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APPLICATION OF TOPOLOGY IN ALGEBRA AND DIFFERENTIAL GEOMETRY

Matemaatika- ja mehaanikaalaseid töid

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International Autumn School
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On September 19-22, 1991 the third autumn school "Applications of Topology in Algebra and Differential Geometry" organized by the Laboratory of Applied Mathematics of Tartu University in cooperation with the Estonian Mathematical Society took place in the Sports Centre of Tartu University at Kääriku. The first two schools of the same kind took place at Kääriku in 1985 and 1988. Differently from the previous schools the present gathering was international, having gests from Russia (6), Latvia (4), Kyrgystan (3), Byelorussia (1), Georgia (1), Moldova (1) and Norway (1).

In the programme of the school there were 14 lectures and 6 short communications. The new research results were presented and some open problems were being discussed in the following fields:

general topology,
dimension theory,
fuzzy topology,
bitopological spaces,
topological algebras, groups and semigroups,
category theory,
spectral theory,
topological quantum field theory,
homogeneous spaces and manifolds,
Lie groups,
geometry of differential equations,

fixed point theorems,
topological lattices,
functors, kernels and jets.

In the present issue of the journal "Acta et Commentationes Universitatis Tartuensis" most of the lectures delivered at the school are published.

M.Abel
R.Roomeldi

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SOME OPEN PROBLEMS IN THE THEORY OF TOPOLOGICAL ALGEBRAS

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The theory of topological algebras began in 1939 with the publication of Gelfand's paper [9]. During the last fifty years, in addition to Banach algebras, the main attention has been devoted to locally convex (in particular, to locally m-convex) algebras and locally bounded algebras. At this, locally pseudoconvex algebras (which include both locally convex and locally bounded algebras), Fréchet algebras and several (more general) classes of topological algebras have not been studied so intensively. Working in the field of topological algebras, some open problems have arisen. Below we should like to introduce some of them.

a) Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, A be a topological $\mathbb{K}$-algebra, $\mathrm{hom}_o A$ be the set of all continuous homomorphisms from A into $\mathbb{K}$, homA be the set of non-trivial homomorphisms in $\mathrm{hom}_o A$ and M(A) be the set of all closed regular two-sided ideals of A which are maximal as left (or as right) ideals in A. If for each $M \in M(A)$ the quotient algebra A/M is topologically isomorphic to $\mathbb{K}$ then A is called a *Gelfand-Mazur* $\mathbb{K}$*-algebra*. This class of topological algebras was introduced at the beginning of the 1980s. Until now 10 different classes of Gelfand-Mazur $\mathbb{K}$-algebras (see [4-6]) and some examples of topological $\mathbb{C}$-algebras, which are not Gelfand-Mazur $\mathbb{C}$-algebras (see [8], p. 214-217, [11], p. 75, [19], p. 141-148, [20], [21], p. 83-86; or [22], p.127), are known.

P R O B L E M 1. *How to describe all Gelfand-Mazur* $\mathbb{K}$*-algebras?*

b) A topological $\mathbb{K}$-algebra A with separately continuous multiplication is a *Waelbroeck* $\mathbb{K}$*-algebra* if the set of all quasi-invertible elements of A is open and the quasi-invertion is continuous in A. If the topology of A is complete and metrizable then A is called *a Fréchet* $\mathbb{K}$*-algebra*. Moreover, if A has a base of neighbourhoods of zero consisting of balanced pseodoconvex sets then A is called a *locally pseudoconvex* $\mathbb{K}$*-algebra*. It is known (see [19], p. 3-6) that a topological $\mathbb{K}$-algebra A is *locally pseudoconvex* if the topology of A has been given by the family $\{p_\alpha : \alpha \in \mathfrak{U}\}$ of k_α-homogeneous ($k_\alpha \in (0,1]$ for each $\alpha \in \mathfrak{U}$) seminorms. In the case when $k_\alpha \equiv k$ ($k_\alpha \equiv 1$) for each $\alpha \in \mathfrak{U}$, then A is called a *locally k-convex* (respectively a *locally convex*) $\mathbb{K}$*-algebra*. In addition of this, if for each $\alpha \in \mathfrak{U}$ and $a \in A$ there exist $M(\alpha, a) > 0$ and $N(\alpha, a) > 0$ such that

$$p_\alpha(ab) \le M(\alpha, a)p_\alpha(b) \tag{1}$$

and

$$p_\alpha(ba) \le N(\alpha, a)p_\alpha(b) \tag{2}$$

for each $b \in A$, then A is called *a absorbingly pseudoconvex* $\mathbb{K}$*-algebra* (*a absorbingly k-convex* $\mathbb{K}$*-algebra* if $k_\alpha \equiv k$ for each $\alpha \in \mathfrak{U}$). In the case, when $M(\alpha, a) = N(\alpha, a) = p_\alpha(a)$ for each $\alpha \in \mathfrak{U}$ and $a \in A$ in (1) and (2), then A is called *a locally m-pseudoconvex* $\mathbb{K}$*-algebra (a locally m-(k-convex)* $\mathbb{K}$*-algebra* if $k_\alpha \equiv k$ for each $\alpha \in \mathfrak{U}$). It is easy to see that the center of many Gelfand-Mazur $\mathbb{K}$-algebras is also a Gelfand-Mazur $\mathbb{K}$-algebra, for example, in the case of locally pseudoconvex Fréchet $\mathbb{C}$-algebras, absorbingly pseudoconvex $\mathbb{C}$-algebras and some locally pseudoconvex $\mathbb{R}$-algebras.

P R O B L E M 2. *Does there exist a Gelfand-Mazur* $\mathbb{K}$*-algebra the center of which is not a Gelfand-Mazur* $\mathbb{K}$*-algebra?*

c) A topological $\mathbb{K}$-algebra A is called *a normal* $\mathbb{K}$*-algebra* if every closed regular left (right) ideal of A is contained in a closed maximal regular left (respectively, right) ideal of A. It is known that every Q-algebra is a normal $\mathbb{K}$-algebra and every commutative Hausdorff locally m-(k-convex) $\mathbb{C}$-algebra with unit is a normal $\mathbb{C}$-algebra. In [3], p. 7, it has been shown that every dense subalgebra of a commutative normal Gelfand-Mazur $\mathbb{K}$-algebra is a Gelfand-Mazur $\mathbb{K}$-algebra.

P R O B L E M 3. *Does there exist a non-commutative normal Gelfand-Mazur* $\mathbb{K}$*-algebra having a dense subalgebra which is not a Gelfand-Mazur* $\mathbb{K}$*-algebra?*

P R O B L E M 4. *How to describe all normal* $\mathbb{K}$*-algebras?*

d) Let A be a topological $\mathbb{C}$-algebra and $sp_A(a)$ be the specter of $a \in A$. It is known (see [1] and [2], p. 22) that

$$sp_A(a) = \{ \varphi(a) : \varphi \in hom_o A \} \qquad (3)$$

for each $a \in A$ if A is a commutative Gelfand-Mazur Q-algebra over $\mathbb{C}$ or a commutative complete Hausdorff locally m-(k-convex) $\mathbb{C}$-algebra with $k \in (0,1]$ (which is also a Gelfand-Mazur $\mathbb{C}$-algebra (see [6], Theorem 3.3)). Moreover, in [14], p. 67, has been given a commutative absorbingly convex $\mathbb{C}$-algebra (consequently, a Gelfand-Mazur $\mathbb{C}$-algebra (see [6], Theorem 3.3) which does not have the property (3).

P R O B L E M 5. *Which Gelfand-Mazur* $\mathbb{C}$*-algebras A (besides of mentioned above) have the property (3) for each* $a \in A$*?*

e) Let A be an exponentially galbed $\mathbb{K}$-algebra, that is a topological $\mathbb{K}$-algebra with separately continuous multiplication in which for each neighbourhood of zero $\mathcal{U}$ there exists a neighbourhood of zero $\mathcal{V}$ such that

$$\left\{ \sum_{k=0}^{n} \frac{a_k}{2^k} : \; a_0, \dots, a_n \in \mathcal{V} \right\} \subset \mathcal{U}$$

for each $n \in \mathbb{N}$, and let A_o be the set of all $a \in A$ for which there exists $\lambda \in \mathbb{K}\setminus\{0\}$ such that the set $\{(a/\lambda)^n : \; n \in \mathbb{N}\}$ is bounded in A. It is known (see [4,5]) that every exponentially galbed $\mathbb{C}$-algebra with $A_o = A$ is a Gelfand-Mazur $\mathbb{C}$-algebra and every commutative exponentally galbed $\mathbb{R}$-algebra with continuous multiplication for which $A_o = A$ is a Gelfand-Mazur $\mathbb{R}$-algebra if A satisfies the property

(α) *if* $a, b \in A$ *and* $a^2 + b^2 \in M$ *then* $a, b \in M$ *for each* $M \in \mathcal{M}(A)$.

P R O B L E M 6. *Does there exist a non-commutative exponentially galbed* $\mathbb{R}$*-algebra with* $A_o = A$ *which is not a Gelfand-Mazur* $\mathbb{R}$*-algebra but has the property* (α)*?*

f) It is known (see [5]) that both every Waelbroeck $\mathbb{C}$-algebra and every commutative Waelbroeck $\mathbb{R}$-algebra with the property (α) are Gelfand-Mazur $\mathbb{K}$-algebras if their dual spaces are not trivial.

P R O B L E M 7. *Does there exist a non-commutative Waelbroeck $\mathbb{R}$-algebra with non-trivial dual space which is not a Gelfand-Mazur $\mathbb{R}$-algebra but has the property (α)?*

g) It is noted in [3] that the projective limit of Gelfand-Mazur $\mathbb{K}$-algebras is not necessarily a Gelfand-Mazur $\mathbb{K}$-algebra.

P R O B L E M 8. *When the injective limit of Gelfand-Mazur $\mathbb{K}$-algebras is a Gelfand-Mazur $\mathbb{K}$-algebra?*

h) Let A be a locally pseudoconvex $\mathbb{K}$-algebra the topology of which is given by the family $\{p_\alpha : \alpha \in \mathfrak{U}\}$ of k_α-homogeneous seminorms with $k_\alpha \in (0,1]$ for each $\alpha \in \mathfrak{U}$ and let

$$A_b = \{\, a \in A : \sup_{\alpha \in \mathfrak{U}} p_\alpha(a) \text{ is finite } \}.$$

It is known (see [7], Theorem 2.3, or [15], p.174) that A_b is dense in A if A is a locally C^*-algebra, i. e. a complete locally m-convex $\mathbb{C}$-algebra with involution in which every seminorm p satisfies the condition $p(aa^*) = p(a)^2$ for each $a \in A$.

P R O B L E M 9. *In which locally pseudoconvex $\mathbb{K}$-algebras A (besides locally C^*-algebras) is the set A_b dense in A?*

i) Let A be a topological $\mathbb{K}$-algebra for which the set homA is not empty.

P R O B L E M 10. *For which topological $\mathbb{K}$-algebras A is the Lebesque dimension of the space homA (in the Gelfand topology) finite?*

j) Let X be a completely regular Hausdorff space and υX be the realcompactification of X (see [10], p. 116). In the case when $\upsilon X = X$ the space X is called *a realcompact space*. It is known that homA is a hemicompact space (consequently, a realcompact space (see [8], p. 26)) if A is a full

Fréchet locally convex ℂ-algebra (see [12], p. 269) or a metrizable spectrally barreled ℂ-algebra with unit (see [12], p. 170 and 183).

P R O B L E M 11. *For which topological* 𝕂*-algebra A is the space homA realcompact?*

k) It is known (see [16] or [19], p.123) that a commutative locally convex Waelbroeck 𝕂-algebra is locally m-convex.

P R O B L E M 12. *Does there exist a commutative locally k-convex Waelbroeck* 𝕂*-algebra with* $k \neq 1$ *which is not a locally m-(k-convex)* 𝕂*-algebra?*

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ON SYMPLECTIC STRUCTURE OF THE TOPOLOGICAL QUANTUM FIELD THEORY.

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1. The Lagrangian of the topological quantum field theory ([4]) on four dimensional compact Riemannian manifold M has the form

$$\mathcal{L} = \int_M de\ \mathrm{Tr}\Big(\sum_{i=1}^{3}\zeta_i + \sum_{i=1}^{3}\xi_i + \zeta_0 + \xi_0\Big), \tag{1.1}$$

where de is the volume element on manifold M and

$$\begin{aligned} \zeta_1 &= \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}, & \xi_1 &= \tfrac{1}{2}\phi D_\alpha D^\alpha\lambda, \\ \zeta_2 &= iD_\alpha\psi_\beta\cdot\chi^{\alpha\beta}, & \xi_2 &= -i\eta D_\alpha\psi^\alpha, \\ \zeta_3 &= -\tfrac{i}{8}\phi[\chi_{\alpha\beta},\chi^{\alpha\beta}], & \xi_3 &= -\tfrac{i}{2}\lambda[\psi_\alpha,\psi^\alpha], \end{aligned} \tag{1.2}$$

$$\zeta_0 = \tfrac{i}{2}\phi[\eta,\eta],\ \xi_0 = -\tfrac{1}{8}[\phi,\lambda]^2.$$

Let us explain the notations used in (1.2). There is a principal fiber bundle P(M,G) over manifold M with compact semisimple Lie group G, where the Lie algebra of this group is denoted by $\underline{G}$. The invariant Killing form of this algebra is denoted by Tr. The local coefficients of the connection 1-form ω define the gauge field $A_\alpha(x)$ the strength tensor of which is $F_{\alpha\beta}(x)$. So, the term ζ_1 is the Yang-Mills Lagrangian. We will denote the curvature 2-form of the connection ω by Θ^ω, so the local coefficients of the form Θ^ω are the components of the strength tensor F. The ϕ and λ are the auxiliary spinless fields in the adjoint representation of the gauge group G. The covariant derivative with respect to gauge field $A_\alpha(x)$ is denoted by $D_\alpha = \partial_\alpha + [A_\alpha, \ ]$ and we will denote the corresponding covariant differential by D^ω. It is convenient to introduce the associated vector bundle $E = P\times_G\underline{G}$, where G acts on $\underline{G}$ by adjoint representation. Then the fields ϕ and λ can be considered as sections of the E. If we denote the space of all smooth sections of E by $\mathcal{F}(E)$ then $\phi,\lambda \in \mathcal{F}(E)$. The space of the smooth k-form on M with coefficients in E will be denoted by $\Omega^k(E)$. So, we have $\Theta^\omega \in \Omega^2(E)$.

Now, we will consider the fields η, ψ_α, $\chi_{\alpha\beta}$, where α,β = = 1,...,dimM. They are fermion fields of this theory and take their values in the vector bundle E. It is convenient to consider the 0-form η, 1-form $\psi = \psi_\alpha dx^\alpha$ and 2-form χ = $= \chi_{\alpha\beta} dx^\alpha \wedge dx^\beta$ so that the 2-form χ is self-dual $\chi = *\chi$. We shall call such forms fermion forms. It has been shown in [1] how one can construct the infinite dimensional Grassmann algebra with scalar product and involution on vector bundle using the theory of de Rham currents. This construction has allowed us to interpret fermion type forms such as η, ψ, χ as generators of the Grassmann algebra. So, the correct mathematical approach to fermion forms η, ψ, χ has been described in [1]. It is well known ([4]) that the form's ψ coefficients can be interpreted as differentials on the space of all connections. So, it is naturally to suppose that all quantities of the quantum topological field theory which contain fermion fields can be interpreted as differential forms on infinite dimensional proper space. In this paper we study the Lagrangian and show how it can be described by the language of differential forms on some infinite dimensional vector bundle. Also, we will write the Lagrangian in a more general form using the differential forms. It is shown that the term ζ, is closely related to the symplectic structure on the orbits of co-adjoint representation.

2. Let $\mathfrak{U}$ be the affine space of all irreducible smooth connections on principal fiber bundle P(M,G). The group of gauge transformations $\mathfrak{G}$ acts on $\mathfrak{U}$ and $\mathfrak{U}$ is the principal fiber bundle with respect to this action. Let $\mathfrak{U}_o = \mathfrak{U}/\mathfrak{G}$ be the base space of this principal fiber bundle or the space of all classes of gauge equivalent connections. Let us denote by $\Omega^2_+(E)$ the space of self-dual 2-forms on M with coefficients in E.

In this paper we suppose that the main geometrical space of the quantum field theory in dimension 4 is the space $\mathfrak{B}$ = $= \mathfrak{U} \times \underline{\mathfrak{G}}^* \times \Omega^{*2}_+(E)$, where $\underline{\mathfrak{G}}^*$ is the dual space to the Lie algebra $\underline{\mathfrak{G}}$ and the $\Omega^{*2}_+(E)$ is the dual space to $\Omega^2_+(E)$. In other words, one can say, using the terminology of [1], that $\underline{\mathfrak{G}}^*$ is the space of homogeneous de Rham currents on the vector bundle E and $\Omega^{*2}_+(E)$ is the space of self-dual homogeneous de Rham currents on E. The fixed point of this space will be denoted by $z_o = (\omega_o, \theta_o, \Theta_o)$. It is obvious that the tangent space $T_{z_o}\mathfrak{B}$

at the point z_o has the form of the sum

$$T_{z_o}\mathfrak{B} = T_{\omega_o}\mathfrak{U} + T_{\Theta_o}\underline{\mathfrak{G}}^* + T_{\Theta_o}\Omega_+^{*2}(E). \tag{2.1}$$

Thus $\mathfrak{U}$ is the affine space, it is natural to identify the space $T_{\omega_o}\mathfrak{U}$ with the space $\Omega^1(E)$ of the 1-form on E. Since the spaces $\underline{\mathfrak{G}}^*$, $\Omega_+^{*2}(E)$ are linear vector spaces one can make the identifications

$$T_{\Theta_o}\underline{\mathfrak{G}}^* = \underline{\mathfrak{G}}^*, \; T_{\Theta_o}\Omega_+^{*2}(E) = \Omega_+^{*2}(E). \tag{2.3}$$

So, the tangent vector $\mathcal{X}$ at the point z_o will be defined by components

$$\mathcal{X} = (\sigma, \theta, \Theta),$$

where $\sigma \in \Omega'(E)$, $\theta \in \underline{\mathfrak{G}}^*$, $\Theta \in \Omega_+^{*2}(E)$.

The definition of cosymplectic structure on manifold was given in [3]. In this paper, we give the analogous definition in the case of vector fiber bundles.

D E F I N I T I O N. *Cosymplectic structure on the vector fiber bundle* E *is the* $\mathcal{F}(E)$*-bilinear, skewsymmetric form* μ_E *on the space* $\Omega^1(E)$ *with values in* $\mathcal{F}(E)$*, i.e.*

$$\mu_E : \Omega^1(E) \times \Omega^1(E) \to \mathcal{F}(E). \tag{2.5}$$

It should be noted that in the case of the vector bundle $E = P \times_G \underline{G}$ the cosymplectic form μ_E is the 2-form on tangent space $T_\omega\mathfrak{U}$ with values in the Lie algebra $\underline{\mathfrak{G}}$.

Let L_ω be the differential operator on the vector bundle E such that

$$L_\omega : \Omega'(E) \to \mathcal{F}(E) \approx \underline{\mathfrak{G}}. \tag{2.6}$$

This operator depends on the connection ω. We say that the operator L_ω is consistent with the cosymplectic structure μ if the following consistency condition is satisfied

$$L_{\omega+\sigma}(\tau) - L_\omega(\tau) = \mu_E(\sigma,\tau). \tag{2.7}$$

Then the pair (μ_E, L_ω) allows us to define the differential 2-forms on the space $\mathfrak{B}$ which are closely related to the terms ξ_2 and ξ_3 in the Lagrangian $\mathcal{L}$. Let $\mathcal{X} = (\sigma, \theta, \Theta)$, $\mathcal{X}' = (\sigma', \theta', \Theta')$ be the pair of tangent vectors to $\mathfrak{B}$ at the point z_o. The 2-forms ξ_2, ξ_3 on the space $\mathfrak{B}$ are defined by the following formulas

$$\xi_2(\mathcal{X},\mathcal{X}') = \frac{1}{2}(\theta(L_{\omega_o}\sigma') - \theta'(L_{\omega_o}\sigma)) \tag{2.8}$$

$$\xi_3(\mathcal{X},\mathcal{X}') = \Theta_o(\mu_E(\sigma,\sigma')).$$

It should be noticed that the forms ξ_2, ξ_3 are defined at the point $z_0 = (\omega_0, e_0, \Theta_0)$.

P R O P O S I T I O N 1. *The 2-form* $\xi = \xi_2 + \xi_3$ *on the space* $\mathfrak{B}$ *is closed , i. e.*

$$d\xi = 0. \tag{2.9}$$

In order to prove (2.9), one can use the formula

$$\frac{1}{2}\varepsilon^{ijk}(\xi_{z_0+x_i} - \xi_{z_0})(x_j, x_k) = d\xi_{z_0}(x_1, x_2, x_3) + \\ + O(|x|), \tag{2.10}$$

where ε^{ijk} is a totally skewsymmetric tensor. The left side of (2.10) in the case of the 2-forms (2.8) has the form

$$\frac{1}{2}\varepsilon^{ijk}[L_{\omega_0+\sigma_i}(\sigma_k) - \frac{1}{2}e_k L_{\omega_0+\sigma_i}(\sigma_j) + (\Theta_0 + e_i)\mu_E(\sigma_j,\sigma_k) - \\ - \frac{1}{2}e_j(L_{\omega_0}(\sigma_k)) + \frac{1}{2}e_k(L_{\omega_0}(\sigma_j)) - \Theta_0(\mu_E(\sigma_j,\sigma_k))].$$

Taking into account (2.7), one can reduce this expression to the following one

$$\frac{1}{2}\varepsilon^{ijk}[\frac{1}{2}e_j(\mu_E(\sigma_i,\sigma_k)) - \frac{1}{2}e_k(\mu_E(\sigma_i,\sigma_j)) + e_i(\mu_E(\sigma_j,\sigma_k))] =$$

$$\frac{1}{2}[\varepsilon^{ijk}e_j(\mu_E(\sigma_i,\sigma_k)) + \varepsilon^{ijk}e_i(\mu_E(\sigma_j,\sigma_k))] = 0$$

There is natural cosymplectic structure in the case of the vector bundle $E = P \times_G \mathfrak{G}$ with $\dim M = 4$. Let us define this structure by the formula

$$de \cdot \mu_E(\sigma,\tau) = [\sigma, *\tau], \tag{2.11}$$

where *de* is the volume element on the Riemannian manifold M. Then the differential operator δ^ω (adjoint operator to covariant differential D^ω) is consistent with cosymplectic structure (2.11).

D E F I N I T I O N. *The* $\mathcal{F}(E)$*-bilinear, symmetric form* $\nu_E: \Omega^1(E) \times \Omega^1(E) \to \Omega^2(E)$ *on the space* $\Omega^1(E)$ *is called symplectic structure of second order on the vector bundle* E.

Let R_ω be a differential operator which depends on the connection $\omega \in \mathfrak{U}$ so that

$$1)\ R_\omega: \Omega^1(E) \to \Omega^2(E), \tag{2.12}$$
$$2)\ R_{\omega+\sigma}(\tau) - R_\omega(\tau) = \nu_E(\sigma,\tau), \qquad \sigma,\tau \in \Omega^1(E).$$

If the conditions (2.12) are satisfied, we say that operator R_ω is consistent with symplectic structure of the second order. The pair (ν_E, R_ω) allows us to define the differential 2-form on the space $\mathfrak{B}$ by the formula

$$\zeta_2(\mathcal{X},\mathcal{X}') = \frac{1}{2}[\Theta(R_{\omega_0}\sigma') - \Theta'(R_{\omega_0}(\sigma))], \tag{2.13}$$

where $\mathcal{X},\mathcal{X}'$ are the tangent vectors to $\mathfrak{B}$.

P R O P O S I T I O N 2. *The differential 2-form ζ_2 is closed, i.e.*

$$d\zeta_2 = 0. \tag{2.14}$$

Proposition 2 can be proved in the same way as the Proposition 1 by means of the formula (2.10). It should be noticed that 2-form ζ_2 is closely related to the term ζ_2 in the Lagrangian (1.1). One can define the natural symplectic form ν_E of the second order on the vector bundle $E = P \times_G \mathfrak{G}$ by the formula

$$\nu_E(\sigma,\tau) = *[\sigma,\tau]. \tag{2.15}$$

Then the operator $R_\omega = *D^\omega$ is consistent with symplectic structure (2.15).

3. In this section the terms ζ_0 and ξ_0 of the Lagrangian (1.2) will be studied. It will be shown that ζ_0 is closely connected with symplectic 2-form of Kirillov and ξ_0 is the curvature 2-form of the invariant Riemannian metric on Lie algebra.

Firstly, let us consider the finite dimensional semisimple Lie algebra $\mathfrak{G}$ with Killing form k. The Killing form k induces the bilaterally invariant Riemannian metric $\tilde{k}$ whose curvature in the direction of the two orthonormal vectors $a,b \in \mathfrak{G}$ is ([2])

$$K_{(a,b)} = \frac{1}{4}k([a,b],[a,b]). \tag{3.1}$$

Let $\mathfrak{G}^*$ be the dual space to $\mathfrak{G}$. The form k allows us to identify the spaces $\mathfrak{G}$ and $\mathfrak{G}^*$. Namely, if $\Theta \in \mathfrak{G}^*$, then

$$\Theta(b) = k(a_\Theta,b), \tag{3.2}$$

and we identify the elements $\Theta \in \mathfrak{G}^*$ and $a_\Theta \in \mathfrak{G}$. Let $\mathfrak{G}^*_\Theta$ be the orbit of co-adjoint representation of the Lie group G passing through the element Θ and B be the corresponding symplectic 2-form of Kirillov. Let b_ρ be element of the Lie algebra $\mathfrak{G}$ and ρ be the corresponding (in the sense of (3.2)) element of the dual space $\mathfrak{G}^*$. Then the element $[b_\rho,b']$ is the tangent vector to the orbit $\mathfrak{G}^*_\rho$ at the point ρ. If there are two tangent vectors $\mathcal{X}' = [b_\rho,b']$, $\mathcal{X}'' = [b_\rho,b'']$, then

$$B_\rho(\mathcal{X}',\mathcal{X}'') = k(b_\rho,[b',b'']). \tag{3.3}$$

The Lie algebra element $[b_\rho,a_\Theta]$ induces the vector field $\mathcal{X}^\Theta_\rho = [b_\rho,[b_\rho,a_\Theta]]$ for every $\Theta \in \mathfrak{G}^*$ which is the tangent vector to the orbit $\mathcal{G}^*_\rho$. One can consider the curvature (3.1) as a function on the dual space $\mathcal{G}^*$

$$K(\theta,\rho) = \frac{1}{4}k([a_\theta,b_\rho],\ [a_\theta,b_\rho]). \qquad (3.4)$$

Let us remind that the vector field $\mathcal{X}$ is called strictly Hamiltonian if there exists such a function F that the equation

$$i(\mathcal{X})B = -dF, \qquad (3.5)$$

is satisfied. We say that the vector field $\mathcal{X}_t$ which depends on some parameters $t = (t^1,\dots,t^N)$, is the quasi-Hamiltonian vector field, if there exists such a function F_t which also depends on the same parameters $t = (t^1,\dots,t^N)$ so that the equation

$$i(\mathcal{X}_t)B = -d_t F_t, \qquad (3.6)$$

is satisfied.

P R O P O S I T I O N 3. *The vector field* $\mathcal{X}^\theta_\rho$ *considered along the orbit* $\mathcal{G}^*_\rho$ *is quasi-Hamiltonian vector field, i.e. the 1-form* $i(\mathcal{X}^\theta_\rho)B_\rho$ *is exact with respect to* θ *at each point* $\rho \in \mathcal{G}^*_\rho$ *and, moreover,*

$$\frac{1}{2}i(\mathcal{X}^\theta_\rho)B_\rho = -d_\theta K(\theta,\rho), \qquad (3.7)$$

where d_θ *means exterior differentiation with respect to* θ *(*ρ *is the fixed point).*

So, we are able now to give the geometrical description of the terms ζ_o and ξ_o in Lagrangian (1.2). Obviously, the term ζ_o is the symplectic 2-form of Kirillov if one consider it along the orbits of co-adjoint representation of the Lie group $\mathcal{G}$ of the gauge transformations. The term ξ_o is the curvature in the direction of the elements $\phi,\lambda \in \underline{\mathcal{G}}$. Equation (3.7) asserts that the sum $\zeta_o + \xi_o$ is invariant under the supersymmetry:

$Q\lambda = 2i\eta$ (this is the analog of exterior differentiation),
$Q\phi = 0$ (it means that ϕ is the fixed point),
$Q\eta = \frac{1}{2}[\phi,\lambda]$ (the interior product with vector field $[\phi,\lambda]$).

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LORENTZIAN SPACE-HOMOGENEOUS MANIFOLDS

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Standard cosmological models are based on the assumption that our Universe is a space-homogeneous 4-dimensional Lorentzian manifold, that is a 4-dimensional Lorentzian manifold with the isometry group G whose orbits are space-like hypersurfaces. We give the classification of all such manifolds.

P R O P O S I T I O N 1. *Let* (M,g) *be a Lorentzian manifold. Assume that a connected isometry group* G *of* (M,g) *has an orbit that is a space-like hypersurface. Then the following conditions are equivalent:*

1) *Almost all orbits of* G *(up to countable numbers ones) are space-like submanifolds.*

2) *Any neighbourhood of a point of* M *has a non-trivial intersection with an orbit that is a space-like hypersurface.*

3) *All orbits of* G *are space-like hypersurfaces.*

A Lorentzian manifold (M,g) that satisfies one of the equivalent conditions of Proposition 1 is called **space-homogeneous Lorentzian G-manifold.**

Let (M,g) be a space-homogeneous Lorentzian G-manifold. Changing the sign of the metric g in the directions tangent to G-orbits we obtain a Riemannian G-invariant metric g^+. This leads to the following

P R O P O S I T I O N 2. *There exists natural one-to-one correspondence*

$$(M,g) \longleftrightarrow (M,g^+)$$

between space-homogeneous Lorentzian G-manifolds and Riemannian manifolds that have an isometry group G with all orbits of codimension one.

R e m a r k. This correspondence was established by L.Berard-Bergery [1]. We note that it breaks for arbitrary Lorentzian manifolds with an isometry group that has a space-like hypersurface as an orbit (and, hence, one-dimensional orbit space). A good counterexample is the Minkowski space with the Lorentz group G whose orbit space is a crux.

Proposition 2 reduces the problem of classification of space-homogeneous Lorentzian manifolds to the problem of classification of Riemannian manifolds with an isometry group G that has codimension one orbits. The structure of such manifolds was determined in [2,3].

P R O P O S I T I O N 3 [2,3]. *Let (M,g) be a Riemannian manifold with an isometry group* G *that has codimension one orbits. Then the orbit space* $\Omega = M/G$ *is isomorphic to one of the spaces:*

i) $\mathbb{R}$ *ii)* S^1 *iii)* $\mathbb{R}^+$ *iv)* $[0,\pi]$.

In the case i) the manifold M can be identified with the direct product $M = \mathbb{R} \times G/K$ equipped with a metric of the form

$$g_0 = dt^2 + g_t \, , \tag{i}$$

where g_t is a one-parameter family of invariant metrics on a homogeneous space G/K. Any manifold (M,g) with the orbit space S^1 or $\mathbb{R}$ may be obtained by factorizing some manifold $(\mathbb{R} \times G/K, g_0)$ by the discrete isometry group $\Gamma_\tau = \{T_\tau^n \, , \, n \in \mathbb{Z}\}$ or, respectively, $\Gamma_\sigma = \{R_\sigma^n \, , \, n \in \mathbb{Z}\}$, where

$$T_\tau : (t,bK) \longrightarrow (t + 2\pi, b\tau K)$$

$$R_\sigma : (t,bK) \longrightarrow (-t, b\sigma K)$$

and τ,σ are elements of the group $N = N_G(K)/K$ with $\sigma^2 = 1$.

Any manifold (M,g) with the orbit space $[0,\pi]$ may be obtained by G-equivariant gluing together two G-manifolds with the boundary of the form

$$M_+ = \big([-\pi/4,\pi/4] \times G/K\big)/\Gamma_\sigma \quad ,$$

$$M_- = \big([-\pi/4,\pi/4] \times G/K\big)/\Gamma_{\sigma'} \, .$$

This result reduces the problem of classification of 4-dimensional space-homogeneous Lorentzian manifolds to the following problems:

1. Determine all 3-dimensional homogeneous Riemannian manifolds (G/K,g);

2. For each such manifold G/K, describe elements of the group $N = N_G(K)/K$ up to a conjugation;

3. Describe all involutions in N up to a conjugation.

All necessary information for the solution of these problems in the case when G is a connected compact Lie group is contained in the Table 1. We indicate the dimension d of the manifold of invariant metrics on G/K, the possible order ordτ of elements $\tau \in N = N_G(K)/K$, the set Inv(N) of involutions in N and the number n of the involutions and also the possible values tw = ord($\sigma \cdot \sigma'$) of the order of the product $\sigma \cdot \sigma'$ of two involutions. This invariant is called a twist and it has an important geometrical interpretation [3].

