DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

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Morita equivalence of partially ordered semigroups

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1 Introduction

1.1 Background

The notion and study of Morita equivalence go back to ring theory and the now classical paper by Kiiti Morita ([37]). Therein he presented a general framework of duality and equivalence for unital rings, drawing on Pontryagin duality and the Artin-Wedderburn Theorem as precursors. While most of his paper dealt with duality (it is worth noting that in the same time frame, G. Azuyama ([6]) and E. Matlis ([30]) developed similar ideas of duality for right artinian rings and commutative complete noetherian local rings, respectively), today it is mostly used and cited for its results on what is now called “Morita equivalence“.

Two unital rings R and S are said to be *Morita equivalent* if their categories of modules Mod_R and Mod_S are equivalent. The main results of Morita's paper are known as Morita I-III. They can be stated as follows:

Theorem 1.1 (Morita I) *Let P be a finitely generated projective generator (a* progenerator*) in Mod_R, the category of right modules over a unital ring R, and let $(R, P, Q, S, \alpha, \beta)$ be the Morita context associated with P. Then*

(1) *$-\otimes_R Q : \mathsf{Mod}_R \to \mathsf{Mod}_S$ and $-\otimes_S P : \mathsf{Mod}_S \to \mathsf{Mod}_R$ are mutually inverse equivalences of categories;*

(2) *$P \otimes_R - : {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ and $Q \otimes_S - : {}_S\mathsf{Mod} \to {}_R\mathsf{Mod}$ are mutually inverse equivalences of categories.*

Theorem 1.2 (Morita II) *Let R and S be unital rings and let*

$$F : \mathsf{Mod}_R \to \mathsf{Mod}_S, \quad G : \mathsf{Mod}_S \to \mathsf{Mod}_R$$

be mutually inverse equivalences of categories. Take $Q = F(R_R) \in {}_R\mathsf{Mod}_S$ and $P = G(S_S) \in {}_S\mathsf{Mod}_R$. Then we have natural isomorphisms $F \cong -\otimes_R Q$ and $G \cong -\otimes_S P$.

Theorem 1.3 (Morita III) *Let R and S be unital rings. The isomorphism classes of equivalences of categories of type $\mathsf{Mod}_S \to \mathsf{Mod}_R$ are in one-to-one correspondence with the isomorphism classes of (S, R)-progenerators. Composition of equivalences corresponds to tensor products of progenerators.*

As we see, the basic question in Morita theory is to give necessary and sufficient conditions for two rings to be Morita equivalent and to explicitly describe how those equivalences can be realized. There is a large number of important ring-theoretic properties that are Morita invariant: being right primitive, prime, simple, right self-injective, quasi-Frobenius etc. Yet there are also many others that are lost in Morita equivalence: being commutative, local, a domain, a division ring, right Goldie and more.

The Morita equivalence theory was simultaneously adapted to monoids in 1972 by U. Knauer ([24]) and B. Banaschewski ([7]). Two monoids S and T were taken to be Morita equivalent if the categories of (right) S-acts and T-acts are equivalent. Their results largely coincide and the main observation to come out of it all was a description theorem analogous to the Morita equivalence between a unital ring R and an $n \times n$ matrix ring over R.

Theorem 1.4 (Thm 6.1 of [24], cf. Prop 4 of [7]) *The monoids A and B are Morita equivalent iff $B \cong aAa$, where $a^2 = a \in A$ and there exist $l, l' \in A$ such that $al = l$ and $l'l = 1$.*

A crucial observation of Banaschewski was that the notion of Morita equivalence for monoids does not carry over to semigroups.

Theorem 1.5 (cf. Prop 5 of [7]) *Let S and T be semigroups. If the categories of right S-acts and right T-acts are equivalent, then S is isomorphic to T.*

Due to this, any Morita theory for (partially ordered) semigroups has to proceed along a route similar to Morita duality and consider equivalences between some subcategories of the categories of (one-sided) acts.

While there was some precedence in developing a Morita theory for non-unital rings (by Abrams in [1], Ánh and Márki in [5], García and Simón in [20] and Abrams, Ánh and Márki in [2]), the first serious results on Morita equivalence for semigroups come from S. Talwar. In his first paper ([44]), he modified the Hom-sets used for rings so that the Morita theory in [5] could be carried over to semigroups with local units. He did this by defining a representation of such a semigroup S, deriving a functor from that and using the subcategories $\mathbf{FS-Act}$ of fixed objects of that functor to define Morita equivalence. This resulted in an analogue of Morita III:

Theorem 1.6 (Thm 6.1 of [44]) *Let R and S be equivalent semigroups via inverse equivalences $G : \mathbf{FR-Act} \to \mathbf{FS-Act}$ and $H : \mathbf{FS-Act} \to \mathbf{FR-Act}$.*

Set $P = H({}_SS)$ and $Q = G({}_RR)$. Then P and Q are unitary biacts ${}_RP_S$ and ${}_SQ_R$ respectively such that

(1) *${}_RP$ and ${}_SQ$ are generators for $\mathbf{FR-Act}$ and $\mathbf{FS-Act}$ respectively;*

(2) *$R \cong R \otimes_R \mathrm{End}_S Q$, $S \cong S \otimes_S \mathrm{End}_R P$ as semigroups;*

(3) *$G \approx S \otimes_S \mathrm{Hom}_R(P, -)$, $H \approx R \otimes_R \mathrm{Hom}_S(Q, -)$;*

(4) *${}_SQ_R \cong S \otimes_S \mathrm{Hom}_R(P, R)$, ${}_RP_S \cong R \otimes_R \mathrm{Hom}_S(Q, S)$.*

Talwar was also the first to introduce the notion of Morita context (a concept originating from Bass's lecture notes ([8]) and in widespread use in studies related to the Morita theory of rings) to semigroups and prove that Morita equivalence for semigroups can be described by Morita contexts. In effect, he proved that

Theorem 1.7 *Two semigroups R and S with local units are Morita equivalent if and only if there exists a Morita context $\langle R, S, {}_RP_S, {}_SQ_R, \langle,\rangle, [,]\rangle$, such that ${}_RP \in R-$* **FxAct**, *${}_SQ \in S-$* **FxAct** *and the maps $\langle,\rangle, [,]$ are surjective.*

Finally, he described those semigroups with local units that are Morita equivalent to a monoid, a group with zero or a regular bisimple monoid (with or without zero). The case for a group with zero can be considered to be the semigroup analogue for the Morita equivalence between unital simple artinian rings and division rings.

The second paper by Talwar ([45]) furthered the use of Morita contexts and defined what he called "strong Morita equivalence" on the class of all factorizable semigroups. It turned out that Morita equivalence always follows from strong Morita equivalence (cf. Theorem 1.1).

Theorem 1.8 (Thm 3 of [45]) *Let R and S be strongly Morita equivalent and let $\langle R, S, {}_RP_S, {}_SQ_R, \langle,\rangle, [,]\rangle$ be a unitary Morita context with $\langle,\rangle : P \otimes_S Q \to R$ and $[,] : Q \otimes_R P \to S$ surjective. Then R and S are Morita equivalent via the functors*

$$S \otimes_S S\mathrm{Hom}_R(P,\ ) : R - \mathbf{FxAct} \to S - \mathbf{FxAct} \quad \text{and}$$

$$R \otimes_R R\mathrm{Hom}_S(Q,\ ) : S - \mathbf{FxAct} \to R - \mathbf{FxAct}.$$

Furthermore, Talwar also introduced the notion of Morita semigroup and used it to construct strongly Morita equivalent semigroups, especially when given a Morita context. Finally, he employed his results to re-prove the Rees-Suschkewitsch theorem, studied what he called Morita semigroups having tensorial coordinate form, Morita equivalent semigroups within finite semigroups and also Morita equivalences of unambiguous semigroups. Some of Talwar's further work (e.g. [46]) deals with applying his theory to unambiguous semigroups and iterative matrix semigroups.

Since strong Morita equivalence requires factorizability, Talwar's work was and is considered to be at the limits of a useful Morita theory for semigroups. There have been a number of further studies of the Morita theory of semigroups ([40], [39], [16], [15], [25]) exploring the finer points of the theory, e.g. equivalences between the subcategories of unitary acts or ideals in Morita semigroups. The developments include a number of results that describe all semigroups (which need to be at least factorizable) that are Morita equivalent to a semigroup from a given class of semigroups (e.g. monoids, sandwich semigroups, semigroups with local units).

The next substantial advancement of the theory came in a recent paper ([29]) by Lawson. He restricts his theory to just semigroups with local units, but is able to reformulate Talwar's theory in a more straightforward form and give a number of additional algebraic characterizations of Morita equivalence. His main result is the following:

Theorem 1.9 (Thms 1.1 and 1.2 of [29]) *Let S and T be semigroups with local units. Then the following are equivalent.*

1. *S and T are Morita equivalent.*
2. *The categories $C(S)$ and $C(T)$ are equivalent.*
3. *S and T have a joint enlargement which can be chosen to be regular if S and T are both regular.*
4. *There is a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective mappings.*
5. *There is a consolidation q on $C(S)$ and a local isomorphism $\psi : C(S)^q \to T$.*

As an application of this theorem, Lawson provided a few Morita invariants and reproved the necessary and sufficient conditions for Morita equivalence with a monoid or group. More importantly, he was able to link his theory with the results of a series of papers by McAlister ([35], [32], [33], [34], [31]) that investigate various classes of regular semigroups which can be described by the structure of their local submonoids. In the end, Lawson managed to establish necessary and sufficient conditions for a semigroup with local units to be Morita equivalent to a semigroup from one of those classes of regular semigroups (e.g. inverse, orthodox, union of groups).

Lawson's theorem was developed further by Laan and Márki in [28], who were able to show that there were two additional necessary and sufficient conditions for Morita equivalence. To wit,

Theorem 1.10 (Thm 8 and Cor 12 of [28]) *Let S and T be two semigroups with local units. Then the following are equivalent.*

1. *S and T are strongly Morita equivalent.*
2. *There are a Rees matrix semigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ with $S = S\,\mathrm{Im}(M)S$ and a strict local isomorphism $\tau : \mathcal{M} \to T$ along which idempotents lift.*
3. *There exist a surjectively defined unitary Morita semigroup $Q \otimes_S P$ and a strict local isomorphism : $\tau : Q \otimes_S P \to T$ along which idempotents lift.*

They were also able to use Morita contexts in [28] to derive a number of Morita invariants, including the validity of identities, the lattice of ideals and the lattice of congruences.

In the last twenty years, there has also been some interest in generalizing certain parts of semigroup theory to partially ordered monoids and semigroups, specifically those aspects that deal with partially ordered acts and their tensor products, beginning with Fakhruddin ([18], [19], [43], [13], [12], [42]). A relatively recent overview of such studies can be found in [11]. As one part of this program, Laan has generalized the Morita theory for monoids due to Knauer and Banaschewski ([24], [7]) to pomonoids in [26]. His description of Morita equivalence is strikingly similar to the same result for semigroups.

Theorem 1.11 (Thm 7 of [26]) *Pomonoids S and T are Morita equivalent if and only if there exists an idempotent $e \in S$ such that $e\mathcal{J}1$ and $T \cong eSe$ as pomonoids.*

This is the current extent of Morita theory for partially ordered semigroups and it is basically the same as it was for semigroups in 1972. The Morita theory of semigroups has seen a number of recent innovations and it remains to be seen how much of it is can be transferred to posemigroups.

1.2 Summary of the thesis

The general goal of the thesis is to develop a theory of Morita equivalence for partially ordered semigroups similar to the Morita theory for semigroups. There are two main lines of research pursued within this general framework. On the one hand, we strive to prove analogues of Theorems 1.9 and 1.10 for posemigroups. On the other hand, we investigate which properties remain invariant under Morita equivalence. The characterization theorems play a crucial role in this investigation. Our starting point is Talwar's observation that strong Morita equivalence implies Morita equivalence and therefore we take the notion of strong Morita equivalence as primary for posemigroups. This turns out to be the more practical solution for most of our work.

The thesis is organized as follows.

Chapter 1 makes a brief historical overview of Morita equivalence and its treatment in semigroup theory, followed by an overview of the thesis.

Next, Chapter 2 familiarizes the reader with the tools and concepts used to study Morita equivalence. We introduce a number of concepts related to partially ordered semigroups such as partially ordered acts, various kinds of local units, partially ordered Rees matrix semigroups, congruences, tensor products, joint enlargements, Cauchy completions, Morita contexts and strict local isomorphisms. We also prove a few simple but useful results.

In Chapter 3 we present the main result of the thesis and prove roughly half of it. It turns out that if we assume that two partially ordered semigroups have certain kinds of local units, Morita equivalence can be described in a number of equivalent ways. Specifically, we establish that under such assumptions categorial Morita equivalence via Cauchy completions, being embedded into a “larger“ partially ordered semigroup and the classical definition using a Morita context are all equivalent. When two partially ordered semigroups satisfy these conditions (or specifically, the strongest of them that uses Morita contexts), we call them “strongly Morita equivalent.“

The rest of the main theorem is proved in Chapter 4. We show that, again under the assumption of having suitable local units, the first three conditions are equivalent to having one of the partially ordered semigroups be a “strictly locally isomorphic“ image of a partially ordered semigroup constructed from the other partially ordered semigroup. The constructions we consider are a partially ordered semigroup derived from the Cauchy completion, a Rees matrix posemigroup and a Morita posemigroup. We also establish a number of interesting properties connected with partially ordered Rees matrix semigroups and Morita posemigroups. Of particular interest is the construction of partially ordered regular Rees matrix semigroups in the case when both posemigroups are regular.

Chapter 5 extends the main theorem to include a version of Morita equivalence that resembles the original, ring-theoretic definition which uses categories of modules. We show that if we assume the existence of certain kinds of local units, then all the first six conditions are equivalent to the existence of an equivalence of categories between certain subcategories of the respective categories of partially ordered acts.

After having proved that all the different characterizations are equivalent, we follow it up with an investigation of what we call “Morita invariants“, properties that are retained by strong Morita equivalence. Chapter 6 is focused on two main lines of study. One is the establishment of necessary and sufficient conditions for a partially ordered semigroup to be strongly Morita equivalent to a partially ordered semigroup from a well-known class, such as partially ordered groups, partially ordered monoids, partially ordered unions of groups etc. We again pay special attention to the case when the partially ordered semigroups in question are regular. The other line of study is concerned with the structural invariants of strongly Morita equivalent partially ordered semigroups such as local subpomonoids, various lattices of ideals, greatest commutative images, the validity of inequalities etc.

Finally, we provide a summary of the work and give an outline of some topics for further research.

Chapter 3 and Chapter 4 are largely based on [49] and [48], respectively, while Chapter 5 is, more or less, an exposition of [51]. Chapter 6 draws on both [47] and [50], with a small section from [49].

2 Preliminaries and notation

This first chapter introduces the reader to the basic concepts, tools and notation of the Morita equivalence of partially ordered semigroups. Additionally, we prove several results that will be used later on.

While a lot of our notation does not have a standard form and is consequently introduced on the spot, we do use the established notation, terminology and results of both semigroup and category theory, as found in e.g. [21] and [10]. Categorial compositions will be written from right to left.

The basis for most of our work is the symmetric monoidal closed category Pos of partial orders and monotone maps (using cartesian product for monoidal tensor product), together with different categories enriched over Pos. Our reference for Pos-categories, Pos-functors and Pos-equivalences is [22]. For the convenience of the reader, we remark that a full and faithful Pos-functor must provide a poset isomorphism instead of a bijection between the corresponding morphism posets.

2.1 Posemigroups with local units

First of all, we define the objects of our study and clarify what we mean by “local units“, “factorizability“ and “local structure“.

Definition 2.1 A *partially ordered semigroup* S (a *posemigroup* for short) is a (nonempty) semigroup that is endowed with a partial order $\leq$ so that its operation is monotone, i.e.

$$(s \leq s') \wedge (t \leq t') \Rightarrow st \leq s't'$$

for every $s, s', t, t' \in S$. A *pomonoid* is a posemigroup with an identity element.

We occasionally construct pomonoids from posemigroups by adjoining an external identity. If we start with a posemigroup S equipped with a partial order $\leq$, then the corresponding pomonoid is denoted by S^1, where 1 is the externally adjoined identity and the order $\leq$ on S^1 is taken to be $\leq \cup \{(1, 1)\}$.

Definition 2.2 A *right S-poset* over a fixed posemigroup S is a set A equipped with a partial order $\leq$ and an S-action $A \times S \to A$, $(a, s) \mapsto as$, such that

$$(as)t = a(st) \quad \text{and} \quad (a \leq a') \wedge (s \leq t) \Rightarrow as \leq a't$$

for every $a, a' \in A$, $s, t \in S$.

The notion of left S-poset is dual.

As usual, $E(S)$ denotes the set of all idempotents of a semigroup S.

Definition 2.3 A right S-poset X is said to be *unitary* if $X = XS$, i.e. for every $x \in X$ there exist $x' \in X$ and $s \in S$ such that $x = x's$.

Definition 2.4 We say that a right S-poset X is Pos-*unitary* if for all $x, y \in X$ such that $x \leq y$ there exist idempotents $s, t \in S$ such that $s \leq t$, $xs = x$ and $yt = y$.

The notions for left S-posets are dual.

Definition 2.5 Let S and T be posemigroups. A poset is called an (S, T)-*biposet* if it is a left S- and a right T-poset and its S- and T-actions commute with each other. (S, T)-biposets are called *unitary (*Pos*-unitary)* if they are unitary (Pos-unitary) as both left S- and right T-posets.

A number of basic facts about S-posets over pomonoids can be found in [11].

Definition 2.6 A *posemigroup homomorphism* is a monotone semigroup homomorphism.

Definition 2.7 Let A and B be right S-posets. A monotone mapping $f : A \to B$ is called a *right S-poset morphism* if $f(as) = f(a)s$ for every $a \in A$ and $s \in S$.

Left S-poset and (S, T)-biposet morphisms are defined analogously. All such morphisms naturally form posets with respect to the pointwise order, e.g. if we have right S-poset morphisms $f, g : A \to B$, then

$$f \leq g \iff f(a) \leq g(a) \text{ for all } a \in A.$$

Since we now have both objects and morphism posets, we can introduce the following notation:

- Pos – the category of posets and monotone maps,
- Pos_S – the Pos-category of right S-posets and right S-poset morphisms,
- ${}_S\mathsf{Pos}$ – the Pos-category of left S-posets and left S-poset morphisms,
- ${}_S\mathsf{Pos}_T$ – the Pos-category of (S, T)-biposets and (S, T)-biposet morphisms,
- UPos_S – the full Pos-subcategory of Pos_S generated by unitary S-posets.

Definition 2.8 A posemigroup S is said to have *local units* if for any $s \in S$ there exist $e \in E(S)$ and $f \in E(S)$ such that

$$es = s = sf.$$

A posemigroup S is said to have *weak local units* if for any $s \in S$ there exist $e \in S$ and $f \in S$ such that $es = s = sf$.

Definition 2.9 A posemigroup is said to have *common (weak) local units* (cf. [27]) if for any $s, s' \in S$ there exist $e \in E(S)$ and $f \in E(S)$ ($e \in S$ and $f \in S$) such that $es = s = sf$ and $es' = s' = s'f$.

Definition 2.10 We say that a posemigroup S has *ordered (weak) local units* if for all $s, s' \in S$, $s \leq s'$, there exist $e, e', f, f' \in E(S)$ ($e, e', f, f' \in S$) such that

$$es = s = sf,\ e's' = s' = s'f',\ e \leq e',\ f \leq f'.$$

A natural example of a posemigroup with ordered local units is a partially ordered band (a *poband*), e.g. $(\mathbb{N}, \min, \leq)$. The last example is noteworthy because it illustrates two important facts: (1) we can choose from many such pairs of local units and (2) not all of the local units are in the required order relation.

Definition 2.11 A posemigroup is called *factorizable* if $S^2 = S$, i.e. every element of S can be written as a product of two elements of S. A *sandwich* posemigroup is a posemigroup S which satisfies the relation $S = SE(S)S$.

Having local units implies having weak local units, which in turn implies factorizability. Also, a posemigroup with common (weak) local units has ordered (weak) local units, and one with ordered (weak) local units has (weak) local units. The converse implications do not hold in general (see Example 2.18). A posemigroup with local units is also a sandwich posemigroup, and the latter is always factorizable. These relationships are captured in the following scheme.

$$\begin{tikzcd}
\text{common local units} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{common weak local units} \arrow[d, Rightarrow] \\
\text{ordered local units} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{ordered weak local units} \arrow[d, Rightarrow] \\
\text{local units} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{weak local units} \arrow[d, Rightarrow] \\
\text{sandwich posemigroup} \arrow[r, Rightarrow] & \text{factorizable}
\end{tikzcd}$$

We call a posemigroup *regular* or *inverse* if its underlying semigroup is regular or inverse. The poset of inverse elements of s will be denoted by $V(s)$.

