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ON THE CAPTURE OF COMETS BY PLANETS, ESPECIALLY THEIR CAPTURE BY JUPITER.

By H. A. NEWTON.

1. Some years ago I obtained and published* a formula expressing in simple terms the total result of the action of a planet in increasing or diminishing the velocity of a comet or small body that passes near the planet. This formula is practically a modification of the integral of energy, the smaller terms in the perturbing function being omitted. A very brief and partial treatment of it was presented to the British Association for the Advancement of Science in 1879, at its Sheffield meeting.† Within the last two or three years several astronomers have made special study of the manner of Jupiter's action in changing the orbits of comets that pass very near him. M. Tisserand has given us an expression connecting the major axis, inclination and parameter of the orbit described before coming near to Jupiter with the corresponding elements of the orbit after leaving the neighborhood of the planet.‡ M. Schulhof has applied the formula of M. Tisserand as a criterion for determining the possible identity of various comets whose orbits pass near to Jupiter's orbit.§ Messrs. Seeliger, Callandreaux, and others have continued these investigations. The interest thus shown in the problem has led me to resume the study of the subject, and to work out the results of the formula obtained by me in 1878 more fully than they have been hitherto developed.

2. One of the remarkable distinctions between the comets of long (or infinite) periods, and those of short periods, is that the orbits of the latter have almost without exception direct motions and small inclinations to the plane of the ecliptic, while the orbits of the former have all possible inclinations between 0° and 180° . At first sight this seems to imply that the two groups of comets are radically distinct in origin or nature one from the other. The most natural line of investigation therefore is the effect of perturbations in bringing or not bringing the comets to move with the planet after the perturbation.

3. The algebraic processes by which was obtained the formula for the change of energy which a small body experiences from passing near a planet were given in the article cited, and they need not be here reproduced. The following was the resulting equation, viz:

$$\Delta = -\frac{4mfa^2v \cos \varphi \sin \alpha}{pv_0}, \quad (1)$$

and it was obtained from the general differential equations of motion by making assumptions not greatly differing from those used in obtaining Laplace's well-known theorem, that a sphere of suitable magnitude may be described about the planet as a center and that for a tolerable first approximation the comet may be regarded as moving when without this sphere in a conic section of which the sun is the focus, and as moving when within the sphere in a conic section (an hyperbola) of which the planet is the focus. In other words, only perturbations of the first order of

* American Journal of Science, III, Vol. xvi, p. 175, 1878.

† Report, 1879, p. 274.

‡ Sur la théorie de la capture des comètes périodiques, Bull. Astron., Tome vi, juin and juillet, 1889.

§ Notes sur quelques Comètes à courte période, Astron. Nachrichten, No. 2964.

magnitude are taken account of. A comet is treated throughout this paper as a small indivisible body whose mass may be neglected.

4. *Notation.*—The symbols used in (1) and also other symbols which I shall have occasion to use may be thus defined:

Let $\mathcal{C}_1$ = orbit of the comet about the sun before the comet comes under the appreciable action of the planet;

$\mathcal{C}$ = orbit of the comet about the sun after perturbation by the planet;

C = hyperbolic orbit of the comet relative to Jupiter when near the planet;

$\mathcal{J}$ = elliptic orbit of Jupiter about the sun;

A = point on $\mathcal{C}_1$ which is nearest to $\mathcal{J}$;

E = point on $\mathcal{J}$ which is nearest to $\mathcal{C}_1$;

d = length of the straight line EA being the perpendicular distance between the orbits at their nearest approach;

ω = angle between the tangent of $\mathcal{C}_1$ at A and the tangent to $\mathcal{J}$ at E;

h = distance which the planet has yet to pass over to reach E when the comet is at A (h may be negative);

m = mass of the planet, sun's mass = unity;

a = unit of distance, in general the mean distance of the earth from the sun;

f = sun's attractive force at the unit of distance;

v_1 = planet's velocity in its orbit at E;

v_0 = comet's velocity in its orbit C when the comet enters the sphere of Jupiter's perceptible influence;

v = comet's velocity at A relative to the sun;

$s = v_0/v_1$;

ω_1 = semiaxis major of $\mathcal{C}_1$ (negative if $\mathcal{C}_1$ is an hyperbola);

ω = semiaxis major of $\mathcal{C}$ (negative if $\mathcal{C}$ is an hyperbola);

p = perpendicular from the planet upon asymptote to C;

α = acute angle between the transverse axis of C and the asymptote to C.

φ = angle between the tangent to $\mathcal{J}$ at O (drawn in the direction of the planet's motion) and the line from the planet to the vertices and center of C;

A = semitransverse axis of C;

B = semiconjugate axis of C (hence equal to p);

r = distance of the planet from the sun;

r_1 = distance of the comet from the sun;

r_0 = distance of the comet from the planet;

ρ , and ρ = distances of the comet from the sun at selected epochs before and after perturbation;

u , and u = velocities of the comet at the selected epochs;

Δ = increase which $v^2 - \frac{2fa^2}{r_1} - \frac{2mfa^2}{r_0}$ receives by the planet's action during the whole period in which the comet is passing near to Jupiter.

5. If we assume two epochs, one before and one after the perturbation, at which the comet is equally distant from the planet, the term $2mfa^2/r_0$ is the same at both instants, and it disappears from the value of Δ . Therefore

$$\Delta = u^2 - \frac{2fa^2}{\rho} - u_1^2 + \frac{2fa^2}{\rho_1}.$$

But the well-known formulas from the law of gravitation,

$$u_1^2 = 2fa^2 \left(\frac{1}{\rho_1} - \frac{1}{2\omega_1} \right),$$

and

$$u^2 = 2fa^2 \left(\frac{1}{\rho} - \frac{1}{2\omega} \right);$$

give

$$\Delta = fa^2 \left(\frac{1}{\omega_1} - \frac{1}{\omega} \right),$$

that is, from (1) $\frac{1}{\omega_1} - \frac{1}{\omega} = -\frac{4m \cos \varphi \sin \alpha}{ps}$.

This equation is valid whatever be $@$, the major axis of the orbit $\mathcal{C}$, and may be used to determine the major axis of either orbit from the elements of the other. My present purpose is, however, to study the action of Jupiter in changing orbits that are originally parabolas, and hence in general $@$, will be taken infinite. In that case

$$a = \frac{ps}{4m \cos \varphi \sin \alpha}. \quad (2)$$

It will be found that the second number of (2) depends on ω , d and h , and these are known quantities when the elements of $\mathcal{C}$, and $\mathcal{H}$ are given. The use of the equation is moreover greatly simplified and enhanced by the fact that the plane of the planet's orbit is involved only in so far as that it must contain the tangent to $\mathcal{H}$ at E.

6. In the second member of (2) all the factors are positive except $\cos \varphi$, hence, if $\varphi < \frac{1}{2}\pi$, $@$ is positive and the orbit $\mathfrak{C}$ is an ellipse; but, if $\varphi > \frac{1}{2}\pi$, $@$ is negative and $\mathfrak{C}$ is an hyperbola. This result may be thus expressed; *if the comet passes in front of Jupiter the kinetic energy of the comet is diminished; if it passes behind the planet the kinetic energy of the comet is increased.* The reason for this may also be given in general language. If the comet passes in front of the planet the comet's attraction increases the velocity, and hence increases the kinetic energy of the planet and *vice versa*. But the total energy of the two bodies is constant; so that when that of the planet is increased, that of the comet is diminished and *vice versa*.

7. It is desirable now to transform the value of $@$ given in equation (2) so as to be able to determine the major axis of the new orbit of the comet directly from the circumstances of its initial approach to the planet before perturbation; in other words, to find $@$ in terms of ω , d and h . For this we must find in terms of ω , d and h , values for s , p , α and φ .