We denote by Γ the symmetry group of a perfect polyhedra in $\mathbb{R}^3$, $\tilde{\Gamma}$ is it preimage under the projection $SU_2 \to SO_3$, T is a maximal torus of SU_2, $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

T a b l e 1

G	K	d	N	ordτ	Inv(N)	n	tw = ord($\sigma \cdot \sigma'$)
T^3	1	6	T^3	$\forall n \in \hat{\mathbb{N}}$	$(\pm1, \pm1, \pm1)$	7	1,2
SU_2	1	6	SU_2	$\forall n \in \hat{\mathbb{N}}$	-id	1	1
SU_2	$\mathbb{Z}_k$	2	$\mathbb{Z}_2 : T^1$	$\forall n \in \hat{\mathbb{N}}$	-id	1	1
SU_2	$\mathbb{Z}_2 \cdot \mathbb{Z}_k$	2	$\mathbb{Z}_2$	1,2	-id	1	1
SU_2	$\tilde{\Gamma}$	1	1	1	$\emptyset$	0	–
SO_3	1	6	SO_3	$\forall n \in \hat{\mathbb{N}}$	$\{S_v, v \in \mathbb{R}^3\}$	∞	$\forall n \in \hat{\mathbb{N}}$
SO_3	$\mathbb{Z}_2$	2	$T^1 \times \mathbb{Z}_2$	$\forall n \in \hat{\mathbb{N}}$	(-1,1),(1,-1),(-1,-1)	3	1,2
SO_3	$\mathbb{Z}_k$ k>2	2	O_2	$\forall n \in \hat{\mathbb{N}}$	T^1	∞	$\forall n \in \hat{\mathbb{N}}$
SO_3	D_k	2	T^1	$\forall n \in \hat{\mathbb{N}}$	-1	1	1
SO_3	Γ	1	1	1	$\emptyset$	0	–
$T^1 \times SU_2$	$\{e^{in\varphi}, e^{i\varphi}\}$	2	$\mathbb{Z}$	∞	-1	1	1
$T^1 \times SO_3$	$(t^n, t) \approx T^1$	2	$T^1 \times \mathbb{Z}_2$	$\forall n \in \hat{\mathbb{N}}$	$(\pm1, \pm1) \neq (1,1)$	3	1,2
$T^1 \times SO_3$	$O_2 \subset SO_3$	2	T^1	$\forall n \in \hat{\mathbb{N}}$	-1	1	1
$T^1 \times SO_3$	$O_2 \not\subset SO_3$	2	T^1	$\forall n \in \hat{\mathbb{N}}$	-1	1	1
$\frac{T^1 \times SU_2}{\mathbb{Z}_2}$	(t^n, t) n=2m+1	2	$N_{SU_2}(T)$	$\forall n \in \hat{\mathbb{N}}$	-id	1	1
SO_4	SO_3	1	$\mathbb{Z}_2$	1,2	-1	1	1
$SO_4/\mathbb{Z}_2$	SO_3	1	1	1	$\emptyset$	0	–

3*

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ON A HOMEOMORPHISM OF CONTINUOUS MAPPINGS

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The results of section 1 have been proved by V. I. Varankina and by B. A. Pasynkov jointly, section 2 is due to B. A. Pasynkov and section 3 is due to I. V. Bludova.

Below a space is a topological space and a continuous mapping is a continuous mapping of spaces.

1. In case of spaces we have two following statements (see for example [1], 3.12.21):

(A) *For any Tychonoff space X the space I(X) of all maximal ideals in the ring $C^*(X)$ (of all bounded continuous real-valued functions defined on X) is the Stone-Cech compactification of X.*

(B) *Compact Hausdorff spaces X and Z are homeomorphic iff the rings $C^*(X)$ and $C^*(Z)$ are isomorphic.*

We are going to prove generalizations of these statements in the case of continuous mappings.

$f: X \to Y$ is (see[9]):

completely regular if for any point $x \in X$ and any closed in X set F, $x \notin F$, there exists a neighbourhood O of fx and a continuous function $a: f^{-1}O \to [0,1]$ such that $x \in f^{-1}O \cap a^{-1}(0)$, $F \cap f^{-1}O \subset a^{-1}(1)$;

T_0-mapping if for any $y \in Y$ and $x, x' \in f^{-1}y$, $x \neq x'$, $x \in \mathrm{int}(X \setminus \{x'\})$ or $x' \in \mathrm{int}(X \setminus \{x\})$;

Tychonoff if f is completely regular T_0-mapping.

It is well-known (see for example [3]) that for any space X there exists a Tychonoff space τX and a continuous mapping $\tau_X : X \longrightarrow \tau X$ such that for any continuous mapping $f : X \longrightarrow Y$ there exists a continuous mapping $\tau f : \tau X \longrightarrow \tau Y$ with following property: $\tau_Y \circ f = (\tau f) \circ \tau_X$.

Let $f : X \longrightarrow Y$ be a Tychonoff mapping, Θ be the topology of Y and $X_O = f^{-1}O$, $O \in \Theta$.

Let us take the system $C^*(f)$ of the rings

$$C^*_O(f) = C^*(X_O) = C^*(\tau X_O), \qquad O \in \Theta,$$

and their homomorphisms

$$i^*_{UVf} : C^*_U(f) \longrightarrow C^*_V(f)$$

(induced by identical embeddings $i_{UVf} : f^{-1}U \longrightarrow f^{-1}V$ when $U \subset V$, $U, V \in \Theta$) as an analogue of the ring $C^*(Z)$ for a space Z. It is clear that $C^*(f)$ is a sheaf of rings.

Let us define a continuous mapping $J(f) : J_f X \longrightarrow Y$ which is an analogue of the space of all maximal ideals in the ring $C(Z)$ for a space Z.

For any $O \in \Theta$ the space β_{Of} of all maximal ideals in the ring $C^*_O(f)$ is the Stone-Cech compactification of τX_O. Let

$$j_{Of} = i_{Of} \circ \tau_{Of}$$

where $\tau_{Of} = \tau_{X_O}$ and i_{Of} is the natural embedding of τX_O into β_{Of} (see [1], 3.12.21(c)). It would be important for us that the set $j_{Of} X_O$ is dense in β_{Of}. The mapping f and j_{Of} define continuous mapping

$$l_{Of} : X \longrightarrow P_{Of}$$

to the partial topological product (see [8], [2])

$$P_{Of} = (Y, \beta_{Of}, O \in \Theta)$$

such that

$$f = p_{Of} \circ l_{Of}, \quad j_{Of} = q_{Of} \circ l_{Of}/X_O \tag{1}$$

where $p_{Of} : P_{Of} \longrightarrow Y$ is the projection of the partial product P_{Of} onto the base Y (see [8], [2]) and q_{Of} is the projection of the product $O \times \beta_{Of}$ onto its factor β_{Of}. The diagonal product $l(f) = \Delta(l_{Of} : O \in \Theta)$ is continuous and the image $l(f)X$ is a subset of the partial topological product ([8], [2])

$$P(f) = (Y, \{\beta_{Of}\}, \{O\};\ O \in \Theta).$$

If $p(f):P \longrightarrow Y$, $\pi_{of}:P(f) \longrightarrow P_{of}$, $O \in @$, are the projections ([8],[2]) then ([8],[2])

$$f = p(f) \circ l(f) \ , \ l_{of} = \pi_{of} \circ l(f) \ , \ O \in @. \qquad (2)$$

Let $J_f X$ be the closure of $l(f)X$ in $P(f)$ and $J(f)$ be the restriction of $p(f)$ to $J_f X$. As all spaces β_{of}, $O \in @$ are Tychonoff and compact so the projection $p(f)$ and the mapping $J(f)$ are Tychonoff and compact (=perfect) (see [9], [8]).

If $x \in X$, a set F is closed in X and $x \notin F$ then there exist $O \in @$ and a continuous function $a:X_o \longrightarrow [0,1]$ such that $x \in X_o \cap a^{-1}(0)$ and $F \cap X_o \subset a^{-1}(1)$. Then

$$j_{of}x \notin [j_{of}(X_o \cap F)], \ l_{of}x \notin [l_{of}F] \text{ and } l(f)x \notin [l(f)F].$$

If $fx \neq fx'$ and $x \neq x'$ then $x \notin F = [\{x'\}]$ and so $l(f)x \neq l(f)x'$. We have proved that $l(f)$ is a topological embedding. So $J(f)$ is a Tychonoff compactification of f. If $O \in @$ and $a:X_o \longrightarrow [0,1]$ is a continuous function then there exists a continuous function $\bar{a}:\beta_{of} \longrightarrow [0,1]$ such that $1 = \bar{a} \circ j_{of}$. Then (see (1), (2))

$$a = \bar{a} \circ q_{of} \circ l_{of}/X_o = \bar{a} \circ q_{of} \circ \pi_{of} \circ l(f)/X_o \ .$$

Thus the mapping $\bar{a} \circ q_{of} \circ \pi_{of}$ is a continuous extension of the mapping a over $(J(f))^{-1}O \subset (p(f))^{-1}O$ if one identify X and $l(f)X$. So (see [9]) $J_f X$ coincides with the maximal Tychonoff compactification $\beta f:\beta_f X \longrightarrow Y$ of f (see [9]). We have proved

T H E O R E M 1. *$J(f) \equiv \beta f$ for any Tychonoff mapping f.*

C O R O L L A R Y 1. *$J(f) \equiv f$ for any Tychonoff compact mapping f (more exactly, $l(f)$ is a homeomorphism of X and $J(X)$ such that $f = J(f) \circ l(f)$).*

R E M A R K 1. Theorem 1 is a generalization of the statement (A) because for any Tychonoff space X and its mapping f onto the one-point space Y we have $C^*(X) \equiv C^*(f)$ and $J(X) \equiv J(f)$.

For two continuous mappings $f:X \longrightarrow Y$ and $g:Z \longrightarrow Y$ a homeomorphism $l:X \longrightarrow Z$ is called *a homeomorphism of f and g* if $f = g \circ l$. Two continuous mappings are *homeomorphic* if there exists their homeomorphism.

T H E O R E M 2. *If for Tychonoff mappings $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ the sheafs of rings $C^*(f)$ and $C^*(g)$ are isomorphic then $\beta f \equiv \beta g$ (more exactly, βf and βg are homeomorphic).*

P r o o f. We can identify $C^*(f)$ and $C^*(g)$. Let

$C^* = C^*(f) \equiv C^*(g)$, $C^*_o = C^*_o(f) \equiv C^*_o(g)$, $\beta_o = \beta_{of} \equiv \beta_{og}$,

$P_o = P_{of} \equiv P_{og}$, $p_o = p_{of} \equiv p_{og}$, $q_o = q_{of} \equiv q_{og}$,

$P = P(f) \equiv P(g)$, $\pi_o = \pi_{of} \equiv \pi_{og}$, $O \in \Theta$,

$i^*_{UV} = i^*_{UVf} \equiv i^*_{UVg}$, $U, V \in \Theta$, $U \subset V$.

Then we have

$$\beta\tau i_{UV} = \beta\tau i_{UVf} = \beta\tau i_{UVg}, \quad U, V \in \Theta, \ U \subset V, \tag{3}$$

where the mapping $\beta\tau i_{UV}: \beta_U \longrightarrow \beta_V$ is generated by the homomorphism i^*_{UV} and $\beta\tau i_{UVf}: \beta\tau X_U \longrightarrow \beta\tau X_V$, $\beta\tau i_{UVg}: \beta\tau Z_U \longrightarrow \beta\tau Z_V$ are the unique continuous extensions of $\tau i_{UVf}: \tau X_U \longrightarrow \tau X_V$ and $\tau i_{UVg}: \tau Z_U \longrightarrow \tau Z_V$ correspondently.

It is enough for us to prove that the subspaces $J_f X$ and $J_g Z$ of $P = P(f) \equiv P(g)$ coincide. Let $J_f X \setminus J_g Z \neq \emptyset$. Then there exists a point $x \in X$ such that $t = l(f)x \notin J_g Z$. As $J_g Z$ is closed in P so there exist $O(k) \in \Theta$, a neighbourhood O of pt and the sets $U(k)$, $k = 1, \ldots, s$, open in $\beta_k = \beta_{O(k)}$, such that

$$\begin{aligned} t &\in \bigcap\{\pi^{-1}_{O(k)}(O \times U(k)): k = 1, \ldots, s\} = \\ &= p^{-1}O \cap \bigcap\{\pi^{-1}_{O(k)} q^{-1}_{O(k)} U(k): k = 1, \ldots, s\} \subset P \setminus J_g Z. \end{aligned} \tag{4}$$

The set $W = \bigcap\{(\beta\tau i_{OO(k)})^{-1} U(k): k = 1, \ldots, s\}$ is open in β_o. It is not empty because (see (3))

$$\begin{aligned} &\beta\tau i_{UV} \circ j_{Uf} = j_{Vf} \circ i_{UVf}, \\ &\beta\tau i_{UV} \circ j_{Ug} = j_{Vg} \circ i_{UVg}, \quad U, V \in \Theta, \ U \subset V \end{aligned} \tag{5}$$

and (see (1), (2), (4), (5))

$$\begin{aligned} &\beta\tau i_{OO(k)} \circ q_o \circ \pi_o(t) = \beta\tau i_{OO(k)} \circ q_o \circ \pi_o \circ l(f)(x) = \\ &= \beta\tau i_{OO(k)} \circ q_o \circ l_{of}(x) = \beta\tau i_{OO(k)} \circ j_{of}(x) = \end{aligned}$$

$$= j_{O(k)f} \circ i_{OO(k)}(x) \equiv j_{O(k)f}(x) =$$

$$= q_{O(k)} \circ 1_{O(k)f}(x) = q_{O(k)} \circ \pi_{O(k)} \circ 1(f)(x) =$$

$$= q_{O(k)} \circ \pi_{O(k)}(t) \in U(k)\ , \quad k = 1,\dots,s.$$

So $q_O \circ \pi_O(t) \in W$ and $W \neq \emptyset$. It follows from this that $j_{Og}Z_O \cap W = \emptyset$. But if $z \in g^{-1}O$ and $j_{Og}(z) \in W$ then (see (2)) $1(g)(z) \in p^{-1}O$ and (see (5), (1), (2))

$$\beta\pi i_{OO(k)} \circ j_{Og}(z) = q_{O(k)g} \circ i_{OO(k)g}(z) \equiv j_{O(k)g}(z) =$$

$$q_{O(k)} \circ 1_{OO(k)g}(z) = q_{O(k)} \circ \pi_{O(k)} \circ 1(g)(z) \in U,\ k = 1,\dots,s.$$

So (see (4)) $1(g)(z) \in P \setminus J_O Z$. It is impossible. Thus $J_f X \subset J_g Z$. Analogously, $J_g Z \subset J_f X$. So $J_g Z = J_f X$. Theorem is proved.

C O R O L L A R Y 2. *Compact Tychonoff mapping $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ are homeomorphic iff the sheafs of rings $C^*(f)$ and $C^*(g)$ are isomorphic.*

Corollary 2 is a generalization of the statement (B) (the case of the one-point space Y).

R E M A R K 2. In [6] a different variant of an analogue for continuous mapping of the ring $C^*(X)$ for a space X has been given and in [7] it has been used in order to obtain analogues of statements (A) and (B).

2. For a space X the symbol $C_p(X)$ denotes the ring of all continuous real-valued functions defined on X with the topology of pointwise convergence.

In [4] J. Nagata proved the following statement.

(C) *Tychonoff spaces X and Z are homeomorphic iff the topological rings $C_p(X)$ and $C_p(Z)$ are isomorphic.*

We would give a generalization of this statement over continuous mappings now.

Let $f: X \longrightarrow Y$ be a Tychonoff mapping, Θ be the topology of Y and $X_O = f^{-1}O$, $O \in \Theta$.

The system $C_p(f)$ of topological rings $C_{pO}(f) = C_p(X_O) \equiv$ $\equiv C_p(\tau X_O)$, $O \in \Theta$, and their continuous homomorphisms

$$i^*_{pUVf}: C_{pU}(f) \longrightarrow C_{pV}(f)$$

(induced by the identical embeddings $i_{UVf}: f^{-1}U \longrightarrow f^{-1}V$ when $U \subset V$, $U, V \in \Theta$) forms a sheaf of topological rings.

T H E O R E M 3. *Let continuous mappings $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ be such that Y is a T_0-space and for any $y \in Y$ there exists a neighbourhood O_y for which the inverse images $f^{-1}O_y$ and $g^{-1}O_y$ are Tychonoff spaces. Then f and g are homeomorphic iff the sheafs of topological rings $C_p(f)$ and $C_p(g)$ are isomorphic.*

We can give another approach to the question eliminating demands from the spaces X,Z,Y in Theorem 3.

Let we have a continuous mapping $f: X \longrightarrow Y$ again (we use cited above designations). For any $O \in \Theta$ we have the partial topological product $P_O = P(Y, \tau X_O, O)$, the projections

$$p_O: P_O \longrightarrow Y, \quad q_O: O \times \tau X_O \longrightarrow \tau X_O$$

and a continuous mapping $\mu_O: X \longrightarrow P_O$ such that $f = p_O \circ \mu_O$ and $\tau_{X_O} = q_O \circ 1_O / X_O$. The mappings μ_O define a continuous mapping $\mu(f): X \longrightarrow P$ into the partial topological product

$$P = P(Y, (\tau X_O), (O); \quad O \in \Theta)$$

such that

$$f = p \circ \mu(f) \quad \text{and} \quad \mu_O = \pi_O \circ \mu(f), \quad O \in \Theta,$$

where $p: P \longrightarrow Y$ and $\pi_O: P \longrightarrow P_O$, $O \in \Theta$, are the projections. Let $T_f X = [\mu(f)X]$ and Tf be the restriction of p to $T_f X$. The mappings p and Tf are Tychonoff because all τX_O are Tychonoff spaces (see[9]).

L E M M A 1. *If the mapping f is Tychonoff then $\mu(f)$ is a topological embedding and so $T(f)$ is a Tychonoff extension of f.*

For Tychonoff f the mapping T(f) would be called the *T-(≡Tychonoff) completion* of f. If $\mu(f): X \longrightarrow T_f X$ is a homeomorphism then f would be called *T-complete*.

T H E O R E M 4. *T-complete mappings $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ are homeomorphic iff the sheafs of topological rings $C_p(f)$ and $C_p(g)$ are isomorphic. More exactly, if for Tychonoff mappings $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ the sheafs $C_p(f)$ and*

$C_p(g)$ *are isomorphic then the mappings* Tf *and* Tg *are homeomorphic.*

P R O P O S I T I O N 1. *Any continuous mapping of any Tychonoff space to any Hausdorff space is T-complete.*

R E M A R K 3. Partially the results of section 2 were published in [10].

3. It is very simple to construct nonhomeomorphic mappings $f: X \longrightarrow Y$ and $g: Z \longrightarrow Y$ such that the sheafs $C_p(f)$ and $C_p(g)$ are isomorphic and either 1) X and Z are Tychonoff spaces or 2) Y is T_0-space.

T H E O R E M 5. [5] *For a Tychonoff mapping* $f: X \longrightarrow Y$ *the following statements are equivalent:*

(1) f *is T-complete;*

(2) there exist a partial topological product $P = P(Y, (T_\alpha), (O_\alpha); \alpha \in A)$ *and a topological embedding* $\pi: X \longrightarrow P$ *such that all spaces* T_α *are Tychonoff, the set* πX *is closed in* P *and* $f = p \circ \pi$ *where* $p: P \longrightarrow Y$ *is the projection of partial topological product;*

(3) for any point $x \in \beta f X \setminus X$ *there exist a neighbourhood* O *of the point* $\beta f(x)$ *in* Y *and a functionally open cover* w *of* $f^{-1}O$ *such that* x *does not belong to the closure in* $(\beta f)^{-1}U$ *of any* $U \in w$.

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ON FACTORIZATION OF UNIFORMLY CONTINUOUS MORPHISMS

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We transfer some notions and results concerning topological and uniform spaces to the class of uniform mappings and uniformly continuous morphisms.

Let f be a mapping from a set X into a topological space (Y,τ). A system $U = \{U_O : O \in \tau\}$ of pseudouniformities U_O on the sets $f^{-1}(O)$ is called *a pseudopreuniformity on the mapping f* [2] if for every $O,O' \in \tau$ such that $O' \subseteq O$ the identity embedding of $f^{-1}(O')$ into $f^{-1}(O)$ is a uniformly continuous mapping with respect to the pseudouniformities $U_{O'}$ and U_O. Moreover, if for every two distinct points $x,x' \in f^{-1}(y)$, $y \in Y$, there exist a neighbourhood O of y and a cover $\alpha \in U_O$ such that x' does not belong to $\alpha(x) = \cup\{A \in \alpha : x \in A\}$, then the pseudopreuniformity U on f is called *a preuniformity on f* and the pair (f,U) is called *a preuniform mapping*.

Let X' be another set. Let $U' = \cup\{U'_O : O \in \tau\}$ be a preuniformity on a mapping $f' : X' \rightarrow Y$. A mapping $\lambda : X \rightarrow X'$ satisfying condition $f = f'\cdot\lambda$ is called a *(U,U')-uniformly continuous morphism of the preuniform mapping f to the preuniform mapping f'* [2] if for every $O' \in \tau$ and every cover $\alpha' \in U'_{O'}$ there exists $O \in \tau$ such that $O \subseteq O'$ and $\lambda^{-1}(\alpha') \cap f^{-1}(O) \in U_O$.

Let $\alpha = \{A\}$ be a uniform cover of a set $f^{-1}(O)$. *The star of a set M in $f^{-1}(O)$ with respect to* α is the set $\alpha(M) = \cup\{A \in \alpha : A \cap M \neq \emptyset\}$. A set M is called *$\alpha$-dense* in $(f^{-1}(O),U_O)$ if for every $x \in f^{-1}(O)$ there exists $x' \in M$ such that $x \in \alpha(x')$, i.e. $\alpha(M) = f^{-1}(O)$. Let μ be a cardinal number. A pseudopreuniformity U on f is called *μ-bounded* if for every $O \in \tau$ and every $\alpha \in U_O$ there exists a set M which is α-dense in $f^{-1}(O)$ and $|M| = \mu$; if μ is finite, then (f,U) is called *totally bounded*.

P R O P O S I T I O N 1. *If $\lambda : X \to X'$ is a (U,U')-uniformly continuous morphism of a μ-bounded pseudopreuniform mapping $(f,U) : X \to Y$ to a pseudopreuniform mapping $(f',U') : X' \to Y$ then the mapping (f',U') is also μ-bounded.*

P R O P O S I T I O N 2. *If $(f,U) : X \to Y$ is a μ-bounded pseudopreuniform mapping, then for every $X' \subseteq X$ the induced pseudopreuniformity $U' = U|_{f'}$ on the mapping $f' = f|_{X'} : X' \to Y$ is μ-bounded.*

P R O P O S I T I O N 3. *Let X' be α-dense set in X with respect to the topology induced by the pseudopreuniformity U on f. If the mapping $(f',U') : X' \to Y$ is μ-bounded, then $(f,U) : X \to Y$ is also μ-bounded.*

Let U be a pseudopreuniformity on a mapping f. If the space Y contains a base B such that for every $O \in B$ the (dimension of the pseudopreuniformity U_O on $f^{-1}(O)$) $\dim(U_O) \le n$ (where the integer $n \ge -1$), then it is said that *(the dimension of the pseudopreuniformity U) $\dim(U) \le n$.* If in Y exists a base B such that for each $O \in B$ (the weight of U_O) $w(U_O) \le m$ (where m is an arbitrary cardinal), then it is said that *(the weight of U) $w(U) \le m$.*

Let $(f,U) : X \to Y$, $(f',U') : X' \to Y$ are preuniform mappings and let $\dim(U) = n$ and $w(U) = m$.

T H E O R E M. *Let $\phi : X \to X'$ be (U,U')-uniformly continuous morphism of a mapping f to μ-bounded mapping f' and Y be a space satisfying $w(Y) \le m$. Then there exist a set Z, μ-bounded preuniform mapping $(g,V) : Z \to Y$ and (U,V)- and (V,U')-uniformly continuous morphisms ψ of f to g and $\varkappa$ of g to f' respectively, such that $\dim(V) \le n$, $w(V) \le m$ and $\phi = \varkappa \cdot \psi$.*

If X, X′ are pseudouniform spaces and $f : X \to X'$ is a uniform mapping then W.Kulpa [1] proved the existence of a uniform space Z, such that $\dim(Z) \le \dim(X)$, $w(Z) \le w(X')$, and a factorization $X \xrightarrow{g} Z \xrightarrow{h} X'$ of f into uniform mappings. The main assertions of the Kulpa result follow from Theorem when (Y,τ) is a one-point space.

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ON A FUZZY UNIFORM STRUCTURE

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The aim of this note is to introduce and to study the concept of a fuzzy (quasi-)uniformity which would be consistent with a fuzzy topology as it was defined by the second author [6]. Our approach contains in itself the approach to quasi-uniformities developed by B.Hutton [3] as an important but very special (in a known sense crisp) case.
Hutton's definition of a fuzzy (quasi-)uniformity is given in Section 0 ; in the sequel such structures are called Hutton (quasi-)uniformities.

0. Preliminaries.

For a set X let I^X denote the family of all its fuzzy subsets (i.e. the family of all mappings M: $X \to I := [0,1]$).
Let D_X or just D denote the subset of $(I^X)^{I^X}$ such that

(D1) $U(M) \geq M$ for each $U \in D$ and each $M \in I^X$;

(D2) $U(\bigvee_\gamma M_\gamma) = \bigvee_\gamma U(M_\gamma)$ for each family $\{M_\gamma : \gamma \in \Gamma\} \subset I^X$.

It is easy to notice that $U_1, U_2 \in D$ implies $U_1 \wedge U_2 \in D$.
Besides, if $U \in D$, then $U^{-1} \in D$ where $U^{-1}(M) = \wedge\{N \in I^X : U(N^c) \leq M^c\}$ $(M^c := 1-M)$.

(0.1) D E F I N I T I O N [3]. A Hutton quasi-uniformity on a set X is a non-void subset $\mathcal{U} \subset D$ such that

(HU1) if $U \in \mathcal{U}$, $V \in D$ and $U \leq V$, then $V \in \mathcal{U}$;

(HU2) if $U_1, U_2 \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \leq U_1 \wedge U_2$;

(HU3) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \leq U$.

A Hutton quasi-uniformity is called a Hutton uniformity if

(HU4) $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$.

A pair $(X, \mathcal{U})$ will be called a Hutton (quasi-)uniform space.

(0.2) D E F I N I T I O N [3]. Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be Hutton (quasi-)uniform spaces and f: $X \to Y$ be a mapping. Then f is called uniformly continuous if for every $V \in \mathcal{V}$ there exists $U\in\mathcal{U}$ such that $U\le f^{-1}(V)$ (here $f^{-1}(V)(M) :=$ $:= f^{-1}(V(f(M)))$).

The theory of Hutton (quasi-)uniform spaces was developed by Hutton and other authors (see e.g. [1],[3],[5]). In particular, in [3] it was shown that given a Hutton quasi-uniformity $\mathcal{U}$, the formula $\mathcal{T}_{\mathcal{U}} = \{M\in I^X: M = \bigvee\{N: N\in I^X, \exists U\in\mathcal{U}, U(N)\le M\}\}$ defines a Chang fuzzy topology [2] $\mathcal{T}_{\mathcal{U}}$ on X and that if $f: (X,\mathcal{U})\to(Y,\mathcal{V})$ is uniformly continuous, then the mapping $f: (X,\mathcal{T}_{\mathcal{U}})\to(Y,\mathcal{T}_{\mathcal{V}})$ is continuous.

1. Fuzzy (quasi-)uniform spaces: basic notions.

(1.1) D E F I N I T I O N. A fuzzy quasi-uniformity on a set X is a non-zero mapping $\mathcal{U}: D\to I$ satisfying the following axioms:

(FU1) if $U,V\in D$ and $U\le V$, then $\mathcal{U}(U)\le\mathcal{U}(V)$;

(FU2) if $U,V\in D$, then $\mathcal{U}(U\wedge V) = \mathcal{U}(U)\wedge\mathcal{U}(V)$;

(FU3) $\mathcal{U}(U) = \sup\{\mathcal{U}(V): V\in D, V\circ V\le U\}$ for each $U\in D$.

A fuzzy quasi-uniformity is called a fuzzy uniformity if it satisfies additionally the following axiom:

(FU4) $\mathcal{U}(V) = \mathcal{U}(V^{-1})$ for each $V\in D$.

A pair $(X,\mathcal{U})$ is called a fuzzy (quasi-)uniform space.

(1.2) R E M A R K. Under assumption that (FU1) holds, the axioms (FU2) and (FU3) can be reformulated as follows:

(FU2') if $U,V\in D$, then $\mathcal{U}(U\wedge V)\ge\mathcal{U}(U)\wedge\mathcal{U}(V)$;

(FU3') for each $U\in D$ and each $\varepsilon>0$ there exists $V\in D$ such that $V\circ V\le U$ and $\mathcal{U}(V)>\mathcal{U}(U)-\varepsilon$.

(1.3) T H E D E C O M P O S I T I O N O F A F U Z Z Y (Q U A S I-) U N I F O R M I T Y I N T O A S Y S T E M O F H U T T O N (Q U A S I-)U N I F O R M I T I E S. Let $\mathcal{U}$ be a fuzzy (quasi-)uniformity on a set X and $h := \sup\{\mathcal{U}(U): U\in D\}$ $(=: h_{\mathcal{U}})$. Then for each $\alpha\in[0,h)$ $\mathcal{U}_\alpha := \{U\in D: \mathcal{U}(U)>\alpha\}$ is a Hutton (quasi-)uniformity on X. Thus a system $\Phi(\mathcal{U}) := \{\mathcal{U}_\alpha : \alpha\in[0,h)\}$ of Hutton (quasi-)uniformities on the set X is obtained. Besides this system is uppersemicontinuous in the sense that for each $\alpha\in[0,h)$ $\mathcal{U}_\alpha = \bigcup\{\mathcal{U}_\beta: \beta\in(\alpha,h)\}$ Conversely, given an uppersemicontinuous system $\Omega = \{\mathcal{U}_\alpha: \alpha\in[0,h)\}$ of Hutton (quasi-)uniformities on X and defining a mapping $\psi(\Omega): D\to I$ by the equality $\psi(\Omega)(U) = \sup\{\alpha: U\in\mathcal{U}_\alpha\}$ (or, equivalently, $\psi(\Omega)(U) = \inf\{\beta: U\notin\mathcal{U}_\beta\}$) we obtain a fuzzy (quasi-)uniformity $\psi(\Omega)$ on X such that $h_{\psi(\Omega)} = h$ and $(\psi(\Omega))_\alpha = \mathcal{U}_\alpha$. It is easy to conclude that the operators

Φ and Ψ thus defined are reciprocal: $\Psi(\Phi(\mathcal{U})) = \mathcal{U}$ for each fuzzy (quasi-)uniformity $\mathcal{U}$ and $\Phi(\Psi(\Omega)) = \Omega$ for each uppersemicontinuous system Ω of Hutton (quasi-)uniformities.

(1.4) DEFINITION. Let $(X,\mathcal{U}),(Y,\mathcal{V})$ be two fuzzy (quasi-)uniform spaces. A mapping $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ is called uniformly continuous if $\mathcal{U}(f^{-1}(V)) \geq \mathcal{V}(V)$ for each $V \in D_Y$.

(1.5) PROPOSITION. A mapping $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous iff for each $\alpha \in [0,h_{\mathcal{V}})$ the mapping $f: (X,\mathcal{U}_\alpha) \to (Y,\mathcal{V}_\alpha)$ of the corresponding Hutton (quasi-) uniform spaces is uniformly continuous.

(1.6) PROPOSITION. If the mappings $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ and $g: (Y,\mathcal{V}) \to (Z,\mathcal{W})$ of fuzzy (quasi-) uniform spaces are uniformly continuous, then the composition $g \circ f: (X,\mathcal{U}) \to (Z,\mathcal{W})$ is uniformly continuous.

(1.7) THE CATEGORY QFU AND SOME OF ITS SUBCATEGORIES. Let QFU denote the category the objects of which are fuzzy quasi-uniform spaces and the morphisms are uniformly continuous mappings between these spaces. Its full subcategory consisting of fuzzy uniform spaces will be denoted FU.

The category of Hutton (quasi-)uniform spaces and uniformly continuous mappings between them in an obvious way can be identified with the full subcategory H(Q)FU of the category (Q)FU the objects $(X,\mathcal{U})$ of which satisfy the following additional axiom:

(H) $\mathcal{U}(D) \subset 2 := \{0,1\} \subset I$.

2. Fuzzy topologies generated by fuzzy (quasi-)uniformities.

Recall that by a fuzzy topology on a set X we call [6] a mapping $\mathcal{T}: I^X \to I$ such that

(FT1) $\mathcal{T}(0) = \mathcal{T}(1) = 1$;

(FT2) $\mathcal{T}(M \wedge N) \geq \mathcal{T}(M) \wedge \mathcal{T}(N)$ for all $M,N \in I^X$;

(FT3) $\mathcal{T}(\bigvee_\gamma N_\gamma) \geq \bigwedge_\gamma \mathcal{T}(N_\gamma)$ for each family $\{N_\gamma : \gamma \in \Gamma\} \subset I^X$.

A mapping $f: (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ of fuzzy topological spaces is called continuous if $\mathcal{T}_X(f^{-1}(V)) \geq \mathcal{T}_Y(V)$ for each $V \in I^Y$ [6]. Let $(X,\mathcal{U}),(Y,\mathcal{V})$ be fuzzy (quasi-)uniform spaces. It is easy to establish the validity of the next statements:

(2.1) PROPOSITION. The mapping $\mathcal{T}_{\mathcal{U}} : I^X \to I$ defined by the equality

$\mathcal{T}_{\mathcal{U}}(M) = \sup\{\alpha : \bigvee\{N \in I^X : \exists U \in \mathcal{U}_\alpha \text{ s.t. } U(N) \leq M\} = M\}$
is a fuzzy topology on X.

(2.2) R E M A R K. Following [7], for a fuzzy topology $\mathcal{T}$ on X and $\alpha \in [0,1]$ let $\mathcal{T}_\alpha = \{U \in I^X : \mathcal{T}(U) \geq \alpha\}$. (Obviously, $\mathcal{T}_\alpha$ is a Chang fuzzy topology.) Then $(\mathcal{T}_{\mathcal{U}})_\alpha = \bigcap\{\tau_{\mathcal{U}_\beta} : \beta < \alpha\}$ where $\tau_{\mathcal{U}_\beta} := \{M \in I^X : \bigvee\{N \in I^X : \exists U \in \mathcal{U}_\beta, \mathcal{U}(N) \leq M\} = M\}$ are Chang fuzzy topologies generated by the Hutton quasi-uniformities $\mathcal{U}_\beta$.

(2.3) P R O P O S I T I O N. If a mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous, then the mapping $f : (X, \mathcal{T}_{\mathcal{U}}) \to (Y, \mathcal{T}_{\mathcal{V}})$ of the corresponding fuzzy topological spaces is continuous.

(2.4) F U Z Z Y (Q U A S I-) U N I F O R M I Z A T I O N O F A F U Z Z Y T O P O L O G Y. A fuzzy topology $\mathcal{T}$ on X is called (quasi-)uniformizable if there exists a fuzzy (quasi-)uniformity $\mathcal{U}$ on X such that $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$.