Definition 2.12 A posemigroup S is said to have a property *locally* (e.g. is locally inverse) if every local subpomonoid eSe, $e \in E(S)$, has that property (e.g. is inverse).

2.2 Partially ordered Rees matrix semigroups

We will make use of the following partially ordered versions of Rees matrix semigroups. First, let us recall the definition of Rees semigroup for convenience and then introduce our notation.

Definition 2.13 Take a semigroup S, two nonempty sets U and V and a mapping $M : V \times U \to S$. Consider the set $\mathcal{M}(S, U, V, M) = U \times S \times V$ with the multiplication rule

$$(u, s, v)(u', s', v') = (u, sM(v, u')s', v).$$

Then $\mathcal{M} = \mathcal{M}(S, U, V, M)$ is called the $U \times V$ *Rees matrix semigroup over the semigroup S with sandwich matrix M*.

For a Rees matrix semigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over a semigroup S, let $\mathrm{Im}(M)$ denote the image of the mapping $M : V \times U \to S$. It is easy to see that the Rees matrix semigroup $\mathcal{M}$ is factorizable iff $S = S\,\mathrm{Im}(M)S$.

Definition 2.14 We say that $\mathcal{M} = \mathcal{M}(S, U, V, M)$ is a *partially ordered Rees matrix semigroup* over a posemigroup S if it is a Rees matrix semigroup over S with any kind of order that is compatible with multiplication.

Definition 2.15 We call $\mathcal{M} = \mathcal{M}(S, U, V, M)$ a *Rees matrix posemigroup* over a posemigroup S when its order is the specific one defined by

$$(u, s, v) \leq (u', s', v') \iff u = u', v = v', s \leq s'.$$

By Lemma 2.1 of [35], the set $\mathcal{R} = \mathcal{R}(S, U, V, M)$ of all the regular elements of $\mathcal{M}$ is a subsemigroup of $\mathcal{M}$.

Definition 2.16 The posemigroup $\mathcal{R}$ as the subposemigroup of the corresponding Rees matrix posemigroup $\mathcal{M}$ will be called the *regular Rees matrix posemigroup* over S and will be denoted by $\mathcal{R}_m$.

Alternatively, as it is done in [35], we can assume that U and V are also partially ordered, M is monotone and consider the cartesian order on $\mathcal{R}$.

Definition 2.17 We say that the posemigroup $\mathcal{R}$ with componentwise order is the *cartesian regular Rees matrix posemigroup* over S and denote it by $\mathcal{R}_c$.

Example 2.18 All lower semilattices (with their natural order) that lack upwards direction have ordered local units but not common local units. The smallest example of such is the three-element lower semilattice that is not a chain. Take posets $U = \{1\}$, $V = \{1 < 2\}$, the linearly ordered group $(\mathbb{Z}, +, \leq)$, $M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and construct the Rees matrix semigroup $\mathcal{M} = \mathcal{M}(\mathbb{Z}, U, V, M)$ with cartesian order. The elements $(1, z, 1)$, $z \in \mathbb{Z}$, have only one local right unit, namely $(1, 0, 1)$. Elements $(1, z, 2)$, $z \in \mathbb{Z}$, also have only $(1, -1, 2)$ as a local right unit. But then $(1, 0, 1) \leq (1, 0, 2)$, yet $(1, 0, 1) \nleq (1, -1, 2)$, so $\mathcal{M}$ has local units but not ordered local units.

2.3 Congruences

Congruences, admissible preorders and tensor products play a central role in much of the theory of Morita equivalence. While most of the notions are not new and exist for any ordered universal algebras (see e.g. [17]), we will still explicitly write them out for posemigroups and S-posets.

Definition 2.19 The *directed kernel* of a posemigroup homomorphism $f : S \to T$ is the relation

$$\overrightarrow{\mathrm{Ker}}\, f = \{(a,b) \in S^2 | \, f(a) \leq f(b) \text{ in } T\}.$$

Given a morphism $f : S \to T$, we can construct a quotient posemigroup $S/\overrightarrow{\mathrm{Ker}}\, f$ if we factorize S by the semigroup congruence $\mathrm{Ker}\, f = \overrightarrow{\mathrm{Ker}}\, f \cap (\overrightarrow{\mathrm{Ker}}\, f)^{-1}$ and set

$$[s] \leq [s'] \text{ in } S/\overrightarrow{\mathrm{Ker}}\, f \iff (s,s') \in \overrightarrow{\mathrm{Ker}}\, f. \tag{2.1}$$

We have the following theorem for any ordered universal algebras $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.20 (Theorem 1.3 of [17]) *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective homomorphism, then $\mathcal{A}/\overrightarrow{\mathrm{Ker}}\,\varphi \cong \mathcal{B}$, an isomorphism is given by $[a]\Phi \mapsto a\varphi$, where Φ denotes the order-congruence $\overrightarrow{\mathrm{Ker}}\,\varphi \cap (\overrightarrow{\mathrm{Ker}}\,\varphi)^{-1}$.*

If $f : A \to B$ is a posemigroup homomorphism or an S-poset ((S,T)-biposet) morphism, then we will denote by $\mathrm{Im} f$ the subset $\{b \in B | \, \exists a \in A : b = f(a)\} \subseteq B$ together with order and multiplication or action(s) that are the restrictions of those of B. So the theorem implies that $S/\overrightarrow{\mathrm{Ker}}\, f \cong \mathrm{Im} f$ as posemigroups.

Definition 2.21 An *admissible preorder* on a posemigroup S is a preorder on S that is compatible with S-multiplication and which contains the order on S.

Definition 2.22 The *admissible preorder generated by a relation $H \subseteq S \times S$* is the smallest admissible preorder on S that contains H.

By Theorem 1.2 of [17], admissible preorders are exactly the directed kernels of posemigroup homomorphisms. Therefore we can use any admissible preorder ρ on S to construct the quotient posemigroup S/ρ as above.

If S is a posemigroup and ρ is a reflexive relation on S, one can define a preorder $\leq_\rho$ on S by setting $s \leq_\rho t$ if there exist $n \in \mathbb{N}$, $s_i, t_i \in S$, $1 \leq i \leq n$ such that

$$s \leq s_1 \,\rho\, t_1 \leq s_2 \,\rho\, t_2 \leq \ldots \leq s_n \,\rho\, t_n \leq t.$$

Definition 2.23 A *posemigroup congruence* on S is a semigroup congruence θ on S such that the *closed chains condition* holds:

$$\text{if } s \leq_\theta t \leq_\theta s, \text{ then } s\theta t.$$

Definition 2.24 The *posemigroup congruence generated by a relation* $H \subseteq S \times S$ (denoted by $\theta(H)$) is the smallest posemigroup congruence on S that contains H.

For a more detailed description, the reader is referred to [11], but we remark that if H is a semigroup congruence, then $\theta(H) = \leq_H \cap \geq_H$. The notions of admissible preorder and congruence for S-posets and (S,T)-biposets are analogous.

Definition 2.25 The *tensor product* $A \otimes_S B$ of a right S-poset A and a left S-poset B is the quotient poset $(A \times B)/\sim$, where $(a,b) \sim (a',b')$ iff $(a,b) \leq (a',b')$ and $(a',b') \leq (a,b)$, and $(a,b) \leq (a',b')$ iff there exist $s_1,\dots,s_n,t_1,\dots,t_n \in S^1$, $a_1,\dots,a_n \in A$ and $b_2,\dots,b_n \in B$ such that

$$\begin{array}{rclrcl} a & \leq & a_1 s_1 & & & \\ a_1 t_1 & \leq & a_2 s_2 & s_1 b & \leq & t_1 b_2 \\ \vdots & & & & \vdots & \\ a_n t_n & \leq & a' & s_n b_n & \leq & t_n b', \end{array} \tag{2.2}$$

where $xu = x$ for every element $x \in \{a_1,\dots,a_n\}$ and $uy = y$ for every element $y \in \{b,b'\} \cup \{b_2,\dots,b_n\}$, if $u \in S^1$ is the externally adjoined identity.

For $(a,b) \in A \times B$, the equivalence class $[(a,b)]_\sim$ is denoted by $a \otimes b$. The order relation on $A \otimes_S B$ is defined by setting

$$a \otimes b \leq a' \otimes b' \iff (a,b) \leq (a',b')$$

for $a \otimes b, a' \otimes b' \in A \otimes_S B$. A general result for congruences from which the above is derived can be found in Proposition 1.9 of [17]. Note that the tensor products of posets over posemigroups are slightly different (smaller) from the tensor products of posets over pomonoids, even if the posemigroup is a pomonoid.

If A is a (T,S)-biposet, then $A \otimes_S B$ is a left T-poset, with the left T-action $t(a \otimes b) = (ta) \otimes b$. Similarly, if B is an (S,T)-biposet then $A \otimes_S B$ is a right T-poset.

Let C be an (S,T)-biposet. Then we get a Pos-functor $- \otimes_S C : \mathsf{Pos}_S \to \mathsf{Pos}_T$ by taking $(- \otimes_S C)(A) = A \otimes_S C$. If $f : A \to B$ is an S-poset morphism, then $f \otimes_S C : A \otimes_S C \to B \otimes_S C$ is defined by $(f \otimes_S C)(a \otimes c) = f(a) \otimes c$ for any $a \in A$ and $c \in C$. Likewise, we obtain another Pos-functor $\mathsf{Pos}_T(C,-) : \mathsf{Pos}_T \to \mathsf{Pos}_S$ if we put $\mathsf{Pos}_T(C,-)(D) = \mathsf{Pos}_T(C,D)$, with the right S-action defined by $(h,s) \mapsto hs$, $(hs)(c) = h(sc)$ for any $h \in \mathsf{Pos}_T(C,D)$, $s \in S$, $c \in C$. Again, whenever $g : D \to E$ is a T-poset morphism, we get $\mathsf{Pos}_T(C,g) : \mathsf{Pos}_T(C,D) \to \mathsf{Pos}_T(C,E)$ by putting $\mathsf{Pos}_T(C,g)(h) = gh$. It is easy to prove that these Pos-functors are a Pos-adjoint pair, similar to how it is done in Proposition 2.5.19 of [23].

Definition 2.26 A right S-poset X is unitary iff the canonical S-poset morphism

$$\mu_X : X \otimes_S S \to X, \; x \otimes s \mapsto xs,$$

is surjective. If it is also an order embedding (implying it is an order isomorphism) then X is said to be *closed*.

Closed left S-posets are defined dually. All closed right S-posets generate a full Pos-subcategory of Pos_S, which we will denote by FPos_S.

2.4 Morita equivalence

We now introduce the notion of strong Morita equivalence and several related concepts (joint enlargements, Cauchy completions, consolidations).

Definition 2.27 A posemigroup R is called an *enlargement* of a subposemigroup S (with inherited order) if

$$S = SRS \quad \text{and} \quad R = RSR.$$

For posemigroups S, T and R, R is said to be a *joint enlargement* (cf. [29]) of S and T if it is an enlargement of its subposemigroups $S' \cong S$ and $T' \cong T$.

In general, if it does not cause any confusion, we will ignore the isomorphisms $S' \cong S$ and $T' \cong T$ and write S instead of S' and T instead of T'.

For a Pos-category C, we will use the notation C_0 for its class of objects instead of C in cases when it is not immediately clear from the context whether we mean objects or morphisms. By $C(A,B)$ we denote the poset of morphisms in the Pos-category C from object A to object B.

Definition 2.28 The *Cauchy completion* of a posemigroup S (cf. [29]) is the (small) Pos-category $C(S)$ that has $C(S)_0 = E(S)$, morphism posets

$$C(S)(f,e) = \{(e,s,f) | \, s \in S, esf = s\},$$

with the order

$$(e,s,f) \leq (e,s',f) \Longleftrightarrow s \leq s' \text{ in } S \tag{2.3}$$

and the composition rule

$$(e,s,f) \circ (f,s',g) = (e,ss',g). \tag{2.4}$$

Definition 2.29 A category C is called *strongly connected* if for all $A, B \in C_0$ there always exists a morphism $f : A \to B$.

Definition 2.30 A *consolidation* on a strongly connected category C is a map

$$p : C_0 \times C_0 \to C$$

(denoted by $p_{B,A} := p(B,A) : A \to B$) such that $p_{A,A} = 1_A$.

A small Pos-category C (actually its set of morphisms, which we will again denote by C) with a consolidation p can be made into a posemigroup if we define a multiplication $\diamond$ by

$$g \diamond f = g \circ p_{\mathrm{dom}g,\mathrm{cod}f} \circ f$$

(then $h \diamond (g \diamond f) = h \circ p_{\mathrm{dom}h,\mathrm{cod}g} \circ g \circ p_{\mathrm{dom}g,\mathrm{cod}f} \circ f = (h \diamond g) \diamond f$) and taking the order from the morphism-poset order of C. So

$$f \leq g \Longleftrightarrow \mathrm{dom}f = \mathrm{dom}g \wedge \mathrm{cod}f = \mathrm{cod}g \wedge f \leq g \text{ in } C(\mathrm{dom}f, \mathrm{cod}f).$$

This posemigroup will be denoted by C^p.

Remark 2.31 It is easy to see that if the composite $g \circ f$ exists for morphisms f and g in C, then $g \diamond f = g \circ f$.

Let $\mathcal{IP}_S$ be the full Pos-subcategory of Pos_S generated by objects of the form eS, $e \in E(S)$.

Proposition 2.32 *Let S be a posemigroup with local units. Then the* Pos*-category $C(S)$ is* Pos*-equivalent to $\mathcal{IP}_S$.*

Proof. Take a right S-poset morphism $h : eS \to fS$ for some $e, f \in E(S)$. Observe that $h(es) = h(e)es$ for every $s \in S$, wherefore $h = \lambda_{h(e)}$. Furthermore, we have $fh(e) = h(e) = h(e^2) = h(e)e$, so $(f, h(e), e) \in C(S)(e, f)$. It is clear that the map $\lambda_s : eS \to fS$ is a right S-poset morphism if $(f, s, e) \in C(S)(e, f)$. Define a functor $F : C(S) \to \mathcal{IP}_S$ by taking $F(e) = eS$ and $F(f, s, e) = \lambda_s$. It is easy to see that we obtain a Pos-functor, that it provides poset isomorphisms $\mathcal{IP}_S(eS, fS) \cong C(S)(e, f)$ and that F is (essentially) surjective on objects. □

Definition 2.33 Let S and T be two posemigroups. We say that a sextuple

$$(S, T, P, Q, \langle -, - \rangle, [-, -])$$

is a *Morita context* if the following conditions hold:

(M1) P is an (S, T)-biposet and Q is a (T, S)-biposet;

(M2) $\langle -, - \rangle : P \otimes_T Q \to S$ is an (S, S)-biposet morphism and
$[-, -] : Q \otimes_S P \to T$ is a (T, T)-biposet morphism;

(M3) the following two conditions hold for all $p, p' \in P$ and $q, q' \in Q$:
(i) $\langle p, q \rangle p' = p[q, p']$,
(ii) $q\langle p, q' \rangle = [q, p]q'$.

Definition 2.34 A Morita context is called *unitary* if the biposets P and Q are unitary.

Definition 2.35 We say that posemigroups S and T are *strongly Morita equivalent* (cf. [45]) if there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ such that the mappings $\langle -, - \rangle$ and $[-, -]$ are surjective.

Lemma 2.36 (cf. Lemma 3.15 of [29]) *Let S and T be two posemigroups with ordered weak local units. If $(S, T, P, Q, \langle -, - \rangle, [-, -])$ is a unitary Morita context with surjective maps then $\langle -, - \rangle : P \otimes_T Q \to S$ and $[-, -] : Q \otimes_S P \to T$ are biposet isomorphisms.*

Proof. We will prove only the claim for $[-, -]$, since the proof for $\langle -, - \rangle$ is symmetric. It is sufficient to show that $[-, -]$ reflects order. Let $q \otimes p, q' \otimes p' \in Q \otimes P$

be such that $[q,p] \leq [q',p']$ in T. As Q is unitary, T has ordered weak local units and $[-,-]$ is surjective, we can find $t'' \in T$, $q_0, q_0', q'', q_1, q_2 \in Q$, $p_0, p_0', p_1, p_2 \in P$ such that $q = tq'' = [q_0,p_0]tq'' = [q_0,p_0]q$, $q' = [q_0',p_0']q'$, $[q,p] = [q,p][q_1,p_1]$, $[q',p'] = [q',p'][q_2,p_2]$ and $[q_1,p_1] \leq [q_2,p_2]$. Then in $Q \otimes_S P$

$$\begin{aligned}
q \otimes p &= [q_0,p_0]q \otimes p = q_0\langle p_0,q\rangle \otimes p = q_0 \otimes \langle p_0,q\rangle p = q_0 \otimes p_0[q,p][q_1,p_1] \\
&\leq q_0 \otimes p_0[q,p][q_2,p_2] = q_0 \otimes \langle p_0,q\rangle p[q_2,p_2] = q_0\langle p_0,q\rangle \otimes p[q_2,p_2] \\
&= [q_0,p_0]q \otimes \langle p,q_2\rangle p_2 = q\langle p,q_2\rangle \otimes p_2 = [q,p]q_2 \otimes p_2 \leq [q',p']q_2 \otimes p_2 \\
&= q'\langle p',q_2\rangle \otimes p_2 = [q_0',p_0']q' \otimes \langle p',q_2\rangle p_2 = q_0'\langle p_0',q'\rangle \otimes p'[q_2,p_2] \\
&= q_0' \otimes \langle p_0',q'\rangle p'[q_2,p_2] = q_0' \otimes p_0'[q',p'][q_2,p_2] = q_0' \otimes p_0'[q',p'] \\
&= q_0' \otimes \langle p_0',q'\rangle p' = q_0'\langle p_0',q'\rangle \otimes p' = [q_0',p_0']q' \otimes p' = q' \otimes p'. \qquad \square
\end{aligned}$$

Corollary 2.37 *Let S and T be strongly Morita equivalent posemigroups with ordered weak local units. Then we have biposet isomorphisms $\langle -,-\rangle : P \otimes_T Q \cong S$ and $[-,-] : Q \otimes_S P \cong T$.*

Proposition 2.38 (cf. Prop 14 of [28]) *Let $(S,T,P,Q,\langle -,-\rangle,[-,-])$ be a (not necessarily unitary) posemigroup Morita context with factorizable posemigroups S, T and surjective mappings $\langle -,-\rangle, [-,-]$. Then there exists a unitary posemigroup Morita context $(S,T,P',Q',\overline{\langle -,-\rangle},\overline{[-,-]})$ with surjective $\overline{\langle -,-\rangle}, \overline{[-,-]}$.*

Proof. The proof is exactly the same as in [28], which uses only the properties of tensor products and Morita contexts, not explicit descriptions, to demonstrate that it is viable to choose $P' = SPT$, $Q' = TQS$, $\overline{\langle tqs, s'pt'\rangle} = \langle tqs, spt\rangle$ and $\overline{[tqs, s'pt']} = [tqs, s'pt']$. $\square$

The previous proposition shows that if we were to define context equivalence similarly to how it is done in [25] then for factorizable posemigroups it would be the same as strong Morita equivalence.

Lemma 2.39 (cf. Lemma 7 of [25]) *Let $(S,T,P,Q,\langle -,-\rangle,[-,-])$ be a unitary Morita context with surjective $\langle -,-\rangle$ and $[-,-]$. Then S and T are factorizable.*

Proof. This is a repeat of Lemma 7 of [25]. Take $s \in S$, then $s = \langle p,q\rangle$ for some $p \in P$, $q \in Q$. Since P is unitary, there exist $p' \in P$ and $t \in T$ such that $p't = p$. Moreover, $t = [q',p'']$ for some $p'' \in P$, $q' \in Q$. Thus

$$s = \langle p,q\rangle = \langle p'[q',p''],q\rangle = \langle p', q'\langle p'',q\rangle\rangle = \langle p',q'\rangle\langle p'',q\rangle \in S^2. \qquad \square$$

Using Lemma 2.39, we can state Proposition 2.38 more precisely as

Proposition 2.40 (cf. Cor 15 of [28]) *Posemigroups S and T are strongly Morita equivalent if and only if they are factorizable and context equivalent.*

Due to this, we will refrain from using the notion of context equivalence.

2.5 Strict local isomorphisms

The central theme of Chapter 4 will be realizing strong Morita equivalences as "strict local isomorphisms" from some constructed posemigroup. Below, we will introduce the notion and make some immediate remarks about its properties.

Definition 2.41 A posemigroup homomorphism $\tau : S \to T$ between arbitrary posemigroups S and T is called a *strict local isomorphism* (cf. [29] and [28]) if the following conditions hold:

(LI1) τ restricted to aSb is a posemigroup isomorphism with $\tau(a)T\tau(b)$ for any $a \in Sa, b \in bS$;

(LI2) *idempotents lift along* τ, i.e. if $e' = \tau(s) \in E(T)$ for some $s \in S$, then there exists $e \in E(S)$ such that $e' = \tau(e)$;

(LI3) for any $e \in E(T)$ there exists $f \in E(T)$ such that $f = \tau(s)$ for some $s \in S$ and $e\mathcal{D}f$.