8. *To find s.*—In Fig. 1 let A and E represent the two points A and E as defined above (Art. 4), and the line AE represent d . Let AY be the tangent to $\mathcal{C}$, at A, and EO the tangent to $\mathcal{M}$ at E. It is an admissible supposition that the planet is describing the *straight* line OE, and that the comet in its unperturbed orbit is describing the *straight* line YA. At some certain moment the line joining the planet and the unperturbed comet must evidently be perpendicular to OE. Let OY be the line joining the bodies at that moment, so that the planet is at O when the comet is at Y, and EOY is a right angle. Instead, however, of supposing the planet to move from O toward E we may apply an equal, opposite motion to the comet, and consider the planet to remain at rest at O. Draw AC parallel to EO and make AB equal to the distance described by the planet during the time that the comet is moving from Y to A. Join YB. Then since YA and BA represent in direction and magnitude the motions of the two bodies in a given interval, the third side YB of the triangle represents in magnitude and direction the motion of the comet relative to the planet. The angle YAB is the angle ω , and the three sides of the triangle YA, YB, and BA are proportional to v , v_0 and v_* . Let the angle YBC be θ ; then from the triangle YAB we have

$$v_0^2 = v_1^2 - 2v_1 v \cos \omega + v^2,$$

and

$$v:v_i:v_o::\sin \theta:\sin (\theta-\omega):\sin \omega. \quad (3)$$

Since v and v_r can be computed from the given elements of the orbits of the planet and comet, we may readily compute from ω the value of s , or v_0/v_r . But if the planet is at its mean distance from the sun, and the comet's orbit is parabolic, $v^2 = 2v_r^2$, and we have

$$s^2 = 3 - 2\sqrt{2} \cos \omega. \quad (4)$$

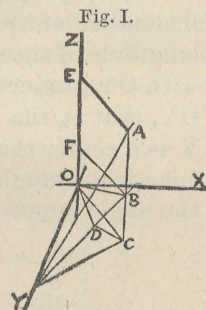
Also from the triangle

$$2v_i^2 = v_o^2 + 2v_o v_i \cos \theta + v_i^2,$$

or

$$2s \cos \theta = 1 - s^2. \quad (5)$$

9. To find p .—The planet being regarded at rest at O and the *relative* unperturbed motion of the comet being along YB, this line may within admissible limits of error be treated as one asymptote of the relative orbit C. The perpendicular from O upon YB will then be by definition (Art. 4) the line p . Draw OX from O perpendicular to OY and OE, and let these three lines be coordinate



axes. Let the line AB meet the plane XOY in C. Join OC, let fall OD perpendicular to YB, and join CD. Since EA is perpendicular to AY and also to EO, and so to its parallel line AC, therefore it is perpendicular to the plane YAC. Hence OC, parallel to EA is perpendicular to the plane, and so perpendicular to CD. Again, CDY is a right angle; for $OD^2 + DY^2 = OY^2 = OC^2 + CY^2$, and $OD^2 = OC^2 + DC^2$. Hence, $DC^2 + DY^2 = CY^2$, and consequently CDY is a right angle.

The quantity h is the line BC; for h is the distance which the planet, when the comet is at A, has yet to pass over before reaching E. But the comet was at Y when the planet was at O, and the planet describes BA, while the comet describes YA, leaving BC as the distance yet to be described, or h . But the angle CBD is θ , so that we have

$$p^2 = OD^2 = OC^2 + CD^2 = d^2 + h^2 \sin^2 \theta. \quad (6)$$

10. To find α .—The angle α is the acute angle between the asymptote and the transverse axis of the hyperbola, and hence from the nature of the hyperbola $\tan \alpha = B/A$. By known formulas we have, if the planet is at its mean distance

$$v_i^2 = 2fa^2 \left(\frac{1}{r} - \frac{1}{2r} \right),$$

$$v_o^2 = 2mfa^2 \left(\frac{1}{\alpha} + \frac{1}{2A} \right).$$

Therefore

$$\left. \begin{aligned} \frac{v_o^2}{v_i^2} &= \frac{mr}{A}, \text{ or } A = \frac{mr}{s^2}. \\ \tan \alpha &= \frac{B}{A} = \frac{p}{A} = \frac{s^2 (d^2 + h^2 \sin^2 \theta)^{\frac{1}{2}}}{mr}. \end{aligned} \right\} \quad (7)$$

11. To find φ .—The orbit of the comet relative to Jupiter lies in the plane YOB. Let i be the inclination of the plane YOB to YOX, measured positive from x positive to z positive; let l be the longitude of the direction YC, measured in the plane YOX from OY, that is, the angle made by YC with OY produced; let λ be the longitude of the direction YB measured in the plane YOB from OY, that is, the angle made by YB with OY produced. Imagine now a sphere described about Y as a center that shall cut the three planes XOY, BOY and BCY in three sides of a right-angled spherical triangle. The hypotenuse of this triangle is λ , the base l , the perpendicular $\frac{1}{2}\pi - \theta$, and the angle opposite to the perpendicular is i ; hence we have

$$\cos \lambda = \cos l \sin \theta, \quad (8)$$

$$\cos \theta = \sin i \sin \lambda, \quad (9)$$

$$\cot i = \sin l \tan \theta. \quad (10)$$

Also from the triangles OCY and BCY

$$\tan l = \tan OYC = - \frac{OC}{YC} = - \frac{d}{h \tan \theta}. \quad (11)$$

The angle φ is by definition the angle between the direction OE, and a line in the plane YOB that makes with YB an angle α . Hence we have readily

$$\cos \varphi = \sin i \sin (\lambda \pm \alpha). \quad (12)$$

These equations enable us to compute φ in terms of d , h and ω ; for in succession θ may be computed by (3), l by (11), λ by (8), i by (10), and φ by (12).

12. These values of s , p , x and α give by equation (2) the value of ω . The suppositions that the planet is at its mean distance, and that $\mathcal{C}$ is a parabola, are involved in that equation, but they are not necessary to the determination of ω when no such hypotheses are made, and changes in the equation that are not serious would make it applicable without these limitations. The quantities in the several equations may be regarded as having values:—

$$\begin{aligned} d &\text{ positive,} \\ h &\text{ positive or negative,} \\ \alpha &\text{ positive and less than } \frac{1}{2}\pi, \\ \omega, \theta, \varphi \text{ and } i &\text{ positive and less than } \pi, \\ l \text{ and } \lambda &\text{ positive and less than } 2\pi. \end{aligned}$$

13. We may, however, also find directly the value of ω in terms of d , h , and the known functions of α .

From (12)

$$\cos \varphi \sin \alpha = \sin i \sin \lambda \cos \alpha \sin \alpha \pm \sin i \cos \lambda \sin^2 \alpha.$$

From (7)

$$\cos \alpha \sin \alpha = \frac{AB}{A^2+B^2}, \text{ and } \sin^2 \alpha = \frac{B^2}{A^2+B^2}.$$

From (10) and (8)

$$\sin i \cos \lambda = \pm \frac{\cos l \sin \theta}{(1+\sin^2 l \tan^2 \theta)^{\frac{1}{2}}} = \pm \frac{\cot l \sin \theta}{(\sec^2 \theta + \cot^2 l)^{\frac{1}{2}}},$$

hence from (6) and (11)

$$\sin i \cos \lambda = \pm \frac{h \sin^2 \theta}{(d^2 + h^2 \sin^2 \theta)^{\frac{1}{2}}} = \pm \frac{h \sin^2 \theta}{B}.$$

From these and (9)

$$\cos \varphi \sin \alpha (A^2 + B^2) = AB \cos \theta \pm hB \sin^2 \theta,$$

and hence from (2)

$$\omega = \frac{s}{4m} \cdot \frac{A^2+B^2}{A \cos \theta \pm h \sin^2 \theta} = \frac{s}{4m} \cdot \frac{A^2+d^2+h^2 \sin^2 \theta}{A \cos \theta \pm h \sin^2 \theta}. \quad (13)$$

Since m is the known mass of the planet, and θ , s and A are known functions of ω , equation (13) gives directly the value of ω , the semi-axis major of the new orbit $\mathcal{C}$ in terms of d , h and ω .