To describe the procedure of (quasi-)uniformization of a fuzzy topology effectively, consider the decomposition of $\mathcal{T}$ into its level Chang fuzzy topologies $\mathcal{T}_\alpha = \{M : \mathcal{T}(M) \geq \alpha\}$ $a \in [0,1]$ and let $\mathcal{U}_{\mathcal{T}_\alpha}$ be a Hutton quasi-uniformity generating $\mathcal{T}_\alpha$. For each $\alpha \in [0,1)$ let $\mathcal{U}^\alpha = \bigcup_{\beta > \alpha} \mathcal{U}_{\mathcal{T}_\beta}$; then $\{\mathcal{U}_\alpha : \alpha \in [0,1)\}$ is uppersemicontinuous family of Hutton (quasi-)uniformities on the set X. Hence the formula $\mathcal{U}(U) = \sup\{\alpha : U \in \mathcal{U}^\alpha\}$ defines a fuzzy (quasi-)uniformity $\mathcal{U}(:= \mathcal{U}_{\mathcal{T}})$ on X. Besides $\mathcal{T}_{\mathcal{U}}(M) = \sup\{\alpha : \bigvee\{N \in I^X : \exists U \in \mathcal{U}_\alpha, U(N) \leq M\} = M\} = \sup\{\beta : \bigvee\{N \in I^X : \exists U \in \mathcal{U}_{\mathcal{T}_\beta}, U(N) \leq M\} = M\} = \sup\{\beta : M \in \mathcal{T}_\beta\} = \sup\{\beta : \mathcal{T}(M) \geq \beta\} = \mathcal{T}(M)$, i.e. $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$.

This construction and [3, Theorem 7] imply

(2.5) P R O P O S I T I O N. Each fuzzy topological space is quasi-uniformizable.

(2.6) D E F I N I T I O N. A fuzzy topology $\mathcal{T}$ on X is called completely regular if there exists a decreasing system $\{\tau_\alpha : \alpha \in [0,1)\}$ of completely regular Chang fuzzy topologies [4],[3] such that $\mathcal{T}(M) = \bigvee\{\alpha : M \in \tau_\alpha\}$. (The system $\{\tau_\alpha : \alpha \in [0,1)\}$ will be called a completely regular decomposition of $\mathcal{T}$ in this case.)

(2.7) P R O P O S I T I O N. A fuzzy topology $\mathcal{T}$ is uniformizable iff it is completely regular.

(Indeed, if $\{\mathcal{T}_\alpha : \alpha\in[0,1)\}$ is a completely regular decomposition of a fuzzy topology $\mathcal{T}$, then reasoning in the same way as in (2.4) one can construct a Hutton uniformity $\mathcal{U}$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$. Conversely, given a fuzzy uniformity $\mathcal{U}$, consider Chang fuzzy topologies $\mathcal{T}_{\mathcal{U}_\alpha}$ induced by the level Hutton uniformities $\mathcal{U}_\alpha$. Then, noticing that $\mathcal{T}_{\mathcal{U}}(M) = \bigvee\{\alpha : M\in\mathcal{T}_{\mathcal{U}_\alpha}\}$ we conclude that $\mathcal{T}_{\mathcal{U}}$ is completely regular.)

3. Initial fuzzy (quasi-)uniformities.

(3.1) CONSTRUCTION OF INITIAL FUZZY (QUASI-) UNIFORMITIES. Let X be a set, $(Y,\mathcal{V})$ be a fuzzy (quasi-)uniform space and $f: X\to Y$ be a mapping. Define the mapping $f^{-1}(\mathcal{V}): D_X\to I$ by the equality $(f^{-1}(\mathcal{V}))(U) = \sup\{\mathcal{V}(V): f^{-1}(V)\le U,\ V\in D_Y\}$ for each $U\in D_X$. Then $f^{-1}(\mathcal{V})$ is the weakest fuzzy (quasi-)uniformity on X for which the mapping $f: (X,\mathcal{U})\to(Y,\mathcal{V})$ is uniformly continuous; it is called the initial fuzzy (quasi-)uniformity for the mapping f.

Let now $\{(Y_\gamma,\mathcal{V}^\gamma): \gamma\in\Gamma\}$ be a family of fuzzy (quasi-)uniform spaces and $f_\gamma : X\to Y_\gamma$ be a family of mappings. Let $f_\gamma^{-1}(\mathcal{V}^\gamma) =: \mathcal{U}^\gamma$ be the initial for f_γ fuzzy (quasi-)uniformity on X and let $\mathcal{U}^\gamma_\alpha$ be the corresponding α-level Hutton (quasi-)uniformity. Then $\bigcup_\gamma \mathcal{U}^\gamma_\alpha$ is the subbase for the Hutton (quasi-)uniformity $\mathcal{U}_\alpha = \sup_\gamma \mathcal{U}^\gamma_\alpha$ on X (cf [5]). Besides, since $\{\mathcal{U}^\gamma_\alpha : \alpha\in[0,1)\}$ is uppersemicontinuous, it easily follows that the system $\Omega = \{\mathcal{U}_\alpha : \alpha\in[0,1)\}$ is uppersemicontinuous, too. Hence the formula $\mathcal{U}(U) = \sup\{\alpha : U\in\mathcal{U}_\alpha\}$ defines a fuzzy (quasi-)uniformity on X which is obviously the weakest one for which all $f_\gamma : (X,\mathcal{U})\to(Y_\gamma,\mathcal{V}^\gamma)$ are uniformly continuous; $\mathcal{U}$ is called initial for this family of mappings.

(3.2) PRODUCTS. Let $(X_\gamma,\mathcal{U}^\gamma)$, $\gamma\in\Gamma$ be a family of fuzzy (quasi-)uniform spaces. The pair $(X,\mathcal{U})$ where $X = \prod_\gamma X_\gamma$ and $\mathcal{U}$ is the initial fuzzy (quasi-)uniformity for the family of all projections $p_\gamma: X\to(X_\gamma,\mathcal{U}^\gamma)$, $\gamma\in\Gamma$, is the product of these fuzzy (quasi-)uniform spaces in the category (Q)FU.

(3.3) P R O P O S I T I O N. If $\mathcal{U}$ is the initial fuzzy (quasi-)uniformity on X for the family $f_\gamma : X \to (Y_\gamma, \mathcal{V}^\gamma)$, $\gamma \in \Gamma$, and $\mathcal{T}$ is the initial fuzzy topology on X for the family of mappings $f_\gamma : X \to (Y_\gamma, \mathcal{T}^\gamma)$, where $\mathcal{T}^\gamma = \mathcal{T}_{\mathcal{V}^\gamma}$, then $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$.

(Indeed, from (2.3) it follows that all $f_\gamma : (X, \mathcal{T}_{\mathcal{U}}) \to (Y_\gamma, \mathcal{T}^\gamma)$ are continuous. Therefore $\mathcal{T}_{\mathcal{U}} \leq \mathcal{T}$. Assume that $\mathcal{T}_{\mathcal{U}}(M) < \alpha < \mathcal{T}(M)$ for some $M \in I^X$. Without loss of generality one can assume that $M = f_\gamma^{-1}(P)$ for some $\gamma \in \Gamma$ and some $P \in I^{Y_\gamma}$ such that $\mathcal{T}(M) = \mathcal{T}^\gamma(P)$. However, then $\mathcal{T}_{\mathcal{U}}(M) = \sup\{\beta : \bigvee\{N \in I^X : \exists\, U \in \mathcal{U}_\beta,\ U(N) \leq M\} = M\} \geq \sup\{\beta : \bigvee\{S \in I^{Y_\gamma} : \exists\, U^\gamma \in \mathcal{V}^\gamma_\beta \text{ s.t. } U^\gamma(S) \leq P\} = P\} = \mathcal{T}^\gamma(P) = \mathcal{T}(M) > \alpha$.)

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EXISTENCE OF A COMMON FIXED POINT FOR A FAMILY OF NONEXPANSIVE MAPPINGS ON A METRIC SPACE

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1. I N T R O D U C T I O N. The central concept of our work is a closure operator used as a convexity form in a metric space. The existence of a common fixed point for a family of nonexpansive mappings is proved. In [1, Theorem 2, 3] L.P.Belluce and W.A.Kirk examine the existence of a fixed point for nonexpansive mappings if the condition of diminisching orbital diameters or the condition of normal structure is satisfied. Generalizing these results for a case of a metric space with the closure operator for a family of nonexpansive mappings these conditions are weekened.

2. B A S I C D E F I N I T I O N S. Let X be a metric space with a distance d. Let PX be a set of all subsets of X.

Definition 1.

A closure operator on X is a mapping S: $PX \to PX$ for each $A, B \in PX$ satisfying:

1) $A \subset B \to S(A) \subset S(B)$;
2) $A \subset S(A)$;
3) $S(S(A))=S(A)$.

Definition 2.

A closure operator S on X is said to be algebraic if for each $A \in PX$ and $x \in S(A)$ there exists a finite set $F \subset A$ such that $x \in S(F)$.

Let S be a closure operator on X. A subset A of X is said to be S-closed if $A=S(A)$.

Definition 3.

A space X is said to be S-compact if each centered system of S-closed subsets of X has a nonempty intersection.

Note that the intersection of S-closed subsets of X is S-closed.

Definition 4.

A convex subset K of a normed vector space X has normal structure if for each bounded and convex subset H of K containing more than one element there exists a point $y \in H$ such that :

$$\sup\{ \| y-x \| \ / \ x \in H \} < \operatorname{diam} H.$$

In our Theorem 2 we use the condition of normal structure in a weekened form.

3. R E S U L T S. [1] and [2] inspire the following result:

Theorem 1.

Suppose (X,d) is q metric space and S is an algebraic closure operator on X. Suppose $\overline{S(A)}=S(\overline{S(A)})=:S'(A)$ for each $A \in PX$ and X is S'-compact. Let each closed ball $B(x,r)$ $(x\in X,\ r\in R^+)$ be S-closed. F is a family of selfmaps of X satisfying:

$$1)\ \forall x,y\in X\ \forall f,g\in F:\ d(f(x),g(y))\leq d(x,y)\ -$$

nonexpansivity condition;

$$2)\ \forall x\in X\ (\ \exists v\in F\ :\ v(x)\neq x\ )\ \exists y\in A(x):$$

$$\sup\{\inf\{\sup\{d(y,f^m(x)/m\geq n\}n\in Z^+\}f\in F\}<\operatorname{diam} A(x),$$

where

$$A(x):=\bigcap\{\ A\in PX\ /\ x\in A\ \&\ A=S'(A)\ \&\ \forall f\in F:f(A)\subset A\ \}.$$

Then F has a common fixed point.

Proof.

Using Zorn's Axiom and S'-compactness of X we conclude that there exists a minimal nonempty S'-closed and invariant under F subset M of X.

Let $a\in M$ and there exists $f\in F$ such that $f(a)\neq a$. Since $M=A(a)$, by 2) there exists a point $a_o\in M$ such that:

$$\sup\{\inf\{\sup\{d(a_o,f^m(a))/m\geq n\}n\in Z^+\}f\in F\}=:q<\operatorname{diam} M.$$

Let $r\in]q;\ \operatorname{diam} M[$ and

$$A:=\bigcup\{\ \bigcap\{\ \bigcap\{\ B(f^m(a),r)\cap M\ /\ m\geq n\}f\in F\}n\in Z^+\}.$$

A is nonempty ($a_o \in A$) and S-closed as a union of an increasing sequence of S-closed subsets of X (S is algebraic!). We prove that $f: A \to A$ for each $f \in F$.

Let

$$y \in \bigcap \{ \bigcap \{B(f^m(a),r) \cap M \;/\; m \geq n\} f \in F\} \text{ for some } n \in Z^m.$$

Then

$$d(y,f^m(a)) \leq r, \forall m \geq n, \forall f \in F.$$

We conclude that

$$d(f(y),f(f^m(a))) \leq d(y,f^m(a)) \leq r, \; \forall m \geq n, \forall f \in F.$$

Hence,

$$f(y) \in \bigcap \{ \bigcap \{B(f^m(a),r) \cap M/m \geq n+1\} f \in F\}$$

and $f(A) \subset A$, $\forall f \in F$. By continuity of $f: \overline{A} \to \overline{A}, \forall f \in F$. Minimality of M implies: $M=\overline{A}$. We choose $p \in M$ freely. Then $p \in \overline{A}$ and for each $\varepsilon \in R^+$ there exists $p_o \in A$ such that:

$$d(p,p_o) < \varepsilon.$$

Then there exists $n_o \in Z^+$ such that

$$d(p_o,f^m(a)) \leq r, \; m \geq n_o, \forall f \in F.$$

Hence, $\quad d(p,f^m(a)) \leq r+\varepsilon, \; m \geq n_o, \forall f \in F,$

and $\quad S(\bigcup \{ \bigcup \{f^m(a)/m \geq n_o\} f \in F\}) \subset B(p,\varepsilon+r).$

Since ε is arbitrary,

$$\bigcap \{S(\bigcup \{ \bigcup \{f^m(a)/m \geq n\} f \in F\}) n \in Z^+\} \subset B(p,r).$$

Let $\quad z \in \bigcap \{S(\bigcup \{ \bigcup \{f^m(a)/m \geq n\} f \in F\}) n \in Z^+\}.$

Then $\quad z \in B(p,r)$ and $z \in \bigcap \{B(p,r)/p \in M\}$

because p was choosen freely.

Thus:

$$z \in \bigcap \{B(p,r) \cap M/p \in M\} =: A_1,$$

where A_1 is nonempty and S-closed as an intersection of S-closed sets. We prove that A_1 is invariant under F. Let us assume that there exists $h \in F$ and $x \in A_1$ such that $h(x) \notin A_1$. Then there exists $y \in M$ such that $y \notin B(h(x),r)$.

Hence

$$A_2 := B(h(x),r) \cap M$$

is a proper subset of M.

But: 1) $A_2 \neq \emptyset$ ($h(x) \in A_2$);

2) A_2 is S'-closed;

3) A_2 is invariant under F because for each $z \in A_2$ and each $g \in F$: $d(h(x),g(z)) \leq d(x,z) \leq r$ (condition 2).

Hence $A_2=M$. Therefore $f(A_1) \subset A_1$ for each $f \in F$. Minimality of M implies $M=A_1$.

However : diam A_1 $\le$ r < diam M. The contradiction obtained completes the proof.

In the following theorem the second condition of Theorem 1 is strengthened. It allows to simplify the conditions for the closure operator.

Theorem 2.

Suppose (X,d) is a metric space and S is a closure operator on X. Suppose $\overline{S(A)}=S(\overline{S(A)})=:S'(A)$ for each $A\in PX$ and X is S'-compact. Let each closed ball $B(x,r)$ $(x\in X,\ r\in R^+)$ be S-closed. F is a family of selfmaps of X satisfying:

1) $\forall x,y\in X\ \forall f,g\in F:\ d(f(x),g(y))\le d(x,y)$ - *nonexpansivity condition;*

2) $\forall x\in X\ (\exists v\in F:\ v(x)\ne x)\ \exists y\in A(x):$

$\sup\{d(y,z)/z\in A(x)\}<\operatorname{diam} A(x)$,

where $A(x):=\bigcap\{A\in PX/x\in A\ \&\ A=S'(A)\ \&\ \forall f\in F:\ f(A)\subset A\}$- *"normal structure" condition.*

Then F has a common fixed point.

Proof.

Define M as in the proof of Theorem 1. Let $a\in M$ and there exists $f\in F$ such that $f(a)\ne a$. Since $M=A(a)$, by 2) there exists $y\in M$ such that:

$$\sup\{d(y,z)/z\in M\}=:r<\operatorname{diam} M.$$

Define

$$A_1:=\left(\bigcap\{B(x,r)/x\in M\}\right)\cap M.$$

A_1 is nonempty ($y\in A_1$) and S-closed sets. The further proof of our theorem repeats the proof of Theorem 1.

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ABOUT THE FUNDAMENTAL GROUP G_n^{p-1} OF THE CONNECTION GENERATED BY THE DIFFERENTIAL SYSTEM OF HIGHER ORDER

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The system of ordinary differential equations is considered:

$$\frac{d^p x^a}{(dx^1)^p} = s^a\left(x^k, \frac{dx^b}{dx^1}, \dots, \frac{d^{p-1}x^b}{(dx^1)^{p-1}}\right), \quad \begin{array}{l} p = 3,4,\dots \\ i;k = \overline{1,n};\ a;b = \overline{2,n} \end{array} \tag{1}$$

defined on the differentiable manifold as a cross-section

$$s:\ S^{p-1}(V_n) \longrightarrow S^p(V_n) \tag{2}$$

of the bundle of tangent p-elements $S^p(V_n)$ above the space of (p-1)-elements of tangency of the space V_n.

The cross-section (2) is characterized by the structural equations:

$$\begin{gathered} \omega^a_{(p)} = V^{a0}_{pk}\ \omega^k + V^{a1}_{pb}\ \omega^b_{(1)} + \dots + V^{a\ p-1}_{p\ \ b}\ \omega^b_{(p-1)}, \\ \omega^a_{(p)} = \omega^a_{\underbrace{1\dots 1}_{p}} \end{gathered} \tag{3}$$

which connect the canonical forms of the bundle $S^p(V_n)$. As a result of the p-steps partial prolongation of the equations (3), attended by the reduction of the principal frame bundle $H^p(V_n)$ at each stage, combined with the scope operations, we obtain the following structural equations:

$$d\omega^1 = \omega^1 \wedge \omega^1_1 + \Omega^1$$

$$d\omega^a = \omega^1 \wedge \omega^a_1 + \omega^b \wedge \omega^a_b$$

$$d\omega^a_{(1)} = \omega^b_{(1)} \wedge (\omega^a_b - \delta^a_b \omega^1_1) + \frac{p-1}{2}\, \omega^a \wedge \omega^1_{11} + \omega^1 \wedge \omega^a_{(2)} + \Omega^a_{(1)}$$

$$d\omega^a_{(s)} = \omega^b_{(s)} \wedge (\omega^a_b - s\delta^a_b \omega^1_1) + \frac{s(p-s)}{2} \omega^a_{(s-1)} \wedge \omega^1_{11} + \tag{4}$$

$$+ \omega^1 \wedge \omega^a_{(s+1)} + \Omega^a_{(s)}, \qquad s = \overline{2, p-2}$$

$$d\omega^a_{(p-1)} = \omega^b_{(p-1)} \wedge (\omega^a_b - (p-1)\delta^a_b \omega^1_1) + \frac{p-1}{2} \omega^a_{(p-2)} \wedge \omega^1_{11} + \Omega^a_{(p-1)}$$

$$d\omega^1_1 = \omega^1 \wedge \omega^1_{11} + \Omega^1_1$$

$$d\omega^1_{11} = \omega^1_1 \wedge \omega^1_{11} + \Omega^1_{11}$$

$$d\omega^a_b = \omega^c_b \wedge \omega^a_c + \frac{p-1}{2} \delta^a_b \omega^1 \wedge \omega^1_{11} + \Omega^a_b;$$

the forms (Ω) are semi-base in $S^{p-1}(V_n)$.

It is shown that the following assertion is valid:

P R O P O S I T I O N 1. *The equations (4) are the structural equations of Cartan's fundamentally-group connection in some principal fibre bundle above $S^{p-1}(V_n)$.*

So the principal fibre bundle with the fundamentally-group connection associated with the system (1) is constructed. In other words we obtain the complete system of invariants of differential equations (1).

P R O P O S I T I O N 2. *The fundamental group G^{p-1}_n is represented in R_n by the transformations of the following type:*

$$\tilde{x}^1 = \frac{y^1_1 x^1 + y^1}{y^1_{11} x^1 + y}, \qquad \begin{vmatrix} y^1_1 & y^1 \\ y^1_{11} & y \end{vmatrix} = 1$$

$$\tilde{x}^a = \frac{y^a_b x^b + \sum\limits_{q=0}^{p-1} \frac{1}{q!} y^a_{(q)} (x^1)^q}{(y^1_{11} x^1 + y)^{p-1}}, \qquad y^a_{(q)} = y^a_{\underbrace{1\ldots1}_{q}} \tag{5}$$

The proof of this fact is based on the results of Lemma 1, Lemma 2 and on the consideration of the representation of G^{p-1}_n equivalent to the parametric one.

L E M M A 1.

a) $$\sum_{r=v}^{k+1} \frac{r!}{v!(r-v)!} \prod_{s=v+1}^{r} (p-s)_s \left[(-1)^{r-1+k} \cdot \frac{k!}{(r-1)!(k-(r-1))!} \cdot \prod_{s=r}^{k} (p-s)_s + (-1)^{r+k} \frac{k!}{r!(k-r)!} \prod_{s=r+1}^{k} (p-s)_s (r-p+k+1) \right] = 0,$$

$$k = \overline{0, p-2}; \quad v = \overline{0, k}$$

b) $$\sum_{r=v}^{p-1} \frac{r!}{v!(r-v)!} \cdot \prod_{s=v+1}^{r} (p-s)_s \Big[(-1)^{r+p-2} \frac{(p-1)!}{(r-1)!(p-r)!} \cdot$$

$$\cdot \prod_{s=r}^{p-1} (p-s)_s + (-1)^{r+p-1} \frac{(p-1)!}{r!(p-1-r)!} \prod_{s=r+1}^{p-1} (p-s)_s \cdot r \Big] = 0,$$

$$v = \overline{0,p-1}$$

c) The following conditions are equivalent $(\forall k = \overline{0,p-1})$:

$$\overset{*}{\tilde{x}}{}^c_b (\tilde{x}^1_1)^k (d\tilde{P}^b_{(k)} - \tilde{P}^b_{(k+1)} d\tilde{x}^1) =$$

$$= \overset{*}{x}{}^c_b \cdot \sum_{t=0}^{k} (-1)^{k+t} (\tfrac{1}{2})^{k-t} \frac{k!}{t!(k-t)!} \prod_{s=t+1}^{k} (p-s)_s (x^1_1)^t \cdot$$

$$\cdot (dP^b_{(t)} - P^b_{(t+1)} dx^1)(x^1_{11} \overset{*}{x}{}^1_1 - \tilde{x}^1_{11} \overset{*}{\tilde{x}}{}^1_1)^{k-t}$$

$$\overset{*}{\tilde{x}}{}^c_b \cdot \sum_{t=0}^{k} (-1)^{k+t} (\tfrac{1}{2})^{k-t} \frac{k!}{t!(k-t)!} \cdot \prod_{s=t+1}^{k} (p-s)_s (\tilde{x}^1_1)^t \cdot$$

$$\cdot (d\tilde{P}^b_{(t)} - \tilde{P}^b_{(t+1)} d\tilde{x}^1)(\tilde{x}^1_{11} \overset{*}{\tilde{x}}{}^1_1)^{k-t} =$$

$$= \overset{*}{x}{}^c_b \cdot \sum_{t=0}^{k} (-1)^{k+t} (\tfrac{1}{2})^{k-t} \frac{k!}{t!(k-t)!} \cdot \prod_{s=t+1}^{k} (p-s)_s (x^1_1)^t \cdot$$

$$\cdot (dP^b_{(t)} - P^b_{(t+1)} dx^1)(x^1_{11} \overset{*}{x}{}^1_1)^{k-t}$$

L E M M A 2. $(\forall k = \overline{0,p-1})$:

a) $$\sum_{t=u}^{k-1} (-1)^{k+t} \frac{k!}{t!(k-t)!} \prod_{s=t+1}^{k} (p-s)_s (k-t) \frac{t!}{u!(t-u)!} \prod_{s=u+1}^{t} (p-s)_s = 0,$$

$$u = \overline{0,k-2}$$

b) $$\sum_{t=u}^{k} (-1)^{k+1+t} \frac{(k+1)!}{t!(k+1-t)!} \prod_{s=t+1}^{k+1} (p-s)_s \frac{t!}{u!(t-u)!} \prod_{s=u+1}^{t} (p-s)_s +$$

$$+ \sum_{t=u}^{k-1} (-1)^{k+t} \frac{k!}{t!(k-t)!} \prod_{s=t+1}^{k} (p-s)_s (k-t) \frac{t!}{u!(t-u)!} \prod_{s=u+1}^{t} (p-s)_s +$$

$$+ \sum_{t=u}^{k-1} (-1)^{k+t} \frac{k!}{t!(k-t)!} \prod_{s=t+1}^{k} (p-s)_s \sum_{v=u}^{t} (-1)^{t+1+v} \frac{(t+1)!}{v!(t+1-v)!} \cdot$$

$$\cdot \prod_{s=v+1}^{t+1} (p-s)_s \frac{v!}{u!(v-u)!} \prod_{s=u+1}^{v} (p-s)_s -$$

$$- (-1)^{k+u-1} \frac{k!}{(u-1)!(k-(u-1))!} \prod_{s=u}^{k} (p-s)_s = 0, \qquad u = \overline{1,k-1}$$

c) $\sum_{t=0}^{k}(-1)^{k+1+t}\frac{(k+1)!}{t!(k+1-t)!}\prod_{s=t+1}^{k+1}(p-s)_s\prod_{s=1}^{t}(p-s)_s+$

$$+\sum_{t=u}^{k-1}(-1)^{k+t}\frac{k!}{t!(k-t)!}\prod_{s=t+1}^{k}(p-s)_s(k-t)\prod_{s=1}^{t}(p-s)_s+$$

$$+\sum_{t=0}^{k-1}(-1)^{k+t}\frac{k!}{t!(k-t)!}\prod_{s=t+1}^{k}(p-s)_s\sum_{v=0}^{t}(-1)^{t+1+v}\frac{(t+1)!}{v!(t+1-v)!}\cdot$$

$$\cdot\prod_{s=v+1}^{t+1}(p-s)_s\prod_{s=1}^{v}(p-s)_s=0.$$

Introducing the variables:

$$x_1^1=\left.\frac{\partial x^1}{\partial X^1}\right|_{X^1=0},\quad x_{11}^1=\left.\frac{\partial^2 x^1}{(\partial X^1)^2}\right|_{X^1=0},\quad x_b^a=\left.\frac{\partial x^a}{\partial X^b}\right|_{X^b=0}$$

$$P_{(1)}^a=\frac{dx^a}{dx^1},\quad P_{(2)}^a=\frac{d^2x^a}{(dx^1)^2},\quad\ldots\quad,P_{(p-1)}^a=\frac{d^{p-1}x^a}{(dx^1)^{p-1}},$$

$$P_{(p-1)}^a=P^a_{\underbrace{1\ldots1}_{p-1}};\quad(X^1,\ldots,X^n)\in R_n,$$

we shall consider the representation of G_n^{p-1} equivalent to the parametric one:

$$\tilde{x}^1=\frac{y_1^1x^1+y^1}{y_{11}^1x^1+y}$$

$$\tilde{x}^a=\frac{y_b^ax^b+\sum_{q=0}^{p-1}\frac{1}{q!}y_{(q)}^a(x^1)^q}{(y_{11}^1x^1+y)^{p-1}}$$

$$\tilde{x}_1^1=\frac{x_1^1}{(y_{11}^1x^1+y)^2},\quad\tilde{x}_b^a=\frac{y_c^ax_b^c}{(y_{11}^1x^1+y)^{p-1}}\qquad(6)$$

$$\tilde{x}_{11}^1=\frac{x_{11}^1(y_{11}^1x^1+y)-2y_{11}^1(x_1^1)^2}{(y_{11}^1x^1+y)^3}$$

$$\tilde{P}_{(s)}^a=\frac{1}{(y_{11}^1x^1+y)^{p-(s+1)}}\cdot\sum_{t=0}^{s}(-1)^{t+s}\frac{s!}{t!(s-t)!}\prod_{k=t+1}^{s}(p-k)_k\cdot$$

$$\cdot\left[y_b^aP_{(t)}^b+\sum_{q=t}^{p-1}\frac{1}{(q-t)!}y_{(q)}^a(x^1)^{q-t}\right](y_{11}^1)^{s-t}(y_{11}^1x^1+y)^t$$

$$s=\overline{1,p-1}$$

The values $\langle y\rangle$ are considered to be the group parameters. Differentiating (6), considering $\langle y\rangle$ invariable and taking into account Lemmas 1 and 2 we obtain the forms over the variables x^1, x^a, x^1_1, x^1_{11}, x^a_b, $P^a_{(1)}$, $P^a_{(2)}$, ... ,$P^a_{(p-1)}$, invariant under the transformations (6):

$$\theta^1 = \overset{*}{x}{}^1_1 dx^1; \quad \theta^1_1 = \overset{*}{x}{}^1_1 (dx^1_1 - \overset{*}{x}{}^1_1 x^1_{11} dx^1), \quad \overset{*}{x}{}^1_1 x^1_1 = 1$$

$$\theta^1_{11} = \overset{*}{x}{}^1_1 (dx^1_{11} - 2\overset{*}{x}{}^1_1 x^1_{11} dx^1_1 + \frac{1}{2}(\overset{*}{x}{}^1_1 x^1_{11})^2 dx^1)$$

$$\theta^a_b = \overset{*}{x}{}^a_c (dx^c_b - \frac{p-1}{2} x^c_b (\overset{*}{x}{}^1_1)^2 x^1_{11} dx^1), \quad \overset{*}{x}{}^a_c x^c_b = \delta^a_b$$

$$\theta^a = \overset{*}{x}{}^a_b (dx^b - P^b_{(1)} dx^1); \tag{7}$$

$$\theta^a_{(k)} = \overset{*}{x}{}^a_b (x^1_1)^k \sum_{t=0}^{k} (-1)^{k+t} \frac{k!}{t!(k-t)!} \prod_{s=t+1}^{k} (p-s)_s \cdot$$

$$\cdot (\frac{1}{2}(\overset{*}{x}{}^1_1)^2 x^1_{11})^{k-t} (dP^b_{(t)} - P^b_{(t+1)} dx^1), \qquad \overline{k = 1, p-2};$$

$$\theta^a_{(p-1)} = \overset{*}{x}{}^a_b (x^1_1)^{p-1} \Big\{ dP^b_{(p-1)} + \sum_{t=0}^{p-2} (-1)^{p-1+t} \frac{(p-1)!}{t!(p-1-t)!} \cdot$$

$$\cdot \prod_{s=t+1}^{p-1} (p-s)_s (\frac{1}{2}(\overset{*}{x}{}^1_1)^2 x^1_{11})^{p-1+t} (dP^b_{(t)} - P^b_{(t+1)} dx^1) \Big\}$$

The exterior differentiation of (7) shows us that the forms $\langle\theta\rangle$ obey the structural equations (4) under the condition of identical vanishing the curvature-torsion forms $\langle\Omega\rangle$. Thus Proposition 2 is completely proved.

The results obtained show that the fundamental group G^{p-1}_n is the group of invariableness of the cross-section s, written in the suitable local coordinate system as follows:

$$\frac{d^p x^a}{(dx^1)^p} = 0 \tag{8}$$

and the set of integral curves of (8):

$$x^a = \sum_{q=0}^{p=1} \frac{1}{q!} C^a_{(q)} (x^1)^q, \quad C^a_{(q)} = C^a_{\underbrace{1\ldots1}_{q}}$$

determined by the Pfaff's system also:

$$\omega^a = 0, \; \omega^a_{(1)} = 0, \; \dots \;, \; \omega^a_{(p-1)} = 0 \qquad (9)$$

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CONDENSATIONS OF TOPOLOGICALLY INVERSE SEMI-GROUPS

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Cardinal-valued topological invariants play an important role in our considerations. We consider $l(X)$, $\chi(X)$, $\psi(X)$, $t(X)$, $d(X)$: the Lindelöf number, the character, the pseudocharacter, the tightness, and the density of a topological space X respectively. One can find the definitions of these functions in [3,7]. Let G be a topological group. We write $b(G) \le \tau$, if G is a τ-bounded topological group [4]. All the spaces under consideration are assumed to be Hausdorff. By condensation we mean one-to-one continuous surjective map.

Let S be a semi-group. The elements a and b are said to be inverse if $aba = a$ and $bab = b$. If every element of a semi-group S has a unique inverse, then a semi-group S is called *an inverse semi-group* [5]. We shall say that a topological semi-group S is *topologically inverse* if S is an inverse semi-group and the operations of multiplication and of invertion are continuous. The inverse element of a is denoted by a^{-1}.

Let S be a topologically inverse semi-group and $E(S)$ be the set of idempotents of S. The map $i: S \longrightarrow E(S)$, $i(x) = xx^{-1}$, is continuous and $i(x) = x$ for every $x \in E(S)$. Hence, the map $i: S \longrightarrow E(S)$ is a retraction and the space $E(S)$ is retract of S. Therefore, $E(S)$ is a closed subset of S.

Let $\le$ be the natural partial order on $E(S)$: $e \le f$ if and only if $ef = fe = e$. Let $l_f(S) = \{x \in E(S) \mid x \le f\}$ and $r_f(S) = \{x \in E(S) \mid f \le x\}$ be the left and right rays of idempotent $f \in E(S)$ respectively.

L E M M A 1. *Every left ray and every right ray are closed subspaces of a topologically inverse semi-group.*

P r o o f. The map $\varphi_x: E(S) \longrightarrow l_x(S)$, $\varphi_x(y) = xy$, $x \in E(S)$, is a retraction. Hence, a space $l_x(S)$ is closed in $E(S)$, and therefore in S.

Now, assume that $\overline{r_x(S)} \ne r_x(S)$ and $z \in \overline{r_x(S)} \setminus r_x(S)$. If

$z < x$, then by virtue of Hausdorffness of S there exist neighbourhoods $U(z)$ of z and $U(x)$ of x such that $U(z) \cap U(x) = \Lambda$. Since $xz = z$, by continuity of multiplication, there exist neighbourhoods $W(x)$ and $W(z)$ such that $W(x)V(z) \subset U(z)$. However, $z \in \overline{r_x(S)}$ and therefore $V(z) \cap r_x(S) \neq \Lambda$. Consequently, there exists $p \in r_x(S)$, $p \geq x$, and $x = xp \in W(x)V(z) \subset U(z)$, and it contradicts to the assumption that $U(z) \cap U(x) = \Lambda$.

If the points z and x are not comparable, then $zx < x$ and $zx < z$. Since S is a Hausdorff space, there exists a neighbourhood $U(zx)$ of zx such that $x \in U(zx)$. But again, by virtue of continuity of multiplication, there exist neighbourhoods $U(z)$ and $U(x)$ such that $U(z)U(x) \subset U(zx)$. Since $z \in \overline{r_x(S)}$, then there exists $p > x$ and $p \notin U(z)$. Hence, $x = px \in U(z)U(x) \subset U(zx)$, and we get a contradiction with the assumption that $x \notin U(zx)$.

