$.

Given a metric space M, a point $x \in M$, and $r \geqslant 0$, we denote by

$B(x,r)$ the closed ball with center x and radius r. For every $u, v \in M$ and $\delta > 0$, we define a metric segment

$$[u,v] := \{p \in M \colon d(u,p) + d(v,p) = d(u,v)\}$$

and an approximate metric segment

$$[u,v]_\delta := \{p \in M \colon d(u,p) + d(v,p) < d(u,v) + \delta\}\,.$$

1.4 Preliminaries

In this section, we introduce the main notation and some general results that are used throughout the thesis.

Diametral points

Daugavet and Δ-points were first defined in [AHLP20]. There are several ways to define these points; we first present the one from [AHLP20]: given a Banach space X and $x \in S_X$, we say that x is

(1) a *Daugavet point* if $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$,

(2) a *Δ-point* if $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$,

where

$$\Delta_\varepsilon(x) = \big\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\big\}.$$

Both Daugavet and Δ-points admit equivalent definitions in the terms of slices (see [AHLP20, Lemmas 2.1, 2.2]). We first recall the definition of a slice.

Definition 1.1. Let X be a Banach space, and let A be a non-empty bounded subset of X. For $x^* \in S_{X^*}$ and $\alpha > 0$, the set

$$S(A, x^*, \alpha) := \left\{x \in A \colon x^*(x) > \sup_{y \in A} x^*(y) - \alpha\right\},$$

is called a *slice* of the set A.

We primarily work with slices of the unit ball B_X, and in this case we usually write $S(x^*, \alpha)$ instead of $S(B_X, x^*, \alpha)$.

We now introduce the equivalent definitions of Daugavet and Δ-points, as well as their super versions and $\mathfrak{D}$-points. Both definitions are used throughout the thesis.

Definition 1.2. Let X be a Banach space, and let $x \in S_X$. We say that x is

(1) a *Daugavet point* if $\sup_{y\in S} \|x - y\| = 2$ for every slice S of B_X;

(2) a *Δ-point* if $\sup_{y\in S} \|x - y\| = 2$ for every slice S of B_X that contains x;

(3) a *super Daugavet point* if $\sup_{y\in V} \|x - y\| = 2$ for every non-empty relatively weakly open subset V of B_X;

(4) a *super Δ-point* if $\sup_{y\in V} \|x - y\| = 2$ for every relatively weakly open subset V of B_X that contains x;

(5) a *$\mathfrak{D}$-point* if $\sup_{y\in S} \|x - y\| = 2$ for every slice $S = S(x^*, \varepsilon)$ of B_X with $x^*(x) = 1$ and $\varepsilon > 0$.

It is immediate from the definition of weak convergence that $x \in S_X$ is a super Daugavet point if and only if for every $y \in S_X$ there exists a net $(y_\alpha) \subseteq S_X$ such that $y_\alpha \to y$ weakly and $\|x - y_\alpha\| \to 2$. Similarly, $x \in S_X$ is a super Δ-point if and only if there exists a net $(y_\alpha) \subseteq S_X$ such that $y_\alpha \to x$ weakly and $\|x - y_\alpha\| \to 2$.

The following implications can be derived directly from the definitions.

$$\begin{array}{ccccc} \text{super Daugavet point} & \Longrightarrow & \text{Daugavet point} & & \\ \Downarrow & & \Downarrow & & \\ \text{super } \Delta\text{-point} & \Longrightarrow & \Delta\text{-point} & \Longrightarrow & \mathfrak{D}\text{-point} \end{array}$$

None of the reverse implications hold, meaning that these are indeed all distinct notions. We refer the reader to [AHLP20] and [MPRZ24] for concrete examples.

Similarly, one can define the weak* slice and weak* versions of Daugavet and Δ-points.

Definition 1.3. Let X be a Banach space, and let A be a non-empty bounded subset of X^*. For $x \in S_X$ and $\alpha > 0$, the set

$$S(A, x, \alpha) := \left\{ x^* \in A \colon x^*(x) > \sup_{y^*\in A} y^*(x) - \alpha \right\},$$

is called a *weak* slice* of the set A.

We primarily work with weak* slices of the unit ball B_{X^*}, and in these cases we usually denote the weak* slice by $S(x, \alpha)$.

Definition 1.4. Let X be a Banach space, and let $x^* \in S_{X^*}$. We say that x^* is

(1) a *weak* Daugavet point* if $\sup_{y^* \in S} \|x^* - y^*\| = 2$ for every weak* slice S of B_{X^*};

(2) a *weak* Δ-point* if $\sup_{y^* \in S} \|x^* - y^*\| = 2$ for every weak* slice S of B_{X^*} that contains x^*;

(3) a *weak* super Daugavet point* if $\sup_{y^* \in V} \|x^* - y^*\| = 2$ for every non-empty relatively weak* open subset V of B_{X^*};

(4) a *weak* super Δ-point* if $\sup_{y^* \in V} \|x^* - y^*\| = 2$ for every relatively weak* open subset V of B_{X^*} that contains x^*.

In Chapters 4 and 5, we will also work with certain stronger versions of super Δ-points.

Definition 1.5. Let X be a Banach space. We say that

(1) a point $x \in S_X$ is a *sequential super Δ-point* if there exists a sequence (x_n) in S_X that converges weakly to x and satisfies $\|x - x_n\| \to 2$.

(2) a point $x^* \in S_{X^*}$ is a *weak* sequential super Δ-point* if there exists a sequence (x_n^*) in S_{X^*} that converges weak* to x^* and satisfies $\|x^* - x_n^*\| \to 2$.

Throughout the thesis, when working with a slice $S(x^*, \alpha)$, we often want to assume that α is as small as needed. The following lemma allows us to do so.

Lemma 1.6 (see [IK04, Lemma 2.1])**.** *Let X be a Banach space, $x^* \in S_{X^*}$, and $\alpha > 0$. For every $x \in S(x^*, \alpha) \cap S_X$ and every $\beta \in (0, \alpha)$, there exists $y^* \in S_{X^*}$ such that*

$$x \in S(y^*, \beta) \subseteq S(x^*, \alpha).$$

We first focus on Daugavet points. To this end, we recall the definitions of extreme points and denting points.

Definition 1.7. Let A be a bounded, closed, convex subset of a Banach space X. We say that $x \in A$ is

(1) an *extreme point* of A if $x_1 = x_2 = x$ whenever $x_1, x_2 \in A$ and $x = \frac{1}{2}(x_1 + x_2)$;

(2) a *denting point* of A if for every $\varepsilon > 0$, there exists a slice S of A such that $x \in S$ and $\operatorname{diam}(S) < \varepsilon$.

We denote the set of all denting points of A by $\operatorname{dent}(A)$.

Furthermore, in Chapter 2, we also briefly work with preserved extreme points. Recall that an extreme point of the closed unit ball of a Banach space is said to be a *preserved extreme point* if it is also an extreme point of the closed unit ball of the bidual space. Next proposition shows that there is a close relation between Daugavet points and denting points.

Proposition 1.8 (see [JRZ22, Proposition 3.1])**.** *Let X be a Banach space, and let $x \in S_X$ be a Daugavet point. Then for every $y \in \operatorname{dent}(B_X)$, we have $\|x - y\| = 2$.*

It was also shown in [JRZ22] that the converse of Proposition 1.8 does not hold. Indeed, the space ℓ_∞ has no denting points in its unit ball, but at the same time does not have the Daugavet property. On the other hand, it was shown in [JRZ22] that for Lipschitz-free spaces over compact metric spaces, the converse does hold (see [JRZ22, Theorem 3.2]). We say that a Banach space X satisfies the *distance-to-denting property for Daugavet points* if for every $x \in S_X$, the point x is a Daugavet point if and only if x is at distance 2 from all the denting points of B_X. One of the aims of the thesis is to show that the distance-to-denting property for Daugavet points holds for all Lipschitz-free spaces.

To conclude the introduction of Daugavet points, we also state the following result.

Proposition 1.9 (see [AAL$^+$24b, Proposition 3.15])**.** *Let X be a Banach space, let $x \in S_X$, and let $(z_n)_{n\in I}$ be a family of elements in B_X for some non-empty subset I of $\mathbb{N}$. Suppose there exists a family of scalars $(\lambda_n)_{n\in I} \subseteq (0, 1]$ with $\sum_{n\in I} \lambda_n = 1$ such that $x = \sum_{n\in I} \lambda_n z_n$. If x is a Daugavet point, then each z_n is also a Daugavet point.*

Proof. Fix $k \in I$ and $\varepsilon > 0$, and pick any slice S of B_X. Since x is a Daugavet point, we can find $y \in S$ such that $\|x - y\| > 2 - \lambda_k \varepsilon$. We have

$$\begin{aligned}
\lambda_k \|z_k - y\| &\geqslant \left\| \sum_{n\in I} \lambda_n z_n - \sum_{n\in I} \lambda_n y \right\| - \left\| \sum_{n\in I\setminus\{k\}} \lambda_n z_n - \sum_{n\in I\setminus\{k\}} \lambda_n y \right\| \\
&\geqslant \|x - y\| - \sum_{n\in I\setminus\{k\}} \lambda_n \|z_n - y\| \\
&> 2 - \varepsilon\,\lambda_k - 2(1 - \lambda_k) \\
&= (2 - \varepsilon)\lambda_k,
\end{aligned}$$

so $\|z_k - y\| > 2 - \varepsilon$, and it follows that z_k is a Daugavet point. □

Next we will focus on Δ-points. In particular, we introduce the following result, which is a useful tool for examining Δ-points.

Proposition 1.10 (see [Vee25a, Proposition 2.1])**.** *Let $x_0 \in S_X$ be a Δ-point. Then for every slice $S(x_0^*, \alpha)$ of B_X with $x_0 \in S(x_0^*, \alpha)$, there exist sequences (x_i) in $S(x_0^*, \alpha)$ and (x_i^*) in B_{X^*} such that $x_i^* \in S(x_0, 1/i)$ for all $i \in \mathbb{N}$, and*

$$\|x_i - x_j\| \geqslant 2 - \alpha \quad \text{and} \quad \|x_i^* - x_j^*\| \geqslant 2 - \alpha$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i \neq j$.

Furthermore, if $A \subseteq B_X$ and $C \subseteq B_{X^*}$ are such that every subslice of $S(x_0^*, \alpha)$ contains a point from A, and for some $\delta > 0$ every subslice of $S(x_0, \delta)$ contains a point from C, then we may additionally assume that the sequence (x_i) is contained in A and the sequence (x_i^*) is contained in C.

Proof. It is enough to prove the "Furthermore..." statement, as the first statement follows by taking $A = B_X$ and $C = B_{X^*}$. Let $A \subseteq B_X$, $C \subseteq B_{X^*}$, and $\delta > 0$ be as in the statement of the proposition. Fix a slice $S(x_0^*, \alpha)$ with $x_0 \in S(x_0^*, \alpha)$. Let $\gamma \in (0, \alpha)$ be such that $x_0 \in S(x_0^*, \alpha - 2\gamma)$, $\gamma/2 < \delta$, and $\gamma < 1$. We will construct the sequences (x_i) and (x_i^*) recursively. Since x_0 is a Δ-point, by [JRZ22, Lemma 2.2] we can find a subslice T of $S(x_0^*, \alpha - \gamma)$ such that $\|x_0 - y\| > 2 - \gamma/2$ for every $y \in T$. Choose $x_1 \in T \cap A \subseteq S(x_0^*, \alpha - \gamma) \cap A$, then also $\|x_0 - x_1\| > 2 - \gamma/2$. Note that

$$S\left(\frac{x_0 - x_1}{\|x_0 - x_1\|}, 1 - \frac{2 - \gamma/2}{\|x_0 - x_1\|}\right) \subseteq S(x_0, \delta).$$

Indeed, if $y^* \in S\left(\frac{x_0-x_1}{\|x_0-x_1\|}, 1-\frac{2-\gamma/2}{\|x_0-x_1\|}\right)$, then $y^*(x_0-x_1) > 2-\gamma/2$, and thus

$$y^*(x_0) > 2-\gamma/2+y^*(x_1) \geqslant 1-\gamma/2 > 1-\delta.$$

Choose $x_1^* \in S\left(\frac{x_0-x_1}{\|x_0-x_1\|}, 1-\frac{2-\gamma/2}{\|x_0-x_1\|}\right) \cap C$. Then $x_1^*(x_0-x_1) > 2-\gamma/2$, which implies both $x_1^*(x_0) > 1-\gamma/2$ and $x_1^*(-x_1) > 1-\gamma/2$. Furthermore,

$$\|x_0-x_1\| > 2-\frac{\gamma}{2} > 2-\alpha$$

and

$$\|x_0^*-x_1^*\| \geqslant x_0^*(x_1)-x_1^*(x_1) > 1-\alpha+\gamma+1-\frac{\gamma}{2} > 2-\alpha.$$

Assume that we have found $x_1,\dots,x_{n-1} \in A$ and $x_1^*,\dots,x_{n-1}^* \in C$ such that

$$x_i \in S(x_0^*, \alpha-\gamma), \quad x_i^*(x_0) > 1-\gamma/2^i, \quad \text{and} \quad x_i^*(-x_i) > 1-\gamma/2^i$$

for every $i \in \{1,\dots,n-1\}$, and

$$\|x_i-x_j\| \geqslant 2-\alpha \quad \text{and} \quad \|x_i^*-x_j^*\| \geqslant 2-\alpha$$

for every $i,j \in \{0,\dots,n-1\}$ with $i \neq j$. Let

$$y^* = \sum_{i=0}^{n-1} x_i^*.$$

Then

$$y^*(x_0) = x_0^*(x_0) + \sum_{i=1}^{n-1} x_i^*(x_0) > 1-\alpha+2\gamma+\sum_{i=1}^{n-1}(1-\gamma/2^i) > n-\alpha+\gamma.$$

Since x_0 is a Δ-point, as above, we can find $x_n \in A$ such that

$$y^*(x_n) > n-\alpha+\gamma$$

and $\|x_0-x_n\| > 2-\gamma/2^n$. Similarly to the base step, we find $x_n^* \in C$ such that $x_n^*(x_0-x_n) > 2-\gamma/2^n$. Then $x_n^*(x_0) > 1-\gamma/2^n$ and $x_n^*(-x_n) > 1-\gamma/2^n$. Furthermore,

$$x_i^*(x_n) = y^*(x_n) - \sum_{j\in\{0,\dots,n-1\}\setminus\{i\}} x_j^*(x_n) > n-\alpha+\gamma-(n-1) = 1-\alpha+\gamma$$

for every $i \in \{0, \dots, n-1\}$. Thus,

$$\|x_n - x_i\| \geqslant x_i^*(x_n) - x_i^*(x_i) > 1 - \alpha + \gamma + 1 - \frac{\gamma}{2^i} > 2 - \alpha$$

for every $i \in \{1, \dots, n-1\}$, and

$$\|x_n^* - x_i^*\| \geqslant x_i^*(x_n) - x_n^*(x_n) > 1 - \alpha + \gamma + 1 - \frac{\gamma}{2^n} > 2 - \alpha$$

for every $i \in \{0, \dots, n-1\}$. □

By using Proposition 1.10, we can show that the Kuratowski measure of non-compactness is equal to 2 for each slice containing a Δ-point, thus providing a stronger version of [ALMP22, Theorem 4.2], as well as an alternative proof of [ALMP22, Theorem 3.5]. Recall that for a bounded subset $A \subseteq X$, the *Kuratowski measure of non-compactness* $\alpha(A)$ is the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of sets with diameter less than ε.

Corollary 1.11 (see [Vee25a, Corollary 2.2])**.** *Let $x \in S_X$ be a Δ-point. Then $\alpha(S) = 2$ for every slice S of B_X with $x \in S$. Furthermore, $\alpha\big(S(x, \delta)\big) = 2$ for every $\delta > 0$.*

Proof. By Proposition 1.10, any slice $S(x^*, \delta)$ of B_X with $x \in S(x^*, \delta)$ contains countable number of points such that any two points are at least at distance $2-\delta$ from each other. Therefore, covering $S(x^*, \delta)$ with a finite number of subsets means that at least one of these subsets must contain two points whose distance is at least $2 - \delta$. By Lemma 1.6, for every $\beta \in (0, \delta)$ there exists $y^* \in S_X$ such that $x \in S(y^*, \beta) \subseteq S(x^*, \delta)$, and thus

$$\alpha(S(x^*, \delta)) \geqslant \alpha(S(y^*, \beta)) \geqslant 2 - \beta.$$

Since β was arbitrary, it follows that $\alpha(S(x^*, \delta)) = 2$.

Next we show the equality $\alpha(S(x, \delta)) = 2$ for every $\delta > 0$. Fix $\delta > 0$. For any $\beta > 0$, by Proposition 1.10, we can find a sequence (x_i^*) in S_{X^*} such that $x_i^* \in S(x, 1/i)$ and $\|x_i^* - x_j^*\| \geqslant 2 - \beta$ for any distinct $i, j \in \mathbb{N}$. Then, countably many of these functionals must belong to the slice $S(x, \delta)$, and thus $\alpha(S(x, \delta)) \geqslant 2 - \beta$. As β was arbitrary, we get $\alpha(S(x, \delta)) = 2$. □

In [ALMP22], these results were used to show that asymptotically uniformly smooth and reflexive asymptotically uniformly convex spaces do not contain Δ-points (for more information about these types of

spaces, see, e.g., [JLPS02]). At the end of [ALMP22, Section 4], it was noted that it is not clear whether quasi-denting points can be Daugavet or Δ-points. Recall that $x \in S_X$ is a *quasi-denting point* of B_X if for every $\varepsilon > 0$ there exists a slice S of B_X with $x \in S$ such that $\alpha(S) < \varepsilon$. From Corollary 1.11, it is clear that a quasi-denting point cannot be a Δ-point.

Similarly to Proposition 1.10 and Corollary 1.11, we can prove the two following statements for weak* Δ-points.

Proposition 1.12 (see [Vee25a, Proposition 2.3])**.** *Let $x_0^* \in S_{X^*}$ be a weak* Δ-point. Then for every weak* slice $S(x_0, \alpha)$ of B_{X^*} with $x_0^* \in S(x_0, \alpha)$, there exist sequences (x_i^*) in $S(x_0, \alpha)$ and (x_i) in B_X such that $x_i \in S(x_0^*, 1/i)$ for every $i \in \mathbb{N}$, and*

$$\|x_i^* - x_j^*\| \geqslant 2 - \alpha \quad \text{and} \quad \|x_i - x_j\| \geqslant 2 - \alpha$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i \neq j$.

Furthermore, if $A \subseteq B_{X^*}$ and $C \subseteq B_X$ are such that every subslice of $S(x_0, \alpha)$ contains a point from A, and for some $\delta > 0$ every subslice of $S(x_0^*, \delta)$ contains a point from C, then we may additionally assume that the sequence (x_i^*) is contained in A and the sequence (x_i) is contained in C.

Corollary 1.13 (see [Vee25a, Corollary 2.4])**.** *Let $x^* \in S_{X^*}$ be a weak* Δ-point. Then $\alpha(S) = 2$ for every weak* slice S of B_{X^*} with $x^* \in S$. Furthermore, $\alpha\big(S(x^*, \delta)\big) = 2$ for every $\delta > 0$.*

The Radon–Nikodým property and the Schur property

On the opposite end of the spectrum from the Daugavet property lies the Radon–Nikodým property. This property is well studied and admits several equivalent definitions. For the purpose of this thesis, we use the following.

Definition 1.14. A Banach space has the *Radon–Nikodým property* if every non-empty bounded closed convex set is the closed convex hull of its denting points.

From Proposition 1.8, we already know that a Daugavet point must be at distance two from every denting point. Consequently, a

Banach space cannot have both the Daugavet property and the Radon–Nikodým property. It is not trivial to find examples of Banach spaces with the Radon–Nikodým property while also containing a Daugavet point. This topic is further discussed in Section 2.3 and Chapter 4.

Another property considered in Chapter 4 is the Schur property.

Definition 1.15. A Banach space X has the *Schur property* if every weakly convergent sequence in X is norm convergent.

It is clear that the Schur property is incompatible with the existence of sequential super Δ-points. However, there does exist a (highly non-trivial) example of a Banach space with both the Daugavet property and the Schur property (see [KW04]).

Spaces of Lipschitz functions and Lipschitz-free spaces

Let M be a metric space with metric d and a fixed base point 0. We denote by $\mathrm{Lip}_0(M)$ the Banach space of all Lipschitz functions $f\colon M \to \mathbb{R}$ with $f(0) = 0$ equipped with the natural linear structure and the norm

$$\|f\| := \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)}\colon x, y \in M, x \neq y\right\}.$$

Let $\delta\colon M \to \mathrm{Lip}_0(M)^*$ be the canonical isometric embedding of M into $\mathrm{Lip}_0(M)^*$, given by $x \mapsto \delta_x$, where $\delta_x(f) = f(x)$. The norm-closed linear span of $\delta(M)$ in $\mathrm{Lip}_0(M)^*$ is called the *Lipschitz-free space over* M and is denoted by $\mathcal{F}(M)$ (see [God15] and [Wea18] for background). An element in $\mathcal{F}(M)$ of the form

$$m_{xy} := \frac{\delta_x - \delta_y}{d(x,y)}$$

for $x, y \in M$ with $x \neq y$ is called a *molecule*. Clearly $m_{xy} \in S_{\mathcal{F}(M)}$ for all $x, y \in M$ with $x \neq y$, and it is well known that

$$B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}\,\{m_{xy}\colon x, y \in M, x \neq y\}$$

and

$$\mathcal{F}(M)^* = \mathrm{Lip}_0(M).$$

We also require the definition of the support of elements in Lipschitz-free spaces, as introduced in [AP20] and [APPP20].

Definition 1.16. For an element $\mu \in \mathcal{F}(M)$, we say that the *support* of μ, denoted by $\operatorname{supp}(\mu)$, is the smallest closed set $K \subseteq M$ such that $\mu \in \mathcal{F}(K \cup \{0\})$.

It is know that any finitely supported element can be represented as convex combination of molecules (see, e.g., [Wea18, Proposition 3.16]). Next, we introduce the notation of a local Lipschitz function.

Definition 1.17. A Lipschitz function f is called *local* if for every $\varepsilon > 0$ there exist $u, v \in M$ with $u \neq v$ such that $\langle f, m_{uv} \rangle > \|f\| - \varepsilon$ and $d(u, v) < \varepsilon$.

We conclude this subsection by introducing the de Leeuw's map. Let

$$\widetilde{M} = (M \times M) \backslash \{(x, x) \colon x \in M\}.$$

The mapping $\Phi : M \to \ell_\infty(\widetilde{M})$, defined by

$$f \mapsto \tilde{f}, \quad \text{where} \quad \tilde{f}(x, y) = \frac{f(x) - f(y)}{d(x, y)},$$

sometimes called the *de Leeuw's map* or the *de Leeuw's transform*, is linear and isometric. Recall that the dual of $\ell_\infty(\widetilde{M})$ is the Banach space $ba(\widetilde{M})$ of all bounded, finitely additive signed measures on $\widetilde{M}$, where the norm is given by the total variation. Thus, for every $F \in S_{\operatorname{Lip}_0(M)^*}$, by the Hahn–Banach extension theorem, there exists a measure $\mu \in ba(\widetilde{M})$ with $|\mu|(\widetilde{M}) = 1$ such that

$$F(f) = \int_{\widetilde{M}} \tilde{f} d\mu$$

for every $f \in \operatorname{Lip}_0(M)$ (see, e.g., [Wea18] for background).

Unconditional basis

In Chapters 4 and 5, we will be working with Banach spaces that have a basis. We introduce here a few necessary definitions.

Let X be a Banach space with a Schauder basis (e_n) and the corresponding biorthogonal functionals (e_n^*). The projections defined by

$$P_k(x) = \sum_{n=1}^{k} e_n^*(x) e_n$$

are called the *partial sum projections*.

We are particularly interested in unconditional basis.

Definition 1.18. A basis (e_n) is said to be *unconditional* if the series $x := \sum_{n=1}^{\infty} a_n e_n$ converges unconditionally for every $x \in X$. The basis (e_n) is said to be *k-unconditional* for $k \geqslant 1$, if for every $N \in \mathbb{N}$ and for any scalars $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ with $|a_n| \leqslant |b_n|$ for all $n = 1, \ldots, N$, we have

$$\left\| \sum_{n=1}^{N} a_n e_n \right\| \leqslant k \left\| \sum_{n=1}^{N} b_n e_n \right\|.$$

For more information on basis, see, e.g., [AK16].

It is known that a Banach space with an unconditional basis cannot have the Daugavet property (see [Kad96, Corollary 2.3]). However, there exists a Banach space with a 1-unconditional basis that contains a Daugavet point (see [ALMT21, Section 4]). This naturally motivates further study of Daugavet and Δ-points in this setting.

Chapter 2

Daugavet and Delta-points in Lipschitz-free spaces

The study of Daugavet and Δ-points in Lipschitz-free spaces was initiated in [JRZ22], where the following characterizations were established:

- If M is a compact metric space, then $\mathcal{F}(M)$ satisfies the distance-to-denting property for Daugavet points, i.e., $\mu \in S_{\mathcal{F}(M)}$ is a Daugavet-point if and only if $\|\mu - \nu\| = 2$ for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$ (see [JRZ22, Theorem 3.2]).

- Let $x, y \in M$ with $x \neq y$. Then $m_{xy} \in S_{\mathcal{F}(M)}$ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $m_{xy} \in S$, there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$ (see [JRZ22, Theorem 4.7]).

In this chapter, we aim to generalize both of these characterizations. In addition, we provide a metric condition for molecules to be Δ-points, as well as a characterization of when finitely supported elements are Δ-points. Furthermore, we construct a Lipschitz-free space that has both a Daugavet point and the Radon–Nikodým property; in fact, the space is a dual space isomorphic to ℓ_1. We conclude the chapter by proving that Daugavet and Δ-points coincide in Lipschitz-free spaces over $\mathbb{R}$-trees. This chapter is based on [AAL$^+$24a], [AAL$^+$24b], [Vee23], [Vee25a], and [Vee25b].

To begin, let us recall a few general results that will be used throughout this chapter.

Theorem 2.1 (see [JRZ22, Theorem 2.6]). *Let (u_n) and (v_n) be two sequences in M such that $u_n \neq v_n$ for every $n \in \mathbb{N}$ and $d(u_n, v_n) \to 0$. Then for every $\mu \in S_{\mathcal{F}(M)}$, we have*

$$\|\mu + m_{u_n v_n}\| \to 2.$$

As an immediate consequence of Theorem 2.1 we obtain the following proposition. While this result was not stated in [JRZ22], it was proved in the case of a molecule (see [JRZ22, Theorem 4.7]). However, the proof of that case did not use the fact that m_{xy} is a molecule, and thus the same argument applies to arbitrary unit sphere elements of a Lipschitz-free space.

Proposition 2.2 (cf. [JRZ22, Theorem 4.7]). *Let $\mu \in S_{\mathcal{F}(M)}$ be such that for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$ there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$. Then μ is a Δ-point.*

Next, we present a lemma for computing the norm between two molecules, which will be useful in several results. Note that the "Furthermore..." part is formulated slightly different from how it appears in [Vee23], but it follows by the same reasoning.

Lemma 2.3 (see [Vee23, Lemma 1.2]). *Let $x, y, u, v \in M$ with $x \neq y, u \neq v$, and let $\varepsilon > 0$. The following statements are equivalent:*

(i) $\|m_{xy} + m_{uv}\| \geqslant 2 - \varepsilon$;

(ii) $d(x, v) + d(u, y) \geqslant d(x, y) + d(u, v) - \varepsilon \max\{d(x, y), d(u, v)\}$.

Furthermore,

$$\|m_{xy} + m_{uv}\| = \min\left\{2, \frac{d(x, v) + d(u, y) + |d(x, y) - d(u, v)|}{\max\{d(x, y), d(u, v)\}}\right\}.$$

Proof. Without loss of generality, assume that y is the fixed point 0 and that $d(x, y) \geqslant d(u, v)$.

(i)⇒(ii). Assume that $\|m_{xy} + m_{uv}\| \geqslant 2 - \varepsilon$. Then

$$\begin{aligned} 2 - \varepsilon &\leqslant \|m_{xy} + m_{uv}\| \\ &= \frac{\|(\delta_x - \delta_v)d(u, v) + (\delta_u - \delta_y)d(u, v) + (\delta_u - \delta_v)\,(d(x, y) - d(u, v))\|}{d(x, y)d(u, v)} \\ &\leqslant \frac{d(x, v) + d(u, y) + d(x, y) - d(u, v)}{d(x, y)}. \end{aligned}$$

Therefore

$$\begin{aligned} d(x,v)+d(u,y) &\geqslant (1-\varepsilon)d(x,y)+d(u,v) \\ &= d(x,y)+d(u,v)-\varepsilon\max\{d(x,y),d(u,v)\}. \end{aligned}$$

(ii)⇒(i). Assume that

$$d(x,v)+d(u,y)\geqslant d(x,y)+d(u,v)-\varepsilon\max\{d(x,y),d(u,v)\},$$

which is equivalent to

$$d(v,x)+d(y,u)-d(u,v)\geqslant (1-\varepsilon)d(x,y).$$

Define the function $f\colon M\to\mathbb{R}$ by

$$f(p)=\min\{d(y,p),d(v,p)+d(y,u)-d(u,v)\},\quad p\in M.$$

By [Wea18, Proposition 1.32], we have $f\in \mathrm{Lip}_0(M)$ and $\|f\|\leqslant 1$. Let us note that

$$\begin{aligned} f(x)&=\min\{d(y,x),d(v,x)+d(y,u)-d(u,v)\}\geqslant(1-\varepsilon)d(x,y),\\ f(y)&=\min\{0,d(v,y)+d(y,u)-d(u,v)\}=0,\\ f(u)&=\min\{d(y,u),d(v,u)+d(y,u)-d(u,v)\}=d(y,u),\\ f(v)&=\min\{d(y,v),0+d(y,u)-d(u,v)\}=d(y,u)-d(u,v). \end{aligned}$$

Hence

$$\|m_{xy}+m_{uv}\|\geqslant\langle f,m_{xy}\rangle+\langle f,m_{uv}\rangle\geqslant 1-\varepsilon+1=2-\varepsilon.$$

Now observe that equality in (i) holds if and only if equality holds in (ii). Suppose such an $\varepsilon>0$ exists where both equalities (i) and (ii) hold. Then

$$\begin{aligned} \|m_{xy}+m_{uv}\| &= 2-\varepsilon \\ &= 2-\frac{d(x,y)+d(u,v)-d(x,v)-d(u,y)}{\max\{d(x,y),d(u,v)\}} \\ &= \frac{d(x,v)+d(u,y)+|d(x,y)-d(u,v)|}{\max\{d(x,y),d(u,v)\}}. \end{aligned}$$

On the other hand, if equality is not attained for any $\varepsilon>0$, then necessarily $\|m_{xy}+m_{uv}\|=2$, and

$$d(x,v)+d(u,y)\geqslant d(x,y)+d(u,v).$$

In this case,

$$\frac{d(x,v)+d(u,y)+|d(x,y)-d(u,v)|}{\max\{d(x,y),d(u,v)\}} \geqslant \frac{d(x,y)+d(u,v)+|d(x,y)-d(u,v)|}{\max\{d(x,y),d(u,v)\}} = 2.$$

Thus, in either case,

$$\|m_{xy}+m_{uv}\| = \min\left\{2, \frac{d(x,v)+d(u,y)+|d(x,y)-d(u,v)|}{\max\{d(x,y),d(u,v)\}}\right\}. \quad \square$$

2.1 The distance-to-denting property for Daugavet points in Lipschitz-free spaces

This section is dedicated to the study of Daugavet points in Lipschitz-free spaces. Specifically, we show that Daugavet points in these spaces satisfy the distance-to-denting property. The section combines some of the results from [Vee23] and [Vee25a].

Throughout the section, let M be a metric space with a fixed point 0. By combining [AG19, Theorem 4.1] and [GLPPRZ18, Theorem 2.4], we obtain the following characterization for denting points.

Theorem 2.4 (see [AG19, Theorem 4.1], [GLPPRZ18, Theorem 2.4])**.** *Let $u, v \in M$ with $u \neq v$. The molecule m_{uv} is a denting point of $B_{\mathcal{F}(M)}$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$[u,v]_\delta \subseteq B(u,\varepsilon) \cup B(v,\varepsilon).$$

Our first goal is to prove that any slice of the unit ball of a Lipschitz-free space, whose defining functional is not local, contains a denting point – a result previously shown only for Lipschitz-free spaces over compact metric spaces (see [CGLMRZ21, Lemma 3.13]). The proof will be broken into several lemmas.

Lemma 2.5 (see [Vee23, Lemma 2.2])**.** *Let $u, v \in M$, $\delta > 0$, and let $x \in [u,v]_\delta$. There exists $\delta' > 0$ such that*

$$[u,x]_{\delta'} \cup [v,x]_{\delta'} \subseteq [u,v]_\delta.$$

Figure 2.1: Illustration of Lemma 2.5

Proof. Since $x \in [u,v]_\delta$, there exists $\delta' > 0$ such that

$$d(u,x) + d(v,x) + \delta' < d(u,v) + \delta.$$

Let $p \in [u,x]_{\delta'}$. Then

$$\begin{aligned} d(u,p) + d(v,p) &< d(u,x) + \delta' - d(x,p) + d(v,p) \\ &\leqslant d(u,x) + \delta' + d(v,x) \\ &< d(u,v) + \delta, \end{aligned}$$

so $p \in [u,v]_\delta$. Hence $[u,x]_{\delta'} \subseteq [u,v]_\delta$. The inclusion $[v,x]_{\delta'} \subseteq [u,v]_\delta$ follows by a symmetric argument. For an illustration, see Figure 2.1. □

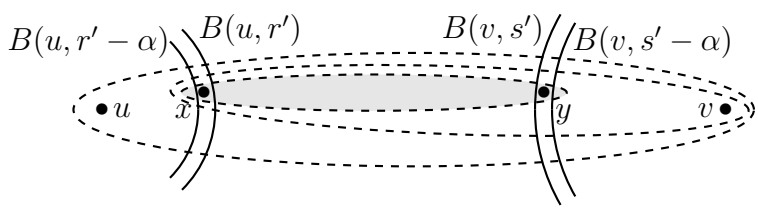

Figure 2.2: Illustration of Lemma 2.6

Lemma 2.6 (see [Vee23, Lemma 2.3])**.** *Let $u, v \in M$ and $r, s, \delta > 0$ be such that*

$$[u,v]_\delta \subseteq B(u,r) \cup B(v,s).$$

For every $\varepsilon > 0$ there exist $\delta' > 0$ and points $x \in B(u,r)$ and $y \in B(v,s)$ such that the following hold:

(1) $d(u,x) + d(v,y) + d(x,y) < d(u,v) + \delta$*;*

(2) $[x,y]_{\delta'} \subseteq [u,v]_\delta$*;*

(3) $d(x,y) \leqslant d(u,v)$;

(4) $[x,y]_{\delta'} \subseteq B(x,\varepsilon) \cup B(y,\varepsilon)$.