When S and T have trivial order, a strict local isomorphism $S \to T$ is a local isomorphism in the sense of [29] and every surjective strict local isomorphism that lifts idempotents (as in [28]) is a strict local isomorphism.

Lemma 2.42 *Let S and T be posemigroups with local units and let $\tau : S \to T$ be a surjective posemigroup homomorphism. Then* (LI1) *is equivalent to*

(LI1') *τ restricted to eSf is a posemigroup isomorphism with $\tau(e)T\tau(f)$ for any $e, f \in E(S)$.*

Proof. It is clear that (LI1) implies (LI1'). For the converse, take $a \in Sa$ and $b \in bS$. As S has local units, $a = ea$ and $b = bf$ for some $e, f \in E(S)$. Therefore the map $\tau|_{eSf} : eSf \to \tau(e)T\tau(f)$ is a posemigroup isomorphism by (LI1'). Since $aSb \subseteq eSf$, $\tau|_{aSb} = (\tau|_{eSf})|_{aSb} : aSb \to \tau(a)T\tau(b)$ is a surjective posemigroup embedding, i.e. a posemigroup isomorphism. □

Remark 2.43 Note that by Proposition 2.3.5 of [21], the requirement $e\mathcal{D}f$ in condition (LI3) can be rephrased as follows: there exist $x \in S$ and $x' \in V(x)$ such that $xx' = e$ and $x'x = f$. Also, (LI3) is obviously satisfied for a surjective homomorphism τ.

3 Strong Morita equivalence

This chapter is used to present our characterization theorem for strong Morita equivalence, make some immediate remarks and prove the equivalence of the first three conditions. Namely, we establish that if S and T are posemigroups with local units, then they are strongly Morita equivalent if and only if their Cauchy completions are Pos-equivalent if and only if S and T have a joint enlargement R. It turns out that only one of the implications used in the proof requires the existence of local units and the other steps can be completed in a more general setting (for factorizable or even arbitrary posemigroups). The chapter is based on [49].

3.1 Strong Morita equivalence

The lynchpin of all our subsequent work is the following result which establishes a number of equivalent descriptions of (strong) Morita equivalence.

Theorem 3.1 (cf. Thms 1.9 and 1.10) *Let S and T be posemigroups with local units. Then the following are equivalent:*

(1) *S and T are strongly Morita equivalent;*

(2) *S and T have a joint enlargement;*

(3) *the Pos-categories $C(S)$ and $C(T)$ are Pos-equivalent;*

(4) *there exist a consolidation q on the Pos-category $C(S)$ and a strict local isomorphism $C(S)^q \to T$;*

(5) *there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over S with $S = S\,\mathrm{Im}(M)S$ and a surjective strict local isomorphism $\mathcal{M} \to T$.*

Proof. We use Proposition 3.22 for (1) ⇒ (3), Theorem 3.17 for (3) ⇒ (2), Proposition 3.21 for (2) ⇒ (1), Proposition 4.4 for (2) ⇒ (4), Proposition 4.3 for (4) ⇒ (3) and Theorem 4.8 for (1) ⇔ (5). □

Remark 3.2 Due to Proposition 2.40, it is not necessary for the Morita context in condition (1) to be unitary.

Remark 3.3 By Theorem 4.25, there is a sixth equivalent condition for posemigroups with ordered local units:

(6) there exists a surjectively defined unitary Morita posemigroup $Q \otimes_S P$ and a surjective strict local isomorphism $Q \otimes_S P \to T$.

Remark 3.4 Using Theorem 5.14 and Corollary 5.18, we have another equivalent condition for posemigroups with common local units (cf. Theorem 1.7):

(7) the Pos-categories FPos_S and FPos_T are Pos-equivalent.

Corollary 3.5 (cf. Prop 4 of [25]) *Strong Morita equivalence is an equivalence relation on the class of posemigroups with local units.*

Proof. Because condition (3) clearly provides an equivalence relation between such posemigroups and it is equivalent to condition (1), the result follows. □

The following is an important observation about the structure of strongly Morita equivalent posemigroups, namely that they have the same local structure.

Corollary 3.6 (cf. Cor 6 of [25], Prop 5.1 (1) of [29]) *Let S and T be strongly Morita equivalent posemigroups with local units. Then each local subpomonoid of S is isomorphic to a local subpomonoid of T.*

Proof. By Theorem 3.1 (2), $C(S)$ and $C(T)$ are Pos-equivalent. Any local subpomonoid eSe of S is isomorphic to the pomonoid $C(S)(e, e)$, which is isomorphic to some pomonoid $C(T)(f, f)$, which in turn is isomorphic to the subpomonoid fTf of T for some $f \in E(T)$. □

3.2 From Cauchy completions to enlargements

In this section, we prove implication (3) ⇒ (2) of Theorem 3.1 and examine some related properties.

Lemma 3.7 (cf. Lemma 3.5 of [29]) *Let C be a small strongly connected Pos-category with a consolidation p. Then C^p is a posemigroup with local units. Moreover, if C is regular then C^p is regular.*

Proof. The same as in [29]. For $f : A \to B$, $f \diamond 1_A = f p_{A,A} 1_A = f 1_A = f$ and $1_B \diamond f = 1_B p_{B,B} f = 1_B f = f$. For $f : A \to B$ in C, if we have $f' : B \to A$ such that $f = ff'f$ and $f' = f'ff'$, then we also have $f \diamond f' \diamond f = f p_{A,A} f' p_{B,B} f = f$ and $f' \diamond f \diamond f' = f' p_{B,B} f p_{A,A} f' = f'$. □

Let C be a Pos-category. We say that C is *bipartite* with left part $\mathcal{A}$ and right part $\mathcal{B}$ (cf. [41]) and write $C = [\mathcal{A}, \mathcal{B}]$ if the following conditions hold:

- $\mathcal{A}$ and $\mathcal{B}$ are disjoint full Pos-subcategories of C and $\mathcal{A}_0 \cup \mathcal{B}_0 = C_0$;
- for every $A \in \mathcal{A}_0$ there is an isomorphism $f : A \to C$ with $C \in \mathcal{B}_0$, and for each $B \in \mathcal{B}_0$ there is an isomorphism $g : D \to B$ with $D \in \mathcal{A}_0$.

The Pos-category C is therefore made up of four kinds of morphisms: those of $\mathcal{A}$, those of $\mathcal{B}$, and those with either domain in $\mathcal{A}_0$ and codomain in $\mathcal{B}_0$ or domain in $\mathcal{B}_0$ and codomain in $\mathcal{A}_0$. The latter two are called *heteromorphisms*. We will denote the classes of such morphisms by $\mathcal{A}\mathcal{A}$, $\mathcal{B}\mathcal{B}$, $\mathcal{B}\mathcal{A}$ and $\mathcal{A}\mathcal{B}$, respectively. If $\mathcal{A}$ and $\mathcal{B}$ are strongly connected, then so is C.

Proposition 3.8 (cf. Thm 2.2 of [41]) *Two Pos-categories $\mathcal{A}$ and $\mathcal{B}$ are Pos-equivalent if and only if there exists a bipartite Pos-category C with left part isomorphic to $\mathcal{A}$ and right part isomorphic to $\mathcal{B}$.*

Proof. Assume that $\mathcal{A}$ and $\mathcal{B}$ are Pos-equivalent via Pos-equivalence $F : \mathcal{A} \to \mathcal{B}$. By [22], Section 1.11 a Pos-equivalence can be described as a fully faithful and essentially surjective (dense) Pos-functor. Take a skeleton $\overline{\mathcal{A}}$ of $\mathcal{A}$, let $\bar{F} : \overline{\mathcal{A}} \to \overline{\mathcal{B}}$ be the Pos-isomorphism obtained by restricting F to $\overline{\mathcal{A}}$ (cf. Corollary 4.15 of [3]), so $\overline{\mathcal{B}}$ is a skeleton of of $\mathcal{B}$, and fix an isomorphism $\xi_A : A \to A_0$ with $A_0 \in \overline{\mathcal{A}}_0$ for every $A \in \mathcal{A}_0$. Put $C_0 = \mathcal{A}_0 \coprod \mathcal{B}_0$. For any $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$, define heteromorphisms by taking

$$C(A, B) = \{\langle A, \alpha\rangle | \, \alpha \in \mathcal{B}(\bar{F}(A_0), B)\}$$

and

$$C(B, A) = \{\langle \alpha, A\rangle | \, \alpha \in \mathcal{B}(B, \bar{F}(A_0))\}.$$

These inherit morphism orders from $\mathcal{B}(\bar{F}(A_0), B)$ and $\mathcal{B}(B, \bar{F}(A_0))$, respectively, i.e. for $\langle A, \alpha\rangle, \langle A, \alpha'\rangle \in C(A, B)$ we set

$$\langle A, \alpha\rangle \leq \langle A, \alpha'\rangle \text{ in } C(A, B) \iff \alpha \leq \alpha' \text{ in } \mathcal{B}(\bar{F}(A_0), B).$$

The order on $C(B, A)$ is defined analogously. We compose as follows:

1. for $f : A \to A'$ in $\mathcal{A}$, $\begin{array}{l}\langle A', \alpha'\rangle \circ f = \langle A, \alpha' \bar{F}(\xi_{A'} f \xi_A^{-1})\rangle, \\ f \circ \langle \alpha, A\rangle = \langle \bar{F}(\xi_{A'} f \xi_A^{-1})\alpha, A'\rangle;\end{array}$
2. for $g : B \to B'$, $h : B' \to B$ in $\mathcal{B}$, $\begin{array}{l}g \circ \langle A, \alpha\rangle = \langle A, g\alpha\rangle, \\ \langle \alpha, A\rangle \circ h = \langle \alpha h, A\rangle;\end{array}$
3. $\langle A, \alpha'\rangle \circ \langle \alpha, A\rangle = \alpha'\alpha$;
4. $\langle \alpha, A\rangle \circ \langle A', \alpha'\rangle = \xi_A^{-1} \bar{F}^{-1}(\alpha\alpha')\xi_{A'}$;
5. the compositions on $\mathcal{A}$ and $\mathcal{B}$ remain the same.

We will see that the identity morphisms of A and B are still identities for the heteromorphisms. The cases for $C(A, B)$ and $C(B, A)$ are clearly symmetric. If we take $\langle A, \alpha\rangle \in C(A, B)$ then

$$\langle A, \alpha\rangle = \langle A, 1_B\alpha\rangle = 1_B \circ \langle A, \alpha\rangle, \tag{3.1}$$

$$\langle A, \alpha\rangle = \langle A, \alpha 1_{\bar{F}(A_0)}\rangle = \langle A, \alpha\bar{F}(1_{A_0})\rangle = \langle A, \alpha\bar{F}(\xi_A 1_A \xi_A^{-1})\rangle = \langle A, \alpha\rangle \circ 1_A. \tag{3.2}$$

The associativity of this composition is easy to check. For example, if we compose $\langle A, \alpha'\rangle : A \to B$, $g : B \to B'$ and $\langle \alpha, A'\rangle : B' \to A'$, then

$$\begin{aligned}(\langle \alpha, A'\rangle \circ g) \circ \langle A, \alpha'\rangle &= \langle \alpha g, A'\rangle \circ \langle A, \alpha'\rangle = \xi_{A'}^{-1}\bar{F}^{-1}(\alpha g\alpha')\xi_A \\ &= \langle \alpha, A'\rangle \circ \langle A, g\alpha'\rangle = \langle \alpha, A'\rangle \circ (g \circ \langle A, \alpha'\rangle).\end{aligned}$$

Since the compositions on $\mathcal{A}$ and $\mathcal{B}$ preserve order, as do the Pos-functors $\bar{F}$ and $\bar{F}^{-1}$, the composition is monotone, as required. For example, if we take $\langle A, \alpha\rangle, \langle A, \alpha'\rangle : A \to B$ such that $\langle A, \alpha\rangle \leq \langle A, \alpha'\rangle$, i.e. $\alpha \leq \alpha'$ in $\mathcal{B}$, and $f : A' \to A$ in $\mathcal{A}$, then

$$\langle A, \alpha\rangle \circ f = \langle A', \alpha\bar{F}(\xi_A f \xi_{A'}^{-1})\rangle \leq \langle A', \alpha'\bar{F}(\xi_A f \xi_{A'}^{-1})\rangle = \langle A, \alpha'\rangle \circ f.$$

Therefore if we take the disjoint union $\mathcal{A} \sqcup \mathcal{B}$ equipped with the above composition and order, it will be a Pos-category.

Moreover, $\mathcal{A} \sqcup \mathcal{B}$ is a bipartite Pos-category with left part isomorphic to $\mathcal{A}$ and right part isomorphic to $\mathcal{B}$. For objects $A \in \mathcal{A}_0$, the necessary isomorphisms are $\langle A, 1_{\bar{F}(A_0)}\rangle : A \to \bar{F}(A_0)$. If we take $B \in \mathcal{B}_0$, then due to essential surjectivity there is a $C \in \mathcal{A}_0$ such that $F(C) \cong B$. But $\bar{F}(C_0) = F(C_0) \cong F(C)$ as well. So there is an isomorphism $i : \bar{F}(C_0) \to B$. The second family of isomorphisms required for $\mathcal{A} \sqcup \mathcal{B}$ to be bipartite thus consists of morphisms $\langle i^{-1}, C\rangle : B \to C$.

For the converse, assume that we have a bipartite $C = [\mathcal{A}, \mathcal{B}]$. Define the functor $F : \mathcal{A} \to \mathcal{B}$ by fixing an isomorphism $\xi_A : A \to C_A$ with $C_A \in \mathcal{B}_0$ for each $A \in \mathcal{A}_0$ and taking $F(A) = C_A$. For $f : A \to A'$, we then put $F(f) = \xi_{A'} f \xi_A^{-1}$. Analogously, define $G : \mathcal{B} \to \mathcal{A}$ by fixing an isomorphism $\eta_B : B \to D_B$ with $D_B \in \mathcal{A}_0$ for all $B \in \mathcal{B}_0$, taking $G(B) = D_B$ and $G(g) = \eta_{B'} g \eta_B^{-1}$ for $g : B \to B'$ in $\mathcal{B}$. Such definitions clearly yield Pos-functors. Moreover, for each $B \in \mathcal{B}_0$ we have an isomorphism $\mu_B = \xi_{G(B)} \circ \eta_B : B \to (FG)(B)$ and similarly for all $A \in \mathcal{A}_0$ there is an isomorphism $\varepsilon_A = \eta_{F(A)} \circ \xi_A : A \to (GF)(A)$. Then

$$(FG)(g) = \xi_{G(B')}\eta_{B'} g \eta_B^{-1}\xi_{G(B)}^{-1} = \mu_{B'} g \mu_B^{-1}$$

and

$$(GF)(f) = \eta_{F(A')}\xi_{A'} f \xi_A^{-1}\eta_{F(A)}^{-1} = \varepsilon_{A'} f \varepsilon_A^{-1}$$

for every $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$. Therefore $\mu : 1_{\mathcal{B}} \cong FG$ and $\varepsilon : 1_{\mathcal{A}} \cong GF$. □

A consolidation r on a small bipartite Pos-category C induces consolidations on its parts $\mathcal{A}$ and $\mathcal{B}$, making $\mathcal{A}^r$ and $\mathcal{B}^r$ into subposemigroups of C^r.

Lemma 3.9 (cf. Lemma 3.7 of [29]) *Let* $C = [\mathcal{A}, \mathcal{B}]$ *be a small bipartite* Pos-*category with a consolidation* r. *Let* p *be the restriction of* r *to* $\mathcal{A}$ *and let* q *be the restriction of* r *to* $\mathcal{B}$. *Then* C^r *is an enlargement of both* $\mathcal{A}^p$ *and* $\mathcal{B}^q$.

Proof. Again the same as in [29]. We only need to prove this for, say, $\mathcal{A}$. So we have to show that $\mathcal{A}^p \diamond C^r \diamond \mathcal{A}^p = \mathcal{A}^p$ and $C^r \diamond \mathcal{A}^p \diamond C^r = C^r$. By Lemma 3.7, $\mathcal{A}^p$ has local units, so $\mathcal{A}^p \subseteq \mathcal{A}^p \diamond C^r \diamond \mathcal{A}^p$. For the converse, take morphisms $a, a' \in \mathcal{A}$ and $c \in C$. Then $a \diamond c \diamond a' = ar_{\mathrm{dom}(a),\mathrm{cod}(c)} c r_{\mathrm{dom}(c),\mathrm{cod}(a')} a'$ is in $\mathcal{A}$, since its domain and codomain are in $\mathcal{A}_0$ and $\mathcal{A}$ is a full Pos-subcategory of C. Therefore $\mathcal{A}^p \diamond C^r \diamond \mathcal{A}^p = \mathcal{A}^p$.

In any case, $C^r \diamond \mathcal{A}^p \diamond C^r \subseteq C^r$. Now suppose we have $c \in C$. There are four possibilities:

1. $c \in \mathcal{A}$, in which case there exist $e, f \in \mathcal{A}$ such that $c = e \diamond c \diamond f$.
2. $c \in \mathcal{B}$, so we can find isomorphisms $\alpha : A \to \mathrm{dom}(c)$ and $\beta : \mathrm{cod}(c) \to A'$ for some $A, A' \in \mathcal{A}_0$ because C is a bipartite Pos-category. Then we get that $c = \beta^{-1}(\beta c \alpha)\alpha^{-1} = \beta^{-1} \diamond (\beta c \alpha) \diamond \alpha^{-1}$ and $\beta c \alpha$ is in $\mathcal{A}$ since the latter is a full Pos-subcategory of C.
3. $A = \mathrm{dom}(c) \in \mathcal{A}_0$ and $\mathrm{cod}(c) \in \mathcal{B}_0$, whence $c = c1_A 1_A = c \diamond 1_A \diamond 1_A$.
4. $\mathrm{dom}(c) \in \mathcal{B}_0$ and $A = \mathrm{cod}(c) \in \mathcal{A}_0$, whence $c = 1_A 1_A c = 1_A \diamond 1_A \diamond c$.