C O R O L L A R Y. *Let $E(S)$ be a set of idempotents of a topologically inverse semi-group S. If $\leq$ is linear order, then the intervals*

$$(e,f) = \{x \in E(S) \mid e < x < f\}, \quad e,f \in E(S)$$

are open sets in $E(S)$. Consequently, the topology induced on $E(S)$ by the linear order $\leq$ is coarser than the original topology on $E(S)$.

Toplogically inverse semi-group S is called L-semi-group if the subset $E(S)$ is linearly ordered with respect to the relation $\leq$.

T H E O R E M 1. *If S is a compact L-semi-group, then $E(S)$ is linearly ordered compactum. In particular, if S is a metrizable connected L-semi-group, then $E(S)$ is homeomorphic to segment $[0,1]$ or S is a topological group.*

The proof follows from the fact that the condensation of compactum is homeomorphism, and from the corollary.

L E M M A 2. *Let S be a topologically inverse semi-group and*

$$H(e,k) = \{x \in S \mid xx^{-1} = e \text{ and } x^{-1}x = k\}, \quad e,k \in E(S).$$

Then every non-empty subspace $H(e,k)$ is homeomorphic to closed maximal subgroups $H(e,e)$ and $H(k,k)$. Besides,

$$S = \bigcup \{H(e,k) \mid e,k \in E(S)\}, \quad H(e,k) \cap H(f,l) = \Lambda,$$

if $e \neq f$ *or* $k \neq l$ *and every subset* $H(e,k)$ *is a closed subspace of* S.

The cardinality of space X is denoted by $|X|$. Let q be a cardinal invariant. We denote by $gq(S)$ the following cardinal function

$$gq(S) = \sup \{ \tau \mid q(H(e,e)) \le \tau, \quad e \in E(S)\}.$$

T H E O R E M 2. *Let* S *be a L-semi-group. Then*

$$|S| \le gb(S) g\psi(S) \exp c^{+}(S),$$

where $c^{+}(S)$ *is the smallest cardinal number larger than the Souslin number* $c(S)$.

P r o o f. Since there exists the condensation of space $E(S)$ onto a linearly ordered space X, we see that

$$|E(S)| \le |X| \le \exp d(X) \le \exp c^{+}(X) \le \exp c^{+}(S).$$

If G is a topological group, then $|G| \le b(G)\psi(G)$ by [4]. By virtue of Lemma 2 the cardinality of the family of all subspaces $H(e,k)$ is $|E(S)|$. Therefore

$$|S| \le |E(S)| gb(S) g\psi(S) \le gb(S) g\psi(S) \exp c^{+}(S).$$

Note that there exists topologically inverse L-semi-group X which has the Souslin property but the Souslin and Lindelöf numbers of a maximal closed subgroup are equal to continuum.

The following corollary can be also deduced directly from Theorem 2 and results of the articles [1,4].

C O R O L L A R Y. *Let* S *be a L-semi-group. Then*

a) $|S| \le gwl(S)\psi(S) \exp d(S)$,

b) $|S| \le gl(S)\psi(S) \exp d(S)$,

c) $|S| \le gc(S)\psi(S) \exp d(S)$.

T H E O R E M 3. *Let* S *be a compact L-semi-group. Then*

$$|S| \le \exp t(S).$$

P r o o f. It is well known [5] that a compact topologically inverse semi-group S is a union of pairwise disjoint compact subgroups [6]. By [2] for a compact topological group G we have $t(G) = w(G)$. Hence

$$|H(e,e)| \le \exp t(H(e,e)) \le \exp t(S)$$

and

$$|S| \le |E(S)| \exp t(S).$$

By virtue of Lemma 1 and equality $t(X) = \chi(X)$ for every linearly ordered space X [7, 3.12.4(d)], we have $|E(S)| \le \le \exp \chi(S) = \exp t(S)$. Consequently,

$$|S| \le \exp t(S) \exp t(S) = \exp t(S).$$

Topologically inverse semi-group S is called *ω-semi-group* if the subset E(S) has the order type of the set of negative integers. It is easy to verify that in this case the only topology on E(S) which makes E(S) to be a topological semi-group is the discrete topology. Thus, E(S) is a discrete subspace of any topological semi-group containing it.

P R O P O S I T I O N. *Let S be a ω-topologically inverse semi-group. Then S is the sum of the spaces H(e,k). Hence, the topological space of ω-semi-group S is completely regular.*

T H E O R E M 4. *Topologically inverse ω-semi-group S is metrizable if and only if S satisfies the first countability axiom.*

Note that in the last theorem the assumption that S is ω-semi-group cannot be replaced by the assumption that S is L-semi-group and $\chi(S) \le \tau$.

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ON BEHAVIOUR OF TWO-SIDED SKEW PRODUCTS OF CATEGORIES BY HOMOMORPHISMS

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In [2] we first considered the two-sided skew products of three monoids and noted that they are useful for describing the monoids of endomorphisms of bipolygons over two monoids. Then we generalized this product, defined the two-sided skew product of one category with two systems of categories and got some properties of such skew products.

In this paper we continue the study of the skew products of categories. We study how these products behave by homomorphisms of corresponding categories. The results of this paper are also presented in [3] (with some other results), but here they are given with proofs while [3] contains no proofs. Some denotations and ideas for getting our results are going from [4].

1. Basic definitions.

First we recall the definitions of the left net of categories, of the inner left net and of the two-sided skew product of categories from [2] and also the definitions of the two-sided versions of these notions.

Suppose that $C = (R,S,C_r^s,\cdot,e_{Cr})$ is a (small) category [3]. Let besides the category C a category $M = (P,V,M_p^q,\cdot,e_{Mp})$ be given. A *homomorphism* $h : C \to M$ (a covariant functor $h : C \to M$) is a family of mappings $h = (h_0 : R \to P \,;\, h_r^s : C_r^s \to M_{rh_0}^{sh_0},\ (r,s) \in S)$ satisfying the conditions

$$\text{if } (r,s) \in S \text{ then } (rh_0, sh_0) \in V, \tag{1}$$

$$(c\cdot d)h_r^t = (ch_r^s)\cdot(dh_s^t), \tag{2}$$

$$e_{Cr}h_r^r = e_{M,rh_0} \tag{3}$$

for each $r \in R$, $(r,s),(s,t) \in S$, $c \in C_r^s$, $d \in C_s^t$.

If C is a category, a system (A_r,C) is called a *left C-net of categories* if

LNC1. to each $r \in R$ a (small) category $A_r = (I_r, J_r, A^j_{ri}, \cdot, e_{A_r i})$ corresponds, for different r the basic sets A^j_{ri} of the categories A_r are disjuncted;

LNC2. to any $c \in C^r_s$ and each $i \in I_r$ an element $c \triangleright i \in I_s$ corresponds;

LNC3. if moreover $d \in C^s_t$, then $d \triangleright (c \triangleright i) = (d \cdot c) \triangleright i$ is satisfied;

LNC4. for each $i \in I_r$ the equality $e_{Cr} \triangleright i = i$ holds;

LNC5. to any $c \in C^r_s$, $x \in A^j_{ri}$ an element $c \triangleright x \in A^{c \triangleright j}_{s, c \triangleright i}$ corresponds;

LNC6. if moreover $d \in C^s_t$ then $d \triangleright (c \triangleright x) = (d \cdot c) \triangleright x$ holds;

LNC7. for each $x \in A^j_{ri}$ the equality $e_{Cr} \triangleright x = x$ is satisfied;

LNC8. if moreover $y \in A^k_{rj}$ then $c \triangleright (x \cdot y) = (c \triangleright x) \cdot (c \triangleright y)$ holds;

LNC9. the equality $c \triangleright e_{A_r i} = e_{A_s, c \triangleright i}$ holds for any $i \in I_r$.

Here C is called the *active category* and A_r the *passive categories* of the net. The conditions LNC2, LNC5 and LNC8 mean that any $c \in C^r_s$ determines a homomorphism (covariant functor) of the category A_r to A_s. The notion of the left C-net (A_r,C), shortly (A,C) is equivalent to the notion of a contravariant functor $A : C \to cat$ from the category C to the category cat of small categories.

We get an example of a net of categories in the following way. Take a category $G = (R,S,G^s_r,\cdot,e_{Gr})$ for which the graph (R,S) is symmetric, i.e. $(r,s) \in S$ implies $(s,r) \in S$. Suppose that we have a subgroupoid [1] $C = (R,S,C^s_r,\cdot,e_{Gr})$ in it, i.e. each element $c \in C^s_r$ is invertible in C. Moreover, suppose that a subcategory [1] $A = (R,\Delta_R,A^r_r,\cdot,e_{Gr})$ in G is given for which only the diagonal subsets A^r_r are non-empty and which is C-invariant ($c \in C^r_s$, $a \in A^r_r$ imply $c \cdot a \cdot c^{-1} \in A^s_s$). We put now $A_r = (\{r\},\{(r,r)\},A^r_r,\cdot,e_{Gr})$ (so that all A_r are simply monoids), $c \triangleright r = s$ for $c \in C^r_s$, $c \triangleright a = c \cdot a \cdot c^{-1}$ for any $a \in A^r_r$. It is easy to see that if we take C for the active category and the monoids A_r for passive categories then such a system satisfies all axioms LNC1-LNC9. We call such a net the *left inner net of categories* and denote it by $I(G,C,A) = (I_{1r}(G,C,A), I_2(G,C,A))$ where $I_{1r}(G,C,A) = A_r$, $I_2(G,C,A) = C$.

Now suppose that besides the left C-net (A_r,C) another left C-net (B_r,C) of categories with the same active category C is given where $B_r = (K_r,L_r,B^l_{rk},\cdot,e_{B_r k})$. Then we say that a *two-sided C-net* (A_r,C,B_r), shortly (A,C,B) is given. We get an example of a two-sided net of categories if we take the categories G,C,A as in the last example, take another C-invariant subcategory $B = (R,\Delta_R,B^r_r,\cdot,e_{Gr})$ in G and put $c \triangleright b = c\cdot b\cdot c^{-1}$ for each $b \in B^r_r$ and $B_r = (\{r\},\{(r,r)\},B^r_r,\cdot,e_{Gr})$. If we take C for the active category and monoids A_r and B_r for the systems of passive categories we get a two-sided C-net (A_r,C,B_r). We denote it by $I(G,C,A,B) = (I_{1r}(G,C,A,B),I_2(G,C,A,B),I_{3r}(G,C,A,B))$ where $I_{1r}(G,C,A,B) = A_r$, $I_2(G,C,A,B) = C$, $I_{3r}(G,C,A,B) = B_r$ and name it the *two-sided inner net.*

Now let a two-sided C-net (A_r,C,B_r) be given. Denote by M the set of all triples (i,r,k) where $r \in R$, $i \in I_r$, $k \in K_r$. Denote by N the set of all pairs $((i,r,k),(j,s,l))$ where $(i,r,k),(j,s,l) \in M$, $(r,s) \in S$ and $c \in C^s_r$ exists so that

$$(i,c \triangleright j) \in J_r \;,\; (c \triangleright l,k) \in L_r \;.$$

If $((i,r,k),(j,s,l)) \in N$ we denote by G^{jsl}_{irk} the set of all ninuples

$$(i,r,k,a,c,b,j,s,l) \tag{4}$$

where

$$c \in C^s_r \;,\; a \in A^{c\triangleright j}_{ri} \;,\; b \in B^k_{r,c\triangleright l} \;. \tag{5}$$

Define the equality of ninuples componentwise. We define the product of two ninuples from G^{jsl}_{irk} and G^{mtn}_{jsl} by the formula

$$\begin{aligned}(i,r,k,a,c,b,j,s,l)\cdot(j,s,l,d,g,f,m,t,n) &= \\ = (i,r,k,a\cdot(c \triangleright d),c\cdot g,(c \triangleright f)\cdot b,m,t,n).\end{aligned} \tag{6}$$

We define additionally

$$e_{Girk} = (i,r,k,e_{A_r i},e_{Cr},e_{B_r k},i,r,k).$$

P R O P O S I T I O N 1 ([2], p.24). *Let (A_r,C,B_r) be a two-sided net of categories. Then the family $G = (M,N,G^{jsl}_{irk},\cdot,e_{Girk})$ is a category. We denote $G = \{A_r\} \rightthreetimes C \leftthreetimes \{B_r\}$ and name it the two-sided skew product of the net (A_r,C,B_r).*

We note that if a left net (A_r,C) is given then one can define analogically the left skew product $\{A_r\} \rightthreetimes C$ of the net (A_r,C) using sixtuples (i,r,a,c,j,s).

Now let besides a left net (A_r,C) a category M and a left net (D_p,M) be given where $D_p = (N_p,F_p,D^o_{pn},\cdot,e_{D_p n})$. A *homomorphism of left nets* $(f_r,h) : (A_r,C) \rightarrow (D_p,M)$ is a family of homomorphisms of categories $h : C \rightarrow M$ and $f_r : A_r \rightarrow D_{rk_0}$ $(r \in R)$ where

$$f_r = (f_r : I_r \to N_{rh_0} \ ; \ f^j_{ri} : A^j_{ri} \to D^{jf_{r0}}_{rh_0, if_{r0}} \ , \ (i,j) \in J_r)$$

and the conditions

$$(c \triangleright i)f_{s0} = (ch^r_s) \triangleright (if_{r0}), \tag{7}$$

$$(c \triangleright a)f^{c \triangleright j}_{s, c \triangleright i} = (ch^r_s) \triangleright (af^j_{ri}), \tag{8}$$

are satisfied for any $i \in I_r$, $a \in A^j_{ir}$, $(s,r) \in S$, $c \in C^r_s$.

One can show that all the nets of small categories and their homomorphisms form a category LN.

Now suppose that the two-sided nets (A_r,C,B_r) and (D_p,M,E_p) are given. Then we have the left nets of categories (A_r,C), (B_r,C), (D_p,M), (E_p,M). We say that a *homomorphism* $(f_r,h,g_r) : (A_r,C,B_r) \to (D_p,M,E_p)$ is given if $(f_r,h) : (A_r,C) \to (D_p,M)$ and $(g_r,h) : (B_r,C) \to (E_p,M)$ are homomorphisms of the nets of categories.

It can be shown that all two-sided nets of small categories and their homomorphisms form a category TN.

2. Behaviour of two-sided skew products by homomorphisms.

P R O P O S I T I O N 2. *To each homomorphism $(f_r,h,g_r) : (A_r,C,B_r) \to (D_p,M,E_p)$ of two-sided nets of categories a homomorphism $d : \{A_r\} \leftthreetimes C \rightthreetimes \{B_r\} \to \{D_p\} \leftthreetimes M \rightthreetimes \{E_p\}$ of the two-sided skew products corresponds.*

If we denote the skew product $\{A_r\} \leftthreetimes C \rightthreetimes \{B_r\}$ by S then Proposition 2 means that S is a covariant functor from TN to cat.

For the p r o o f we denote $\{A_r\} \leftthreetimes C \rightthreetimes \{B_r\} = G$, $\{D_p\} \leftthreetimes M \rightthreetimes \{E_p\} = H$ and denote the object sets of E_p, G and H correspondingly by Q, T and U. By the definition the elements of T are the triples (i,r,k) where $i \in I_r$, $r \in R$, $k \in K_r$. First we define the mapping $d_0 : T \to U$ by the formula

$$(i,r,k)d_0 = (if_{r0}, rh_0, kg_{r0}) \tag{9}$$

and the mappings

$$d^{jsl}_{irk} : G^{jsl}_{irk} \to H^{jf_{s0}, sh_0, lg_{s0}}_{if_{r0}, rh_0, kg_{r0}}$$

by the formula

$$(i,r,k,a,c,b,j,s,l)\, d^{jsl}_{irk} =$$

$$= (if_{r0}, rh_0, kg_{r0}, af^{c \triangleright j}_{ri}, ch^s_r, bg^k_{r, c \triangleright e}, jf_{s0}, sh_0, lg_{s0}). \tag{10}$$

First we control that the mappings d_0 and d^{jsl}_{irk} are defined correctly. As by the definition of the homomorphisms f_r, h, g_r the relations $h_0: R \to P$, $f_{r0}: I_r \to N_{rk_0}$, $g_{r0}: K_r \to Q_{rh_0}$ hold, we have $rh_0 \in P$, $if_{r0} \in N_{rh_0}$, $kg_{r0} \in Q_{rh_0}$ for each $r \in R$, $i \in I_r$, $k \in K_r$, so that $(i,r,k) \in T$ always implies $(i,r,k)d_0 \in U$. It follows from the definition of the homomorphism f_r that $f^{c\triangleright j}_{ri}$ maps $A^{c\triangleright j}_{ri}$ into $D^{(c\triangleright j)f_{r0}}_{rh_0, if_{r0}}$. This is the demanded condition for the fourth element of the right hand side ninuple. From the definition of the homomorphism h we have $ch^s_r \in M^{sh_0}_{rh_0}$ what is the condition for the fifth member of the right hand side. Analogically we get the demanded condition for the sixth element of the right side. The conditions for the first three and the last three elements of the right side are checked already.

Now we compute

$$[(i,r,k,a,c,b,j,s,l)\cdot(j,s,l,p,u,q,m,t,n)]\, d^{mtn}_{irk} =$$
$$= (i,r,k,a\cdot(c \triangleright p),c\cdot u,(c \triangleright q)\cdot b,m,t,n)\, d^{mtn}_{irk} =$$
$$= (if_{r0},rh_0,kg_{r0},[a\cdot(c \triangleright p)]f^{(c\cdot u)\triangleright m}_{ri},(c\cdot u)h^t_r,$$
$$[(c \triangleright q)\cdot b]g^k_{r,(c\cdot u)\triangleright n},mf_{t0},th_0,kg_{t0}) =$$
$$= (if_{r0},rh_0,kg_{r0},(af^{c\triangleright j}_{ri})\cdot[(c \triangleright p)f^{c\triangleright(u\triangleright m)}_{r,c\triangleright j}],(ch^s_r)\cdot(uh^t_s),$$
$$[(c \triangleright q)g^{c\triangleright l}_{r,c\triangleright(u\triangleright n)}]\cdot(bg^k_{r,c\triangleright l}),mf_{t0},th_0,kg_{t0}) =$$
$$= (if_{r0},rh_0,kg_{r0},(af^{c\triangleright j}_{ri})\cdot[(ch^s_r) \triangleright (pf^{u\triangleright m}_{sj})],(ch^s_r)\cdot(uh^t_s),$$
$$[(ch^s_r) \triangleright (qg^l_{s,u\triangleright n})]\cdot(bg^k_{r,c\triangleright l}),mf_{t0},th_0,ng_{t0}) =$$
$$= (if_{r0},rh_0,kg_{r0},af^{c\triangleright j}_{ri},ch^s_r,bg^k_{r,c\triangleright l},jf_{s0},sh_0,lg_{s0})\cdot$$
$$\cdot(jf_{s0},sh_0,lg_{s0},pf^{u\triangleright m}_{sj},uh^t_s,qg^l_{s,u\triangleright n},mf_{t0},th_0,ng_{t0}) =$$
$$= [(i,r,k,a,c,b,j,s,l)d^{jsl}_{irk}]\cdot[(j,s,l,p,u,q,m,t,n)d^{mtn}_{jsl}].$$

Here the first equality holds by the definition (6) of the product in G and the second by the definition (10) of the mapping d^{mtn}_{irk}. In the third equality we use that h, f_r and g_r are homomorphisms of the corresponding categories (the conditions (1)-(3)). The fourth equality is based on the condition (8). In the fifth equality we use the definition (6) of the product in H and in the sixth one the definition (10) of the mapping d^{jsl}_{irk} and d^{mtn}_{jsl}. The proposition is proved.

P R O P O S I T I O N 3. *Let $G = (R,S,G^s_r,\cdot,e_{Gr})$ be a category, $C = (R,S,C^s_r,\cdot,e_{Gr})$ be a subgroupoid in G and let $A = (R,\Delta_R,A^r_r,\cdot,e_{Gr})$ and*

$B = (R,\Delta_R,B^r_r,\cdot,e_{Gr})$ be two C-invariant subcategories in G. Suppose that $H = (P,Q,H^p_p,\cdot,e_{Hp})$ is also a category and $d : G \to H$, $d = (d_0,d^s_r,(r,s) \in S)$ is a homomorphism on categories where d_0 maps R onto P and S onto Q. Denote $C^s_r d^s_r = M^{sd_0}_{rd_0}$, $A^r_r d^r_r = D^{rd_0}_{rd_0}$, $B^r_r d^r_r = E^{rd_0}_{rd_0}$, $M = (P,Q,M^{sd_0}_{rd_0},\cdot,e_{Hrd_0})$, $D = (P,\Delta_p,D^{rd_0}_{rd_0},\cdot,e_{Hrd_0})$, $E = (P,\Delta_p,E^{rd_0}_{rd_0},\cdot,e_{Hrd_0})$ and put $h_0 = d_0$, $h^s_r = d^s_r|_{C^s_r}$ (restriction of d^s_r on C^s_r), $h = (h_0,h^s_r, (r,s) \in S)$, $f_r = d^r_r|_{A^r_r}$, $g_r = d^r_r|_{B^r_r}$. Then M is a subgroupoid in H, D and E are M-invariant subcategories in H and (f_r,h,g_r) is a homomorphism of the inner two-sided net I(G,A,C,B) into the inner two-sided net I(H,D,M,E).

It is easy to see that the condition (7) has now form $sd_0 = (ch^r_s) \triangleright \triangleright (rd_0)$ and is trivially satisfied. We must only check (8) which in our case has the form

$$(c \triangleright a)f_s = (ch^r_s) \triangleright (af_r) \tag{11}$$

and its analogue

$$(c \triangleright b)g_s = (ch^r_s) \triangleright (bg_r) \tag{12}$$

for any $c \in C^r_s$, $a \in A^r_r$, $b \in B^r_r$. But by the definition of the operation $\triangleright$ in the inner net we have

$$(c \triangleright a)f = (c\cdot a\cdot c)^{-1}f_s = (c\cdot a\cdot c)^{-1}d^s_s = (cd^r_s)\cdot(ad^r_r)\cdot(c^{-1}d^s_r) =$$
$$= (cd^r_s)(ad^r_r)(cd^r_s)^{-1} = (cd^r_s) \triangleright (af_r).$$

Analogically we get (12). This completes the proof.

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APPLICATION OF BITOPOLOGY TO ALGEBRA

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1. The notion of bitopological structure on X means here a topological structure on $X\times X$ thus (X, β) is a bitopological space if and only if $(X\times X, \beta)$ is a topological space. This notion proved to be [4] a natural generalization of the old notion [6] of bitopological space in Kelly's sense understood as triplet (X,τ_1,τ_2), where τ_1 and τ_2 are topological structures on X. Since (X,τ_1,τ_2) and $(X\times X, \tau_1\times\tau_2)$ define each other so introducing (X,τ_1,τ_2) or $(X,\tau_1\times\tau_2)$ is as a matter of fact the same procedure. We call (X,β) where $\beta=\tau_1\times\tau_2$ decomposable [4] bitopological space.

2. The prominent role of topologization of algebraic objects for example groups is well known, the theory of topological groups contains outstanding results. As to bitopological groups their theory is in the very beginning. Birsan T. has introduced [1] the following notion of bitopological in the Kelly's sense groups : a group G and two topologies τ_1,τ_2 on G that is a triplet (G,τ_1,τ_2) is a bitopological group if both $p:(G\times G,\tau_1\times\tau_1)\to(G,\tau_1)$ and $p:(G\times G,\tau_2\times\tau_2)\to(G,\tau_2)$ are continuous mappings, $p(x,y)=xy$ is a multiplication on G, and the mapping $q(x)=x^{-1}$ is a homeomorphism of (G,τ_1) onto (G,τ_2). Birsan considered there some initial statements of a theory of bitopological groups in that sense. One can prefers another approach of the same kind to the notion, for example by the following definition: a triplet (G,τ_1,τ_2) is a bitopological in Kelly's sense group if $\tilde{p}(G\times G,\tau_1\times\tau_2)\to(G,\tau_1)$, $\tilde{p}(x,y)=xy^{-1}$, is continuous mapping and the mapping $q(x)=x^{-1}$ is a homeomorphism of (G,τ_1) onto (G,τ_2). It is easy to see that the mapping $\bar{p}:(G\times G,\tau_1\times\tau_2)\to(G,\tau_2)$, where $\bar{p}(x,y)=yx^{-1}$, is continuous too. This definition of bitopological group is not new in reality since it is equivalent to the Birsan's definition.

3. General notion of bitopological group has to be based on the general notion of bitopological space. In [5] there are different variants of corresponding definitions but now it seems the most natural the following: bitopological group is a triplet (G,β,τ), where G is a group, β is a bitopological structure on G, τ is a topological structure on G, mapping $\bar{p}:(G\times G,\beta)\rightarrow(G,\tau)$ being continuous, $\bar{p}(x,y)=xy^{-1}$. The reason of such preference is clear from the above considerations. There is no general connection between β and τ and one of possible directions of research is discovering of such connections for special classes of bitopological groups . We have considered already the case with $\beta=\tau_1\times\tau_2$, $\tau=\tau_1$. We shall say that homomorphism $\phi : G_1\rightarrow G_2$ is bicontinuous homomorphism of bitopological group (G_1,β_1,τ_1) into bitopological group (G_2,β_2,τ_2) if $\phi :(G_1,\tau_1)\rightarrow(G_2,\tau_2)$ and $\phi\times\phi :(G_1\times G_1 ,\beta_1)\rightarrow(G_2\times G_2 ,\beta_2)$ are continuous mappings, $(\phi\times\phi)(x,y)=(\phi x,\phi y)$.

4. Another variant of definition of bitopological group has been proposed in [2] .The authors have formulated there the following definition : bitopological group is a pair (G,β) ,where G is a group, β is a bitopological structure on G, if the mappings $\bar{m}(x,y)=(xy, y^{-1}x)$, $\bar{i}(x,y)=(x^{-1},y^{-1})$ are bicontinuous mappings. Here in correspondence with [4] a mapping $f:X_1\rightarrow X_2$ is called bicontinuous mapping of (X_1,β_1) into (X_2,β_2) if the mapping $(f\times f):(G_1\times G_1, \beta_1)\rightarrow(G_2\times G_2 , \beta_2)$, $(f\times f)(x,y)=(fx,fy)$, is continuous mapping .

The notion of bitopological group in [2] is rather attractive since it has pure bitopological character ,here is no necessity to use additional topological structure. On the other hand it has essential lack since it is not a natural generalization of a corresponding notion of bitopological in Kelly's sense group namely the authors prove in [2] that if $\beta=\tau_1\times\tau_2$ then $\tau_1=\tau_2$.

5.Some authors have examined in fact bitopological groups having no idea about general notion of bitopological group. In [3] for example the author has considered topological groups (G,τ) in such sense that G is a group, τ is a topological structure on G and multiplication $p(x,y)=xy$ is not necessary continuous as a function of two variables but it is continuous as a function of any variable separately . In other words multiplication $\rho:(G\times G,\tau\times\tau)\rightarrow(G,\tau)$ is not necessary continuous mapping but it is continuous mapping on every horizontal set $G\times\{y\}$ and on every vertical set $\{x\}\times G$.

Let us define a topological structure β on $G\times G$ by the following condition: any set U is open in $(G\times G,\beta)$ if and only if its intersection with any horizontal set and with any vertical set is open in topology induced on this set by $\tau\times\tau$. It is easy to see that ρ will be continuous mapping of $(G\times G,\beta)$ into (G,τ) that is we have here a bitopological group (G,β,τ).

6. From now on we shall consider bitopological groups only in the sense of the item 3 that is triplets (G,β,τ) with continuous multiplications $\bar{\rho}(x,y)=xy^{-1}$. As it was said in the same item a problem of discovering of general connections between β and τ for different classes of bitopological groups not taking into account multiplication $\bar{\rho}$ is the problem of great importance and of great interest. In [4] the following connection between β and τ has been proposed. Let (X,β) be a bitopological space that is $(X\times X,\beta)$ be a topological space , $j:X\rightarrow X\times X$ be the diagonal mapping, $jx=(x,x)$, $j^{-1}\beta$ be a topological structure on X induced by β and j. We can suppose now that $\tau=j^{-1}\beta$ for any bitopological group (G,β,τ) . Such approach looks very attractive but again as in [2] from $\beta=\tau_1\times\tau_2$ follows $\tau_1=\tau_2$.

7. Another way is to fix on G the discrete topology τ_0. Then on any group G there exists the minimal bitopology β_0 ensuring continuity of $\bar{\rho}$. This topological structure on $G\times G$ consists of inverse images of all subsets of G and has a base consisting of inverse image of all elements of G, which proved to be minimal open sets in $(G\times G,\beta)$, they make a partition of $G\times G$. Each element G_c of the partition corresponds to some $c\in G$ namely $G_c=\{(cx,x)\,|\,x\in G\}$, in particular if $c=1_G$ then $G_c=\Delta=\{(x,x)\,|\,x\in G\}$ is the diagonal of $G\times G$. It is easy to prove that if we fix β_0 and the unit of G then multiplication on G will be fully defined. It means that in this situation such bitopological structure almost defines a corresponding algebraic structure on G. Moreover any bicontinuous mapping of a bitopological space (G,β_0) into bitopological

space (G',β_0') which transfers 1_G to $1_{G'}$ has to be homomorphism of G into G'

It is of interest to investigate different properties of above defined bitopological structure β_0 .Such structure for example will be always symmetric [4] that is the mapping $\phi(x,y)=(y,x)$ will be homeomorphism of $(G\times G,\beta_0)$ onto itself. It will be also transitive [4] that is convergences $((x_\lambda,y_\lambda)|\lambda\in\Lambda)\rightarrow(x,y)$ and $((y_\lambda,z_\lambda)|\lambda\in\Lambda)\rightarrow(y,z)$ imply convergence $((x_\lambda,z_\lambda)|\lambda\in\Lambda)\rightarrow(x,z)$.

8.We shall say that a bitopological structure β on X is co-ordinated [4] with topological structure τ on X if any convergence $((x_\lambda,y_\lambda)|\lambda\in\Lambda)\xrightarrow[\beta]{}(x,y)$ implies convergences $(x_\lambda|\lambda\in\Lambda)\xrightarrow[\tau]{}x$ and $(y_\lambda|\lambda\in\Lambda)\xrightarrow[\tau]{}y$ in other words if $\tau\times\tau\subset\beta$. So we can consider a class of bitopological groups (G,β,τ) for which β is co-ordinated with τ .It is easy to see that if $\beta=\tau_1\times\tau_2$ then $\tau\subset\tau_1\cap\tau_2$ so the condition of co-ordinatedness is too strong condition. Let us consider another not so strong condition. We shall say that a bitopological structure β on X is horizontally co-ordinated with topological structure τ on X if any convergence $((x_\lambda,y)|\lambda\in\Lambda)\xrightarrow[\beta]{}(x,y)$ implies convergence $(x_\lambda|\lambda\in\Lambda)\xrightarrow[\tau]{}x$. It is easy to see that bitopological structure $\beta=\tau_1\times\tau_2$ is horizontally co-ordinated with τ_1 .The condition of horizontal co-ordinatedness can be expressed in another way. Bitopological structure β being a topological structure on $X\times X$ induces a topological structure $\beta(\ ,y)$ on $X\times\{y\}$ for any $y\in X$. So we have the system $(\beta(\ ,y)|y\in X)$ topological structures on horizontal sets. Let π be the first projection of $X\times X$ onto X, $\pi(x,y)=x$, $\pi^{-1}(\tau)$ be a topological structure on $X\times X$. It induces on $X\times\{y\}$ a topological structure $\pi^{-1}(\tau)\wedge(X\times\{y\})$. Bitopological structure β will be horizontally co-ordinated with τ if $\pi^{-1}(\tau)\wedge(X\times\{y\})\subset\beta(\ ,y)$ for every $y\in X$. By the way bitopological structure β_0 of the item 7 is horizontally co-ordinated with τ_0 but of course it is not co-ordinated with τ_0.

9. The item 5 was devoted to a transition from continuity of multiplication $\rho:(G\times G,\ \tau\times\tau)\rightarrow(G,\tau)$, $\rho(x,y)=xy$, in regard to any variable, the other being fixed, to continuity of $\rho:(G\times G,\ \beta)\rightarrow(G,\tau)$ in regard to both variables simultaneously. The corresponding bitopological structure β can be

constructed such that $\tau\times\tau\subset\beta$, $\tau\times\tau$ and β induce coincided topology on every horizontal and every vertical set of $G\times G$. The same procedure is possible when we consider some bitopological structure β' instead of $\tau\times\tau$. The corresponding bitopological structure β can be constructed for $\bar{p}(x,y)=xy^{-1}$ such that $\beta'\subset\beta$, β' and β induce coincided topology on every horizontal and on every vertical set of $G\times G$. As a rule this bitopological structure β will not be locally decomposable [4] but one can imagine a situation when just a locally decomposable bitopological structure on a group G will be necessary. Locally decomposable bitopological groups have their peculiarity.

10. Bitopological structure β on X is called local [4] if any subset of $(X\times X)\setminus\Delta$ is open set in $(X\times X,\beta)$. Bitopological group (G,β,τ) is called local if β is local bitopological structure on G. Since $\bar{p}(\Delta)=1_G$ we could suppose that any subset of $G\setminus\{1_G\}$ is open set in (G,τ) however the condition of co-ordinatedness of β and τ implies some kind of homogeneity of τ.

11. We see that there is a lot of problems concerning properties of structures β and τ of bitopological groups. Another set of problems arises in connection with combining of bitopological groups: subgroups, quotient groups, products of groups, characters groups and so on.

12. The notion of bitopological subgroups and the notion of bitopological quotient group are quite natural. A bitopological group (G',β',τ') is a bitopological subgroup of (G,β,τ) if G' is a subgroup of G, β' and τ' are induced by β and τ on $G'\times G'\subset G\times G$ and on $G'\subset G$ respectively. A bitopological group $(\hat{G},\hat{\beta},\hat{\tau})$ is a bitopological quotient group of (G,β,τ) if $\hat{G}$ is a quotient group of G, $j:G\to\hat{G}$ is the qoutient mapping, $\hat{\beta}$ and $\hat{\tau}$ are maximal topological structures on $\hat{G}\times\hat{G}$ and $\hat{G}$ respectively for which $(j\times j):(G\times G,\beta)\to(\hat{G}\times\hat{G},\hat{\beta})$ and $j:(G,\tau)\to(\hat{G},\hat{\tau})$ become continuous mappings. Corresponding theory can be created without any difficulty.