14. For a particular case of approach, equation (13) is convenient for computation. We may, however, now treat d , h and ω as independent variables whose varying values may express all the different possible cases of approach of the comet to the planet so far as change of periodic time of the comet is concerned. The dependence of ω upon the *three* variables can not be very easily represented graphically in a single plane diagram. But by giving to ω successive values in multiples of 10° , viz, $\omega=10^\circ, 20^\circ, 30^\circ$, etc., to 170° , I have prepared a series of diagrams to exhibit in each case in succession the relation of ω to the other two variables. The values of θ , s and A for the several values of ω were needed in making the diagrams, and they are given in Table I. Equations (4), (5), and (7) are used in making the table. The disturbing planet is assumed to be Jupiter, so that m was taken equal to $1/1050$ and $r=5.2$.

TABLE I.

ω	θ		s	A	ω	θ		s	A
$^\circ$	$^\circ$	$'$			$^\circ$	$^\circ$	$'$		
0	0	0	0.414	.02886	100	131	48	1.868	.00142
10	32	1	0.463	.02309	110	138	9	1.992	.00125
20	55	47	0.585	.01448	120	144	21	2.101	.00112
30	72	22	0.742	.00900	130	150	26	2.195	.00103
40	84	46	0.913	.00594	140	156	26	2.273	.00096
50	94	47	1.087	.00419	150	162	22	2.334	.00091
60	103	27	1.259	.00312	160	168	16	2.379	.00088
70	111	14	1.426	.00244	170	174	8	2.405	.00086
80	118	27	1.584	.00197	180	180	0	2.414	.00085
90	125	16	1.732	.00165					

15. Using these values of θ , s and A we may now represent graphically the dependence of ω upon the other two variables d and h for each specified value of ω . Let d and h be Cartesian coördinates, then for each point of the coördinate plane there is a value of ω . The ambiguous sign will be fully satisfied by giving positive and negative values to h . For an assumed value of ω we shall have a curve whose equation is (13), and each point of this curve represents values of d and h for which the total action of the planet upon the comet will be to reduce the energy of the comet a constant amount. This locus will be called an *isergonal* curve.

16. *Faisceau of isergonal ellipses.*—The equation (13) of the isergonal curve may be written

$$4m\omega (A \cos \theta + h \sin^2 \theta) = s(A^2 + d^2 + h^2 \sin^2 \theta),$$

and this is the equation of an ellipse. As ω changes its value we may treat it as a parameter and we have a faisceau of similar isergonal ellipses, each ellipse symmetrical with the axis of h . The radical axis of the faisceau $A \cos \theta + h \sin^2 \theta = 0$, and the imaginary ellipse $A^2 + d^2 + h^2 \sin^2 \theta = 0$,

are theoretically two members of the faisceau. For points on the radical axis $\omega = \alpha$ and therefore for this locus there is no change in the energy of the comet.

17. *Center and area of the isergonal ellipse.*—The center of the isergonal ellipse is upon the axis of h ; making $d=0$, and solving for h we have

$$h = \frac{2m\omega}{s} \pm \frac{2m\omega}{s \sin \theta} \left(1 - \left(\cos \theta - \frac{As}{2m\omega} \right)^2 \right)^{\frac{1}{2}} \quad (14)$$

The first term of the second member of (14) is the ordinate of the center, and the second term is the semi-axis major of the ellipse. The ratio of the axes being $1 : \sin \theta$, and As^2 being $=mr$, the area of the ellipse will be equal to

$$\frac{4\pi m^2 \omega^2}{s^2 \sin \theta} \left(1 - \left(\cos \theta - \frac{r}{2\omega s} \right)^2 \right).$$

18. *Maximum action of the planet.*—For two particular values of ω the isergonal ellipses become points. These values of ω result if the maximum effect of the planet in increasing and in decreasing the energy of the comet takes place, and they are obtained by making the two values of h equal to each other in (14); that is, by making $\cos \theta - \frac{As}{2m\omega} = \pm 1$. Since at the same time $h = 2m\omega/s$, we obtain

$$h = \frac{A}{\cos \theta \pm 1}, \text{ and } \omega = \frac{As}{2m(\cos \theta \pm 1)}. \quad (15)$$

Let h' and h'' , and ω' and ω'' be the positive and negative values of h and ω in (15) and we may construct the following table of their values. As in Table I, Jupiter is assumed to be the perturbing planet:

TABLE II.

ω	h'	h''	ω'	ω''	ω	h'	h''	ω'	ω''
0	.01443	α	3.14	$-\alpha$	100	.00426	-.00085	4.17	-0.33
10	.01250	-.15174	3.04	-36.90	110	.00489	-.00072	5.12	-0.75
20	.00927	-.03307	2.85	-10.15	120	.00598	-.00062	6.60	-0.68
30	.00690	-.01290	2.69	-5.03	130	.00789	-.00055	9.09	-0.63
40	.00544	-.00654	2.61	-3.13	140	.01149	-.00050	13.71	-0.60
50	.00457	-.00387	2.61	-2.21	150	.01934	-.00047	23.70	-0.57
60	.00407	-.00253	2.69	-1.68	160	.04192	-.00044	52.36	-0.55
70	.00382	-.00179	2.86	-1.34	170	.16336	-.00043	206.30	-0.54
80	.00377	-.00134	3.14	-1.11	180	α	-.00043	α	-0.54
90	.00390	-.00105	3.55	-0.95					

19. *Explanation of Table II.*—The meaning of the numbers in this table may be explained by an example. If a comet moving in a parabola passes near to Jupiter, and the directions of the two original motions at nearest points of the orbits make an angle of 10° , then the greatest action of Jupiter (during the whole period of transit) in diminishing the velocity of the comet in its orbit about the sun will take place if the two orbits actually intersect ($d=0$), and if the comet in its unperturbed orbit arrives first at the point of intersection at the instant when Jupiter is distant therefrom .01250 (the earth's mean distance from the sun being unity). The resulting semi-axis major of the comet's orbit about the sun will be 3.04.

On the other hand, the greatest effect in increasing the velocity of the comet will take place when the two orbits actually intersect, the comet in its unperturbed orbit reaches the point of intersection later than the planet, and when the planet is distant therefrom 0.15174. The semi-transverse axis of the resulting hyperbolic orbit about the sun will be 36.90.

20. *Resulting orbits of maximum perturbation.*—The position of the relative orbit about Jupiter in these cases of maximum perturbation for given values of ω is easily determined. From the equations (7), (6), and (15)

$$\tan \alpha = B/A = h \sin \theta / A = \sin \theta / (\cos \theta \pm 1).$$

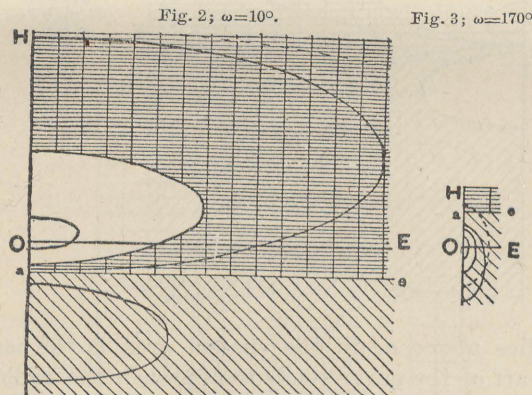
The positive sign gives $2\alpha = \theta$, and the negative sign gives $2\alpha = \pi + \theta$. But the angle 2α in the first case is the angle of the asymptotes inclosing the branch of the hyperbola described about Jupiter by the comet. Since the two original orbits intersect, the plane of the relative orbit contains the planet's path, so that the comet passes directly in front of the planet and being turned backward leaves Jupiter exactly in the direction of Jupiter's quit.* The place of encounter with Jupiter will be near an apse of the comet's resulting orbit about the sun. The comet leaves the planet with the relative velocity v_o , so that if $s < 1$ the motion about the sun in the new orbit will be direct; if $s > 1$ the motion in the new orbit will be retrograde. That is, by (4) when $\omega < \frac{1}{4}\pi$ the resulting motion is direct; when $\omega > \frac{1}{4}\pi$ the resulting motion is retrograde.