Proof. Fix $\varepsilon > 0$. We may assume that $\delta < \varepsilon$. Let $\alpha > 0$ be such that $\delta + \alpha < \varepsilon$. Define

$$r' = \min\left\{t \geqslant 0 \colon [u,v]_\delta \subseteq B(u,t) \cup B(v,s)\right\}.$$

Clearly, $0 \leqslant r' \leqslant r$. If $r' \leqslant \varepsilon$, we take $x = u$. Otherwise, choose

$$x \in [u,v]_\delta \backslash \left(B(u,r'-\alpha) \cup B(v,s)\right),$$

which is possible by minimality of r' (see Figure 2.2). Then $x \in B(u,r') \subseteq B(u,r)$. By Lemma 2.5, there exists $\gamma > 0$ such that

$$[x,v]_\gamma \subseteq [u,v]_\delta.$$

We may assume that γ is small enough to satisfy

$$d(u,x) + d(x,v) + \gamma < d(u,v) + \delta.$$

Hence,

$$[x,v]_\gamma \subseteq [u,v]_\delta \subseteq B(u,r') \cup B(v,s).$$

Define

$$s' = \min\left\{t \geqslant 0 \colon [x,v]_\gamma \subseteq B(u,r') \cup B(v,t)\right\}.$$

Then $0 \leqslant s' \leqslant s$. If $s' \leqslant \varepsilon$, then set $y = v$. Otherwise, choose

$$y \in [x,v]_\gamma \backslash \left(B(u,r') \cup B(v,s'-\alpha)\right)$$

(see Figure 2.2). Note that $y \in B(v,s') \subseteq B(v,s)$. We now verify the desired properties. Condition (1) follows from

$$d(u,x) + d(v,y) + d(x,y) < d(u,x) + d(x,v) + \gamma < d(u,v) + \delta.$$

By Lemma 2.5, there exists $\delta' > 0$ such that

$$[x,y]_{\delta'} \subseteq [x,v]_\gamma.$$

Additionally, we choose δ' small enough so that

$$d(u,x) + d(v,y) + d(x,y) + \delta' < d(u,v) + \delta.$$

Then (2) holds since $[x,v]_\gamma \subseteq [u,v]_\delta$. If $r' > \varepsilon$, then

$$d(u,x) > r' - \alpha > \varepsilon - \alpha > \delta,$$

and therefore

$$d(x,y) < d(u,v) + \delta - d(u,x) - d(v,y) < d(u,v).$$

Analogously, $d(x,y) < d(u,v)$, if $s' > \varepsilon$. On the other hand, if $r' \leqslant \varepsilon$ and $s' \leqslant \varepsilon$, then $x = u$ and $y = v$, hence $d(x,y) \leqslant d(u,v)$. Thus we have (3).

In order to show (4), let $p \in [x,y]_{\delta'}$. Then

$$p \in [x,y]_{\delta'} \subseteq [x,v]_\gamma \subseteq B(u,r') \cup B(v,s').$$

Assume that $p \in B(u,r')$ (the case $p \in B(v,s')$ is analogous). If $r' \leqslant \varepsilon$, then $x = u$ and $p \in B(u,r') \subseteq B(x,\varepsilon)$. Otherwise $d(u,x) > r' - \alpha$ and then

$$\begin{aligned} d(x,p) &< d(x,y) + \delta' - d(y,p) \\ &< d(u,v) + \delta - d(u,x) - d(v,y) - d(y,p) \\ &\leqslant \delta - d(u,x) + d(u,p) \\ &\leqslant \delta - d(u,x) + r' \\ &< \delta + \alpha \\ &< \varepsilon. \end{aligned}$$

This gives us $p \in B(x,\varepsilon) \cup B(y,\varepsilon)$ and therefore (4) holds. □

The following lemma appeared in [Vee23] without the final inequality. A strengthened version, including the inequality, was later stated in [Vee25a], where the original proof was reused and only the final estimate was added. For completeness, we present the full argument here by combining both proofs into a single unified version.

Lemma 2.7 (see [Vee23, Lemma 2.4], [Vee25a, Lemma 2.6])**.** *Assume that M is complete, and let $u, v \in M$ and $r, s, \delta > 0$ with $r + s < d(u,v)$ be such that*

$$[u,v]_\delta \subseteq B(u,r) \cup B(v,s).$$

Then there exist $x \in B(u,r)$ and $y \in B(v,s)$ such that m_{xy} is a denting point of $B_{\mathcal{F}(M)}$ and

$$d(u,x) + d(v,y) + d(x,y) < d(u,v) + \delta.$$

Proof. We will construct sequences (x_n) and (y_n) of elements in the sets

$$B(u,r) \cap [u,v]_\delta \quad \text{and} \quad B(v,s) \cap [u,v]_\delta,$$

convergent to points $x \in B(u,r) \cap [u,v]_\delta$ and $y \in B(v,s) \cap [u,v]_\delta$, respectively, such that the molecule m_{xy} is a denting point and the required inequality is satisfied. Alongside, we define two sequences of positive numbers, (δ_n) and (ε_n), both converging to zero. Let us begin by choosing $\varepsilon_1 > 0$ such that $r + s + 2\varepsilon_1 < d(u,v)$. By applying Lemma 2.6 with $u = u$, $v = v$, $r = r$, $s = s$, $\delta = \delta/2$ and $\varepsilon = \varepsilon_1$, we obtain x_1 $(= x)$, y_1 $(= y)$ and δ_1 $(= \delta')$. We may additionally assume that $\delta_1 < \varepsilon_1$. In particular, we have

$$d(u,x_1) + d(v,y_1) + d(x_1,y_1) < d(u,v) + \frac{\delta}{2}. \tag{2.1}$$

Now assume that x_n, y_n, δ_n and ε_n have been constructed for some $n \in \mathbb{N}$. Let $\varepsilon_{n+1} \in (0, \delta_n/6)$. By applying Lemma 2.6 with $u = x_n$, $v = y_n$, $r = \varepsilon_n$, $s = \varepsilon_n$, $\delta = \varepsilon_{n+1}$ and $\varepsilon = \varepsilon_{n+1}$, we obtain x_{n+1} $(= x)$, y_{n+1} $(= y)$ and δ_{n+1} $(= \delta')$. We may additionally assume that $\delta_{n+1} < \varepsilon_{n+1}$.

Clearly, for every $n \in \mathbb{N}$, we have

(1) $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n+1}, y_{n+1}) < d(x_n, y_n) + \varepsilon_{n+1}$;

(2) $[x_n, y_n]_{\delta_n} \subseteq [u,v]_\delta$;

(3) $d(x_{n+1}, y_{n+1}) \leqslant d(x_n, y_n)$;

(4) $[x_n, y_n]_{\delta_n} \subseteq B(x_n, \varepsilon_n) \cup B(y_n, \varepsilon_n)$.

Furthermore, $\varepsilon_{n+1} < \delta_n/6 < \varepsilon_n/6$ for every $n \in \mathbb{N}$. Then for every $m, n \in \mathbb{N}$ with $m > n$ we have

$$d(x_n, x_m) \leqslant \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{m-1} \varepsilon_i < 2\varepsilon_n \to 0$$

and thus (x_n) converges to some $x \in M$. Analogously, (y_n) converges to some $y \in M$. Next we verify that $x \in B(u,r)$. Note that

$$d(v, x_n) \geqslant d(u,v) - d(u,x_1) - d(x_1,x_n) > d(u,v) - r - 2\varepsilon_1 > s,$$

and thus $x_n \notin B(v,s)$. Since

$$x_n \in [x_n, y_n]_{\delta_n} \subseteq [u,v]_\delta \subseteq B(u,r) \cup B(v,s)$$

it must be that $x_n \in B(u,r)$, and hence $x \in B(u,r)$. Similarly, $y \in B(v,s)$. Also, $x \neq y$, because $r + s < d(u,v)$.

Now we verify that m_{xy} is a denting point using Theorem 2.4. Fix $\varepsilon > 0$, and choose $n \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon/2$, $d(y_n, y) < \varepsilon/2$, and $\varepsilon_n < \varepsilon/2$. Then from (1) and (3), for all $m > n$,

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\leqslant d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \\ &\quad + d(x_{n+1}, x_m) + d(y_{n+1}, y_m) \\ &< d(x_n, y_n) + \varepsilon_{n+1} - d(x_{n+1}, y_{n+1}) \\ &\quad + 2\varepsilon_{n+1} + 2\varepsilon_{n+1} \\ &\leqslant d(x_n, y_n) + 5\varepsilon_{n+1} - d(x, y), \end{aligned}$$

which yields

$$d(x_n, x) + d(y_n, y) \leqslant d(x_n, y_n) + 5\varepsilon_{n+1} - d(x, y). \tag{2.2}$$

If $p \in [x,y]_{\varepsilon_{n+1}}$, then

$$\begin{aligned} d(x_n, p) + d(y_n, p) &\leqslant d(x_n, x) + d(y_n, y) + d(x, p) + d(y, p) \\ &< d(x_n, y_n) + 5\varepsilon_{n+1} - d(x, y) + d(x, y) + \varepsilon_{n+1} \\ &< d(x_n, y_n) + \delta_n. \end{aligned}$$

So

$$[x,y]_{\varepsilon_{n+1}} \subseteq [x_n, y_n]_{\delta_n}.$$

Since

$$d(x_n, x) + \varepsilon_n < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

we have $B(x_n, \varepsilon_n) \subseteq B(x, \varepsilon)$, and similarly $B(y_n, \varepsilon_n) \subseteq B(y, \varepsilon)$. Therefore,

$$[x,y]_{\varepsilon_{n+1}} \subseteq [x_n, y_n]_{\delta_n} \subseteq B(x_n, \varepsilon_n) \cup B(y_n, \varepsilon_n) \subseteq B(x, \varepsilon) \cup B(y, \varepsilon),$$

and thus by Theorem 2.4, m_{xy} is a denting point. To complete the proof, observe that we could choose $\varepsilon_2 > 0$ arbitrary small, so assume that $5\varepsilon_2 < \delta/2$. Then from (2.1) and (2.2) we obtain

$$\begin{aligned} d(u, x) + d(v, y) + d(x, y) &\leqslant d(u, x_1) + d(v, y_1) \\ &\quad + d(x_1, x) + d(y_1, y) + d(x, y) \\ &< d(u, v) - d(x_1, y_1) + \frac{\delta}{2} + d(x_1, y_1) + 5\varepsilon_2 \\ &< d(u, v) + \delta. \end{aligned}$$

□

Proposition 2.8 (see [Vee25a, Proposition 2.7]). *Let $f \in S_{\mathrm{Lip}_0(M)}$, and assume that f is not local. Then for every $\alpha > 0$, the slice $S(f, \alpha)$ contains a denting point of $B_{\mathcal{F}(M)}$.*

Proof. We may assume that M is complete, since the Lipschitz-free space remains the same under completion: $\mathcal{F}(M) = \mathcal{F}(M')$ for the completion M' of M.

Since f is not local, there exists $\varepsilon > 0$ such that for every $m_{uv} \in S(f, \varepsilon)$, we have $d(u, v) \geqslant \varepsilon$. Fix $\alpha > 0$. We may assume that $\alpha < \varepsilon$.

Let $m_{u_0 v_0} \in S(f, \alpha/2)$. Choose $n \in \mathbb{N}$ and $\delta > 0$ such that

$$\left(\frac{3}{4} + \delta\right)^n d(u_0, v_0) < \varepsilon$$

and $f(u_0) - f(v_0) > (1 - \alpha/2 + \delta)(1 + \delta)^n d(u_0, v_0)$.

Suppose

$$[u_0, v_0]_{\delta d(u_0, v_0)} \subseteq B\left(u_0, d(u_0, v_0)/4\right) \cup B\left(v_0, d(u_0, v_0)/4\right).$$

Then by Lemma 2.7, there exist points $x \in B\left(u_0, d(u_0, v_0)/4\right)$ and $y \in B\left(v_0, d(u_0, v_0)/4\right)$ such that m_{xy} is a denting point of $B_{\mathcal{F}(M)}$ and

$$d(u_0, x) + d(v_0, y) + d(x, y) < d(u_0, v_0) + \delta d(u_0, v_0).$$

This implies that $d(x, y) \geqslant d(u_0, v_0)/2$. Then

$$\begin{aligned} f(x) - f(y) &= f(x) - f(u_0) + f(v_0) - f(y) + f(u_0) - f(v_0) \\ &> -d(x, u_0) - d(v_0, y) + (1 - \alpha/2 + \delta) d(u_0, v_0) \\ &> d(x, y) - \alpha/2 d(u_0, v_0) \\ &\geqslant (1 - \alpha) d(x, y). \end{aligned}$$

Thus $m_{xy} \in S(f, \alpha)$, and since m_{xy} is a denting point of $B_{\mathcal{F}(M)}$, the proof is complete.

Otherwise, there exists

$$p \in [u_0, v_0]_{\delta d(u_0, v_0)} \setminus \left(B\left(u_0, d(u_0, v_0)/4\right) \cup B\left(v_0, d(u_0, v_0)/4\right)\right).$$

Then

$$\begin{aligned} f(u_0) - f(p) + f(p) - f(v_0) &= f(u_0) - f(v_0) \\ &> (1 - \alpha/2 + \delta)(1 + \delta)^n d(u_0, v_0) \\ &> (1 - \alpha/2 + \delta)(1 + \delta)^{n-1} \left(d(u_0, p) + d(v_0, p)\right), \end{aligned}$$

which implies that at least one of the following inequalities holds:

$$f(u_0) - f(p) > (1 - \alpha/2 + \delta)(1 + \delta)^{n-1} d(u_0, p)$$

or

$$f(p) - f(v_0) > (1 - \alpha/2 + \delta)(1 + \delta)^{n-1} d(v_0, p).$$

Additionally, since $p \notin B\left(u_0, d(u_0, v_0)/4\right) \cup B\left(v_0, d(u_0, v_0)/4\right)$, we have

$$d(u_0, p) < (1 + \delta) d(u_0, v_0) - d(v_0, p) < \left(\frac{3}{4} + \delta\right) d(u_0, v_0),$$

and analogously $d(v_0, p) < (3/4 + \delta)\, d(u_0, v_0)$. Therefore, we can define new points $u_1, v_1 \in M$ such that

$$f(u_1) - f(v_1) > (1 - \alpha/2 + \delta)(1 + \delta)^{n-1} d(u_1, v_1)$$

and $d(u_1, v_1) < (3/4 + \delta) d(u_0, v_0)$.

We now repeat this argument inductively. Suppose that for some $k \in \{1, \ldots, n-1\}$, we have constructed pairs u_k, v_k satisfying

$$f(u_k) - f(v_k) > (1 - \alpha/2 + \delta)(1 + \delta)^{n-k} d(u_k, v_k)$$

and $d(u_k, v_k) < (3/4 + \delta)^k d(u_0, v_0)$.

If

$$[u_k, v_k]_{\delta d(u_k, v_k)} \subseteq B\left(u_k, d(u_k, v_k)/4\right) \cup B\left(v_k, d(u_k, v_k)/4\right),$$

then we can find a denting point m_{xy} of $B_{\mathcal{F}(M)}$ in the slice $S(f, \alpha)$ as we did before.

Otherwise, we can find $u_{k+1}, v_{k+1} \in M$ such that

$$f(u_{k+1}) - f(v_{k+1}) > (1 - \alpha/2 + \delta)(1 + \delta)^{n-k-1} d(u_{k+1}, v_{k+1})$$

and $d(u_{k+1}, v_{k+1}) < (3/4 + \delta)^{k+1} d(u_0, v_0)$.

By the n-th step we must have found a denting point m_{xy} of $B_{\mathcal{F}(M)}$ in the slice $S(f, \alpha)$, because otherwise we would have

$$f(u_n) - f(v_n) > (1 - \alpha/2 + \delta) d(u_n, v_n) > (1 - \varepsilon) d(u_n, v_n)$$

and

$$d(u_n, v_n) < \left(\frac{3}{4} + \delta\right)^n d(u_0, v_0) < \varepsilon,$$

which contradicts the choice of ε. Therefore the slice $S(f, \alpha)$ contains a denting point of $B_{\mathcal{F}(M)}$. □

By applying Proposition 2.8, we are now ready to prove the distance-to-denting property for Lipschitz-free spaces. Since our goal is also to obtain metric characterizations of Daugavet points in Lipschitz-free spaces, we will introduce an alternative characterization for Daugavet points as well. The proof of this theorem differs from the one provided in [Vee23], since the use Proposition 2.8 allows us to obtain a shorter and simpler proof.

Theorem 2.9 (see [Vee23, Theorem 2.1]). *Let $\mu \in S_{\mathcal{F}(M)}$. The following statements are equivalent:*

(i) μ is a Daugavet point;

(ii) for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$ we have $\|\mu - \nu\| = 2$;

(iii) for every $u, v \in M$ with $u \neq v$, and every $r, s > 0$, if there exists $\delta > 0$ such that

$$[u, v]_\delta \subseteq B\left(u, rd(u, v)\right) \cup B\left(v, sd(u, v)\right),$$

then $\|\mu - m_{uv}\| \geqslant 2 - 2r - 2s$.

Proof. (i)⇒(ii) is exactly Proposition 1.8.

(ii)⇒(i). Assume that $\|\mu - \nu\| = 2$ for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$. Fix a slice $S(f, \alpha)$. If f is local, then there exists a sequence of molecules $(m_{u_i v_i})$ in $S(f, \alpha)$ with $d(u_i, v_i) \to 0$, and Theorem 2.1 yields $\|\mu - m_{u_i v_i}\| \to 2$. If f is not local, then by Proposition 2.8 there exists a denting point $\nu \in S(f, \alpha)$, and by assumption $\|\mu - \nu\| = 2$. In both cases, we find elements in the slice $S(f, \alpha)$ arbitrarily close to being at distance 2 from μ. Hence μ is a Daugavet point.

It is sufficient to prove the equivalence (ii)⇔(iii) under the assumption that M is complete, since both properties are preserved when passing to the completion M' of M (since $\mathcal{F}(M) = \mathcal{F}(M')$).

(ii)⇒(iii). Assume that $\|\mu - \nu\| = 2$ for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$. This implication is trivial if $r + s \geqslant 1$, so assume that $r + s < 1$. Let $u, v \in M$ with $u \neq v$ and suppose there exists $\delta > 0$ such that

$$[u, v]_\delta \subseteq B\left(u, rd(u, v)\right) \cup B\left(v, sd(u, v)\right).$$

By Lemma 2.7, we can find $x \in B\left(u, rd(u, v)\right)$ and $y \in B\left(v, sd(u, v)\right)$ such that m_{xy} is a denting point of $B_{\mathcal{F}(M)}$. By (ii) we have $\|\mu - m_{xy}\| = 2$.

Moreover, Lemma 2.3 implies

$$\begin{aligned}\|m_{xy} - m_{uv}\| &= \min\left\{2, \frac{d(u,x) + d(v,y) + |d(u,v) - d(x,y)|}{\max\{d(u,v), d(x,y)\}}\right\} \\ &\leqslant \frac{2d(u,x) + 2d(v,y)}{d(u,v)} \leqslant \frac{2rd(u,v) + 2sd(u,v)}{d(u,v)} \leqslant 2(r+s),\end{aligned}$$

and thus,

$$\|\mu - m_{uv}\| \geqslant \|\mu - m_{xy}\| - \|m_{xy} - m_{uv}\| \geqslant 2 - 2r - 2s.$$

(iii)⇒(ii). Assume that (iii) holds. By [Wea18, Corollary 3.44] every preserved extreme point, and thus also every denting point, is a molecule. Let $m_{uv} \in \operatorname{dent}(B_{\mathcal{F}(M)})$. Let $\varepsilon > 0$ be arbitrary. By Theorem 2.4 there exists $\delta > 0$ such that

$$[u,v]_\delta \subseteq B(u, \varepsilon d(u,v)) \cup B(v, \varepsilon d(u,v)).$$

Then by (iii) we have $\|\mu - m_{uv}\| \geqslant 2 - 4\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $\|\mu - m_{uv}\| = 2$. □

2.2 Delta-points in Lipschitz-free spaces

This section is dedicated to characterizing Δ-points in Lipschitz-free spaces. Proposition 2.2, together with the fact that the converse holds for molecules, suggests a potential characterization of Δ-points in Lipschitz-free spaces.

Conjecture 2.10. *Let $\mu \in S_{\mathcal{F}(M)}$. Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$, there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$.*

Although we do not know whether this conjecture holds in full generality, we will verify it for several important cases. First we prove it for finitely supported elements in arbitrary Lipschitz-free spaces. We then extend the result to all elements in certain classes of Lipschitz-free spaces, among these Lipschitz-free spaces over bounded subsets of ℓ_1. In addition, we provide a metric characterization for when a molecule is a Δ-point. We conclude the section by showing that a finitely supported element is a Δ-point if and only if it can be written as a convex combination of molecules that are Δ-points. This section is based on [AAL+24a], [AAL+24b], [Vee23], [Vee25a], and [Vee25b].

2.2.1 Characterization of Delta-points for finitely supported elements

Our first aim is to prove Conjecture 2.10 for certain elements of Lipschitz-free spaces – most notably for finitely supported elements. More specifically, we will prove the following result.

Theorem 2.11 (see [Vee25b, Theorem 1.1])**.** *Let $\mu := \sum_{i\in I} \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$ be a convex combination of molecules $m_{x_i y_i}$, $i \in I$. Assume that $\max_{i,j\in I} d(x_i, y_j) < \infty$ and that there exists $\delta > 0$ such that for every collection of pairwise distinct indices $k_1, \dots, k_m \in I$ and $k_{m+1} := k_1$ the following implication holds*

$$\sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) > \sum_{i=1}^{m} d(x_{k_i}, y_{k_i}) \Rightarrow (1-\delta) \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) > \sum_{i=1}^{m} d(x_{k_i}, y_{k_i}).$$

Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$, there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$.

In particular, if $\mu \in S_{\mathcal{F}(M)}$ is finitely supported, then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$, there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$.

This result generalizes [Vee23, Theorem 4.4] and was proved in [Vee25b]. We will break the proof into several parts, starting with some observations, originally noted in [Vee23].

Throughout the subsection, let M be a metric space with a base point 0. Let $I \subseteq \mathbb{N}$, and suppose that $\lambda_i > 0$, $i \in I$ with $\sum_{i\in I} \lambda_i = 1$. Let $m_{x_i y_i} \in S_{\mathcal{F}(M)}$, $i \in I$ be such that $\mu := \sum_{i\in I} \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$. By [ARZ20, Theorem 2.4], for every sequence $k_1, \dots, k_{m+1} \in I$ with $k_1 = k_{m+1}$, we have the inequality

$$\sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) \geqslant \sum_{i=1}^{m} d(x_{k_i}, y_{k_i}). \tag{2.3}$$

Moreover, equality in (2.3) implies that for every $i \in \{1, \dots, m\}$ with $x_{k_i} \neq y_{k_{i+1}}$, there exist $\nu \in S_{\mathcal{F}(M)}$ and $\lambda \in (0, 1]$ such that $\mu = \lambda m_{x_{k_i} y_{k_{i+1}}} + (1-\lambda)\nu$, or equivalently, $\left\| \mu - \lambda m_{x_{k_i} y_{k_{i+1}}} \right\| = 1 - \lambda$. To see this, assume that the equality holds. Without loss of generality, we may additionally assume that $k_1, \dots, k_m$ are pairwise distinct. Define

$$\lambda_0 := \min_{i\in I} \frac{\lambda_i}{d(x_i, y_i)}$$

and

$$l_i := \begin{cases} \lambda_i - \lambda_0 d(x_i, y_i), & \text{if } i \in \{k_1, \dots, k_m\}, \\ \lambda_i, & \text{if } i \in I \setminus \{k_1, \dots, k_m\}, \end{cases}$$

for every $i \in I$. Then $l_i \geqslant 0$ for all $i \in I$, and we have

$$\begin{aligned} \mu &= \sum_{i \in I} l_i m_{x_i y_i} + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_i}) m_{x_{k_i} y_{k_i}} \\ &= \sum_{i \in I} l_i m_{x_i y_i} + \lambda_0 \sum_{\substack{i=1 \\ x_{k_i} \neq y_{k_{i+1}}}}^{m} d(x_{k_i}, y_{k_{i+1}}) m_{x_{k_i} y_{k_{i+1}}}. \end{aligned}$$

Also,

$$\sum_{i \in I} l_i + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) = \sum_{i \in I} l_i + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_i}) = 1.$$

Since $d(x_{k_i}, y_{k_{i+1}}) > 0$ and $\lambda_0 > 0$, we may choose $\lambda = \lambda_0 d(x_{k_i}, y_{k_{i+1}})$, yielding $\left\| \mu - \lambda m_{x_{k_i} y_{k_{i+1}}} \right\| = 1 - \lambda$.

Next, we define numbers a_{ij}, that measure how close we are to attaining equality in (2.3). For $i, j \in I$, let

$$a_{ij} = \sup \left\{ \frac{\sum_{n=1}^{m} d(x_{k_n}, y_{k_n})}{\sum_{n=1}^{m} d(x_{k_n}, y_{k_{n+1}})} \colon k_1, \dots, k_{m+1} \in I, k_1 = k_{m+1} = i, k_2 = j, \right. \\ \left. k_1, \dots, k_m \text{ are pairwise distinct} \right\}. \tag{2.4}$$

By inequality (2.3), we have $a_{ij} \leqslant 1$ for all $i, j \in I$. If I is finite, then $a_{ij} = 1$ if and only if there exist indices $k_1, \dots, k_{m+1} \in I$, with $k_1 = k_{m+1} = i$, $k_2 = j$, and

$$\sum_{n=1}^{m} d(x_{k_n}, y_{k_n}) = \sum_{n=1}^{m} d(x_{k_n}, y_{k_{n+1}}).$$

We continue with the main proof, breaking it into several parts.

Lemma 2.12 (see [Vee25b, Lemma 2.1])**.** *Let $\mu := \sum_{i \in I} \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$ be a convex combination of molecules $m_{x_i y_i}$, $i \in I$, and let a_{ij}, $i, j \in I$ be as defined in* (2.4)*. Then there exists a function $f \in S_{\mathrm{Lip}_0(M)}$ such that $\langle f, \mu \rangle = 1$ and for all $i, j \in I$,*

$$f(x_i) - f(y_j) \leqslant a_{ij} d(x_i, y_j).$$

Proof. The proof follows the same line as the implication $(iv) \Rightarrow (i)$ in the proof of [ARZ20, Theorem 2.4], with minor modifications. For $i, j \in I$, define

$$\beta_{ij} = a_{ij} d(x_i, y_j) - d(x_i, y_i).$$

We aim to apply either [ARZ20, Lemma 2.3] or [ARZ20, Lemma 2.2], depending on whether I is finite or infinite. Note that $a_{ii} = 1$, hence $\beta_{ii} = 0$ for every $i \in I$. Let $k_1, \dots, k_m \in I$ be pairwise distinct and set $k_{m+1} := k_1$. Then we estimate

$$\sum_{i=1}^{m} a_{k_i k_{i+1}} d(x_{k_i}, y_{k_{i+1}}) \geqslant \min_{i \in \{1,\dots,m\}} a_{k_i k_{i+1}} \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) \geqslant \sum_{i=1}^{m} d(x_{k_i}, y_{k_i})$$

and therefore

$$\sum_{i=1}^{m} \beta_{k_i k_{i+1}} \geqslant 0.$$

The inequality remains valid even if the indices $k_1, \dots, k_m$ are not pairwise distinct. By [ARZ20, Lemma 2.3] or [ARZ20, Lemma 2.2], we obtain real numbers α_i, $i \in I$, such that $\alpha_i \leqslant \alpha_j + \beta_{ij}$ for every $i, j \in I$. Now we define a function f on the set $\{x_i, y_i \colon i \in I\}$ by $f(y_i) = \alpha_i$ and $f(x_i) = \alpha_i + d(x_i, y_i)$. Then for all $i, j \in I$, we compute

$$\begin{aligned}
f(x_i) - f(y_j) &= \alpha_i - \alpha_j + d(x_i, y_i) \leqslant \beta_{ij} + d(x_i, y_i) = a_{ij} d(x_i, y_j), \\
f(y_i) - f(x_j) &= \alpha_i - \alpha_j - d(x_j, y_j) \leqslant \beta_{ij} - d(x_j, y_j) \\
&\leqslant d(x_i, y_j) - d(x_i, y_i) - d(x_j, y_j) \leqslant d(y_i, x_j), \\
f(x_i) - f(x_j) &= \alpha_i - \alpha_j + d(x_i, y_i) - d(x_j, y_j) \\
&\leqslant \beta_{ij} + d(x_i, y_i) - d(x_j, y_j) \\
&\leqslant d(x_i, y_j) - d(x_j, y_j) \leqslant d(x_i, x_j), \\
f(y_i) - f(y_j) &= \alpha_i - \alpha_j \leqslant \beta_{ij} \leqslant d(x_i, y_j) - d(x_i, y_i) \leqslant d(y_i, y_j).
\end{aligned}$$

Thus f is well defined, and its Lipschitz constant is at most 1. Clearly $\langle f, m_{x_i y_i} \rangle = 1$ for every $i \in I$, and thus $\langle f, \mu \rangle = 1$. We extend f to the entire M using the McShane–Whitney extension theorem and adjust by a constant so that $f(0) = 0$. The resulting function $f \in S_{\mathcal{F}(M)}$ satisfies the required properties. □

Before introducing the next lemma, let us recall the Lipschitz function

$$f_{xy}(p) = \frac{d(x, y)}{2} \cdot \frac{d(y, p) - d(x, p)}{d(x, p) + d(y, p)},$$

where $x, y \in M$ with $x \neq y$. This function played an important role in the proof of [JRZ22, Theorem 4.7]. A key property of the function f_{xy}, which is stated in [GLPRZ18, Lemma 3.6], is the following: for every $u, v \in M$ with $u \neq v$ and every $\alpha \in (0,1)$, if $m_{uv} \in S(f_{xy}, \alpha)$ then

$$(1-\alpha)\max\{d(x,v)+d(y,v), d(x,u)+d(y,u)\} < d(x,y).$$

Our goal with the next lemma is to construct a function with a similar property, but adapted to an arbitrary convex combination of molecules in a Lipschitz-free space.

Lemma 2.13 (see [Vee25b, Lemma 2.2]). *Let $\mu := \sum_{i\in I} \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$ be a convex combination of molecules $m_{x_i y_i}$, $i \in I$, and let a_{ij}, $i, j \in I$, be defined as in (2.4). Then there exists a function $f_\mu \in S_{\mathrm{Lip}_0(M)}$ such that the following conditions are satisfied:*

(1) $\langle f_\mu, \mu\rangle = 1$;

(2) For every $u, v \in M$ with $u \neq v$ and every $\alpha \in (0,1)$, if $m_{uv} \in S(f_\mu, \alpha)$, then there exist $i, j \in I$ with $x_i \neq y_j$ and $a_{ij} > 1-\alpha$ such that

$$(1-\alpha)\max\{d(x_i,v)+d(y_j,v), d(x_i,u)+d(y_j,u)\} < d(x_i,y_j).$$

Proof. Without loss of generality, we assume that $0 \in \{x_i, y_i \colon i \in I\}$. By Lemma 2.12, there exists $g \in S_{\mathrm{Lip}_0(M)}$ such that $\langle g, \mu\rangle = 1$ and for all $i, j \in I$,

$$g(x_i) - g(y_j) \leqslant a_{ij} d(x_i, y_j).$$

For every $i \in I$, define the function $h_i \colon M \to \mathbb{R}$ by

$$h_i(p) = \sup\left\{ \frac{g(x_i)-g(y_j)}{d(x_i,p)+d(y_j,p)} d(x_i,p) \colon j \in I, x_i \neq y_j \right\}$$

for every $p \in M$. Since $g(x_i) - g(y_i) = d(x_i, y_i)$, h_i is nonnegative for every $i \in I$. For all $i, j \in I$, we have $d(x_i, y_j) \geqslant g(x_i) - g(y_j)$, and using the function f_{xy} defined above, we obtain

$$\begin{aligned} \frac{d(x_i,y_j)}{2} - f_{x_i y_j}(p) &= \frac{d(x_i,y_j)}{2} \frac{d(x_i,p)+d(y_j,p)-(d(y_j,p)-d(x_i,p))}{d(x_i,p)+d(y_j,p)} \\ &= \frac{d(x_i,y_j)}{g(x_i)-g(y_j)} \frac{g(x_i)-g(y_j)}{d(x_i,p)+d(y_j,p)} d(x_i,p). \end{aligned} \tag{2.5}$$

Hence from [Wea18, Proposition 1.32] and [GLPRZ18, Lemma 3.6], we conclude that h_i is a Lipschitz function with Lipschitz constant at most 1. Define $f_\mu \colon M \to \mathbb{R}$ by

$$f_\mu(p) = \sup_{i \in I} \{g(x_i) - h_i(p)\}.$$

By [Wea18, Proposition 1.32], f_μ is a Lipschitz function with Lipschitz constant at most 1. For every $i \in I$,

$$f_\mu(x_i) \geqslant g(x_i) - h_i(x_i) = g(x_i).$$

Fix $j \in I$ and $\delta > 0$, and choose $i \in I$ such that $f_\mu(y_j) < g(x_i) - h_i(y_j) + \delta$. If $x_i = y_j$, then $f_\mu(y_j) < g(y_j) - h_i(y_j) + \delta = g(y_j) + \delta$. If $x_i \neq y_j$, then

$$\begin{aligned} f_\mu(y_j) &< g(x_i) - h_i(y_j) + \delta \\ &\leqslant g(x_i) - \frac{g(x_i) - g(y_j)}{d(x_i, y_j) + d(y_j, y_j)} d(x_i, y_j) + \delta = g(y_j) + \delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, it follows that $f_\mu(y_j) \leqslant g(y_j)$ for every $j \in I$. Therefore, for every $i, j \in I$, we have $f_\mu(x_i) - f_\mu(y_j) \geqslant g(x_i) - g(y_j)$. Moreover, since

$$d(x_i, y_i) \geqslant f_\mu(x_i) - f_\mu(y_i) \geqslant g(x_i) - g(y_i) = d(x_i, y_i),$$

we conclude $f_\mu(x_i) = g(x_i)$ and $f_\mu(y_i) = g(y_i)$ for every $i \in I$, i.e., f_μ agrees with g on $\{x_i, y_i \colon i \in I\}$. Thus $f_\mu \in S_{\mathrm{Lip}_0(M)}$ and $\langle f_\mu, \mu \rangle = \langle g, \mu \rangle = 1$. This completes the verification of condition (1).

We now prove that condition (2) holds. Fix $u, v \in M$ with $u \neq v$ and $\alpha \in (0, 1)$ such that $m_{uv} \in S(f_\mu, \alpha)$. Let $\delta > 0$ be such that $m_{uv} \in S(f_\mu, \alpha - 2\delta)$ and let $i \in I$ be such that $f_\mu(u) \leqslant g(x_i) - h_i(u) + \delta d(u, v)$. Then $f_\mu(v) \geqslant g(x_i) - h_i(v)$, which yields

$$(1 - \alpha + 2\delta) d(u, v) < f_\mu(u) - f_\mu(v) \leqslant h_i(v) - h_i(u) + \delta d(u, v),$$

so we obtain $(1 - \alpha + \delta) d(u, v) < h_i(v) - h_i(u)$. By the definition of h_i, there exists $j \in I$ with $x_i \neq y_j$ such that

$$h_i(v) < \frac{g(x_i) - g(y_j)}{d(x_i, v) + d(y_j, v)} d(x_i, v) + \delta d(u, v).$$

By (2.5) we get that

$$\begin{aligned}(1-\alpha)d(u,v) &< h_i(v) - h_i(u) - \delta d(u,v)\\ &< \frac{g(x_i)-g(y_j)}{d(x_i,v)+d(y_j,v)}d(x_i,v) - \frac{g(x_i)-g(y_j)}{d(x_i,u)+d(y_j,u)}d(x_i,u)\\ &= \frac{g(x_i)-g(y_j)}{d(x_i,y_j)}\left(f_{x_iy_j}(u) - f_{x_iy_j}(v)\right)\\ &\leqslant a_{ij}\left(f_{x_iy_j}(u) - f_{x_iy_j}(v)\right)\\ &\leqslant \min\left\{a_{ij}d(u,v), f_{x_iy_j}(u) - f_{x_iy_j}(v)\right\}.\end{aligned}$$

Thus $a_{ij} > 1-\alpha$. Now applying [GLPRZ18, Lemma 3.6], we conclude

$$d(x_i,y_j) > (1-\alpha)\max\left\{d(x_i,u)+d(y_j,u), d(x_i,v)+d(y_j,v)\right\}.$$

This verifies condition (2) and completes the proof. □

Proof of Theorem 2.11. Necessity follows from Proposition 2.2.