13. Let us consider now the problem of bitopological groups of homomorphisms. The situation here is not so definitive as for example the corresponding situation in the problem of subgroups. The thing is that we can take all

homomorphisms or only continuous homomorphisms or even only bicontinuous homomorphisms. We stop on the last variant. So we consider two bitopological groups (G',β',τ'), (G,β,τ) and the set Homb(G',G) of their bicontinuous homomorphisms. It is necessary first of all to prove that Homb(G',G) is a group that is to prove that hg^{-1} is bicontinuous if h and g are bicontinuous homomorphisms. After that we have to introduce on $\tilde{H}$=Homb(G',G) needed topological structure $\tilde{\tau}$ and bitopological structure $\tilde{\beta}$ and to investigate correspondence between them to prove that $(\tilde{H},\tilde{\beta},\tilde{\tau})$ is the bitopological group.

14. It is necessary not only to create some general theory of bitopological groups but also to examine some concrete bitopological groups. The first example of such groups is bitopological real line. There are some variants of bitopological real line namely the followings. Let R is the real line, $\tau_1$and τ_2 are topological structures on R consisting of sets $R_{+t}=\{x|x > t\}$ and $R_{-t}=\{x|x < t\}$ respectively then $(R,\tau_1\times\tau_2,\tau_1)$ will be the decomposable bitopological real line considered as a group according to operation $\bar{p}(x,y)=x-y$. On the other hand we can introduce on $R\times R$ a topological structure δ induced by the mapping $\bar{p}:R\times R\rightarrow R$ and the usual topological structure τ on real line R. So we shall have another variant of bitopological real line.

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AN EXAMPLE IN THE MULTIVARIABLE SPECTRAL THEORY

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1. Introduction

In [3] C.-K. Fong and A. Sołtysiak proved that a complex, unital Banach algebra has a non-trivial multiplicative functional if and only if the Harte joint spectrum of an arbitrary finite family of its elements is non-empty. In addition, A. Sołtysiak proved in [7] that a similar result is also available when considering the approximate point joint spectrum . Even more, as it is shown in [6, 8], a unital Banach algebra A possesses non-trivial linear multiplicative functional if and only if such a functional exists on every closed finitely generated subalgebra of A.

In [4] we provided a characterization of joint spectra in the setting of arbitrary complex unital algebras and obtained, among other things, the following result.

T H E O R E M 1. *The following assertions on a complex unital spectrally bounded algebra A are equivalent:*

(a) the Harte joint spectrum $\sigma^{A}(a_1,a_2,\dots,a_n)$ *is non-empty for every n-tuple* $(a_1,a_2,\dots,a_n) \in A^n$*;*

(b) algebra A has a non-trivial linear multiplicative functional.

It is the object of this note to show that spectrally boundedness is an essential assumption on A in the above theorem. Namely, we present an example of a complex commutative algebra A with Harte joint spectrum non-empty for every finite family of elements of A, but without non-zero linear multiplicative functionals.

2. Preliminaries

Besides the standard terminology and notation [1, 5] we shall need the following.

Let A be a complex associative algebra. In the sequel by HomA we denote the set of all non-trivial linear multiplicative functionals on A equipped with the weak *-topology and by c(A) the set of all n-tuples $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of elements of $a_i \in A$ with arbitrary finite length n.

The *left* (resp. *right*) *joint spectrum* $\sigma_l(\mathbf{a})$ (resp. $\sigma_r(\mathbf{a})$) of an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in c(A)$ with respect to A is defined to be the set of all those $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ for which the n-tuple $(a_1 - \alpha_1 e_A, a_2 - \alpha_2 e_A, \ldots, a_n - \alpha_n e_A)$ generates a proper left (resp. right) ideal in A. The *Harte joint spectrum* $\sigma^{\wedge}(\mathbf{a})$ of $\mathbf{a} \in c(A)$ with respect to A is the set $\sigma_l^{\wedge}(\mathbf{a}) \cup \sigma_r^{\wedge}(\mathbf{a})$.

Finally, recall that A is said to be *spectrally bounded* if $\sigma^{\wedge}(a) \subset \mathbb{C}$ is bounded for each $a \in A$.

Now suppose we are given an inductive system $\{A_\alpha, f_{\beta\alpha}, \mathfrak{A}\}$ of unital algebras A_α together with algebra morphisms $f_{\beta\alpha} : A_\alpha \longrightarrow A_\beta$ for any $\alpha \le \beta$ in a directed index set $\mathfrak{A}$. Moreover, suppose $f_{\beta\alpha}(e_\alpha) = e_\beta$ for any $\alpha \le \beta$, where e_α (resp. e_β) stands for the identity of A_α (resp. A_β), that $\mathrm{Hom}A_\alpha$ is non-empty for every $\alpha \in \mathfrak{A}$, and consider for each pair $\alpha, \beta \in \mathfrak{A}$ such that $\alpha \le \beta$ the corresponding transpose map

$$\pi_\alpha^\beta : \mathrm{Hom}A_\beta \longrightarrow \mathrm{Hom}A_\alpha$$

defined by

$$\pi_\alpha^\beta(\Lambda) = \Lambda \circ f_{\beta\alpha}$$

for all $\Lambda \in \mathrm{Hom}A_\beta$. Clearly the family $\{\mathrm{Hom}A_\alpha, \pi_\alpha^\beta, \mathfrak{A}\}$ constitutes an inverse system of topological spaces, so that we may consider the corresponding inverse limit $H = \varprojlim\{\mathrm{Hom}A_\alpha, \pi_\alpha^\beta, \mathfrak{A}\}$.

For the proof of the following useful fact we refer to [5, p. 153].

T H E O R E M 2. *Let* $\{A_\alpha, f_{\beta\alpha}, \mathfrak{A}\}$ *be an inductive system of unital algebras and let* $A = \varinjlim A_\alpha$ *be the respective inductive limit algebra. If* $f_{\beta\alpha}(e_\alpha) = e_\beta$ *for any* α, β *in* $\mathfrak{A}$, *with* $\alpha \le \beta$, *then*

$$\mathrm{Hom}A = \varprojlim\{\mathrm{Hom}A_\alpha, \pi_\alpha^\beta, \mathfrak{A}\}$$

within a homeomorphism of the respective topological spaces.

3. Example

Let S be an uncountable set and denote by $\mathfrak{A}$ the set of all finite subsets of the set S directed by inclusion. For each $\alpha \in \mathfrak{U}$ let X_α be the set of all injections of α into the integers and denote for any $\alpha \le \beta$ $(\alpha,\beta \in \mathfrak{U})$ by p_α^β the restriction map $X_\beta \longrightarrow X_\alpha$. Then each X_α is countable, p_α^β is surjective for any $\alpha \le \beta$ in $\mathfrak{U}$, and the inverse limit $\varprojlim\{ X_\alpha, p_\alpha^\beta, \mathfrak{U} \}$ is empty [8, p. 618].

Further, let A_α $(\alpha \in \mathfrak{U})$ be the algebra of complex-valued continuous functions on the discrete topological space X_α, endowed with the compact-open topology, and define for any $\alpha \le \beta$ $(\alpha,\beta \in \mathfrak{U})$ an algebra morphism $f_{\beta\alpha}: A_\alpha \longrightarrow A_\beta$ by

$$f_{\beta\alpha}(g) = g \circ p_\alpha^\beta \qquad (g \in A_\alpha).$$

It is easily to be seen that $\{ A_\alpha, f_{\beta\alpha}, \mathfrak{U} \}$ is an inductive system and, as X_α is locally compact and hemicompact, every evaluation map $\delta_\alpha: X_\alpha \longrightarrow \mathrm{Hom}A_\alpha$ is, in fact, an onto homeomorphism [1, p. 29]. Consequently, since $\pi_\alpha^\beta = \delta_\alpha \circ p_\alpha^\beta \circ (\delta_\beta)^{-1}$ for any $\alpha \le \beta$, $\{\mathrm{Hom}A_\alpha, \pi_\alpha^\beta, \mathfrak{U} \}$ is an inverse system with all π_α^β surjective and with empty inverse limit [2, p. 90]. Thus, if $A = \varinjlim A_\alpha$ is the inductive limit of the system $\{ A_\alpha, f_{\beta\alpha}, \mathfrak{U} \}$, then, according to Theorem 2, HomA is an empty set.

Next we shall show that $\sigma^A(a)$ is non-empty for any $a \in c(A)$. To this end let $a_1, a_2, \dots, a_n \in A$ and choose an index $\alpha \in \mathfrak{U}$ and elements $x_1, x_2, \dots, x_n \in A_\alpha$ so that $f_\alpha(x_k) = a_k$ $(k = 1,2,\dots n)$, where f_α is the canonical morphism of A_α into A. Further, fix any $\lambda \in \mathrm{Hom}A_\alpha$ and suppose that there exist $b_1, b_2, \dots, b_n \in A$ satisfying

$$\sum_{k=1}^{n} (a_k - \lambda(x_k)e_A)b_k = e_A .$$

By the well-known properties of inductive systems we now can choose indices $\beta, \gamma \in \mathfrak{U}$ and $y_1, y_2 \dots, y_n, z_1, z_2 \dots, z_n \in A_\beta$ so that $\gamma \ge \alpha$, $\gamma \ge \beta$, $f_\beta(y_k) = a_k$, $f_\beta(z_k) = b_k$ and $f_{\gamma\alpha}(x_k) = f_{\gamma\beta}(y_k)$ for each $k = 1,2,\dots,n$. Besides, since π_α^γ is onto, there exists $\Lambda \in \mathrm{Hom}A_\gamma$ such that $\Lambda \circ f_{\gamma\alpha} = \lambda$ and, using this, we get

$$\Lambda(f_{\gamma\beta}(\sum_{k=1}^{n} (y_k - \lambda(x_k)e_\beta)z_k)) =$$

$$= \Lambda(\sum_{k=1}^{n} (f_{\gamma\beta}(y_k) - \lambda(x_k)e_\gamma)f_{\gamma\beta}(z_k)) =$$

$$= \sum_{k=1}^{n} (\lambda(x_k) - \lambda(x_k))\Lambda(f_{\gamma\beta}(z_k)) = 0.$$

On the other hand,

$$f_\gamma(f_{\gamma\beta}(\sum_{k=1}^{n} (y_k - \lambda(x_k)e_\beta)z_k)) =$$

$$= \sum_{k=1}^{n} (f_\beta(y_k) - \lambda(x_k)f_\beta(e_\beta))f_\beta(z_k) =$$

$$= \sum_{k=1}^{n} (a_k - \lambda(x_k)e_A)b_k = e_A .$$

But this, in turn, implies that

$$f_{\gamma\beta}(\sum_{k=1}^{n} (y_k - \lambda(x_k)e_\beta)z_k) = e_\gamma .$$

Hence

$$\Lambda(f_{\gamma\beta}(\sum_{k=1}^{n} (y_k - \lambda(x_k)e_\beta)z_k)) = \Lambda(e_\gamma) = 1.$$

This contradiction proves that

$$\sum_{k=1}^{n} (a_k - \lambda(x_k)e_A)b_k \neq e_A$$

for every $b_1,b_2,\dots,b_n \in A$. Equivalently, the Harte joint spectrum $\sigma^A(a_1,a_2,\dots,a_n)$ of the n-tuple $(a_1,a_2,\dots,a_n)$ is non-empty.

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SOME FUZZY FIXED POINT THEOREMS

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Fuzzy generalization of the famous Caristi-Kirk fixed point theorem [1] is obtained.

Let X be a metric space with a distance d, and let F be a set of all fuzzy subsets A of X ($A: X \to [0,1]$).

Suppose $S \subset F$.

D E F I N I T I O N 1. We shall say that S *tries to be like a system of closed fuzzy subsets of* X if for each $A, B \in S$:

$$\min(A,B) \in S.$$

Let $A \in F$ ($A \neq 0$, i.e. $\exists x \in X$: $x \in A$, i.e. $A(x) > 0$).

We define

$$\operatorname{diam} A := \sup \{d(x,y) \mid x,y \in A\}.$$

D E F I N I T I O N 2. We shall say that X *tries to be like a complete metric space* if for each descending sequence (A_n), $n \in N$ of closed fuzzy subsets of X

$$\inf\{A_n \mid n \in N\} \neq 0$$

if $\lim_{n \to +\infty} \operatorname{diam} A_n = 0$.

Let $O: R \times R \to [0,1]$.

D E F I N I T I O N 3. We shall say that O *tries to be like a fuzzy order on* R if for each $x, y \in R$:

1) $O(0,0) > 0$;

2) $O(x,y) > 0 \Rightarrow x \leq y$.

Let $G: X \to R$ ($\inf\{G(x) \mid x \in X\} \in R$).

D E F I N I T I O N 4. We shall say that G *tries to be like a continuous (or better - lower semi-continuous) function* if for each $x \in X$:

$$O(d(x,*),G(x)-G(*)) \in S.$$

Let $f: X \longrightarrow X$.

D E F I N I T I O N 5. We shall say that f *tries to be like a G-Cinderella* if for each $x, y \in X$:

$$O(d(x,y),G(x)-G(y)) \leq O(d(x,f(y)),G(x)-G(f(y))).$$

T H E O R E M 1. *Let X be a metric space which tries to be like a complete space, and let G be a function which tries to be like a continuous function.*

Then G-Cinderella f has a fixed point.

P r o o f. Let $x_0 \in X$. For each $x \in X$ we define

$$A_0(x) := O(d(x_0,x),G(x_0)-G(x)).$$

Since $A_0(x_0) = O(d(x_0,x_0),G(x_0)-G(x_0)) = O(0,0) > 0$, $x_0 \in A_0$ and $A_0 \neq \emptyset$. Since G tries to be like a continuous function, $A_0 \in S$.

Let $m_0 := \inf\{G(x) \mid x \in A_0\}$.

There exists $x_1 \in A_0$ such that $G(x_1) < m_0 + 1$.

For each $x \in X$ we define

$$B_1(x) := O(d(x_1,x),G(x_1)-G(x)).$$

Since $B_1(x_1) = O(d(x_1,x_1),G(x_1)-G(x_1)) = O(0,0) > 0$, $x_1 \in B_1$. Since G tries to be like a continuous function, $B_1 \in S$.

Let $A_1 := \min(A_0, B_1)$. Since

$$A_1(x_1) = \min(A_0(x_1),B_1(x_1)) > 0,$$

$x_1 \in A_1$ and $A_1 \neq \emptyset$. Since $A_0, B_1 \in S$ and S tries to be like a system of closed fuzzy subsets of X, $A_1 \in S$.

Inductively we construct a descending sequence (A_n), $n \in N$ of closed fuzzy subsets of X:

$$An \le Bn,$$

$$Bn(*) := \alpha(d(x_n,*), G(x_n)-G(*)),$$

$$G(x_n) < m_{n-1} + \frac{1}{n},$$

$$m_n := \inf\{G(x) \mid x \in An\}.$$

Let $n \in N$ and $x \in An$.
We have $0 < An(x) \le Bn(x) = \alpha(d(x_n,x), G(x_n)-G(x))$. Therefore

$$d(x_n,x) \le G(x_n)-G(x) <$$

$$< m_{n-1} + \frac{1}{n} - G(x) = \inf\{G(y) \mid y \in An\} + \frac{1}{n} - G(x) \le \frac{1}{n}.$$

The last inequality holds because $x \in A_{n-1}$.

Let $y \in An$.

Hence, $d(x,y) \le d(x_n,x) + d(x_n,y) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$, $\operatorname{diam} An \le \frac{2}{n}$ and $\lim_{n \to +\infty} \operatorname{diam} An = 0$.

Since X tries to be a complete space, we conclude that $\inf\{An \mid n \in N\} \ne 0$. Let $x \in X$: $\inf\{An(x) \mid n \in N\} > 0$. Since f tries to be like a G-Cinderella,

$$\inf\{An(f(x)) \mid n \in N\} > 0.$$

Therefore

$$d(x,f(x)) \le \operatorname{diam} An \le \frac{2}{n}$$

for each $n \in N$ and $x = f(x)$.

Theorem 1 can be slightly generalized using the concepts of illusionlike properties of G and f.

D E F I N I T I O N 6. We shall say that *G tries to be illusionlike continuous* and *f tries to be illusionlike G-Cinderella* if for each $x \in X$ there exists $A \in S$ $(x \in A)$ such that

$$A(*) \le \alpha(d(x,*), G(x)-G(*))$$

and $A(y) \in A(f(y))$ for each $y \in X$.

T H E O R E M 2. *Let X be a metric space which tries to be like a complete space. Let G try to be illusionlike continuous, and let f try to be illusionlike G-Cinderella.*

Then f has a fixed point.

P r o o f. Let $x_0 \in X$.

For each $x \in X$ we define

$$B_0(x) := \alpha(d(x_0,x), G(x_0) - G(x)).$$

Since G tries to be illusionlike continuous and f tries to be illusionlike G-Cinderella, there exists $A_0 \in S$ $(x_0 \in A_0)$ such that $A_0 \le B_0$ and $A_0(x) \le A_0(f(x))$ for each $x \in X$.

Let $m_0 := \langle G(x) \mid x \in A_0 \rangle$.

There exists $x_1 \in A_0$ such that $G(x_1) < m_0 + 1$.

For each $x \in X$ we define $B_1(x) := \alpha(d(x_1,x), G(x_1) - G(x))$. Again, since G tries to be illusionlike continuous and f tries to be illusionlike G-Cinderella, there exists $C_1 \in S$ $(x_1 \in C_1)$ such that $C_1 \le B_1$ and $C_1(x) \le C_1(f(x))$ for each $x \in X$.

Let $A_1 := \min\langle A_0, C_1 \rangle$. Since

$$A_1(x_1) = \min\langle A_0(x_1), C_1(x_1) \rangle,$$

$A_1(x_1) > \emptyset$, $x_1 \in A_1$ and $A_1 \ne \emptyset$. Since $A_0, C_1 \in S$ and S tries to be like a system of closed fuzzy subsets of X, $A_1 \in S$.

Inductively we construct a descending sequence (An), $n \in N$ of closed fuzzy subsets of X and complete the proof of Theorem 2 by the same argument as proving the previous one.

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TOPOLOGICAL INVERTIBILITY IN TOPOLOGICAL ALGEBRAS

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1. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and A be a topological $\mathbb{K}$-algebra with unit and with separately continuous multiplication. Topological $\mathbb{K}$-algebra A is called

a) *a Q-algebra* if the set *InvA* of all invertible elements of A is open;

b) *a locally pseudoconvex algebra* if the topology of A is given by the family $\{p_\alpha: \alpha \in \mathfrak{U}\}$ of k_α-homogeneous seminorms where $k_\alpha \in (0,1]$ for all $\alpha \in \mathfrak{U}$;

c) *a locally k-convex algebra* if A is a locally pseudoconvex algebra for which $k_\alpha = k$ for each $\alpha \in \mathfrak{U}$;

d) *a locally m-(k-convex) algebra* if A is a locally k-convex algebra every seminorm p_α of which satisfies the condition

$$p_\alpha(ab) \le p_\alpha(a)p_\alpha(b)$$

for each $a,b \in A$.

In the case, when $k = 1$, locally m-(k-convex) algebra is called *a locally m-convex algebra*.

2. Let $\mathfrak{U}$ be a directed set and $\{A_\alpha: \alpha \in \mathfrak{U}\}$ be the family of topological $\mathbb{K}$-algebras. If for all pairs $(\alpha,\beta) \in \mathfrak{U} \times \mathfrak{U}$ with $\alpha \le \beta$ there exists a continuous homomorphism h_α^β from A_β into A_α such that

1) h_α^α is the identity mapping on A_α

and

2) $h_\alpha^\beta \circ h_\beta^\gamma = h_\alpha^\gamma$ for all $\alpha,\beta,\gamma \in \mathfrak{U}$ with $\alpha \le \beta \le \gamma$,

then the triple $(A_\alpha, h_\alpha^\beta; \mathfrak{U})$ is called *a projective system of* *topological* $\mathbb{K}$*-algebras A* over $\mathfrak{A}$. The set

$$limA_{\alpha} = \{(a_{\alpha}) \in \prod_{\alpha\in\mathfrak{A}} A_{\alpha}: h_{\alpha}^{\beta}(a_{\beta}) = a_{\alpha} \text{ if } \alpha \leq \beta \text{ in } \mathfrak{A}\}$$

is called a *projective limit of this system*. In the case when every algebra A_{α} has the unit e_{α}, we assume that $h_{\alpha}^{\beta}(e_{\beta}) = e_{\alpha}$ for each $\alpha, \beta \in \mathfrak{A}$ with $\alpha \leq \beta$. Then the set $limA_{\alpha}$ is not empty.

3. Let A be a topological $\mathbb{K}$-algebra. We shall say that an element $a \in A$ is *topologically invertible in A* if there exists a net $(a_{\lambda})_{\lambda\in\Lambda}$ in A such that the nets $(aa_{\lambda})_{\lambda\in\Lambda}$ and $(a_{\lambda}a)_{\lambda\in\Lambda}$ converge to the unit e_{A} of A in the topology of A. We shall denote this set by *TopinvA*.

In a general case (see [1], p. 1309) the set *TopinvA* is wider than the set *InvA*. At this, *TopinvA* = *InvA* if A is a *Q*-algebra over $\mathbb{K}$ with unit or a Fréchet locally m-convex $\mathbb{C}$-algebra with unit (see [1], p. 1309).

4. In the present notice we shall show that *TopinvA* = *InvA* in more general case, when A is a projective limit of Hausdorff *Q*-algebras over $\mathbb{K}$ with unit.

T H E O R E M. *Let* $limA_{\alpha}$ *be a projective limit of Hausdorff Q-algebras* A_{α} *over* $\mathbb{K}$ *with unit. Then*

$$Topinv\,limA_{\alpha} = Inv\,limA_{\alpha} .$$

P r o o f. Let $a = (a_{\alpha}) \in limA_{\alpha}$ be a topologically invertible element. Then there exists a net $(a(\lambda))_{\lambda\in\Lambda}$ in $limA_{\alpha}$ such that the nets $(aa(\lambda))_{\lambda\in\Lambda}$ and $(a(\lambda)a)_{\lambda\in\Lambda}$ converge to $(e_{\alpha})_{\alpha\in\mathfrak{A}}$. Let π_{α} be the projection from $\prod\{A_{\alpha}: \alpha \in \mathfrak{A}\}$ onto A_{α} and $\mu_{\alpha} = \pi_{\alpha}|limA_{\alpha}$ for each $\alpha \in \mathfrak{A}$. As μ_{α} is continuous for each $\alpha \in \mathfrak{A}$, the nets $(\mu_{\alpha}(aa(\lambda)))_{\lambda\in\Lambda}$ and $(\mu_{\alpha}(a(\lambda)a))_{\lambda\in\Lambda}$ converge to $\mu_{\alpha}((e_{\alpha})_{\alpha\in\mathfrak{A}}) = e_{\alpha}$ for each $\alpha \in \mathfrak{A}$ (here e_{α} is the unit of A_{α}). Hence $\mu_{\alpha}(a) \in TopinvA_{\alpha}$ for each $\alpha \in \mathfrak{A}$. As A_{α} is a *Q*-algebra then $\mu_{\alpha}(a) \in InvA_{\alpha}$ for each $\alpha \in \mathfrak{A}$. Consequently, $a \in Inv\,limA_{\alpha}$ (see [4], p. 7).

C O R O L L A R Y. *Let* $k \in (0,1]$ *and* A *be a complete Hausdorff locally m-(k-convex)* $\mathbb{K}$*-algebra with unit. Then*

$$\mathrm{Topinv}A = \mathrm{Inv}A.$$

P r o o f. Let A be a complete locally m-(k-convex) $\mathbb{K}$-algebra with unit the topology of which is given by the family $\{p_\alpha : \alpha \in \mathfrak{U}\}$ of k-homogeneous seminorms, $A_\alpha = A/\ker p_\alpha$ and π_α be the canonical mapping from A onto A_α for each $\alpha \in \mathfrak{U}$. Let $\tilde{A}_\alpha$ be the completion of A_α, τ_α be the topological isomorphism from A_α onto a dense subset of $\tilde{A}_\alpha$, $\mu_\alpha = \pi_\alpha | \lim \tilde{A}_\alpha$ and Φ be the mapping from A into $\Pi\{\tilde{A}_\alpha : \alpha \in \mathfrak{U}\}$ defined by

$$\Phi(a) = (\tau_\alpha \circ \pi_\alpha(a))_{\alpha \in \mathfrak{U}}$$

for each $a \in A$. Then Φ is a topological isomorphism from A onto $\lim \tilde{A}_\alpha$ (see [4], p. 22).

Let $a \in \mathrm{Topinv}A$. Then there exists a net $(a_\lambda)_{\lambda \in \Lambda}$ in A such that the nets $(aa_\lambda)_{\lambda \in \Lambda}$ and $(a_\lambda a)_{\lambda \in \Lambda}$ converge in A to e_A. As μ_α and Φ are continuous mappings then the nets $(\mu_\alpha \circ \Phi(aa_\lambda))_{\lambda \in \Lambda}$ and $(\mu_\alpha \circ \Phi(a_\lambda a))_{\lambda \in \Lambda}$ converge to the element $\mu_\alpha \circ \Phi(e_A) = \tilde{e}_\alpha$ (the unit of $\tilde{A}_\alpha$). Hence $(\mu_\alpha \circ \Phi(a)) \in \mathrm{Topinv}\tilde{A}_\alpha$ for each $\alpha \in \mathfrak{U}$. As $\tilde{A}_\alpha$ is a k-Banach $\mathbb{K}$-algebra with unit (hence a Q-algebra (see [2], p. 10, or [3], p. 17)) for each $\alpha \in \mathfrak{U}$, then $(\mu_\alpha \circ \Phi(a)) \in \mathrm{Inv}\tilde{A}_\alpha$ for each $\alpha \in \mathfrak{U}$. Taking this into account it is clear that $\Phi(a) \in \mathrm{Inv}\lim \tilde{A}_\alpha$ (see [4], p. 7). As the mapping Φ is an isomorphism then $a \in \mathrm{Inv}A$.

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CATEGORIES OF HAUSDORFF AND K-SPACES

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In memory of my daughter Josefina.

In this paper we investigate the category of Hausdorff topological spaces TH and k-right maps and full subcategory of k-spaces TK of TH, which is analogous to the theory for TOP_k and K_k in [5].

The reflecto-coreflective functor F_k : TH -> TK, which stabilizes the equivalence between category TH and subcategory TK, was constructed.

D.E.Cohen in [2] and J.Kelly, in [3] investigated the k-spaces : A topological space X is called a k-space if the subset $A \subset X$ is closed iff $A \cap C \in \mathcal{C}$ for each $C \in \mathcal{C}$, where $\mathcal{C}$ is a family of closed compact sets.

They established that for every Hausdorff topological space X there exists a k-space $\tilde{X}$ so that there exists a one-to-one continuous k-map φ of $\tilde{X}$ onto X.

The map f : X -> Y is called a k-map if for each compact $F \subset X$ the image fF of F under f is compact, at the some time the inverse image $f^{-1}\Phi$ of each compact $\Phi \subseteq Y$ under f is compact as well. A.V.Arhangel'skii in [1] proved that for every Hausdorff topological space X there exists a unique k-space $\tilde{X}$ that $\varphi : \tilde{X} \to X$, where φ is a one-to-one onto continuous k-map.

D E F I N I T I O N 1.

The map f : X -> Y is called a k-right map if for each compact $F \subset X$ the image f $\subset Y$ of F. is compact.

D E F I N I T I O N 2. TH-CATEGORY.

The objects of the category TH *are all Hausdorff topological spaces, the morphisms are k-right maps (def. 1) while the composition is a usual composition of maps.*

D E F I N I T I O N 3. TK-CATEGORY.

The objects of the category TK *are all Hausdorff topological k-spaces, the morfisms are k-right maps.*

It is clear that the composition of two k-right maps is again a k-right map. The TK-category is a full subcategory of category TH.

D E F I N I T I O N 4. FUNCTOR F_k.

Functor F_k : TH -> TK, $\forall X \in$ TH, $F_k(X) = \tilde{X}$, *id est*,

Functor F_k *assigns every* X *its underlying the k-spaces* $\tilde{X}$, *and for* $\mu \in Hom_{TH}(X,Y)$ *assigns* $\tilde{\mu}$, *id est* $F_k(\mu) = \tilde{\mu}$, *where the map* $\tilde{\mu} = \varphi_y^{-1} \circ \mu \circ \varphi_x$ *is defined so that the commutativity* $\mu \circ \varphi_x = \tilde{\varphi}_y \circ \tilde{\mu}$ holds in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ \varphi_x \uparrow & \tilde{\mu} & \uparrow \varphi_y \\ \tilde{X} & \longrightarrow & \tilde{Y} \end{array}$$

D E F I N I T I O N 5. [4].

Let X *and* Y *be topological spaces. A function* $f : X \longrightarrow Y$ *is called a partially continuous at a point* $x_0 \in X$ *if for every neighbourhood* $U(x_0)$ *of* x_0 *and every neighbourhood* $V(f(x_0))$ *of* $f(x_0)$ *there exists a point* $x \in U(x_0)$, $x \neq x_0$ *such that* $f(x) \in V(f(x_0))$.

T H E O R E M 1.

The functor F : TH$\longrightarrow$TK *is simultaneously reflective and coreflective.*

P r o o f. The functor F_k, which assigns to every space X its underlying the k-space $\tilde{X}$ possesses a front adjunction map, which is

$$\varphi_x^{-1} : X \longrightarrow \tilde{X} \; , \; \varphi_x^{-1} \in Hom_{TH}(X, F_k(X))$$

(map φ_x^{-1} is partially continuous).

One of the caracteristic properties of this map φ_x^{-1} consists in the fact that the commutativity $\varphi_y^{-1} \circ \mu = F_k(\mu) \circ \varphi_x^{-1}$ holds in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;\mu\;} & Y \\ \varphi_x^{-1} \downarrow & & \downarrow \varphi_y^{-1} \\ \tilde{X} & \xrightarrow{\tilde{\mu}=F_k(\mu)} & \tilde{Y} \end{array}$$

for all $\mu \in Hom_{TH}(X,Y)$ and φ_x^{-1} the defining map

$$(\varphi_x^{-1})^{*}(\eta) = \eta \circ \varphi_x^{-1} \in Hom_{TH}(X,\tilde{Y}) \text{ for all } \eta \in$$

$Hom_{Tk}(F_k(X),\tilde{Y})$ and in view of φ_x^{-1} commutativity also holds in the diagram :

$$\begin{array}{ccc} X & & \\ \varphi_x^{-1} \downarrow & \searrow^{\eta \circ \varphi_x^{-1}} & \\ F_k(X) & \xrightarrow{\;\eta\;} & \tilde{Y} \end{array}$$

The map $(\varphi_x^{-1})^{*} : Hom_{TK}(F_k(X),\tilde{Y}) \longrightarrow Hom_{TH}(X, i\tilde{Y})$ is one-to-one and onto whenever $\tilde{Y} \in TH$. Hence, the functor F_k is a reflector and it is left adjoint of the injection functor

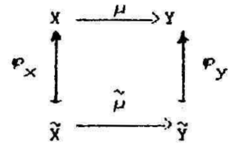
$$i : TK \longrightarrow TH \; , \quad \tilde{X} \in Tk \; , \; i\tilde{X} = \tilde{X}$$

$$F_k \vdash i$$

consequently a category TK is reflective in the category TH.

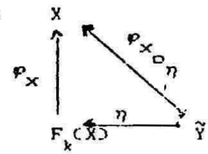
The functor F_k is dually, i.e., it is coreflector as the functor F_k has the end adjunction map $\varphi_x \in Hom_{TH}(F_k(X),X)$.

One of the characteristics of φ_x consists in the fact that the commutativity $\mu \circ \varphi_x = \varphi_y \circ F_k(\mu)$ holds in the diagram :



for all $\mu \in Hom_{TH}(X,Y)$ and defining map $(\varphi_X)_*(\eta) = \varphi_X \circ \eta \in Hom_{TH}(\tilde{Y},X)$ for all $\eta \in Hom_{TK}(\tilde{Y},F_k(X))$ and in view of φ_X.

Commutativity also holds in the diagram



The map $(\varphi_X)_* : Hom_{TK}(\tilde{Y}, F_K(X)) \longrightarrow Hom_{TK}(i\tilde{Y},X)$ is one-to-one and onto whenever $\tilde{Y} \in TK$ and it is right adjoint of the injection functor i, $i \dashv F_k$. Therefore, we stated that the category TK is coreflective in category TH.

T H E O R E M 2 .

The category TH is a categorial equivalent to subcategory TK.

Really, the maps φ_X and φ_X^{-1} are categorial isomorphisms , because

$$\varphi_X \circ \varphi_X^{-1} = 1_X \qquad \varphi^{-1}_X \circ \varphi = 1_{\tilde{X}}$$

Further 1_{TH} , $i \circ F_K : TH \longrightarrow TH$

$\forall X \in TH \; (i \circ F_k)(X) = \tilde{X}$, $1_{TH}(X) = X$

Consequently, we have functorial isomorphism

$$\varphi^{-1} : 1_{TH} \longrightarrow i \circ F_k \quad (\varphi : i \circ F_k \longrightarrow 1_{TH})$$

because $\varphi_X^{-1} : X \longrightarrow \tilde{X}$ $(\varphi_X : \tilde{X} \longrightarrow X)$ is a categorial isomorphism.

Analogously for functors

$$1_{TK} \longrightarrow F_k \circ i : TK \longrightarrow TK$$

we have $1_{TK}(\tilde{X})=\tilde{X}$, $(F_k \circ i)(\tilde{X})=\tilde{X}$, so

$1_{\tilde{x}} : \tilde{X} \longrightarrow \tilde{X}$ is a categorial isomorphism,

hence $1 : 1_{TK} \longrightarrow F_k \circ i$ is a functorial isomorphism.

Further $(F_k . \varphi^{-1})(X) = F_k(\varphi_x^{-1}) = 1_{\tilde{X}}$ it follows from the diagram:

$$\begin{array}{ccc} X & \xleftarrow{\varphi_x} & F_k(X) \\ \downarrow{\scriptstyle \varphi_x^{-1}} & & \downarrow{\scriptstyle F_k(\varphi_X^{-1})} \\ \tilde{X} & \xleftarrow{1_{\tilde{X}}} & F_k(X) \end{array}$$

$$\varphi_x^{-1} \circ \varphi_x = 1_{\tilde{X}} \quad , \quad 1_{\tilde{X}} \circ 1_{\tilde{X}} = 1_{\tilde{X}} \; .$$

Analogous $(1.F_k)(X) = 1.(F_k(X)) = 1_{\tilde{X}}$, hence $F_k . \varphi^{-1} = 1.F_k$.