In the second case the angle 2α , being greater than 180° , stands for the angle between the asymptotes exterior to the orbit. Hence the comet passing behind the planet will be turned forward and will leave the planet in the direction of Jupiter's goal, and have a velocity that will send it permanently out of the solar system.

21. The results of article 20 assume that ω is given. To find for what value of ω the period of the resulting orbit is the shortest possible we may put $\Delta s^2 = mr$ and $1 - s^2 = 2s \cos \theta$ in (15) so that

$$\omega = \frac{r}{1 - s^2 \pm 2s}.$$

To find the minimum for ω place $\frac{d\omega}{ds} = 0$ in this equation. This gives $s = \pm 1$, in which result since s is inherently positive only the positive sign is used. But when $s = 1$, $\omega = \frac{1}{2}r$, $h = mr$ and $\omega = \frac{1}{4}\pi$. Hence the greatest effect of perturbation of a planet moving in a circular orbit in shortening the periodic time of a comet originally moving in a parabola is obtained if the comet's original orbit actually intersects the planet's orbit at an angle of 45° , and if the comet is due first at the point of intersection at the instant when the planet's distance therefrom is equal to the planet's distance from the sun multiplied by the ratio of the mass of the planet to the mass of the sun.

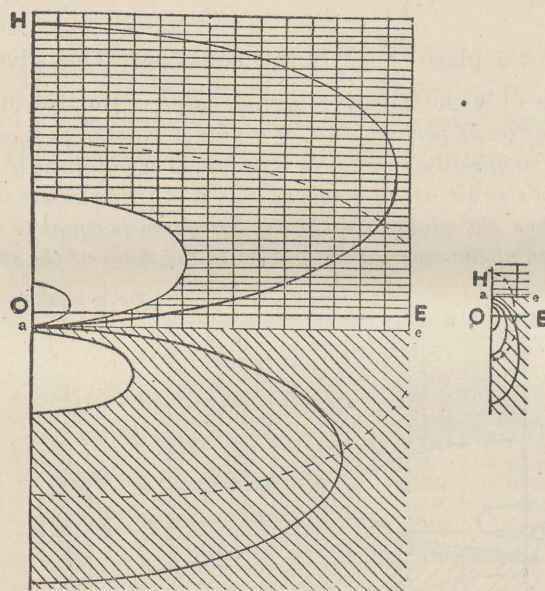


The relative velocity of the comet on leaving the planet's sphere of action would be equal to and directly opposite to the planet's velocity ($s = 1$), and the comet would be left entirely at rest to fall to the sun. This case could not happen for planets like the earth where mr is less than the semidiameter of the planet. In the case of the earth mr is less than 300 miles, and actual collision would result. But for Jupiter mr is greater than the distance of the second satellite from the planet. The nearest approach of the comet to the planet would be $mr(\sqrt{2} - 1)$, which is more than four times the radius of Jupiter. Hence, this case of maximum diminution of major axis could occur near Jupiter.

22. *Isergonal ellipses for $\omega = 10^\circ$.*—If we make $\omega = 10^\circ$ the vanishing points of the isergonal ellipses will be (Table II) at $d = 0$, $h = .01250$, and $d = 0$, $h = -.15174$. In Fig. 2 let OE and OH be

* The goal and the quit of a moving body are those two points on the celestial sphere towards which and from which the body is moving.

the axes of d and h respectively. The vanishing points will be on the axis OH at distances h' and h'' above and below O . Upon this diagram are shown the halves of four isergonal ellipses. The scales used for d and h are not equal to each other, since the use of the same scale for both coördinates would make the figures of inconvenient shape. In this, and in all the Figs. 2-18, the unit in d is to the unit in h , as 1 to $\sin \omega$. But to indicate more clearly this scale, and at the same time to give a kind of shading to a part of the area, there are drawn above the radical axis ae lines parallel to OE and parallel to OH , at intervals of $\cdot 01$; that is, the sides of each of the small rectangles in the quadrant HOE are $\cdot 01$, or about 925,000 miles. Only the positive values of d are represented in the figures. The positive vanishing point being $1\cdot 250$ of these divisions above O , and the negative vanishing point $15\cdot 174$ below O , we lay off $Oa = \frac{1}{2}(h' + h'') = -6\cdot 962$ divisions, and draw ae for the radical axis. The smallest positive value of ω is (Table II) $3\cdot 04$. As ω increases from $3\cdot 04$ the ellipse increases in size, and the innermost curve represents what it becomes when $\omega = 5$. The second curve (separating the blank and shaded areas) corresponds to $\omega = 20$. Any parabolic comet passing Jupiter with an original angle of $\alpha = 10^\circ$, and having d and h such as to be represented by a point within the blank area of Fig. 2 will leave the vicinity of the planet in an elliptic orbit whose semiaxis major is less than 20, and whose period, therefore, is less than ninety years.

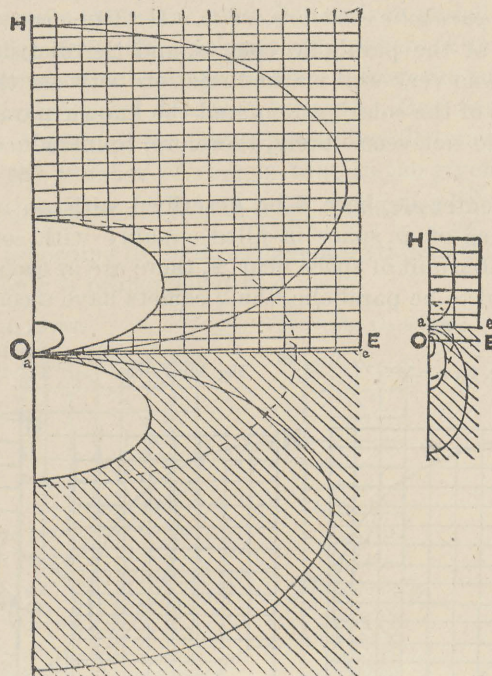
Fig 4; $\omega = 20^\circ$.Fig 5; $\omega = 160^\circ$.

The larger curve that lies above ae in the shaded area is the isergonal ellipse for $\omega = 50$. As ω increases the lower part of the curve tends to approach the radical axis ae , with which it coincides when $\omega = \infty$. For points in the area below ae (distinguished by the oblique-line shading), the planet increases the velocity of the comets, and the comet would be thrown permanently out of the solar system. The smallest semitransverse axis, the one corresponding to the vanishing ellipse is (Table II) $36\cdot 90$, and the isergonal curve for $\omega = -50$ is drawn in the figure.