Assume that μ is a Δ-point. Fix $\varepsilon > 0$ and a slice $S := S(f,\alpha)$ with $\mu \in S$. By [IK04, Lemma 2.1], we can assume that $\alpha < \delta$ and

$$\left(\frac{1}{(1-\alpha)^2} - 1\right)\max_{i,j\in I} d(x_i,y_j) < \varepsilon.$$

Choose $\gamma > 0$ such that $\mu \in S(f,\alpha-\gamma)$, and let $J \subseteq I$ be such finite subset that

$$\left\|\mu - \frac{\sum_{i\in J}\lambda_i m_{x_iy_i}}{\sum_{i\in J}\lambda_i}\right\| < \gamma.$$

Define $\nu = \sum_{i\in J}\lambda_i m_{x_iy_i}/\sum_{i\in J}\lambda_i \in S_{\mathcal{F}(M)}$ and

$$b_{ij} = \sup\left\{\frac{\sum_{n=1}^m d(x_{k_n},y_{k_n})}{\sum_{n=1}^m d(x_{k_n},y_{k_{n+1}})} \colon k_1,\dots,k_{m+1}\in J, k_1 = k_{m+1} = i, k_2 = j,\right.$$
$$\left. k_1,\dots,k_m \text{ are pairwise distinct}\right\}.$$

By Lemma 2.13, there exists $f_\nu \in S_{\mathrm{Lip}_0(M)}$ such that $\langle f_\nu,\nu\rangle = 1$ and for every $m_{uv} \in S(f_\nu,\alpha)$ there exist $i,j\in J$ such that $x_i \neq y_j$, $b_{ij} > 1-\alpha$, and

$$(1-\alpha)\max\left\{d(x_i,v)+d(y_j,v), d(x_i,u)+d(y_j,u)\right\} < d(x_i,y_j).$$

Observe that if $b_{ij} > 1-\alpha$, then there exist $k_1, \dots, k_{m+1} \in J$ with $k_1 = k_{m+1} = i$, $k_2 = j$, and $k_1, \dots, k_m$ pairwise distinct such that

$$\frac{\sum_{n=1}^{m} d(x_{k_n}, y_{k_n})}{\sum_{n=1}^{m} d(x_{k_n}, y_{k_{n+1}})} > 1-\alpha > 1-\delta.$$

Thus, by the assumption of the theorem and inequality (2.3),

$$\sum_{n=1}^{m} d(x_{k_n}, y_{k_n}) = \sum_{n=1}^{m} d(x_{k_n}, y_{k_{n+1}}).$$

Therefore, by the discussion at the beginning of the subsection, for every $i, j \in J$ with $b_{ij} > 1-\alpha$ there exists $l_{ij} \in (0,1]$ such that $\left\|\mu - l_{ij} m_{x_i y_j}\right\| \leqslant 1 - l_{ij}$.

We now aim to find $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$. Set $g = f + f_\mu$. Then

$$\langle g, \mu \rangle = \langle f_\nu, \mu \rangle + \langle f, \mu \rangle > \langle f_\nu, \nu \rangle - \gamma + \langle f, \mu \rangle > 2 - \alpha,$$

so $\mu \in S\left(g/\|g\|, 1 - (2-\alpha)/\|g\|\right)$. Since μ is a Δ-point, by [JRZ22, Remark 2.4], there exist $u, v \in M$ with $u \neq v$ such that $\langle g, m_{uv} \rangle > 2-\alpha$ and

$$\|\mu - m_{uv}\| \geqslant 2 - \alpha \min\left\{l_{ij} \colon i, j \in J, b_{ij} > 1-\alpha\right\}.$$

It follows that $\langle f_\nu, m_{uv} \rangle > 1-\alpha$ and $\langle f, m_{uv} \rangle > 1-\alpha$, hence $m_{uv} \in S$. Moreover, as $\langle f_\nu, m_{uv} \rangle > 1-\alpha$, there exist $i, j \in J$ such that $x_i \neq y_j$, $b_{ij} > 1-\alpha$, and

$$(1-\alpha) \max\left\{d(x_i, v) + d(y_j, v), d(x_i, u) + d(y_j, u)\right\} < d(x_i, y_j). \quad (2.6)$$

Since $\|\mu - m_{uv}\| \geqslant 2 - \alpha l_{ij}$ and $\left\|\mu - l_{ij} m_{x_i y_j}\right\| \leqslant 1 - l_{ij}$, we have

$$\begin{aligned} 2 - \alpha l_{ij} &\leqslant \|\mu - m_{uv}\| \\ &\leqslant l_{ij} \|m_{x_i y_j} - m_{uv}\| + \left\|\mu - l_{ij} m_{x_i y_j}\right\| + (1 - l_{ij}) \|m_{uv}\| \\ &\leqslant l_{ij} \|m_{x_i y_j} - m_{uv}\| + 2 - 2 l_{ij}, \end{aligned}$$

which implies $\|m_{x_i y_j} - m_{uv}\| \geqslant 2 - \alpha$. By Lemma 2.3, this yields

$$\begin{aligned} d(x_i, u) + d(y_j, v) &\geqslant d(x_i, y_j) + d(u, v) - \alpha \max\left\{d(x_i, y_j), d(u, v)\right\} \\ &> (1-\alpha)\left(d(x_i, y_j) + d(u, v)\right). \end{aligned}$$

Since $b_{ij} > 1-\alpha$ we have $b_{ij} = 1$ and thus $\left\langle f_\nu, m_{x_i y_j} \right\rangle = 1$. Hence

$$\|m_{x_i y_j} + m_{uv}\| \geqslant \left\langle f_\nu, m_{x_i y_j} \right\rangle + \langle f_\nu, m_{uv} \rangle > 2 - \alpha.$$

Applying Lemma 2.3 again, we obtain

$$d(x_i, v) + d(y_j, u) > (1 - \alpha)\left(d(x_i, y_j) + d(u, v)\right).$$

Combining these inequalities, we conclude that

$$\begin{aligned}\min\{d(x_i, v) + d(y_j, u), d(x_i, u) + d(y_j, v)\} \\ > (1 - \alpha)\left(d(x_i, y_j) + d(u, v)\right).\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we deduce

$$\begin{aligned}d(u, v) &< \frac{d(x_i, v) + d(y_j, u) + d(x_i, u) + d(y_j, v)}{2(1 - \alpha)} - d(x_i, y_j) \\ &< \frac{2d(x_i, y_j)}{2(1 - \alpha)^2} - d(x_i, y_j) \\ &\leqslant \left(\frac{1}{(1 - \alpha)^2} - 1\right) \max_{i', j' \in I} d(x_{i'}, y_{j'}) \\ &< \varepsilon.\end{aligned}$$

Therefore, we have found $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$. □

To conclude this subsection, we verify Conjecture 2.10 for two additional cases. First, by combining Lemma 2.13 with Proposition 1.10, we obtain the following result. This generalizes [Vee25a, Proposition 3.2] and was proved in [AAL$^+$24b].

Proposition 2.14 (see [AAL$^+$24b, Proposition 4.3])**.** *Let $\mu \in S_{\mathcal{F}(M)}$ be such that*

$$\lim_{\delta \to 0} \sup_{x, y \in \operatorname{supp}(\mu) \cup \{0\}} \alpha\left(\{p \in M \,:\, d(p, x) + d(p, y) < (1 + \delta) d(x, y)\}\right) = 0.$$

Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$, there exists $m_{uv} \in S$ such that $d(u, v) < \varepsilon$.

Proof. One implication follows from Proposition 2.2.

Now fix $\mu \in S_{\mathcal{F}(M)}$ and assume that it is a Δ-point. Fix a slice $S(f, \delta)$ of $B_{\mathcal{F}(M)}$ with $\mu \in S(f, \delta)$ and fix $\varepsilon > 0$. By Lemma 1.6, we may additionally assume that $\delta < 1/2$ and

$$\begin{aligned}\sup_{x, y \in \operatorname{supp}(\mu) \cup \{0\}} \alpha\left(\{p \in M \,:\, d(p, x) + d(p, y) < (1 + 2\delta) d(x, y)\}\right) \\ < \varepsilon(1 - \delta).\end{aligned} \tag{2.8}$$

Let $\gamma > 0$ be such that $\mu \in S(f, \delta - \gamma)$. Then there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in \operatorname{supp}(\mu) \cup \{0\}$ and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$ and

$$\left\| \mu - \sum_{i=1}^n \lambda_i m_{x_i y_i} \right\| < \gamma.$$

Let $\nu = \sum_{i=1}^n \lambda_i m_{x_i y_i}$ and let f_ν be the function given by Lemma 2.13. Then $\langle f + f_\nu, \mu \rangle > 2 - \delta$. Since every slice of $B_{\mathcal{F}(M)}$ contains a molecule and μ is a Δ-point, then by Proposition 1.10 there exists a sequence $(m_{u_k v_k})$ such that $\langle f + f_\nu, m_{u_k v_k} \rangle > 2 - \delta$ and

$$\|m_{u_k v_k} - m_{u_l v_l}\| \geqslant 2 - \delta$$

for all $k, l \in \mathbb{N}$ with $k \neq l$. Thus $(m_{u_k v_k}) \subseteq S(f, \delta)$ and $(m_{u_k v_k}) \subseteq S(f_\nu, \delta)$. Therefore, there exist $i, j \in \{1, \ldots, n\}$ such that

$$(1 - \delta) \max \{d(x_i, v_k) + d(y_j, v_k), d(x_i, u_k) + d(y_j, u_k)\} < d(x_i, y_j)$$

for infinitely many $k \in \mathbb{N}$. Since $\delta < 1/2$, we have $(1 - \delta)^{-1} \leqslant 1 + 2\delta$. Hence by (2.8) there exist distinct $k, l \in \mathbb{N}$ such that

$$\max \{d(u_k, u_l), d(v_k, v_l)\} < \varepsilon(1 - \delta).$$

By Lemma 2.3, we also have

$$(1 - \delta)(d(u_k, v_k) + d(u_l, v_l)) \leqslant d(u_k, u_l) + d(v_k, v_l) < 2\varepsilon(1 - \delta).$$

It follows that $d(u_k, v_k) < \varepsilon$ or $d(u_l, v_l) < \varepsilon$. Therefore, there exists $m_{uv} \in S(f, \delta)$ with $d(u, v) < \varepsilon$, completing the proof. □

Proposition 2.14 applies, for instance, to any norm-1 element of a Lipschitz-free space over a compact metric space or over a bounded subset of ℓ_1. To conclude this subsection, we observe that by applying Proposition 2.8, any slice of the unit ball $B_{\mathcal{F}(M)}$ contains either a denting point or a molecule m_{uv} with $d(u, v)$ arbitrarily small. This observation allows us to prove Conjecture 2.10 for yet another class of spaces, including the Lipschitz-free space introduced in Section 2.3.

Proposition 2.15 (see [Vee25a, Proposition 3.3])**.** *Assume that M is a complete metric space such that for every $\varepsilon > 0$ and every bounded subset $M_0 \subseteq M$, the set*

$$\left\{ m_{uv} \in \operatorname{dent}\left(B_{\mathcal{F}(M)}\right) : u, v \in M_0, d(u, v) \geqslant \varepsilon \right\}$$

is finite. Let $\mu \in S_{\mathcal{F}(M)}$. Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$, there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$.

Proof. The necessity direction follows from Proposition 2.2.

Now assume that μ is a Δ-point. Fix $\varepsilon > 0$ and a slice $S(f, \alpha)$ of $B_{\mathcal{F}(M)}$ with $\mu \in S(f, \alpha)$. We consider two cases.

Case 1. Suppose that there exists a subslice $S(g, \beta) \subseteq S(f, \alpha)$, such that the functional g is local. Then by the definition of locality, there exists $m_{uv} \in S(g, \beta) \subseteq S(f, \alpha)$ such that $d(u, v) < \varepsilon$, as desired.

Case 2. Suppose that for every subslice $S(g, \beta) \subseteq S(f, \alpha)$ the functional g is not local. Then by Proposition 2.8 every subslice of $S(f, \alpha)$ contains a denting point of $B_{\mathcal{F}(M)}$, and by [Wea18, Corollary 3.44], every denting point is a molecule.

Let $\gamma > 0$ be such that $\mu \in S(f, \alpha - \gamma)$. Since elements with finite support are dense in $\mathcal{F}(M)$, we can find $\nu \in S_{\mathcal{F}(M)}$ with finite support such that $\|\mu - \nu\| < \gamma$. Let $g \in S_{\mathrm{Lip}_0(M)}$ be such that $\langle g, \nu \rangle = 1$. Define a function h on the set $\operatorname{supp}(\nu) \cup \{0\} \cup \big(M \backslash B\big(0, R\big)\big)$ in such way that $h(p) = g(p)$ for $p \in \operatorname{supp}(\nu) \cup \{0\}$ and $h(p) = 0$ for $p \in M \backslash B\big(0, R\big)$, where

$$R = 2 \max_{x \in \operatorname{supp}(\nu)} d(0, x).$$

This ensures that h is Lipschitz with constant 1 and it can be extended to M by the McShane–Whitney extension theorem, preserving the Lipschitz constant. Also $\langle h, \nu \rangle = 1$ and

$$\langle f, \mu \rangle + \langle h, \mu \rangle > 1 - \alpha + \gamma + \langle h, \nu \rangle - \gamma = 2 - \alpha.$$

Define

$$S := S\left(\frac{f + h}{\|f + h\|}, 1 - \frac{2 - \alpha}{\|f + h\|} \right).$$

Then $\mu \in S \subseteq S(f, \alpha)$, and by assumption, every subslice of S contains a denting point. By Proposition 1.10, we find a sequence of molecules $(m_{u_i v_i})$ in $S \cap \operatorname{dent}(B_{\mathcal{F}(M)})$ such that $\|m_{u_i v_i} - m_{u_j v_j}\| \geqslant 1$ for all $i, j \in \mathbb{N}$ with $i \neq j$. Then $(m_{u_i v_i}) \subseteq S(f, \alpha)$ and $(m_{u_i v_i}) \subseteq S(h, \alpha)$. Moreover, since h is constant outside of the ball $B(0, R)$, the set $\{u, v \in M \colon m_{uv} \in S(h, \alpha)\}$ is bounded. Therefore the set

$$\left\{ m_{uv} \in \operatorname{dent}(B_{\mathcal{F}(M)}) \colon u, v \in \{u_j, v_j \colon j \in \mathbb{N}\}, d(u, v) \geqslant \varepsilon \right\}$$

is finite by assumption. So there must exist $i \in \mathbb{N}$ with $d(u_i, v_i) < \varepsilon$, concluding the proof. □

Let us mention that in Section 2.4, we study Daugavet and Δ-points in Lipschitz-free spaces over subsets of $\mathbb{R}$-trees, and among other results, we confirm Conjecture 2.10 for this class of Lipschitz-free spaces (see Proposition 2.33).

2.2.2 Discretely connectable points and their relation to Delta-molecules

In this subsection, we aim to provide a purely metric characterization of Δ-points in Lipschitz-free spaces. The subsection is based on [AAL$^+$24a].

In [JRZ22, Proposition 4.2], Jung and Rueda Zoca provided a sufficient metric condition for a molecule m_{xy} to be a Δ-point: namely that the points x and y are *connectable*, i.e., they can be joined by Lipschitz paths in M whose lengths are arbitrarily close to $d(x, y)$. In particular, if x and y are connected by a geodesic, then m_{xy} is a Δ-point. Moreover, the converse is true when M is compact under some additional assumptions (see [JRZ22, Theorem 4.13]). We will show below that these extra hypotheses are, in fact, superfluous and the existence of a geodesic characterizes Δ-molecules for proper M (see Corollary 2.23). For general metric spaces, the notion of connectability is unnecessarily strong and it may be relaxed to allow for discrete paths as follows.

Definition 2.16. Let $x, y \in M$. Given $\varepsilon > 0$, we say that x and y are *ε-discretely connectable (in M)* if there exists a finite sequence of points $p_0, p_1, \ldots, p_n, p_{n+1}$ in M with $p_0 = x$ and $p_{n+1} = y$, such that:

(1) $d(p_i, p_{i+1}) < \varepsilon$ for each $i = 0, \ldots, n$;

(2) $\displaystyle\sum_{i=0}^{n} d(p_i, p_{i+1}) < d(x, y) + \varepsilon$.

We say that x and y are *discretely connectable* if they are ε-discretely connectable for every $\varepsilon > 0$.

In the case where M is proper, this notion is equivalent to the existence of geodesics between points.

Proposition 2.17 (see [AAL$^+$24a, Proposition 6.2])**.** *If M is a proper metric space and $x \neq y \in M$, then x and y are discretely connectable if and only if they are connected by a geodesic.*

Proof. Denote $I = [0, d(x,y)] \subseteq \mathbb{R}$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$.

For every $n \in \mathbb{N}$ we can find points $p_0^n, p_1^n, \ldots, p_{m(n)+1}^n$ in M with $p_0^n = x$ and $p_{m(n)+1}^n = y$ such that $d(p_i^n, p_{i+1}^n) < 1/n$ for $0 \leqslant i \leqslant m(n)$ and

$$\sum_{i=0}^{m(n)} d(p_i^n, p_{i+1}^n) < d(x,y) + \frac{1}{n}.$$

Fix $n \in \mathbb{N}$. For every $a \in I$, choose the smallest $k = k_n(a)$ with $1 \leqslant k \leqslant m(n)+1$ such that

$$a < \sum_{i=0}^{k-1} d(p_i^n, p_{i+1}^n) \leqslant a + \frac{1}{n}.$$

Define $p_a^n = p_k^n$. Then

$$d(x, p_a^n) \leqslant \sum_{i=0}^{k-1} d(p_i^n, p_{i+1}^n) \leqslant a + \frac{1}{n}$$

and

$$\begin{aligned} d(p_a^n, y) &\leqslant \sum_{i=k}^{m(n)} d(p_i^n, p_{i+1}^n) \\ &= \sum_{i=0}^{m(n)} d(p_i^n, p_{i+1}^n) - \sum_{i=0}^{k-1} d(p_i^n, p_{i+1}^n) < d(x,y) + \frac{1}{n} - a. \end{aligned}$$

Thus

$$p_a^n \in B\left(x, a + \frac{1}{n}\right) \cap B\left(y, d(x,y) - a + \frac{1}{n}\right).$$

Since all points p_a^n belong to the compact set $B(x, d(x,y)+1)$, we can define $p_a = \lim_{\mathcal{U}} p_a^n$. In particular, $p_0 = x$ and $p_{d(x,y)} = y$.

We now show that the map $a \mapsto p_a$ from I to M is an isometric embedding. Let $a, b \in I$ with $a < b$ and fix $\delta > 0$. Since U is a free ultrafilter, there exist sets $A, B \in \mathcal{U}$ such that $p_a^n \in B(p_a, \delta)$ for all $n \in A$ and $p_b^n \in B(p_b, \delta)$ for all $n \in B$. Let $C = A \cap B \in \mathcal{U}$. Since $\mathcal{U}$ is free we know that C is infinite. Therefore, we can choose $N \in C$ such that $1/N < \delta$. If $k_N(a) = k_N(b)$, then $p_a^N = p_b^N \in B(p_a, \delta) \cap B(p_b, \delta)$

and $d(p_a, p_b) \leqslant 2\delta$. Otherwise, $k_N(a) < k_N(b)$, and we have

$$\begin{aligned} d(p_a, p_b) &\leqslant \delta + d(p_a^N, p_b^N) + \delta \leqslant 2\delta + \sum_{i=k_N(a)}^{k_N(b)-1} d(p_i^N, p_{i+1}^N) \\ &= 2\delta + \sum_{i=0}^{k_N(b)-1} d(p_i^N, p_{i+1}^N) - \sum_{i=0}^{k_N(a)-1} d(p_i^N, p_{i+1}^N) \\ &< 2\delta + b + \frac{1}{N} - a < b - a + 3\delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, we get $d(p_a, p_b) \leqslant |b - a|$ for all $a, b \in I$. Now, we also have

$$\begin{aligned} d(x, y) &\leqslant d(x, p_a) + d(p_a, p_b) + d(p_b, y) \\ &\leqslant (a - 0) + (b - a) + (d(x, y) - b) = d(x, y) \end{aligned}$$

so all inequalities must be equalities and $d(p_a, p_b) = |b - a|$ for all $a, b \in I$. Thus $a \mapsto p_a$ is an isometric embedding of I into M, i.e., a geodesic from x to y. □

The argument used in [JRZ22, Proposition 4.2] can be discretized to show that m_{xy} is a Δ-point whenever x and y are discretely connectable. A slight variation of that argument yields the following, more general result.

Proposition 2.18 (see [AAL+24a, Proposition 6.3])**.** *Let $\mu \in S_{\mathcal{F}(M)}$. Suppose that for every $\eta > 0$, the element μ can be written as a series of molecules*

$$\mu = \sum_{k=1}^{\infty} a_k m_{x_k y_k} \quad \textit{with} \quad \sum_{k=1}^{\infty} |a_k| < 1 + \eta, \tag{2.9}$$

where each pair (x_k, y_k) is discretely connectable in M. Then μ is a Δ-point in $\mathcal{F}(M)$.

The hypothesis in Proposition 2.18 is clearly satisfied when $\mu = m_{xy}$ for some discretely connectable x and y. More generally, the condition also holds if every pair of points in $\operatorname{supp}(\mu) \cup \{0\}$ is discretely connectable in M, since every $\mu \in S_{\mathcal{F}(M)}$ admits a representation of the form (2.9) (see, e.g., [AP20, Lemma 2.1]).

Proof of Proposition 2.18. Let $S := S(f, \alpha)$ be a slice of $B_{\mathcal{F}(M)}$ with $\mu \in S$, for some $f \in S_{\mathrm{Lip}_0(M)}$ and $\alpha > 0$. Fix $\eta > 0$ such that $\langle f, \mu \rangle > (1-\alpha)(1+\eta)$, and choose a representation of μ of the form (2.9) where every pair (x_k, y_k) is discretely connectable. Without loss of generality, we may assume that $a_k \geqslant 0$ for all k, by swapping x_k with y_k if needed. By convexity, there exists an index k such that $\langle f, m_{x_k y_k} \rangle > 1 - \alpha$, i.e. there are $x = x_k, y = y_k$ in M such that $m_{xy} \in S$ and x, y are discretely connectable.

Now fix $\delta > 0$ such that $\langle f, m_{xy} \rangle > (1-\alpha)(1+\delta)$, and let $\varepsilon < \delta \cdot d(x,y)$ be arbitrary. Choose a sequence of points $p_1, \ldots, p_n \in M$ as in Definition 2.16, and denote $p_0 = x$, $p_{n+1} = y$. Then we have

$$\begin{aligned}
\max_{i \in \{0,\ldots,n\}} \left\langle f, m_{p_i, p_{i+1}} \right\rangle &= \max_{i \in \{0,\ldots,n\}} \frac{f(p_i) - f(p_{i+1})}{d(p_i, p_{i+1})} \\
&\geqslant \frac{\sum_{i=0}^{n} (f(p_i) - f(p_{i+1}))}{\sum_{i=0}^{n} d(p_i, p_{i+1})} > \frac{f(x) - f(y)}{d(x,y) + \varepsilon} \\
&> \frac{\langle f, m_{xy} \rangle}{1+\delta} > 1 - \alpha.
\end{aligned}$$

Therefore we may choose $u = p_k, v = p_{k+1}$ with $d(u,v) < \varepsilon$ and $\langle f, m_{uv} \rangle > 1 - \alpha$, i.e. $m_{uv} \in S$. By Proposition 2.2, μ is a Δ-point. $\square$

For simplicity, in this section, molecules that are Δ-points will be referred to as Δ-*molecules*. Discrete connectability does, in fact, characterize Δ-molecules in Lipschitz-free spaces. In order to prove this, we now construct a family of alternative metrics on M that provide information about how "well connected" (in the sense of Definition 2.16) a given pair of points is, by reducing their distance whenever there is a partial discrete path between them. We define them precisely as the shortest possible distance when giving a preference to discrete paths with sufficiently small step.

Fix $\alpha \in (0,1)$. For any $\varepsilon > 0$ and $x, y \in M$, define

$$w_{\alpha,\varepsilon}(x,y) := \begin{cases} d(x,y), & \text{if } d(x,y) \geqslant \varepsilon, \\ (1-\alpha) d(x,y), & \text{if } d(x,y) < \varepsilon, \end{cases}$$

and

$$b_{\alpha,\varepsilon}(x,y) := \inf \left\{ \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) \,:\, p_0, \ldots, p_{n+1} \in M, p_0 = x, p_{n+1} = y \right\}.$$

Note that $w_{\alpha,\varepsilon}$ and $b_{\alpha,\varepsilon}$ increase as ε decreases. This allows us to define

$$b_\alpha(x,y) := \sup_{\varepsilon>0} b_{\alpha,\varepsilon}(x,y) = \lim_{\varepsilon\to 0} b_{\alpha,\varepsilon}(x,y).$$

Lemma 2.19 (see [AAL$^+$24a, Lemma 6.4])**.** *Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. Then the following hold:*

(1) b_α and $b_{\alpha,\varepsilon}$ are bi-Lipschitz equivalent metrics on M;

(2) For any $x, y \in M$, we have

$$(1-\alpha)d(x,y) \leqslant b_{\alpha,\varepsilon}(x,y) \leqslant b_\alpha(x,y) \leqslant d(x,y);$$

(3) If $x, y \in M$ are not ε-discretely connectable, then

$$b_{\alpha,\varepsilon}(x,y) \geqslant (1-\alpha)d(x,y) + \varepsilon \cdot \min\{\alpha, 1-\alpha\};$$

(4) Two points $x, y \in M$ are discretely connectable if and only if

$$b_\alpha(x,y) = (1-\alpha)d(x,y).$$

Proof. For any $x, y \in M$, we have $b_{\alpha,\varepsilon}(x,y) \leqslant w_{\alpha,\varepsilon}(x,y) \leqslant d(x,y)$. Also, for any finite sequence $p_0 = x, p_1, \ldots, p_n, p_{n+1} = y$ in M, we have

$$\sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) \geqslant \sum_{i=0}^{n} (1-\alpha)d(p_i, p_{i+1}) \geqslant (1-\alpha)d(x,y),$$

so $b_{\alpha,\varepsilon}(x,y) \geqslant (1-\alpha)d(x,y)$. This proves (2) for $b_{\alpha,\varepsilon}$, and thus also for b_α by taking limits.

It is clear that $w_{\alpha,\varepsilon}$ and $b_{\alpha,\varepsilon}$ are symmetric. To verify the triangle inequality, fix $x, y, z \in M$ and $\delta > 0$. Choose two finite sequences $p_1, \ldots, p_n$ and $p'_1, \ldots, p'_m$ of points in M such that

$$\begin{aligned} w_{\alpha,\varepsilon}(x,p_1) + w_{\alpha,\varepsilon}(p_1,p_2) + \cdots + w_{\alpha,\varepsilon}(p_n,z) &< b_{\alpha,\varepsilon}(x,z) + \delta, \\ w_{\alpha,\varepsilon}(z,p'_1) + w_{\alpha,\varepsilon}(p'_1,p'_2) + \cdots + w_{\alpha,\varepsilon}(p'_m,y) &< b_{\alpha,\varepsilon}(z,y) + \delta. \end{aligned}$$

Therefore

$$\begin{aligned} b_{\alpha,\varepsilon}(x,y) &\leqslant w_{\alpha,\varepsilon}(x,p_1) + \cdots + w_{\alpha,\varepsilon}(p_n,z) + \\ &\qquad w_{\alpha,\varepsilon}(z,p'_1) + \cdots + w_{\alpha,\varepsilon}(p'_m,y) \\ &< b_{\alpha,\varepsilon}(x,z) + b_{\alpha,\varepsilon}(z,y) + 2\delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, this proves the triangle inequality for $b_{\alpha,\varepsilon}$, completing the proof that $b_{\alpha,\varepsilon}$ is a metric. Combining with (2), we conclude that $b_{\alpha,\varepsilon}$ is an equivalent metric on M, and letting $\varepsilon \to 0$ we get (1).

For part (3), assume that x and y are not ε-discretely connectable. Then for every finite sequence $p_0 = x, p_1, \ldots, p_{n+1} = y$ in M, at least one of the two following statements holds:

(a) $\sum_{i=0}^{n} d(p_i, p_{i+1}) \geqslant d(x,y) + \varepsilon$.

(b) $d(p_k, p_{k+1}) \geqslant \varepsilon$ for some $k \in \{0, \ldots, n\}$.

In case (a), we estimate

$$\begin{aligned}\sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) &\geqslant \sum_{i=0}^{n} (1-\alpha) d(p_i, p_{i+1}) \geqslant (1-\alpha)(d(x,y) + \varepsilon) \\ &\geqslant (1-\alpha) d(x,y) + (1-\alpha)\varepsilon.\end{aligned}$$

In case (b), we have

$$\begin{aligned}\sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) &\geqslant (1-\alpha) \sum_{i=0}^{k-1} d(p_i, p_{i+1}) + d(p_k, p_{k+1}) \\ &\qquad + (1-\alpha) \sum_{i=k+1}^{n} d(p_i, p_{i+1}) \\ &= (1-\alpha) \sum_{i=0}^{n} d(p_i, p_{i+1}) + \alpha d(p_k, p_{k+1}) \\ &\geqslant (1-\alpha) d(x,y) + \alpha\varepsilon.\end{aligned}$$

In either case, taking the infimum over all choices of p_i yields the desired lower bound for $b_{\alpha,\varepsilon}(x,y)$, proving (3).

Finally, one of the implications in (4) is already established by (3). For the converse, assume that x and y are discretely connectable. Let $\varepsilon' > 0$ and $\delta \in (0, \varepsilon')$. Then there exists a finite sequence of points $p_0 = x, p_1, \ldots, p_n, p_{n+1} = y$ in M such that $d(p_i, p_{i+1}) < \delta$ for all $i \in \{0, \ldots, n\}$ and $\sum_{i=0}^{n} d(p_i, p_{i+1}) < d(x,y) + \delta$. Thus

$$\begin{aligned}b_{\alpha,\varepsilon'}(x,y) &\leqslant \sum_{i=0}^{n} w_{\alpha,\varepsilon'}(p_i, p_{i+1}) \\ &= (1-\alpha) \sum_{i=0}^{n} d(p_i, p_{i+1}) < (1-\alpha)(d(x,y) + \delta).\end{aligned}$$

Letting $\delta \to 0$ followed by $\varepsilon' \to 0$ yields $b_\alpha(x,y) \leqslant (1-\alpha)d(x,y)$, and an appeal to (2) ends the proof. □

Notice that Lemma 2.19 (2) implies that the Lipschitz-free spaces $\mathcal{F}(M,d)$ and $\mathcal{F}(M,b_\alpha)$ are linearly isomorphic completions of $\operatorname{span}\delta(M)$. In particular, the norm $\|\cdot\|_{\mathcal{F}(M,b_\alpha)}$ is equivalent to the original norm on $\mathcal{F}(M)$. The same holds for $b_{\alpha,\varepsilon}$ in place of b_α. More precisely, for every $\mu \in \mathcal{F}(M)$, we have

$$(1-\alpha)\|\mu\|_{\mathcal{F}(M)} \leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} \leqslant \|\mu\|_{\mathcal{F}(M,b_\alpha)} \leqslant \|\mu\|_{\mathcal{F}(M)}.$$

The corresponding inclusions hold for the unit balls of the dual spaces

$$(1-\alpha)B_{\mathrm{Lip}_0(M,d)} \subseteq B_{\mathrm{Lip}_0(M,b_{\alpha,\varepsilon})} \subseteq B_{\mathrm{Lip}_0(M,b_\alpha)} \subseteq B_{\mathrm{Lip}_0(M,d)}.$$

We now state the following result.

Lemma 2.20 (see [AAL+24a, Lemma 6.5])**.** *For any $\mu \in \mathcal{F}(M)$ and $\alpha \in (0,1)$ we have*

$$\lim_{\varepsilon\to 0} \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} = \|\mu\|_{\mathcal{F}(M,b_\alpha)}.$$

Proof. Suppose first that μ has finite support and set $S = \operatorname{supp}(\mu)\cup\{0\}$. Let $\eta > 0$. Since S is finite, we can find $\varepsilon_0 > 0$ such that $b_\alpha(x,y) \leqslant (1+\eta)b_{\alpha,\varepsilon}(x,y)$ for all $x,y \in S$ and all $\varepsilon \in (0,\varepsilon_0)$. For any such ε we can, by e.g. [Wea18, Proposition 3.16], write μ as a finite sum of molecules in $\mathcal{F}(M,b_{\alpha,\varepsilon})$ in the form

$$\mu = \sum_{k=1}^{n} a_k \frac{\delta(x_k)-\delta(y_k)}{b_{\alpha,\varepsilon}(x_k,y_k)}$$

where $x_k \neq y_k \in S$ and $\sum_k |a_k| = \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})}$. Then

$$\begin{aligned}\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} \leqslant \|\mu\|_{\mathcal{F}(M,b_\alpha)} &= \left\| \sum_{k=1}^{n} a_k \frac{b_\alpha(x_k,y_k)}{b_{\alpha,\varepsilon}(x_k,y_k)} \frac{\delta(x_k)-\delta(y_k)}{b_\alpha(x_k,y_k)} \right\|_{\mathcal{F}(M,b_\alpha)} \\ &\leqslant \sum_{k=1}^{n} |a_k| \frac{b_\alpha(x_k,y_k)}{b_{\alpha,\varepsilon}(x_k,y_k)} \leqslant (1+\eta)\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})}\end{aligned}$$

for $\varepsilon < \varepsilon_0$. Thus the lemma holds for this μ.

Now let $\mu \in \mathcal{F}(M)$ be arbitrary, and let $\eta > 0$. Choose a finitely supported element $\nu \in \mathcal{F}(M)$ such that $\|\mu-\nu\|_{\mathcal{F}(M,b_\alpha)} \leqslant \eta$. By the

previous case, $\|\nu\|_{\mathcal{F}(M,b_\alpha)} \leqslant \|\nu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + \eta$ when ε is small enough. For such ε, we have

$$\begin{aligned}\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} \leqslant \|\mu\|_{\mathcal{F}(M,b_\alpha)} &\leqslant \|\nu\|_{\mathcal{F}(M,b_\alpha)} + \eta \\ &\leqslant \|\nu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + 2\eta \\ &\leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + \|\mu-\nu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + 2\eta \\ &\leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + \|\mu-\nu\|_{\mathcal{F}(M,b_\alpha)} + 2\eta \\ &\leqslant \|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} + 3\eta.\end{aligned}$$

Since $\eta > 0$ was arbitrary, this proves the lemma. □

The relevance of the next lemma lies in the fact that condition (i) characterizes Δ-points when μ is a molecule (see [JRZ22, Theorem 4.7]) or, more generally, a finitely supported element of $S_{\mathcal{F}(M)}$ (see Theorem 2.11).

Lemma 2.21 (see [AAL+24a, Lemma 6.6])**.** *Let $\mu \in S_{\mathcal{F}(M)}$. Then the following are equivalent:*

(i) Every slice of $B_{\mathcal{F}(M)}$ that contains μ also contains molecules m_{uv} for arbitrarily small $d(u,v)$.

(ii) $\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} = 1-\alpha$ for all $\alpha \in (0,1)$ and all $\varepsilon > 0$.

(iii) $\|\mu\|_{\mathcal{F}(M,b_\alpha)} = 1-\alpha$ for all $\alpha \in (0,1)$.