Thus $C^r \subseteq C^r \diamond \mathcal{A}^p \diamond C^r$, as required. □

Lemma 3.10 (cf. Lemma 3.8 of [29]) *Let* C *be a* Pos-*category with the property that the idempotents of each endomorphism pomonoid generate a regular subposemigroup. Then the regular morphisms of* C *form a* Pos-*subcategory.*

Proof. Yet again, the same as in [29]. Let f and g be regular morphisms such that $\mathrm{dom}(f) = \mathrm{cod}(g)$, with inverses f' and g', respectively. Idempotents $f'f$ and gg' belong to the endomorphism pomonoid of $\mathrm{dom}(f)$. Let E be the poset of idempotents of the pomonoid $\mathrm{End}\,(\mathrm{dom} f)$. Because of our assumption, we can use Proposition 2.5.1 of [21] to see that the sandwich set

$$S(f'f, gg') = \{h \in V((f'f)(gg')) \cap E \,|\, h(f'f) = (gg')h = h\}$$

is not empty. Take $h \in S(f'f, gg')$ and consider the element $g'hf'$. Then fg is regular because

$$fg(g'hf')fg = f(gg')h(f'f)g = f(f'f)(gg')h(f'f)(gg')g = f(f'f)(gg')g = fg.$$

□

Since isomorphisms are regular, we can use Lemma 3.10 to drop heteromorphisms which are not regular and get the following:

Corollary 3.11 (cf. Cor 3.9 of [29]) *If $\mathcal{A}$ and $\mathcal{B}$ are two regular* Pos*-equivalent* Pos*-categories then there exists a regular bipartite* Pos*-category C with left part isomorphic to $\mathcal{A}$ and right part isomorphic to $\mathcal{B}$.*

Let $C = [\mathcal{A}, \mathcal{B}]$ be a small bipartite Pos-category with strongly connected parts $\mathcal{A}$ and $\mathcal{B}$. Let p be a consolidation on $\mathcal{A}$ and q be a consolidation on $\mathcal{B}$. Fix an isomorphism $\xi : B_0 \leftarrow A_0 \in \mathcal{B}\mathcal{A}$ and define a consolidation r on C as follows:

$$r_{Y,X} = \begin{cases} p_{Y,X} & \text{if } X, Y \in \mathcal{A}_0, \\ q_{Y,X} & \text{if } X, Y \in \mathcal{B}_0, \\ q_{Y,B_0}\xi p_{A_0,X} & \text{if } X \in \mathcal{A}_0, Y \in \mathcal{B}_0, \\ p_{Y,A_0}\xi^{-1}q_{B_0,X} & \text{if } X \in \mathcal{B}_0, Y \in \mathcal{A}_0. \end{cases}$$

Then r is said to be the *natural extension of p and q to C via ξ.*

Lemma 3.12 (cf. Lemma 3.10 of [29]) *Let $C = [\mathcal{A}, \mathcal{B}]$ and r be as above.*

(1) *For $f : A' \leftarrow A \in \mathcal{A}\mathcal{A}$ and $g : D' \leftarrow D \in \mathcal{B}\mathcal{B}$ we have $f \diamond \xi^{-1} \diamond g = f \diamond g$.*

(2) *For $f : B' \leftarrow B \in \mathcal{B}\mathcal{B}$ and $g : C' \leftarrow C \in \mathcal{A}\mathcal{A}$ we have $f \diamond \xi \diamond g = f \diamond g$.*

(3) *For $f : C \leftarrow D \in \mathcal{A}\mathcal{B}$ and $g : A' \leftarrow A \in \mathcal{A}\mathcal{A}$ we have $f \diamond \xi \diamond \xi^{-1} \diamond g = f \diamond g$.*

(4) *For $f : A' \leftarrow A \in \mathcal{A}\mathcal{A}$ and $g : D \leftarrow C \in \mathcal{B}\mathcal{A}$ we have $f \diamond \xi \diamond \xi^{-1} \diamond g = f \diamond g$.*

Proof. Simple calculations show that

$$\begin{array}{lrcl}
(1): & f \diamond \xi^{-1} \diamond g & = & f r_{A,A_0} \xi^{-1} r_{B_0,D'} g \\
 & & = & f p_{A,A_0} \xi^{-1} q_{B_0,D'} g = f r_{A,D'} g = f \diamond g, \\
(2): & f \diamond \xi \diamond g & = & f r_{B,B_0} \xi^{-1} r_{A_0,C'} g \\
 & & = & f q_{B,B_0} \xi^{-1} p_{A_0,C'} g = f r_{B,C'} g = f \diamond g, \\
(3): & f \diamond \xi^{-1} \diamond \xi \diamond g & = & f r_{D,A_0} \xi^{-1} r_{B_0,B_0} \xi r_{A_0,A'} g = f r_{D,A_0} p_{A_0,A'} g \\
 & & = & f q_{D,B_0} \xi p_{A_0,A_0} p_{A_0,A'} g = f r_{D,A'} g = f \diamond g, \\
(4): & f \diamond \xi \diamond \xi^{-1} \diamond g & = & f r_{A,B_0} \xi r_{A_0,A_0} \xi^{-1} r_{B_0,D} g = f r_{A,B_0} q_{B_0,D} g \\
 & & = & f p_{A,A_0} \xi^{-1} q_{B_0,B_0} q_{B_0,D} g = f r_{A,D} g = f \diamond g.
\end{array}$$
□

Lemma 3.13 *Let H be a binary relation on a posemigroup S and let ρ be the admissible preorder generated by H. Then $s \,\rho\, t$ iff there exist $n \in \mathbb{N}$ and for $1 \leq i \leq n$ elements $s_i, t_i, u_i, v_i \in S$, $x_i, y_i \in S^1$ such that $s = s_1$, $s_i \leq t_i$, $t_n = t$, and if $1 \leq i < n$ then $t_i = x_i u_i y_i$, $s_{i+1} = x_i v_i y_i$ and $(u_i, v_i) \in H$.*

Proof. Take s and t such that there are $n \in \mathbb{N}$ and elements $s_i, t_i, u_i, v_i, x_i, y_i \in S$ for $1 \leq i \leq n$ such that $s = s_1$, $s_i \leq t_i$, $t_n = t$, $(u_i, v_i) \in H$ and if $1 \leq i < n$ then $t_i = x_i u_i y_i$, $s_{i+1} = x_i v_i y_i$. Then we have for $u, v \in S$ that

$$usv = us_1v \leq ut_1v = (ux_1)u_1(y_1v), (ux_1)v_1(y_1v) = us_2v \leq ut_2v, \ldots, ut_nv = utv,$$

so the above relation is compatible with multiplication. It clearly includes $\leq$ and is therefore reflexive as well. A standard argument shows that it is also transitive, making it into an admissible preorder.

Take s and t as above. Since ρ is an admissible preorder containing H, we have $u_i \,\rho\, v_i$, implying $t_i \,\rho\, s_{i+1}$ due to compatibility. Also, $t_i \,\rho\, s_i$ since ρ extends $\leq$. But ρ is transitive, so $s \,\rho\, t$. Thus the above relation is an admissible preorder extending H that is contained in ρ, so it must be ρ. □

Proposition 3.14 (cf. Prop 3.11 of [29]) *Let $C = [\mathcal{A}, \mathcal{B}]$ be a small bipartite* Pos*-category with strongly connected parts $\mathcal{A}$ and $\mathcal{B}$. Let p be a consolidation on $\mathcal{A}$ and q be a consolidation on $\mathcal{B}$. Take an isomorphism $\xi \in \mathcal{B}\mathcal{A}$ and let r be the natural extension of p and q to C via ξ.*

Let ρ_1 be an admissible preorder on the posemigroup $\mathcal{A}^p$ and let ρ_2 be an admissible preorder on the posemigroup $\mathcal{B}^q$. Let ρ be the admissible preorder on the posemigroup C^r generated by $\rho_1 \cup \rho_2$. Then

(i) $\rho \cap (\mathcal{A} \times \mathcal{A}) = \rho_1$ *if*

  (1) $a \,\rho_1\, a' \Rightarrow [(\xi^{-1} \diamond a) \,\rho_1\, (\xi^{-1} \diamond a')] \wedge [(a \diamond \xi) \,\rho_1\, (a' \diamond \xi)]$,

  (2) *for all isomorphisms $\alpha \in \mathcal{A}\mathcal{B}$ and $\beta \in \mathcal{B}\mathcal{A}$, having $b\rho_2 b'$ implies $\alpha \diamond b \diamond \beta \leq \alpha \diamond b' \diamond \beta$;*

(ii) $\rho \cap (\mathcal{B} \times \mathcal{B}) = \rho_2$ *if*

  (1) $b \,\rho_2\, b' \Rightarrow [(\xi \diamond b) \,\rho_2\, (\xi \diamond b')] \wedge [(b \diamond \xi^{-1}) \,\rho_2\, (b' \diamond \xi^{-1})]$,

  (2) *for all isomorphisms $\alpha \in \mathcal{B}\mathcal{A}$ and $\beta \in \mathcal{A}\mathcal{B}$, having $a\rho_1 a'$ implies $\alpha \diamond a \diamond \beta \leq \alpha \diamond a' \diamond \beta$.*

Proof. We will only show that $\rho \cap (\mathcal{A} \times \mathcal{A}) = \rho_1$, the proof that $\rho \cap (\mathcal{B} \times \mathcal{B}) = \rho_2$ is symmetric.

Obviously $\rho_1 \subseteq \rho \cap (\mathcal{A} \times \mathcal{A})$. Take $a, a' \in \mathcal{A}$ such that $a \,\rho\, a'$. We need to show that $a \,\rho_1\, a'$. By Lemma 3.13, there exist $n \in \mathbb{N}$ and for $1 \leq i \leq n$ morphisms $z_i, w_i \in C$ such that $a = z_1$, $z_i \leq w_i$ in the posemigroup C^r and $w_n = a'$. Furthermore, for $1 \leq i < n$ we have $w_i = x_i \diamond u_i \diamond y_i$, $z_{i+1} = x_i \diamond v_i \diamond y_i$ for some $(u_i, v_i) \in \rho_1 \cup \rho_2$, $u_i, v_i \in C$, $x_i, y_i \in (C^r)^1$, where 1 is the externally adjoined identity. As the order on the posemigroup C^r is the morphism-poset order of the Pos-category C, two morphisms can only be comparable if they share domains and codomains. Because $a = z_1 \in \mathcal{A}$ and $z_1 \leq w_1$, we also have $w_1 \in \mathcal{A}$. But then $x_1 \in \mathcal{A}* \cup \{1\}$ and $y_1 \in *\mathcal{A} \cup \{1\}$, where $*$ is either $\mathcal{A}$ or $\mathcal{B}$. If $x_1 = 1$ or $y_1 = 1$, then $u_1, v_1 \in \mathcal{A}*$ or $u_1, v_1 \in *\mathcal{A}$, respectively, whence $u_1, v_1 \in \mathcal{A}$ because $(u_1, v_1) \in \rho_1 \cup \rho_2$. In any case, because the domain and codomain of z_2 are in $\mathcal{A}$ which is a full Pos-subcategory of C, we must also have $z_2 \in \mathcal{A}$. This argument can be repeated to get all $x_i \in \mathcal{A}* \cup \{1\}$, $y_i \in *\mathcal{A} \cup \{1\}$ and $z_i, w_i \in \mathcal{A}$.

We will show that actually $w_i \,\rho_1\, z_{i+1}$ for all $1 \leq i \leq n-1$. There are two possibilities: either $u_i \,\rho_1\, v_i$ or $u_i \,\rho_2\, v_i$.

First, suppose that $u_i \,\rho_1\, v_i$. Note that then $u_i, v_i \in \mathcal{A}$. As $x_i \in \mathcal{A}{*} \cup \{1\}$ and $y_i \in {*}\mathcal{A} \cup \{1\}$, we have the following possibilities for x_i and y_i:

(1) $x_i \in \mathcal{A}\mathcal{A} \cup \{1\}$ and $y_i \in \mathcal{A}\mathcal{A} \cup \{1\}$,

(2) $x_i \in \mathcal{A}\mathcal{B} \cup \{1\}$ and $y_i \in \mathcal{A}\mathcal{A} \cup \{1\}$,

(3) $x_i \in \mathcal{A}\mathcal{A} \cup \{1\}$ and $y_i \in \mathcal{B}\mathcal{A} \cup \{1\}$,

(4) $x_i \in \mathcal{A}\mathcal{B} \cup \{1\}$ and $y_i \in \mathcal{B}\mathcal{A} \cup \{1\}$.

We will examine each of those possibilities in turn.

(1) Since ρ_1 is an admissible preorder on $\mathcal{A}^p$ and $x_i, y_i \in \mathcal{A} \cup \{1\}$, multiplying yields

$$w_i = x_i \diamond u_i \diamond y_i \;\; \rho_1 \;\; x_i \diamond v_i \diamond y_i = z_{i+1}.$$

(2) Because $y_i \in \mathcal{A} \cup \{1\}$, we again get that $u_i \diamond y_i \,\rho_1\, v_i \diamond y_i$. If $x_i = 1$, then we already have that $w_i \,\rho_1\, z_{i+1}$. Otherwise, due to assumption (i)(1), we have $\xi^{-1} \diamond u_i \diamond y_i \,\rho_1\, \xi^{-1} \diamond v_i \diamond y_i$. As $\mathcal{A}$ is a full Pos-subcategory, $x_i \diamond \xi \in \mathcal{A}$, so

$$(x_i \diamond \xi \diamond \xi^{-1} \diamond u_i \diamond y_i) \,\rho_1\, (x_i \diamond \xi \diamond \xi^{-1} \diamond v_i \diamond y_i).$$

Due to Lemma 3.12 (3), this is equivalent to $w_i \,\rho_1\, z_{i+1}$.

(3) Symmetric, using Lemma 3.12 (4) and the other half of condition (i)(1).

(4) If $x_i = 1$ or $y_i = 1$, the proof reduces to the latter parts of cases (3) and (2), respectively. So we can assume that $x_i \neq 1$ and $y_i \neq 1$. Because of our assumption (i)(1), we have that $\xi^{-1} \diamond u_i \diamond \xi \,\rho_1\, \xi^{-1} \diamond v_i \diamond \xi$. Again using the fact that $\mathcal{A}$ is a full Pos-subcategory of C, we get $x_i \diamond \xi, \xi^{-1} \diamond y_1 \in \mathcal{A}$. Thus

$$(x_i \diamond \xi \diamond (\xi^{-1} \diamond u_i \diamond \xi) \diamond \xi^{-1} \diamond y_i) \,\rho_1\, (x_i \diamond \xi \diamond (\xi^{-1} \diamond v_i \diamond \xi) \diamond \xi^{-1} \diamond y_i).$$

By associativity and Lemma 3.12 (3), (4), this again reduces to $w_i \,\rho_1\, z_{i+1}$.

Now suppose that $u_i \,\rho_2\, v_i$. If $x_i = 1$ or $y_i = 1$, then we have already seen that $u_i, v_i \in \mathcal{A}$, which is impossible. Therefore we have the following possibilities for x_i and y_i:

(1) $x_i \in \mathcal{A}\mathcal{A}$ and $y_i \in \mathcal{A}\mathcal{A}$,

(2) $x_i \in \mathcal{A}\mathcal{B}$ and $y_i \in \mathcal{A}\mathcal{A}$,

(3) $x_i \in \mathcal{A}\mathcal{A}$ and $y_i \in \mathcal{B}\mathcal{A}$,

(4) $x_i \in \mathcal{A}\mathcal{B}$ and $y_i \in \mathcal{B}\mathcal{A}$.

We will look at each of them in turn.

(1) Let $x_i, y_i \in \mathcal{A}$. By our assumption in (i)(2), we have $\xi^{-1} \diamond u_i \diamond \xi \leq \xi^{-1} \diamond v_i \diamond \xi$. Therefore in C we get

$$x_i \diamond \xi^{-1} \diamond u_i \diamond \xi \diamond y_i \leq x_i \diamond \xi^{-1} \diamond v_i \diamond \xi \diamond y_i.$$

By Lemma 3.12 (1) and (2) the above simplifies to $w_i \leq z_{i+1}$ in $\mathcal{A}$.

(2) Let $x_i \in \mathcal{AB}$ and $y_i \in \mathcal{A}$. Since C is bipartite, there exists an isomorphism $\alpha \in \mathcal{AB}$ such that $\mathrm{dom}(x_i) = \mathrm{dom}(\alpha)$. By (i)(2) we have $\alpha \diamond u_i \diamond \xi \leq \alpha \diamond v_i \diamond \xi$. So $\alpha \diamond u_i \diamond \xi \diamond y_i \leq \alpha \diamond v_i \diamond \xi \diamond y_i$. By Lemma 3.12 (2), this simplifies to $\alpha \diamond u_i \diamond y_i \leq \alpha \diamond v_i \diamond y_i$. Consequently, we have in C that

$$x_i \diamond \alpha^{-1} \diamond \alpha \diamond u_i \diamond y_i \leq x_i \diamond \alpha^{-1} \diamond \alpha \diamond v_i \diamond y_i.$$

Because $x_i \diamond \alpha^{-1} \diamond \alpha = x_i \circ \alpha^{-1} \circ \alpha = x_i$ by Remark 2.31, this implies $w_i \leq z_{i+1}$ in $\mathcal{A}$.

(3) Symmetric to (2) and uses Lemma 3.12 (1) instead.

(4) Let $x_i \in \mathcal{AB}$ and $y_i \in \mathcal{BA}$. Since C is bipartite, there exist an isomorphism $\alpha \in \mathcal{AB}$ such that $\mathrm{dom}(x_i) = \mathrm{dom}(\alpha)$ and an isomorphism $\beta \in \mathcal{BA}$ such that $\mathrm{cod}(y_i) = \mathrm{cod}(\beta)$. Then by (i)(2) we have $\alpha \diamond u_i \diamond \beta \leq \alpha \diamond v_i \diamond \beta$. Therefore

$$x_i \diamond \alpha^{-1} \diamond \alpha \diamond u_i \diamond \beta \diamond \beta^{-1} \diamond y_i \leq x_i \diamond \alpha^{-1} \diamond \alpha \diamond v_i \diamond \beta \diamond \beta^{-1} \diamond y_i.$$

As before, this simplifies to $w_i \leq z_{i+1}$ in $\mathcal{A}$.

Since ρ_1 must extend the order on $\mathcal{A}^p$, all four cases imply $w_i \, \rho_1 \, z_{i+1}$. For the same reason, $z_i \, \rho_1 \, w_i$ and thus $a = z_1 \, \rho_1 \, w_n = a'$ by transitivity of ρ_1. Therefore we have demonstrated that if $a \, \rho \, a'$, then actually $a \, \rho_1 \, a'$. □

Proposition 3.15 *Let* $\tau : S \to T$ *be a surjective posemigroup homomorphism. If* S *is an enlargement of a subposemigroup* $R \subseteq S$*, then* T *is an enlargement of* $\tau(R)$*.*

Proof. We will show that $T \subseteq T\tau(R)T$. The converse and the two inclusions of $\tau(R)T\tau(R) = \tau(R)$ either hold in general or are proved in the same way. Take $t \in T$. As τ is surjective, there exists an element $s \in S$ such that $\tau(s) = t$. Because S is an enlargement of R, there also exist $s', s'' \in S$ and $r \in R$ such that $s = s'rs''$. But then

$$t = \tau(s) = \tau(s'rs'') = \tau(s')\tau(r)\tau(s''). \qquad \square$$

Proposition 3.16 (cf. Prop 3.12 of [29]) *Let* $C = [\mathcal{A}, \mathcal{B}]$ *be a small bipartite* Pos-*category with strongly connected parts* $\mathcal{A}$ *and* $\mathcal{B}$*. Let* p *be a consolidation on* $\mathcal{A}$ *and* q *be a consolidation on* $\mathcal{B}$*. Take an isomorphism* $\xi \in \mathcal{B}\mathcal{A}$ *and let* r *be the natural extension of* p *and* q *to* C *via* ξ*. Let* S *be a homomorphic image of the posemigroup* $\mathcal{A}^p$ *by a morphism with directed kernel* ρ_1 *and let* T *be a homomorphic image of the posemigroup* $\mathcal{B}^q$ *by a morphism with directed kernel* ρ_2*. Let* ρ *be the admissible preorder on the posemigroup* C^r *generated by* $\rho_1 \cup \rho_2$ *and let conditions (i) and (ii) from Proposition 3.14 hold. Take* $R = C^r/\rho$*. Then* R *is a joint enlargement of* S *and* T*.*

Proof. By Lemma 3.9, C^r is a joint enlargement of its subposemigroups $\mathcal{A}^p$ and $\mathcal{B}^q$. According to Proposition 3.15, $R = C^r/\rho$ is a joint enlargement of its subposemigroups $\mathcal{A}^p/\rho = \{[a]_{\rho\cap\rho^{-1}} | a \in \mathcal{A}\}$ and $\mathcal{B}^q/\rho = \{[b]_{\rho\cap\rho^{-1}} | b \in \mathcal{B}\}$. Due to Proposition 3.14, $\rho \cap (\mathcal{A} \times \mathcal{A}) = \rho_1$ and $\rho \cap (\mathcal{B} \times \mathcal{B}) = \rho_2$. This implies that the correspondence $[a]_{\rho\cap\rho^{-1}} \longleftrightarrow [a]_{\rho_1\cap\rho_1^{-1}}$ is a posemigroup isomorphism, since

$$[a]_{\rho\cap\rho^{-1}} \leq [a']_{\rho\cap\rho^{-1}} \Leftrightarrow a\rho a' \Leftrightarrow a\rho_1 a' \Leftrightarrow [a]_{\rho_1\cap\rho_1^{-1}} \leq [a']_{\rho_1\cap\rho_1^{-1}}$$

for $a, a' \in \mathcal{A}$. Since the same can be said for $\mathcal{B}^q$, we have posemigroup isomorphisms $\mathcal{A}^p/\rho \cong \mathcal{A}^p/\rho_1$ and $\mathcal{B}^q/\rho \cong \mathcal{B}^q/\rho_2$. By Theorem 2.20, $S \cong \mathcal{A}^p/\rho_1$ and $T \cong \mathcal{B}^q/\rho_2$, so R is a joint enlargement of (isomorphic copies of) S and T. □

Theorem 3.17 (cf. Thm 3.13 of [29]) *Let* S *and* T *be posemigroups with local units. If their Cauchy completions* $C(S)$ *and* $C(T)$ *are* Pos-*equivalent, then* S *and* T *have a joint enlargement* R*. Moreover, if* S *and* T *are both regular, then* R *can be chosen to be regular. Furthermore, if both* S *and* T *have common local units, then* R *also has common local units.*

Proof. The overall scheme is the same as in [29]. Let $C(S)$ and $C(T)$ be Pos-equivalent. Due to Proposition 3.8, there exists a small bipartite Pos-category $C = [C(S), C(T)]$. As Cauchy completions are strongly connected (for idempotents e and f, $eef = ef = eff$, so $(e, ef, f) : f \to e$), C is also strongly connected. Define consolidations p on $C(S)$ and q on $C(T)$ by taking

$$p_{e,f} = (e, ef, f) \quad \text{and} \quad q_{i,j} = (i, ij, j)$$

for all idempotents $e, f \in S$ and all idempotents $i, j \in T$. Furthermore, define two maps $\Pi_1 : C(S)^p \to S$ and $\Pi_2 : C(T)^q \to T$ by

$$\Pi_1((e, s, f)) = s \quad \text{and} \quad \Pi_2((i, t, j)) = t.$$

These are clearly surjective and monotone. Moreover,

$$\begin{aligned}
\Pi_1((e, s, f) \diamond (g, s', h)) &= \Pi_1((e, s, f)p_{f,g}(g, s', h)) \\
&= \Pi_1((e, s, f)(f, fg, g)(g, s', h)) = \Pi_1((e, sfgs', h)) \\
&= \Pi_1((e, ss', h)) = ss' = \Pi_1((e, s, f))\Pi_1((g, s', h)).
\end{aligned}$$

In the same way $\Pi_2((i,t,j) \diamond (k,t',l)) = \Pi_2((i,t,j))\Pi_2((k,t',l))$.