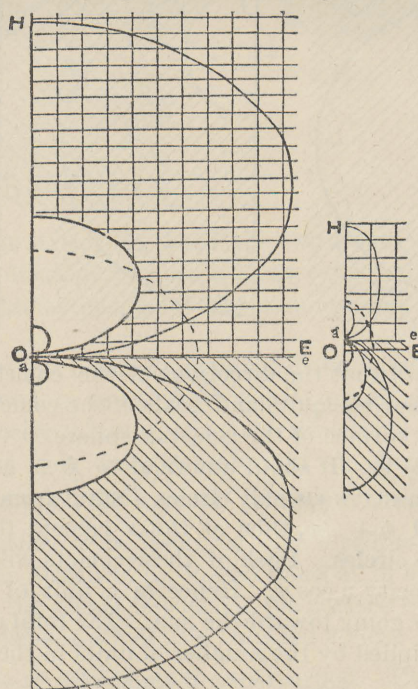
23. *Isergonal ellipses for $\omega = 170^\circ$.*—In Fig. 3 are drawn the three ellipses corresponding to the values of ω , -5 , -20 , and -50 . The ellipses above ae do not appear, inasmuch as the smallest possible elliptic orbit has a semiaxis major of $206\cdot 3$ (Table II), and a period of about 3,000 years. The radical axis ae is $\cdot 08146$ (or over 8 divisions) above OE .

24. Figures 4 and 5 are like diagrams for $\omega = 20^\circ$ and $\omega = 160^\circ$. With altered numbers the explanations of arts. 22 and 23 apply with slight change to these figures. The line ae in Figs. 4 and 5 is nearer to OE than is the same line in Figs. 2 and 3. In Fig. 4 the line for $\omega = -20$ appears below ae , while above ae are the three curves for $+5$, $+20$, and $+50$, respectively. In Fig. 5 the ellipse for $\omega = 50$ is wanting since the minimum ellipse has a semiaxis major $52\cdot 36$ (Table II), while below ae the three curves are present.

In Figs. 6 and 7 are contrasted in like manner the isergonal curves for the angles $\omega=30^\circ$, and $\omega=\text{the supplement of } 30^\circ$. In Fig. 6 the curve $\omega=-5$ is wanting, and in Fig. 7 the two curves $\omega=5$, and $\omega=20$ are both wanting.

Fig. 6; $\omega=30^\circ$.Fig. 7; $\omega=150^\circ$.

In like manner are to be explained the Figs. 8-18. The numbers needed for drawing the figures are furnished by equation (13). The curves that in each figure separate the shaded area from

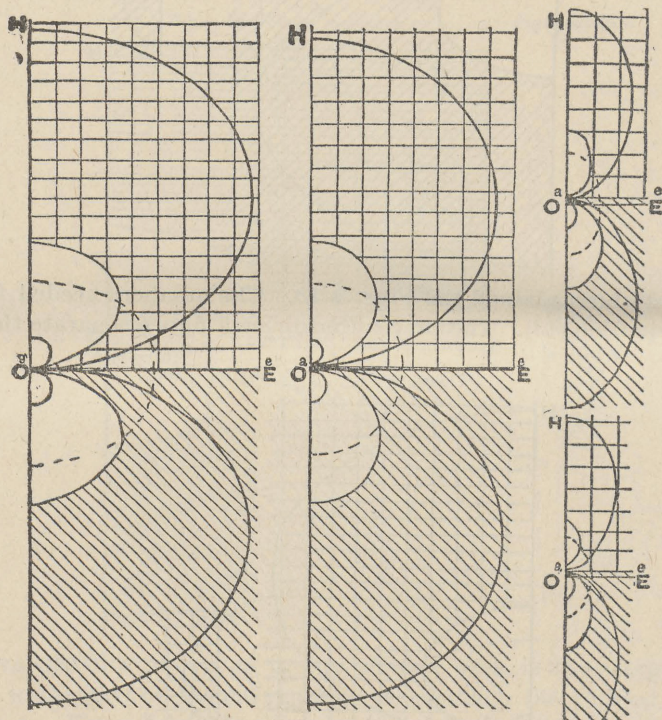
Fig. 8; $\omega=40^\circ$.Fig. 9; $\omega=140^\circ$.

the nonshaded area are the ellipses for $\omega=20$, and $\omega=-20$. The shading is introduced in order to compare more readily the corresponding curves in the figures.

25. The dotted curve in the several figures represents those values of d and h for which the total change of direction in the relative orbit is 10° ; that is, $\alpha = 85^\circ$. It is that curve whose equation is $A \tan 85^\circ = B$, or $d^2 + h^2 \sin^2 \theta = A^2 \tan^2 85^\circ$. It is therefore an ellipse whose center is the origin of coördinates, and it is similar in each figure to the isergonal ellipses.

26. *Hypotheses about the parabolic cometary orbits.*—It will be convenient to make two assumptions about the distribution of the parabolic comets, and the distribution of the goals of their motions. There seems to be no very well marked relation between the ecliptic, or to speak more strictly, the invariable plane of the solar system, and the known parabolic cometary orbits. The following two assumptions do not seem likely, therefore, to introduce any very serious error into our reasonings.

If about the sun as a center a sphere $\mathbb{S}$ be described with an arbitrary radius r , it will be assumed that near the surface of $\mathbb{S}$, space is filled equably with comets. We may express this by supposing that in each cubic unit of space near $\mathbb{S}$, there are at each and every instant n comets. As the orbits are all assumed to be parabolic, the n comets have a common velocity v .

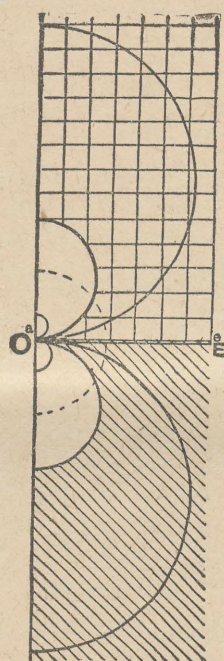
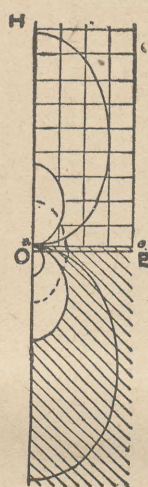
Fig. 10; $\omega = 50^\circ$.Fig. 12; $\omega = 60^\circ$.Fig. 13, $\omega = 120^\circ$.
Fig. 11; $\omega = 130^\circ$.

It will be furthermore assumed that the directions of the comets in each cubic unit of space near $\mathbb{S}$ are at random, that is, that the quits and goals of the comet's motions relative to the sun are distributed equably over the surface of the celestial sphere.

27. *Number of comets entering $\mathbb{S}$.*—If about a normal to $\mathbb{S}$ as an axis there be described two cones cutting the celestial sphere in two small circles distant from the point where the normal meets the celestial sphere ψ and $\psi + d\psi$, then of the n comets there will be $\frac{1}{2}n \sin \psi d\psi$ comets whose quits are between the two circles. Each of these comets will move perpendicularly to the spherical surface $\mathbb{S}$ with the velocity $v \cos \psi$. Hence in a unit of times $\frac{1}{2}nv \cos \psi \sin \psi d\psi$ comets will cross a unit of the surface $\mathbb{S}$ going toward the sun. The total entering the sphere in the unit of time will be this number multiplied by the number of units in the surface of $\mathbb{S}$, or

$$2\pi r^2 \int_0^{\frac{1}{2}\pi} nv \cos \psi \sin \psi d\psi = \pi n v r^2.$$

28. *Distribution of parabolic comets as to perihelion distance.*—This supposition of equable distribution of the goals of comets as they cross the spherical surface $\mathbb{S}$ involves also a law of distribution of comets as to perihelion distance. The number of comets that enter the sphere in a given time whose motions make with the normal angles between ψ and $\psi + d\psi$ is proportional to $\sin\psi \cos\psi d\psi$. If N be the number of comets that enter $\mathbb{S}$ in a given period of time with an angle with the normal less than ψ , we may write $dN = k \sin\psi \cos\psi d\psi$, where k is some constant. But if q is the perihelion distance of a comet which at the distance r from the sun moves at an angle with the radius equal to ψ , then $q = r \sin^2\psi$, and $dq = 2r \sin\psi \cos\psi d\psi$. But comets that enter $\mathbb{S}$ with angles to the normal between ψ and $\psi + d\psi$, have perihelion distances between q and $q + dq$. Hence N may also represent the number of comets that in the given period of time pass their perihelia, and whose perihelion distances are less than q . Therefore $\frac{dN}{dq}$ is a constant, and we conclude that if comets be grouped according to their perihelion distances the number of comets whose perihelion distances are less than q is proportional to q .