Proof. (i) ⇒ (ii). Assume that (i) holds, and suppose $\|\mu\|_{\mathcal{F}(M,b_{\alpha,\varepsilon})} > 1-\alpha$ for some $\varepsilon > 0$ and $\alpha \in (0,1)$. Then there exists $h \in S_{\mathrm{Lip}_0(M,b_{\alpha,\varepsilon})} \subseteq B_{\mathrm{Lip}_0(M)}$ such that $\langle h,\mu\rangle > 1-\alpha$. By assumption, there exists a molecule m_{uv} in $\mathcal{F}(M)$ such that $\langle h, m_{uv}\rangle > 1-\alpha$ and $d(u,v) < \varepsilon$. Thus

$$b_{\alpha,\varepsilon}(u,v) \geqslant h(u) - h(v) > (1-\alpha)d(u,v) = w_{\alpha,\varepsilon}(u,v) \geqslant b_{\alpha,\varepsilon}(u,v),$$

which is a contradiction. This proves (ii).

(ii) ⇒ (iii) follows directly from Lemma 2.20.

(iii) ⇒ (i). Assume that (iii) holds. Fix $\varepsilon > 0$ and a slice $S(f,\alpha)$ such that $\mu \in S(f,\alpha)$. By Lemma 1.6, we may assume $\alpha \in (0,1)$. From (iii), we have

$$\|f\|_{\mathrm{Lip}_0(M,b_\alpha)} \geqslant \frac{\langle f,\mu\rangle}{\|\mu\|_{\mathcal{F}(M,b_\alpha)}} > \frac{1-\alpha}{1-\alpha} = 1.$$

Therefore, there exist $x, y \in M$ such that

$$f(x) - f(y) > b_\alpha(x, y) \geqslant b_{\alpha,\varepsilon}(x, y).$$

By the definition of $b_{\alpha,\varepsilon}(x, y)$, we can find points $p_0, p_1, \ldots, p_{n+1} \in M$ such that $p_0 = x, p_{n+1} = y$, and

$$f(x) - f(y) > \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}).$$

Define the sets $I_1 = \{i \in \{0, \ldots, n\} : d(p_i, p_{i+1}) < \varepsilon\}$ and $I_2 = \{0, \ldots, n\} \setminus I_1$. Then

$$\begin{aligned}(1-\alpha)\sum_{i\in I_1} d(p_i, p_{i+1}) + \sum_{i\in I_2} d(p_i, p_{i+1}) &= \sum_{i=0}^{n} w_{\alpha,\varepsilon}(p_i, p_{i+1}) \\ &< f(x) - f(y) \\ &= \sum_{i=0}^{n} (f(p_i) - f(p_{i+1})) \\ &\leqslant \sum_{i\in I_1} (f(p_i) - f(p_{i+1})) + \sum_{i\in I_2} d(p_i, p_{i+1})\end{aligned}$$

and therefore there exists $i \in I_1$ such that $f(p_i) - f(p_{i+1}) > (1 - \alpha)d(p_i, p_{i+1})$. Since $d(p_i, p_{i+1}) < \varepsilon$, we conclude that (i) holds with $u = p_i$, $v = p_{i+1}$. □

We are now in a position to prove our characterization of Δ-molecules.

Theorem 2.22 (see [AAL$^+$24a, Theorem 6.7])**.** *Let $x \neq y \in M$. Then m_{xy} is a Δ-point of $\mathcal{F}(M)$ if and only if x and y are discretely connectable in M.*

Proof. One implication follows immediately from Proposition 2.18. For the converse, suppose that m_{xy} is a Δ-point, and fix $\alpha \in (0, 1)$. Then $\mu = m_{xy}$ satisfies condition (i) of Lemma 2.21 by [JRZ22, Theorem 4.7], and hence also satisfies condition (iii). In particular,

$$1 - \alpha = \|m_{xy}\|_{\mathcal{F}(M,b_\alpha)} = \left\| \frac{\delta(x) - \delta(y)}{d(x, y)} \right\|_{\mathcal{F}(M,b_\alpha)} = \frac{b_\alpha(x, y)}{d(x, y)}.$$

That is, $b_\alpha(x, y) = (1 - \alpha)d(x, y)$. Now Lemma 2.19 (4) shows that x and y are discretely connectable. □

Corollary 2.23 (see [AAL+24a, Corollary 6.8]). *Let M be a proper metric space, and let $x \neq y \in M$. Then m_{xy} is a Δ-point of $\mathcal{F}(M)$ if and only if x and y are connected by a geodesic.*

By [Vee23, Corollary 4.5], every convex combination (finite or infinite) of Δ-molecules of $\mathcal{F}(M)$ is again a Δ-point. It is then natural to ask whether the converse holds. Note that the question only makes sense for elements of $S_{\mathcal{F}(M)}$ that can actually be written as convex combinations of molecules, which is not the case for all elements in general (see [APS24, Section 4]). This question was explicitly raised in [Vee23, Problem 3] for those Δ-points $\mu \in S_{\mathcal{F}(M)}$ with finite support. In that case, we have:

- μ can always be written as a convex combination of molecules (see, e.g., [Wea18, Proposition 3.16]); and
- μ also satisfies property (i) from Lemma 2.21, by Theorem 2.11.

The techniques developed in this section allow us to answer the question in the positive.

Theorem 2.24 (see [AAL+24a, Theorem 6.9]). *Suppose that $\mu \in S_{\mathcal{F}(M)}$ is a Δ-point with finite support. Then μ can be written as a finite convex combination of Δ-molecules in $S_{\mathcal{F}(M)}$.*

Proof. Fix $\alpha \in (0,1)$. By Lemma 2.21 and Theorem 2.11, we have $\|\mu\|_{\mathcal{F}(M,b_\alpha)} = 1-\alpha$. Since μ is a finitely supported element of $\mathcal{F}(M,b_\alpha)$, we may write $\mu/\|\mu\|_{\mathcal{F}(M,b_\alpha)}$ as a finite convex combination of b_α-molecules (see, e.g., [Wea18, Proposition 3.16]). That is,

$$\frac{\mu}{1-\alpha} = \sum_{i=1}^{n} \lambda_i \frac{\delta(x_i) - \delta(y_i)}{b_\alpha(x_i, y_i)}$$

for some $x_i \neq y_i \in M$ and $\lambda_i > 0$ with $\sum_{i=1}^n \lambda_i = 1$. Then

$$\begin{aligned} 1 = \|\mu\|_{\mathcal{F}(M,d)} &= (1-\alpha) \left\| \sum_{i=1}^{n} \lambda_i \frac{d(x_i,y_i)}{b_\alpha(x_i,y_i)} \frac{\delta(x_i)-\delta(y_i)}{d(x_i,y_i)} \right\|_{\mathcal{F}(M,d)} \\ &\leqslant \sum_{i=1}^{n} \lambda_i \cdot (1-\alpha) \frac{d(x_i,y_i)}{b_\alpha(x_i,y_i)} \leqslant \sum_{i=1}^{n} \lambda_i \cdot \frac{1-\alpha}{1-\alpha} = 1. \end{aligned}$$

Therefore all inequalities must be equalities. In particular, $\mu = \sum_{i=1}^n \lambda_i m_{x_i y_i}$ is a finite convex combination of molecules such that $b_\alpha(x_i,y_i) = (1-\alpha)d(x_i,y_i)$ for all i. By Lemma 2.19 (4) and Theorem 2.22, it follows that each $m_{x_i y_i}$ is a Δ-point. □

We conclude by noting that the existence of Δ-points in $\mathcal{F}(M)$ does not, in general, imply the existence of Δ-molecules. For instance, if M is the Smith–Volterra–Cantor set, then $\mathcal{F}(M)$ is isometric to $L_1 \oplus_1 \ell_1$ by the proof of [God10, Corollary 3.4]. Since L_1 contains Δ-points, so does $\mathcal{F}(M)$. However, M is compact and totally disconnected, so $\mathcal{F}(M)$ cannot contain Δ-molecules. In particular, the converse of Proposition 2.18 fails in general.

2.3 Example of a dual Lipschitz-free space that is equivalent to ℓ_1 and contains a Daugavet point

This section is dedicated to constructing and studying an example of a Lipschitz-free space with the Radon–Nikodým property whose unit sphere contains a Daugavet point. The example was first introduced in [Vee23], where the aforementioned properties were shown. Later, in [AAL+24a], even stronger properties of this space were established, more specifically, the space is isomorphic to ℓ_1 and is also a dual Banach space. We now proceed to define the underlying metric space.

We construct a metric space $\mathcal{M}$ as a subset of $\mathbb{R}^2$ as follows. Let $x := (0,0)$, $y := (1,0)$, and set $S_0 := \{x, y\}$. For every $n \in \mathbb{N}$, define

$$S_n := \left\{(2^{-n}k, 2^{-n}) \,:\, k = 0, 1, \ldots, 2^n\right\},$$

and let $\mathcal{M} := \bigcup_{n=0}^{\infty} S_n$ (see Figure 2.3). Endow $\mathcal{M}$ with the metric

$$d((a_1,b_1),(a_2,b_2)) := \begin{cases} |a_1 - a_2|\,, & \text{if } b_1 = b_2; \\ |b_1 - b_2| + \min\{a_1 + a_2, 2 - (a_1 + a_2)\}\,, & \text{if } b_1 \neq b_2; \end{cases}$$

for $(a_1,b_1),(a_2,b_2) \in \mathcal{M}$, and take x as its base point. It is straightforward to verify that $\mathcal{M}$ is complete. Indeed, let (u_n) be a Cauchy sequence in $\mathcal{M}$. If there exists $m \in \mathbb{N}$ such that the sequence (u_n) is in the finite set $\bigcup_{n=0}^{m} S_n$, then it must eventually be constant. Otherwise, for every $m \in \mathbb{N}$, there exist $k > m$ and $n \in \mathbb{N}$ such that $u_n \in S_k$. In this case, we can extract a subsequence (u_{n_k}) with $u_{n_k} \in S_{m_k}$, where $m_1 < m_2 < m_3 < \cdots$. By construction, the distance between any two different elements $u_{n_k} = (a_{n_k}, b_{n_k})$ and $u_{n_l} = (a_{n_l}, b_{n_l})$ is

$$\min\{a_{n_k} + a_{n_l}, 2 - a_{n_k} - a_{n_l}\} + |b_{n_k} - b_{n_l}|.$$

Since (u_{n_k}) is a Cauchy sequence, either $u_{n_k} \to x$ or $u_{n_k} \to y$. Thus $\mathcal{M}$ is complete.

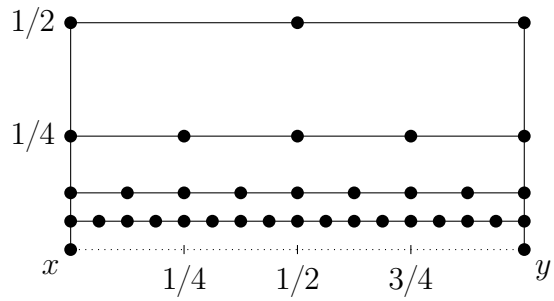

Figure 2.3: The sets $S_0, \ldots, S_4$

To see that $\mathcal{F}(\mathcal{M})$ has the Radon–Nikodým property, recall that by [AGPP22, Theorem 4.6], a Lipschitz-free spaces over a metric space M has the Radon–Nikodým property if and only if the completion of M is purely 1-unrectifiable. A metric space M is said to be *purely 1-unrectifiable* if for every $A \subseteq \mathbb{R}$ and every Lipschitz function $f\colon A \to M$, the 1-dimensional Hausdorff measure of $f(A)$ is 0. The *1-dimensional Hausdorff* measure is defined by

$$\mathcal{H}^1(M) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i) \colon M \subseteq \bigcup_{i=1}^{\infty} E_i, \operatorname{diam}(E_i) < \delta \right\}.$$

It follows immediately that any countable metric space is purely 1-unrectifiable. Since $\mathcal{M}$ is countable and complete, its completion is purely 1-unrectifiable, which implies that $\mathcal{F}(M)$ has the Radon–Nikodým property. We now show that the molecule m_{xy} is a Daugavet point.

Proposition 2.25 (see [Vee23, Example 3.1])**.** *Let $\mathcal{M}$ and $x, y \in \mathcal{M}$ be as defined above. Then the molecule m_{xy} is a Daugavet point.*

Proof. By Theorem 2.9, it suffices to show that $\|m_{xy} - \nu\| = 2$ for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(\mathcal{M})})$. Note that all denting points are preserved extreme points and according to [Wea18, Corollary 3.44] all preserved extreme points are molecules in Lipschitz-free spaces over complete metric spaces, hence $\operatorname{dent}(B_{\mathcal{F}(\mathcal{M})})$ consists only of molecules. By Theorem 2.4 a molecule m_{uv} is a denting point of $B_{\mathcal{F}(\mathcal{M})}$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[u, v]_\delta \subseteq B(u, \varepsilon) \cup B(v, \varepsilon).$$

Therefore, neither m_{xy} nor m_{yx} is a denting point of $B_{\mathcal{F}(\mathcal{M})}$, since for every $n \in \mathbb{N}$, we have

$$z := \left(1/2, 1/2^{n+2}\right) \in [x,y]_{1/2^n} \setminus \left(B(x,1/2) \cup B(y,1/2)\right),$$

because $d(x,z) = d(y,z) = 1/2 + 1/2^{n+2}$.

Now fix $m_{uv} \in \operatorname{dent}(B_{\mathcal{F}(\mathcal{M})})$, and write $u = (a_1, b_1)$ and $v = (a_2, b_2)$. Since m_{uv} is a denting point, we must have $[u,v] = \{u,v\}$. Let us show that $\|m_{xy} - m_{uv}\| = 2$.

If $b_1 = b_2$, then $|a_1 - a_2| = b_1$, because otherwise either $(a_1 + b_1, b_1)$ or $(a_1 - b_1, b_1)$ is in $[u,v] \setminus \{u,v\}$. Hence

$$d(x,u) + d(y,v) = a_1 + b_1 + 1 - a_2 + b_2 \geqslant 1 + |a_1 - a_2| = d(x,y) + d(u,v)$$

and by Lemma 2.3 we get $\|m_{xy} - m_{uv}\| = 2$.

If $b_1 \neq b_2$, then either $a_1 = a_2 = 0$ or $a_1 = a_2 = 1$, because otherwise one of the four points $(0,b_1), (0,b_2), (1,b_1), (1,b_2)$ is in $[u,v] \setminus \{u,v\}$. Hence

$$\begin{aligned} d(x,u) + d(y,v) &= a_1 + b_1 + 1 - a_2 + b_2 = 1 + b_1 + b_2 \\ &\geqslant 1 + |b_1 - b_2| = d(x,y) + d(u,v) \end{aligned}$$

and again Lemma 2.3 gives $\|m_{xy} - m_{uv}\| = 2$.

Now we have shown that for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(\mathcal{M})})$ we have $\|m_{xy} - \nu\| = 2$. Therefore, by Theorem 2.9 the molecule m_{xy} is a Daugavet point. □

Next, let us introduce our candidate for a predual of $\mathcal{F}(\mathcal{M})$. Define

$$V := \{(a,b) \in \mathcal{M} : a = 0 \text{ or } a = 1\},$$

and consider the function $h : \mathcal{M} \to \mathbb{R}$ given by

$$h(a,b) := a.$$

Observe that $h \in S_{\operatorname{Lip}_0(\mathcal{M})}$. Define

$$Y := \Big\{ f \in \operatorname{Lip}_0(\mathcal{M}) \,:\, \lim_n \Big\| \big(f - f(y)\cdot h\big)\big|_{S_n} \Big\|_L \to 0, \; f\big|_V \text{ is locally flat} \Big\}.$$

Recall that a function $f \in \operatorname{Lip}_0(M)$ is called *locally flat* if

$$\lim_{x,y \to z} \frac{f(x) - f(y)}{d(x,y)} = 0$$

for every $z \in M$. We are now ready to state the main theorem of this section.

Theorem 2.26 (see [AAL+24a, Theorem 2.1]). *Let $\mathcal{M}$ and Y be as defined above. Then the following hold:*

(1) The Banach space Y satisfies $Y^* = \mathcal{F}(\mathcal{M})$;

(2) The Lipschitz-free space $\mathcal{F}(\mathcal{M})$ is isomorphic to ℓ_1.

In particular, there exists a separable dual space isomorphic to ℓ_1 that contains a Daugavet point.

Proof of Theorem 2.26 (1). According to a theorem by Petunin and Plichko [PP74, Theorem 4], given a separable Banach space X, a subspace Y of X^* is an isometric predual of X if it satisfies the following conditions:

(a) Y is norm closed;

(b) Y separates points of X;

(c) every $f \in Y$ attains its norm on S_X.

We will verify these conditions for $X := \mathcal{F}(\mathcal{M})$ and the subspace Y of $\mathrm{Lip}_0(\mathcal{M})$.

We start with (a). Let (f_n) be a sequence in Y converging in norm to some $f \in \mathrm{Lip}_0(\mathcal{M})$. We show that $f \in Y$. Fix $\varepsilon > 0$. Choose k such that $\|f - f_k\|_L < \varepsilon$. Then for every n,

$$\begin{aligned}\left\|(f - f(y)\cdot h)|_{S_n}\right\|_L &\leqslant \left\|(f - f_k)|_{S_n}\right\|_L + \left\|(f_k - f_k(y)\cdot h)|_{S_n}\right\|_L \\ &\quad + \left\|((f_k(y) - f(y))\cdot h)|_{S_n}\right\|_L \\ &< 2\varepsilon + \left\|(f_k - f_k(y)\cdot h)|_{S_n}\right\|_L.\end{aligned}$$

Since $f_k \in Y$, this will be less than 3ε for n large enough. Moreover, V is compact and so the space $\mathrm{lip}_0(V)$ is closed (see e.g. [Wea18, Corollary 4.5]). Since $f_k|_V \to f|_V$ in norm, we conclude $f|_V \in \mathrm{lip}_0(V)$. Hence $f \in Y$, and thus Y is closed.

Let us now verify (b). Fix $\mu \in \mathcal{F}(\mathcal{M})$, $\mu \neq 0$. We show that there exits $f \in Y$ such that $\langle f, \mu\rangle \neq 0$. Suppose first that $\mathrm{supp}(\mu)$ contains no isolated points of $\mathcal{M}$. Then $\mathrm{supp}(\mu) = \{y\}$, since by definition the base point cannot be an isolated point of $\mathrm{supp}(\mu)$. Thus μ is a nonzero multiple of δ_y, and so $\langle h, \mu\rangle \neq 0$ and we have $h \in Y$. Now suppose that $\mathrm{supp}(\mu)$ contains some isolated point u of $\mathcal{M}$. Then $\{u\}$ is an open

neighborhood of u, and by [APPP20, Proposition 2.7] there exists a function $f \in \mathrm{Lip}_0(\mathcal{M})$, supported on $\{u\}$, such that $\langle f, \mu \rangle \neq 0$. In other words, $\langle \chi_{\{u\}}, \mu \rangle \neq 0$, where $\chi_{\{u\}}$ is the characteristic function of the set $\{u\}$. But $\chi_{\{u\}} \in Y$, so this finishes the proof of (b).

Finally we check (c). We show that every $f \in Y$ attains its norm on $S_{\mathcal{F}(\mathcal{M})}$. In fact, we show that f attains its Lipschitz constant between two points of $\mathcal{M}$. Let $f \in Y$ and assume that $\|f\|_L = 1$. We will find a sequence of pairs of points (u_n, v_n) in $\mathcal{M}$ such that $\langle f, m_{u_n v_n} \rangle \to 1$. Note that $p \in [u, v]$ implies that m_{uv} is a convex combination of m_{up} and m_{pv}, hence

$$\max \{ \langle f, m_{up} \rangle, \langle f, m_{pv} \rangle \} \geqslant \langle f, m_{uv} \rangle .$$

Thus, by replacing each pair u_n, v_n with other points in $[u_n, v_n]$ and passing to a subsequence if necessary, we may assume that either $u_n, v_n \in V$ for all n or there exists a sequence (k_n) in $\mathbb{N}$ such that $u_n, v_n \in S_{k_n}$ for all n.

In the first case, all u_n, v_n belong to the compact set V, so by passing to a subsequence we get $u_n \to u$, $v_n \to v$ for some $u, v \in V$. Note that $u \neq v$, since $f|_V$ is locally flat. Thus $\langle f, m_{uv} \rangle = 1$, and so f attains its norm on $S_{\mathcal{F}(\mathcal{M})}$.

In the second case, assume that $u_n, v_n \in S_{k_n}$ for all n. If (k_n) is bounded, say by N, then this implies that f restricted to the finite set $S_1 \cup \cdots \cup S_N$ has Lipschitz constant 1, and so it must attain its Lipschitz constant in that set. Otherwise we may assume that $k_n \to \infty$. Then we have $\left\| f|_{S_{k_n}} \right\|_L \geqslant \langle f, m_{u_n v_n} \rangle$ and, given that $\left\| f|_{S_{k_n}} \right\|_L \to |f(y)|$, we obtain $|f(y)| = 1$. So f attains its Lipschitz constant between x and y.

Since all three conditions in the Petunin–Plichko theorem are satisfied, it follows that $Y^* = \mathcal{F}(\mathcal{M})$. □

Remark 2.27 (see [AAL$^+$24a, Remark 2.2]). The definition of Y is not equivalent if we ask that f is locally flat instead of $f|_V$. For example, the function h is not locally flat while $h|_V$ is. It is conjectured that whenever a Lipschitz-free space is a separable dual, it admits a predual consisting entirely of locally flat functions (see, e.g., [AGP24, Question 3.3]). Since preduals of Lipschitz-free spaces are not unique in general, an alternative predual Y satisfying this stronger condition may exist.

Remark 2.28 (see [AAL$^+$24a, Remark 2.3]). The argument in the proof of Theorem 2.26 (1) can be easily adapted to show that the Lipschitz-free space from [ANPP21, Example 4.2] is also a dual space. The corresponding predual has a simpler description, as the local flatness condition can be omitted. Note that the molecule m_{0q} in that example is a Δ-point (this follows, for instance, from Theorem 2.22) but it is not a Daugavet point as the unit ball of this space contains denting points that are at distance strictly less than 2 from m_{0q}.

Before proceeding with the proof of Theorem 2.26 (2) let us observe that the argument given below can be adapted to show that the Lipschitz-free space over other similar metric spaces, such as those from [ANPP21, Examples 4.2 and 4.3], is also isomorphic to ℓ_1. It is based on the following general lemma, which may be viewed as a finite version of the approach developed in [AACD21].

Lemma 2.29 (see [AAL$^+$24a, Lemma 2.4])**.** *Let M be a complete pointed metric space, and let $\varphi_1, \dots, \varphi_n$ be non-negative Lipschitz functions on M with bounded support such that $\varphi_1 + \cdots + \varphi_n = 1$. Suppose that $A_1, \dots, A_n$ are subsets of M containing the base point, and that $\operatorname{supp}(\varphi_k) \subseteq A_k$ for all k. Then $\mathcal{F}(M)$ is isomorphic to a complemented subspace of $\mathcal{F}(A_1) \oplus \cdots \oplus \mathcal{F}(A_n)$.*

Proof. For each $k = 1, \dots, n$, we define the weighting operator $W_k \colon \mathcal{F}(M) \to \mathcal{F}(M)$ by

$$\langle W_k \mu, f \rangle := \langle \mu, f \cdot \varphi_k \rangle$$

for all $\mu \in \mathcal{F}(M)$ and $f \in \operatorname{Lip}_0(M)$. By results from [APPP20, Section 2], each W_k is a well-defined bounded operator. Moreover, its range is contained in $\mathcal{F}(A_k)$, which we naturally identify with a subspace of $\mathcal{F}(M)$. Define the operators

$$\begin{aligned} T &\colon \mathcal{F}(M) \to \mathcal{F}(A_1) \oplus \cdots \oplus \mathcal{F}(A_n) \\ S &\colon \mathcal{F}(A_1) \oplus \cdots \oplus \mathcal{F}(A_n) \to \mathcal{F}(M) \end{aligned}$$

by $T\mu := (W_1\mu, \dots, W_n\mu)$ and $S(\mu_1, \dots, \mu_n) := \mu_1 + \cdots + \mu_n$. Both T and S are clearly bounded. For every $\mu \in \mathcal{F}(M)$ and $f \in \operatorname{Lip}_0(M)$, we have

$$\begin{aligned} \langle ST\mu, f \rangle &= \langle W_1\mu, f \rangle + \cdots + \langle W_n\mu, f \rangle \\ &= \langle \mu, f \cdot \varphi_1 \rangle + \cdots + \langle \mu, f \cdot \varphi_n \rangle \\ &= \langle \mu, f \rangle, \end{aligned}$$

by the choice of φ_k. Thus ST is the identity on $\mathcal{F}(M)$. Therefore, $P := TS$ is a projection from $\mathcal{F}(A_1) \oplus \cdots \oplus \mathcal{F}(A_n)$ onto the subspace $T(\mathcal{F}(M))$, which is isomorphic to $\mathcal{F}(M)$. □

The following result provides a simple sufficient condition under which Lemma 2.29 can be applied.

Lemma 2.30 (see [AAL+24a, Lemma 2.5])**.** *Let M be a bounded complete metric space, and let $U_1, \ldots, U_n$ be an open cover of M. Suppose that*

$$\inf_{u \in M} \sum_{k=1}^{n} d(u, M \backslash U_k) > 0.$$

Then there exist non-negative Lipschitz functions $\varphi_1, \ldots, \varphi_n$ on M such that $\varphi_1 + \cdots + \varphi_n = 1$ and each φ_k vanishes outside of U_k.

Proof. Define for each $u \in M$,

$$\varphi_k(u) := \frac{d(u, M \backslash U_k)}{\sum_{i=1}^{n} d(u, M \backslash U_i)}.$$

By assumption, the denominator is Lipschitz and bounded below, so each φ_k is Lipschitz (see, e.g., [Wea18, Proposition 1.30]). The other conditions are satisfied trivially. □

Proof of Theorem 2.26 (2). Fix real numbers α, β with $0 < \beta < \alpha < \frac{1}{2}$ and consider the sets

$$\begin{aligned} A &:= \{(a,b) \in \mathcal{M} \,:\, a < \alpha\}, \\ B &:= \{(a,b) \in \mathcal{M} \,:\, \beta < a < 1 - \beta\}, \\ C &:= \{(a,b) \in \mathcal{M} \,:\, a > 1 - \alpha\}, \end{aligned}$$

which form an open cover of $\mathcal{M}$. We aim to apply Lemma 2.30 to this cover. Note that $A \cap C = \varnothing$, while B intersects both A and C. For $z = (a,b) \in \mathcal{M}$, define

$$D(z) := d(z, \mathcal{M} \backslash A) + d(z, \mathcal{M} \backslash B) + d(z, \mathcal{M} \backslash C).$$

Let us see that $D(z) \geqslant \alpha - \beta$ for all z. By symmetry, it suffices to verify this when $a \leqslant \frac{1}{2}$. Then we consider three cases.

Case 1. Assume that $z \in A \backslash B$. Then $D(z) = d(z, \mathcal{M} \backslash A)$. So either $z = x$, in which case $D(z) \geqslant \alpha$, or $a \leqslant \beta$ and $b > 0$. Then the

closest point to z in $\mathcal{M}\backslash A$ has the form (a_1, b) with $a_1 \geqslant \alpha$, yielding $D(z) = a_1 - a \geqslant \alpha - \beta$.

Case 2. Assume that $z \in A \cap B$. Then $D(z) = d(z, \mathcal{M}\backslash A) + d(z, \mathcal{M}\backslash B)$, and we have $b > 0$ and $\beta < a < \alpha$. Thus the closest points in $\mathcal{M}\backslash A$ and $\mathcal{M}\backslash B$ are (a_1, b) and (a_2, b), where $a_1 \geqslant \alpha$ and $a_2 \leqslant \beta$, respectively, so $D(z) = (a_1 - a) + (a - a_2) \geqslant \alpha - \beta$.

Case 3. Assume that $z \in B\backslash(A \cup C)$. Then $D(z) = d(z, \mathcal{M}\backslash B)$, and we have $b > 0$ and $\alpha \leqslant a \leqslant \frac{1}{2}$. Thus the closest point in $\mathcal{M}\backslash B$ is (a_2, b) with $a_2 \leqslant \beta$, and so $D(z) = a - a_2 \geqslant \alpha - \beta$.

Thus $\inf D(z) > 0$, and we may apply Lemma 2.30, followed by Lemma 2.29, to conclude that $\mathcal{F}(\mathcal{M})$ is isomorphic to a complemented subspace of

$$\mathcal{F}(\overline{A}) \oplus \mathcal{F}(\overline{B} \cup \{x\}) \oplus \mathcal{F}(\overline{C} \cup \{x\}).$$

We now verify that each of these three Lipschitz-free spaces is isomorphic to ℓ_1, and then the result will follow by Pełczyński's classical theorem that complemented infinite-dimensional subspaces of ℓ_1 are isomorphic to ℓ_1 (see, e.g, [AK16, Theorem 2.2.4]).

The set $\overline{A}$ is an infinite weighted tree, i.e., a connected graph with no cycle. In particular, it is isometric to a subset of an $\mathbb{R}$-tree that contains all of its branching points $(0, 2^{-n})$, $n \in \mathbb{N}$. Thus $\mathcal{F}(\overline{A})$ is isometric to ℓ_1 by [God10, Corollary 3.4].

Let $K_0 := \{x\}$ and $K_n := \left\{(a, b) \in \overline{B} : b = 2^{-n}\right\}$ for each $n \in \mathbb{N}$. Each $\mathcal{F}(K_n)$ is isometric to a finite-dimensional ℓ_1-space. Moreover, there is a bound above and below on the distance between elements in distinct K_n's. Using [God10, Proposition 5.1] we get that $\mathcal{F}(\overline{B} \cup \{x\})$ is isomorphic to ℓ_1.

Finally, notice that $\mathcal{F}(\overline{C}) = \mathcal{F}(\overline{A})$ as $\overline{C}$ and $\overline{A}$ are isometric. Furthermore, by [AACD21, Lemma 2.8], $\mathcal{F}(\overline{C})$ and $\mathcal{F}(\overline{C} \cup \{x\})$ are isomorphic. Thus $\mathcal{F}(\overline{C} \cup \{x\})$ is also isomorphic to ℓ_1, and this completes the proof. □

2.4 Equivalence of Daugavet points and Delta-points in Lipschitz-free spaces over subsets of $\mathbb{R}$-trees

Recall that it was proved in [AHLP20, Theorem 3.1] that Daugavet and Δ-points coincide in $L_1(\mu)$ spaces when μ is a σ-finite measure.

The result was extended to arbitrary measures in [MPRZ24], and it was actually shown in [MPRZ24, Corollary 4.11] that these points also coincide with their super versions in this context. This naturally rises the question of whether Δ- and Daugavet points coincide in all subspaces of $L_1[0,1]$. While it was proved in [AAL$^+$24b, Theorem 4.16] that the answer is negative in general, it remains natural to look for classes of subspaces of $L_1[0,1]$, for which it is true. By [God10, Theorem 4.2], Lipschitz-free spaces over subsets of $\mathbb{R}$-trees are precisely the Lipschitz-free spaces which embed isometrically into an $L_1(\mu)$ space. This section is dedicated to showing that, for this class of spaces, Daugavet points and Δ-points, do indeed coincide. The results presented here are based on [AAL$^+$24b]

Recall that an $\mathbb{R}$-*tree* is an arc-connected metric space (T,d) such that there is a unique arc connecting any pair of points $x \neq y \in T$ which is isometric to the real interval $[0, d(x,y)] \subseteq \mathbb{R}$. This arc is called a *segment* of T, and is denoted by $[x,y]$. It coincides with the metric segment

$$[x,y] = \{p \in T : d(x,p) + d(p,y) = d(x,y)\}.$$

Let T be an $\mathbb{R}$-tree. Then for every $x, y, p \in T$, there exists a unique element $z_p \in [x,y]$ such that $d(p,z_p) = \min_{q\in[x,y]} d(p,q)$. Define the map $\mathsf{Y}_{xy}\colon T \to T$ by

$$\mathsf{Y}_{xy}p := z_p.$$

Then Y_{xy} is a 1-Lipschitz retraction from T onto the segment $[x,y]$. Moreover, if $\mathsf{Y}_{xy}p \neq \mathsf{Y}_{xy}q$ for some $p, q \in T$, then

$$[p,q] = [p, \mathsf{Y}_{xy}p] \cup [\mathsf{Y}_{xy}p, \mathsf{Y}_{xy}q] \cup [\mathsf{Y}_{xy}q, q], \tag{2.10}$$

so that $\mathsf{Y}_{xy}p, \mathsf{Y}_{xy}q \in [p,q]$ and thus

$$d(p,q) = d(p, \mathsf{Y}_{xy}p) + d(\mathsf{Y}_{xy}p, \mathsf{Y}_{xy}q) + d(\mathsf{Y}_{xy}q, q). \tag{2.11}$$

Arguing by contradiction and using (2.11), we obtain the following consequence:

$$\text{If } \mathsf{Y}_{xy}p = \mathsf{Y}_{xy}q, \text{ then } \mathsf{Y}_{xy}r = \mathsf{Y}_{xy}p \text{ for all } r \in [p,q]. \tag{2.12}$$

When M is a (complete) subset of an $\mathbb{R}$-tree, there exists a unique smallest $\mathbb{R}$-tree T containing M. For $x, y \in M$, the notation $[x,y] \subseteq M$ means that the corresponding segment in T is entirely contained in M, or equivalently, that x and y are connected by a geodesic in M. In this case, Y_{xy} is still a well-defined 1-Lipschitz retraction in M.

We start by proving that the notions of Daugavet and Δ-points coincide for molecules in Lipschitz-free spaces over subsets of $\mathbb{R}$-trees and provide a concrete metric characterization of such points.

Proposition 2.31 (see [AAL+24b, Proposition 4.1])**.** *Let M be a complete subset of an $\mathbb{R}$-tree T, and let $x \neq y \in M$. The following assertions are equivalent:*

(i) The molecule m_{xy} is a super Daugavet point in $\mathcal{F}(M)$;

(ii) The molecule m_{xy} is a Δ-point in $\mathcal{F}(M)$;

(iii) $[x, y] \subseteq M$.

Proof. (i)⇒(ii) is obvious, and (ii)⇒(iii) follows from [JRZ22, Corollary 4.9] and the completeness of M since if $\varepsilon > 0$ and $z \in [x, y]$ with $r = d(x, z)$, then for all $p \in B(x, r + \varepsilon) \cap B(y, d(x, y) - r + \varepsilon)$ we have

$$d(z, p) \leqslant d(z, \mathsf{Y}_{xy}p) + d(\mathsf{Y}_{xy}p, p) \leqslant 2\varepsilon.$$

Let us show (iii)⇒(i). Suppose $[x, y] \subseteq M$, i.e., x and y are joined by a geodesic in M. Then one can show that m_{xy} is a Daugavet point in $\mathcal{F}(M)$ by Theorem 2.9. However, we aim to prove the stronger conclusion that m_{xy} is a super Daugavet point. We will show that $\mathcal{F}([x, y])$ is an L-summand in $\mathcal{F}(M)$, where we fix x as the base point. Then as $\mathcal{F}([x, y])$ is isometrically isomorphic to $L_1[0, 1]$, and as this space has the Daugavet property, every point on the unit sphere of $\mathcal{F}([x, y])$ is a super Daugavet point (see [Shv00, Lemma 3]). In particular, m_{xy} is a super Daugavet point in $\mathcal{F}([x, y])$ and since these points transfer through ℓ_1-sums by [MPRZ24, Remark 3.28], we get that m_{xy} is a super Daugavet point in $\mathcal{F}(M)$.