Consequently, an important property was established. This is expressed by the following categorial equivalence between the category TH and the subcategory TK, i.e. TH $\simeq$ TK.

Now we establish some properties of the functor F_k. The definition 7 was established in [5].

D E F I N I T I O N 6. *A map* $f : X \longrightarrow Y$ *will be called perfect if and only if the inverse image* f_y^{-1} *of the point* $y \in Y$ *is compact in* X.

D E F I N I T I O N 7. *Let* $\mathcal{C}_1$, $\mathcal{C}_2$ *be categories of topological spaces. The functor* $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ *is called perfect if a map* $f \in Hom_{\mathcal{C}_1}(X,Y)$ *is perfect, then the map* $F(f) \in Hom_{\mathcal{C}}(F(X),F(Y))$ *is also a perfect map.*

The functor F_k is perfect because F_k preserves the perfect property, that is if a map $\mu \in Hom_{TH}(X,Y)$ is perfect then $F_k(f)=\tilde{f} \in Hom_{TK}(\tilde{X},\tilde{Y})$ is also a perfect map, it follows from the definition map $\tilde{f}$.

The next definitions were established in [4].

Let $F : T \longrightarrow P$ be a reflector defined on the category topological spaces (P is a full subcategory in T) F is a simple reflector if the front adjoint map

$e_x : X \longrightarrow F(X)$ is one-to-one and onto for all X.

F is an identifying reflector if the e_x map X is onto F(X) for all X.

D E F I N I T I O N . *The full subcategory* P *is simple (resp. identifying) iff there exists a simple (resp. identifying) reflector* $F : T \longrightarrow P$.

D E F I N I T I O N . *A coreflector* $G : T \longrightarrow P$ *is cosimple if the end adjunction map* $\varepsilon_x : G(X) \longrightarrow X$ *is one-to-one and onto for all X.*

Remark 1. The functor F_k is simultaneously a simple, identifying a cosimple as the full subcategory TK is simple and identifying.

D E F I N I T I O N 8. *Let* (X, τ_x), (Y, τ_y) *be topological spaces. The map* $f : X \longrightarrow Y$ *is called a k-left map if for each compact* $F \subset Y$ *the inverse image* $f^{-1}F \subset X$ *of F under f is compact.*

Remark 2.

If in categories TH, TK instead of k-right maps take k-left maps then an analogous theory for TH, TK, as in case k-right maps, can be constructed.

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CUMULANTS OF K-STATISTICS UNDER AN ELLIPTICAL POPULATION

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In this paper some programs in REDUCE will be described to compute higher cumulants of k-statistics under an elliptical population.

1. The assumption of elliptically distributed population has been given great attention among statisticians in recent years [1,5,7], because in this case it is possible to get quite general results which are almost as simple as in the normal distribution case.

We recall that the random p-vector $x = (x_1,...,x_p)$ is said to have an elliptical distribution $E_p(\mu,V)$ if its density function is of the form

$$f(x) = a_p|V|^{-1/2}h((x-\mu)^T V^{-1}(x-\mu)) \qquad (1)$$

for some function h, a positive definite matrix V and a normalizing constant a_p.

As a special case ($a_p = (2\pi)^{-p/2}$, $h(u) = e^{-u/2}$) we get the density function for the multivariate normal distribution

$$f_1(x) = (2\pi)^{-p/2}|V|^{-1/2}e^{-(x-\mu)^T V^{-1}(x-\mu)/2} \qquad (2)$$

with covariance matrix V.

The characteristic function of the p-dimensional elliptical distribution has the form

$$\varphi(t) = \psi(t^T Vt)\, e^{it^T\mu} \qquad (3)$$

for some function ψ. The k-order cumulants $\varkappa_{ij...s}$ [3,4] of $E_p(\mu,V)$ can be derived as the partial derivatives of $\ln\varphi(t)$

$$\left.\frac{\partial^k \ln \varphi(t_1,...,t_p)}{\partial t_i \partial t_j \ldots \partial t_s}\right|_{t=0} = i^k \varkappa_{ij...s}, \qquad (4)$$

where i,j,...,s are k distinct indices. Hence, the characteristic function can be expressed as an asymptotic series through cumulants. The cumulants of an elliptical distribution up to the 4-th order are given in [5], higher cumulants in [7]:

$$Ex = \mu, \quad cov(x) = (\varkappa_{ij}) = k_2 V, \tag{5}$$

$$\varkappa_{ijkl} = k_4 \sum^{3} \varkappa_{ij}\varkappa_{kl} = k_4(\varkappa_{ij}\varkappa_{kl} + \varkappa_{ik}\varkappa_{jl} + \varkappa_{il}\varkappa_{jk}), \tag{6}$$

$$\varkappa_{ijklab} = k_6 \sum^{15} \varkappa_{ij}\varkappa_{kl}\varkappa_{ab}, \tag{7}$$

$$\varkappa_{ijklabcd} = k_8 \sum^{105} \varkappa_{ij}\varkappa_{kl}\varkappa_{ab}\varkappa_{cd}, \tag{8}$$

where k_2, k_4, k_6, k_8 are the constants depending on the derivatives of the function ψ [7]. It is easy to see that the number of summands for $\varkappa_{i_1 i_2 \ldots i_{2k-1} i_{2k}}$ is $(2k-1)!!$.

The <u>k-order k-statistic</u> is the sample function, the mean value of which is the k-order cumulant. k-statistics are widely used in mathematical statistics. In many cases, for instance in asymptotic theory, their higher cumulants are needed.

2. Let us consider the general expressions of higher cumulants of k-statistics, given by Kaplan [2],[3,P.441] and transform them to the case of elliptical population (in this case all the $\varkappa$-s with the odd number of indices are equal to zero). For example, the fourth cumulant of the second order k-statistics is

$$\varkappa(ab,ij,pq,uv) = \varkappa_{abijpquv}/n^3 + \sum^{24} \varkappa_{ai}\varkappa_{bjpquv}/\{n^2(n-1)\} +$$
$$+ \sum^{8} \varkappa_{aipu}\varkappa_{bjqv}(n^2-3n+3)/\{n^2(n-1)^3\} + 3\sum^{24} \varkappa_{abpu}\varkappa_{ijqv}/\{n^2(n-1)\} +$$
$$+ \left[\sum^{96} \varkappa_{ai}\varkappa_{bp}\varkappa_{jquv} + \sum^{48} \varkappa_{ai}\varkappa_{pu}\varkappa_{bjqv}\right]/\{n(n-1)^2\} +$$
$$+ \sum^{48} \varkappa_{bi}\varkappa_{jp}\varkappa_{qu}\varkappa_{va}/(n-1)^3 . \tag{9}$$

The $\varkappa$-s in products are commuting with each other and the indices by each $\varkappa$ are also commuting. The sums in (9) are generated according to the partition of indices (ab,ij,pq,uv), where the rules for taking indices are determined by the given product of $\varkappa$-s. The total number of summands is shown above the summation sign. For example,

$$\sum^{8} \varkappa_{aipu}\varkappa_{bjqv} = \varkappa_{aipu}\varkappa_{bjqv} + \varkappa_{aipv}\varkappa_{bjqu} + \varkappa_{aiqu}\varkappa_{bjpv} + \varkappa_{aiqv}\varkappa_{bjpu} +$$
$$+ \varkappa_{ajpu}\varkappa_{biqv} + \varkappa_{ajpv}\varkappa_{biqu} + \varkappa_{ajqu}\varkappa_{bipv} + \varkappa_{ajqv}\varkappa_{bipu}.$$

Using the accuracy $o(n^{-3})$ the exact formula (9) may be expressed in the form

$$\varkappa(ab,ij,pq,uv) = 1/n^3\Big\{\varkappa_{abijpquv} + \sum^{24} \varkappa_{ai}\varkappa_{bjpquv} + \sum^{8} \varkappa_{aipu}\varkappa_{bjqv} +$$

$$+ 3\sum^{24} \varkappa_{abpu}\varkappa_{ijqv} + \sum^{96} \varkappa_{ai}\varkappa_{bp}\varkappa_{jquv} + \sum^{48} \varkappa_{ai}\varkappa_{pu}\varkappa_{bjqv} +$$

$$+ \sum^{48} \varkappa_{bi}\varkappa_{jp}\varkappa_{qu}\varkappa_{va}\Big\} . \qquad (10)$$

3. In the elliptical population case it is possible to give the expressions of higher cumulants of k-statistics as the polynomials of $\varkappa_{ij}$. For instance, in our example (10) the $\varkappa$-s with 4,6 and 8 indices must be replaced by the corresponding formulae (6)-(8) and the result may be given in the form

$$\varkappa(ab,ij,pq,uv) = \sum_{i=1}^{5} \alpha_i S_i \qquad (11)$$

where S_i are the sums of all the possible products of four $\varkappa_{ij}$-s for the given paritition of indices (ab,ij,pq,uv):

$$S_1 = \sum^{1} \varkappa_{ab}\varkappa_{ij}\varkappa_{pq}\varkappa_{uv} ,$$

$$S_2 = \sum^{12} \varkappa_{ai}\varkappa_{bj}\varkappa_{pq}\varkappa_{uv} ,$$

$$S_3 = \sum^{32} \varkappa_{ai}\varkappa_{bp}\varkappa_{jq}\varkappa_{uv} , \qquad (12)$$

$$S_4 = \sum^{12} \varkappa_{ai}\varkappa_{bj}\varkappa_{pu}\varkappa_{qv} ,$$

$$S_5 = \sum^{48} \varkappa_{ai}\varkappa_{bp}\varkappa_{ju}\varkappa_{qv} .$$

We note that all the products in these sums are pairwise distinct and the total number of such summands is 1+12+32+12+48=105 which is equal to the number of summands in the formula (8) for the eighth cumulant $\varkappa_{ijklabcd}$ of an elliptical distribution.

For comparison we give an analogous to (11) formula for the third cumulant of the second order k-statistics (see [7])

$$\varkappa(ab,ij,pq) = n^{-2}\Big\{k_6\varkappa_{ab}\varkappa_{ij}\varkappa_{pq} + (k_6+3k_4+1)\sum^{8} \varkappa_{ai}\varkappa_{bp}\varkappa_{jq} +$$

$$+ (k_6+2k_4)\sum^{6} \varkappa_{ab}\varkappa_{ip}\varkappa_{jq}\Big\} + o(n^{-2}) . \qquad (13)$$

4. We have made the following programs in REDUCE on IBM PC XT:

1) MAIN - the main program;

2) GENKAPA - gives the rules for creating the cumulants $\varkappa_{ijkl}$, $\varkappa_{ijklab}$, $\varkappa_{ijklabcd}$, ... correspondingly to the expressions (6)-(8);

3) DISTSUM - gives the main partition of indices in the cumulant (the matrix Distr) and the partition of indices in the i-th sum of the cumulant (the matrix Sum).

For example, in the case (9) for the second sum the program DISTSUM begins as follows:

```
Distr:=Mat((a,b,0,i,j,0,p,q,0,u,v))$
Distproc()$
Sum:=Mat((a,i,0,b,j,p,q,u,v))$
Sumproc()$:
```

Here zeros are used as delimiters. In the general case the procedure Distproc shares the blocks of indices of the cumulant into N sets $A_1, A_2, \dots, A_N$ such that all the blocks with the equal number of indices belong to the one set A_j. It computes

V_j - the number of indices in each block of A_j;

M_j - the number of blocks of A_j.

Thus, $A_j = \left\{G_{j1}, \dots, G_{jM_j}\right\}$ and $|G_{jk}| = V_j$ for every $k=1,\dots,M_j$. In our example (9) $N = 1$, $V_1 = 2$, $M_1 = 4$ and

$$A_1 = \{\{a,b\},\{i,j\},\{p,q\},\{u,v\}\} .$$

4) MAKESUM - transforms the i-th sum in (9) to the form of polynomial of $\varkappa$-s with 2 indices and saves it on the disk. More precisely, let Perm(n) denotes the set of all permutations of n indices. Then MAKESUM generates the products of $\varkappa$-s correspondingly to the matrices Sum and Distr for every set of permutations $\sigma_1 \in \mathrm{Perm}(M_1), \dots, \sigma_N \in \mathrm{Perm}(M_N)$ of the corresponding blocks in $A_1, \dots, A_N$ and for every set of permutations $\sigma_{jk} \in \mathrm{Perm}(V_k)$ of indices in each block G_{jk}. Hence, it generates

$$h = V_1! \cdot V_2! \cdot \dots \cdot V_N! \cdot (M_1!)^{V_1} \cdot (M_2)^{V_2} \cdot \dots \cdot (M_N)^{V_N} \tag{14}$$

products of $\varkappa$-s. Some of them are equal and MAKESUM devides the result by the great common divisor d. In our example (9)

$$h = 4!(2!)^4 = 384,$$

for the second summand d=16 and we obtain 384:16=24 summands of type $\varkappa_{ai}\varkappa_{bjpquv}$.

Finally, using GENKAPA, the program MAKESUM computes the i-th sum of the cumulant in the form of the polynomial of $\varkappa$-s of two indices and saves it on the disk.

As in our example h=384, it took MAKESUM about 10 minutes to compute one sum of (9) in REDUCE on PC XT. While the indices may be integers as well, it will be better to write MAKESUM in a faster programming language as PASCAL or C.

The second difficulty is connected with the overflow of the computer memory and we must reload REDUCE after each processing of MAKESUM (the result is saved on the disk).

5) TRANSAR - transfers polynomials of $\varkappa$-s to the array-form. It is needed because we cannot add the sums of higher cumulants immediately as polynomials, since the memory will be overflowed. Therefore we must transfer these polynomials to the array-form in the following way. We use the natural order of monomials, applied in REDUCE for numbering the products of $\varkappa$-s .

In our example there are 105 monomials:

$$Mo_1 = \varkappa_{ab}\varkappa_{ij}\varkappa_{pq}\varkappa_{uv} \ , \ Mo_2 = \varkappa_{ab}\varkappa_{ij}\varkappa_{pu}\varkappa_{qv} \ , \ \dots \ ,$$

$$Mo_{104} = \varkappa_{av}\varkappa_{bu}\varkappa_{ip}\varkappa_{jq} \ , \ Mo_{105} = \varkappa_{av}\varkappa_{bu}\varkappa_{iq}\varkappa_{jp} \ .$$

For every polynomial of $\varkappa$-s TRANSAR creates the array $Co_1,\dots,Co_K$ of distinct coefficients of monomials and the array $Ind_1,\dots,Ind_{(2k-1)!!}$ of indices of coefficients of the corresponding monomials.

6) SUMAR - adds all the array-form sums of the observed cumulant together and saves the result in the array-form on the disk. In our example there are 5 distinct coefficients in the result

$$\begin{aligned}
Co_1 &= k_8/n^3,\\
Co_2 &= 2(k_6+3k_4^2)/\{n^2(n-1)\} + k_8/n^3,\\
Co_3 &= 3k_4/\{n(n-1)^2\} + 3(3k_4^2+k_6)/\{n^2(n-1)\} + k_8/n^3,\\
Co_4 &= 2k_4^2(n^2-3n+3)/\{n^2(n-1)^3\} + 4k_4/\{n(n-1)^2\} + \qquad (15)\\
&\quad + 4k_6/\{n^2(n-1)\} + k_8/n^3 = 2(2k_4-3k_4^2)\{n(n-1)^2\} +\\
&\quad + 2(3k_4^2+2k_6)/\{n^2(n-1)\} + 2k_4^2/(n-1)^3 + k_8/n^3,\\
Co_5 &= k_4^2(n^2-3n+3)/\{n^2(n-1)^3\} + 6k_4/\{n(n-1)^2\} +\\
&\quad + 2(3k_4^2+2k_6)/\{n^2(n-1)\} + k_8/n^3 + 1/(n-1)^3 =\\
&= 3(2k_4-k_4^2)\{n(n-1)^2\} + (9k_4^2+4k_6)/\{n^2(n-1)\} +\\
&\quad + (k_4^2+1)/(n-1)^3 + k_8/n^3.
\end{aligned}$$

In addition, $Ind_1 = 1$, $Ind_2 = Ind_3 = Ind_4 = 2$, $Ind_5 = Ind_6 = 3$, $Ind_7 = 4, \dots$

Using $o(n^{-3})$ as in (10), we get

$$
\begin{aligned}
Co_1' &= k_8/n^3 , \\
Co_2' &= (6k_4^2+2k_6+k_8)/n^3 , \\
Co_3' &= (3k_4+9k_4^2+3k_6+k_8)/n^3 , \\
Co_4' &= (4k_4+2k_4^2+4k_6+k_8)/n^3 , \\
Co_5' &= (6k_4+7k_4^2+4k_6+k_8+1)/n^3 .
\end{aligned}
\tag{16}
$$

From equation (7) and the array Ind it is easy to see that in (11)

$$\alpha_l = Co_l \ , \ l = 1,\dots,5 \tag{17}$$

which gives the representation of the cumulant $\varkappa(ab,ij,pq,uv)$, analogous to (11)

$$
\begin{aligned}
\varkappa(ab,ij,pq,uv) = n^{-3}\Big\{ & k_8\sum^{1} \varkappa_{ab}\varkappa_{ij}\varkappa_{pq}\varkappa_{uv} + \\
& + (6k_4^2+2k_6+k_8)\sum^{12} \varkappa_{ai}\varkappa_{bj}\varkappa_{pq}\varkappa_{uv} + \\
& + (3k_4+9k_4^2+3k_6+k_8)\sum^{32} \varkappa_{ai}\varkappa_{bp}\varkappa_{jq}\varkappa_{uv} + \\
& + (4k_4+2k_4^2+4k_6+k_8)\sum^{12} \varkappa_{ai}\varkappa_{bj}\varkappa_{pu}\varkappa_{qv} + \\
& + (6k_4+7k_4^2+4k_6+k_8+1)\sum^{48} \varkappa_{ai}\varkappa_{bp}\varkappa_{ju}\varkappa_{qv}\Big\} + o(n^{-3}).
\end{aligned}
$$

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MARTIN'S AXIOM AND SOME QUESTIONS OF THE DIMENSION THEORY OF TYCHONOFF'S SPACES.

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All topological spaces in this paper are assumed to be Tychonoff's (≡Hausdorff and completely regular) spaces, T denotes a class of all Tychonoff's spaces; T_{CB}-class of all metrizable separable spaces, T_{FCB}-class of spaces, expressed as a union of finitely many separable metrizable subsets; $T_{\sigma C}$-class of all σ-compact spaces and T_{LC}-class of all locally compact spaces.

I^n, where $n=-1,0,1,\cdots$ denotes n-cube (with natural topology), besides $I^{-1}=\emptyset$ (≡ empty set), $I^0=\{1\}$ (≡singleton), and $I^1=I$ (≡the closed real interval); $I^{\aleph_1}$ denotes the product of $\aleph_1$ copies of I(with Tychonoff's topology), βX-Stone-Čech compactification of X and $\beta X\setminus X$-Stone-Čech remainder of X .

Finally, N denotes the set of all natural numbers and $N^*=\{-1\}\cup\{0\}\cup N\cup\{+\infty\}$ (with usual arithmetical operations).

Definition 1. Let $d(X,T)$ be a N^*-valued function, such that for each $X\in T$ one has $d(X,T)\in N^*$. A function $d(X,T)$ is called a general dimensional-like function on T (abbreviated GDLF on T) if the following is satisfied: i) $d(\emptyset,T)=-1$; ii) $d(\{p\}, T)=0$, where $\{p\}$ is singleton; iii) if X and Y are homeomorphic, then $d(X,T)=d(Y,T)$.

Definition 2. We say, that the GDLF $d(X,T)$ on T has the property of normability (abbreviated property P_1) if

$d(I^n,T)=n$, where $n=-1,0,1,\cdots$.

Definition 3. We say, that the GDLF $d(X,T)$ on T has the property of c-monotonity (abbreviated property P_2) if for every $X\in T$ and every closed subset A of X we have: $d(A,T)\leqslant d(X,T)$.

Definition 4. We say, that the GDLF $d(X,T)$ on T has the property of o-monotonity (abbreviated property P_3) if for every $X\in T$ and every open subset G of X we have: $d(G,T)\leqslant d(X,T)$.

Definition 5. We say, that the GDLF $d(X,T)$ on T has the property of compactification (abbreviated property P_4) if for every $X\in T$ there exists the compactification bX such that $d(bX,T)\leqslant d(X,T)$.

Definition 6. We say, that the GDLF $d(X,T)$ on T has the property of subadditivity (abbreviated property P_5) if for every $X\in T$, where $X=A\cup B$ we have: $d(X,T)\leqslant d(A,T)+d(B,T)+1$.

In this paper we solve the problem of existence (and non-existence) of such a GDLF on T, which possesses all the possible combinations of the system properties $P_1,P_2,P_3,$ P_4,P_5.

Further we shall write " Theorem (MA+$\neg$CH) " if in the proof of this theorem is applied the Martin's Axiom + the Negation of the Continuum Hypothesis.

We shall begin with the following.

Theorem 1. (MA+$\neg$CH). There exists Mrowka-Isbell space [1] $N\cup R$ (≡ space of the form $N\cup R$, where R is some infinite maximal almost disjoint family of N) such that $\beta(N\cup R)\setminus(N\cup R)$ is homeomorphic $I^{\aleph_1}$.

From Theorem 1 we have the following

Theorem 2. (MA+¬CH). The GDLF on T, which has the properties of normability, c-monotonity, o-monotonity, compactification and subadditivity does not exist.

From Theorem 2 it follows, that the GDLF on T, which has the all properties P_1, P_2, P_3, P_4, P_5 does not exist.

Here we shall show, that for other all combinations of properties P_1-P_5 there exist corresponding general dimensional-like functions on T.

For this we need to define some GDLF on T.

$$d_1(X,T)=\begin{cases}-1 \Leftrightarrow X=\emptyset; \\ 0 \Leftrightarrow X\neq\emptyset.\end{cases};$$

$$d_2(X,T)=\sup\{\dim Y: Y\subseteq X, Y\in T_{CB} \text{ and } Y \text{ is open in } X\}.$$

$$d_3(X,T)=\begin{cases}\dim X \Leftrightarrow X\in T_{\sigma C}; \\ +\infty \Leftrightarrow X\in T\setminus T_{\sigma C}.\end{cases};$$

$$d_4(X,T)=\begin{cases}\dim X \Leftrightarrow X\in T_{FCB}; \\ +\infty \Leftrightarrow X\in T\setminus T_{FCB}.\end{cases};$$

$$d_5(X,T)=\begin{cases}\gamma(X) \Leftrightarrow X\in T_{LC}; \\ +\infty \Leftrightarrow X\in T\setminus T_{LC}.\end{cases};$$

where $\gamma(X)$ is dimensional-like function, which is defined and studied in [2] (see, also [3] and [4])[1].

Theorem 3. The GDLF $d_1(X,T)$ on T has the P_2, P_3, P_4, P_5 properties.

Theorem 4. The GDLF $d_2(X,T)$ on T has the P_1, P_3, P_4, P_5 properties.

---

[1] We shall give the definition of the dimension function γX: i) $\gamma X=-1 \Leftrightarrow X=\emptyset$; ii) $\gamma X\leq n$ if for every compact subset $A\subseteq X$ and any open set $V\subset X$, which contains A there exists an open set $U\subset X$ such that $A\subset U\subset V$ and $\gamma(F_rU)\leq n-1$, where F_rU denotes the boundary of U in X.

Theorem 5. The GDLF $d_3(X,T)$ on T has the P_1, P_2, P_4, P_5 properties.

Theorem 6. The GDLF $d_4(X,T)$ on T has the P_1, P_2, P_3, P_5 properties.

Theorem 7. The GDLF $d_5(X,T)$ on T has the P_1, P_2, P_3, P_4 properties.

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OPEN NORMAL FUNCTORS WITH COUNTABLE KERNELS

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All the definitions and results concerning the notion of normal functor in the categories Comp and Tych can be found in [1] and [2]. In [3] a characterization of G-symmetric power functors SP^n_G is given. Here we extend this characterization onto the class of normal functors of infinite degree.

Definition [2]. Normal functor F:Tych→Tych is said to be a functor with countable kernels if for each $a \in FX$ there exists a map $f:Y\to X$ of a countable Tychonoff space Y and $b \in FY$ such that $Ff(b)=a$. Normal functor F:Comp→ Comp is said to be a functor with countable kernels if there exists a normal extension $\hat{F}$:Tych→Tych of F with countable kernels.

Theorem 1. Let F:Comp→Comp be a normal functor with countable kernels. The following is equivalent:

1) F is open functor;

2) F is bicommutative functor;

3) there exists an open natural transformation $\xi:(-)^\omega\to F$;

4) $F \cong \prod\left\{ SP^{n(i)}_{G(i)} \mid i \in \omega\right\}$ for some subgroups G(i) of symmetric groups $S_{n(i)}$.

Proof. 1)====>2) is proved in [1], 4)====>1) is an easy consequence of a result of [4].

2)====>3).All thespaces under consideration are Tychonoff, all the maps are continuous. For a map $f:X\to Y$ into a compact space Y we denote by $\hat{f}:\beta X\to Y$ the natural extension of f onto Čech-Stone compactification βX of X .

Lemma 1.For bicommutative functor F there exists $a \in F\beta\omega$, $\ker(a)=\omega$, with the following property: for each $b \in FX$ there exists $f:\omega\to X$ such that $F\hat{f}(a)=b$.

Proof. First, remark that there exists $a \in F\beta\omega$ with the following property: for every map $g:\omega\to\omega$ and $a' \in F\beta\omega$ with $F\beta g(a')=a$ and $\ker(a')=\omega$ we have that g is bijection. Indeed, assuming the co...

we can obtain the inverse system $S = \{A_\alpha, p_{\alpha\gamma}, \omega_1\}$, where A_α are countable sets, $A=\underleftarrow{\lim}\ S$ is uncountable and for each $\alpha<\omega_1$ there exists $a_\alpha \in F\beta A_\alpha$ such that $\ker(a_\alpha) = A_\alpha$, and $F\beta p_{\alpha\gamma}(a_\alpha) = a_\gamma$ whenever $\gamma\leq\alpha$. Then there exists $c \in F\beta A$ such that $F\beta p_\alpha(c) = a_\alpha$, $\alpha<\omega_1$ ($p_\alpha: A\to A_\alpha$ denotes the limit projection of S). But then obviously $\ker(c)=A$, and we get the contradiction.

Now, it follows from the bicommutativity of F that for each $b \in FX$ there exists $d \in F(\beta\omega\times X)$ with $Fpr_1(d)=a$ and $Fpr_2(d)=b$. Evidently, pr_1 maps bijectively $\ker(d)$ onto ω. Denote by $f': \omega\to\ker(d)$ the inverse map to $pr_1 | \ker(d)$ and let $f=pr_2\circ f'$. Then $Ff(a)=b$.

Fix $a \in F\beta\omega$ satisfying the condition of Lemma 1.

Lemma 2. For each X the map

$$\xi X: X^\omega = C(\omega,X)\to FX$$

defined by the formula $\xi X(f)=F\hat{f}(a)$, $f \in C(\omega,X)$, is continuous.

Proof. By continuity of functors $(-)^\omega$ and F it is sufficient to consider the case of metrizable compactum X. Let $< f_i | i \in \omega>$ be a sequence of elements of $C(\omega,X)$ and $f=\lim< f_i | i \in \omega>$. Let $b_i =F(1_{\beta\omega},\hat{f}_i)(a)$, $i \in \omega$. Assuming that the sequence $< b_i | i \in \omega>$ has two limit points, c_1 and c_2, we can easily obtain from the lower semicontinuity of supports [1] that

$$\ker(c_1)=\ker(c_2)=\{(i,f(i)) \mid i \in \omega\}.$$

Hence, $c_1 =c_2 =F(1_{\beta\omega},\hat{f})(a)$ and

$$\lim<F\hat{f}_i(a) \mid i \in \omega>=F\hat{f}(a).$$

It is easy to see that

$$\xi =<\xi X>:(-)^\omega\to F$$

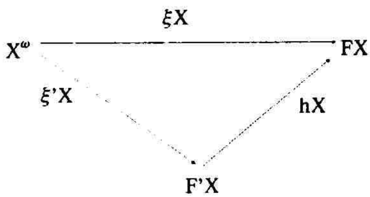
is a natural transformation. It follows from Lemma 1 that ξX is surjective for each X .

3)====>4). Let $G=\{\sigma \in C(\omega,\omega) \mid F\beta\sigma(a)=a\}$.

Lemma 3. Each orbit of natural action of (discrete) group G onto ω is finite.

Proof. Assuming that there exists an infinite orbit A fix $x_0 \in A$ and let $h: \omega\to\{0,1\}$ be the map defined by: $h(x)=0$ if $x=x_0$, $h(x)=1$ otherwise. Then for each $\sigma \in G$ we have $|\operatorname{supp} Fh\circ F\beta\sigma(a)| =2$. It is easy to see that the constant function h_0 , $h_0(x)=1$, $x\in \beta\omega$, lies in the closure of the set $\{h\circ\beta\sigma | \sigma \in G\}$ and it follows from Lemma 1 that $Fh_0(a)=Fh \circ F\beta\sigma(a)=Fh(a)$ for each $\sigma \in G$. But $|\mathrm{supp}Fh_0(a)|=1$ and we get a contradiction.

Let F'X be the orbit space of X^ω by the action of G . We have the commutative triangle



natural by X . By virtue of bicommutativity of F the set

$$\{a \in FX \mid \deg(a)=n\}$$

is dense in FX for every X without isolated points. Since hX is one-to-one on the set of points of degree n, we obtain that hX is homeomorphism for each X without isolated points. By naturality, we obtain that hX is homeomorphism for each X and $FX=X^\omega/G$. Now, it is sufficient to apply Lemma 3. The theorem is proved.

Recall that a triple $\mathcal{T}=(T,\eta,\mu)$ on a category $\mathcal{C}$ consists of an endofunctor T in $\mathcal{C}$ and natural transformations $\eta:1\to T, \mu:T^2\to T$ such that:

$$\mu\circ\mu T = \mu\circ T\mu,\ \mu\circ\eta T = \mu\circ T\eta = 1_T.$$

Theorem 2. Let F be a normal functor with countable kernels and F generates a triple $\mathcal{T}=(F,\eta,\mu)$ in Comp. Then F is a power functor $(-)^n$, $1\leqslant n\leqslant\omega$.

Proof. Recall (see [3]) that for each compact spaces X and Y and $c \in FX$, $d \in FY$ the tensor product of c and d is defined by the formula:

$$c \otimes d=F\mu(X\times Y)\circ Ff_d(c)$$

(here, the map $f_d:X\to F(X\times Y)$ acts by the formula: $f_d(x)= Fi_x(d)$, where the map $i_x:Y\to X\times Y$ acts by the formula: $i_x(y)=(x,y)$, $y \in Y$).

We proceed as in the proof of Theorem 1 and fix $a\in F\beta\omega$ satisfying the condition of Lemma 1.

For each compact space X and $b \in FX$ the set $\mathrm{supp}(a\otimes b) \cap (\omega\times X)$ can be considered as the graph of a map $h_b:\omega\to X$. It is easily to see that $F\hat{h}_b(a)=b$. Define the natural transformation $\xi:(-)^\omega\to F$ by $\xi X(h)=F\hat{h}(a)$, $h \in C(\omega,X)=X^\omega$. It is remarked above that ξX is surjective, for each X .

To prove that ξX is injective, note that

$$\mathrm{supp}(a\otimes Fh_1(a)) = \mathrm{supp}(a\otimes Fh_2(a))$$

whenever $h_1 \neq h_2$, $h_1,h_2 \in C(\omega,X)=X^\omega$. Therefore, $\xi=<\xi X>$ is functorial isomorphism. The theorem is proved.

The following problems are open.

1. Characterize uncountable products of G-symmetric power functors within the class of open normal functors.

2. Describe all the extensions of functors with countable kernels onto the Kleisli categories of triples generated by the hyperspace functor and probability measure functor (see [3]).

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О СЛАБОЙ ПОЛНОТЕ ТОПОЛОГИЧЕСКИХ ГРУПП

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Слабая τ-полнота равномерных пространств определена и изучена в работе [2]. В данной заметке показано, что достаточно большой класс топологических групп удовлетворяет слабой τ-полноте, именно, все замкнутые подгруппы произведения групп характера ≤ τ. Имеет место, также, аналог известной теоремы Е.В.Щепина [7] для слабо τ-полных τ-уравновешенных групп.

О П Р Е Д Е Л Е Н И Е 1 ([1,3]). Пусть $\mathcal{U}$ окрестность единицы топологической группы G. Система $\{\mathcal{U}_\alpha : \alpha \in A\}$ окрестностей единицы называется *квазиинвариантной базой относительно окрестности* $\mathcal{U}$, если для каждого g из G существует индекс $\alpha \in A$, $g^{-1}\cdot\mathcal{U}_\alpha\cdot g \subseteq \mathcal{U}$.

Топологическая группа G, в которой у каждой окрестности единицы существует квазиинвариантная база мощности ≤τ, называется τ-*уравновешенной*.

В работе [3] приведена характеристика τ-уравновешенных групп.

Фильтр $\mathcal{F}$, в котором пересечение каждого подсемейства мощности ≤τ снова является элементом фильтра $\mathcal{F}$, называется τ-*центрированным*. Всякий τ-центрированный фильтр Коши, относительно левой равномерности топологической группы G, будет называться τ-*фильтром Коши*. Топологическая группа, в которой сходится каждый τ-фильтр Коши, будет называться *слабо* τ-*полной топологической группой*.

Имеет место следующая характеристика слабо τ-полных τ-уравновешенных топологических групп.

Т Е О Р Е М А 1. *Для топологической группы G следующие условия эквивалентны:*

1. группа G τ-уравновешена и слабо τ-полна;

2. группа G является пределом обратного спектра $S = \{G_\alpha, f_\alpha^\beta, M\}$, где $\chi(G_\alpha) \le \tau$ для каждого $\alpha \in M$, f_α^β - непрерывные гомоморфизмы и M τ-полное индексное множество;

3. группа G замкнуто и топологически изоморфно вкладывается в произведение $\prod\{G_\alpha : \alpha \in M\}$, где $\chi(G_\alpha) \le \tau$ для каждого $\alpha \in M$.