Since C is bipartite, we can take an isomorphism $\xi : e_0 \to i_0 \in C(T)C(S)$ and define the natural extension r of p and q to C via ξ. Let $\rho_i = \overrightarrow{\mathrm{Ker}}\,\Pi_i$, i.e.

$$\begin{aligned} (e,s,f)\ \rho_1\ (e',s',f') &\iff s \leq s' \text{ in } S, \\ (i,t,j)\ \rho_2\ (i',t',j') &\iff t \leq t' \text{ in } T, \end{aligned}$$

for $e,e',f,f' \in E(S)$, $s,s' \in S$, $i,i',j,j' \in E(T)$, $t,t' \in T$. By Theorem 2.20, we have posemigroup isomorphisms $C(S)^p/\rho_1 \cong S$ and $C(T)^q/\rho_2 \cong T$, since Π_1 and Π_2 are surjective posemigroup homomorphisms. Now take ρ as the admissible preorder generated by $\rho_1 \cup \rho_2$.

We show that the assumptions of conditions (i) and (ii) of Proposition 3.14 are fulfilled. We only need to show (i), as (ii) follows by symmetry.

To verify (i)(1), take $(e,s,f)\rho_1(e',s',f')$, whence $s \leq s'$. Then

$$\begin{aligned} \xi^{-1} \diamond (e,s,f) &= \xi^{-1} r_{i_0,e}(e,s,f) = \xi^{-1} q_{i_0,i_0} \xi p_{e_0,e}(e,s,f) \\ &= (e_0, e_0 e, e)(e,s,f) = (e_0, e_0 s, f). \end{aligned}$$

Similar calculations yield $\xi^{-1} \diamond (e',s',f') = (e_0, e_0 s', f')$, $(e,s,f) \diamond \xi = (e, s e_0, e_0)$ and $(e',s',f') \diamond \xi = (e', s' e_0, e_0)$. Since $e_0 s \leq e_0 s'$ and $s e_0 \leq s' e_0$, we get that $[\xi^{-1} \diamond (e,s,f)]\ \rho_1\ [\xi^{-1} \diamond (e',s',f')]$ and $[(e,s,f) \diamond \xi]\ \rho_1\ [(e',s',f') \diamond \xi]$, as required.

For (i)(2), take $(i,t,j)\rho_2(i',t',j')$, in which case $t \leq t'$ (and thus $itj \leq i't'j'$). Now let $\alpha : e_1 \leftarrow i_1 \in C(S)C(T)$ and $\beta : i_2 \leftarrow e_2 \in C(T)C(S)$ be isomorphisms in C. Then

$$\begin{aligned} &\alpha \diamond (i,t,j) \diamond \beta = \alpha r_{i_1,i}(i,t,j) r_{j,i_2} \beta = \alpha q_{i_1,i}(i,t,j) q_{j,i_2} \beta \\ = \ &\alpha(i_1, i_1 i, i)(i,t,j)(j, j i_2, i_2)\beta = \alpha(i_1, i_1 itj i_2, i_2)\beta \\ \leq \ &\alpha(i_1, i_1 i't'j' i_2, i_2)\beta = \alpha(i_1, i_1 i', i')(i',t',j')(j', j' i_2, i_2)\beta \\ = \ &\alpha q_{i_1,i'}(i',t',j') q_{j',i_2}\beta = \alpha r_{i_1,i'}(i',t',j') r_{j',i_2}\beta = \alpha \diamond (i',t',j') \diamond \beta. \end{aligned}$$

By Proposition 3.16, $R = C^r/\rho$ is a joint enlargement of S and T via the isomorphic copies $C(S)^r/\rho \cong C(S)^p/\rho_1 \cong S$ and $C(T)^r/\rho \cong C(T)^q/\rho_2 \cong T$. If both $C(S)$ and $C(T)$ were regular, then C could also be taken to be regular by Corollary 3.11. Then C^r would be regular by Lemma 3.7 and therefore its homomorphic image R would be regular as well.

Let S and T have common local units and take comparable $[c]_{\rho\cap\rho^{-1}} \leq [c']_{\rho\cap\rho^{-1}}$ in $R = C^r/\rho$, i.e. $c\,\rho\,c'$. As ρ is generated by ρ_1 and ρ_2, we can use Lemma 3.13 to find $n \in \mathbb{N}$ and for $1 \leq i \leq n$ elements $c_i, c_i', u_i, v_i \in C^r$, $x_i, y_i \in (C^r)^1$ such that $c = c_1$, $c_i \leq c_i'$, $c_n' = c'$, and if $1 \leq i < n$ then

$$c_i' = x_i \diamond u_i \diamond y_i, \quad c_{i+1} = x_i \diamond v_i \diamond y_i \quad \text{and} \quad (u_i, v_i) \in \rho_1 \cup \rho_2.$$

We have four possibilities:

(1) $c \in C(S)$,

(2) $c \in C(T)$,

(3) $c \in C(T)C(S)$,

(4) $c \in C(S)C(T)$.

It is sufficient to consider only cases (1) and (3). If $c \in C(S)$, then $c'_1 \in C(S)$ as well, because the order on C^r is derived from the morphism-poset order of C. Now there are four subcases:

(a) $x_1 \neq 1, y_1 \neq 1$;

(b) $x_1 = 1, y_1 \neq 1$;

(c) $x_1 \neq 1, y_1 = 1$;

(d) $x_1 = 1, y_1 = 1$.

For (a), we have $x_1 \in C(S)*$ and $y_1 \in *C(S)$, where $* \in \{C(S), C(T)\}$, so we get $c_2 = x_1 \diamond v_1 \diamond y_1 \in C(S)$. In subcase (b), we have $u_1 \in C(S)*$, $y_1 \in *C(S)$, whence $u_1 \rho_1 v_1$ in $C(S)$ and consequently $c_2 = v_1 \diamond y_1 \in C(S)$. Subcases (c) and (d) similarly yield $u_1 \rho_1 v_1$ in $C(S)$ (and $x_1 \in C(S)*$ in subcase (c)), so $c_2 \in C(S)$ again. These two steps can be repeated to get $c_1, c'_1, c_2, \ldots, c_n, c'_n = c' \in C(S)$.

For case (3), we have the same subcases (a)-(d) and can again perform a similar analysis to get $c'_1 \in C(T)C(S)$, $x_1 \in C(T)* \cup \{1\}$, $y_1 \in *C(S) \cup \{1\}$. Once more we have $u_1 \rho_j v_1$ for subcases (c) (with $j = 1$) and (b) (with $j = 2$). Subcase (d) never arises. Thus $c_2 \in C(T)C(S)$ as well and we can again repeat the two steps until we have $c_1, c'_1, c_2, \ldots, c_n, c'_n = c' \in C(T)C(S)$.

Since we can still apply Proposition 3.14 and get that $\rho \cap (C(S) \times C(S)) = \rho_1$, case (1) reduces to $c \,\rho_1\, c'$, i.e. the relation $[c]_{\rho\cap\rho^{-1}} \leq [c']_{\rho\cap\rho^{-1}}$ in the subposemigroup $C^p/\rho_1 \cong S$. Since S has common local units, this concludes case (1).

For case (3), we have already shown that $c = c_1, c' = c'_n \in C(T)C(S)$, i.e. they are heteromorphisms. Thus there are elements $e, e' \in E(S)$ and $j, j' \in E(T)$ such that $c \in C(e, j)$ and $c' \in C(e', j')$. Therefore we can write $c = \langle e, (j, t, i) \rangle$ and $c' = \langle e', (j', t', i') \rangle$, where $(j, t, i) \in C(T)(i, j)$, $(j', t', i') \in C(T)(i', j')$ and $i = \bar{F}(e_0)$, $i' = \bar{F}(e'_0)$ in the notation of Proposition 3.8. As S and T have common local units, there are $e'' \in E(S)$ and $j'' \in E(T)$ such that $j''t = t$, $j''t' = t'$, $ee'' = e$ and $e'e'' = e'$. Then we have $(j'', t, i) \in C(T)(i, j'')$ and $(j'', t', i') \in C(T)(i', j'')$, implying that $\langle e, (j'', t, i) \rangle \in C(e, j'')$ and $\langle e', (j'', t', i') \rangle \in C(e', j'')$. Moreover, clearly $(e, e, e'') \in C(S)(e'', e)$ and $(e', e', e'') \in C(S)(e'', e')$, so we also have $\xi_e(e, e, e'')\xi_{e''}^{-1} \in C(S)(e''_0, e_0)$ and $\xi_{e'}(e', e', e'')\xi_{e''}^{-1} \in C(S)(e''_0, e'_0)$. Denoting $\bar{F}(e''_0) = i''$, there exist elements $t'' \in iTi''$ and $t''' \in i'Ti''$ such that we can write $\bar{F}(\xi_e(e, e, e'')\xi_{e''}^{-1}) = (i, t'', i'')$ and $\bar{F}(\xi_{e'}(e', e', e'')\xi_{e''}^{-1}) = (i', t''', i'')$.

Observe that we have relations $(j,t,i)\,\rho\cap\rho^{-1}\,(j'',t,i)$, $(j',t',i')\,\rho\cap\rho^{-1}\,(j'',t',i')$, $(e,e,e)\,\rho\cap\rho^{-1}\,(e,e,e'')$ and $(e',e',e')\,\rho\cap\rho^{-1}\,(e',e',e'')$. So

$$\begin{array}{rcl} c & = & \langle e,(j,t,i)\rangle = (j,t,i)\diamond\langle e,(i,i,i)\rangle\diamond(e,e,e) \\ & \rho\cap\rho^{-1} & (j'',t,i)\diamond\langle e,(i,i,i)\rangle\diamond(e,e,e'') = \langle e,(j'',t,i)\rangle\diamond(e,e,e'') \\ & = & \langle e'',(j'',t,i)\bar{F}(\xi_e(e,e,e'')\xi_{e''}^{-1})\rangle = \langle e'',(j'',t,i)(i,t'',i'')\rangle \\ & = & \langle e'',(j'',tt'',i'')\rangle \end{array}$$

and

$$\begin{array}{rcl} c' & = & \langle e',(j',t',i')\rangle = (j',t',i')\diamond\langle e',(i',i',i')\rangle\diamond(e',e',e') \\ & \rho\cap\rho^{-1} & (j'',t',i')\diamond\langle e',(i',i',i')\rangle\diamond(e',e',e'') = \langle e',(j'',t',i')\rangle\diamond(e',e',e'') \\ & = & \langle e'',(j'',t',i')\bar{F}(\xi_{e'}(e',e',e'')\xi_{e''}^{-1})\rangle = \langle e'',(j'',t',i')(i',t''',i'')\rangle \\ & = & \langle e'',(j'',t't''',i'')\rangle. \end{array}$$

Therefore

$$c\,\rho\cap\rho^{-1}\,\langle e'',(j'',tt'',i'')\rangle \text{ and } c'\,\rho\cap\rho^{-1}\,\langle e'',(j'',t't''',i'')\rangle$$

in C^r. Equations (3.2) and (3.1) yield that

$$\begin{array}{rcl} [c]_{\rho\cap\rho^{-1}}[(e'',e'',e'')]_{\rho\cap\rho^{-1}} & = & [\langle e'',(j'',tt'',i'')\rangle\diamond(e'',e'',e'')]_{\rho\cap\rho^{-1}} \\ & = & [\langle e'',(j'',tt'',i'')\rangle]_{\rho\cap\rho^{-1}} = [c]_{\rho\cap\rho^{-1}} \end{array}$$

and

$$\begin{array}{rcl} [(j'',j'',j'')]_{\rho\cap\rho^{-1}}[c]_{\rho\cap\rho^{-1}} & = & [(j'',j'',j'')\diamond\langle e'',(j'',tt'',i'')\rangle]_{\rho\cap\rho^{-1}} \\ & = & [\langle e'',(j'',tt'',i'')\rangle]_{\rho\cap\rho^{-1}} = [c]_{\rho\cap\rho^{-1}}. \end{array}$$

Similarly

$$[c']_{\rho\cap\rho^{-1}}[(e'',e'',e'')]_{\rho\cap\rho^{-1}} = [c']_{\rho\cap\rho^{-1}},$$

and

$$[(j'',j'',j'')]_{\rho\cap\rho^{-1}}[c']_{\rho\cap\rho^{-1}} = [c']_{\rho\cap\rho^{-1}}.$$

This concludes case (3) and our proof is complete. □

Corollary 3.18 *Let S and T be two posemigroups with common local units. If their Cauchy completions $C(S)$ and $C(T)$ are* Pos*-equivalent and R is the joint enlargement of S and T constructed in Theorem 3.17, then $P = SRT \in {}_S\mathsf{Pos}_T$ and $Q = TRS \in {}_T\mathsf{Pos}_S$ are* Pos*-unitary.*

Proof. We merely need to observe that $Q = TRS$ is a subposet of $C(T)C(S)/\rho$. The last part of the proof of Theorem 3.17 demonstrates that the elements of $C(T)C(S)/\rho$ have common local units, which are a stronger version of the ordered local units required in the definition of Pos-unitarity. The case for P is symmetrical via $C(S)C(T)/\rho$. □

For illustrative purposes, we present a simple example of the construction of a joint enlargement. This example also demonstrates that the requirement for both S and T to have common local units is not necessary to get Pos-unitary biposets SRT and TRS.

Example 3.19 Let $(S, \leq)$ be a left zero posemigroup and $(T, \leq)$ be another left zero posemigroup. Take the rectangular poband $C = (S \coprod T) \times (S \coprod T)$ with the discrete order $\leq$. Consider the binary relation

$$(r_1, r_2)\rho(r_1', r_2') \iff (r_1 \leq r_1') \wedge (r_2, r_2' \in S \vee r_2, r_2' \in T).$$

It is easy to see that ρ is an admissible preorder. Consider $R = C/\rho$. Then the posemigroup $R \cong (S \coprod T) \times \{S, T\}$ is a joint enlargement of its subposemigroups $S \cong S \times \{S\} \subseteq (S \coprod T) \times \{S, T\}$ and $T \cong T \times \{T\} \subseteq (S \coprod T) \times \{S, T\}$. This is easy to verify directly because every rectangular band satisfies the identity $xyx = x$ and we have the required equalities $R = RSR$, $S = SRS$, $R = RTR$ and $T = TRT$. The biposet SRT can be written as $SRT \cong S \times \{T\} \subseteq (S \coprod T) \times \{S, T\}$. If S has only ordered local units but not common local units, then SRT is still Pos-unitary. Similarly, if T has ordered local units but not common local units, then $TRS \cong T \times \{S\} \subseteq (S \coprod T) \times \{S, T\}$ is also Pos-unitary.

3.3 From enlargements to Morita contexts

We now show that implication (2) ⇒ (1) of Theorem 3.1 holds even if S and T are only factorizable.

Lemma 3.20 *Let T, P and Q all be subposemigroups of some posemigroup R. Furthermore, let P be a right T-poset and Q a left T-poset with respect to actions defined by multiplication in R. If $p \otimes q \leq p' \otimes q'$ in $P \otimes_T Q$, then $pq \leq p'q'$ in R.*

Proof. Assume that $p \otimes q \leq p' \otimes q'$ in $P \otimes_T Q$. Then by (2.2) we have $u_1, \ldots, u_n$, $v_1, \ldots, v_n \in T^1$, $p_1, \ldots, p_n \in P$ and $q_2, \ldots, q_n \in Q$ such that

$$\begin{array}{rclcrcl} p & \leq & p_1u_1 & & & & \\ p_1v_1 & \leq & p_2u_2 & & u_1q & \leq & v_1q_2 \\ & \vdots & & & & \vdots & \\ p_nv_n & \leq & p' & & u_nq_n & \leq & v_nq'. \end{array}$$

Therefore in R we get that

$$\begin{aligned} pq &\leq (p_1u_1)q = p_1(u_1q) \leq p_1(v_1q_2) = (p_1v_1)q_2 \leq (p_2u_2)q_2 \\ &= p_2(u_2q_2) \leq \ldots \leq p_n(u_nq_n) \leq p_n(v_nq') = (p_nv_n)q' \leq p'q'. \end{aligned}$$

□

Proposition 3.21 (cf. Prop 3.14 of [29]) *Let S and T be two factorizable posemigroups with a joint enlargement R. Then S and T are strongly Morita equivalent.*

Proof. Without loss of generality, assume that S and T are subposemigroups of R. Take $P = SRT$ and $Q = TRS$. Then P is an (S, T)-biposet and Q is a (T, S)-biposet, inheriting their actions from the multiplication of R (note that the S- and T-actions are thus monotone). Due to factorizability, $SP = SSRT = SRT = P$, $PT = SRTT = SRT = P$, $TQ = TTRS = TRS = Q$, $QS = TRSS = TRS = Q$, so P and Q are unitary biposets. Define the mappings

$$\langle -, - \rangle : P \otimes_T Q \to S$$

and

$$[-, -] : Q \otimes_S P \to T$$

by

$$\langle p, q \rangle = pq \quad \text{and} \quad [q, p] = qp$$

for $p \in P = SRT$ and $q \in Q = TRS$. First of all, due to R being a joint enlargement and the factorizability of both S and R,

$$pq \in (SRT)(TRS) = S(RTR)S = SRS = S \tag{3.3}$$

and

$$qp \in (TRS)(SRT) = T(RSR)T = TRT = T. \tag{3.4}$$

Because of (3.3) and (3.4), $\langle -, - \rangle$ and $[-, -]$ are obviously surjective. It is easy to see that they are also biact morphisms. Due to Lemma 3.20, $\langle -, - \rangle$ and $[-, -]$ are both well-defined and monotone, which makes them biposet morphisms. Finally, for $p, p' \in P$, $q, q' \in Q$ we have

$$\langle p, q \rangle p' = (pq)p' = p(qp') = p[q, p']$$

and

$$q\langle p, q' \rangle = q(pq') = (qp)q' = [q, p]q'. \qquad \square$$

3.4 From Morita contexts to Cauchy completions

In this section we complete the proof of the first part of Theorem 3.1 by proving the implication (1) ⇒ (3) for arbitrary posemigroups. While in general this passes beyond strong Morita equivalence, strongly Morita equivalent posemigroups are factorizable by Lemma 2.39 and so our interest in the result is restricted to the case of factorizable posemigroups.

Proposition 3.22 *Let S and T be posemigroups and let $(S, T, P, Q, \langle -, - \rangle, [-, -])$ be a Morita context with surjective maps. Then the Cauchy completions $C(S)$ and $C(T)$ are* Pos*-equivalent* Pos*-categories.*

Proof. For every $e \in E(S)$ we choose $p_e \in P$ and $q_e \in Q$ such that

$$e = \langle p_e, q_e \rangle \tag{3.5}$$

and define

$$u_e := [q_e, p_e]^2 \in T. \tag{3.6}$$

Because

$$\begin{aligned} u_e^2 &= [q_e, p_e]^4 = [q_e, p_e[q_e, p_e]^3] = [q_e, \langle p_e, q_e \rangle^3 p_e] = [q_e, e^3 p_e] \\ &= [q_e, ep_e] = [q_e, \langle p_e, q_e \rangle p_e] = [q_e, p_e[q_e, p_e]] = [q_e, p_e]^2 = u_e, \end{aligned}$$

u_e are always idempotents.

We define a Pos-functor $F : C(S) \to C(T)$ by the assignment

$$\begin{tikzcd} e \arrow[r, mapsto] & u_e \\ f \arrow[u, "{(e,a,f)}"] \arrow[r, mapsto] & u_f \arrow[u, "{(u_e,[q_e,ap_f],u_f)}"'] \end{tikzcd} .$$

If $eaf = a$ then

$$\begin{aligned} u_e[q_e, ap_f]u_f &= \Big[q_e, p_e[q_e, p_e][q_e, ap_f][q_f, p_f]^2\Big] \\ &= \Big[q_e, \langle p_e, q_e \rangle^2 \langle ap_f, q_f \rangle \langle p_f, q_f \rangle p_f\Big] \\ &= \Big[q_e, \langle p_e, q_e \rangle^2 a \langle p_f, q_f \rangle^2 p_f\Big] = [q_e, eafp_f] \\ &= [q_e, ap_f], \end{aligned}$$

thus we indeed have a morphism $(u_e, [q_e, ap_f], u_f) : u_f \to u_e$ in $C(T)$.