So it only remains to prove that $\mathcal{F}([x, y])$ is an L-summand in $\mathcal{F}(M)$. Clearly Y_{xy} is a 1-Lipschitz retraction of M onto $[x, y]$, so its linearization $\widehat{\mathsf{Y}}_{xy} : \mathcal{F}(M) \to \mathcal{F}([x, y])$ is a norm-one projection. To see that it is an L-projection, it suffices to check that $\|\mu\| = \left\|\widehat{\mathsf{Y}}_{xy}\mu\right\| + \left\|\mu - \widehat{\mathsf{Y}}_{xy}\mu\right\|$ when $\mu \in \mathcal{F}(M)$ has finite support. Write

$$\mu = \sum_{i=1}^{n} a_i \delta(p_i) + \sum_{j=1}^{m} b_j \delta(q_j),$$

where $p_i \in [x, y]$ and $q_j \notin [x, y]$. Then $\widehat{\mathsf{Y}}_{xy}\mu = \sum_i a_i \delta(p_i) + \sum_j b_j \delta(\mathsf{Y}_{xy}q_j)$ and $\mu - \widehat{\mathsf{Y}}_{xy}\mu = \sum_j b_j(\delta(q_j) - \delta(\mathsf{Y}_{xy}q_j))$. Choose

$f, g \in B_{\mathrm{Lip}_0(M)}$ such that $\left\langle f, \widehat{\mathsf{Y}}_{xy}\mu \right\rangle = \left\|\widehat{\mathsf{Y}}_{xy}\mu\right\|$ and $\left\langle g, \mu - \widehat{\mathsf{Y}}_{xy}\mu \right\rangle = \left\|\mu - \widehat{\mathsf{Y}}_{xy}\mu\right\|$. Define a function $h : M \to \mathbb{R}$ by

$$h(p) := f(\mathsf{Y}_{xy}p) + g(p) - g(\mathsf{Y}_{xy}p).$$

We now verify that $h \in B_{\mathrm{Lip}_0(M)}$. For $p, q \in M$, we either have $\mathsf{Y}_{xy}p = \mathsf{Y}_{xy}q$ and thus $h(p) - h(q) = g(p) - g(q) \leqslant d(p, q)$, or $\mathsf{Y}_{xy}p \neq \mathsf{Y}_{xy}q$ and by (2.12)

$$\begin{aligned} h(p) - h(q) &= g(p) - g(\mathsf{Y}_{xy}p) + f(\mathsf{Y}_{xy}p) - f(\mathsf{Y}_{xy}q) + g(\mathsf{Y}_{xy}q) - g(q) \\ &\leqslant d(p, \mathsf{Y}_{xy}p) + d(\mathsf{Y}_{xy}p, \mathsf{Y}_{xy}q) + d(\mathsf{Y}_{xy}q, q) = d(p, q). \end{aligned}$$

Hence $h \in B_{\mathrm{Lip}_0(M)}$, and

$$\begin{aligned} \|\mu\| \geqslant \langle h, \mu\rangle &= \sum_{i=1}^{n} a_i \big(f(\mathsf{Y}_{xy}p_i) + g(p_i) - g(\mathsf{Y}_{xy}p_i)\big) \\ &\quad + \sum_{j=1}^{m} b_j \big(f(\mathsf{Y}_{xy}q_j) + g(q_j) - g(\mathsf{Y}_{xy}q_j)\big) \\ &= \sum_{i=1}^{n} a_i f(p_i) + \sum_{j=1}^{m} b_j f(\mathsf{Y}_{xy}q_j) + \sum_{j=1}^{m} b_j \big(g(q_j) - g(\mathsf{Y}_{xy}q_j)\big) \\ &= \left\langle f, \widehat{\mathsf{Y}}_{xy}\mu \right\rangle + \left\langle g, \mu - \widehat{\mathsf{Y}}_{xy}\mu \right\rangle = \left\|\widehat{\mathsf{Y}}_{xy}\mu\right\| + \left\|\mu - \widehat{\mathsf{Y}}_{xy}\mu\right\|. \end{aligned}$$

This proves that Y_{xy} is a L-projection and concludes the proof. □

Next, we show that the Conjecture 2.10 holds for arbitrary elements of Lipschitz-free spaces over subsets of $\mathbb{R}$-trees. Note that by Proposition 2.14, it already holds for bounded subsets of $\mathbb{R}$-trees. To extend this result we construct a Lipschitz function that serves the role of the functional f_ν used in the proof of Proposition 2.14.

Lemma 2.32 (see [AAL+24b, Lemma 4.4])**.** *Let M be a complete subset of an $\mathbb{R}$-tree T, and let $\mu \in S_{\mathcal{F}(M)}$. If μ can be expressed as a convex combination of molecules $m_{x_iy_i}$ with $i \in \{1, \dots, n\}$, then there exists a function $g_\mu \in S_{\mathrm{Lip}_0(M)}$ such that the following hold:*

(1) $\langle g_\mu, \mu\rangle = 1$;

(2) For every $u, v \in M$ with $u \neq v$ and every $\alpha > 0$, if $m_{uv} \in S(g_\mu, \alpha)$, then there exist $i, j \in \{1, \dots, n\}$ with $x_i \neq y_j$ such that

$$(1 - \alpha)d(u, v) < d(\mathsf{Y}_{x_iy_j}u, \mathsf{Y}_{x_iy_j}v).$$

Proof. Let $f \in S_{\mathrm{Lip}_0(M)}$ be such that $\langle f, \mu \rangle = 1$. Then $\langle f, m_{x_i y_i} \rangle = 1$ for every $i \in \{1, \dots, n\}$. Define $g_\mu \colon M \to \mathbb{R}$ by

$$g_\mu(p) := \max_{i \in \{1,\dots,n\}} \big(f(x_i) - \max_{j \in \{1,\dots,n\}} d(x_i, \mathsf{Y}_{x_i y_j} p) \big).$$

For every $p, q \in M$ we have

$$\big| d(x_i, \mathsf{Y}_{x_i y_j} p) - d(x_i, \mathsf{Y}_{x_i y_j} q) \big| \leqslant d(\mathsf{Y}_{x_i y_j} p, \mathsf{Y}_{x_i y_j} q) \leqslant d(p, q),$$

and thus, by [Wea18, Proposition 1.32], we conclude that $\|g_\mu\| \leqslant 1$. Furthermore, for every $k \in \{1, \dots, n\}$, we have

$$\begin{aligned} g_\mu(x_k) &\geqslant f(x_k) - \max_{j \in \{1,\dots,n\}} d(x_k, \mathsf{Y}_{x_k y_j} x_k) \\ &= f(x_k) - \max_{j \in \{1,\dots,n\}} d(x_k, x_k) = f(x_k), \end{aligned}$$

and similarly,

$$\begin{aligned} g_\mu(y_k) &\leqslant \max_{i \in \{1,\dots,n\}} \big(f(x_i) - d(x_i, \mathsf{Y}_{x_i y_k} y_k) \big) \\ &= \max_{i \in \{1,\dots,n\}} \big(f(x_i) - d(x_i, y_k) \big) \leqslant f(y_k). \end{aligned}$$

Therefore $\langle g_\mu, m_{x_k y_k} \rangle \geqslant \langle f, m_{x_k y_k} \rangle = 1$ for every $k \in \{1, \dots, n\}$ and thus $\langle g_\mu, \mu \rangle = 1$. This proves (1).

Now we prove (2). Fix $u, v \in M$ with $u \neq v$ and $\alpha > 0$ such that $m_{uv} \in S(g_\mu, \alpha)$. By definition of g_μ, there exist $i, j \in \{1, \dots, n\}$ such that

$$g_\mu(u) = f(x_i) - \max_{k \in \{1,\dots,n\}} d(x_i, \mathsf{Y}_{x_i y_k} u)$$

and

$$g_\mu(v) = \max_{k \in \{1,\dots,n\}} \big(f(x_k) - d(x_k, \mathsf{Y}_{x_k y_j} v) \big).$$

Then

$$\begin{aligned} (1 - \alpha) d(u, v) &< g_\mu(u) - g_\mu(v) \\ &\leqslant f(x_i) - d(x_i, \mathsf{Y}_{x_i y_j} u) - f(x_i) + d(x_i, \mathsf{Y}_{x_i y_j} v) \\ &= d(x_i, \mathsf{Y}_{x_i y_j} v) - d(x_i, \mathsf{Y}_{x_i y_j} u) \leqslant d(\mathsf{Y}_{x_i y_j} u, \mathsf{Y}_{x_i y_j} v), \end{aligned}$$

as desired. □

Proposition 2.33 (see [AAL+24b, Proposition 4.5])**.** *Let M be a subset of an $\mathbb{R}$-tree T, and let $\mu \in S_{\mathcal{F}(M)}$. Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$, there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$.*

Proof. One implication is Proposition 2.2.

Assume that $\mu \in S_{\mathcal{F}(M)}$ is a Δ-point. Fix a slice $S(f, \delta)$ with $\mu \in S(f, \delta)$ and fix $\varepsilon > 0$. By Lemma 1.6, we may assume $\delta < 1/2$. Choose $\gamma > 0$ such that $\mu \in S(f, \delta - \gamma)$. Then there exist $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, and $m_{x_1y_1}, \ldots, m_{x_ny_n} \in S_{\mathcal{F}(M)}$ such that $\sum_{i=1}^n \lambda_i m_{x_iy_i} \in S_{\mathcal{F}(M)}$ and

$$\left\| \mu - \sum_{i=1}^n \lambda_i m_{x_iy_i} \right\| < \gamma.$$

Let $\nu := \sum_{i=1}^n \lambda_i m_{x_iy_i}$ and let $g_\nu \in S_{\mathrm{Lip}_0(M)}$ be the function from Lemma 2.32. Then $\langle g_\nu, \mu \rangle > 1 - \gamma$, giving us $\langle f + g_\nu, \mu \rangle > 2 - \delta$.

Since μ is a Δ-point, by Proposition 1.10 there exists a sequence $(m_{u_iv_i})$ such that $\langle f + g_\nu, m_{u_iv_i} \rangle > 2 - \delta$ and

$$\|m_{u_iv_i} - m_{u_jv_j}\| \geqslant 2 - \delta$$

for all $i, j \in \mathbb{N}$ with $i \neq j$. Thus $(m_{u_iv_i}) \subseteq S(f, \delta)$ and $(m_{u_iv_i}) \subseteq S(g_\nu, \delta)$. For every $k \in \mathbb{N}$ there exist $i, j \in \{1, \ldots, n\}$ such that

$$(1 - \delta) d(u_k, v_k) < d(\mathsf{Y}_{x_iy_j} u_k, \mathsf{Y}_{x_iy_j} v_k).$$

By passing to a subsequence of $(m_{u_kv_k})$, if necessary, we may assume that the same x_i and y_j work for all k. Define $x := x_i$ and $y := y_j$. We aim to use the compactness of the segment $[x, y]$ in T to show that $d(u_k, v_k) < \varepsilon$ for some $k \in \mathbb{N}$.

Notice that $\mathsf{Y}_{xy} u_k \neq \mathsf{Y}_{xy} v_k$, since $0 < (1 - \delta) d(u_k, v_k) < d(\mathsf{Y}_{xy} u_k, \mathsf{Y}_{xy} v_k)$. From (2.11) we get

$$d(u_k, v_k) = d(u_k, \mathsf{Y}_{xy} u_k) + d(\mathsf{Y}_{xy} u_k, \mathsf{Y}_{xy} v_k) + d(\mathsf{Y}_{xy} v_k, v_k).$$

Since $\|m_{u_kv_k} - m_{u_lv_l}\| \geqslant 2 - \delta$ we have, by Lemma 2.3,

$$d(u_k, u_l) + d(v_k, v_l) \geqslant d(u_k, v_k) + d(u_l, v_l) - \delta \max\big\{d(u_k, v_k), d(u_l, v_l)\big\}$$

for all $k \neq l$. Therefore

$$\begin{aligned}
\delta \max\big\{d(u_k,v_k),&d(u_l,v_l)\big\} \geqslant d(u_k,v_k)+d(u_l,v_l)-d(u_k,u_l)-d(v_k,v_l)\\
&\geqslant d(u_k,\mathsf{Y}_{xy}u_k)+d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}v_k)+d(\mathsf{Y}_{xy}v_k,v_k)\\
&\quad+d(u_l,\mathsf{Y}_{xy}u_l)+d(\mathsf{Y}_{xy}u_l,\mathsf{Y}_{xy}v_l)+d(\mathsf{Y}_{xy}v_l,v_l)\\
&\quad-\big(d(u_k,\mathsf{Y}_{xy}u_k)+d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}u_l)+d(\mathsf{Y}_{xy}u_l,u_l)\big)\\
&\quad-\big(d(v_k,\mathsf{Y}_{xy}v_k)+d(\mathsf{Y}_{xy}v_k,\mathsf{Y}_{xy}v_l)+d(\mathsf{Y}_{xy}v_l,v_l)\big)\\
&=d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}v_k)+d(\mathsf{Y}_{xy}u_l,\mathsf{Y}_{xy}v_l)\\
&\quad-d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}u_l)-d(\mathsf{Y}_{xy}v_k,\mathsf{Y}_{xy}v_l)\\
&>(1-\delta)\big(d(u_k,v_k)+d(u_l,v_l)\big)\\
&\quad-d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}u_l)-d(\mathsf{Y}_{xy}v_k,\mathsf{Y}_{xy}v_l),
\end{aligned}$$

and thus

$$(1-2\delta)\big(d(u_k,v_k)+d(u_l,v_l)\big)<d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}u_l)+d(\mathsf{Y}_{xy}v_k,\mathsf{Y}_{xy}v_l).$$

Since the segment $[x,y]$ in T is compact, there exist $k,l\in I$ such that

$$d(\mathsf{Y}_{xy}u_k,\mathsf{Y}_{xy}u_l)+d(\mathsf{Y}_{xy}v_k,\mathsf{Y}_{xy}v_l)<(1-2\delta)\varepsilon.$$

Then $d(u_k,v_k)<\varepsilon$ and $m_{u_kv_k}\in S(f,\delta)$, and we have found the element we were looking for. □

Theorem 2.34 (see [AAL$^+$24b, Theorem 4.6])**.** *Let M be a subset of an $\mathbb{R}$-tree T. Then $\mu\in S_{\mathcal{F}(M)}$ is a Daugavet point if and only if it is a Δ-point.*

Proof. One direction is trivial, so we assume that $\mu\in S_{\mathcal{F}(M)}$ is a Δ-point. Without loss of generality, we may assume that M is complete. Choose $f\in S_{\mathrm{Lip}_0(M)}$ such that $\langle f,\mu\rangle=1$.

By Theorem 2.9, it is enough to show that for every $\nu\in\operatorname{dent}(B_{\mathcal{F}(M)})$ we have $\|\mu-\nu\|=2$. So let ν be a denting point in $B_{\mathcal{F}(M)}$. By [Wea18, Corollary 3.44], we have $\nu=m_{xy}$ for some $x,y\in M$ with $x\neq y$.

Let $0<\varepsilon<1/2$. It suffices to show that $\|\mu-m_{xy}\|\geqslant 2-4\varepsilon$. If $\langle f,m_{yx}\rangle\geqslant 1-4\varepsilon$, then $\|\mu-m_{xy}\|\geqslant\langle f,\mu-m_{xy}\rangle\geqslant 2-4\varepsilon$, and we are done.

Now assume $\langle f,m_{yx}\rangle<1-4\varepsilon$. Extend f to T by the McShane–Whitney extension theorem. For simplicity, assume that x is the base

point 0. Then our assumption becomes $f(y) < (1-4\varepsilon)d(x,y)$. Define $g\colon M \to \mathbb{R}$ by

$$g(p) := \min\big\{ \max\big\{d(x, \mathsf{Y}_{xy}p) - f(\mathsf{Y}_{xy}p) - 2\varepsilon d(x,y), 0\big\}, \\ (1-4\varepsilon)d(x,y) - f(y)\big\}.$$

Observe that $g(p) \geqslant 0$ for all $p \in M$, and that $g(p) = g(q)$ whenever $\mathsf{Y}_{xy}p = \mathsf{Y}_{xy}q$. By [Wea18, Proposition 1.32], we have $g \in \mathrm{Lip}_0(M)$ since, for every $p, q \in M$,

$$|d(x, \mathsf{Y}_{xy}p) - f(\mathsf{Y}_{xy}p) - d(x, \mathsf{Y}_{xy}q) + f(\mathsf{Y}_{xy}q)| \leqslant 2d(\mathsf{Y}_{xy}p, \mathsf{Y}_{xy}q) \leqslant 2d(p,q).$$

Moreover, when p is close to either x or y, we have complete control over $g(p)$. Specifically, if $p \in B\big(x, \varepsilon d(x,y)\big)$, then

$$d(x, \mathsf{Y}_{xy}p) - f(\mathsf{Y}_{xy}p) \leqslant 2d(x, \mathsf{Y}_{xy}p) \leqslant 2\varepsilon d(x,y),$$

and thus $g(p) = 0$. Similarly, if $p \in B\big(y, \varepsilon d(x,y)\big)$, then

$$d(x, \mathsf{Y}_{xy}p) - f(\mathsf{Y}_{xy}p) \geqslant \big(d(x,y) - d(y, \mathsf{Y}_{xy}p)\big) - \big(f(y) + d(y, \mathsf{Y}_{xy}p)\big) \\ \geqslant d(x,y) - f(y) - 2\varepsilon d(x,y),$$

and therefore $g(p) = (1-4\varepsilon)d(x,y) - f(y) > 0$.

Since m_{xy} is a denting point, by [AG19, Theorem 4.1], there exists $\delta > 0$ such that for all $p \in M$,

$$p \notin B\big(x, \varepsilon d(x,y)\big) \cup B\big(y, \varepsilon d(x,y)\big) \Rightarrow d(x,p) + d(y,p) \geqslant (1+\delta)d(x,y).$$

We may assume $\delta < 1 - 2\varepsilon$.

We now show that if $\langle g, m_{uv}\rangle > 0$ for some $u, v \in M$ with $u \neq v$, then $d(u,v) \geqslant \delta d(x,y)/2$. First assume $u, v \in B\big(x, \varepsilon d(x,y)\big) \cup B\big(y, \varepsilon d(x,y)\big)$. From the properties of g it follows that $d(y,u) \leqslant \varepsilon d(x,y)$ and $d(x,v) \leqslant \varepsilon d(x,y)$, hence

$$d(u,v) \geqslant d(x,y) - 2\varepsilon d(x,y) \geqslant \delta d(x,y).$$

Next, the two cases $u \notin B\big(x, \varepsilon d(x,y)\big) \cup B\big(y, \varepsilon d(x,y)\big)$ and $v \notin B\big(x, \varepsilon d(x,y)\big) \cup B\big(y, \varepsilon d(x,y)\big)$ are analogous, so we only consider the case $u \notin B\big(x, \varepsilon d(x,y)\big) \cup B\big(y, \varepsilon d(x,y)\big)$. Then

$$(1+\delta)d(x,y) \leqslant d(x,u) + d(y,u) = d(x,y) + 2d(\mathsf{Y}_{xy}u, u)$$

and thus $\delta d(x,y) \leqslant 2d(\mathsf{Y}_{xy}u, u)$. Furthermore, since $g(u) \neq g(v)$, we must have $\mathsf{Y}_{xy}u \neq \mathsf{Y}_{xy}v$, and thus, by (2.11),

$$\begin{aligned} d(u,v) &= d(u, \mathsf{Y}_{xy}u) + d(\mathsf{Y}_{xy}u, \mathsf{Y}_{xy}v) + d(\mathsf{Y}_{xy}v, v) \\ &\geqslant d(\mathsf{Y}_{xy}u, u) \geqslant \delta d(x,y)/2. \end{aligned}$$

Hence, we have proven that if $\langle g, m_{uv}\rangle > 0$, then $d(u,v) \geqslant \delta d(x,y)/2$. Therefore, by Proposition 2.33, the slice $S(g/\|g\|, 1)$ cannot contain any Δ-points. Since both μ and $-\mu$ are Δ-points, we conclude that $\mu, -\mu \notin S(g/\|g\|, 1)$, and thus $\langle g, \mu\rangle = 0$.

Let $h = f + g$. Then $\langle h, \mu\rangle = 1$, and

$$\langle h, m_{yx}\rangle = \frac{f(y) + g(y)}{d(x,y)} = \frac{f(y) + (1-4\varepsilon)d(x,y) - f(y)}{d(x,y)} = 1 - 4\varepsilon,$$

so that $\langle h, \mu\rangle - \langle h, m_{xy}\rangle = 2 - 4\varepsilon$. Therefore, it remains to show that $\|h\| \leqslant 1$. Let $u, v \in M$ with $u \neq v$. If $g(u) \leqslant g(v)$, then

$$\langle h, m_{uv}\rangle \leqslant \langle f, m_{uv}\rangle \leqslant 1.$$

If $g(u) > g(v)$, then $\mathsf{Y}_{xy}u \neq \mathsf{Y}_{xy}v$. Moreover, $g(u) > 0$ and $g(v) < (1-4\varepsilon)d(x,y) - f(y)$, since $0 \leqslant g(p) \leqslant (1-4\varepsilon)d(x,y) - f(y)$ for all $p \in M$. Thus

$$\begin{aligned} &g(u) - g(v) \\ &\quad = \min\big\{d(x, \mathsf{Y}_{xy}u) - f(\mathsf{Y}_{xy}u) - 2\varepsilon d(x,y), (1-4\varepsilon)d(x,y) - f(y)\big\} \\ &\qquad - \max\big\{d(x, \mathsf{Y}_{xy}v) - f(\mathsf{Y}_{xy}v) - 2\varepsilon d(x,y), 0\big\} \\ &\quad \leqslant d(x, \mathsf{Y}_{xy}u) - f(\mathsf{Y}_{xy}u) - d(x, \mathsf{Y}_{xy}v) + f(\mathsf{Y}_{xy}v) \\ &\quad \leqslant d(\mathsf{Y}_{xy}u, \mathsf{Y}_{xy}v) - f(\mathsf{Y}_{xy}u) + f(\mathsf{Y}_{xy}v), \end{aligned}$$

and by (2.11) we get

$$\begin{aligned} h(u) - h(v) &= f(u) - f(v) + g(u) - g(v) \\ &\leqslant f(u) - f(\mathsf{Y}_{xy}u) + d(\mathsf{Y}_{xy}u, \mathsf{Y}_{xy}v) + f(\mathsf{Y}_{xy}v) - f(v) \\ &\leqslant d(u, \mathsf{Y}_{xy}u) + d(\mathsf{Y}_{xy}u, \mathsf{Y}_{xy}v) + d(\mathsf{Y}_{xy}v, v) \\ &= d(u,v). \end{aligned}$$

Therefore $\|h\| \leqslant 1$, and thus

$$\|\mu - m_{xy}\| \geqslant \langle h, \mu\rangle - \langle h, m_{xy}\rangle = 2 - 4\varepsilon.$$

Since $\varepsilon \in (0, 1/2)$ was arbitrary, we get $\|\mu - m_{xy}\| = 2$, as desired. $\square$

For a complete subset M of an $\mathbb{R}$-tree, we can now combine this result with Proposition 1.9 to obtain a characterization of those Daugavet points in $\mathcal{F}(M)$ that are convex combinations or convex series of molecules. We can actually say a bit more about such elements: they are all super Daugavet points. To see this, we will need the following lemma.

Lemma 2.35 (see [AAL$^+$24b, Lemma 4.7])**.** *Let M be a complete subset of an $\mathbb{R}$-tree T, and let $\mu \in S_{\mathcal{F}(M)}$. If μ can be written as a convex combination of molecules $m_{x_iy_i} \in S_{\mathcal{F}(M)}$ with $i \in \{1,\dots,n\}$, and if $[x_i, y_i] \subseteq M$ for every i, then μ can be written as a convex combination of molecules $m_{u_jv_j}$ with $j \in \{1,\dots,m\}$ such that*

(1) $[u_j, v_j] \subseteq M$ for every $j \in \{1,\dots,m\}$;

(2) for every $j,k \in \{1,\dots,m\}$ with $j \neq k$, we have either $\mathsf{Y}_{u_jv_j}p = u_k$ for every $p \in [u_k, v_k]$ or $\mathsf{Y}_{u_jv_j}p = v_k$ for every $p \in [u_k, v_k]$.

Proof. For each i, let $B_i = \mathsf{Y}_{x_iy_i}(\{x_j, y_j : j = 1,\dots,n\})$ and write $B_i = \{p_1,\dots,p_l\}$ where $p_1 = x_i$, $p_l = y_i$, and $p_k \in [p_{k-1}, p_{k+1}]$ for all $1 < k < l$. Then

$$m_{x_iy_i} = \sum_{k=1}^{l-1} \frac{d(p_k, p_{k+1})}{d(x_i, y_i)} m_{p_kp_{k+1}}$$

is a convex combination of molecules. By substituting this expression into the original convex combination of μ, and grouping identical terms, we obtain a convex combination of distinct molecules $m_{u_jv_j}$ such that for each j, the pair u_j and v_j are consecutive points in some B_i. In particular, $[u_j, v_j] \subseteq M$.

Now (2) can be derived from (2.12). Indeed, since every pair $\{u_j, v_j\}$ is composed of consecutive points of some B_i, it follows that $\mathsf{Y}_{u_jv_j}u_k, \mathsf{Y}_{u_jv_j}v_k \in \{u_k, v_k\}$ for all j,k. Moreover, if $\mathsf{Y}_{u_jv_j}u_k \neq \mathsf{Y}_{u_jv_j}v_k$, then (2.11) shows that $u_j, v_j \in [u_k, v_k]$. Again, because u_k and v_k are consecutive, we get $\{u_j, v_j\} = \{u_k, v_k\}$ and therefore $j = k$. □

Theorem 2.36 (see [AAL$^+$24b, Theorem 4.8])**.** *Let M be a complete subset of an $\mathbb{R}$-tree T, and let $\mu \in S_{\mathcal{F}(M)}$. If μ can be written as a convex combination or as a convex series of molecules $m_{x_iy_i}$ with $i \in I$, for some non-empty subset I of $\mathbb{N}$, then the following are equivalent*

(i) μ is a super Daugavet point;

(ii) μ *is a* Δ*-point;*

(iii) $m_{x_iy_i}$ *is a super Daugavet point for every* $i \in I$*;*

(iv) $m_{x_iy_i}$ *is a* Δ*-point for every* $i \in I$*;*

(v) $[x_i, y_i] \subseteq M$ *for every* $i \in I$*.*

Proof. The equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow from Proposition 2.31; the implication (i) $\Rightarrow$ (ii) is clear; and (ii) $\Rightarrow$ (iv) follows by combining Theorem 2.34 and Proposition 1.9.

We now prove (v) $\Rightarrow$ (i). Since the set of all super Daugavet points in any Banach space is closed, it suffices to prove the implication for finite I. Furthermore, by Lemma 2.35, we may assume that the molecules $m_{x_iy_i}$ satisfy

(a) $[x_i, y_i] \subseteq M$ for every $i \in I$;

(b) for every $i, j \in I$ with $i \neq j$, we have either $\mathsf{Y}_{x_iy_i}p = x_j$ for every $p \in [x_j, y_j]$ or $\mathsf{Y}_{x_iy_i}p = y_j$ for every $p \in [x_j, y_j]$.

We proceed by induction on $|I|$. If $|I| = 1$, then μ is a super Daugavet point by Proposition 2.31. Let $n \in \mathbb{N}$ and assume that the assumption holds when $|I| \leqslant n$. Assume that $|I| = n + 1$, and fix $i \neq j \in I$. Then either $\mathsf{Y}_{x_iy_i}x_j = \mathsf{Y}_{x_iy_i}y_j = x_i$ or $\mathsf{Y}_{x_iy_i}x_j = \mathsf{Y}_{x_iy_i}y_j = y_i$. We will only consider the first case, the second case is analogous. For simplicity, assume that x_i is the fixed point 0. Define

$$M_1 = \{p \in M : \mathsf{Y}_{x_iy_i}p = x_i\},$$

and $M_2 = (M \backslash M_1) \cup \{x_i\}$. By [Wea18, Proposition 3.9], the space $\mathcal{F}(M)$ is isometric to $\mathcal{F}(M_1) \oplus_1 \mathcal{F}(M_2)$, since $d(p, q) = d(p, x_i) + d(x_i, q)$ for all $p \in M_1$ and $q \in M_2$. Indeed, this equality is obvious for $q = x_i$; otherwise $x_i = \mathsf{Y}_{x_iy_i}p \neq \mathsf{Y}_{x_iy_i}q$ and so by (2.11)

$$d(p, q) = d(p, x_i) + d(x_i, \mathsf{Y}_{x_iy_i}q) + d(\mathsf{Y}_{x_iy_i}q, q) = d(p, x_i) + d(x_i, q).$$

Now notice that $m_{x_iy_i} \in \mathcal{F}(M_2)$, and for every $k \in I \backslash \{i\}$, we have $\mathsf{Y}_{x_iy_i}x_k = \mathsf{Y}_{x_iy_i}y_k$, and thus either $m_{x_ky_k} \in \mathcal{F}(M_1)$ or $m_{x_ky_k} \in \mathcal{F}(M_2)$. Let

$$J = \{k \in I : m_{x_ky_k} \in \mathcal{F}(M_1)\}.$$

Since $m_{x_jy_j} \in \mathcal{F}(M_1)$ and $m_{x_iy_i} \notin \mathcal{F}(M_1)$ it follows that $|J| \leqslant n$ and $|I \backslash J| \leqslant n$. By assumption, both $\sum_{k\in J} \lambda_k m_{x_ky_k} / \sum_{k\in J} \lambda_k$ and

$\sum_{k\in I\setminus J}\lambda_k m_{x_k y_k}/\sum_{k\in I\setminus J}\lambda_k$ are super Daugavet points in $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$, respectively. Then, by [MPRZ24, Proposition 3.27], the element $(\sum_{k\in J}\lambda_k m_{x_k y_k}, \sum_{k\in I\setminus J}\lambda_k m_{x_k y_k})$ is a super Daugavet point. Thus μ is a super Daugavet point, as desiered. □

This leaves us with a natural question.

Question 2.1. Does every Daugavet point (respectively, every Δ-point) in a Lipschitz-free space over a subset of an $\mathbb{R}$-tree have to be a super Daugavet point (respectively, a super Δ-point)?

Chapter 3

Daugavet and Delta-points in spaces of Lipschitz functions

The study of Daugavet points in spaces of Lipschitz functions was initiated in [JRZ22], where the authors proved that every local function is a weak* Daugavet point (see [JRZ22, Theorem 3.6]). This was generalized later in [HOP24], where it was shown that every local function is, in fact, a Daugavet point (see [HOP24, Theorem 1.5]). In this chapter, we will extend that knowledge. In particular, we prove that in spaces of Lipschitz functions over proper metric spaces, the notions of Daugavet and Δ-points coincide. We also show that this equivalence does not hold in general, i.e., there exist Δ-points that are not Daugavet. This chapter is based on [Vee25a].

3.1 Equivalence of Daugavet and Delta-points in spaces of Lipschitz functions over proper metric spaces

We begin by showing that in the space of Lipschitz functions over a compact metric space, every weak* Δ-point (and, in particular, every Daugavet point) must be a local function. Since every slice of $B_{\mathcal{F}(M)}$ contains a molecule, the following result is immediate from Proposition 1.12.

Proposition 3.1 (see [Vee25a, Proposition 4.1])**.** *Let $f \in S_{\mathrm{Lip}_0(M)}$ be a weak* Δ-point. Then for every $\varepsilon > 0$, there exists a sequence $(m_{u_i v_i})$*

in $S(f,\varepsilon)$ *such that*

$$\|m_{u_iv_i} - m_{u_jv_j}\| \geqslant 2-\varepsilon$$

for all $i,j \in \mathbb{N}$ *with* $i \neq j$.

Proposition 3.2 (see [Vee25a, Proposition 4.2])**.** *Assume that* M *is compact and let* $f \in S_{\mathrm{Lip}_0(M)}$ *be a weak* \Delta-point. Then* f *is local.*

Proof. Fix $\varepsilon > 0$. We may assume $\varepsilon < 2$. By Proposition 3.1 there exists a sequence $(m_{u_iv_i}) \subseteq S(f,\varepsilon)$ such that

$$\|m_{u_iv_i} - m_{u_jv_j}\| \geqslant 2-\varepsilon$$

for all $i,j \in \mathbb{N}$ with $i \neq j$. Since M is compact, by moving to subsequences we may assume that the sequences (u_i) and (v_i) are convergent. Let $u,v \in M$ be such that $u_i \to u$ and $v_i \to v$. If $u \neq v$, then $m_{u_iv_i} \to m_{uv}$, which leads to a contradiction, since the sequence $(m_{u_iv_i})$ does not converge. Therefore $u = v$ and there exist $i \in \mathbb{N}$ such that $d(u_i,v_i) < \varepsilon$. By assumption we also have $\langle f, m_{u_iv_i}\rangle > 1-\varepsilon$, so f is local. □

Our next step is to generalize this result to proper metric spaces. To do so, we first present a more general version of [HOP24, Proposition 4.1].

Proposition 3.3 (see [Vee25a, Proposition 4.3])**.** *Let* $\varepsilon > 0$ *and let* $(m_{u_iv_i})$ *be a sequence in* $S_{\mathcal{F}(M)}$ *such that there exists a sequence* (A_i) *of pairwise disjoint subsets of* M *such that*

$$d(u_i,x) + d(v_i,y) \geqslant (1-\varepsilon)\,(d(u_i,v_i) + d(x,y)) \tag{3.1}$$

for all $i \in \mathbb{N}$ *and* $x,y \in M\backslash A_i$. *Then* $\limsup_i \|F + m_{u_iv_i}\| \geqslant 2-2\varepsilon$ *for every* $F \in S_{\mathrm{Lip}_0(M)^*}$.

Proof. The case $\varepsilon \geqslant 1$ is trivial, so we may assume $\varepsilon < 1$. Note that for every $i \in \mathbb{N}$ we must have $u_i \in A_i$ or $v_i \in A_i$, since otherwise we could take $x = u_i$ and $y = v_i$ in the inequality (3.1), which leads to a contradiction. By passing to a subsequence, we may assume that $u_i \in A_i$ for every $i \in \mathbb{N}$ or $v_i \in A_i$ for every $i \in \mathbb{N}$. Without loss of generality, we assume that $u_i \in A_i$ for every $i \in \mathbb{N}$.

Fix $F \in S_{\mathrm{Lip}_0(M)^*}$. By de Leeuw's transform, there exists a measure $\mu \in ba(\widetilde{M})$ with $|\mu|(\widetilde{M}) = 1$, where

$$\widetilde{M} = (M \times M) \setminus \{(x, x) \colon x \in M\},$$

such that

$$F(f) = \int_{\widetilde{M}} \tilde{f} d\mu \quad \text{where} \quad \tilde{f}(x, y) = \frac{f(x) - f(y)}{d(x, y)}$$

for every $f \in \mathrm{Lip}_0(M)$ (see, e.g., [Wea18]). Following the notation used in [HOP24], define

$$\Gamma_{1,A} = \left\{(x, y) \in \widetilde{M} \colon x \in A\right\} \quad \text{and} \quad \Gamma_{2,A} = \left\{(x, y) \in \widetilde{M} \colon y \in A\right\}.$$

Let $\gamma > 0$ be arbitrary and let $f \in S(F, \gamma)$. Since the sets $A_1, A_2, \ldots$ are pairwise disjoint, there exists $n \in \mathbb{N}$ such that for every $i \geqslant n$,

$$|\mu|(\Gamma_{1,A_i}) < \gamma \quad \text{and} \quad |\mu|(\Gamma_{2,A_i}) < \gamma.$$

Fix $i \geqslant n$. To complete the proof, it is sufficient to construct a function $g \in B_{\mathrm{Lip}_0(M)}$ such that $g|_{M \setminus A_i} = (1-\varepsilon) f|_{M \setminus A_i}$ and $\langle g, m_{u_i v_i} \rangle \geqslant 1 - \varepsilon$. Indeed, since $g|_{M \setminus A_i} = (1 - \varepsilon) f|_{M \setminus A_i}$ and $\mu(\Gamma_{1,A_i} \cup \Gamma_{2,A_i}) < 2\gamma$, we obtain

$$(1 - \varepsilon) F(f) - F(g) = \int_{\Gamma_{1,A_i} \cup \Gamma_{2,A_i}} ((1 - \varepsilon)\tilde{f} - \tilde{g}) d\mu < 4\gamma,$$

and thus

$$F(g) > (1 - \varepsilon) F(f) - 4\gamma > (1 - \varepsilon)(1 - \gamma) - 4\gamma > 1 - \varepsilon - 5\gamma.$$

Furthermore, since $\langle g, m_{u_i v_i} \rangle = 1 - \varepsilon$, we conclude

$$\|F + m_{u_i v_i}\| \geqslant 2 - 2\varepsilon - 5\gamma.$$

As γ was arbitrary, this implies $\limsup_j \|F + m_{u_j v_j}\| \geqslant 2 - 2\varepsilon$. To construct the function g, we now consider two cases.