Fig. 14; $\omega = 70^\circ$.Fig. 15; $\omega = 110^\circ$.

29. It follows as a corollary to Article 28 that if the two assumptions of Article 26 be made for the spherical surface $\mathbb{S}$, the like distributions are true for every smaller concentric spherical surface. It would be but a reasonable extension of the assumptions to make them apply to larger spheres, if finite.

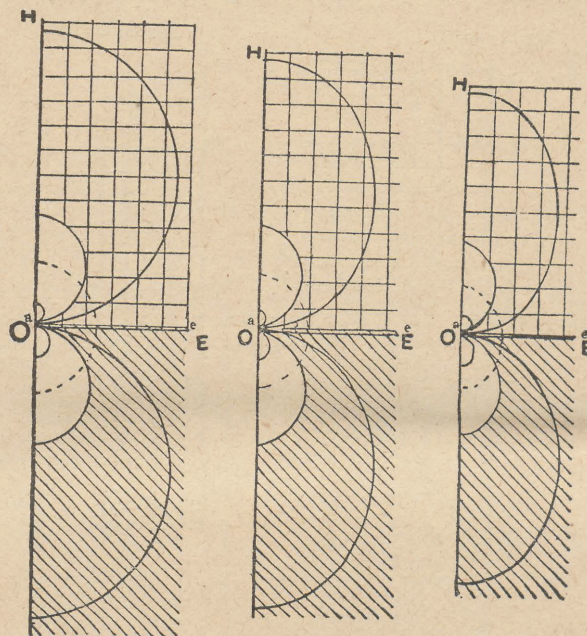
30. If there are assumed to be n comets equably distributed in each unit of the space near and through which a planet is moving, and if these comets are all assumed to be moving in parabolas about the sun with the velocity v , having also their directions of motion equably distributed, then the number that are moving from quits lying within an element dS of the surface of the celestial sphere will be $\frac{ndS}{4\pi}$. Let v_0 be the common velocity of these comets relative to the planet. Then suppose that a spherical surface S' is described with a radius r' about the planet as center; r' being small relative to the sun's distance, yet not so small as to forbid the omission of the planet's perturbing action so long as the comet is without the surface S' . In each unit of time out of these comets directed from the element dS of the celestial sphere there would pass nearer than r' to the planet $n \frac{dS}{4\pi} \cdot \pi r'^2 v_0 = \frac{1}{4} n v_0 r'^2 dS$ comets if unperturbed. Evidently an equal number cross the surface S' entering the sphere in each unit of time.

If now ω be the angle which the comet's unperturbed motion is making with the planet's motion, and if v_1 or its equal, $v/\sqrt{2}$, be the planet's velocity in its orbit about the sun, then $v_0^2 = \frac{1}{2}v^2[3 - 2\sqrt{2}\cos\omega]$. The element dS may be taken to be the elemental zone between the two small circles, whose common pole is the planet's quit, and whose distances from the planet's quit are ω and $\omega + d\omega$. Then $dS = 2\pi \sin \omega d\omega$. The number of comets entering S' in a unit of time with quits within that elemental zone will be

$$\frac{1}{4}nvv_0r'^2 \times 2\pi \sin \omega d\omega = \frac{\pi nvr'^2}{2\sqrt{2}} (3 - 2\sqrt{2}\cos\omega)^{\frac{1}{2}} \sin \omega d\omega.$$

The integral of this,

$$\frac{\pi nvr'^2}{2\sqrt{2}} \int_0^\pi (3 - 2\sqrt{2}\cos\omega)^{\frac{1}{2}} \sin \omega d\omega = \frac{7}{6}\pi nvr'^2,$$

Fig. 16; $\omega = 80^\circ$.Fig. 17; $\omega = 90^\circ$.Fig. 18; $\omega = 100^\circ$.

expresses the total number of comets that, under the hypotheses that have been made, would, in a unit of time, enter the sphere S' .

31. If we compare the two expressions obtained in articles 27 and 30 we find that the number of comets which, in a given period of time, come nearer to the sun than r is to the number that (unperturbed) come nearer to the planet than r' as $6r^2$ is to $7r'^2$. The factor $\frac{7}{6}$ expresses the increase of numbers caused by the planet's motion in its circular orbit. The value of r' , as has been said, must not be too small, nor yet must it be very large.

32. In order to determine the number N of comets which, in a unit of time will have their periodic times reduced below a given period, we may make use of the isergonal curves represented in Figs. 2-18. Although the diagrams were not constructed to exhibit the motions of the bodies, yet they may be utilized for that purpose. Let OH be the tangent to the planet's orbit, O the place of the planet considered at rest, and let the plane HOE contain the shortest line d between the two orbits. This d will be the abscissa of the point at which the comet's unperturbed orbit will cut the plane. The ordinate of the same point, produced if necessary, will be the projection of the comet's path upon the plane HOE , and the comet's path makes with the plane the angle θ . The velocity of the comet perpendicular to the plane will be $v_0 \sin \theta$. By reason of the hypothesis that the comets are equably distributed, the points of intersection with the plane HOE will be equably distributed over the plane. Hence the number of comets whose quits are in the element dS of the celestial sphere and that will pass the planet in a unit of time in such a way as to have

their periodic times reduced below a given period will be equal to the area inclosed in the corresponding isergonal curve multiplied by the velocity perpendicular to the plane $v_0 \sin \theta$, and by the factor $\frac{ndS}{4\pi}$. If ω is the semimajor axis of the orbit for the limiting periodic time, the area of the corresponding isergonal curve will be (article 17).

$$\frac{\pi}{\sin \theta} \left(\frac{4m^2 \omega^2}{s^2} - \left(\frac{2m\omega \cos \theta}{s} - \frac{mr}{s^2} \right)^2 \right).$$

For dS we may, as before, take $2\pi \sin \omega d\omega$, and we shall then have

$$N = \frac{\pi n}{2} \int v_0 \sin \omega \left[\frac{4m^2 \omega^2}{s^2} - \left(\frac{2m\omega \cos \theta}{s} - \frac{mr}{s^2} \right)^2 \right] d\omega.$$

The integration must extend through the positive values of the quantity in square brackets beginning at $\omega = 0$. [In case $\omega = 0$ gives a negative value for the quantity in square brackets we must integrate between the two values of ω corresponding to the zero value of the bracketed quantity.] We may make s the independent variable by the equations $s ds = \sqrt{2} \sin \omega d\omega$, $v_0 \sqrt{2} = sv$, and $2s \cos \theta = 1 - s^2$.

These give:

$$N = \frac{1}{4} \pi n m^2 v \int \left[4\omega^2 - \left(\frac{\omega - r - \omega s^2}{s} \right)^2 \right] ds.$$

33. If, now, we require the number of comets which, in each unit of time, shall pass the planet in such way as that they shall have, after the passage, respectively, less than one-half, once, three-halves, and twice the planet's period of revolution

we may place $\omega = rT^{\frac{2}{3}}$, and make T equal, successively, to $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2, and compute in each case the value of N as given in the last article. The results are found to be $\pi n m^2 r^2 v$ multiplied severally by the coefficients 0.139, 0.925, 1.875, and 2.943.