For every $e \in E(S)$ we have

$$\begin{aligned} F(e, e, e) &= (u_e, [q_e, ep_e], u_e) = (u_e, [q_e, \langle p_e, q_e \rangle p_e], u_e) \\ &= (u_e, [q_e, p_e[q_e, p_e]], u_e) = (u_e, u_e, u_e). \end{aligned}$$

For composable morphisms $(e, a, f), (f, b, g)$ in $C(S)$ we get that

$$\begin{aligned} F(e, a, f)F(f, b, g) &= (u_e, [q_e, ap_f], u_f)(u_f, [q_f, bp_g], u_g) \\ &= (u_e, [q_e, ap_f[q_f, bp_g]], u_g) \\ &= (u_e, [q_e, a\langle p_f, q_f \rangle bp_g], u_g) \\ &= (u_e, [q_e, afbp_g], u_g) \\ &= (u_e, [q_e, abp_g], u_g) \\ &= F(e, ab, g) = F((e, a, f)(f, b, g)). \end{aligned}$$

And for comparable morphisms $(e, a, f) \leq (e, b, f)$ in $C(S)$, i.e. $a \leq b$ in S, we easily see that

$$F(e, a, f) \quad = \quad (u_e, [q_e, ap_f], u_f) \leq (u_e, [q_e, bp_f], u_f) = F(e, b, f)$$

since $[-, -]$ and $\otimes$ are monotone in both arguments. So F is indeed a Pos-functor.

If $eaf = a$ and $ea'f = a'$, $e, f \in E(S)$, $a, a' \in S$, and

$$(u_e, [q_e, ap_f], u_f) = F(e, a, f) \leq F(e, a', f) = (u_e, [q_e, a'p_f], u_f)$$

then

$$\begin{aligned} a &= eaf = ea\langle p_f, q_f\rangle = \langle eap_f, q_f\rangle = \langle\langle p_e, q_e\rangle ap_f, q_f\rangle \\ &= \langle p_e[q_e, ap_f], q_f\rangle \leq \langle p_e[q_e, a'p_f], q_f\rangle = \langle\langle p_e, q_e\rangle a'p_f, q_f\rangle \\ &= \langle ea'p_f, q_f\rangle = ea'\langle p_f, q_f\rangle = ea'f = a', \end{aligned}$$

so $(e, a, f) \leq (e, a', f)$. Furthermore, if we take $(u_e, t, u_f) : u_f \to u_e$ in $C(T)$ then

$$\begin{aligned} t &= u_e t u_f = [q_e, p_e]^2 t [q_f, p_f]^2 = [q_e, p_e[q_e, p_e] t [q_f, p_f]^2] \\ &= [q_e, \langle p_e, q_e\rangle\langle p_e, tq_f\rangle\langle p_f, q_f\rangle p_f] = [q_e, e\langle p_e, tq_f\rangle f p_f], \end{aligned}$$

so $(u_e, t, u_f) = F(e, e\langle p_e, tq_f\rangle f, f)$ and the Pos-functor F is fully faithful.

Finally, we verify that F is essentially surjective. Take $u = [q, p] \in E(T)$. Then $e := \langle p, q\rangle^2 \in E(S)$ and the morphisms

$$\begin{tikzcd} F(e) = u_e \arrow[r, shift left, "{(u, u[q, p_e]u_e, u_e)}"] & u \arrow[l, shift left, "{(u_e, u_e[q_e, p]u, u)}"] \end{tikzcd}$$

are mutually inverse isomorphisms, because

$$\begin{aligned} (u[q, p_e]u_e)(u_e[q_e, p]u) &= u[q, p_e]u_e[q_e, p]u = u[q, p_e[q_e, p_e]^2[q_e, p]]u \\ &= u[q, \langle p_e, q_e\rangle^3 p]u = u[q, e^3 p]u = u[q, ep]u \\ &= u[q, \langle p, q\rangle^2 p]u = u[q, p[q, p]^2]u = u^5 = u \end{aligned}$$

and

$$\begin{aligned} (u_e[q_e, p]u)(u[q, p_e]u_e) &= u_e[q_e, p][q, p][q, p_e]u_e = u_e[q_e, p[q, p][q, p_e]]u_e \\ &= u_e[q_e, \langle p, q\rangle^2 p_e]u_e = u_e[q_e, ep_e]u_e \\ &= u_e[q_e, \langle p_e, q_e\rangle p_e]u_e = u_e[q_e, p_e]^2 u_e = u_e^3 = u_e. \quad \square \end{aligned}$$

4 Strong Morita equivalence via strict local isomorphisms

This chapter is focused on proving the rest of Theorem 3.1 and the equivalence mentioned in Remark 3.3. Moreover, we establish some related results dealing with partially ordered Rees matrix semigroups and Morita posemigroups. Finally, we verify that if S and T are both regular, then the Rees matrix posemigroup in condition (6) of Remark 3.3 can be taken to be regular as well. The chapter is based on [48].

4.1 Consolidations and strict local isomorphisms

First, we transfer more of Lawson's work on semigroups in [29] to posemigroups and prove that condition (4) of Theorem 3.1 is equivalent to conditions (1)-(3).

Lemma 4.1 (cf. Lemma 4.1 of [29]) *Let S and T be factorizable posemigroups and let T be an enlargement of S. Then each idempotent of T is $\mathcal{D}$-related to an idempotent of S.*

Proof. If a posemigroup R is an enlargement of a posemigroup S, then the underlying semigroup of R is also an enlargement of the underlying semigroup of S. Therefore the result follows from Lemma 4.1 of [29]. We repeat the argument for convenience. Take $e \in E(T)$. As $T = TST$, we can find $u, v \in T$ and $s \in S$ such that $e = usv$. But $S^2 = S$, so we can also choose $a, b \in S$ so that $s = ab$. Now take $x = ua$, $y = bv$ and $f = yex = (ye)(ex)$. Clearly $f^2 = yexyex = ye^3x = f$ and $ye \in V(ex)$. As $e, f \in E(T)$ and $e = xy = (ex)(ye)$, we have $e\mathcal{D}f$ by Remark 2.43. Moreover, $f = yex = (bv)e(ua) = b(veu)a \in S(TTT)S = STS = S$. □

Lemma 4.2 (cf. Lemma 4.2 of [29]) *(1) Let $\tau : S \to T$ be a surjective strict local isomorphism and $\tau' : T \to U$ a strict local isomorphism between posemigroups S, T and U, all with local units. Then $\tau'\tau : S \to U$ is also a strict local isomorphism.*

(2) Let T be a posemigroup with local units and let S be a subposemigroup of T which also has local units. Then T is an enlargement of S if and only if the posemigroup embedding of S in T is a strict local isomorphism.

(3) Let S and T be posemigroups with local units and let $\tau : S \to T$ be a strict local isomorphism. Then the posemigroup T is an enlargement of $\tau(S)$.

Proof. We follow the overall scheme from Lemma 4.2 of [29].

(1) The map $\tau'\tau$ is a posemigroup homomorphism. For verifying (LI1), take $a, b \in S$, with no restrictions because S has local units. Since T also has local units, we have posemigroup isomorphisms $aSb \cong \tau(a)T\tau(b) \cong \tau'(\tau(a))U\tau'(\tau(b))$. For (LI2), if $e \in E(U)$ is such that $e = \tau'(\tau(s))$ for some $s \in S$, then $e = \tau'(e_1)$ for some $e_1 \in E(T)$. As τ is onto, $e_1 \in \tau(S)$, thus there exists $e_2 \in E(S)$ so that $e_1 = \tau(e_2)$, whence $e = (\tau'\tau)(e_2)$. For (LI3), take $e \in E(U)$. Then there exist $f = \tau'(f_1) \in E(U)$, $f_1 \in E(T)$ such that $e\mathcal{D}f$ in U by (LI2) and (LI3) for τ'. Moreover, there exist $f_2 = \tau(f_3) \in E(T)$, $f_3 \in E(S)$ such that $f_1\mathcal{D}f_2$ in T by (LI2) and (LI3) for τ. By Remark 2.43, $\tau'(f_1)\mathcal{D}\tau'(f_2)$ in U. So we have $f_3 \in E(S)$ such that $(\tau'\tau)(f_3) = \tau'(f_2)\mathcal{D}\tau'(f_1) = f\mathcal{D}e$.

(2) Suppose that T is an enlargement of S. Then for any $a, b \in S$ we have $e, f \in S$ such that $ae = a$ and $fb = b$. Thus we get $etf \in STS = S$ for any $t \in T$, so $atb = a(etf)b \in aSb$ and $aTb = aSb$. This proves (LI1) for the embedding $S \hookrightarrow T$. Since the latter is essentially the identity, (LI2) follows as well. And (LI3) is deduced from Lemma 4.1.

Conversely, assume that the embedding $S \hookrightarrow T$ is a strict local isomorphism. We show that T is an enlargement of S. Take $a \in STS$. Then certainly there exist $e, f \in E(S)$ such that $a = eaf$. By (LI1), $a \in eTf = eSf$, so $a \in S$. As S is a subposemigroup of T with local units, we have that $S = STS$. Now, take $b \in T$ and let $i \in E(T)$ be such that $bi = b$. Due to (LI3), we have $e \in E(S)$ so that $i\mathcal{D}e$ in T. By Remark 2.43, there exist $x \in T$ and $x' \in V(x)$ such that $x'x = i$ and $xx' = e$. Thus

$$b = bi = bx'x = bx'xx'x = (bx')ex \in TST$$

and we have shown that $T = TST$.

(3) Denote $\tau(S) = T'$. Clearly T' is a subposemigroup of T with local units. Take $a' = \tau(a)$, $b' = \tau(b) \in T'$ and $c' \in T$. We have $e, f \in E(S)$ such that $ae = a$ and $fb = b$. Using (LI1), we can find $c \in S$ such that $\tau(ecf) = \tau(e)c'\tau(f)$. Therefore

$$a'c'b' = a'\tau(e)c'\tau(f)b' = a'\tau(ecf)b' = \tau(aecfb) = \tau(abc) \in T'.$$

As T' is a subposemigroup of T with local units, we have shown that $T'TT' = T'$.

Now, take $t \in T$. Then $t = ti$ for some $i \in E(T)$. By (LI2) and (LI3), we get $i\mathcal{D}\tau(e)$ for some $e \in E(S)$. Remark 2.43 lets us find such $x \in T$ and $x' \in V(x)$ that $x'x = i$ and $xx' = \tau(e)$. Thus

$$t = ti = tx'xx'x = (tx')\tau(e)x \in TT'T.$$

Hence $T = TT'T$ and T is an enlargement of T'. □

Proposition 4.3 *Let S and T be posemigroups with local units. If there exist a consolidation q on $C(S)$ and a strict local isomorphism $\tau : C(S)^q \to T$, then $C(S)$ and $C(T)$ are* Pos*-equivalent* Pos*-categories.*

Proof. Define a Pos-functor $\Psi : C(S) \to C(T)$ by $\Psi(e) = \tau(e, e, e)$ and

$$\Psi(e, s, f) = (\tau(e, e, e), \tau(e, s, f), \tau(f, f, f)).$$

As $q_{e,e} = e$, $(e, e, e) \in E(C(S)^q)$ for any $e \in E(S)$ and we get that $\tau(e, e, e) \in E(T)$. The mapping Ψ is well-defined because

$$(e, e, e) \diamond (e, s, f) \diamond (f, f, f) = (e, e, e) \circ (e, s, f) \circ (f, f, f) = (e, s, f),$$

so

$$\tau(e, e, e)\tau(e, s, f)\tau(f, f, f) = \tau((e, e, e) \diamond (e, s, f) \diamond (f, f, f)) = \tau(e, s, f).$$

We can easily check that

$$\Psi(e, e, e) = (\tau(e, e, e), \tau(e, e, e), \tau(e, e, e))$$

and

$$\Psi(e, s, f) \circ \Psi(f, s', g) = (\tau(e, e, e), \tau(e, ss', g), \tau(g, g, g)).$$

Since $C(S)(f, e) = (e, e, e) \diamond C(S)^q \diamond (f, f, f)$ and there exists an isomorphism $C(T)(\tau(e, e, e), \tau(f, f, f)) \cong \tau(e, e, e)T\tau(f, f, f)$ in Pos (which is actually even a posemigroup isomorphism), condition (LI1) for τ implies that Ψ is a fully faithful Pos-functor.

We will make an intermediary step and prove that if we take $i \in E(T)$ then there exists $j \in E(T)$ such that $i\mathcal{D}j$ and $\tau(g, g, g) = j$ for some $g \in E(S)$. Due to (LI2) and (LI3) we can find $e' \in E(T)$ such that $i\mathcal{D}e'$ and $e' = \tau(e, s, f)$ for an idempotent $(e, s, f) \in E(C(S)^q)$. The latter implies that $sq_{f,e}s = s$, so s is regular. Take $s'' \in V(s)$ and define $s' = fs''e$. Then $s' \in V(s)$ and $fs'e = s'$, meaning that $(f, s', e) \in C(S)$ and $(f, s', e) \in V(e, s, f)$ in $C(S)^q$. We now get that $(f, s's, f) = (f, s', e) \diamond (e, s, f) \in E(C(S)^q)$ and $(e, s, f)\mathcal{L}(f, s's, f)$ in $C(S)^q$. Put $g = s's \in E(S)$, then we have just shown that $(e, s, f)\mathcal{D}(f, g, f)$. Since clearly $(f, g, g) \in V(g, g, f)$,

$$(f, g, f) = (f, g, g) \diamond (g, g, f) \text{ and } (g, g, g) = (g, g, f) \diamond (f, g, g),$$

we have $(f, g, f)\mathcal{D}(g, g, g)$ in $C(S)^q$ by Remark 2.43. Put $\tau(g, g, g) = j$. Then

$$j = \tau(g, g, g)\mathcal{D}\tau(f, g, f)\mathcal{D}\tau(e, s, f) = e'\mathcal{D}i$$

in T, as required.

Finally, we need to show that Ψ is essentially surjective. Take an identity morphism (i, i, i) in $C(T)$, $j \in E(T)$ and $g \in E(S)$ such that $i\mathcal{D}j$ and $\tau(g, g, g) = j$. We can choose $x \in T$ and $x' \in V(x)$ so that $x'x = j$ and $xx' = i$ by Remark 2.43. Thus (i, x, j) is an isomorphism between $(j, j, j) = \Psi(g, g, g)$ and (i, i, i). □

Proposition 4.4 *Let S and T be two posemigroups with local units and a joint enlargement R. Then there exist a consolidation q on $C(S)$ and a strict local isomorphism $\tau : C(S)^q \to T$.*

Proof. Without loss of generality, assume that S and T are subposemigroups of R. We will construct a subposemigroup T' of T such that T is an enlargement of T', a consolidation q on $C(S)$ and a surjective strict local isomorphism $\tau : C(S)^q \to T'$. Then Lemma 4.2 (1) and (2) imply that we have a strict local isomorphism from $C(S)^q$ to T.

For each $e \in E(S) \subseteq E(R)$ there exists $i \in E(T)$ such that $e\mathcal{D}i$ in R due to Lemma 4.1. Therefore Remark 2.43 provides $x_e \in R$ and $x'_e \in V(x_e) \subseteq R$ such that

$$x'_e x_e = e \in E(S) \quad \text{and} \quad x_e x'_e = i \in E(T). \tag{4.1}$$

Define the consolidation q on $C(S)$ by

$$q_{e,f} = (e, x'_e x_f, f).$$

Such a definition is correctly typed because

$$x'_e x_f = (x'_e x_e)(x'_e x_f)(x'_f x_f) \in SRS = S.$$

Note that $q_{e,e} = (e, x'_e x_e, e) = (e, e, e)$ and $e(x'_e x_f)f = x'_e x_e x'_e x_f x'_f x_f = x'_e x_f$, so the definition is also valid. Now define a mapping $\tau : C(S)^q \to T$ by

$$\tau(e, s, f) = x_e s x'_f.$$

Observe that $x_e s x'_f = (x_e x'_e)(x_e s x'_f)(x_f x'_f) \in TRT = T$. Furthermore,

$$\begin{aligned} \tau((e_1, s_1, f_1) \diamond (e_2, s_2, f_2)) &= \tau(e_1, s_1 x'_{f_1} x_{e_2} s_2, f_2) \\ &= (x_{e_1} s_1 x'_{f_1})(x_{e_2} s_2 x'_{f_2}) = \tau(e_1, s_1, f_1)\tau(e_2, s_2, f_2). \end{aligned}$$

If $(e_1, s_1, f_1) \leq (e_2, s_2, f_2)$ in $C(S)$, then $e_1 = e_2$, $f_1 = f_2$ and $s_1 \leq s_2$ in S. Thus

$$\tau(e_1, s_1, f_1) = x_{e_1} s_1 x'_{f_1} \leq x_{e_2} s_2 x'_{f_2} = \tau(e_2, s_2, f_2).$$

So τ is a posemigroup homomorphism. Take $T' = \mathrm{Im}\tau$.

We show that $\tau : C(S)^q \to T'$ is a strict local isomorphism. By Lemma 2.42, we can replace condition (LI1) with (LI1'). So we only need to examine the restriction $\overline{\tau} : (e_0, e, e_1)C(S)^q(f_0, f, f_1) \to \tau(e_0, e, e_1)T'\tau(f_0, f, f_1)$ for arbitrary $(e_0, e, e_1), (f_0, f, f_1) \in E(C(S)^q)$. We already know that $\overline{\tau}$ is a surjective posemigroup homomorphism. Thus we only need to verify that it reflects order. Take $(s_0, s, s_1), (z_0, z, z_1) \in C(S)$ and assume that

$$\tau(e_0, e, e_1)\tau(s_0, s, s_1)\tau(f_0, f, f_1) \leq \tau(e_0, e, e_1)\tau(z_0, z, z_1)\tau(f_0, f, f_1)$$

in T, i.e.

$$\tau(e_0, ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}f, f_1) \leq \tau(e_0, ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}f, f_1).$$

Then by (4.1)

$$\begin{aligned} &ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}f = e_0ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}ff_1 = x'_{e_0}(x_{e_0}ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}fx'_{f_1})x_{f_1} \\ &\leq x'_{e_0}(x_{e_0}ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}fx'_{f_1})x_{f_1} = e_0ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}ff_1 = ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}f \end{aligned}$$

in S. So

$$(e_0, e, e_1) \diamond (s_0, s, s_1) \diamond (f_0, f, f_1) \leq (e_0, e, e_1) \diamond (z_0, z, z_1) \diamond (f_0, f, f_1)$$

in $C(S)^q$, as required.

Moreover, idempotents lift along τ. Take $\tau(e, s, f) = x_esx'_f = f \in E(T')$. Then $x_esx'_fx_esx'_f = x_esx'_f$, so

$$sx'_fx_es = esx'_fx_esf = x'_ex_esx'_fx_esx'_fx_f = x'_ex_esx'_fx_f = esf = s$$

and $(e, s, f) \in E(C(S)^q)$. So (LI2) holds as well. (LI3) is trivial because τ is surjective.

Finally, we need to show that T is an enlargement of T'. First, take $t \in T$. Then we have $i, j \in E(T)$ such that $t = itj$. We can again use Lemma 4.1 to find $e, f \in E(S)$, $y_i, y_j \in R$ and $y'_i \in V(y_i)$, $y'_j \in V(y_j)$ so that

$$y'_iy_i = i,\ y_iy'_i = e,\ y'_jy_j = j \text{ and } y_jy'_j = f.$$

Take two new elements $s = y_ity'_j = (y_iy'_i)(y_ity'_j)(y_jy'_j) = esf \in SRS = S$ and $t' = x_ey_ity'_jx'_f = \tau(e, s, f) \in T'$. Then we can use (4.1) and get that

$$\begin{aligned} t &= i^2tj^2 = y'_iey_ity'_jfy_j = y'_ix'_e(x_ey_ity'_jx'_f)x_fy_j = y'_ix'_et'x_fy_j \\ &= [(y'_iy_i)(y'_ix'_e)(x_ex'_e)]t'[(x_fx'_f)(x_fy_j)(y'_jy_j)] \in TRTT'TRT = TT'T. \end{aligned}$$

Obviously $TT'T \subseteq T$ and therefore $T = TT'T$.

Since $C(S)^q$ has local units by Lemma 3.7, $T' = \mathrm{Im}\tau$ has local units as well. Thus $T' \subseteq T'TT'$. Take $e_1, e_2, f_1, f_2 \in E(S)$, $s_1 = e_1s_1f_1 \in S$, $s_2 = e_2s_2f_2 \in S$ and $t \in T$. If we take $z = s_1x'_{f_1}tx_{e_2}s_2 \in SRS = S$, then

$$\tau(e_1, s_1, f_1)t\tau(e_2, s_2, f_2) = (x_{e_1}s_1x'_{f_1})t(x_{e_2}s_2x'_{f_2}) = x_{e_1}zx'_{f_2}.$$

Since $e_1zf_2 = z$, we have that $\tau(e_1, s_1, f_1)t\tau(e_2, s_2, f_2) = \tau(e_1, z, f_2) \in T'$. So $T'TT' = T'$ and T is an enlargement of T'. □

Theorem 4.5 (cf. Thm 1.2 of [29]) *Let S and T be two posemigroups with local units. Then S and T satisfy one of the equivalent conditions of Theorem 3.1 if and only if there exist a consolidation q on $C(S)$ and a strict local isomorphism $\tau : C(S)^q \to T$.*

Proof. Sufficiency is proved in Proposition 4.3 and necessity in Proposition 4.4. □

4.2 Rees matrix covers

We will now investigate how the existence of a Rees matrix cover connection relates to strong Morita equivalence.