Case 1. Assume that $v_i \notin A_i$. Define a function g on $(M \setminus A_i) \cup \{u_i\}$ by

$$g|_{M \setminus A_i} = (1 - \varepsilon) f|_{M \setminus A_i} \quad \text{and} \quad g(u_i) = g(v_i) + (1 - \varepsilon) d(u_i, v_i).$$

For every $x \in M\setminus A_i$, we have

$$\begin{aligned}|g(u_i) - g(x)| &= |(1-\varepsilon)f(v_i) + (1-\varepsilon)d(u_i, v_i) - (1-\varepsilon)f(x)| \\ &\leqslant (1-\varepsilon)\left(d(u_i, v_i) + |f(x) - f(v_i)|\right) \\ &\leqslant (1-\varepsilon)\left(d(u_i, v_i) + d(x, v_i)\right) \\ &\leqslant d(x, u_i),\end{aligned}$$

where the last inequality follows from (3.1) by choosing $y = v_i$. Therefore, the Lipschitz constant of g is at most 1. Furthermore, $\langle g, m_{u_i v_i}\rangle = 1 - \varepsilon$. Extend g to M by using the McShane–Whitney extension theorem. This yields the desired function g.

Case 2. Assume that $v_i \in A_i$. The proof of this case is similar to the proof of [HOP24, Theorem 3.3]. Define a function $g\colon M \to \mathbb{R}$ such that

$$g|_{M\setminus A_i} = (1-\varepsilon)f|_{M\setminus A_i}, \quad g(u_i) = \inf_{x\in M\setminus A_i}\left(g(x) + d(x, u_i)\right),$$

and for every $y \in A_i\setminus\{u_i\}$,

$$g(y) = \sup_{x\in (M\setminus A_i)\cup\{u_i\}}\left(g(x) - d(x, y)\right).$$

Then g is 1-Lipschitz. Moreover, we have $g(v_i) = g(u_i) - d(u_i, v_i)$ or

$$\begin{aligned}g(u_i) - g(v_i) &= \inf_{x,y\in M\setminus A_i}\left(g(x) + d(x, u_i) - g(y) + d(y, v_i)\right) \\ &\geqslant \inf_{x,y\in M\setminus A_i}\left(d(x, u_i) + d(y, v_i) - (1-\varepsilon)d(x, y)\right) \\ &\geqslant (1-\varepsilon)d(u_i, v_i).\end{aligned}$$

In both cases, we obtain $\langle g, m_{u_i v_i}\rangle \geqslant 1 - \varepsilon$. Thus, we have constructed a suitable function g. □

Our main interest in Proposition 3.3 is to establish the following result.

Proposition 3.4 (see [Vee25a, Proposition 4.4])**.** *Let $(m_{u_i v_i})$ be a sequence in $S_{\mathcal{F}(M)}$ such that at least one of the sequences (u_i) and (v_i) is unbounded. Then $\limsup_i \|F + m_{u_i v_i}\| = 2$ for every $F \in S_{\mathrm{Lip}_0(M)^*}$.*

To prove this, we first state a geometric lemma.

Lemma 3.5 (see [Vee25a, Lemma 4.5])**.** *Let $a, \varepsilon > 0$. Then*

$$d(u,x) + d(v,y) \geqslant (1-\varepsilon)\left(d(u,v) + d(x,y)\right)$$

for all points $u \in B(0,8a)$, $v \in B(0,8a)\backslash B(0,4a)$, and $x, y \in (M\backslash B(0,32a/\varepsilon)) \cup B(0,a\varepsilon)$.

Proof. Fix $u \in B(0,8a)$, and $v \in B(0,8a)\backslash B(0,4a)$. If $x, y \in B\left(0,a\varepsilon\right)$, then

$$d(v,y) \geqslant d(v,0) - d(0,y) > 4a - a\varepsilon \geqslant a(4-\varepsilon)\frac{d(x,y)}{2a\varepsilon} > 2\frac{1-\varepsilon}{\varepsilon}d(x,y)$$

and we get

$$\begin{aligned} d(u,x) + d(v,y) &\geqslant (1-\varepsilon)\left(d(u,x) + d(v,y)\right) + \varepsilon d(v,y) \\ &\geqslant (1-\varepsilon)\left(d(u,v) - d(x,y)\right) + 2(1-\varepsilon)d(x,y) \\ &= (1-\varepsilon)\left(d(u,v) + d(x,y)\right). \end{aligned}$$

On the other hand, if $x \in M\backslash B(0,32a/\varepsilon)$ or $y \in M\backslash B(0,32a/\varepsilon)$, then

$$\begin{aligned} d(u,x) + d(v,y) &\geqslant d(x,0) - d(0,u) + d(y,0) - d(0,v) \\ &\geqslant d(x,0) + d(y,0) - 16a \\ &\geqslant (1-\varepsilon)d(x,y) + \varepsilon\left(d(x,0) + d(y,0)\right) - 16a \\ &> (1-\varepsilon)d(x,y) + 32a - 16a \\ &\geqslant (1-\varepsilon)\left(d(u,v) + d(x,y)\right). \end{aligned}$$

□

Proof of Proposition 3.4. We construct a sequence $(k_i) \subseteq \mathbb{N}$ and a sequence of pairwise disjoint subsets (A_i) of M such that

$$d(u_{k_i},x) + d(v_{k_i},y) \geqslant (1-1/i)\left(d(u_{k_i},v_{k_i}) + d(x,y)\right)$$

for all $i \in \mathbb{N}$ and all $x, y \in M\backslash A_i$. Then by Proposition 3.3, we obtain $\limsup_i \|F + m_{u_{k_i}v_{k_i}}\| = 2$, which implies that $\limsup_i \|F + m_{u_iv_i}\| = 2$ for every $F \in S_{\mathrm{Lip}_0(M)^*}$.

Let $k_1 = 1$, and choose $a_1 > 0$ such that $u_{k_1}, v_{k_1} \in B(0,8a_1)$ and either $u_{k_1} \notin B(0,4a_1)$ or $v_{k_1} \notin B(0,4a_1)$ (or both). Define $A_1 = B(0,32a_1)\backslash B(0,a_1)$. Then for all $x, y \in M\backslash A_1$,

$$d(u_{k_1},x) + d(v_{k_1},y) \geqslant (1-1/1)\left(d(u_{k_1},v_{k_1}) + d(x,y)\right).$$

Suppose that we have constructed $k_1, \dots, k_{n-1} \in \mathbb{N}$, and pairwise disjoint bounded sets $A_1, \dots, A_{n-1}$ such that

$$d(u_{k_i}, x) + d(v_{k_i}, y) \geqslant (1 - 1/i)\,(d(u_{k_i}, v_{k_i}) + d(x, y))$$

for all $i \in \{1, \dots, n-1\}$ and all $x, y \in M \backslash A_i$. Choose $a_n > 0$ and $k_n \in \mathbb{N}$ such that $\bigcup_{i=1}^{n-1} A_i \subseteq B(0, a_n/n)$, $u_{k_n}, v_{k_n} \in B(0, 8a_n)$ and either $u_{k_n} \notin B(0, 4a_n)$ or $v_{k_n} \notin B(0, 4a_n)$ (or both). Define $A_n = B(0, 32a_n n) \backslash B(0, a_n/n)$. Then $A_n \cap A_i = \varnothing$ for every $i \in \{1, \dots, n-1\}$, and by Lemma 3.5, we have

$$d(u_{k_n}, x) + d(v_{k_n}, y) \geqslant (1 - 1/n)\,(d(u_{k_n}, v_{k_n}) + d(x, y))$$

for all $x, y \in M \backslash A_n$. □

By combining Propositions 3.3 and 3.4, we obtain the following sufficient condition for Daugavet points.

Proposition 3.6 (see [Vee25a, Proposition 4.6])**.** *Let $f \in S_{\mathrm{Lip}_0(M)}$ be such that for every $\varepsilon > 0$, there exists a sequence $(m_{u_i v_i})$ in $S(f, \varepsilon)$ and a sequence (A_i) of pairwise disjoint subsets of M such that*

$$d(u_i, x) + d(v_i, y) \geqslant (1 - \varepsilon)\,(d(u_i, v_i) + d(x, y))$$

for all $i \in \mathbb{N}$ and all $x, y \in M \backslash A_i$. Then f is a Daugavet point.

In particular, if $f \in S_{\mathrm{Lip}_0(M)}$ is such that for every $\varepsilon > 0$, there exists a sequence $(m_{u_i v_i})$ in $S(f, \varepsilon)$ such that at least one of the sequences (u_i) and (v_i) is unbounded, then f is a Daugavet point.

Proof. We prove only the first statement; the second follows analogously by applying Proposition 3.4 instead of Proposition 3.3. Fix $\varepsilon > 0$ and a slice $S(F, \alpha)$ of $B_{\mathrm{Lip}_0(M)^*}$. Without loss of generality, we may assume that $\varepsilon < \alpha$. By assumption, there exists a sequence $(m_{u_i v_i})$ in $S(f, \varepsilon/4)$ and a sequence (A_i) of pairwise disjoint subsets of M such that

$$d(u_i, x) + d(v_i, y) \geqslant \left(1 - \frac{\varepsilon}{4}\right)(d(u_i, v_i) + d(x, y))$$

for all $i \in \mathbb{N}$ and all $x, y \in M \backslash A_i$. By Proposition 3.3, we have $\limsup_i \| -F + m_{u_i v_i} \| \geqslant 2 - \frac{\varepsilon}{2}$. Choose $i \in \mathbb{N}$ such that $\| -F + m_{u_i v_i} \| > 2 - 3\varepsilon/4$. Then there exists $g \in S_{\mathrm{Lip}_0(M)}$ such that $(F - m_{u_i v_i})(g) > 2 - 3\varepsilon/4$. This implies that $g \in S(F, \varepsilon) \subseteq S(F, \alpha)$, and

$$\|f - g\| \geqslant \langle f, m_{u_i v_i} \rangle - \langle g, m_{u_i v_i} \rangle > 1 - \frac{\varepsilon}{4} + 1 - \frac{3\varepsilon}{4} = 2 - \varepsilon.$$

Hence, f is a Daugavet point. □

Now let us present the main result of this section.

Theorem 3.7 (see [Vee25a, Theorem 4.7])**.** *Assume that M is a proper metric space and let $f \in S_{\mathrm{Lip}_0(M)}$. Then the following conditions are equivalent:*

(i) f is a Daugavet point;

(ii) f is a Δ-point;

(iii) f is a weak Daugavet point;*

(iv) f is a weak Δ-point;*

(v) for every $\varepsilon > 0$ there exists a sequence $(m_{u_i v_i}) \subseteq S(f, \varepsilon)$ such that

$$\|m_{u_i v_i} - m_{u_j v_j}\| \geqslant 2 - \varepsilon$$

for every $i, j \in \mathbb{N}$ with $i \neq j$;

(vi) for every $\varepsilon > 0$ there exists a sequence $(m_{u_i v_i}) \subseteq S(f, \varepsilon)$ such that $d(u_i, v_i) \to 0$ or at least one of the sequences (u_i) and (v_i) is unbounded.

Proof. The implications (i)⇒(ii)⇒(iv) and (i)⇒(iii)⇒(iv) follow directly from the definitions. Implication (iv)⇒(v) is given by Proposition 3.1, and (vi)⇒(i) follows from [HOP24, Theorem 1.5] and Proposition 3.6.

It remains to show (v)⇒(vi). Suppose that for $\varepsilon > 0$, there exists a sequence $(m_{u_i v_i}) \subseteq S(f, \varepsilon)$ such that

$$\|m_{u_i v_i} - m_{u_j v_j}\| \geqslant 2 - \varepsilon$$

for every $i, j \in \mathbb{N}$ with $i \neq j$. If both sequences (u_i) and (v_i) are bounded, then, since M is proper, we may pass to convergent subsequences. It follows that $u_i \to u$ and $v_i \to v$ for some $u, v \in M$. If $u \neq v$, then $m_{u_i v_i} \to m_{uv}$, which is a contradiction. Hence $u = v$ and thus $d(u_i, v_i) \to 0$. □

We conclude this section by establishing the equivalence between Daugavet and Δ-points for another class of spaces of Lipschitz functions. By definition, if $f \in S_{\mathrm{Lip}_0(M)}$ is not local, then there exists $\varepsilon > 0$ such that $m_{uv} \in S(f, \varepsilon)$ only if $d(u, v) \geqslant \varepsilon$. Therefore, by Proposition 2.8, every subslice $S(g, \alpha) \subseteq S(f, \varepsilon)$ contains a denting point, since its defining functional g is also not local. Applying Proposition 1.12, we obtain the following.

Proposition 3.8 (see [Vee25a, Proposition 4.8]). *Let $f \in S_{\mathrm{Lip}_0(M)}$ be a weak* Δ-point that is not local. Then, for every $\varepsilon > 0$, there exists a sequence (μ_i) in $S(f,\varepsilon) \cap \mathrm{dent}(B_{\mathcal{F}(M)})$ such that*

$$\|\mu_i - \mu_j\| \geqslant 2 - \varepsilon$$

for all $i, j \in \mathbb{N}$ with $i \neq j$.

Recall that if M is complete, then all denting points are molecules (see [Wea18, Corollary 3.44]). The proof of the next theorem is analogous to the proof of Theorem 3.7, with the exception that we use Proposition 3.8 instead of Proposition 3.1.

Theorem 3.9 (see [Vee25a, Proposition 4.9]). *Assume that M is a complete metric space such that for every $\varepsilon > 0$ and every bounded subset $M_0 \subseteq M$, the set*

$$\big\{m_{uv} \in \mathrm{dent}(B_{\mathcal{F}(M)})\colon u, v \in M_0, d(u,v) \geqslant \varepsilon\big\}$$

is finite. Let $f \in S_{\mathrm{Lip}_0(M)}$. Then the following are equivalent:

(i) f is a Daugavet point;

(ii) f is a Δ-point;

(iii) f is a weak Daugavet point;*

(iv) f is a weak Δ-point;*

(v) for every $\varepsilon > 0$ there exists a sequence $(m_{u_i v_i}) \subseteq S(f,\varepsilon)$ such that $d(u_i, v_i) \to 0$ or at least one of the sequences (u_i) and (v_i) is unbounded.

Proof. The implications (i)⇒(ii)⇒(iv) and (i)⇒(iii)⇒(iv) follow directly from the definitions. The implication (v)⇒(i) follows from [HOP24, Theorem 1.5] and Proposition 3.6. It only remains to prove (iv)⇒(v). Assume that f is a weak* Δ-point and fix $\varepsilon > 0$. If f is local, then the conclusion follows immediately: for every $\varepsilon > 0$ there exists a sequence $(m_{u_i v_i}) \subseteq S(f,\varepsilon)$ such that $d(u_i, v_i) \to 0$. Otherwise, by Proposition 3.8, there exists a sequence $(m_{u_i v_i}) \subseteq S(f,\varepsilon) \cap \mathrm{dent}(B_{\mathcal{F}(M)})$ such that

$$\|m_{u_i v_i} - m_{u_j v_j}\| \geqslant 2 - \varepsilon$$

for every $i, j \in \mathbb{N}$ with $i \neq j$. If $\varepsilon < 2$, which we may assume, then the molecules $m_{u_i v_i}$ are pairwise distinct. If both sequences (u_i) and (v_i) are bounded, then for every $\gamma > 0$, the set

$$\left\{m_{uv} \in \operatorname{dent}(B_{\mathcal{F}(M)}) \colon u, v \in \{u_i, v_i \colon i \in \mathbb{N}\}, d(u, v) \geqslant \gamma\right\}$$

must be finite by assumption. Therefore, $d(u_i, v_i) \to 0$. □

We can apply Theorem 3.9 to metric spaces that are not proper, for example, to the metric space discussed in Section 2.3.

3.2 Existence of Daugavet and Delta-points in spaces of Lipschitz functions

In this section, we prove that every space of Lipschitz functions over an infinite metric space contains a Δ-point. Furthermore, we show that the same does not hold for Daugavet points, thus showing that there exist Δ-points in spaces of Lipschitz functions that are not Daugavet points.

We start by showing that if M satisfies certain conditions, for example, if M is unbounded or fails to be uniformly discrete, then there exists a Daugavet point in $\operatorname{Lip}_0(M)$.

Proposition 3.10 (see [Vee25a, Proposition 5.1])**.** *Let $(m_{u_i v_i})$ be a sequence in $S_{\mathcal{F}(M)}$ such that there exists a sequence (A_i) of pairwise disjoint subsets of M such that*

$$d(u_i, x) + d(v_i, y) \geqslant (1 - 1/2^{i+1}) \left(d(u_i, v_i) + d(x, y)\right)$$

for all $i \in \mathbb{N}$ and all $x, y \in M \backslash A_i$. Then there exists a Daugavet point in $\operatorname{Lip}_0(M)$.

Proof. By passing to a subsequence, we may assume that

$$A_i \cap \{u_j, v_j \colon j \in \{1, \dots, i-1\}\} = \varnothing$$

for every $i \in \mathbb{N}$, since the sets (A_i) are pairwise disjoint. For simplicity, we assume that u_1 is the base point 0. We define f recursively on the set $\{u_i, v_i \colon i \in \mathbb{N}\}$. Let

$$f(u_1) = f(v_1) = 0.$$

Assume that we have defined $f(u_1), \dots, f(u_{i-1}), f(v_1), \dots, f(v_{i-1})$ such that $\langle f, m_{u_j v_j} \rangle \geqslant 1 - 1/2^{j-1}$ for every $j \in \{1, \dots, i-1\}$ and the Lipschitz constant of f is at most $1 - 1/2^{i-1}$. Let $M_{i-1} = \{u_j, v_j \colon j \in \{1, \dots, i-1\}\}$. Define

$$f(u_i) = \min_{x \in M_{i-1}} \big(f(x) + (1 - 1/2^i) d(x, u_i)\big)$$

and

$$f(v_i) = \max_{x \in M_{i-1} \cup \{u_i\}} \big(f(x) - (1 - 1/2^i) d(x, v_i)\big)\,.$$

Then the Lipschitz constant of f on $M_{i-1} \cup \{u_i, v_i\}$ is at most $1 - 1/2^i$. Furthermore, we have $f(v_i) = f(u_i) - (1 - 1/2^i) d(u_i, v_i)$ or

$$\begin{aligned}
&f(u_i) - f(v_i) \\
&\quad = \min_{x,y \in M_{i-1}} \big(f(x) + (1 - 1/2^i) d(x, u_i) - f(y) + (1 - 1/2^i) d(y, v_i)\big) \\
&\quad \geqslant \min_{x,y \in M_{i-1}} \big((1 - 1/2^i)\,(d(x, u_i) + d(y, v_i)) - (1 - 1/2^{i-1}) d(x, y)\big) \\
&\quad \geqslant (1 - 1/2^i) \min_{x,y \in M_{i-1}} \big(d(x, u_i) + d(y, v_i) - (1 - 1/2^{i+1}) d(x, y)\big) \\
&\quad \geqslant (1 - 1/2^i)(1 - 1/2^{i+1}) d(u_i, v_i) \\
&\quad \geqslant (1 - 1/2^{i-1}) d(u_i, v_i).
\end{aligned}$$

Hence, in both cases, we have $\langle f, m_{u_i v_i} \rangle \geqslant 1 - 1/2^{i-1}$. Proceeding recursively, we define f on the set $\{u_i, v_i \colon i \in \mathbb{N}\}$ in such a way that $\langle f, m_{u_i v_i} \rangle \to 1$. Extend f to M by McShane–Whitney extension theorem. Then, by Proposition 3.6, f is a Daugavet point. □

If M is a complete metric space that is unbounded or fails to be uniformly discrete, then, by following the idea of the proof of [HOP24, Proposition 2.4], it is possible to construct a sequence $(m_{u_i v_i})$ in $S_{\mathcal{F}(M)}$ and a sequence (A_i) of pairwise disjoint subsets of M such that

$$d(u_i, x) + d(v_i, y) \geqslant (1 - 1/2^{i+1})\,(d(u_i, v_i) + d(x, y))$$

for all $i \in \mathbb{N}$ and all $x, y \in M \backslash A_i$. Since a metric space M and its completion generate the same Lipschitz-free space, we may drop the assumption of completeness. This leads to the following corollary.

Corollary 3.11 (see [Vee25a, Corollary 5.2])**.** *Assume that M is unbounded or not uniformly discrete. Then* $\mathrm{Lip}_0(M)$ *contains a Daugavet point.*

We now present another sufficient condition for the existence of Daugavet points, which allows us to verify their presence under assumptions different from those in Proposition 3.10.

Proposition 3.12 (see [Vee25a, Proposition 5.3])**.** *Let $f \in S_{\mathrm{Lip}_0(M)}$. Assume that there exists a sequence (u_i) of pairwise distinct points in M such that for every $i \in \mathbb{N}$, every $\delta > 0$, and every $v \in M \setminus \{u_i\}$, there exists a point $p \in [u_i, v]_\delta \setminus \{u_i\}$ such that $m_{u_i p} \in S(f, \delta)$. Then f is a Daugavet point.*

Proof. If f is local, then by [HOP24, Theorem 1.5], it is a Daugavet point. Thus we we may assume that f is not local. Fix $g \in S_{\mathrm{Lip}_0(M)}$ and $\varepsilon > 0$. We will show that $g \in \overline{\mathrm{conv}}\, \Delta_\varepsilon(f)$. For every $i \in \mathbb{N}$, define a function $h_i : M \to \mathbb{R}$ by $h_i|_{M\setminus\{u_i\}} = g|_{M\setminus\{u_i\}}$ and

$$h_i(u_i) = \sup_{v \in M\setminus\{u_i\}} \left(g(v) - d(v, u_i)\right).$$

Then $\|h_i\| \leqslant 1$. Since f is not local, there exists $\delta > 0$ such that $d(u, v) > \delta$ for every $m_{uv} \in S(f, \delta)$. We may also assume that $2\delta < \varepsilon$ and $\delta < 1$. By definition of $h_i(u_i)$, there exists $v \in M \setminus \{u_i\}$ such that

$$h_i(u_i) < g(v) - d(v, u_i) + \delta^2/2.$$

Then

$$h_i(v) - h_i(u_i) > g(v) - \big(g(v) - d(v, u_i) + \delta^2/2\big) = d(v, u_i) - \delta^2/2.$$

There exists $p \in [u_i, v]_{\delta^2/2}$ such that $m_{u_i p} \in S(f, \delta^2/2)$. Since $\delta^2/2 < \delta$, we get $d(u_i, p) > \delta$. Hence

$$\begin{aligned} h_i(p) - h_i(u_i) &= h_i(p) - h_i(v) + h_i(v) - h_i(u_i) \\ &> -d(p, v) + d(v, u_i) - \delta^2/2 \\ &> d(p, u_i) - \delta^2 \\ &> (1 - \delta) d(p, u_i). \end{aligned}$$

Then for every $i \in \mathbb{N}$ we get

$$\|f - h_i\| \geqslant \langle f, m_{u_i p}\rangle - \langle h_i, m_{u_i p}\rangle > 2 - 2\delta > 2 - \varepsilon.$$

Now fix $k \in \mathbb{N} \setminus \{1\}$ and take $p, q \in M$ with $p \neq q$. Then there exist $i, j \in \{1, \ldots, k\}$ with $i \neq j$ such that $p \notin \{u_n \colon n \in \{1, \ldots, k\} \setminus \{i\}\}$ and

$q \notin \{u_n \colon n \in \{1, \dots, k\} \setminus \{j\}\}$. Since each h_n differs from g only at u_n, we get

$$\begin{aligned}\left| \left(g - \frac{1}{k} \sum_{n=1}^{k} h_n \right)(p) - \left(g - \frac{1}{k} \sum_{n=1}^{k} h_n \right)(q) \right| \\ &= \frac{1}{k} \left| g(p) - h_i(p) - g(q) + h_j(q) \right| \\ &= \frac{1}{k} \left| 2g(p) - 2g(q) + h_i(q) - h_i(p) + h_j(q) - h_j(p) \right| \\ &\leqslant \frac{4}{k} d(p, q).\end{aligned}$$

Thus

$$\left\| g - \frac{1}{k} \sum_{n=1}^{k} h_n \right\| \leqslant \frac{4}{k}.$$

Since each $h_n \in \Delta_\varepsilon(f)$, we conclude that $g \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(f)$, proving that f is a Daugavet point. □

Proposition 3.13 (see [Vee25a, Proposition 5.4])**.** *Let (u_i) be a sequence of pairwise distinct points in M such that for every $i \in \mathbb{N}$, every $\delta > 0$, and every $v \in M \setminus \{u_i\}$, there exists $p \in [u_i, v]_\delta \setminus \{u_i\}$ with $\inf_{j \in \mathbb{N}} d(u_j, p) > (1 - \delta) d(u_i, p)$. Then $\operatorname{Lip}_0(M)$ contains a Daugavet point.*

Proof. Without loss of generality, assume that u_1 is the fixed base point 0. Define a function $f : M \to \mathbb{R}$ by

$$f(p) = -\inf_{i \in \mathbb{N}} d(u_i, p).$$

Then $\|f\| \leqslant 1$ by [Wea18, Proposition 1.32]. Furthermore, for every $i \in \mathbb{N}$, every $\delta > 0$, and every $v \in M \setminus \{u_i\}$, there exists $p \in [u_i, v]_\delta \setminus \{u_i\}$ with $\inf_{j \in \mathbb{N}} d(u_j, p) > (1 - \delta) d(u_i, p)$. Hence

$$f(u_i) - f(p) = \inf_{j \in \mathbb{N}} d(u_j, p) > (1 - \delta) d(u_i, p),$$

which implies that $m_{u_i p} \in S(f, \delta)$. By Proposition 3.12, f is a Daugavet point. □

Proposition 3.13 applies to certain metric spaces for which Proposition 3.10 does not. For example, let M be an infinite set with the

metric

$$d(p,q) = \begin{cases} 0, & p = q, \\ 1, & p \in \{x,y\}\,, q \notin \{x,y\} \text{ or } p \notin \{x,y\}\,, q \in \{x,y\}\,, \\ 2, & \text{otherwise} \end{cases}$$

where $x, y \in M$ are two fixed points. In this case, no sequence $(m_{u_i v_i})$ in $\mathcal{F}(M)$ satisfies the conditions of Proposition 3.10. However, any sequence (u_i) of pairwise distinct points in $M \setminus \{x,y\}$ fulfills the assumptions of Proposition 3.13, and thus $\mathrm{Lip}_0(M)$ contains a Daugavet point.

We now turn our attention to the existence of Δ-points in spaces of Lipschitz functions over infinite metric spaces. First, we will introduce two helpful lemmas.

Lemma 3.14 (see [Vee25a, Lemma 5.5])**.** *Let M be an infinite, bounded, and uniformly discrete metric space. There exist $a > 0$ and a sequence $(u_i) \subseteq M$ such that*

$$a\frac{i-1}{i} \leqslant d(u_i, u_j) \leqslant a\frac{i+1}{i}$$

for all $i, j \in \mathbb{N}$ with $i < j$.

Proof. We construct the sequence (u_i) recursively. First, choose any $u_1 \in M$ and define

$$a_1 = \sup\{b \in \mathbb{R} \colon |\{p \in M \colon d(u_1, p) \leqslant b\}| < \infty\}\,.$$

Since M is bounded and uniformly discrete, it follows that $0 < a_1 < \infty$. Define

$$M_1 = \{p \in M \colon a_1(2-1)/2 < d(u_1, p) < a_1(2+1)/2\}\,.$$

By the definition of a_1, the set M_1 is infinite.

Assume that we have found points $u_1, \ldots, u_{n-1} \in M$, corresponding constants $a_1, \ldots, a_{n-1} > 0$, and infinite sets $M_{n-1} \subseteq \ldots \subseteq M_1 \subseteq M$ such that

$$a_i\frac{2i-1}{2i} \leqslant d(u_i, p) \leqslant a_i\frac{2i+1}{2i} \tag{3.2}$$

for all $i \in \{1, \ldots, n-1\}$ and $p \in M_i$. Now choose $u_n \in M_{n-1}$ arbitratily, and define

$$a_n = \sup\{b \in \mathbb{R} \colon |\{p \in M_{n-1} \colon d(u_n, p) \leqslant b\}| < \infty\}\,,$$

and

$$M_n = \{p \in M_{n-1} \colon a_n(2n-1)/(2n) < d(u_n, p) < a_n(2n+1)/(2n)\}\,.$$

As before, $0 < a_n < \infty$ and $M_n \subseteq M_{n-1}$ is infinite.

The sequence (a_i) is positive and bounded, since M is bounded and uniformly discrete. Therefore, there exists a subsequence (a_{k_i}) and a constant $a > 0$ such that

$$\frac{2i-2}{2i-1}a < a_{k_i} < \frac{2i+2}{2i+1}a$$

for every $i \in \mathbb{N}$. Then, by (3.2), for all $i, j \in \mathbb{N}$ with $i < j$, we have

$$\begin{aligned} a\frac{i-1}{i} < a_{k_i}\frac{2i-1}{2i} \leqslant a_{k_i}\frac{2k_i-1}{2k_i} &\leqslant d(u_{k_i}, u_{k_j}) \\ &\leqslant a_{k_i}\frac{2k_i+1}{2k_i} \leqslant a_{k_i}\frac{2i+1}{2i} < a\frac{i+1}{i}, \end{aligned}$$

since $u_{k_j} \in M_i$. □

Lemma 3.15 (see [Vee25a, Lemma 5.6])**.** *Assume that M is an infinite, bounded, and uniformly discrete metric space. Then there exist a constant $a > 0$ and two sequences (u_i) and (v_i) of pairwise distinct points in M such that*

$$a\frac{i+1}{i} \geqslant d(u_i, v_i) \geqslant a\frac{i-1}{i}$$

and

$$\min\{d(u_i, p), d(v_i, p)\} \geqslant a\frac{i-1}{i}$$

and

$$\min\{d(u_i, q), d(v_i, q)\} \geqslant a\frac{i-1}{2i}$$

for all $i \in \mathbb{N}$, $p \in \{u_j, v_j \colon j > i\}$, and $q \in M \setminus \{u_i, v_i\}$.

Proof. Set

$$a_1 = \sup\left\{b \in \mathbb{R} \colon \left|\bigcup_{p\in M}(B(p,b)\setminus\{p\})\right| < \infty\right\}.$$

Since M is bounded and uniformly discrete, we have $0 < a_1 < \infty$. We now recursively construct two sequences (x_i) and (y_i) in M. Choose

$$x_1 \in \bigcup_{p\in M}(B(p, a_1 + a_1)\setminus\{p\})$$

and select $y_1 \in M$ such that $0 < d(x_1, y_1) \leqslant a_1 + a_1$. Assume that we have selected $x_1, \ldots, x_{n-1} \in M$ and $y_1, \ldots, y_{n-1} \in M$. Then choose

$$x_n \in \bigcup_{p \in M} (B(p, a_1 + a_1/n) \setminus \{p\}) \setminus \left(\bigcup_{p \in M} (B(p, a_1 - a_1/n) \setminus \{p\}) \cup \bigcup_{i=1}^{n-1} \{x_i, y_i\} \right)$$

and select $y_n \in M$ such that $0 < d(x_n, y_n) \leqslant a_1 + a_1/n$. Clearly, the sequence (x_i) is pairwise distinct. By passing to a subsequence, if necessary, we may assume that (y_i) is either pairwise distinct or constant. We now consider each case separately.

Case 1. Assume that (y_i) is pairwise distinct. By passing to a subsequence, we may assume that

$$y_i \notin \bigcup_{p \in M} (B(p, a_1 - a_1/i) \setminus \{p\}) \cup \bigcup_{j=1}^{i-1} \{x_j, y_j\}$$

for every $i \in \mathbb{N}$. Then we may choose a as a_1, (u_i) as (x_i), and (v_i) as (y_i).

Case 2. Assume that (y_i) is constant, i.e., there exists $y \in M$ such that $y = y_i$ for every $i \in \mathbb{N}$. Let

$$M_0 = \{x_i \colon i \in \mathbb{N}\}.$$

By Lemma 3.14, there exist $a > 0$ and a sequence $(z_i) \subseteq M_0$ such that

$$a \frac{i-1}{i} \leqslant d(z_i, z_j) \leqslant a \frac{i+1}{i}$$

for every $i, j \in \mathbb{N}$ with $i < j$. Note that for any $i, j \in \mathbb{N}$ with $i < j$, we have

$$a \frac{i-1}{i} \leqslant d(x_i, x_j) \leqslant d(x_i, y) + d(x_j, y) \leqslant a_1 \frac{i+1}{i} + a_1 \frac{j+1}{j}$$

and therefore $a \leqslant 2a_1$. We may choose (u_i) as (z_{2i-1}) and (v_i) as (z_{2i}). □

Theorem 3.16 (see [Vee25a, Theorem 5.7])**.** *If M is an infinite metric space, then $\mathrm{Lip}_0(M)$ contains a Δ-point.*

Proof. If M is unbounded or not uniformly discrete, then by Corollary 3.11, $\mathrm{Lip}_0(M)$ contains a Daugavet point, and hence a Δ-point.