34. By comparing the results of articles 27 and 33, and making the assumptions of article 26, we have the proposition that *the number of comets which, in a given period of time, pass their perihelia nearer to the sun than a given planet, is to the number of comets whose periodic times are reduced by the perturbing action of the planet so as to be less severally than one-half, once, three halves, and twice, the periodic time of the planet, as unity is to the square of the mass of the planet multiplied severally by 0.139, 0.925, 1.876, and 2.943.*

35. If Jupiter is the planet, $m = \frac{1}{10480}$, and we may express these ratios as

$$1,000,000,000 : 126 : 839 : 1701 : 2670.$$

That is, assuming the hypotheses of article 26, and regarding the planet as without dimension so as to intercept any comets, *if in a given period of time a thousand million comets come in parabolic orbits nearer to the sun than Jupiter, 126 of them will have their orbits changed into ellipses with periodic times less than one-half that of Jupiter; 839 of them will have their orbits changed into ellipses with periodic times less than that of Jupiter; 1,701 of them will have their orbits changed into ellipses with periodic times less than once and a half times that of Jupiter; and 2,670 of them will have their orbits changed into ellipses with periodic times less than twice that of Jupiter.*

36. Another and perhaps a more important inquiry is this. What effect have the perturbations of the planet in bringing or not bringing the comets to move in the same direction that the planet is moving after the comets have by perturbation had their periodic times largely reduced? For simplicity and as a special example I shall consider the action of Jupiter only, and also only his action upon those comets whose periodic times are reduced to be less than Jupiter's period, the original orbits of the comets being parabolic. In other words, how many of the 839 comets which are reduced (article 35) to have periodic times less than Jupiter's period will, after perturbation, have goals distant less than 15° , 30° , 45° , etc., severally from Jupiter's goal?

37. Let BA, Fig. 19, be drawn to represent v_1 and CA to represent $v_1 \sqrt{2}$. With A as a center and AB and AC as radii describe the semicircles BLO and CHG. Let the angle BAH be made equal to ω and BH be drawn; then HA will represent the comet's velocity about the sun, BA the planet's velocity about the sun, and therefore HB the comet's velocity v_0 in its orbit about the planet before perturbation. About B as center describe the semicircle KHT. Since

the relative velocity after as well as before perturbation is equal to HB, therefore the velocity of the comet about the sun after perturbation will evidently be represented by a line drawn from some point in the semicircumference KHT to A. If the velocity is increased, the new velocity will be represented by a line to A from some point in the arc KH; if diminished, by a line to A from some point in the arc HT. If the new velocity is less than the planet's velocity, and so the new cometic period less than the planet's period, the new velocity will be represented by a line to A from some point in the arc ET. If in a diagram constructed for $\omega = \text{BAH}$ the isergonal curve be drawn for $\omega = r$, those comets for which d and h represent points within that isergonal curve will, after perturbation, have velocities represented by lines drawn from points in ET to A, while comets for which d and h represent points outside that isergonal curve will, after perturbation, have directions of motion represented by lines drawn to A from points in EHK. The number of comets having motions represented by lines to A from points in ET will be proportional to the area of the isergonal curve $\omega = r$. Let the angle BAS represent a limiting value ω'' of distance of quits of comets from Jupiter's quit after perturbation. The comets which are thus limited and at the same time have $\omega < r$ will be moving in lines directed to A from points in the area bounded by the straight lines SA and AF, and the arcs FD and DS. Let ω receive an increment $d\omega = \text{Hh}$ and let a new semicircumference be drawn with B h as radius. To the elemental arc H h will cor-

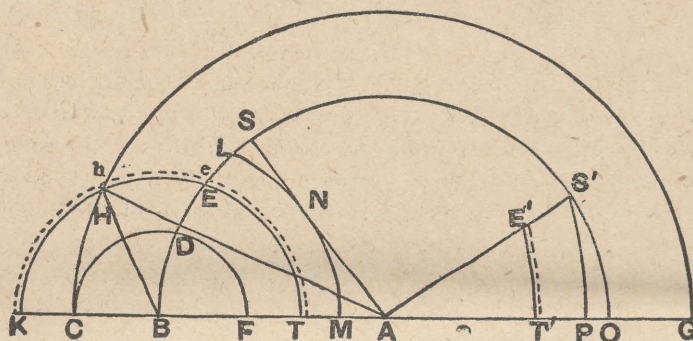


Fig. 19.

respond the elemental area along the semicircumference KET. If ET lies wholly in SAFD the number of comets that pass the planet in a unit of time having initial angles of direction with Jupiter's motion between ω and $\omega + d\omega$ will be equal to the area of the isergonal curve for $\omega = r$ multiplied by the elemental number $\frac{1}{2}n \sin \omega d\omega$, and by the relative velocity $v_0 \sin \theta$ of the comet perpendicular to the isergonal area. If the area of the isergonal curve be represented by $\Phi/s^2 \sin \theta$, then this product will be

$$\frac{\Phi}{s^2 \sin \theta} \cdot v_0 \sin \theta \cdot \frac{n \sin \omega d\omega}{2} = \frac{nv}{4} \Phi ds$$

since $\sqrt{2}v_0 = sv$, and $\sqrt{2} \sin \omega d\omega = sds$.

38. This expresses the elemental number of comets corresponding to the elemental area Te. The integral of this expression, that is, $\frac{1}{4}nv\Phi ds$, so taken as to cover the area AFDS will give the number of comets which in a unit of time will pass the planet in such a way as to have $\omega < r$ and $\omega' < \text{BAS}$. When the elemental area does not extend from the arc DS to the line BA, the area of another appropriate isergonal curve is to be used in determining Φ .

By article 17 we have

$$\Phi = \pi m^2 \left[4\omega^2 - \left(\frac{\omega - r - \omega s^2}{s} \right)^2 \right].$$

For the elemental areas of the surface AFDS which end on the arc DS we make $\omega = r$, and let Φ_0 be the resulting value of Φ ; then $\Phi_0 = \pi m^2 r^2 (4 - s^2)$.

For elemental areas that end on the radius AS the values of ω on that line are functions of s . To compute them let v' be the comet's velocity in its orbit about the sun, and hence equal to the distance of the point on AS from A; then by the triangle of velocities

$$v'^2 + v^2 - 2v'v \cos \omega' = v_0^2 = s^2 v^2.$$

Again by the laws of gravitation,

$$v'^2 = \left(2 - \frac{r}{\omega}\right) v_i^2.$$

Hence

$$s^2 = 3 - \frac{r}{\omega} - 2\sqrt{2 - \frac{r}{\omega}} \cos \omega'',$$

or

$$\frac{\omega}{r} = \frac{3 - s^2 - 2 \cos^2 \omega'' \pm 2 \cos \omega'' (s^2 - \sin^2 \omega'')^{\frac{1}{2}}}{9 - 8 \cos^2 \omega'' - 6s^2 + s^4}.$$

Let Φ' and Φ'' be the two values of Φ obtained by substituting in Φ these values of ω , Φ'' representing the value for the point nearer to A.