Приведем набросок доказательства (1$\to$2). В силу некоторых результатов из [3] мы получаем право на следующие рассуждения. Если $\mathcal{B}$ фильтр окрестностей единицы группы G, тогда для каждой окрестности $\mathcal{U} \in \mathcal{B}$ существует фильтр $\mathcal{B}_\mathcal{U}$ в $\mathcal{B}$ такой, что $|\mathcal{B}_\mathcal{U}| \le \tau$ и $\cap\mathcal{B}_\mathcal{U} = N_\mathcal{U} \subseteq \mathcal{U}$. Итак, существует семейство $\{\mathcal{B}_\alpha : \alpha \in M\}$ фильтров $\mathcal{B}_\alpha \subset \mathcal{B}$, $|\mathcal{B}_\alpha| \le \tau$, $|M| = |\mathcal{B}|$ таких, что $\cap\mathcal{B}_\alpha = N_\alpha$ замкнутый нормальный делитель в G для каждого $\alpha \in M$.

Множество M частично упорядочено: $\alpha < \beta$ тогда и только тогда, когда $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$. Пусть $M_0 \subset M$ и $|M_0| \le \tau$. Положим $\tilde{\mathcal{B}} = \mathrm{Sup}\{\mathcal{B}_\alpha : \alpha \in M\}$, тогда $\tilde{\mathcal{B}} \subset \mathcal{B}$ фильтр, $|\tilde{\mathcal{B}}| \le \tau$ и $\cap\tilde{\mathcal{B}} = N$ - замкнутый нормальный делитель группы G. Это означает, что $\mathrm{Sup}M_0$ существует в M, т.е. M τ-полное индексное множество.

Для каждого $\alpha \in M$ имеем естественное отображение $f_\alpha : G \to G|_{N_\alpha}$, где для каждого $x \in G$ имеем $f_\alpha(x) = xN_\alpha$ и $G|_{N_\alpha}$ снабжено топологией, порожденной фильтром

$$\{f_\alpha(\mathcal{U}) = \mathcal{U}\cdot N_\alpha : \mathcal{U} \in \mathcal{B}_\alpha\}$$

и значит $\chi(G|_{N_\alpha}) \le \tau$. Положим для каждого $\alpha \in M$, $G|_{N_\alpha} = G_\alpha$.

По построению, если $\alpha < \beta$, тогда $N_\beta \subset N_\alpha$ и, следовательно, $f_\beta(x) = xN_\beta \subset xN_\alpha = f_\alpha(x)$, поэтому отображение $f_\alpha^\beta : G_\beta \to G_\alpha$, где $f_\alpha^\beta(x) = xN_\alpha$ определено корректно и является непрерывным гомоморфизмом, так как $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ и

$$f_\alpha^\beta(x\cdot y) = x\cdot yN_\alpha = x\cdot N_\alpha\cdot y\cdot N_\alpha = f_\alpha^\beta(x)\cdot f_\alpha^\beta(y).$$

По построению легко усмотреть, что если $\alpha < \beta < \gamma$, тогда $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$.

Таким образом составлен обратный спектр $S = \{G_\alpha, f_\alpha^\beta, M\}$, состоящий из групп G_α , $\chi(G_\alpha) \le \tau$ для каждого $\alpha \in M$, непрерывных гомоморфизмов f_α^β и τ-полного индексного множества M. Положим $\tilde{G} = \varprojlim S$. Легко видеть, что отображения проектирования являются отображениями "на" для каждого $\alpha \in M$.

Рассмотрим отображение $f : G \to \tilde{G}$, определенное по правилу $f(x) = \{f_\alpha(x) : \alpha \in M\}$ для каждой точки $x \in G$. Отображение f является непрерывным, как диагональное произведение непрерывных отображений f_α , $\alpha \in M$. Если $x, y \in G$ и $x \ne y$, то существует $\mathcal{U} \in \mathcal{B}$ такое, что $x \cdot \mathcal{U} \cap y \cdot \mathcal{U} = \emptyset$, так как $\mathcal{U} \in \mathcal{B}_\alpha$ при некотором $\alpha \in M$, тогда $x \cdot N_\alpha \cap y \cdot N_\alpha = \emptyset$, следовательно, $f_\alpha(x) \ne f_\alpha(y)$, т.е. отображение f - инъективно. Более того, f является топологическим вложением группы G в группу $\tilde{G}$. В силу слабой τ-полноты топологической группы G отображение f оказывается отображением "на", т.е. $f(G) = \tilde{G}$ и, следовательно, G топологически изоморфно $\tilde{G}$.

Действительно, если $y \in \tilde{G}$, тогда семейство

$$\{f_\alpha^{-1}(h_\alpha(y)) : \alpha \in M\}$$

τ-центрировано в силу τ-полноты индексного множества M, и является базой фильтра Коши относительно левой равномерности группы G, а так как G слабо τ-полно, тогда существует точка $x \in G$ такая, что

$$x \in \cap\{f_\alpha^{-1}(h_\alpha(y)) : \alpha \in M\}$$

и ясно, что $f(x) = y$, т.е. $f(G) = \tilde{G}$.

Доказательство ($2 \to 3$) очевидно.

($3 \to 1$). Пусть $\mathcal{F}$ τ-фильтр Коши группы G. Группа G τ-уравновешена, в силу вложенности в произведение $\prod\{G_\alpha : \alpha \in M\}$, где $\chi(G_\alpha) \le \tau$ для каждого $\alpha \in M$ [3]. Тогда $\pi_\alpha(\mathcal{F})$ τ-фильтр Коши в группе G_α, где π_α - естественная проекция для каждого $\alpha \in M$. В силу того, что $\chi(G_\alpha) \le \tau$, фильтр $\pi_\alpha(\mathcal{F})$ имеет предельную точку $x_\alpha \in G_\alpha$. Легко показать, что фильтр $\mathcal{F}$ сходится к нити $x = (x_\alpha : \alpha \in M)$ и, в силу замкнутости G в $\prod\{G_\alpha : \alpha \in M\}$, получаем, что $x \in G$. Итак, G слабо τ-полная группа.

Из работ [2,3] известно, что каждая τ-ограниченная топологическая группа G τ-уравновешена и имеет место неравенство $w(G) \le \Gamma(G)\cdot\chi(G)$, где w - вес, Γ - индекс ограниченности топологической группы G. С учетом этих свойств можно доказать следующую теорему.

Т Е О Р Е М А 2. *Для топологической группы G следующие условия равносильны:*

1. группа G τ-ограничена и слабо τ-полна;

2. группа G является пределом обратного спектра $S = \{G_\alpha, f_\alpha^\beta, M\}$, где $w(G_\alpha) \le \tau$ для каждого $\alpha \in M$, f_α^β - непрерывные гомоморфизмы и M τ-полное индексное множество;

3. группа G замкнуто и топологически изоморфно вкладывается в произведение $\prod\{G_\alpha : \alpha \in M\}$, где $w(G_\alpha) \le \tau$ для каждого $\alpha \in M$.

Б.А.Пасынков [4] ввел класс почти метризуемых групп, а М.М.Чобан [6] рассмотрел подкласс почти метризуемых групп: проективно метризуемые группы. Отталкиваясь от этих определений введем следующие определения.

О П Р Е Д Е Л Е Н И Е 2. Топологическая группа G называется *почти τ-метризуемой*, если в ней существует бикомпактное подмножество B и $\chi(B,G) \le \tau$.

О П Р Е Д Е Л Е Н И Е 3. Топологическая группа G называется *проективно τ-метризуемой*, если для любой окрестности единицы $\mathcal{U}$ существует бикомпактный нормальный делитель $N_{\mathcal{U}} \subset \mathcal{U}$ и $\chi(N_{\mathcal{U}},G) \le \tau$.

П Р Е Д Л О Ж Е Н И Е 3. *Всякая почти τ-метризуемая топологическая группа слабо τ-полна.*

Доказательство следует из [3], так как левая равномерность почти τ-метризуемой группы является τ-перистой.

Т Е О Р Е М А 4. *Для топологической группы G следующие условия равносильны :*

1. группа G - проективно τ-метризуема;

2. группа G является пределом обратного спектра $S = \{G_\alpha, f_\alpha^\beta, M\}$, где $\chi(G_\alpha) \le \tau$ для каждого $\alpha \in M$, M - τ-полное индексное множество, а f_α^β - совершенные гомоморфизмы.

Доказательство основано на теореме 1, так как всякая проективно τ-метризуемая группа является τ-уравновешенной.

Сформулируем аналог спектральной теоремы Е.В.Щепина [7] для класса слабо τ-полных и τ-уравновешенных топологических групп.

Элемент γ частично упорядоченного множества M называется *предельным элементом* множества $D \subset M$, если для любых двух элементов $\beta_1, \beta_2 \in D$ и $\beta_i < \gamma$ существует такое $\alpha \in D$, что $\beta_i < \alpha < \gamma$, i = 1,2. Для предельного элемента γ множества M множество $M|_\gamma = \{\beta \in M : \beta < \gamma\}$ направлено вверх. Если $S = \{G_\alpha, f_\alpha^\beta, M\}$ - спектр и γ - предельный элемент множества M, то $S|_\gamma$ обозначает ограничение спектра S на множество $M|_\gamma$, т.е. $S|_\gamma = \{G_\alpha, f_\alpha^\beta, M\}$.

Спектр $S = \{G_\alpha, f_\alpha^\beta, M\}$ называется *непрерывным,* если для всякого предельного элемента $\gamma \in M$ имеем $G|_\gamma = \varprojlim S|_\gamma$.

Спектр $S = \{G_\alpha, f_\alpha^\beta, M\}$ называется *τ-спектром,* если :

1) S - непрерывный спектр;

2) M τ-полно;

3) $\chi(G_\alpha) \le \tau$ для любого $\alpha \in M$;

4) f_α^β - непрерывные гомоморфизмы "на".

Т Е О Р Е М А 5. *Пусть $S = \{G_\alpha, f_\alpha^\beta, M\}$ и $T = \{H_\alpha, h_\alpha^\beta, M\}$ τ-спектры; G и H пределы этих спектров, причем имеем $\chi(G) > \tau$ и $\chi(H) > \tau$. Тогда для любой пары гомоморфизмов $\varphi : G \to H$ и $\psi : H \to G$ множество $A = \{\alpha \in M$: существуют такие гомоморфизмы $\varphi_\alpha : G_\alpha \to H_\alpha$ и $\psi_\alpha : H_\alpha \to G_\alpha$, что $\varphi_\alpha \circ h_\alpha = P_\alpha \circ \varphi$ и $\psi_\alpha \circ P_\alpha = h_\alpha \circ \psi\}$ конфинально и замкнуто в M, где $h_\alpha : G \to G_\alpha$ и $P_\alpha : H \to H_\alpha$ естественные проекции.*

С Л Е Д С Т В И Е 1. *Топологический изоморфизм двух слабо τ-полных τ-уравновешенных топологических групп, являющихся пределом τ-спектров над одним и тем же направленным множеством, порождается изоморфизмом замкнутых конфинальных подспектров.*

С Л Е Д С Т В И Е 2. *Топологический изоморфизм двух проективно τ-метризуемых топологических групп, являющихся пределом τ-спектров над одним и тем же направленным множеством, порождается изоморфизмом замкнутых конфинальных подспектров.*

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T_0-ПРОСТРАНСТВА И ТОПОЛОГИЧЕСКИЕ РЕШЕТКИ НЕПРЕРЫВНЫХ $\{0, I\}$-ЗНАЧНЫХ ФУНКЦИЙ

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Пусть Х - топологическое пространство и Д = $\{0, I\}$ - решетка с естественными отношением порядка $\leq$ и операциями умножения $\cdot$ /или $\wedge$ / и взятия точной верхней грани $\vee$. На Д будем рассматривать три топологии - дискретную и топологии связного двоеточия. Относительно каждой из них Д становится топологической решеткой. Если Д наделить топологией связного двоеточия с изолированной точкой I, то

С/Х, Д/ - это решетка

всех непрерывных Д-значных функций, определенных на топологическом пространстве Х, с поточечно заданными порядком и операциями. Фактически, С/Х, Д/ является решеткой всех открытых множеств /их характеристических функций/ пространства Х с операциями пересечения и объединения множеств.

На С/Х, Д/ введем три топологии поточечной сходимости. Если С/Х, Д/ рассматривать как подпространство тихоновской степени $Д^X$ дискретного пространства Д, то получается

топологическая решетка C_p/Х, Д/.

Если же С/Х, Д/ считать подпространством $Д^X$ для связного двоеточия Д с изолированной точкой 0, то получается

топологическая решетка C_0/Х, Д/.

Аналогично определяется топологическая решетка C_I/Х, Д/.

В докладе [I] автором отмечено, что произвольное T_0-пространство Х определяется каждой из топологических решеток C_p/Х, Д/ и C_0/Х, Д/, но не обязано определяться топологической решеткой C_I/Х, Д/. В обзоре [2, теорема 3.6] дан набросок доказательства существования двойственности /антиэквивалентности/ между категорией всех T_0-пространств и их непрерывных отображений и категорией всех топологических решеток вида C_p/Х, Д/ с непрерывными гомоморфизмами / категорией всех топологических решеток вида $C_0/X,\ Д/$ и их непрерывных гомоморфизмов, сохраняющих все имеющиеся точные верхние грани/. В данной работе этот результат будет подробно доказан.

Чуть подробнее рассмотрим топологии пространств функций $C_т/X,\ Д/$ при т = р, 0, I. Пусть значениями А и В служат всевозможные конечные множества пространства X. Множества

$$P/A,\ B/ = \{\varphi \in C/X,\ Д/:\ \ A \subseteq \varphi^{-I}/0/ \subseteq X \setminus B\}$$

образуют базу открытых /и замкнутых/ множеств нульмерного хаусдорфова пространства $C_p/X,\ Д/$. Множества $P/A,\ \emptyset/$ /соответственно, $P/\emptyset,\ B//$ образуют базу открытых множеств пространства $C_0/X,\ Д/$ / $C_I/X,\ Д/$ /. T_0-пространства $C_0/X,\ Д/$ и $C_I/X,\ Д/$ никогда не являются T_I-пространствами. Подчеркнем, что множества

$$P_x = \{\varphi \in C/X,\ Д/:\ \ \varphi/x/ = 0\}\ ,\ x \in X,$$

составляют предбазу открытых /замкнутых/ множеств пространства $C_0/X,\ Д/$ /$C_I/X,\ Д//$.

З А М Е Ч А Н И Е I. Как показал еще М.Стоун, для любого топологического пространства X существует T_0-пространство У, такое, что решетки С/Х, Д/ и С/У, Д/ изоморфны. Пространство У получается из X отождествлением точек, не разделяемых никакими функциями из С/Х, Д/. Разделяемость различных точек пространства У функциями из С/У, Д/ как раз и означает, что У - T_0-пространство. При этом естественный алгебраический изоморфизм топологических решеток $C_т/X,\ Д/$ и $C_т/У,\ Д/$, где т = р, 0, I , является и гомеоморфизмом. Поэтому в задаче определяемости пространств X топологическими решетками $C_т/X,\ Д/$ следует ограничиться классом T_0-пространств.

П Р И М Е Р. Пусть X - множество всех натуральных чисел, собственными замкнутыми множествами в котором объявлены лишь начальные отрезки натурального ряда. И пусть $У = X \cup \{c\}$ - пространство, имеющее те же самые собственные замкнутые множества, что и X. Ясно, что компактные T_0-пространства X и У не гомеоморфны, но естественное вложение X в У индуцирует топологический изоморфизм $C_I/X,\ Д/$ и $C_I/У,\ Д/$. Однако, $C_p/X,\ Д/$ гомеоморфно одноточечной компактификации дискретного X, $C_p/У,\ Д/$ гомеоморфно дискретному X, $C_0/X,\ Д/$ гомеоморфно У, а $C_0/У,\ Д/$ гомеоморфно У, в котором дополнительно открыто множество $\{c\}$.

Собственное подмножество P полурешетки /т.е. коммутативной идемпотентной полугруппы/ $\langle C, \cdot \rangle$ называется идеалом в C, если $ав \in P$ для любых $а \in P$ и $в \in C$. Идеал P полурешетки C называется простым, если $ав \in P$ влечет $а \in P$ или $в \in P$ для любых $а, в \in C$. Идеал вида $аC$, $а \in C$, полурешетки C называется главным. Идеалом решетки $\langle C, \cdot, \vee \rangle$ называется идеал полурешетки $\langle C, \cdot \rangle$, являющийся подполурешеткой полурешетки $\langle C, \vee \rangle$. Идеал P решетки C называется простым, если P - простой идеал /нижней/ полурешетки $\langle C, \cdot \rangle$. Когда говорится о полурешетке C/X, Д/, имеется в виду мультипликативная полурешетка $\langle$ C/X, Д/, $\cdot \rangle$.

Л Е М М А. Для произвольных топологического пространства X и собственного подмножества P в C/X, Д/ эквивалентны следующие утверждения:

I/ $P = P_x$ для некоторой точки $x \in X$;

2/ P - открыто-замкнутый простой идеал топологической решетки C_p/X, Д/ ;

3/ P - открытый простой главный идеал топологической полурешетки C_0/X, Д/ ;

4/ P - открытый простой главный идеал топологической полурешетки C_p/X, Д/ .

Причем, если X - T_0-пространство, то в I/ точка x определяется однозначно.

Д о к а з а т е л ь с т в о. Импликации I/ $\Longrightarrow$ 2/ и 3/ $\Longrightarrow$ 4/ очевидны. Поскольку $P_x = ф \cdot C$/X, Д/ , где $x \in X$, ф - характеристическая функция открытого множества $X \setminus \overline{x}$ и $\overline{x}$ - замыкание $\{x\}$ в X, то I/ $\Longrightarrow$ 3/.

2/ $\Longrightarrow$ 4/. Достаточно показать, что идеал P главный. Пусть функция $ф \in C$/X, Д/ определена условием

$$ф^{-I}/0/ = \bigcap \{ а^{-I}/0/ \; : \; а \in P \} .$$

Возьмем произвольную окрестность T функции ф в топологической решетке C_p/X, Д/. Найдутся такие конечные множества A и B в X, что $ф \in P/A, B/ \subseteq T$, т.е. ф = 0 на A и ф = I на B. Из определения ф вытекает, что для каждой точки $x \in B$ существует такая функция $ф^x \in P$, что $ф^x/x/ = I$. Тогда функция $в = \bigvee \{ ф^x \; : \; x \in B \}$ лежит в идеале P решетки C/X, Д/, равна I на B и равна 0 на A. Следовательно, ф содержится в замыкании $\overline{P} = P$. Поскольку $а \leq ф$ для всех $а \in P$, то $P = ф \cdot C$/X, Д/ .

4/ $\Longrightarrow$ I/. Пусть $P = ф \cdot C$/X, Д/ - открытый простой идеал в топологической полурешетке C_p/X, Д/, порожденный ф. Так как $P \neq C$/X, Д/, то $ф^{-I}$/0/ не пусто. Рассмотрим множество $K = C$/X, Д/ $\setminus$ P и покажем, что найдется точка x из $ф^{-I}$/0/, в которой все функции из K равны I. Тогда $P = P_x$, и импликация доказана. Поэтому предположим, что для всякой точки $x \in ф^{-I}$/0/ найдется функция $ф^x \in K$, равная 0 в x. Возьмем произвольно открытое множество P/A, B/ $\ni$ ф для конечных подмножеств $A = \{x_I, \ldots, x_n\}$ и B в X. Тогда ф/A/ = $\{0\}$ и ф/B/ = $\{I\}$. В силу простоты идеала P функция $ф^{x_I} \cdot \ldots \cdot ф^{x_n}$ принадлежит K. Но тогда

$$ф \vee (ф^{x_I} \cdot \ldots \cdot ф^{x_n}) \in P/A, B/ \cap K .$$

Стало быть, ф лежит в замыкании $\overline{K} = K$, что противоречит включению $ф \in P = C$/X, Д/ $\setminus$ K. Концовка леммы очевидна.

Пусть далее Φ: X $\longrightarrow$ У – непрерывное отображение топологических пространств. Ему отвечает отображение

$$\overline{\Phi} : C/У, Д/ \longrightarrow C/X, Д/ , \quad \overline{\Phi}/ф/ = ф \circ \Phi \qquad /I/$$

для всех $ф \in C$/У, Д/. Поскольку операции в решетках функций выполняются поточечно, то Φ является гомоморфизмом решетки C/У, Д/ в решетку C/X, Д/, сохраняющим 0, I и все точные верхние грани /конечные и бесконечные/. Ясно, что при любом $x \in X$ имеет место равенство

$$(\overline{\Phi})^{-I}/P_x/ = P_{\Phi x} .$$

Отсюда следует, что гомоморфизм

$$C_т/\Phi/ = \overline{\Phi} : C_т/У, Д/ \longrightarrow C_т/X, Д/, \; т = p, 0, I,$$

непрерывен.

Таким образом, $C_т$ – это контравариантный функтор из категории всех топологических пространств и непрерывных отображений в категорию топологических решеток и непрерывных гомоморфизмов. Отметим, что для T_0-пространства У различные непрерывные отображения Φ: X $\longrightarrow$ У индуцируют различные гомоморфизмы $C_т/\Phi/$.

Считаем по определению, что гомоморфизмы ограниченных решеток сохраняют 0 и I.

Т Е О Р Е М А. Контравариантный функтор C_p /соответственно, C_0/ устанавливает двойственность /антиэквивалентность/ между категорией всех T_0-пространств с непрерывными отображениями в качестве морфизмов и категорией всех топологических решеток вида C_p/X, Д/ /соответственно, C_0/X, Д/ / и их непрерывных гомоморфизмов /непрерывных гомоморфизмов, сохраняющих всевозможные точные верхние грани/.

Д о к а з а т е л ь с т в о. Уже показано, что C_p и C_0 - функторы. Остается доказать, что каждый непрерывный решеточный /сохраняющий точные верхние грани/ гомоморфизм Г: C_p/У, Д/ $\longrightarrow$ C_p/X, Д/ /C_0/У, Д / $\longrightarrow$ C_0/X, Д// для T_0-пространств X и У индуцируется некоторым /единственным/ непрерывным отображением Ф: X $\longrightarrow$ У.

Пусть сначала дан морфизм Г: C_p/У, Д/ $\longrightarrow$ C_p/X, Д/ . Пусть $x \in X$ и $P = Г^{-1}/P_x/$. Получаем открыто-замкнутый простой идеал топологической решетки C_p/У, Д/. Тогда по лемме $P = P_y$ для единственной точки $y \in У$. Значит, формула

$$Фx = y \Longleftrightarrow Г^{-1}/P_x/ = P_y \qquad /2/$$

при $x \in X$ и $y \in У$ определяет отображение Ф: X $\longrightarrow$ У. Из /2/ непосредственно вытекает, что для любых $ф \in C$/У, Д/ и $x \in X$

$$Ф^{-1}/ф^{-1}/0/\ / = Г/ф/^{-1}/0/$$

и

$$/ф \circ Ф/x = 0 \Longleftrightarrow Г/ф/x = 0 .$$

Первое соотношение доказывает непрерывность Ф /прообразы замкнутых множеств замкнуты/, а из второго соотношения на основании /I/ заключаем: $Г = \overline{Ф} = C_p/Ф/$.

Предположим, наконец, что непрерывный гомоморфизм Г: C_0/У, Д/ $\longrightarrow$ C_0/X, Д/ сохраняет точные верхние грани. Пусть снова $x \in X$ и $P = Г^{-1}/P_x/$. Ясно, что P - открытый простой идеал топологической полурешетки C_0/У, Д/. В силу леммы и сказанного выше достаточно доказать, что идеал P является главным. Для этого рассмотрим функцию

$$ф = \vee P \in C/У, Д/ .$$

По условию

$$Г/ф/ = Г/\vee P/ = \vee Г/P/ \leq ф' \in P_x ,$$

где $ф'$ - характеристическая функция множества $X \setminus \overline{x}$.
Значит, $Г/ф/ = ф' \cdot Г/ф/ \in P_x$, и $ф \in P$. Следовательно, $P = ф \cdot C$/У, Д/ - главный идеал.

Подчеркнем, что если Г - топологический изоморфизм, то отображение Ф из формулы /2/ - гомеоморфизм.

С Л Е Д С Т В И Е I [I]. Для любых T_0-пространств **X и У** равносильны условия :

I/ X и У гомеоморфны ;

2/ топологические полурешетки C_p/X, Д/ и C_p/У, Д/ топологически изоморфны ;

3/ топологические полурешетки C_0/X, Д/ и C_0/У, Д/ топологически изоморфны .

Топологическое пространство X называется $T_Д$-пространством, если для любой точки $x \in X$ множество $\overline{x} \setminus \{x\}$ замкнуто в X. Класс $T_Д$-пространств строго содержится между классами T_I-пространств и T_0-пространств, а конечные T_0-пространства являются $T_Д$-пространствами.

С Л Е Д С Т В И Е 2 /теорема Трона/. Произвольные $T_Д$-пространства X и У гомеоморфны тогда и только тогда, когда изоморфны решетки С/X, Д/ и С/У, Д/ .

З А М Е Ч А Н И Е 2. Нетрудно дать абстрактные характеристики решеток С/X, Д/ и топологических решеток C_p/X, Д/ и C_0/X, Д/. Решетка называется Т-решеткой /T_I-решеткой/, если она изоморфна решетке С/X, Д/ для некоторого топологического /T_I-пространства/ пространства X. Одна из характеризаций Т-решеток получена Троном в 1962 году. Характеризация T_I-решеток приведена в книге Биркгофа [3, глава IX] . Исходя из этих характеризаций и леммы, можно получить абстрактное описание топологических решеток /и полурешеток/ C_p/X, Д/ и C_0/X, Д/.

Заметим также, что определяемость топологических пространств различными алгебрами непрерывных функций рассмотрена в обзоре [2].

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ОБ АППРОКСИМАЦИИ РЕШЕНИЙ ОДНОРОДНОГО УРАВНЕНИЯ q-СТОРОННЕЙ СВЕРТКИ, ХАРАКТЕРИСТИЧЕСКАЯ ФУНКЦИЯ КОТОРОГО УДОВЛЕТВОРЯЕТ ОЦЕНКАМ СНИЗУ

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§ I. Постановка задачи и основной результат.

В данной работе условимся использовать терминологию и обозначения статей [I] - [2].

Пусть G - выпуклая область в комплексной плоскости $\mathbb{C}$, H - пространство голоморфных в этой области функций, наделенное обычной топологией равномерной сходимости на компактах. Локально-выпуклое пространство H будем называть пространством, а с с о ц и и р о в а н н ы м с областью G. Пусть, далее, H^*- пространство, сильное сопряженное к H. Рассматривается однородное уравнение q-сторонней свертки

$$S \circledast f = 0 \qquad (f \in H), \qquad\qquad (I)$$

где $S \in H^*$, q-сторонняя свертка функционала S и функции f определяется по правилу

$$(S \circledast f)(h) \overset{df}{=} \langle S, f(z+\omega^0 h)\rangle + \langle S, f(z+\omega^1 h)\rangle + \ldots$$
$$\ldots + \langle S, f(z+\omega^{q-1} h)\rangle, \ \omega = \exp(2\pi i/q), \ |h| < \delta, \ \delta > 0.$$

Известно [3], что каждое решение $f \in H$ уравнения (I) можно аппроксимировать в топологии H линейными комбинациями элементарных решений этого уравнения, то есть решений, являющихся корневыми элементами оператора D^q кратного дифференцирования. Настоящая работа посвящается исследованию следующего вопроса: пусть решение $f \in H$ уравнения (I) допускает однозначное аналитическое продолжение в невыпуклую односвязную область $\hat{G}$ $(G \subset \hat{G})$. При каких условиях f можно аппроксимировать комбинациями элементарных решений в топологии пространства H', ассоциированного с областью G' $(G' \subset \hat{G})$, где G' - односвязная область, совпадающая с $\hat{G}$ или "мало" отличающаяся от последней. Ниже в предположении удовлетворения характеристической функцией $\varphi(h) = \langle S, e^{hz}\rangle$ уравнения (I) оценкам снизу вдоль некоторых лучей или полной регулярности роста этой

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функции на упомянутых лучах приводится описание областей G', для которых рассматриваемая аппроксимационная задача имеет положительное решение.

Для формулировки основного результата введем в рассмотрение флаговую конфигурацию, описанную в [2]. Пусть $\Gamma_1, \dots, \Gamma_n$ – допустимые [I] относительно сопряженной диаграммы D функции φ дуги на границе области G с ассоциированными интервалами направлений [I]

$[\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n] \quad (0 < \beta_1 - \alpha_1 < \pi, \dots, 0 < \beta_n - \alpha_n < \pi)$ соответственно; $\bar{G}_{\alpha_i, \delta_i}$ – (α_i, δ_i)-сегмент, $\bar{G}_{\beta_i, \gamma_i}$ – (β_i, γ_i)-сегмент ($\delta_i \geq 0$, $\gamma_i \geq 0$, $i = 1, \dots, n$) замыкания $\bar{G}$ области G. Предполагаем, что сегменты попарно не пересекаются и не имеют общих точек с D. Пусть $\sigma_i = \Gamma_i \setminus (\bar{G}_{\alpha_i, \delta_i} \cup \bar{G}_{\beta_i, \gamma_i})$ и $G_0 = G \setminus \bigcup_{i=1}^{n} (\bar{G}_{\alpha_i, \delta_i} \cup \bar{G}_{\beta_i, \gamma_i})$. Область G_0 "мало" отличается от G, если константы δ_i, γ_i ($i = 1, \dots, n$) малы. Для каждого i ($i = 1, \dots, n$) образуем также (α_i, β_i)-направленный флаг S_i, пристроенный к хорде, стягивающей концы дуги σ_i, при этом соблюдем условия: I. $S_1, \dots, S_n$ попарно не пересекаются, 2. Найдутся положительный и отрицательный опорные лучи к G_0, не пересекающиеся с областью

$$\Omega = G_0 \cup \left(\bigcup_{i=1}^{n} S_i \right). \tag{2}$$

Следуя [4], область (2) назовем ф л а г о в о й конфигурацией, порожденной сегментами $\bar{G}_{\alpha_i, \delta_i}$, $\bar{G}_{\beta_i, \gamma_i}$, $i = 1, \dots, n$. Справедлива следующая аппроксимационная

Т Е О Р Е М А. Пусть характеристическая функция $\varphi(h)$ уравнения (I) удовлетворяет оценкам снизу:

$$\begin{aligned} \ln |\varphi(h)| &\geq (h_\varphi(-\alpha_i) - \delta_i - o(1))|h|, \\ \ln |\varphi(h)| &\geq (h_\varphi(-\beta_i) - \gamma_i - o(1))|h|, \end{aligned} \tag{3}$$

$$h_\varphi(\theta) = \overline{\lim_{r \to \infty}}\, r^{-1} \ln |\varphi(re^{i\theta})|,$$

вдоль лучей $R_{-\alpha_i} = \{h : \arg h = -\alpha_i\}$, $R_{-\beta_i} = \{h : \arg h = -\beta_i\}$, $i = 1, \dots, n$, соответственно, за исключением, быть может, множества кружков нулевой линейной плотности, и решение $f \in H$ этого уравнения допускает однозначное аналитическое продолжение из области G_0 в область G', содержащуюся в (2). Тогда f

равномерно внутри G' можно аппроксимировать линейными комбинациями элементарных решений уравнения (1).

Сформулированная теорема есть дословная переформулировка теоремы А работы [2], которая касалась обычного однородного уравнения свертки, то есть уравнения (1) с $q = 1$. Следовательно, доказываемая теорема является развитием цитированной.

Если функция $\varphi(\theta)$ является функцией вполне регулярного роста в направлениях $\theta = -\alpha_i$, $\theta = -\beta_i$, $i = 1, \dots, n$, то выполнятся оценки (3) с константами δ_i, γ_i, равными нулю. Тогда область G_0 совпадает с G, а флаги $S_1, \dots, S_n$ в (2) будут узкими [1]. И мы имеем распространение теоремы Б из [1] на случай уравнения (1) q-сторонней свертки.

§ 2. Доказательство теоремы.

Пусть H_0 - пространство, ассоциированное с областью G_0, W_0 - подпространство решений $f \in H_0$ уравнения (1). В силу наложенных на сегменты $\bar{G}_{\alpha_i, \delta_i}$, $\bar{G}_{\beta_i, \gamma_i}$, $i = 1, \dots, n$, условий, по теореме 4 работы [3] W_0 - нетривиальное инвариантное относительно оператора D^q кратного дифференцирования подпространство, допускающее спектральный синтез. Для доказательства теоремы достаточно показать, что инвариантное подпространство $W' \subset H'$, где H' - пространство, ассоциированное с G', $W' = H' \cap W_0$, также допускает спектральный синтез. В силу теоремы 1 из [3] допустимость спектрального синтеза подпространством W' равносильна обильности аннуляторного подмодуля I' этого подпространства. Поэтому покажем, что I' - обильный подмодуль. Последнее означает, что всякий элемент $\Phi \in P'$, где P' - модуль целых функций, ассоциированный [5] с областью G', принадлежащий I' локально [3], принадлежит I' глобально, то есть $\Phi \in I'$. Локальная принадлежность Φ подмодулю I' характеризуется тем, что Φ является элементом локального подмодуля $I(\tilde{\lambda})$, порождаемого I', для любого $\lambda \in \mathbb{C}$, где $\tilde{\lambda} = \{\lambda_0 = \omega^0\lambda, \lambda_1 = \omega^1\lambda, \dots, \lambda_{q-1} = \omega^{q-1}\lambda\}$. Значит, для любого $\lambda \in \mathbb{C}$ в некоторой q-симметричной окрестности $\tilde{\lambda}$ функция Φ представляется конечной линейной комбинацией элементов из I' с коэффициентами из локального кольца $O^q(\tilde{\lambda})$ ростков q-симметричных функций, локально аналитических в q-симметричных окрестностях $\tilde{\lambda}$. В частности, Φ представляется такой комбинацией в некоторой окрестности точки $\lambda \in \mathbb{C}$:

$$\Phi = \sum_{k=1}^{m} c_k u_k, \quad u_k \in I', \; c_k \in O^q(\tilde{\lambda}), \; m \in \mathbb{N} \quad (4)$$

Покажем, что Φ есть элемент модуля P_0, ассоциированного с областью G_0. Так как подмодуль I' главный с образующей φ [3], то существуют целые функции экспоненциального типа $\psi_1, \dots, \psi_m$ такие, что будут иметь место представления

$$u_1 = \psi_1 \varphi, \dots, u_m = \psi_m \varphi, \tag{5}$$

откуда, в силу (4),

$$\Phi = \left(\sum_{k=1}^{m} c_k \psi_k\right) \varphi. \tag{6}$$

Из (6) следует, что функция $\psi = \Phi/\varphi$ голоморфна в окрестности λ, а значит, в силу произвольности λ, является целой. Таким образом, Φ представляется в виде $\Phi = \psi\varphi$, откуда по лемме работы [4] заключаем, что $\Phi \in P_0$.