A *classical Rees matrix posemigroup* over a pomonoid S means that for every $u \in U$ there exists $v \in V$ such that $M(v, u)$ is invertible and for every $v \in V$ there also exists a $u \in U$ such that $M(v, u)$ is invertible (i.e. every row and every column of M contains an invertible element).

Theorem 4.6 (cf. Thm 3 of [28]) *Let S and T be two strongly Morita equivalent posemigroups. Then there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ with $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau : \mathcal{M} \to T$.*

Proof. Let $(S, T, P, Q, \langle -, - \rangle, [-, -])$ be a unitary Morita context with surjective maps $\langle -, - \rangle$ and $[-, -]$. As T is factorizable due to Lemma 2.40, we can pick subsets $U, V \subseteq T$ such that $T = UT = TV$. For all $x \in U \cup V$, fix $p_x \in P$ and $q_x \in Q$ such that $x = [q_x, p_x]$. Now, define $M : V \times U \to S$ by

$$M(v, u) = \langle p_v, q_u \rangle.$$

This gives us a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$. Since S and T are also strongly Morita equivalent as semigroups (see Proposition 6.4) and we follow the construction in Theorem 3 of [28], we deduce that $S\,\mathrm{Im}(M)S = S$.

We define our strict local isomorphism $\tau : \mathcal{M} \to T$ by

$$\tau(u, s, v) = [q_u, sp_v].$$

Since P is a left S-poset and $[-, -]$ is monotone, τ is also monotone. Again, as we are following the construction of Theorem 3 of [28], τ is a surjective posemigroup homomorphism along which idempotents and regular elements lift, so (LI2) and (LI3) are satisfied. To prove that (LI1) holds as well, we need to show that τ reflects order when restricted to certain subposemigroups.

Take $(u_1, s_1, v_1), (u_2, s_2, v_2) \in \mathcal{M}$ such that

$$(u_1, s_1', v_1')(u_1, s_1, v_1) = (u_1, s_1, v_1) \quad \text{and} \quad (u_2, s_2, v_2)(u_2', s_2', v_2) = (u_2, s_2, v_2)$$

for some $(u_1, s_1', v_1'), (u_2', s_2', v_2) \in \mathcal{M}$. Suppose that $\tau(u_1, s, v_2) \le \tau(u_1, z, v_2)$ for some $(u_1, s, v_2), (u_1, z, v_2) \in (u_1, s_1, v_1)\mathcal{M}(u_2, s_2, v_2)$, i.e.

$$[q_{u_1}, sp_{v_2}] \le [q_{u_1}, zp_{v_2}].$$

As $(u_1, s, v_2), (u_1, z, v_2) \in (u_1, s_1, v_1)\mathcal{M}(u_2, s_2, v_2)$, we get that

$$(u_1, s_1', v_1')(u_1, s, v_2) = (u_1, s, v_2), \qquad (u_1, s, v_2)(u_2', s_2', v_2) = (u_1, s, v_2),$$

$$(u_1, s_1', v_1')(u_1, z, v_2) = (u_1, z, v_2) \quad \text{and} \quad (u_1, z, v_2)(u_2', s_2', v_2) = (u_1, z, v_2).$$

So

$$\begin{aligned} s &= s_1'M(v_1',u_1)sM(v_2,u_2')s_2' = s_1'\langle p_{v_1'},q_{u_1}\rangle s\langle p_{v_2},q_{u_2'}\rangle s_2' \\ &= s_1'\langle p_{v_1'},q_{u_1}\langle sp_{v_2},q_{u_2'}\rangle\rangle s_2' = s_1'\langle p_{v_1'},[q_{u_1},sp_{v_2}]q_{u_2'}\rangle s_2' \\ &\leq s_1'\langle p_{v_1'},[q_{u_1},zp_{v_2}]q_{u_2'}\rangle s_2' = s_1'\langle p_{v_1'},q_{u_1}\langle zp_{v_2},q_{u_2'}\rangle\rangle s_2' \\ &= s_1'\langle p_{v_1'},q_{u_1}\rangle z\langle p_{v_2},q_{u_2'}\rangle s_2' = s_1'M(v_1',u_1)zM(v_2,u_2')s_2' = z. \qquad \square \end{aligned}$$

Theorem 4.6 admits the following converse:

Lemma 4.7 (cf. Lemma 7 of [28]) *Let S and T be factorizable posemigroups, let $\mathcal{M} = \mathcal{M}(S,U,V,M)$ be a Rees matrix posemigroup over S with $S\,\mathrm{Im}(M)S = S$ and let there exist a surjective strict local isomorphism $\tau : \mathcal{M} \to T$. Then $C(S)$ and $C(T)$ are* Pos*-equivalent.*

Proof. Again, we make use of the fact that surjective strict local isomorphisms of posemigroups are idempotent-lifting strict local isomorphisms of their underlying semigroups and make use of the construction in Lemma 7 of [28] to get a full essentially surjective functor $F : C(S) \to C(T)$. The construction itself is as follows. Take $e \in E(S)$ and pick $a_e, b_e \in S$, $u_e \in U$ and $v_e \in V$ such that

$$e = a_eM(v_e,u_e)b_e.$$

Then $\overline{e} = (u_e, b_eea_e, v_e) \in E(\mathcal{M})$ and $\tau(\overline{e}) \in E(T)$. The assignment

$$\begin{tikzcd} e \arrow[r, mapsto] \arrow[d, "{(f,s,e)}"'] & \tau(\overline{e}) \arrow[d, "{(\tau(\overline{f}),\tau(u_f,b_fsa_e,v_e),\tau(\overline{e}))}"] \\ f \arrow[r, mapsto] & \tau(\overline{f}) \end{tikzcd}$$

defines a functor $F : C(S) \to C(T)$. As τ is monotone, F is a Pos-functor. We now only need to verify that it reflects the order of morphisms. Take two morphisms $(f,s,e),(f,s',e) : e \to f$ in $C(S)$ such that $\tau(u_f,b_fsa_e,v_e) \leq \tau(u_f,b_fs'a_e,v_e)$. First,

$$\begin{aligned} \overline{f}(u_f,b_fsa_e,v_e)\overline{e} &= (u_f,b_ffa_fM(v_f,u_f)b_fsa_eM(v_e,u_e)b_eea_e,v_e) \\ &= (u_f,b_ffsea_e,v_e) = (u_f,b_fsa_e,v_e) \end{aligned}$$

and likewise

$$\overline{f}(u_f,b_fs'a_e,v_e)\overline{e} = (u_f,b_fs'a_e,v_e).$$

Since $\overline{f},\overline{e} \in E(\mathcal{M})$ and τ is a strict local isomorphism, we use (L1) to get that $(u_f,b_fsa_e,v_e) \leq (u_f,b_fs'a_e,v_e)$ in $\mathcal{M}$ and thus $b_fsa_e \leq b_fs'a_e$ in S. But then

$$\begin{aligned} s &= fse = a_fM(v_f,u_f)b_fsa_eM(v_e,u_e)b_e \\ &\leq a_fM(v_f,u_f)b_fs'a_eM(v_e,u_e)b_e = fs'e = s'. \qquad \square \end{aligned}$$

In conclusion, we have the following description of strong Morita equivalence.

Theorem 4.8 (cf. Thm 8 of [28]) *Let S and T be posemigroups with local units. Then S and T are strongly Morita equivalent if and only if there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ with $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau : \mathcal{M} \to T$.*

Proof. Necessity follows from Theorem 4.6. For sufficiency, we use Lemma 4.7 and Theorem 3.1. □

Corollary 4.9 (cf. Prop 2 of [28]) *Let S be a posemigroup with local units. Then any Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over S that has local units is strongly Morita equivalent to S.*

Proof. This follows from Theorem 4.8, using the identity $1_{\mathcal{M}}$. □

A posemigroup called *unipotent* if it contains exactly one idempotent element. The preceding theorem can be specified to unipotent pomonoids as follows:

Theorem 4.10 *Let T be a posemigroup with local units. Then T is strongly Morita equivalent to a unipotent pomonoid S if and only if there exists a classical Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over S and a surjective strict local isomorphism $\tau : \mathcal{M} \to T$.*

Proof. For necessity, let $(S, T, P, Q, \langle -, - \rangle, [-, -])$ be a unitary Morita context with surjective mappings $\langle -, - \rangle$ and $[-, -]$. We follow the proof of Theorem 4.6 and use the existence of local units to put $U = V = E(T)$. One can find an idempotent $e = [q_1, p_1]^2 \in E(T)$ so that $T = TeT$, where $\langle p_1, q_1 \rangle = 1 \in S$, by repeating a part of the proof of Theorem 9 of [25] (knowing that S and T are Morita equivalent as semigroups, as shown in Proposition 6.4, allows us to use this fact without the repetition, like it is done in Proposition 6.6). Since S is a unipotent pomonoid, its only local subpomonoid is $1S1 = S$. Corollary 3.6 therefore implies that every local subpomonoid of T is isomorphic to S. So eTe is a unipotent pomonoid. For every $u \in U$, fix $p_u \in P$ and $q_u \in Q$ such that $u = [q_u, p_u]$.

We prove only the rows part of being a classical Rees matrix posemigroup, as the columns part is proved similarly. Take $v \in V = E(T)$, then $v^2 = v$ and $v = [q_v, p_v]$. We will show that

$$M(v, v) = \langle p_v, q_v \rangle,$$

as constructed in Theorem 4.6, is invertible in S.

If we take

$$x = e[q_1, p_v]v[q_v, p_1]e \in eTe,$$

then

$$\begin{aligned}
x^2 &= e[q_1,p_v]v[q_v,p_1]e[q_1,p_v]v[q_v,p_1]e \\
&= e[q_1,p_v]v[q_v,\langle p_1,eq_1\rangle p_v]v[q_v,p_1]e \\
&= e[q_1,p_v]v[q_v,\langle p_1,[q_1,p_1]^2q_1\rangle p_v]v[q_v,p_1]e \\
&= e[q_1,p_v]v[q_v,\langle p_1,q_1\rangle^3 p_v]v[q_v,p_1]e \\
&= e[q_1,p_v][q_v,p_v][q_v,1^3p_v]v[q_v,p_1]e \\
&= e[q_1,p_v][q_v,p_v[q_v,1p_v]]v[q_v,p_1]e \\
&= e[q_1,p_v][q_v,(\langle p_v,q_v\rangle 1)p_v]v[q_v,p_1]e \\
&= e[q_1,p_v][q_v,\langle p_v,q_v\rangle p_v]v[q_v,p_1]e \\
&= e[q_1,p_v][q_v,p_v]^2v[q_v,p_1]e = e[q_1,p_v]v^3[q_v,p_1]e = x.
\end{aligned}$$

So $x \in E(T)$. Since eTe only has one idempotent, we must have $x = e$. Hence

$$\begin{aligned}
\langle p_v,q_v\rangle\langle p_v,vq_v\rangle &= \langle p_1,q_1\rangle^3\langle p_v,q_v\rangle\langle p_v,vq_v\rangle\langle p_1,q_1\rangle^3 \\
&= \langle p_1,e[q_1,p_v][q_v,p_v]v[q_v,p_1]eq_1\rangle \\
&= \langle p_1,e[q_1,p_v]v^2[q_v,p_1]eq_1\rangle \\
&= \langle p_1,xq_1\rangle = \langle p_1,eq_1\rangle \\
&= \langle p_1,[q_1,p_1]^2q_1\rangle = \langle p_1,q_1\rangle^3 = 1.
\end{aligned}$$

Similarly, $\langle p_v,vq_v\rangle\langle p_v,q_v\rangle = 1$ as well. For sufficiency, note that we may apply Theorem 4.8 since every $s \in S$ can be factorized as $s = szz^{-1}$ for any invertible $z \in \mathrm{Im}(M)$. □

4.3 Regular Rees matrix posemigroups

Drawing on [35] for inspiration, we convert our results on Rees matrix covers to the special case of regular posemigroups.

Theorem 4.11 (cf. Thm 2.9 of [35] and Thm 4.6) *Let S and T be two strongly Morita equivalent regular posemigroups. Then there exist a regular Rees matrix posemigroup $\mathcal{R} = \mathcal{R}(S,U,V,M)$ with $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau : \mathcal{R} \to T$.*

Proof. By Theorem 4.6, there is a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S,U,V,M)$ with $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau' : \mathcal{M} \to T$ that lifts regular elements. Take $\mathcal{R} = \mathcal{R}(S,U,V,M)$ and $\tau = \tau'|_{\mathcal{R}}$. Then τ is surjective because τ' lifts regular elements. Being a restriction, it still lifts idempotents. Since the idempotents of $\mathcal{R}$ are also idempotents of $\mathcal{M}$, condition (LI1) can be verified by an argument similar to the one used for proving Lemma 2.42. □

Lemma 4.12 (cf. Lemma 4.7) *Let S and T be two regular posemigroups. When $\mathcal{R} = \mathcal{R}(S, U, V, M)$ is a regular Rees matrix posemigroup over S, $S\,\mathrm{Im}(M)S = S$ and there exists a surjective strict local isomorphism $\tau : \mathcal{R} \to T$, then $C(S)$ and $C(T)$ are* Pos*-equivalent.*

Proof. We use the proof of Lemma 4.7 and check that if we assume S to be regular, we can use $\tau : \mathcal{R} \to T$ instead of the more general morphism $\mathcal{M}(S, U, V, M) \to T$. This amounts to checking that if $s = fse$ for some $e, f \in E(S)$ is a regular element of S, then $(u_f, b_f sa_e, v_e) \in \mathcal{M}$ is also a regular element; and if $s = sM(v, u)s$ and $e = sM(v, u) \in E(S)$, then (u, ea_e, v_e) and $(u_e, b_e s, v)$ are regular elements of $\mathcal{M}$. Note that the latter condition is necessary to retain the validity of the proof in Lemma 7 of [28] that we obtain an essentially surjective functor. Indeed, if $s = fse$ then we get that $(u_e, b_e s' a_f, v_f) \in V(u_f, b_f sa_e, v_e)$ for all $s' \in V(s)$, since

$$\begin{aligned}
& (u_f, b_f sa_e, v_e)(u_e, b_e s' a_f, v_f)(u_f, b_f sa_e, v_e) \\
= \;& (u_f, b_f sa_e M(v_e, u_e) b_e s' a_f M(v_f, u_f) b_f sa_e, v_e) \\
= \;& (u_f, b_f ses' fsa_e, v_e) = (u_f, b_f sa_e, v_e)
\end{aligned}$$

and

$$\begin{aligned}
& (u_e, b_e s' a_f, v_f)(u_f, b_f sa_e, v_e)(u_e, b_e s' a_f, v_f) \\
= \;& (u_e, b_e s' a_f M(v_f, u_f) b_f sa_e M(v_e, u_e) b_e s' a_f, v_f) \\
= \;& (u_e, b_e s' fses' a_f, v_f) = (u_e, b_e s' a_f, v_f).
\end{aligned}$$

Furthermore, if $s = sM(v, u)s$ and $e = sM(v, u)$ then $(u_e, b_e s, v) \in V(u, ea_e, v_e)$, because

$$\begin{aligned}
& (u_e, b_e s, v)(u, ea_e, v_e)(u_e, b_e s, v) \\
= \;& (u_e, b_e sM(v, u) ea_e M(v_e, u_e) b_e s, v) \\
= \;& (u_e, b_e sM(v, u) e^2 s, v) = (u_e, b_e s, v)
\end{aligned}$$

and

$$\begin{aligned}
& (u, ea_e, v_e)(u_e, b_e s, v)(u, ea_e, v_e) \\
= \;& (u, ea_e M(v_e, u_e) b_e sM(v, u) ea_e, v_e) \\
= \;& (u, e^2 sM(v, u) ea_e, v_e) = (u, ea_e, v_e).
\end{aligned}$$

The case for $(u_e, b_e s, v)$ is similar. □

Theorem 4.13 (cf. Thm 4.8) *Let S and T be regular posemigroups. Then S and T are strongly Morita equivalent if and only if there exists a regular Rees matrix posemigroup $\mathcal{R} = \mathcal{R}(S, U, V, M)$ with $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau : \mathcal{R} \to T$.*

Proof. Necessity follows from Theorem 4.11. Sufficiency is immediate from Lemma 4.12 and Theorem 3.1. □

Corollary 4.14 (cf. Cor 4.9) *If S is a regular posemigroup, then any regular Rees matrix posemigroup $\mathcal{R} = \mathcal{R}(S, U, V, M)$ is strongly Morita equivalent to S.*

Proof. Like in Corollary 4.9, the identity morphism $1_{\mathcal{R}}$ satisfies Theorem 4.13. □

4.4 Rees matrix posemigroups and extended order

Since the order in Rees matrix posemigroups only features comparisons of the type $(u, s, v) \leq (u, s', v)$, we can generalize Lemmas 4.7 and 4.12 to show the following results, the latter of which is a kind of converse to Theorem 2.9 of [35].

Corollary 4.15 *If S and T are factorizable posemigroups, $\mathcal{M} = \mathcal{M}(S, U, V, M)$ is a Rees matrix semigroup over S with a partial order that coincides with that of the corresponding Rees matrix posemigroup on subsets $\{(u_0, s, v_0) | s \in S\}$, such that $S\,\mathrm{Im}(M)S = S$, and there is a surjective strict local isomorphism $\tau : \mathcal{M} \to T$, then $C(S)$ and $C(T)$ are* Pos*-equivalent.*

Corollary 4.16 *If S and T are two regular posemigroups, $\mathcal{R} = \mathcal{R}(S, U, V, M)$ is a regular Rees matrix semigroup over S with a partial order that coincides with that of the corresponding Rees matrix posemigroup on subsets $\{(u_0, s, v_0) | s \in S\}$, $S\,\mathrm{Im}(M)S = S$ and there is a surjective strict local isomorphism $\tau : \mathcal{R} \to T$, then C(S) and C(T) are* Pos*-equivalent.*

4.5 Morita posemigroups

We extend a construction from [45] to the ordered situation. Let S be a posemigroup and let ${}_SP$ and Q_S be respectively a left and a right S-poset. If we have an (S, S)-biposet morphism

$$\langle -, - \rangle : {}_SP \times Q_S \to S,$$

then it turns out that the assignment

$$(q \otimes p)(q' \otimes p') = q \otimes \langle p, q' \rangle p' \tag{4.2}$$

defines a posemigroup structure on $Q \otimes_S P$. It is simple to check that this operation is associative:

$$\begin{aligned} ((q \otimes p)(q' \otimes p'))(q'' \otimes p'') &= q \otimes \langle \langle p, q' \rangle p', q'' \rangle p'' = q \otimes \langle p, q' \rangle \langle p', q'' \rangle p'' \\ &= (q \otimes p)((q' \otimes p')(q'' \otimes p'')). \end{aligned}$$

Verifying that the assignment is monotone (implying it is well-defined) is a bit more involved. Assume that $q \otimes p \leq \overline{q} \otimes \overline{p}$ and $q' \otimes p' \leq \overline{q}' \otimes \overline{p}'$ in $Q \otimes_S P$. By

(2.2) we have a scheme

$$\begin{array}{rclrcl} q & \leq & q_1s_1 & & & \\ q_1t_1 & \leq & q_2s_2 & s_1p & \leq & t_1p_2 \\ & \vdots & & & \vdots & \\ q_nt_n & \leq & \overline{q} & s_np_n & \leq & t_n\overline{p}, \end{array}$$

where $q_1, \ldots, q_n \in Q$, $p_2, \ldots, p_n \in P$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in S^1$. But then also

$$\begin{array}{rclrcl} q & \leq & q_1s_1 & & & \\ q_1t_1 & \leq & q_2s_2 & s_1\langle p, q'\rangle p' & \leq & t_1\langle p_2, q'\rangle p' \\ & \vdots & & & \vdots & \\ q_nt_n & \leq & \overline{q} & s_n\langle p_n, q'\rangle p' & \leq & t_n\langle \overline{p}, q'\rangle p', \end{array}$$

so

$$(q \otimes p)(q' \otimes p') \quad = \quad q \otimes \langle p, q'\rangle p' \leq \overline{q} \otimes \langle \overline{p}, q'\rangle p' = (\overline{q} \otimes \overline{p})(q' \otimes p').$$

A symmetric argument shows that $(\overline{q} \otimes \overline{p})(q' \otimes p') \leq (\overline{q} \otimes \overline{p})(\overline{q}' \otimes \overline{p}')$, therefore $(q \otimes p)(q' \otimes p') \leq (\overline{q} \otimes \overline{p})(\overline{q}' \otimes \overline{p}')$, as required.