Assume now that M is bounded and uniformly discrete. By Lemma 3.15, there exist $a > 0$ and two sequences (u_i) and (v_i) of pairwise distinct points in M such that

$$a\frac{i+1}{i} \geqslant d(u_i, v_i) \geqslant a\frac{i-1}{i}$$

and

$$\min\{d(u_i, p), d(v_i, p)\} \geqslant a\frac{i-1}{i}$$

and

$$\min\{d(u_i, q), d(v_i, q)\} \geqslant a\frac{i-1}{2i}$$

for all $i \in \mathbb{N}$, $p \in \{u_j, v_j \colon j > i\}$ and $q \in M \setminus \{u_i, v_i\}$. Without loss of generality, assume that u_1 is the fixed base point 0. Define a function $f : M \to \mathbb{R}$ by

$$f(p) = \begin{cases} a(i-2)/(2i), & p = u_i \text{ for } i \in \mathbb{N}\setminus\{1\}, \\ -a(i-2)/(2i), & p = v_i \text{ for } i \in \mathbb{N}\setminus\{1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f\| \leqslant 1$ and for every $i \in \mathbb{N}\setminus\{1\}$,

$$\langle f, m_{u_i v_i}\rangle = \frac{a(i-2)}{id(u_i, v_i)} \geqslant \frac{i-2}{i+1},$$

which gives us $f \in S_{\mathrm{Lip}_0(M)}$. Now for every $i \in \mathbb{N}\setminus\{1\}$, define $g_i : M \to \mathbb{R}$ by

$$g_i(p) = \begin{cases} a(j-2)/(2j), & p = u_j \text{ for } j \in \mathbb{N}\setminus\{1, i\}, \\ -a(j-2)/(2j), & p = v_j \text{ for } j \in \mathbb{N}\setminus\{1, i\}, \\ -a(i-2)/(2i), & p = u_i, \\ a(i-2)/(2i), & p = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Analogously to previous argumentation, $g_i \in S_{\mathrm{Lip}_0(M)}$. Moreover, for every $i \in \mathbb{N}\setminus\{1\}$, we have

$$\|f - g_i\| \geqslant \langle f, m_{u_i v_i}\rangle - \langle g_i, m_{u_i v_i}\rangle \geqslant 2\frac{i-2}{i+1}.$$

Since f and g_i differ only at u_i and v_i, we also get

$$\left\| f - \frac{1}{|I|} \sum_{i \in I} g_i \right\| \leqslant \frac{4}{|I|}$$

for every finite set of indexes $I \subseteq \mathbb{N} \setminus \{1\}$. Therefore $f \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(f)$ for every $\varepsilon > 0$, i.e., f is a Δ-point. □

Remark 3.17 (see [Vee25a, Remark 5.8]). The existence of Δ-points in spaces of Lipschitz functions over infinite metric spaces can also be derived from the fact that such spaces always contain an isometric copy of ℓ_∞ (see [CJ17, Theorem 5]). Indeed, ℓ_∞ contains many Δ-points, and if a subspace Y of a Banach space X contains a Δ-point y, then y is also a Δ-point in X, since the set of points in Y that are almost at distance 2 from y is contained in the set of such points in X.

It is natural to ask whether every space of Lipschitz functions over an infinite metric space M contains a Daugavet point or at least a weak* Daugavet point. We know this holds if M is unbounded or not uniformly discrete (see Corollary 3.11). However, the following example shows that there exists an infinite-dimensional space of Lipschitz functions which does not contain any weak* Daugavet points.

Example 3.18 (see [Vee25a, Example 5.9]). Consider $M = \mathbb{N}$ with the metric

$$d(n,k) = \begin{cases} 0, & n = k, \\ 3 - \left|\frac{1}{n} - \frac{1}{k}\right|, & \text{otherwise.} \end{cases}$$

Note that $2 < d(n,k) < 3$ for every $n, k \in M$ with $n \neq k$. Fix $f \in S_{\operatorname{Lip}_0(M)}$. Our goal is to show that f is not a weak* Daugavet point.

There does not exist $n, k \in M$ such that $\langle f, m_{1n} \rangle > 3/4$ and $\langle f, m_{k1} \rangle > 3/4$, because if such elements did exist, we would get

$$\begin{aligned} f(k) - f(n) &= f(k) - f(1) + f(1) - f(n) \\ &> \frac{3}{4}\left(d(1,k) + d(1,n)\right) > 3 > d(k,n). \end{aligned}$$

Thus we must have $\langle f, m_{1k} \rangle \leqslant 3/4$ for every $k \in \mathbb{N}$, or $\langle f, m_{k1} \rangle \leqslant 3/4$ for every $k \in \mathbb{N}$. As f and $-f$ are weak* Daugavet points simultaneously, it is sufficient to consider the case $\langle f, m_{k1} \rangle \leqslant 3/4$ for every $k \in M \setminus \{1\}$. We may additionally assume there exists $n \in M \setminus \{1\}$ such that $\langle f, m_{1n} \rangle \geqslant 0$, since otherwise we would have $-\langle f, m_{k1} \rangle \leqslant 0 \leqslant 3/4$

for every $k \in M\setminus\{1\}$, and we could consider $-f$ instead of f. Let n be the smallest element in $M\setminus\{1\}$ such that $\langle f, m_{1n}\rangle \geqslant 0$.

Consider $\mu = \sum_{i=1}^{n-1} m_{in}/(n-1)$. It is straightforward to check that $\mu \in S_{\mathcal{F}(M)}$. Let

$$\alpha = \frac{1}{3n} - \frac{1}{3(n+1)} = \frac{1}{3n(n+1)}.$$

Then $\alpha \leqslant 1/18$, since $n \geqslant 2$. Our goal is to show that for every $g \in S_{\mathrm{Lip}_0(M)}$ with $\|f-g\| > 2-\alpha$, we have $g \notin S(\mu, \alpha/n)$. Thus f is not a weak* Daugavet point. Fix $g \in S_{\mathrm{Lip}_0(M)}$ with $\|f-g\| > 2-\alpha$. There exist $k, l \in M$ with $k \neq l$ such that $\langle f, m_{kl}\rangle - \langle g, m_{kl}\rangle > 2-\alpha$ giving us $\langle f, m_{kl}\rangle > 1-\alpha$ and $-\langle g, m_{kl}\rangle > 1-\alpha$. Then $l \neq 1$, since $\langle f, m_{kl}\rangle > 3/4$. If $l < n$, then either $k = 1$ and thus $\langle f, m_{1l}\rangle > 0$ or $k \neq 1$ and from $\langle f, m_{kl}\rangle > 1-\alpha$ we get

$$\begin{aligned} f(1) - f(l) &> f(1) - f(k) + (1-\alpha)d(k,l) \\ &> -d(1,k) + d(k,l) - 3\alpha \\ &= -\left(3 - 1 + \frac{1}{k}\right) + 3 - \left(2\max\left\{\frac{1}{k}, \frac{1}{l}\right\} - \frac{1}{k} - \frac{1}{l}\right) - 3\alpha \\ &= 1 - 2\max\left\{\frac{1}{k}, \frac{1}{l}\right\} + \frac{1}{l} - 3\alpha \\ &\geqslant \frac{1}{l} - 3\alpha > 0. \end{aligned}$$

In both cases we have a contradiction, since n is the smallest element in $M\setminus\{1\}$ such that $\langle f, m_{1n}\rangle \geqslant 0$. Therefore $l \geqslant n$. Furthermore, if $l \neq n$, then

$$f(n) - f(l) \leqslant f(1) - f(l) \leqslant 2 + \frac{1}{l} \leqslant 3 - \frac{1}{n} + \frac{1}{l} - \frac{1}{2} < (1 - 1/6)d(n,l),$$

and thus $k \neq n$.

Lastly, we will show that $g \notin S(m_{in}, \alpha)$ for some $i \in \{1, \ldots, n-1\}$. This implies that $g \notin S(\mu, \alpha/n)$, since $S(\mu, \alpha/n) \subseteq S(m_{in}, \alpha)$. We will consider three cases, and use the fact that $-\langle g, m_{kl}\rangle > 1-\alpha$.

Case 1. Assume $1 \leqslant k < n$. Then

$$\begin{aligned} g(k) - g(n) &< g(l) - g(n) - (1-\alpha)d(l,k) \\ &\leqslant d(l,n) - (1-\alpha)d(l,k) \\ &< 3 - 2(1-\alpha) \\ &< 2(1-\alpha) \\ &< (1-\alpha)d(k,n). \end{aligned}$$

Case 2. Assume $l = n$. Then analogously to Case 1, we get

$$g(1) - g(n) < g(1) - g(k) - (1-\alpha)d(n,k) < (1-\alpha)d(1,n).$$

Case 3. Assume $k, l > n$. Then

$$\begin{aligned} g(1) - g(n) &< g(1) - g(k) + g(l) - g(n) - (1-\alpha)d(l,k) \\ &< d(1,k) + d(l,n) - d(k,l) + 3\alpha \\ &= 2 + \frac{1}{k} + 3 - \frac{1}{n} + \frac{1}{l} - 3 + \left|\frac{1}{k} - \frac{1}{l}\right| + \frac{1}{n} - \frac{1}{n+1} \\ &= 2 + 2\max\left\{\frac{1}{k}, \frac{1}{l}\right\} - \frac{1}{n+1} \\ &\leqslant 2 + \frac{1}{n+1} = 2 + \frac{1}{n} - 3\alpha \leqslant (1-\alpha)d(1,n). \end{aligned}$$

Therefore the space $\mathrm{Lip}_0(M)$ has no weak* Daugavet points.

By Theorem 3.16, there exists a Δ-point in the space of Lipschitz functions introduced in the previous example. Therefore we now know that Daugavet and Δ-points are not equivalent in spaces of Lipschitz functions. We will finish this section by showing that Daugavet and Δ-points also differ from their weak* versions in spaces of Lipschitz functions.

Example 3.19 (see [Vee25a, Example 5.10]). Let $X = \{x_i \colon i \in \mathbb{N}\}$, $Y = \{y_i \colon i \in \mathbb{N}\}$, $U = \{u_i \colon i \in \mathbb{N}\}$, $V = \{v_i \colon i \in \mathbb{N}\}$, and let $M = X \cup Y \cup U \cup V$. Define a metric by

$$d(p,q) = \begin{cases} 0, & p = q, \\ 1, & p = u_i, q \in \{x_j, u_j\} \text{ for some } i > j, \\ 1, & q = u_i, p \in \{x_j, u_j\} \text{ for some } i > j, \\ 1, & p = v_i, q \in \{y_j, v_j\} \text{ for some } i > j, \\ 1, & q = v_i, p \in \{y_j, v_j\} \text{ for some } i > j, \\ 2, & \text{otherwise.} \end{cases}$$

Define $f \colon M \to \mathbb{R}$ by

$$f(p) = \begin{cases} 2, & p \in X \cup U, \\ 0, & p \in Y \cup V. \end{cases}$$

Note that $\langle f, m_{pq}\rangle > 0$ if and only if $p \in X \cup U$ and $q \in Y \cup V$, and in that case $\langle f, m_{pq}\rangle = 1$. Thus $f \in S_{\mathrm{Lip}_0(M)}$.

We first show that f is a weak* Daugavet point. Fix $\alpha > 0$ and $\mu \in S_{\mathcal{F}(M)}$. Our goal is to find $h \in S(\mu, \alpha)$ such that $\|f - h\| = 2$. Since elements with finite support are dense in the unit sphere, it suffices to consider the case when μ has finite support. Let $g \in S(\mu, \alpha)$, and let $n \in \mathbb{N}$ be such that $\operatorname{supp}(\mu) \subseteq \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\}$. Define

$$a = \frac{1}{2}\left(\max\left\{ g(p) \colon p \in \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\} \right\} + \min\left\{ g(p) \colon p \in \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\} \right\} \right)$$

and then define $h \colon M \to \mathbb{R}$ by

$$h(p) = \begin{cases} g(p), & p \in \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\}, \\ a - 1, & p \in \bigcup_{i=n+1}^{\infty} \{x_i\}, \\ a + 1, & p \in \bigcup_{i=n+1}^{\infty} \{y_i\}, \\ a, & p \in \bigcup_{i=n+1}^{\infty} \{u_i, v_i\}. \end{cases}$$

Then $\|h\| \leqslant 1$, because

$$|a - g(p)| \leqslant \frac{1}{2}\left(\max\left\{ g(p) \colon p \in \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\} \right\} - \min\left\{ g(p) \colon p \in \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\} \right\} \right) \leqslant 1$$

for every $p \in \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\}$. Clearly, $\langle h, \mu\rangle = \langle g, \mu\rangle > 1 - \alpha$, and

$$\|f - h\| \geqslant \left\langle f, m_{x_{n+1}y_{n+1}} \right\rangle - \left\langle h, m_{x_{n+1}y_{n+1}} \right\rangle = 2.$$

Therefore f is a weak* Daugavet point.

Finally, we show that $f \notin \overline{\operatorname{conv}}\, \Delta_\varepsilon(f)$ for any $\varepsilon \in (0, 1/2)$, proving that f is not a Δ-point. Fix $\varepsilon \in (0, 1/2)$ and functions $g_1, \ldots, g_k \in \Delta_\varepsilon(f)$. Then for each $i \in \{1, \ldots, k\}$, there exists $p_i, q_i \in M$ such that $\langle f, m_{p_iq_i}\rangle - \langle g_i, m_{p_iq_i}\rangle > 2 - 2\varepsilon$. Then $\langle f, m_{p_iq_i}\rangle > 0$, and therefore $p_i \in X \cup U$ and $q_i \in Y \cup V$ for every $i \in \{1, \ldots, k\}$. Let $n \in \mathbb{N}$ be such

that $\{p_1, \ldots, p_k, q_1, \ldots, q_k\} \subseteq \bigcup_{i=1}^{n} \{x_i, y_i, u_i, v_i\}$. By Lemma 2.3, we have

$$\begin{aligned}\|m_{u_{n+1}v_{n+1}} - m_{p_iq_i}\| &= \frac{d(u_{n+1}, p_i) + d(q_i, v_{n+1}) + |d(u_{n+1}, v_{n+1}) - d(p_i, q_i)|}{\max\{d(u_{n+1}, v_{n+1}), d(p_i, q_i)\}} \\ &= \frac{1 + 1 + |2 - 2|}{\max\{2, 2\}} = 1\end{aligned}$$

for every $i \in \{1, \ldots, k\}$. Since $-\langle g_i, m_{p_iq_i}\rangle > 1 - 2\varepsilon$, we obtain

$$\begin{aligned}\left\langle g_i, m_{u_{n+1}v_{n+1}}\right\rangle &< \left\langle g_i, m_{u_{n+1}v_{n+1}}\right\rangle - \langle g_i, m_{p_iq_i}\rangle - 1 + 2\varepsilon \\ &\leqslant \|m_{u_{n+1}v_{n+1}} - m_{p_iq_i}\| - 1 + 2\varepsilon = 2\varepsilon\end{aligned}$$

for every $i \in \{1, \ldots, k\}$. Hence

$$\left\| f - \sum_{i=1}^{k} \lambda_i g_i \right\| \geqslant \left\langle f, m_{u_{n+1}v_{n+1}}\right\rangle - \sum_{i=1}^{k} \lambda_i \left\langle g_i, m_{u_{n+1}v_{n+1}}\right\rangle > 1 - 2\varepsilon$$

for all $\lambda_1, \ldots, \lambda_k \geqslant 0$ with $\sum_{i=1}^{k} \lambda_i = 1$. Since the functions $g_1, \ldots, g_k \in \Delta_\varepsilon(f)$ were arbitrary, we now get $f \notin \overline{\operatorname{conv}}\, \Delta_\varepsilon(f)$.

Chapter 4

Renormings that admit Daugavet and Delta-points

In this chapter, we focus on renorming Banach spaces so that they contain Daugavet and Δ-points. The motivation stems from the following two questions (see [ALMP22, Question 6.1], [MPRZ24, Question 7.7]):

- Does there exist a reflexive/superreflexive Banach space with a Δ-point/super Δ-point?
- Does there exist a reflexive/superreflexive Banach space with a Daugavet point/super Daugavet point?

We answer both of these questions in the affirmative. Furthermore, we show that every infinite-dimensional Banach space can be renormed to admit a Δ-point. Whether the same is true for Daugavet points remains an open question. However, we prove that if a Banach space has unconditional basis, then it can be renormed to admit a Daugavet point. In particular, we will see that the spaces ℓ_p, for $p \in [1, \infty)$ can be renormed to have a Daugavet point. This chapter is based on [AAL$^+$24a] and [HLPV24].

4.1 Renormings that admit Delta-points

It is known that finite-dimensional Banach spaces cannot admit Δ-points (see [ALMP22, Corollary 6.10]). The aim of this section is to prove that every infinite-dimensional Banach space admits an equivalent norm under which the space contains Δ-points. This highlights that

the question of whether a Banach space contains a Δ-point is primarily an isometric and not an isomorphic question. This section is based on [AAL$^+$24a].

Theorem 4.1 (see [AAL$^+$24a, Theorem 4.1])**.** *Let X be an infinite-dimensional Banach space. Then the following holds:*

(1) If X fails the Schur property, and in particular, if X does not contain a copy of ℓ_1, then there exists an equivalent norm on X with respect to which X contains a sequential super Δ-point;

(2) If X contains a copy of ℓ_1, then there exists an equivalent norm on X with respect to which X contains a Δ-point.

Note that we will show in Section 5.1 (see Corollary 5.3) that the existence of a Δ-point in a Banach space automatically implies the existence of a weak* super Δ-point in its dual. Hence, we will immediately obtain a slightly stronger result than initially stated.

For the proof of Theorem 4.1 (2), we need the following lemma, which is an easy consequence of a classical norm extension result.

Lemma 4.2 (see [AAL$^+$24a, Lemma 4.2])**.** *Let X be a Banach space. If there exists a subspace Y of X that can be renormed to admit a Δ-point, then X can be renormed to admit a Δ-point.*

The same holds for super Δ-points.

Proof. Assume that there exists a norm $|\cdot|$ on Y that admits a Δ- or a super Δ-point. By [DGZ93, Lemma II.8.1], we can extend $|\cdot|$ to an equivalent norm $|||\cdot|||$ on X which coincides with $|\cdot|$ on Y. In other words, $(Y, |\cdot|)$ is isometrically isomorphic to a subspace of $(X, |||\cdot|||)$. Since both Δ- and super Δ-points pass to superspaces, the conclusion follows. □

Proof of Theorem 4.1. We begin by proving (1). That spaces not containing a copy of ℓ_1 fail the Schur property follows from Rosenthal's ℓ_1 theorem. Now if X fails the Schur property, then using classic results on extraction of basic sequences (see e.g. [AK16, Proposition 1.5.4]), we can construct a weakly null basic sequence $(e_n)_{n\geqslant 1} \subseteq S_X$. Let $(e_n^*)_{n\geqslant 1}$ be the sequence of biorthogonal functionals on $\overline{\operatorname{span}}\,(e_n)_{n\geqslant 1}$, and for every $n \geqslant 2$, take a Hahn–Banach extension $f_n \in X^*$ of the

functional $e_1^* - 2e_n^*$. As $(e_n)_{n\geqslant 1}$ is basic, there exists a constant $K \geqslant 1$ such that $\sup_{n\geqslant 2} \|f_n\| \leqslant K$. Define an equivalent norm $|||\cdot|||$ on X by

$$|||x||| := \max\left\{\frac{1}{2}\|x\|,\ \sup_{n\geqslant 2}|f_n(x)|\right\}$$

for every $x \in X$. Then $\frac{1}{2}\|\cdot\| \leqslant |||\cdot||| \leqslant K\|\cdot\|$. By construction, for every $n \geqslant 2$, we have

$$|||e_1||| = |||e_1 + e_n||| = 1 \text{ and } |||e_n||| = 2.$$

As $e_1 + e_n \to e_1$ weakly, it follows that e_1 is a sequential super Δ-point in $(X, |||\cdot|||)$.

Now we prove (2). If X contains a subspace Y isomorphic to ℓ_1, then, by Theorem 2.26, the space Y can be renormed to admit a Δ-point. Finally, Lemma 4.2 ensures that X itself can be renormed to admit a Δ-point. □

Let us end the section with a remark.

Remark 4.3 (see [AAL$^+$24a, Remark 4.3]). Note that it is essential in the proof of Theorem 4.1 (1) to go first trough this process of extraction of a basic sequence in order to have complete control over the values of $f_n(e_m)$ for distinct m, n. It is thus unclear whether a similar construction could be carried out using a weakly null normalized net, and in particular whether it could be implemented in ℓ_1. Furthermore, it is still unknown whether the Daugavet point in the space $\mathcal{F}(\mathcal{M})$ introduced in Section 2.3, is a super Δ-point (see Question 7.1 in [MPRZ24] for further discussion). Thus we do not know whether ℓ_1 can be renormed to admit a super Δ-point.

4.2 Renormings that admit Daugavet points

In this section, our aim is to prove that every infinite-dimensional Banach space with unconditional basis can be renormed so that it contains a Daugavet point. This section is based on [HLPV24].

We start with the following theorem, which is the main result of this section.

Theorem 4.4 (see [HLPV24, Theorem 4.2]). *Let X be an infinite-dimensional Banach space with an unconditional weakly null Schauder basis (e_n) and biorthogonal functionals (e_n^*). Then there exists an equivalent norm $|||\cdot|||$ on X such that*

(1) e_1 is a super Daugavet point in $(X, |||\cdot|||)$;

(2) e_1^* is a weak super Daugavet point in $(E, |||\cdot|||)$, where $E := \overline{\operatorname{span}}\{e_n^*\}$.*

We will prove Theorem 4.4 in several steps. From now on, let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with an unconditional weakly null Schauder basis (e_n) and corresponding biorthogonal functionals (e_n^*). Without loss of generality, we assume that (e_n) is normalized and 1-unconditional with respect to the original norm of X.

Let $Y := \overline{\operatorname{span}}\{e_n\}_{n\geqslant 2}$, and let A be the set of all finitely supported elements in the positive cone of Y. Similarly, let F denote the set of all finitely supported elements in the positive cone of $\overline{\operatorname{span}}\{e_n^*\}_{n\geqslant 2}$.

We consider the equivalent norm $|||\cdot|||$ on X whose unit ball is

$$B_{(X,|||\cdot|||)} := \overline{\operatorname{conv}}\{\pm(e_1 + 2x)\colon x \in A \cap B_X\}.$$

Since (e_n) is 1-unconditional, every finitely supported element $y \in B_Y$ can be written as $y = y_+ - y_-$ with $y_+, y_- \in A \cap B_X$. It immediately follows that B_Y is contained in $B_{(X,|||\cdot|||)}$. Hence,

$$B_{Y\oplus_1 \mathbb{R}\, e_1} = \operatorname{conv}(B_Y \cup \{\pm e_1\}) \subseteq B_{(X,|||\cdot|||)} \subseteq 3B_X,$$

and so $|||\cdot|||$ is indeed an equivalent norm on X.

We begin by providing a geometric description of $B_{(E,|||\cdot|||)}$, and by giving useful formulas for the norm $|||\cdot|||$ and for its dual norm.

Lemma 4.5 (see [HLPV24, Lemma 4.3]). *The spaces $(X, |||\cdot|||)$ and $(E, |||\cdot|||)$ have the following properties:*

(1) For every $z^* \in B_{(E,|||\cdot|||)}$ with finite support and $z^*(e_1) \geqslant 0$, there exist $\lambda \in [0,1]$ and $x^*, y^* \in F \cap B_{X^*}$ with disjoint supports such that

$$z^* = \lambda(e_1^* - y^*) + (1-\lambda)\frac{1}{2}(x^* - y^*);$$

(2) The unit ball of the space $(E, |||\cdot|||)$ is given by

$$B_{(E,|||\cdot|||)} = \overline{\operatorname{conv}}\left\{ \pm\,(e_1^* - x^*), \frac{1}{2}(x^* - y^*)\colon x^*, y^* \in F \cap B_{X^*}\right\};$$

(3) For every $x \in X$,

$$|||x||| = \max\left\{|e_1^*(x)|, |e_1^*(x) - \|x_+\||, |e_1^*(x) + \|x_-\||, \frac{1}{2}\big(\|x_+\| + \|x_-\|\big)\right\},$$

where x_+ and x_- are the positive and negative parts of $x - e_1^*(x)e_1$, respectively;

(4) For every $x^* \in E$,

$$|||x^*||| = \max\left\{\big|x^*(e_1) + 2\|x_+^*\|\big|, \big|x^*(e_1) - 2\|x_-^*\|\big|\right\},$$

where x_+^* and x_-^* are the positive and negative parts of $x^* - x^*(e_1)e_1^*$, respectively;

(5) For every $x \in A \cap S_X$ and $y \in A \cap B_X$ with disjoint supports, and for every $\lambda \in [0,1]$, we have

$$|||\lambda x - (1-\lambda)(e_1 + 2y)||| = 1.$$

Proof. For convenience, we first prove (4), and then use it to prove (1) and (2). Fix $x^* \in E$ and let x_+^* and x_-^* be the positive and negative parts of $x^* - x^*(e_1)e_1^*$, respectively. Then $x^* = x^*(e_1)e_1^* + x_+^* - x_-^*$. Since

$$B_{(X,|||\cdot|||)} = \overline{\operatorname{conv}}\{\pm(e_1 + 2x) \colon x \in A \cap B_X\},$$

we get

$$\begin{aligned} |||x^*||| &= \sup_{x \in A \cap B_X} \big|x^*(e_1 + 2x)\big| \\ &= \sup_{x \in A \cap B_X} \big|x^*(e_1) + 2x_+^*(x) - 2x_-^*(x)\big| \\ &= \max\left\{\big|x^*(e_1) + 2\|x_+^*\|\big|, \big|x^*(e_1) - 2\|x_-^*\|\big|\right\}. \end{aligned}$$

The last equality relies strongly on the fact that (e_n) is 1-unconditional with respect to the original norm. Indeed, the latter implies that

$$\|y^*\| = \sup\{y^*(x) : x \in A \cap B_X,\ \operatorname{supp} x \subseteq \operatorname{supp} y^*\}$$

for every $y^* \in F$. Since x_+^* and x_-^* have disjoint supports, each of these functionals admits a norming set in $A \cap B_X$ on which the other functional vanishes.

Now we prove (1). Fix $z^* \in B_{(E,|||\cdot|||)}$ with finite support and $z^*(e_1) \geqslant 0$. Let z_+^* and z_-^* be the positive and negative parts of $z^* - z^*(e_1)e_1^*$, respectively. If $z^*(e_1) = 1$, then by (4), we get

$$\max\left\{\left|1 + 2\|z_+^*\|\right|, \left|1 - 2\|z_-^*\|\right|\right\} = |||z^*||| \leqslant 1,$$

which implies $z_+^* = 0$ and $\|z_-^*\| \leqslant 1$. Hence $z^* = e_1^* - z_-^*$ and $z_-^* \in F \cap B_{X^*}$.

Now suppose that $z^*(e_1) \in [0, 1)$. Then

$$z^* = z^*(e_1)e_1^* + z_+^* - z_-^* = z^*(e_1)\,(e_1^* - y^*) + \big(1 - z^*(e_1)\big)\frac{1}{2}\,(x^* - y^*)\,,$$

where $x^* = \frac{2}{1-z^*(e_1)}z_+^*$ and $y^* = \frac{2}{1+z^*(e_1)}z_-^*$. Furthermore,

$$\max\left\{\left|z^*(e_1) + 2\|z_+^*\|\right|, \left|z^*(e_1) - 2\|z_-^*\|\right|\right\} = |||z^*||| \leqslant 1,$$

which implies

$$\frac{2\|z_+^*\|}{1 - z^*(e_1)} \leqslant 1 \quad \text{and} \quad \frac{2\|z_-^*\|}{1 + z^*(e_1)} \leqslant 1,$$

and therefore $x^*, y^* \in F \cap B_{X^*}$.

We now prove (2). By (1), we have

$$B_{(E,|||\cdot|||)} \subseteq \overline{\operatorname{conv}}\left\{ \pm\,(e_1^* - x^*), \frac{1}{2}(x^* - y^*)\colon x^*, y^* \in F \cap B_{X^*}\right\}.$$

From (4), we get

$$|||e_1^* - x^*||| = \max\left\{\left|1 + 2\|0\|\right|, \left|1 - 2\|x^*\|\right|\right\} = 1$$

and

$$\frac{1}{2}\,|||x^* - y^*||| = \max\left\{\|x^*\|, \|y^*\|\right\} \leqslant 1$$

for all $x^*, y^* \in F \cap B_{X^*}$, which gives the other inclusion.

Next, we prove (3). To that end, we first show that the basis (e_n) is monotone with respect to the new norm. Then we will obtain from [AK16, Lemma 3.2.3] that

$$|||x||| = \sup_{z^* \in B_{(E,|||\cdot|||)}} \left|z^*(x)\right|$$

for every $x \in X$. Fix $n \in \mathbb{N}$ and let P_n be the partial sum projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$. Since (e_n) is 1-unconditional (hence monotone)

with respect to the original norm, we have $P_n(x) \in A \cap B_X$ for every $x \in A \cap B_X$. Therefore $P_n(e_1 + 2x) = e_1 + 2P_n(x) \in B_{(X,|||\cdot|||)}$, and thus $P_n\big(B_{(X,|||\cdot|||)}\big) \subseteq B_{(X,|||\cdot|||)}$, implying $|||P_n||| \leqslant 1$, as desired.

Now fix $x \in X$, and let x_+ and x_- be the positive and negative parts of $x - e_1^*(x)e_1$, respectively. By (2), we have

$$\begin{aligned}
|||x||| &= \sup_{z^* \in B_{(E,|||\cdot|||)}} \big|z^*(x)\big| \\
&= \max\left\{ \sup_{x^* \in F \cap B_{X^*}} |(e_1^* - x^*)(x)|, \frac{1}{2} \sup_{x^*, y^* \in F \cap B_{X^*}} |(x^* - y^*)(x)| \right\} \\
&= \max\left\{ \sup_{x^* \in F \cap B_{X^*}} \big|e_1^*(x) - x^*(x_+) + x^*(x_-)\big|, \right. \\
&\qquad \left. \frac{1}{2} \sup_{x^*, y^* \in F \cap B_{X^*}} \big(x^*(x_+) + y^*(x_-)\big) \right\} \\
&= \max\left\{ \big|e_1^*(x)\big|, \big|e_1^*(x) - \|x_+\|\big|, \big|e_1^*(x) + \|x_-\|\big|, \frac{1}{2}\big(\|x_+\| + \|x_-\|\big) \right\}.
\end{aligned}$$

Once again, the last two equalities rely strongly on the fact that (e_n) is 1-unconditional with respect to the original norm because the latter implies that every $y \in A$ attains its norm on the set

$$\{x^* \in F \cap B_{X^*} : \operatorname{supp} x^* \subseteq \operatorname{supp} y\},$$

and in particular, that x_+ and x_- both admit a norming functional in $F \cap B_{X^*}$ which vanishes at the other point.

Finally, we prove (5). Fix $x \in A \cap S_X$ and $y \in A \cap B_X$ with disjoint supports, and let $\lambda \in [0, 1]$. Then by (3), we have

$$\begin{aligned}
|||\lambda x - (1 - \lambda)(e_1 + 2y)||| &= \max\left\{1 - \lambda, 1, 1 - \lambda, \frac{1}{2}\big(\lambda + 2(1 - \lambda)\big)\right\} \\
&= 1. \qquad \square
\end{aligned}$$

Before proceeding with the proof of the main theorem, note that $|||e_1||| = 1$, $|||x||| = 1$ for every $x \in A \cap S_X$, and that $|||e_1 + 2y||| = 1$ for every $y \in A \cap B_X$. This provides a striaghtfoeward way to verify that e_1 is a super Δ-point, since (e_n) is weakly null and thus $e_1 + 2e_n \to e_1$ weakly. Analogously, observe that $|||e_1^*||| = 1$, $|||x^*||| = 2$ for every $x^* \in F \cap S_{X^*}$, and that $|||e_1^* - x^*||| = 1$ for every $x^* \in F \cap B_{X^*}$. Moreover, since $e_1^* - e_n^* \to e_1^*$ in the weak* topology, it follows that e_1^* is a weak* super Δ-point.

Proof of Theorem 4.4. Let us first prove (1). Fix $z \in B_{(X,|||\cdot|||)}$ with finite support. We will construct a sequence (w_n) in $B_{(X,|||\cdot|||)}$ such that $\|e_1 - w_n\| \to 2$ and (w_n) converges weakly to z.

Let z_+ and z_- be the positive and negative parts of $z - e_1^*(z)e_1$, respectively, and let $\lambda := \frac{1+e_1^*(z)}{2} \in [0,1]$. Define

$$x := \begin{cases} \frac{1}{2\lambda} z_+ & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0, \end{cases} \quad \text{and} \quad y := \begin{cases} \frac{1}{2(1-\lambda)} z_- & \text{if } \lambda < 1, \\ 0 & \text{if } \lambda = 1. \end{cases}$$

Note that if $\lambda = 0$, then $e_1^*(z) = -1$, and it follows from Lemma 4.5 (3) that $z_+ = 0$. Analogously, if $\lambda = 1$, then $z_- = 0$. So, in either case, we have

$$z = \lambda(e_1 + 2x) - (1-\lambda)(e_1 + 2y).$$

Clearly $\|x\| \leqslant 1$ if $\lambda = 0$, and by Lemma 4.5 (3) we have $\|z_+\| - e_1^*(z) \leqslant |||z||| \leqslant 1$ and thus

$$\|x\| = \frac{1}{2\lambda}\|z_+\| \leqslant \frac{1 + e_1^*(z)}{2\lambda} = 1,$$

if $\lambda \neq 0$. So for every $n \in \mathbb{N}$ large enough, we have $\|x\| \leqslant 1$ and $\|x + e_n\| \geqslant \|e_n\| = 1$. Thus there exists $a_n \in [0,1]$ such that $x + a_n e_n \in S_X$. Let $w_n = z + 2\lambda a_n e_n$. By construction, $w_n \in B_{(X,|||\cdot|||)}$, and since (e_n) is weakly null, we have $w_n \to z$ weakly. Moreover, $x + a_n e_n \in A \cap S_X$ and $\operatorname{supp}(x + a_n e_n) \cap \operatorname{supp} y = \varnothing$ for $n \in \mathbb{N}$ large enough, so by Lemma 4.5 (5), we get

$$|||e_1 - w_n||| = 2\,|||(1-\lambda)(e_1 + y) - \lambda(x + a_n e_n)||| = 2$$

as desired. Finally, as the set of all finitely supported elements is dense in $B_{(X,|||\cdot|||)}$, it follows that for every $w \in B_{(X,|||\cdot|||)}$, there exists a net (w_α) in $B_{(X,|||\cdot|||)}$ that converges weakly to w and satisfies $|||e_1 - w_\alpha||| \to 2$. Hence e_1 is a super Daugavet point in $(X, |||\cdot|||)$.

Next we prove (2). Fix $z^* \in B_{(E,|||\cdot|||)}$ with finite support. We will construct as above a sequence (w_n^*) in $B_{(E,|||\cdot|||)}$ such that $\|e_1^* - w_n^*\| \to 2$ and (w_n^*) converges weak* to z^*. Since the set of all finitely supported functionals is dense in $B_{(E,|||\cdot|||)}$, this proves that e_1^* is a weak* super Daugavet point in $(E, |||\cdot|||)$.

In fact, we will show that $\|e_1^* \pm w_n^*\| \to 2$. This gives us the necessary sequences for both z^* and $-z^*$, and thus we may assume without loss

of generality that $z^*(e_1) \geqslant 0$. By Lemma 4.5 (1), there exist $\lambda \in [0,1]$ and $x^*, y^* \in F \cap B_{X^*}$ such that

$$z^* = \lambda(e_1^* - y^*) + (1-\lambda)\frac{1}{2}(x^* - y^*).$$

Again, since $\|x^*\| \leqslant 1$ and $\|x^* + e_{2n}^*\| \geqslant \|e_{2n}^*\| = 1$ for large enough $n \in \mathbb{N}$, there exists $a_n \in [0,1]$ such that $x^* + a_n e_{2n}^* \in S_{X^*}$. Similarly, there exists $b_n \in [0,1]$ such that $y^* + b_n e_{2n+1}^* \in S_{X^*}$. Let

$$w_n^* = z^* + \frac{1-\lambda}{2} a_n e_{2n}^* - \frac{1+\lambda}{2} b_n e_{2n+1}^*.$$

Since (e_n^*) is weak* null, it follows that $w_n^* \to z^*$ in the weak* topology. Furthermore, $\operatorname{supp}(x^* + a_n e_{2n}^*) \cap \operatorname{supp}(y^* + b_n e_{2n+1}^*) = \varnothing$ for large enough $n \in \mathbb{N}$, so by Lemma 4.5 (4), we get

$$\begin{aligned}
|||e_1^* - w_n^*||| &= \left|\left|\left|(1-\lambda)e_1^* - \frac{1-\lambda}{2}(x^* + a_n e_{2n}^*) + \frac{1+\lambda}{2}(y^* + b_n e_{2n+1}^*)\right|\right|\right| \\
&= \max\{(1-\lambda) + (1+\lambda)\left\|y^* + b_n e_{2n+1}^*\right\|, \\
&\qquad (1-\lambda) - (1-\lambda)\left\|x^* + a_n e_{2n}^*\right\|\} \\
&= \max\{2, 0\} = 2,
\end{aligned}$$

and

$$\begin{aligned}
|||e_1^* + w_n^*||| &= \left|\left|\left|(1+\lambda)e_1^* + \frac{1-\lambda}{2}(x^* + a_n e_{2n}^*) - \frac{1+\lambda}{2}(y^* + b_n e_{2n+1}^*)\right|\right|\right| \\
&= \max\{(1+\lambda) + (1-\lambda)\left\|x^* + a_n e_{2n}^*\right\|, \\
&\qquad (1+\lambda) - (1+\lambda)\left\|y^* + b_n e_{2n+1}^*\right\|\} \\
&= \max\{2, 0\} = 2.
\end{aligned}$$

This concludes the proof. □

It is well known that every shrinking basis is weakly null (also referred to as *semi-shrinking*), and that every unconditional basis in a Banach space not containing a copy of ℓ_1 is shrinking (see, e.g., [AK16, Theorem 3.3.1]). So we immediately obtain the following corollary.