Функция Φ будет локально принадлежать аннуляторному подмодулю I_0 подпространства W_0, так как, в силу (5) и уже цитированной леммы из [4], $u_k \in I_0$.

Таким образом, элемент $\Phi \in P'$, принадлежащий I' локально, принадлежит P_0 и локально принадлежит I_0. Но I_0 – обильный подмодуль [3], значит, $\Phi \in I_0$. Из последнего условия, в силу включения $I_0 \subset I'$, имеем $\Phi \in I'$. Теорема доказана.

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СЦЕПЛЕНИЕ ОПЕРАТОРОВ ПОЛНОГО ДИФФЕРЕНЦИРОВАНИЯ ПРИ СТРУЙНОЙ КОМПОЗИЦИИ

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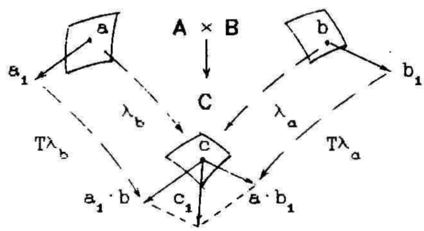
В пространстве струй отображений из одного многообразия в другое известную роль играют операторы полного дифференцирования - ОПД, см. [1],[2]. Если задать три многообразия, первое, второе и третье, то струи из первого многообразия во второе и из второго в третье образуют струйную композицию из первого в третье. Спрашивается: как при этом связаны соответствующие ОПД?

1. Эта связь устанавливается по схеме



Пусть A,B и C - три дифференцируемых многообразия. Отображение

$$\lambda : A \times B \longrightarrow C \quad : \quad (a,b) \longmapsto c = a \cdot b$$

порождает касательное отображение

$$T\lambda : TA \oplus TB \longrightarrow TC \quad : \quad (a,a_1;b,b_1) \longmapsto (c,c_1)$$

где

$$c_1 = a_1 \cdot b + a \cdot b_1 \ .$$

Это значит, что для любой фиксированной точки $b \in B$ имеет место отображение

$$\lambda_b : A \longrightarrow C \quad : \quad a \longmapsto a \cdot b,$$

и, точно так же, для любой фиксированной точки $a \in A$

отображение

$$\lambda_a : B \longrightarrow C \ : \ b \longmapsto a\cdot b,$$

касательные отображения

$$T\lambda_b : TA \longrightarrow TC \quad \text{и} \quad T\lambda_a : TB \longrightarrow TC$$

переносят векторы a_1 и b_1 из точек a и b в точку $c = a\cdot b$, т.е.

$$T\lambda_b(a_1) = a_1\cdot b, \qquad T\lambda_a(b_1) = a\cdot b_1,$$

и сумма этих образов есть образ суммы:

$$T\lambda(a_1 + b_1) = c_1.$$

2. В данном случае роль A, B и C играют расслоения бесконечных струй

$$J(U,V),\ J(V,W) \text{ и } J(U,W),$$

где U, V и W - три гладких многообразия размерностей m, n и p соответственно, $i,j,k = 1,\ldots,m$; $\alpha,\beta,\gamma = 1,\ldots,n$; $\sigma = 1,\ldots,p$.

Координаты $u^i, f^\alpha, f^\alpha_i, f^\alpha_{ij}, f^\alpha_{ijk},\ldots$ порождают локально в J(U,V) натуральный базис, т.е. репер и дуальный корепер

$$\frac{\partial}{\partial u^i},\ \frac{\partial}{\partial f^\alpha},\ \frac{\partial}{\partial f^\alpha_i},\ \frac{\partial}{\partial f^\alpha_{ij}},\ \ldots$$

$$du^i,\ df^\alpha,\ df^\alpha_i,\ df^\alpha_{ij},\ \ldots$$

а если операторы $\frac{\partial}{\partial u^i}$ в нём заменить на ОПД

$$X_i = \frac{\partial}{\partial u^i} + f^\alpha_i \frac{\partial}{\partial f^\alpha} + f^\alpha_{ij} \frac{\partial}{\partial f^\alpha_j} + f^\alpha_{ijk} \frac{\partial}{\partial f^\alpha_{jk}} + \ldots,$$

то адаптированный базис. Эта замена сопровождается заменой дифференциалов df^α, df^α_i, df^α_{ij}, ... формами

$$\omega^\alpha = df^\alpha - f^\alpha_i du^i,\ \omega^\alpha_i = df^\alpha_i - f^\alpha_{ij} du^j,\ \omega^\alpha_{ij} = df^\alpha_{ij} - f^\alpha_{ijk} du^k,\ \ldots$$

Аналогично, имея локальные координаты $v^\alpha, g^\sigma, g^\sigma_\alpha, g^\sigma_{\alpha\beta},\ldots$ в J(V,W) и $u^i, h^\sigma, h^\sigma_i, h^\sigma_{ij},\ldots$ в J(U,W), определяем ОПД и формы

$$Y_\alpha = \frac{\partial}{\partial v^\alpha} + g^\sigma_\alpha \frac{\partial}{\partial g^\sigma} + g^\sigma_{\alpha\beta} \frac{\partial}{\partial g^\sigma_\beta} + g^\sigma_{\alpha\beta\gamma} \frac{\partial}{\partial g^\sigma_{\beta\gamma}} + \ldots,$$

$$\theta^\sigma = dg^\sigma - g^\sigma_\alpha dv^\alpha,\ \theta^\sigma_\alpha = dg^\sigma_\alpha - g^\sigma_{\alpha\beta} dv^\beta,\ \theta^\sigma_{\alpha\beta} = dg^\sigma_{\alpha\beta} - g^\sigma_{\alpha\beta\gamma} dv^\gamma,\ \ldots$$

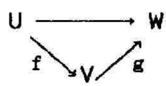
в J(V,W) и

$$Z_i = \frac{\partial}{\partial u^i} + h^\sigma_i \frac{\partial}{\partial h^\sigma} + h^\sigma_{ij} \frac{\partial}{\partial h^\sigma_j} + h^\sigma_{ijk} \frac{\partial}{\partial h^\sigma_{jk}} + \dots ,$$

$$\vartheta^\sigma = dh^\sigma - h^\sigma_i du^i, \quad \vartheta^\sigma_i = dh^\sigma_i - h^\sigma_{ij} du^j, \quad \vartheta^\sigma_{ij} = dh^\sigma_{ij} - h^\sigma_{ijk} du^k, \dots$$

в $J(U,W)$.

3. Струйная композиция индуцируется коммутативной диаграммой



точнее, струйным продолжением отображения

$$\lambda : (f,g) \longmapsto h = gf,$$

которое понимается следующим образом -

$$\Lambda : (u^i, f^\alpha, f^\alpha_i, f^\alpha_{ij}, \dots ; v^\alpha, g^\sigma, g^\sigma_\alpha, g^\sigma_{\alpha\beta}, \dots) \longmapsto (u^i, h^\sigma, h^\sigma_i, h^\sigma_{ij}, \dots),$$

$$\begin{cases} h^\sigma_i = g^\sigma_\alpha f^\alpha_i \\ h^\sigma_{ij} = g^\sigma_{\alpha\beta} f^\alpha_i f^\beta_j + g^\sigma_\alpha f^\alpha_{ij} \\ h^\sigma_{ijk} = g^\sigma_{\alpha\beta\gamma} f^\alpha_i f^\beta_j f^\gamma_k + g^\sigma_{\alpha\beta}\left[f^\alpha_i f^\beta_{jk} + f^\alpha_j f^\beta_{ki} + f^\alpha_k f^\beta_{ij}\right] + g^\sigma_\alpha f^\alpha_{ijk} \\ \dots\dots\dots \end{cases} \qquad (1)$$

Касательное отображение $T\Lambda$ определяется в натуральных базисах системой

$$\begin{cases} dh^\sigma = h^\sigma_i du^i = g^\sigma_\alpha dv^\alpha \\ dh^\sigma_i = dg^\sigma_\alpha f^\alpha_i + g^\sigma_\alpha df^\alpha_i \\ dh^\sigma_{ij} = dg^\sigma_{\alpha\beta} f^\alpha_i f^\beta_i + dg^\sigma_\alpha f^\alpha_{ij} + g^\sigma_{\alpha\beta}\left[df^\alpha_i f^\beta_j + f^\alpha_i df^\beta_j\right] + g^\sigma_\alpha df^\alpha_{ij} \\ \dots\dots\dots\dots \end{cases} \qquad (2)$$

и в адаптированных базисах - системой (первое уравнение не повторяется)

$$\begin{cases} \vartheta^\sigma_i = \theta^\sigma_\alpha f^\alpha_i + g^\sigma_\alpha \omega^\alpha_i \\ \vartheta^\sigma_{ij} = \theta^\sigma_{\alpha\beta} f^\alpha_i f^\beta_j + \theta^\sigma_\alpha f^\alpha_{ij} + g^\sigma_{\alpha\beta}\left[\omega^\alpha_i f^\beta_j + f^\alpha_i \omega^\beta_j\right] + g^\sigma_\alpha \omega^\alpha_{ij} \\ \dots\dots\dots\dots\dots \end{cases}$$

4. ТЕОРЕМА. *ОПД* Z_i *связаны с ОПД* X_i *и* Y_α *соотношением*

$$Z_i = T\Lambda(X_i + Y_\alpha f^\alpha_i) .$$

При этом Z_i *разлагается в сумму*

$$T\Lambda_g X_i + (T\Lambda_f Y_\alpha) f^\alpha_i ,$$

где

$$T\Lambda_g X_i = \frac{\partial}{\partial u^i} + g^\sigma_\alpha f^\alpha_{ij}\frac{\partial}{\partial h^\sigma_j} + \left[g^\sigma_{\alpha\beta}\left(f^\alpha_j f^\beta_{ki} + f^\alpha_k f^\beta_{ij}\right) + g^\sigma_\alpha f^\alpha_{ijk}\right]\frac{\partial}{\partial h^\sigma_{jk}} + \ldots$$

$$T\Lambda_f Y_\alpha = g^\sigma_\alpha \frac{\partial}{\partial h^\sigma} + g^\sigma_{\alpha\beta} f^\beta_j \frac{\partial}{\partial h^\sigma_j} + \left(g^\sigma_{\alpha\beta\gamma} f^\beta_j f^\gamma_k + g^\sigma_{\alpha\beta} f^\beta_{jk}\right)\frac{\partial}{\partial h^\sigma_{jk}} + \ldots$$

Д о к а з а т е л ь с т в о. Вычисляя значения векторного поля $T\Lambda(X_i + Y_\alpha f^\alpha_i)$ в натуральном кобазисе на $J(U,W)$, получаем, с учётом системы (1), компоненты поля Z_i (композицию с Λ опускаем):

$$du^j(X_i + Y_\alpha f^\alpha_i) = du^j(X_i) = du^j\left(\frac{\partial}{\partial u^i}\right) = \delta^j_i ,$$

$$dh^\sigma(X_i + Y_\alpha f^\alpha_i) = h^\sigma_j du^j(X_i) = h^\sigma_i$$

$$\text{или} \quad = g^\sigma_\beta dv^\beta(Y_\alpha f^\alpha_i) = g^\sigma_\alpha f^\alpha_i = h^\sigma_i ,$$

$$dh^\sigma_j(X_i + Y_\alpha f^\alpha_i) = dg^\sigma_\beta(Y_\alpha) f^\alpha_i f^\beta_j + g^\sigma_\alpha df^\alpha_j(X_i) =$$

$$= g^\sigma_{\alpha\beta} f^\alpha_i f^\beta_j + g^\sigma_\alpha f^\alpha_{ij} = h^\sigma_{ij} ,$$

$$dh^\sigma_{jk}(X_i + Y_\alpha f^\alpha_i) = dg^\sigma_{\beta\gamma}(Y_\alpha) f^\alpha_i f^\beta_j f^\gamma_k + dg^\sigma_\beta(Y_\alpha) f^\alpha_i f^\beta_{jk} +$$

$$+ g^\sigma_{\alpha\beta}\left(df^\alpha_j(X_i) f^\beta_k + f^\alpha_j df^\beta_k(X_i)\right) + g^\sigma_\alpha df^\alpha_{jk}(X_i) =$$

$$= g^\sigma_{\alpha\beta\gamma} f^\alpha_i f^\beta_j f^\gamma_k + g^\sigma_{\alpha\beta}\left(f^\alpha_i f^\beta_{jk} + f^\alpha_{ij} f^\beta_k + f^\alpha_j f^\beta_{ik}\right) + g^\sigma_\alpha f^\alpha_{ijk} = h^\sigma_{ijk} ,$$

и т. д.

Отображения $T\Lambda_g$ и $T\Lambda_f$ определяются системой (2), если в ней положить $dg^\sigma_\alpha = dg^\sigma_{\alpha\beta} = \ldots = 0$ и $df^\alpha_i = df^\alpha_{ij} = \ldots = 0$ соответственно. Подстановкой компонент полей X_i и Y_α выявляем слагаемые в разложении Z_i.

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НЕКОТОРЫЕ СВОЙСТВА ТОПОЛОГИИ СВОБОДНЫХ ТОПОЛОГИЧЕСКИХ ГРУПП РАВНОМЕРНЫХ ПРОСТРАНСТВ

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А.Марков (/4/,/5/) для каждого тихоновского пространства X определил свободную топологическую группу $F_M(X)$.

Понятие свободной топологической группы $FN_u(X)$ равномерного пространства $(X,\mathcal{U})$ впервые выделил Накаяма (/7/) Под свободной топологической группой $FN_u(X)$ равномерного пространства $(X,\mathcal{U})$ понимается свободная группа $F(X)$ множества X, наделённая сильнейшей групповой топологией, левая равномерность которой индуцирует на X равномерность $\mathcal{U}$. Нуммела (/8/) модифицировал понятие свободной группы и определил свободную топологическую группу $F_u(X)$ равномерного пространства $(X,\mathcal{U})$ как свободную группу $F(X)$ множества X, наделённую сильнейшей групповой топологией, двусторонняя равномерность которой индуцирует на X равномерность $\mathcal{U}$. Впредь $FN_u(X)$ будет обозначать свободную топологическую группу в смысле Накаяма (/7/), а $F_u(X)$ -свободную топологическую группу в смысле Нуммель (/8/).

Все встречающиеся ниже понятия равномерностей можно найти в книге А.Борубаева (/3/).

Пусть $(sX,s\mathcal{U})$ - Самюэлевская компактификация равномерного пространства $(X,\mathcal{U})$ и $(X^{-1},\mathcal{U}^{-1})$ равномерно гомеоморфная копия пространства $(X,\mathcal{U})$, где $X^{-1}=\{x^{-1}: x\in X\}$. Положим $X^*=X\cup\{e\}\cup X^{-1}$, где X, $\{e\}$ и X^{-1} дизъюнктны и открыто-замкнуты в X^*. Наделим X^* равномерностью $\mathcal{U}^*=\mathcal{U}\oplus\mathcal{U}^{-1}$, которая является дискретной суммой равномерностей $\mathcal{U}$ и $\mathcal{U}^{-1}$. Аналогично определим $\widetilde{X}^*$ как дизъюнктное обьединение

$sX \cup \{e\} \cup (sX)^{-1}$ и наделим $\widetilde{X}^*$ равномерностью $\widetilde{\mathcal{U}}^* = s\mathcal{U} \bullet$ $\bullet (s\mathcal{U})^{-1}$, дискретной суммы $s\mathcal{U}$ и $(s\mathcal{U})^{-1}$. Из определения $(X^*, \mathcal{U}^*)$ и $(\widetilde{X}^*, \widetilde{\mathcal{U}}^*)$ непосредственно следует, что $(sX^*, s\mathcal{U}^*)$ равномерно гомеоморфно $(\widetilde{X}^*, \widetilde{\mathcal{U}}^*)$. Рассмотрим n-степени $(X^*, \mathcal{U}^*)^n$ и $(\widetilde{X}^*, \widetilde{\mathcal{U}}^*)^n$ равномерных пространств $(X^*, \mathcal{U}^*)$ и $(\widetilde{X}^*, \widetilde{\mathcal{U}}^*)$. Имеют место естественные отображения $j_n : (X^*, \mathcal{U}^*)^n \to F_{\mathcal{U}}(X)_n$ и $\widetilde{j}_n : (\widetilde{X}^*, \widetilde{\mathcal{U}}^*)^n \to F_{s\mathcal{U}}(sX)_n$, где $F_{\mathcal{U}}(X)_n$ подмножество в $F_{\mathcal{U}}(X)$ из слов длины $\leq n$, а $F_{s\mathcal{U}}(sX)_n$ подмножество в $F_{s\mathcal{U}}(sX)$ из слов длины $\leq n$. При отображении j_n каждому элементу $(y_1, y_2, \dots, y_n) \in (X^*, \mathcal{U}^*)^n$ сопоставляется слово $y_1 y_2 \dots y_n \in F_{\mathcal{U}}(X)_n$, т.е. $j_n((y_1, y_2, \dots, y_n)) = y_1 y_2 \dots y_n$. Аналогично определяется $\widetilde{j}_n$. Следует отметить, что отображения j_n и $\widetilde{j}_n$ непрерывны относительно топологии, порождённой равномерностью произведения, а отображение $\widetilde{j}_n$-замкнуто, в силу компактности $(\widetilde{X}^*, \widetilde{\mathcal{U}}^*)^n$. Введём следующие обозначения $A_n = \{ y \in X^{*^n} : j_n(y)$ -несократимое слово длины ровно $n \}$ и, аналогично, $\widetilde{A}_n = \{ y \in \widetilde{X}^{*^n} : \widetilde{j}_n(y)$ -несократимое слово длины ровно $n \}$.

Следующая теорема является переносом известной теоремы А.В.Архангельского (/I/) на случай свободных топологических групп равномерных пространств.

Т Е О Р Е М А I. Подпространство A_n пространства X^{*^n} гомеоморфно отображается на подпространство $j_n(A_n)$ топологической группы $F_{\mathcal{U}}(X)$.

Д о к а з а т е л ь с т в о. Рассмотрим $\widetilde{A}_n$ и отображение $\widetilde{j}_n$. Так как $(\widetilde{X}^{*^n}, \widetilde{\mathcal{U}}^{*^n})$ равномерно гомеоморфно $((sX^*)^n, (s\mathcal{U}^*)^n)$, то $\widetilde{X}^{*^n}$ -компакт и $\widetilde{j}_n$-замкнутое непрерывное отображение. Имеем $\widetilde{A}_n = \widetilde{j}_n^{-1} \cdot \widetilde{j}_n(\widetilde{A}_n)$ и $\widetilde{j}_n(y) \neq \widetilde{j}_n(y')$ при $y, y' \in \widetilde{A}_n$, $y \neq y'$. Следовательно, $\widetilde{j}_n|_{\widetilde{A}_n} : \widetilde{A}_n \to \widetilde{j}_n(\widetilde{A}_n)$ - гомеоморфизм. Ясно, что A_n как множество лежит в $\widetilde{A}_n$. Пусть $\mathcal{T}$ -топология, индуцированная на $j_n(A_n)$ из $F_{\mathcal{U}}(X)$, а $\widetilde{\mathcal{T}}$ - топология, индуцированная на $j_n(A_n)$ из группы $F_{s\mathcal{U}}(sX)$. Имеем

$\tilde{\mathcal{T}} \subset \mathcal{T}$, потому, что естественное отображение $i: F_{u}(X) \to F_{\tilde{u}}(X)$ непрерывно. Пусть $\mathcal{T}_1 = j_n^{-1}(\mathcal{T})$ и $\tilde{\mathcal{T}}_1 = j_n^{-1}(\tilde{\mathcal{T}})$. Тогда $\mathcal{T}_1$ и $\tilde{\mathcal{T}}_1$ – топологии на A_n . В силу того, что $\tilde{j}_n|_{\tilde{A}_n}$ – гомеоморфизм и $A_n \subset \tilde{A}_n$, топология $\tilde{\mathcal{T}}_1$ совпадает с топологией $\mathcal{T}^*$, индуцированной на A_n из X^{*^n} . Итак, $\tilde{\mathcal{T}}_1 = \mathcal{T}^*$. Имеем $\tilde{\mathcal{T}}_1 \subset \mathcal{T}_1$, так как $\tilde{\mathcal{T}} \subset \mathcal{T}$. Но j_n – непрерывно отображает A_n на $j_n(A_n)$, значит $\mathcal{T}_1 \subset \mathcal{T}^*$, следовательно $\tilde{\mathcal{T}}_1 = \mathcal{T}_1 = \mathcal{T}^*$, т.е. $\mathcal{T} = \tilde{\mathcal{T}} = j(\mathcal{T}^*)$. Итак, $j_n|_{A_n}$ – гомеоморфизм.

Следующее следствие является переносом на свободные топологические группы равномерных пространств известной леммы Джойнера (/9/).

С Л Е Д С Т В И Е 2. Пусть $x \in j_n(A_n), x = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$. Тогда базу в точке x из $F_u(X)_n$ образуют множества вида $\alpha_1^{\varepsilon_1}(x_1) \cdot \alpha_2^{\varepsilon_2}(x_2) \dots \alpha_n^{\varepsilon_n}(x_n)$, где $\varepsilon_i = \pm 1$ и $\alpha_i \in \mathcal{U}$, $i = 1, 2, \dots, n$.

С Л Е Д С Т В И Е 3. Топология, индуцированная на $j_n(A_n)$ из топологической группы $F_u(X)$, совпадает с топологией, индуцированной из топологической группы $F_M(X)$.

Следующая теорема является аналогом теоремы А.В.Архангельского (/2/).

Т Е О Р Е М А 4. Равномерно непрерывное отображение $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ равномерного пространства $(X, \mathcal{U})$ на равномерное пространство $(Y, \mathcal{V})$ равномерно факторно тогда и только тогда, когда равномерно факторно отображение

$$\bar{f}: FN_{\mathcal{U}}(X) \to FN_{\mathcal{V}}(Y) \; (\bar{f}: F_{\mathcal{U}}(X) \to F_{\mathcal{V}}(Y)) .$$

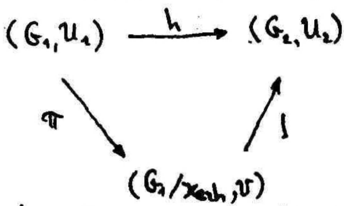
Д о к а з а т е л ь с т в о. Пусть $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ равномерно факторное отображение, и предположим, что отображение $\bar{f}: FN_{\mathcal{U}}(X) \to FN_{\mathcal{V}}(Y)$ не равномерно факторно, тогда на $F(Y)$ существует групповая равномерность $\mathcal{W}$ более сильная, чем левая равномерность группы $FN_{\mathcal{V}}(Y)$ такая, что гомоморфизм $\bar{f}: FN_{\mathcal{U}}(X) \to (F(Y), \mathcal{W})$ равномерно непрерывен. Тогда равномерно непрерывно и отображение $f:(X,\mathcal{U})\to(Y,\mathcal{V}|_Y)$ и, по определению группы $FN_{\mathcal{V}}(Y)$, равномерность $\mathcal{V}$ слабее чем $\mathcal{V}|_Y$, что является противоречием, так как равномерность $\mathcal{V}$ является сильнейшей из равномерностей, при которых f равномерно непрерывно.

Обратно, пусть $\bar{f}:FN_{\mathcal{U}}(X)\to FN_{\mathcal{V}}(Y)$ равномерно факторное отображение и $\mathcal{V}$ такая равномерность на Y, что отображение $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ – равномерно непрерывно. Продолжим это отображение до непрерывного гомоморфизма $\bar{f}:FN_{\mathcal{U}}(X)\to$ $\to FN_{\mathcal{V}}(Y)$. Так как гомоморфизм $\bar{f}$ является равномерно факторным отображением группы $FN_{\mathcal{U}}(X)$ на группу $FN_{\mathcal{V}}(Y)$, относительно левых равномерных структур этих групп, то получаем, что левая равномерность группы $F_{\mathcal{V}}N(Y)$ сильнее левой равномерности группы $FN_{\mathcal{V}}(Y)$, следовательно, равномерность $\mathcal{V}$ слабее чем равномерность $\mathcal{V}$, т.е. f – равномерно факторное отображение.

П Р Е Д Л О Ж Е Н И Е 5. Непрерывный гомоморфизм $h_1:G_1\to G_2$ топологической группы G_1 на топологическую группу G_2 является открытым тогда и только тогда, когда равномерно факторно отображение $h:(G_1,\mathcal{U}_1)\to(G_2,\mathcal{U}_2)$, где $\mathcal{U}_1$, $\mathcal{U}_2$ – левые равномерности групп G_1 и G_2.

Д о к а з а т е л ь с т в о. Если гомоморфизм $h_1:G_1\to$ $\to G_2$ – открыт, тогда он является равномерно открытым отображением в смысле А.Борубаева (/3/) и, поэтому, является равномерно факторным отображением.

Обратно, пусть непрерывный гомоморфизм $h:G_1\to G_2$ является равномерно факторным отображением равномерного пространства $(G_1,\mathcal{U}_1)$ на равномерное пространство $(G_2,\mathcal{U}_2)$, где $\mathcal{U}_1$, $\mathcal{U}_2$ – левые равномерности групп G_1 и G_2, соответственно. Для каждого $\alpha\in\mathcal{U}_1$ положим $h\alpha=\{A\cdot\mathrm{Ker}\,h: A\in\alpha\}$ и $\mathcal{B}=\{h\alpha:\alpha\in\mathcal{U}_1\}$. Тогда $\mathcal{B}$ – база сильнейшей групповой равномерности ϑ на группе $G_1/\ker h$, при которой отображение проектирования $\pi:(G_1,\mathcal{U}_1)\to(G_1/\ker h,\vartheta)$ – равномерно непрерывно. Имеем следующую коммутативную диаграмму :



где отображения h и π равномерно факторны, а j – алгебраический изоморфизм. Покажем, что отображение j – равномерно факторно. Пусть α такое покрытие группы G_2, что $j^{-1}(\alpha)\in\vartheta$. Тогда $\pi^{-1}(j^{-1}(\alpha))=h^{-1}(\alpha)\in\mathcal{U}_1$ и, следовательно, в силу равномерной факторности отображений π и h имеем $h(\pi^{-1}(j^{-1}(\alpha)))\in\mathcal{U}_2$. Итак, j – равномерно факторное отображение, следовательно, j – равномерный изоморфизм группы $(G_1/\ker h,\vartheta)$ и группы $(G_2,\mathcal{U}_2)$. Отсюда непосредственно следует, что гомоморфизм h – открыт.

С учётом предложения 5 и теоремы 4 получаем

С Л Е Д С Т В И Е 6. Равномерно непрерывное отображение $f:(X,\mathcal{U})\to(Y,\vartheta)$ равномерно факторно тогда и только тогда, когда открыт гомоморфизм $\bar{f}:FN_{\mathcal{U}}(X)\to FN_{\vartheta}(Y)$

Пусть $\mathcal{U}_X$ и $\mathcal{U}_Y$ обозначают универсальные равномерности тихоновских пространств X и Y.

С Л Е Д С Т В И Е 7. Непрерывное отображение $f:X\to Y$ тихоновского пространства X на тихоновское пространство Y R – факторно тогда и только тогда, когда равномерно факторно отображение $f:(X,\mathcal{U}_X)\to(Y,\mathcal{U}_Y)$.

Д о к а з а т е л ь с т в о. Пусть непрерывное отображение $f:X\to Y$ R – факторно (см.,например/6/), тогда согласно одному результату О.Окунева (/6/), гомоморфизм $\bar{f}:F_M(X)\to F_M(Y)$ – открыт, т.е. открыт гомоморфизм

$\bar{j}: FN_{\mathcal{U}}(X) \to FN_{\mathcal{V}}(Y)$. Тогда из следствия 6 вытекает равномерная факторность отображения $f:(X,\mathcal{U}_X)\to(Y,\mathcal{U}_Y)$. Аналогично доказывается и обратное утверждение.

Следствие 6 является усилением одного результата О.Окунева (/6/), а следствие 7 показывает, как униформизуется понятие R - факторности.

ОПРЕДЕЛЕНИЕ 8. Равномерные пространства $(X,\mathcal{U})$ и $(Y,\mathcal{V})$ называются N - эквивалентными, если топологически изоморфны их свободные топологические группы $FN_{\mathcal{U}}(X)$ и $FN_{\mathcal{V}}(Y)$.

Вышедоказанные утверждения позволяют сформулировать общий метод доказательства N - эквивалентности различных фактор-пространств заданного равномерного пространства. Следующая теорема является переносом на равномерные пространства известной теоремы О.Окунева (/6/).

ТЕОРЕМА 9. Пусть $(X,\mathcal{U})$, $(Y_1,\mathcal{V}_1)$, $(Y_2,\mathcal{V}_2)$- равномерные пространства, $h_1:(X,\mathcal{U})\to(Y_1,\mathcal{V}_1)$ и $h_2:(X,\mathcal{U})\to(Y_2,\mathcal{V}_2)$ равномерно факторные отображения, и имеем продолжающие их гомоморфизмы $\bar{h}_1: FN_{\mathcal{U}}(X)\to FN_{\mathcal{V}_1}(Y_1)$ и $\bar{h}_2: FN_{\mathcal{U}}(X)\to FN_{\mathcal{V}_2}(Y_2)$. Если существует топологический автоморфизм i группы $FN_{\mathcal{U}}(X)$ такой, что $i(\mathrm{Ker}\,\bar{h}_1)=\mathrm{Ker}\,\bar{h}_2$, то равномерные пространства $(Y_2,\mathcal{V}_2)$ и $(Y_1,\mathcal{V}_1)$ - N - эквивалентны.

Доказательство. Определим отображение $j: FN_{\mathcal{V}_1}(Y_1)\to FN_{\mathcal{V}_2}(Y_2)$ по правилу $j(a)=\bar{h}_2\circ i(\bar{h}_1^{-1}(a))$ для каждого $a\in FN_{\mathcal{U}}(X)$. Это отображение является алгераическим изоморфизмом, так как, если $b\in \mathrm{Ker}\, j$ тогда $j(b)=e_2\in FN_{\mathcal{V}_2}(Y_2)$ и по определению j имеем $i(\bar{h}_1^{-1}(b))\in \mathrm{Ker}\,\bar{h}_2$. По определению автоморфизма i получаем, что $\bar{h}_1^{-1}(b)\subset \mathrm{Ker}\,\bar{h}_1$ или, переходя к образу при $\bar{h}_1$, имеем $b=e\in FN_{\mathcal{V}_1}(Y_1)$. Итак, $\mathrm{Ker}\, j=\{e\}$, что равносильно изоморфности j. Проверим теперь непрерывность отображения j. Имеем $j\circ\bar{h}_1=\bar{h}_2\circ i$ и гомоморфизм $\bar{h}_1$ - открыт, согласно следствию 6. Из непрерывности отображений i и $\bar{h}_2$ следует непрерывность отображения j. Аналогично доказывается непрерывность обратного изоморфизма j^{-1}. Теорема доказана.

Можно указать конкретный автоморфизм топологической группы $FN_{\mathcal{U}}(X)$, который даёт возможность строить примеры не равномерно гомеоморфных пар N - эквивалентных равномерных пространств.

ОПРЕДЕЛЕНИЕ 10. Равномерные ретракции r_1, r_2 равномерного пространства $(X,\mathcal{U})$ называются параллельными, если выполнены соотношения $r_1=r_1\circ r_2$, $r_2=r_2\circ r_1$. Ретракты R_1 и R_2 равномерного пространства $(X,\mathcal{U})$ называются параллельными, если они являются образами равномерного пространства $(X,\mathcal{U})$ при параллельных равномерных ретракциях.

Следующая теорема является переносом на равномерные пространства ещё одной известной теоремы О.Окунева (/6/).

ТЕОРЕМА 11. Пусть R_1 и R_2 - параллельные ретракты равномерного пространства $(X,\mathcal{U})$. Тогда равномерные фактор-пространства X/R_1 и X/R_2 N - эквивалентны.

Доказательство этой теоремы опирается на теорему 9 и следующую алгебраическую лемму.

ЛЕММА 12.(/6/). Пусть X - пространство, R_1 и R_2 - подмножества X, $h_1: X \to X/R_1$ и $h_2: X \to X/R_2$ соответственно R_1- и R_2- тривиальные отображения, $\bar{h}_1: F(X)\to F(X/R_1)$, $\bar{h}_2: F(X)\to F(X/R_2)$ - гомоморфизмы, продолжающие отображения h_1 и h_2. Пусть, далее i - автоморфизм группы $F(X)$ такой, что $i(R_1)=R_2$. Тогда $i(\operatorname{Ker}\bar{h}_1)=\operatorname{Ker}\bar{h}_2$.

Доказательство теоремы 11. Пусть $r_1:(X,\mathcal{U})\to(R_1,\mathcal{U}|_{R_1})$, $r_2:(X,\mathcal{U})\to(R_2,\mathcal{U}|_{R_2})$ - параллельные равномерные ретракции. Рассмотрим отображение $i:(X,\mathcal{U})\to FN_{\mathcal{U}}(X)$, заданное правилом $i(x)=r_1(x)\cdot x^{-1}\cdot r_2(x)$. Это отображение равномерно непрерывно. Пусть $\bar{i}: FN_u(X) \to FN_u(X)$ - гомоморфизм, продолжающий i. Пользуясь параллельностью равномерных ретракций r_1 и r_2 получаем $\bar{i}(R_1)=R_2$ и сужение на (X, U) композиции $\bar{i} \circ \bar{i}$ тождественно. Из этого следует, что и сама композиция $\bar{i} \circ \bar{i}$ тождественна, следовательно $\bar{i}(Ker \bar{h}_1)=Ker \bar{h}_2$ и из теоремы 9 получаем, что фактор-пространства X/R_1 и X/R_2 N - эквивалентны.

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