39. If $\omega'' = 90^\circ$, and therefore $\cos \omega'' = 0$, we have along the limiting line, the two values of ω equal, hence

$$\frac{\omega}{r} = \frac{1}{3 - s^2}, \text{ and } \Phi' = \frac{s^2 - 1}{s^2(3 - s^2)^2},$$

so that the number of comets having quits less than 90° from Jupiter's quit and $\omega < r$ is

$$\frac{nv}{4} \int_{\sqrt{2}-1}^{\sqrt{2}} \Phi_0 ds - \frac{nv}{4} \int_1^{\sqrt{2}} \Phi' ds = \frac{\pi nvm^2 r^2}{4} \left[\int_{\sqrt{2}-1}^{\sqrt{2}} (4 - s^2) ds - 4 \int_1^{\sqrt{2}} \frac{(s^2 - 1) ds}{s^2(3 - s^2)} \right] = \frac{\pi nvm^2 r^2}{12} (7 + \sqrt{2})$$

$$= .7012 \pi nvm^2 r^2.$$

Since the whole number of such comets is (Art. 33) equal to $.925 \pi nvm^2 r^2$, the number of comets the distance of whose quits from Jupiter's quit is between 90° and 180° is $.224 \pi nvm^2 r^2$. The number of the comets for which $\omega < r$ that have inclinations to the ecliptic less than 90° is to the number that have inclinations greater than 90° as 701 is to 224. *Of the 839 comets spoken of in article 36, 203 will after perturbation have retrograde motions, and 636 will have direct motions.*

40. If ω'' is less than 90° the expression to be integrated in order to cover the area SAFD will be

$$\int_{\sqrt{2}-1}^{\sin \omega''} \Phi_0 ds + \int_{\sin \omega''}^{2 \sin \frac{1}{2} \omega''} (\Phi_0 - \Phi') ds + \int_{\sin \omega''}^1 \Phi'' ds.$$

If ω'' is greater than 90° the corresponding expression becomes

$$\int_{\sqrt{2}-1}^{2 \sin \omega''} \Phi_0 ds - \int_1^{2 \sin \frac{1}{2} \omega''} \Phi'' ds.$$

As the value of ω introduces into Φ' and Φ'' only one radical in s , and that a radical of the second degree, these integrations are possible. Finite summation is, however, more convenient. Computing the values for each interval of 15° , we construct the following table. The first column indicates the interval in values of ω'' ; the second column gives that coefficient of $\frac{1}{4} \pi nvm^2 r^2$ that must be used to obtain the number of comets, which in a unit of time will pass perihelion nearer than Jupiter's distance to the sun, shall also have their periodic times reduced to be less than Jupiter's period, and shall also leave Jupiter's vicinity so that the distance between the quits of the two bodies is between the two values in column 1; the third column indicates the distribution of the 839 comets of article 36 through the twelve zones.

TABLE III.

Limiting values of ω'' .	Coefficient of $\frac{1}{4} \pi nvm^2 r^2$.	No. out of 839 comets.
From 0° to 15°	26	6
From 15 to 30	401	91
From 30 to 45	751	170
From 45 to 60	670	152
From 60 to 75	548	124
From 75 to 90	443	101
From 90 to 105	296	67
From 105 to 120	235	53
From 120 to 135	162	37
From 135 to 150	99	23
From 150 to 165	50	11
From 165 to 180	16	4

We see also from the last column of this table that of the 839 comets under consideration 267 have quits less than 45° from Jupiter's quit, while only 38 of them have quits within 45° of Jupiter's goal.

41. Table III gives the distribution of the comet quits relative to Jupiter's quit. It may also be used to determine how many of the comets whose orbits are thus changed shall have an inclination to the plane of Jupiter's orbit less than a given angle.

Let the angle be 30° . Let Q be Jupiter's quit on the celestial sphere, Q' the comet's quit, and S the sun's position as seen from Jupiter. Then in the triangle QQ'S put ω'' for QQ' the distance of the quits. The side QS = 90° , and the QSQ' will be the inclination of the orbits. Represent this angle by i and the angle Q'QS by η . Then $\sin \eta = \cot \omega'' \cot i$.

Let two small circles be drawn about Q at distances ω'' and $\omega'' + d\omega''$ then if $d\omega''$ be made 15° the numbers in the second or third columns of Table III indicate how many quits are in the several zones of 15° on the celestial sphere. These may be distributed at smaller intervals than 15° by known processes. All the quits that lie in the lune between two semicircles drawn through S so as to make angles of 30° with QS will evidently have orbits inclined less than 30° to Jupiter's orbit. From $\omega'' = 0$ to $\omega'' = 30^\circ$ all the quits are included in the lune. From $\omega'' = 30^\circ$ to $\omega'' = 90^\circ$ we compute η from the equation $\sin \eta = \cot \omega'' \cot 30^\circ$; then the portion of the quits in any elemental zone that fall in the lune is to the whole number of quits in that elemental zone as this value of η is to 90° . These may be summed by finite summation, and the result is that among the 839 comets 257 would move in orbits inclined less than 30° to the orbit of Jupiter.

42. If a like summation be made for the equal lune that contains Jupiter's goal we find 51 to be the number out of the 839 comets which move in orbits inclined more than 150° to Jupiter's orbit; that is, *somewhat more than five times as many of these comets move in direct orbits inclined less than 30° to Jupiter's orbit as move in retrograde orbits inclined less than 30° to Jupiter's orbit.*

43. The comet has been thus far considered as approaching Jupiter while moving in a parabolic orbit about the sun. If the comet, however, is moving in any other orbit, and it passes near to the planet, the result of the planet's perturbing action will in general be quite similar to the result when the orbit is parabolic, the other circumstances of the approach being assumed to be alike in the two cases.

44. These are perturbations during one transit past the planet. But the comet, unless the orbit is further changed by another planet, must return at each revolution to the place where it encountered Jupiter. At some time Jupiter will be nigh that place nearly at the same time as the comet, and the comet will suffer a new and perhaps a large perturbation. Its period will again be changed, being shortened or lengthened according as the comet passes before or behind the planet. This process will be repeated again and again, since, after any number of encounters, the new orbit of the comet will still pass near to the orbit of the planet.

This repeated action makes it possible to have an orbit shortened in period by several passages near to Jupiter instead of its being done at one passage. A much larger proportion of comets than 839 out of 1,000,000,000 might therefore have their periodic times reduced below the period of Jupiter.

45. If the comet's orbit is largely inclined to the ecliptic and hence its motion makes a large angle with that of Jupiter the diagrams Figs. 10-18 show that there is nearly an even chance that the velocity will be increased or diminished. A considerable fractional part of the whole number of such comets will at each passage be thrown out of the solar system altogether, or thrown into such long orbits that they will return only at very great intervals of time. This class of comets can not be therefore regarded as permanent members of the family of short period comets, except such of them as happen to come so near to other planets as to have their orbits changed in such wise that they do not have thereafter the near approach to Jupiter's orbit. But when an orbit is greatly inclined to the plane of the solar system the comet passes through the plane in general at a considerable angle and the chance of coming close to another planet is relatively small.

46. On the other hand all the comets which after perturbation are moving in orbits somewhat but not greatly inclined to the ecliptic are liable to meet, in fact are sooner or later almost certain to meet other planets in such a way as to suffer perturbations that will prevent future close

encounters with Jupiter. After such changes those comets must be regarded as tolerably permanent members of the solar system.

47. Comets that have motions not greatly inclined to Jupiter's motion, are, as Figs. 2 and 4 show, more likely in subsequent passages near to Jupiter to have their periodic times shortened than lengthened. On the contrary those passing in nearly opposite direction to Jupiter's motion will as Figs. 3, 5, and 7 show, be much more likely to have their periods lengthened than shortened.

All these causes combine and work together to the one end that those comets which are changed by the perturbing action of Jupiter, or other planets, from parabolic orbits of every possible inclination to the ecliptic into short period ellipses and become permanent members of the solar system, will as a rule (but with exceptions) move in orbits of moderate inclination to the ecliptic, and with direct motions.

We know as a fact that most short period comets do move in orbits having small inclinations and direct motions, while long period and parabolic comets move at all possible inclinations to the ecliptic. If the short period comets have been changed by Jupiter and other planets from parabolic orbits, the preceding investigation shows why their orbits have now small inclinations to the ecliptic, and the comets themselves have direct motions.

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