Corollary 4.6 (see [HLPV24, Corollary 4.8])**.** *Let X be a infinite-dimensional Banach space with an unconditional Schauder basis. If X does not contain a copy of ℓ_1, in particular, if X is reflexive, then X can be renormed so that X contains a super Daugavet point, and X^* contains a weak* super Daugavet point.*

Remark 4.7 (see [HLPV24, Remark 4.9]). If the sequence (e_n^*) of biorthogonal functionals is also assumed to be weakly null in Theorem 4.4, in particular, if the space X is reflexive, then the functional e_1^* is actually also a super Daugavet point in $(E, |||\cdot|||)$.

Note that the basis from [DD67, Examples 2] is unconditonal, boundedly complete, and semi-shrinking; however, as it is not shrinking, the space is not reflexive, so there exist non-trivial examples of this kind. Recall that, in general, diametral properties of points in a dual space are much stronger than their weak* counterparts. For example, every point in $S_{C[0,1]^*}$ is a weak* Daugavet point, while $B_{C[0,1]^*}$ contains denting points.

In particular, let us highlight that Corollary 4.6 yields the following theorem.

Theorem 4.8 (see [HLPV24, Theorem 4.10])**.** *Let (e_n) be the unit vector basis of ℓ_2. Then there exists a renorming of ℓ_2 such that e_1 is a super Daugavet point both in the new norm and in its dual norm (under the identification $\ell_2^* = \ell_2$).*

It was proved above in Theorem 4.1 that every infinite-dimensional Banach space can be renormed to admit a Δ-point. It is thus natural to ask the following question.

Question 4.1 (see [HLPV24, Question 4.12]). Can every infinite-dimensional Banach space be renormed to admit a Daugavet point?

By Theorem 4.4 and by [AAL$^+$24a, Theorem 2.1], the answer is affirmative if X is infinite-dimensional and has a weakly null unconditional Schauder basis, or if $X = \ell_1$. Combining these results, we will now prove that, more generally, the answer is affirmative for every infinite-dimensional Banach space with an unconditional Schauder basis, as well as for any Banach space that contains a complemented copy of such a space.

Note that some of the key ingredients for the construction of Δ-renormings in Theorem 4.1 include a classical norm extension result and the fact that Δ-points pass to superspaces. This is no longer true for Daugavet points; for example, it was shown in [AHLP20] that ℓ_2-sums of Banach spaces never admit Daugavet points. However, we will see that a similar conclusion can still be obtained, up to renorming, whenever the considered point lies in a subspace that is complemented in the superspace.

Proposition 4.9 (see [HLPV24, Proposition 4.13]). *Let X be a Banach space. If X contains a complemented subspace Y that can be renormed to admit a Daugavet point, then X can be renormed to admit a Daugavet point. Moreover, if Y can be renormed so that Y^* contains a (weak*) Daugavet point, then so can X. The same holds for (weak*) super Daugavet points.*

Proof. Since Y is complemented in X, we have that X is isomorphic to $Y \oplus_1 Z$ for some Banach space Z. As this space is also isomorphic to $(Y, |||\cdot|||) \oplus_1 Z$ for every equivalent norm $|||\cdot|||$ on Y, we may assume that Y already contains a Daugavet point. If $y \in S_Y$ is a Daugavet point, then by [HPV21, Proposition 2.3], the element $x := (y, 0)$ is a Daugavet point in $Y \oplus_1 Z$. For super Daugavet points, the result follows analogously using [MPRZ24, Remark 3.28]. Furthermore, since the dual of $(Y, |||\cdot|||) \oplus_1 Z$ is isometric to $(Y^*, |||\cdot|||) \oplus_\infty Z^*$, the dual case follows from known transfer results of Daugavet and super Daugavet points through ℓ_∞-sums (see the next remark). □

Remark 4.10 (see [HLPV24, Remark 4.14]). Had we taken an ℓ_∞-sum instead of an ℓ_1-sum in the proof of the previous result, then we would have obtained a renorming of X with infinitely many (super) Daugavet points. Indeed, by [HPV21, Proposition 2.4], if $y \in S_Y$ is a Daugavet point, then (y, z) is a Daugavet point in $Y \oplus_\infty Z$ for every $z \in B_Z$. The same applies to super Daugavet points, by [MPRZ24, Remark 3.28].

As a corollary, we get that every Banach space with an unconditional basis can be renormed to admit a Daugavet point and a weak* Daugavet point in its dual.

Corollary 4.11 (see [HLPV24, Corollary 4.15]). *Let X be a Banach space with an unconditional basis (e_n). Then X can be renormed with a Daugavet point. More precisely:*

(1) If (e_n) is shrinking, or if (e_n) is neither shrinking nor boundedly complete, then X can be renormed to admit a super Daugavet point and a weak super Daugavet point in its dual;*

(2) If (e_n) is not shrinking, then X can be renormed to admit a Daugavet point in the space and its dual.

Proof. If (e_n) is shrinking, then the claim follows from Corollary 4.6. The remaining cases are direct consequences of Proposition 4.9 together

with classical results by James concerning unconditional bases. Namely if (e_n) is not boundedly complete, then X contains a complemented copy of c_0; and if (e_n) is not shrinking, then X contains a complemented copy of ℓ_1 (see, e.g., [AK16, Theorems 3.3.2 and 3.3.1]). The fact that ℓ_1 can be renormed to admit a Daugavet point follows by combining Theorem 2.26 and Proposition 2.25. Existence of a Daugavet point in the dual space under this renorming follows from Proposition 3.10. □

Using once again Proposition 4.9, we now introduce the following theorem.

Theorem 4.12 (see [HLPV24, Theorem 1.3])**.** *Every infinite-dimensional Banach space that contains a complemented unconditional basic sequence can be renormed to admit a Daugavet point and a weak* Daugavet point in its dual.*

Finally, invoking further classical results from the literature, we obtain the following corollary.

Corollary 4.13 (see [HLPV24, Corollary 4.16])**.** *Let X be a Banach space. Then the following hold:*

(1) If X is separable and contains a copy of c_0, then X can be renormed to admit a super Daugavet point and a weak super Daugavet point in its dual;*

(2) If X^* contains a copy of c_0, then X can be renormed to admit a Daugavet point in the space and its dual;

(3) If X contains a copy of ℓ_∞, then X can be renormed to admit a super Daugavet point in the space and its dual.

Proof. Copies of c_0 in separable Banach spaces are always complemented (see, e.g., [AK16, Corollary 2.5.9]). Furthermore, copies of ℓ_∞ are complemented in arbitrary Banach spaces (see, e.g, [AK16, Proposition 2.5.2]). It is also a well-known result, due to Bessaga and Pełczyński, that if the dual X^* of a Banach space X contains a copy of c_0, then the space X contains a complemented copy of ℓ_1 (see, e.g., [FHHMZ11, Theorem 4.4]).

With these facts at hand, Corollary 4.13 immediately follows from Proposition 4.9 together with Corollary 4.11 and the well-known fact that ℓ_∞, as a $C(K)$-space, admits super Daugavet points (combining,

e.g., [AHLP20, Corollary 5.4] and [MPRZ24, Corollary 4.3]). That ℓ_∞^* also admits super Daugavet points can be obtained as follows. First, it is well-known that ℓ_∞^* contains an isometric copy of the space $L_1[0,1]$. Indeed, ℓ_∞ contains an isometric copy of ℓ_1, so the latter follows, e.g., from the results in [DGH00]. Since Δ-points pass to superspaces, it follows that ℓ_∞^* contains Δ-points. Now, since ℓ_∞^* is isometrically isomorphic to an $L_1(\mu)$-space, it follows from [MPRZ24, Corollary 4.1] that these point are in fact super Daugavet points. □

Observe that Corollary 4.13 (2) applies to any infinite-dimensional Lipschitz-free space. Moreover, a positive answer to the following question would immediately strengthen this result.

Question 4.2 (see [HLPV24, Question 4.17]). Can the space ℓ_1 be renormed to have a super Daugavet point?

Let us recall that it is currently unknown whether the Daugavet molecule from the example in Section 2.3 is a super Δ or a super Daugavet point.

Chapter 5

Properties preventing the existence of Daugavet and Delta-points

In this chapter, our goal is to find classes of Banach spaces that cannot contain Daugavet or Δ-points. From the previous chapter, we know that every infinite-dimensional Banach space can be renormed to admit a Δ-point. Therefore, aside from finite-dimensionality, there is no isomorphic property that prevents the existence of Δ-points. It is currently unknown whether the same holds for Daugavet points. However, since many infinite-dimensional Banach spaces can be renormed to contain a Daugavet point, it is plausible that no isomorphic obstruction exists for them either. Thus we focus on finding isometric conditions that rule out the presence of Daugavet or Δ-points. For example, we show that a Banach space with a shrinking k-unconditional basis for some $k < 2$ cannot contain Δ-points, and a Banach space with the Radon–Nikodým property, in which every extreme point of the unit ball is also a denting point, cannot contain Daugavet points. This chapter is based on [AAL+24a] and [AAL+24b].

5.1 Duality and non-containment of Delta-points

In this section, our main goal is to prove that whenever a Banach space X contains a $\mathfrak{D}$-point, or the dual X^* contains a weak* Δ-point, then

X^* also contains a weak* super Δ-point. As a consequence, we identify several classes of Banach spaces that cannot contain Δ-points. While we will not present all the consequences of Theorems 5.1 and 5.4 as discussed in [AAL$^+$24a], we aim to highlight those most directly related to the topics considered in this work, in particular, those involving Banach spaces with an unconditional basis.

We start by proving our main result of this section.

Theorem 5.1 (see [AAL$^+$24a, Theorem 5.6])**.** *Let X be a Banach space. If X contains a $\mathfrak{D}$-point, or if X^* contains a weak* Δ-point, then X^* contains a weak* super Δ-point.*

Proof. First, assume that $x \in S_X$ is a $\mathfrak{D}$-point. Let

$$D(x) = \{x^* \in S_{X^*} \colon x^*(x) = 1\}.$$

We distinguish between two cases.

Case 1. Assume that the set $D(x)$ contains exactly one element x^*. For every $n \in \mathbb{N}$ there exists $x_n \in S(x^*, 1/n)$, such that $\|x - x_n\| > 2 - 1/n$. Let $x_n^* \in S_{X^*}$ be such that $x_n^*(x) - x_n^*(x_n) > 2 - 1/n$. By weak*-compactness of B_{X^*}, there exists a subnet $(y_\alpha^*)_{\alpha\in\mathcal{A}}$ of (x_n^*) that converges in the weak* topology to some $y^* \in B_{X^*}$. Since $x_n^*(x) > 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$, in follows that $x_n^*(x) \to 1$, and thus also $y_\alpha^*(x) \to 1$. Hence $y^* \in D(x)$, i.e., $y^* = x^*$. Furthermore,

$$\|x^* - x_n^*\| \geqslant x^*(x_n) - x_n^*(x_n) > 2 - \frac{2}{n}$$

for every $n \in \mathbb{N}$. Therefore $\|x^* - x_n^*\| \to 2$ and thus also $\|x^* - y_\alpha^*\| \to 2$, meaning x^* is a weak* super Δ-point.

Case 2. Assume that the set $D(x)$ contains at least two distinct elements. Since $D(x)$ is convex, it msut be infinite. Denote by $\mathcal{A}$ the directed set of all finite subsets of $D(x)$, ordered by inclusion.

For every $A \in \mathcal{A}$, we have $\frac{1}{|A|}\sum_{x^*\in A} x^*(x) = 1$, thus there exists $x_A \in S_X$ such that

$$\frac{1}{|A|}\sum_{x^*\in A} x^*(x_A) > 1 - \frac{1}{|A|^2}$$

and $\|x - x_A\| > 2 - \frac{1}{|A|}$. Therefore there exists $x_A^* \in S_{X^*}$ such that $x_A^*(x - x_A) > 2 - \frac{1}{|A|}$. By weak*-compactness of B_{X^*}, there exists a subnet $(y_B^*)_{B\in\mathcal{B}}$ of $(x_A^*)_{A\in\mathcal{A}}$ that is weak*-convergent to some element

$y^* \in B_{X^*}$. Since $x_A^*(x) > 1 - \frac{1}{|A|}$ for every $A \in \mathcal{A}$, we get $x_A^*(x) \to 1$, and thus also $y_B^*(x) \to 1$. Hence $y^* \in D(x)$.

Fix $\alpha > 0$. Let $A_0 \in \mathcal{A}$ be such that $y^* \in A_0$ and $\frac{2}{|A_0|} < \alpha$. Then for all $A \succeq A_0$, we have $y^* \in A$, and thus

$$1 - \frac{1}{|A|^2} < \frac{1}{|A|} \sum_{x^* \in A} x^*(x_A) \leqslant \frac{1}{|A|} y^*(x_A) + \frac{|A| - 1}{|A|},$$

which implies $y^*(x_A) > 1 - \frac{1}{|A|}$. Hence,

$$\|y^* - x_A^*\| \geqslant y^*(x_A) - x_A^*(x_A) > 1 - \frac{1}{|A|} + 1 - \frac{1}{|A|} \geqslant 2 - \frac{2}{|A_0|} > 2 - \alpha.$$

Therefore $\|y^* - x_A^*\| \to 2$, and thus also $\|y^* - y_B^*\| \to 2$, meaning y^* is a weak* super Δ-point.

Now assume that $x_0^* \in S_{X^*}$ is a weak* Δ-point. Let $x_0 \in S_X$ be such that $x_0^*(x_0) > 1/2$. We construct recursively two sequences (x_i) and (x_i^*) in S_X and S_{X^*}. First, choose $x_1^* \in S(x_0, 1/2)$ such that $\|x_0^* - x_1^*\| > 2 - 1/4$, and choose $x_1 \in S_X$ such that $x_0^*(x_1) > 1 - 1/4$ and $x_1^*(-x_1) > 1 - 1/4$.

Assume that we have found $x_1, \ldots, x_{n-1} \in S_X$ and $x_1^*, \ldots, x_{n-1}^* \in S_{X^*}$ such that

$$-x_k^*(x_k) > 1 - \frac{1}{2^{k+1}} \quad \text{and} \quad \sum_{i=0}^{k-1} x_k^*(x_i) > k - 1 \quad \text{and} \quad x_0^*(x_k) > 1 - \frac{1}{2^{k+1}}$$

for every $k \in \{1, \ldots, n-1\}$. Then

$$x_0^*\Big(\sum_{i=0}^{n-1} x_i\Big) > \sum_{i=0}^{n-1} \Big(1 - \frac{1}{2^{i+1}}\Big) > n - 1.$$

Since x_0^* is a weak* Δ-point, there exist $x_n^* \in S_{X^*}$ such that

$$x_n^*\Big(\sum_{i=0}^{n-1} x_i\Big) > n - 1$$

and $\|x_0^* - x_n^*\| > 2 - 1/2^{n+1}$. Choose $x_n \in S_X$ such that $x_0^*(x_n) > 1 - 1/2^{n+1}$ and $-x_n^*(x_n) > 1 - 1/2^{n+1}$.

So we obtain two sequences (x_i) in S_X and (x_i^*) in S_{X^*} such that $x_i^*(x_i) \to -1$. Furthermore, we have

$$\sum_{i \in I} x_n^*(x_i) \geqslant \sum_{i=0}^{n-1} x_n^*(x_i) - (n - |I|) > n - 1 - (n - |I|) = |I| - 1$$

for every finite set $I \subseteq \{1, \dots, n-1\}$. As B_{X^*} is weak*-compact, there exist subnets $(y_\alpha)_{\alpha\in\mathcal{A}}$ and $(y^*_\alpha)_{\alpha\in\mathcal{A}}$ of the sequences (x_i) and (x^*_i), respectively, such that $(y^*_\alpha)_{\alpha\in\mathcal{A}}$ is weak*-convergent to some element $y^* \in B_{X^*}$. Since

$$x^*_n\Big(\frac{1}{|I|}\sum_{i\in I} x_i\Big) > 1 - \frac{1}{|I|}$$

for every finite set I and $n > \max I$, we get

$$y^*\Big(\frac{1}{|I|}\sum_{i\in I} x_i\Big) > 1 - \frac{1}{|I|}.$$

We now show that there exists a subnet $(z_\beta)_{\beta\in\mathcal{B}}$ of $(y_\alpha)_{\alpha\in\mathcal{A}}$ such that $y^*(z_\beta) \to 1$. Fix $\gamma > 0$ and let $\alpha_0 \in \mathcal{A}$. We can choose a finite set $A \subseteq \{\alpha \in \mathcal{A}\colon \alpha \succeq \alpha_0\}$ such that $1/|A| < \gamma$ and all elements in A correspond to different elements in $\mathbb{N}$. Then

$$y^*\Big(\frac{1}{|A|}\sum_{\alpha\in A} y_\alpha\Big) > 1 - \frac{1}{|A|}$$

and thus there exists $\alpha \in A$ such that $y^*(y_\alpha) > 1 - 1/|A| > 1 - \gamma$. It follows that 1 is a cluster point of the net $(y^*(y_\alpha))_{\alpha\in\mathcal{A}}$, and therefore there exist subnets $(z_\beta)_{\beta\in\mathcal{B}}$ and $(z^*_\beta)_{\beta\in\mathcal{B}}$ of nets $(y_\alpha)_{\alpha\in\mathcal{A}}$ and $(y^*_\alpha)_{\alpha\in\mathcal{A}}$ such that $y^*(z_\beta) \to 1$. As we are working with subnets, $z^*_\beta(z_\beta) \to -1$ by construction of the original sequences, so

$$\|y^* - z^*_\beta\| \geqslant y^*(z_\beta) - z^*_\beta(z_\beta) \to 2.$$

Therefore y^* is a weak* super Δ-point. □

Remark 5.2 (see [AAL$^+$24a, Remark 5.7]). It is known that there exist Δ-points that are not super Δ, and even Banach spaces with large subsets of Daugavet-points that contain no super Δ-points (combine [ALMT21, Proposition 2.12] and [ALMT21, Theorem 3.1] or [ALMT21, Theorem 4.7]). So the previous result is completely specific to the weak* topology, and it is evident from the proof that the key ingredient is the weak* compactness of the dual unit ball.

First, notice that as a direct consequence of Theorem 5.1 and Theorem 4.1, we obtain the following.

Corollary 5.3 (see [AAL$^+$24a, Corollary 5.8]). *Let X be an infinite-dimensional Banach space. Then X can be renormed so that X admits*

a Δ-point and X^* admits a weak super Δ-point. Moreover, if X fails the Schur property, then X can be renormed so that X admits a super Δ-point and X^* admits a weak* super Δ-point.*

Before continuing with the restrictions imposed by Theorem 5.1 on Banach spaces that contain Δ-points, let us note that if a Banach space X is such that B_{X^*} is weak* sequentially compact, then the proof of that theorem could have been carried out using sequences instead of nets. Motivated by this observation, we introduce the following theorem.

Theorem 5.4 (see [AAL+24a, Theorem 5.23])**.** *Let X be a Banach space such that B_{X^*} is weak* sequentially compact. If X contains a $\mathfrak{D}$-point, or if X^* contains a weak* Δ-point, then X^* contains a sequential weak* super Δ-point.*

Proof. If X contains a $\mathfrak{D}$-point, then by Theorem 5.1, the dual space X^* admits a weak* super Δ-point, and hence a weak* Δ-point. Therefore it suffices to prove the second part of the statement.

Assume that X^* contains a weak* Δ-point x^*. Looking back at the second part of the proof of Theorem 5.1, it is clear that we can use weak* sequential compactness to extract sequences instead of nets at each key step. Hence, we obtain a sequential weak* super Δ-point y^* in X^*, and the conclusion follows. □

Combining Theorem 5.4 with the well-known fact that the dual unit ball of a reflexive Banach space is weak sequentially compact, we get the following.

Corollary 5.5 (see [AAL+24a, Corollary 5.25])**.** *Let X be a reflexive Banach space. If X contains a $\mathfrak{D}$-point, then both X and X^* contain a sequential super Δ-point.*

Now we proceed with the following result, which will be used to show that a Banach space with shrinking k-unconditional basis can contain Δ-points only if $k \geqslant 2$.

Theorem 5.6 (see [AAL+24a, Theorem 5.12])**.** *Let X be a Banach space. Assume that there exists a family $(T_\lambda)_{\lambda\in\Lambda}$ of compact operators on X with $\sup_{\lambda\in\Lambda} \|I_X - T_\lambda\| < 2$, such that $T_\lambda^* \to I_{X^*}$ in the strong operator topology. Then X contains no $\mathfrak{D}$-points, and X^* contains no weak* Δ-points.*

Proof. Let $\varepsilon > 0$ be such that

$$\sup_{\lambda\in\Lambda} \|I - T_\lambda\| + 4\,\varepsilon \leqslant 2.$$

By Theorem 5.1, it is enough to show that X^* contains no weak* super Δ-points. Hence it suffices to prove that for any net $(x^*_\alpha)_{\alpha\in\mathcal{A}} \subseteq B_{X^*}$ converging weak* to some $y^* \in B_{X^*}$, and for every $\alpha_0 \in \mathcal{A}$, there exists $\alpha \succeq \alpha_0$ such that

$$\|y^* - x^*_\alpha\| \leqslant 2 - 2\,\varepsilon\,.$$

Pick $\lambda \in \Lambda$ such that $\|y^* - T^*_\lambda y^*\| \leqslant \varepsilon$. By Schauder's theorem, T^*_λ is continuous as a map from (B_{X^*}, w^*) to $(X^*, \|\cdot\|)$ (see, e.g., [BK14, Theorem 6.26]) and it follows that $(T^*_\lambda(x^*_\alpha))_{\alpha\in\mathcal{A}}$ converges in norm to $T^*_\lambda(y^*)$. Thus there exists $\alpha_0 \in \mathcal{A}$ such that $\|T^*_\lambda(y^* - x^*_\alpha)\| \leqslant \varepsilon$ for every $\alpha \succeq \alpha_0$.

Then for every $\alpha \succeq \alpha_0$, we get

$$\begin{aligned}\|y^* - x^*_\alpha\| &\leqslant \|y^* - T^*_\lambda y^*\| + \|T^*_\lambda(y^* - x^*_\alpha)\| + \|x^*_\alpha - T^*_\lambda x^*_\alpha\| \\ &\leqslant \|I^* - T^*_\lambda\| + 2\,\varepsilon \\ &\leqslant 2 - 2\,\varepsilon,\end{aligned}$$

and the conclusion follows. □

Let us now collect a few corollaries from Theorem 5.6. If X is a Banach space with shrinking basis (e_k), then $P^*_k \to I_{X^*}$ in the strong operator topology, where P_k are the partial sum projections. Thus we get the following result.

Corollary 5.7 (see [AAL$^+$24a, Corollary 5.14])**.** *Let X be a reflexive Banach space with a shrinking basis and partial sum projections (P_k). If $\sup_{k\in\mathbb{N}} \|I - P_k\| < 2$, then X and X^* contain no $\mathfrak{D}$-points.*

In particular, if X is a reflexive Banach space with a k-unconditional basis for some $k < 2$, then X and X^* contain no $\mathfrak{D}$-points.

Remark 5.8. Note that the unconditionality constant of the new norm in Section 4.2 is at least 2, since by Lemma 4.5 (3), we have $|||e_1 + e_2||| = 1$ and $|||e_2||| = 2$. This illustrates the interaction between Theorem 4.4 and Corollary 5.7.

In [ALMT21, Theorem 2.17] it was shown that Banach spaces with a subsymmetric basis contain no Δ-points, and this can be applied to show that the Schlumprecht space has no Δ-points. Unlike previous results in this direction, Corollary 5.7 also applies to the Tsirelson space and many of its relatives.

From Theorem 5.6 and the classic results by James on bases in Banach spaces, we obtain the following.

Corollary 5.9 (see [AAL$^+$24a, Corollary 5.15])**.** *Let X be a Banach space with a k-unconditional basis (e_n), where $k < 2$.*

(1) If (e_n) is shrinking, then X contains no $\mathfrak{D}$-points and X^* contains no weak Δ-points.*

(2) If (e_n) is monotone and boundedly complete, then X contains no Δ-points.

In particular, any Banach space with a 1-unconditional basis that admits a Δ-point contains a copy of both c_0 and ℓ_1.

Proof. The first part follows directly from Theorem 5.6, using the partial sum projections.

If X has a monotone, boundedly complete basis (e_n), then the biorthogonal functionals (e_n^*) form a shrinking basis for $Z = \overline{\operatorname{span}}\,(e_n^*)$, and X is isometric to Z^* (see, e.g, [AK16, Theorem 3.2.15]). The unconditionality constants for (e_n) and (e_n^*) are the same, hence the second part follows from the first.

Finally, if (e_n) is an unconditional basis for X, then it fails to be shrinking if and only if X contains a subspace isomorphic to ℓ_1 (see, e.g., [AK16, Theorem 3.3.1]), and it fails to be boundedly complete if and only if X contains a subspace isomorphic to c_0 (see, e.g., [AK16, Theorem 3.3.2]). Since every 1-unconditional basis is monotone, the final statement follows from the previous two. □

We have one final corollary of this section. Recall that $\mathfrak{D}$-points can be lifted from subspaces.

Corollary 5.10 (see [AAL$^+$24a, Corollary 5.17])**.** *Let X be a subspace of a Banach space with a shrinking k-unconditional basis for some $k < 2$. Then X contains no $\mathfrak{D}$-points.*

5.2 Non-containment of Daugavet points

In the final section, we investigate structural restrictions that prevent the existence of Daugavet points. We begin by showing that any Banach space with the Radon–Nikodým property that contains a Daugavet point must also contain a Daugavet point that is an extreme point of the unit ball. In fact, we can establish this result for Banach spaces with the Krein–Milman property, provided they satisfy the distance-to-denting characterization for Daugavet points. This is, at least formally, more general, as it remains unknown whether the Krein–Milman property and the Radon–Nikodým property coincide for Banach spaces. Recall that a Banach space has the *the Krein–Milman property* if every nonempty bounded closed convex set is the closed convex hull of its extreme points.

Proposition 5.11 (see [AAL+24b, Proposition 3.21])**.** *Let X be a Banach space with the Krein–Milman property that satisfies the distance-to-denting characterization for Daugavet points (e.g., a Banach space with the Radon–Nikodým property). If X contains a Daugavet point, then X contains a Daugavet point that is also an extreme point of B_X.*

Proof. Assume that $x \in S_X$ is a Daugavet point, and define

$$F_x := \bigcap_{x^* \in D(x)} \{y \in B_X : x^*(y) = x^*(x) = 1\},$$

where $D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$. Observe that F_x is a non-empty face of B_X.

Let $y \in F_x$. We will first prove that y is a Daugavet point. Since x is a Daugavet point, we have $\|x - z\| = 2$ for every $z \in \operatorname{dent}(B_X)$. By the Hahn–Banach theorem, we can find a functional $z^* \in S_{X^*}$ such that $z^*(x - z) = 2$. Then $z^*(x) = -z^*(z) = 1$, and in particular $z^* \in D(x)$. Since $y \in F_x$, we also have $z^*(y) = 1$, and thus

$$\|y - z\| \geqslant z^*(y - z) = 2.$$

Hence $\|y - z\| = 2$ for all $z \in \operatorname{dent}(B_X)$, and since X satisfies the distance-to-denting characterization for Daugavet points, it follows that y is a Daugavet point.

Now, since F_x is a non-empty, closed, bounded, and convex subset of X, and since X has the Krein–Milman property, F_x admits an

extreme point y. As F_x is a face, y is also an extreme point of B_X. Thus we have found a Daugavet point that is also an extreme point of B_X. □

The above immediately yields the following theorem.

Theorem 5.12 (see [AAL+24b, Theorem 3.22])**.** *Let X be a Banach space. If X has the Krein–Milman property, and if every extreme point of B_X is a denting point, then X contains no Daugavet points.*

Proof. If X has the Krein–Milman property, and if every extreme point of B_X is a denting point, then B_X is the closed convex hull of the set of all denting points of B_X. Therefore, X satisfies the distance-to-denting characterization for Daugavet points. The conclusion then immediately follows from Proposition 5.11. □

The space in Section 2.3 is a separable dual space and hence has the Krein–Milman property. Furthermore, the Daugavet point m_{xy} in the example is also an extreme point by [AP20, Theorem 1.1]. This example shows the necessity of the assumption that every extreme point of the unit ball is a denting point in Theorem 5.12.

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Daugaveti ja Delta punktid Banachi ruumides: Lipschitzi-vabades ruumides, nende kaasruumides ja ümbernormeeringutes

Kokkuvõte

Käesolevas väitekirjas uuritakse Daugaveti ja Δ-punkte, mis on vastavalt Daugaveti omaduse ja diametraalse lokaalse diameeter-kahe omaduse punktilised versioonid, Banach ruumides. Banachi ruumil on Daugaveti omadus parajasti siis, kui iga $x \in S_X$ ja iga $\varepsilon > 0$ korral

$$B_X = \overline{\operatorname{conv}}\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\}.$$

Daugaveti omadus seab Banachi ruumile mitmeid tugevaid geomeetrilisi tingimusi. Näiteks iga Banachi ruum, millel on Daugaveti omadus, sisaldab klassikalist ruumi ℓ_1 ning ei saa omada tingimatut baasi. Samuti on sellise ruumi ühikkera iga viilu diameeter 2. Seetõttu võib Daugaveti omadust käsitleda Radon–Nikodými omaduse vastandina – nimelt eeldab Radon–Nikodými omadus, et leidub ühikkera viile, millel diameeter on kuitahes väike.

Banachi ruumi X elementi $x \in S_X$ nimetatakse Daugaveti punktks, kui $B_X = \overline{\operatorname{conv}}\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\}$ iga $\varepsilon > 0$ korral. Definitsioonist on koheselt selge, et Banachi ruumil on Daugaveti omadus parajasti siis, kui iga ühiksfääri element on Daugaveti punkt. Elementi $x \in S_X$ nimetatakse Δ-punktiks, kui $x \in \overline{\operatorname{conv}}\{y \in B_X \colon \|x - y\| \geqslant 2 - \varepsilon\}$ iga $\varepsilon > 0$ korral. Mõistagi on iga Daugaveti punkt ka Δ-punkt, kuid vastupidine ei pruugi kehtida – leidub näiteid elementidest, mis on Δ-punktid, kuid mitte Daugaveti punktid. Daugaveti ja Δ-punktid on aktiivne uurimissuund, mis alates nende defineerimisest 2020 aastal, on pälvinud märkimisväärset tähelepanu. Neid on uuritud mitmetes klassikalistes ruumides, näiteks $L_1(\mu)$ ruumides ja nende eelruumides, $C(K)$ ruumides ning Banachi ruumide otsesummades. On leitud ka näiteid Banachi ruumidest, millel on tingimatu baas, kuid mis sisaldavad Daugaveti punkte, ning samuti näiteid Radon–Nikodými omadusega ruumidest, mis sisaldavad

Daugaveti punkte. Need näited illustreerivad olulisi erinevusi Daugaveti omaduse ja Daugaveti punktide vahel.

Väitekiri koosneb viiest peatükist, mille eesmärk on uurida Daugaveti ja Δ-punkte erinevates Banachi ruumides.

Esimeses peatükis tutvustatakse põhilisi definitsioone ning üldisi tulemusi, mida kasutatakse ülejäänud peatükkides.

Teises peatükis keskendutakse Daugaveti ja Δ-punktide uurimisele Lipschitzi-vabades ruumides. Tõestatakse, et ühiksfääri element on Daugaveti punkt parajasti siis, kui ta on kaugusel 2 kõigist ühikkera hammaspunktidest. Samuti püstitatakse hüpotees Δ-punkte iseloomustava kriteeriumi kohta ning see tõestatakse mitmetel juhtudel. Lisaks antakse meetriline kriteerium molekulidele, mis on Δ-punktid. Tuuakse näide Lipschitzi-vabast ruumist, millel on Radon–Nikodým omadus ning mis sisaldab Daugaveti punkti. Veelgi enam, tõestatakse, et see ruum on kaasruum ja on isomorfne ruumiga ℓ_1. Peatüki lõpus uuritakse Daugaveti ja Δ-punkte Lipschitzi-vabades ruumides üle $\mathbb{R}$-puude alamhulkade.

Kolmandas peatükis uuritakse Daugaveti ja Δ-punkte Lipschitzi funktsiooniruumides ehk Lipschitzi-vabade ruumide kaasruumides. Näidatakse, et Daugaveti ja Δ-punktid langevad kokku juhul kui vastav meetriline ruum on kompaktne. Tõestatakse, et iga Lipschitzi funktsiooniruum sisaldab Δ-punkti ning suur osa neist ruumidest sisaldab ka Daugaveti punkti. Samas antakse näide sellisest Lipschitzi funktsiooniruumist, mis ei sisalda ühtegi Daugaveti punkti. Seega ei lange Daugaveti ja Δ-punktid selles ruumide klassis kokku.

Neljandas peatükis keskendutakse Banachi ruumide ümbernormeerimisele nii, et saadud ruumid sisaldavad Daugaveti või Δ-punkte. Motivatsiooniks on küsimus, kas leidub refleksiivne Banachi ruum, mis sisaldab Daugaveti või Δ-punkte. Tõestatakse, et igal lõpmatumõõtmelisel Banachi ruumil leidub ekvivalentne norm, mille puhul ruum sisaldab Δ-punkti. Ei ole teada, kas sama väide kehtib Daugaveti punktide kohta. Samas näidatakse, et Banachi ruumidel, millel on tingimatu baas, leidub ekvivalentne norm, mille järgi ruum sisaldab Daugaveti punkti. Seega on iga klassikaline ruum ℓ_p, kus $p \in [0, \infty)$, ümbernormeeritav nii, et see sisaldab Daugaveti punkti, mis annab ka soovitud näite.

Viiendas peatükis keskendutakse Banachi ruumide omadustele, mis välistavad Daugaveti või Δ-punktide olemasolu. Tõestatakse, et kui Banachi ruum sisaldab $\mathfrak{D}$-punkti või kui tema kaasruum sisaldab $*$-

nõrka Δ-punkti, siis kaasruum sisaldab ka $*$-nõrka super Δ-punkti. Selle tulemusena järeldatakse, et Banachi ruum, millel on pingul k-tingimatu baas, kus $k < 2$, ei saa sisaldada $\mathfrak{D}$-punkte ega ka Δ-punkte. Samuti näidatakse, et kui Radon–Nikodými omadusega Banachi ruum sisaldab Daugaveti punkti, siis peab leiduma Daugaveti punkt, mis on samal ajal ka ühikkera ekstreemumpunkt. Sellest tulenevalt ei saa Radon–Nikodými omadusega Banachi ruum, mille kõik ekstreemumpunktid on hammaspunktid, sisaldada Daugaveti punkte.

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