We say that $Q \otimes_S P$ with the multiplication defined by (4.2) is the *Morita posemigroup over* S *defined by* $\langle -, - \rangle$. If Q_S and ${}_SP$ are unitary S-posets, then we call the Morita posemigroup *unitary*; if $\langle -, - \rangle$ is surjective then we say that the Morita posemigroup is *surjectively defined*.

Recall the following auxiliary result from [52].

Lemma 4.17 *Let* A *be an* (S, T)*-biposet and* B *a* (T, S)*-biposet. A* (T, T)*-act morphism* $f : {}_TB \times A_T \to {}_TC_T$ *which preserves the order in both arguments and satisfies the condition*

$$f(b \cdot s, a) = f(b, s \cdot a) \tag{4.3}$$

(usually called a balanced *morphism) yields a well-defined* (T, T)*-biposet morphism* $f' : {}_T(B \otimes_S A)_T \to {}_TC_T$ *by taking*

$$f'(b \otimes a) = f(b, a).$$

Theorem 4.18 (cf. Thm 5 of [45]) *Let* S *be a factorizable posemigroup, let* ${}_SP$ *and* Q_S *be unitary left and right* S*-posets, respectively. Furthermore, let*

$$\langle -, - \rangle : {}_SP \times Q_S \to S$$

be a surjective (S, S)*-biposet morphism. Then the Morita posemigroup* $Q \otimes_S P$ *defined by* $\langle -, - \rangle$ *is strongly Morita equivalent to* S.

Proof. We can define a right $(Q \otimes_S P)$-action on P by

$$p \cdot (q' \otimes p') = \langle p, q' \rangle p'.$$

To show that such an action makes P into a unitary $(S, Q \otimes_S P)$-biposet, we need to verify that this action is monotone (which implies that it is well-defined), associative, satisfies the equality $P(Q \otimes_S P) = P$ and that the S- and $(Q \otimes_S P)$-actions on P commute.

First, let $q' \otimes p' \leq q'' \otimes p''$ in $Q \otimes_S P$ and $p \leq \overline{p}$ in P. Then there are $q_1, \ldots, q_n \in Q$, $p_2, \ldots, p_n \in P$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in S^1$ such that

$$\begin{array}{rclrcl} q' & \leq & q_1 s_1 & & & \\ q_1 t_1 & \leq & q_2 s_2 & s_1 p' & \leq & t_1 p_2 \\ & \vdots & & & \vdots & \\ q_n t_n & \leq & q'' & s_n p_n & \leq & t_n p''. \end{array}$$

Thus

$$\begin{aligned} p \cdot (q' \otimes p') &= \langle p, q' \rangle p' \leq \langle \overline{p}, q' \rangle p' \leq \langle \overline{p}, q_1 s_1 \rangle p' \leq \langle \overline{p}, q_1 \rangle t_1 p_2 \\ &\leq \langle \overline{p}, q_2 s_2 \rangle p_2 \leq \ldots \leq \langle \overline{p}, q_n s_n \rangle p_n \leq \langle \overline{p}, q_n \rangle t_n p'' \\ &\leq \langle \overline{p}, q'' \rangle p'' = \overline{p} \cdot (q'' \otimes p''). \end{aligned}$$

Now, take $p \in P$, $s \in S$, $q' \otimes p', q'' \otimes p'' \in Q \otimes_S P$. Then

$$\begin{aligned} p \cdot ((q' \otimes p')(q'' \otimes p'')) &= \langle p, q' \rangle (\langle p', q'' \rangle p'') = \langle \langle p, q' \rangle p', q'' \rangle p'' \\ &= (p \cdot (q' \otimes p')) \cdot (q'' \otimes p'') \end{aligned}$$

and

$$(sp) \cdot (q' \otimes p') = \langle sp, q' \rangle p' = s \langle p, q' \rangle p' = s \cdot (p \cdot (q' \otimes p')).$$

Finally, P is unitary as a right $(Q \otimes_S P)$-poset since each $p \in P$ can be written as $p = sp'$ for some $s \in S$, $p' \in P$, and $s = \langle p'', q'' \rangle$ for some $p'' \in P$, $q'' \in Q$, so $p = \langle p'', q'' \rangle p' = p'' \cdot (q'' \otimes p')$.

Similarly, if we define a left $(Q \otimes_S P)$-action on Q by

$$(q' \otimes p') \cdot q = q' \langle p', q \rangle,$$

then Q becomes a unitary $(Q \otimes_S P, S)$-biposet.

Then $\langle -, - \rangle : {}_S P \times Q_S \to S$ is a $Q \otimes_S P$-balanced (S, S)-biposet morphism, as

$$\langle p, (q' \otimes p') \cdot q \rangle = \langle p, q' \langle p', q \rangle \rangle = \langle p, q' \rangle \langle p', q \rangle = \langle \langle p, q' \rangle p', q \rangle = \langle p \cdot (q' \otimes p'), q \rangle.$$

Therefore if we define another mapping $|-, -| : P \otimes_{Q \otimes_S P} Q \to S$ by taking $|p, q| = \langle p, q \rangle$, it will turn out to be a well-defined (S, S)-biposet morphism by Lemma 4.17. Taking $[-, -] := 1_{Q \otimes_S P}$, we get that $(S, Q \otimes_S P, P, Q, |-, -|, [-, -])$ is a unitary Morita context with surjective maps, since

$$|p, q| p' = p \cdot (q \otimes p') = p \cdot [q, p'] \quad \text{and} \quad [q, p] \cdot q' = (q \otimes p) \cdot q' = q |p, q'|. \qquad \square$$

Remark 4.19 Note that in the proof of Theorem 5 of [45], Talwar claims that $(R, Q \otimes_R P, P, Q, \langle -,-\rangle, [-,-])$ is the required Morita context, with the mapping $\langle -,-\rangle : {}_RP \times Q_R \to R$.

Proposition 4.20 (cf. Prop 4 of [45]) *Let* $(S, T, P, Q, \langle -,-\rangle, [-,-])$ *be a unitary Morita context of posemigroups. Then the* (S, S)*-biposet* $P \otimes_T Q$ *with multiplication* $(p \otimes q)(p' \otimes q') = p[q, p'] \otimes q'$ *and the* (T, T)*-biposet* $Q \otimes_S P$ *with multiplication* $(q \otimes p)(q' \otimes p') = q \otimes \langle p, q'\rangle p'$ *are posemigroups and the maps* $\langle -,-\rangle : P \otimes_T Q \to S$ *and* $[-,-] : Q \otimes_S P \to T$ *are posemigroup morphisms. If the latter maps are also surjective, then all the posemigroups* $P \otimes_T Q$*,* $Q \otimes_S P$*,* S *and* T *are strongly Morita equivalent.*

Proof. The first assertion follows because $Q \otimes_S P$ is the Morita posemigroup over S defined by $\langle -,-\rangle$ and $P \otimes_T Q$ is the Morita posemigroup over T defined by $[-,-]$.

Since

$$\langle p, q\rangle\langle p', q'\rangle = \langle p, q\langle p', q'\rangle\rangle = \langle p, [q, p']q'\rangle$$

and

$$[q, p][q', p'] = [q, p[q', p']] = [q, \langle p, q'\rangle p']$$

and context maps are monotone, $\langle -,-\rangle$ and $[-,-]$ are indeed posemigroup homomorphisms.

If $\langle -,-\rangle$ and $[-,-]$ are surjective, then S and T are strongly Morita equivalent by definition. Moreover, S and $Q \otimes_S P$, and T and $P \otimes_T Q$, are strongly Morita equivalent by Theorem 4.18. □

Corollary 4.21 (cf. Cor 6 of [45]) *Let* S *be a posemigroup with local units,* ${}_SP$ *a unitary left* S*-poset,* Q_S *a unitary right* S*-poset and* $\langle -,-\rangle : {}_SP \times Q_S \to S$ *a surjective* (S, S)*-biposet morphism. Then* $Q \otimes_S P$ *is a sandwich posemigroup.*

Proof. This is a repeat of Corollary 6 of [45]. □

We now link together Morita posemigroups and Rees matrix posemigroups.

Proposition 4.22 (cf. Prop 10 of [28]) *Let* S *be a posemigroup with ordered weak local units and let* $\mathcal{M} = \mathcal{M}(S, U, V, M)$ *be a Rees matrix posemigroup over* S*. Then* $\mathcal{M}$ *is isomorphic to a unitary Morita posemigroup. If* $S = S\,\mathrm{Im}(M)S$*, then that Morita posemigroup is surjectively defined.*

Proof. Let ${}_SP := {}_S(S \times V)$ and $Q_S := (U \times S)_S$ be the free S-posets with bases V and U, i.e.

$$s(z, v) = (sz, v),\ (s, v_1) \leq (z, v_2) \Leftrightarrow (s \leq z) \wedge (v_1 = v_2),$$

$$(u, s)z = (u, sz) \text{ and } (u_1, s) \leq (u_2, z) \Leftrightarrow (u_1 = u_2) \wedge (s \leq z).$$

for $s, z \in S$, $v, v_1, v_2 \in V$, $u, u_1, u_2 \in U$.

Define a map $\langle -,- \rangle : {}_S P \times Q_S \to S$ by

$$\langle (s,v),(u,z) \rangle = sM(v,u)z$$

for $s,z \in S, u \in U, v \in V$. It is monotone because if $(p,q) \leq (p',q')$, $p = (s,v)$, $p' = (s',v')$, $q = (u,z)$, $q' = (u',z')$ then $v = v'$, $u = u'$, $s \leq s'$ and $z \leq z'$. Thus

$$\langle p,q \rangle = \langle (s,v),(u,z) \rangle = sM(v,u)z \leq s'M(v',u')z' = \langle (s',v'),(u',z') \rangle = \langle p',q' \rangle.$$

Since it is clearly an (S,S)-biposet morphism, we can consider the Morita posemigroup $Q \otimes_S P$ over S defined by $\langle -,- \rangle$. Because S has weak local units, $Q \otimes_S P$ is unitary. Moreover, $Q \otimes_S P$ is surjectively defined iff $S = S\,\mathrm{Im}(M)S$.

We can now define another map $\varphi : \mathcal{M} \to Q \otimes_S P$ by

$$\varphi(u,s,v) = (u,e) \otimes (s,v),$$

where $s = es$, $e \in S$. Suppose that $s = es = e's = sf$, where $e,e',f \in S$. Then

$$(u,e) \otimes (s,v) \;=\; (u,e)s \otimes (f,v) = (u,e's) \otimes (f,v) = (u,e') \otimes (s,v),$$

so φ is well-defined.

The map φ is clearly surjective. To see that it is monotone, take comparable $(u,s,v) \leq (u,s',v)$ in $\mathcal{M}$, whence $s \leq s'$ in S. Then the existence of ordered weak local units in S provides $e,e' \in S$ such that $es = s$, $e's' = s'$ and $e \leq e'$. So

$$\varphi(u,s,v) = (u,e) \otimes (s,v) \leq (u,e') \otimes (s',v) = \varphi(u,s',v).$$

We now show that φ also reflects order. Take $\varphi(u,s,v) \leq \varphi(u',s',v')$, where $s = es$ and $s' = e's'$. Then $(u,e) \otimes (s,v) \leq (u',e') \otimes (s',v')$, whence $u = u'$, $v = v'$ and $e \otimes s \leq e' \otimes s'$ in $S \otimes_S S$. By Lemma 3.20, $s = es \leq e's' = s'$, implying that $(u,s,v) \leq (u',s',v')$ in $\mathcal{M}$.

Finally, suppose again that $s = es$ and $s' = e's'$. Then

$$\begin{aligned}
\varphi((u,s,v)(u',s',v')) &= \varphi(u,sM(v,u')s',v') \\
&= (u,e) \otimes (sM(v,u')s',v') \\
&= (u,e) \otimes (sM(v,u')e's',v') \\
&= (u,e) \otimes \langle (s,v),(u',e') \rangle (s',v') \\
&= ((u,e) \otimes (s,v))((u',e') \otimes (s',v')) \\
&= \varphi(u,s,v)\varphi(u',s',v').
\end{aligned}$$

□

Proposition 4.23 (cf. Prop 11 of [28]) *Let S be an arbitrary posemigroup and let $Q \otimes_S P$ be a unitary Morita posemigroup defined by $\langle -,- \rangle : {}_S P \times Q_S \to S$. Then there exists a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S,U,V,M)$ over S and a surjective strict local isomorphism $\tau : \mathcal{M} \to Q \otimes_S P$. If the mapping $\langle -,- \rangle$ is surjective, then $S = S\,\mathrm{Im}(M)S$.*

Proof. Take $U = Q$, $V = P$ and $M = \langle -,- \rangle$. Then we can define $\tau : \mathcal{M} \to Q \otimes_S P$ by

$$\tau(q,s,p) = q \otimes sp.$$

As ${}_S P$ is unitary, τ is surjective and trivially satisfies (LI3). Also, τ is a posemigroup homomorphism, as it is clearly monotone in s and

$$\begin{array}{rcl} \tau((q,s,p)(q',s',p')) & = & \tau(q, s\langle p,q'\rangle s', p') = q \otimes s\langle p,q'\rangle s'p' = q \otimes \langle sp,q'\rangle s'p' \\ & = & (q \otimes sp)(q' \otimes s'p') = \tau(q,s,p)\tau(q',s',p') \end{array}$$

for all $q, q' \in Q$, $s, s' \in S$, $p, p' \in P$.

To demonstrate (LI1), we need to show that τ reflects order on certain subposets. Fix $(q_1,s_1,p_1),(q_2,s_2,p_2) \in \mathcal{M}$ and $s'_1, s'_2 \in S$, $p'_2 \in P$, $q'_2 \in Q$ such that

$$(q_1,s'_1,p'_2)(q_1,s_1,p_1) = (q_1,s_1,p_1) \quad \text{and} \quad (q_2,s_2,p_2)(q'_2,s'_2,p_2) = (q_2,s_2,p_2).$$

Take $(q_1,s,p_2),(q_1,z,p_2) \in (q_1,s_1,p_1)\mathcal{M}(q_2,s_2,p_2)$ such that

$$q_1 \otimes sp_2 = \tau(q_1,s,p_2) \leq \tau(q_1,z,p_2) = q_1 \otimes zp_2$$

in $Q \otimes_S P$. Then

$$(q_1,s'_1,p'_2)(q_1,s,p_2) = (q_1,s,p_2) = (q_1,s,p_2)(q'_2,s'_2,p_2)$$

and

$$(q_1,s'_1,p'_2)(q_1,z,p_2) = (q_1,z,p_2) = (q_1,z,p_2)(q'_2,s'_2,p_2),$$

implying $s'_1\langle p'_2,q_1\rangle s = s = s\langle p_2,q'_2\rangle s'_2$ and $s'_1\langle p'_2,q_1\rangle z = z = z\langle p_2,q'_2\rangle s'_2$. The inequality $q_1 \otimes sp_2 \leq q_1 \otimes zp_2$ means that there exist $\overline{q}_1,\ldots,\overline{q}_n \in Q$, $\overline{p}_2,\ldots,\overline{p}_n \in P$ and $u_1,\ldots,u_n,v_1,\ldots,v_n \in S^1$ such that

$$\begin{array}{rclcrcl} q_1 & \leq & \overline{q}_1 u_1 & & & & \\ \overline{q}_1 v_1 & \leq & \overline{q}_2 u_2 & & u_1 s p_2 & \leq & v_1 \overline{p}_2 \\ & \vdots & & & & \vdots & \\ \overline{q}_n v_n & \leq & q_1 & & u_n \overline{p}_n & \leq & v_n z p_2. \end{array}$$

Thus

$$\begin{array}{rcl} s & = & s'_1\langle p'_2,q_1\rangle s\langle p_2,q'_2\rangle s'_2 \leq s'_1\langle p'_2,\overline{q}_1 u_1\rangle s\langle p_2,q'_2\rangle s'_2 \\ & \leq & s'_1\langle p'_2,\overline{q}_1\rangle\langle v_1\overline{p}_2,q'_2\rangle s'_2 \leq s'_1\langle p'_2,\overline{q}_2 u_2\rangle\langle \overline{p}_2,q'_2\rangle s'_2 \\ & \leq & \ldots \leq s'_1\langle p'_2,\overline{q}_n u_n\rangle\langle \overline{p}_n,q'_2\rangle s'_2 \\ & \leq & s'_1\langle p'_2,\overline{q}_n\rangle\langle v_n z p_2,q'_2\rangle s'_2 \leq s'_1\langle p'_2,q_1\rangle z\langle p_2,q'_2\rangle s'_2 = z, \end{array}$$

whence $(q_1,s,p_2) \leq (q_1,z,p_2)$.

Now we check that τ lifts idempotents, i.e. it satisfies condition (LI2). Take $q \otimes p = (q \otimes p)(q \otimes p) = q \otimes \langle p,q\rangle p \in E(Q \otimes_S P)$. The equality $q \otimes p = q \otimes \langle p,q\rangle p$ is equivalent to two inequalities $q \otimes p \leq q \otimes \langle p, q\rangle p$ and $q \otimes p \geq q \otimes \langle p, q\rangle p$. The latter means that there are $\overline{q}_1, \ldots, \overline{q}_n \in Q$, $\overline{p}_2, \ldots, \overline{p}_n \in P$ and $u_1, \ldots, u_n, v_1, \ldots, v_n \in S^1$ such that

$$\begin{array}{rclcrcl} q & \leq & \overline{q}_1 u_1 & & & & \\ \overline{q}_1 v_1 & \leq & \overline{q}_2 u_2 & \qquad & u_1\langle p, q\rangle p & \leq & v_1 \overline{p}_2 \\ & \vdots & & & & \vdots & \\ \overline{q}_n v_n & \leq & q & & u_n \overline{p}_n & \leq & v_n p. \end{array}$$

Thus

$$\begin{aligned} q\langle p, q\rangle\langle p, q\rangle &\leq \overline{q}_1 u_1 \langle p, q\rangle\langle p, q\rangle \leq \overline{q}_1 \langle v_1 \overline{p}_2, q\rangle \leq \overline{q}_2 u_2 \langle \overline{p}_2, q\rangle \\ &\leq \ldots \leq \overline{q}_n u_n \langle \overline{p}_n, q\rangle \leq \overline{q}_n \langle v_n p, q\rangle \leq q\langle p, q\rangle. \end{aligned}$$

Analogously, $q \otimes p \leq q \otimes \langle p, q\rangle p$ implies that $q\langle p, q\rangle \leq q\langle p, q\rangle\langle p, q\rangle$, so

$$q\langle p, q\rangle = q\langle p, q\rangle\langle p, q\rangle.$$

Using this equality, we calculate in $\mathcal{M}$ that

$$\begin{aligned} (q, \langle p, q\rangle^2, p)(q, \langle p, q\rangle^2, p) &= (q, \langle p, q\rangle^5, p) = (q, \langle p, q\langle p, q\rangle^4\rangle, p) \\ &= (q, \langle p, q\langle p, q\rangle\rangle, p) = (q, \langle p, q\rangle^2, p). \end{aligned}$$

Therefore $(q, \langle p, q\rangle^2, p) \in E(\mathcal{M})$ and since

$$\tau(q, \langle p, q\rangle^2, p) = q \otimes \langle p, q\rangle^2 p = (q \otimes p)^3 = q \otimes p,$$

(LI2) holds.

If $\langle -, - \rangle$ is surjective, then the unitarity of P and Q allows every $s \in S$ to be written as

$$s = \langle p, q\rangle = \langle s'p', q's''\rangle = s'\langle p', q'\rangle s'' \in S\,\mathrm{Im}(M)S.$$

□

Corollary 4.24 (cf. Cor 12 of [28]) *Let S and T be posemigroups with ordered weak local units. Then the following are equivalent:*

1) *there exists a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ for which $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau : \mathcal{M} \to T$;*
2) *there exists a surjectively defined unitary Morita posemigroup $Q \otimes_S P$ and a surjective strict local isomorphism $\tau : Q \otimes_S P \to T$.*

Proof. Necessity holds due to Proposition 4.22. Sufficiency follows from precomposing with the surjective strict local isomorphism from Proposition 4.23 (see also Lemma 4.2 (1)). □

As a consequence of Corollary 4.24 and Theorem 4.8, we have

Theorem 4.25 (cf. Thm 13 of [28]) *Let S and T be posemigroups with ordered local units. Then S is strongly Morita equivalent to T if and only if there exists a surjectively defined unitary Morita posemigroup $Q \otimes_S P$ and a surjective strict local isomorphism $\tau : Q \otimes_S P \to T$.*

5 The category of closed S-posets

In this chapter we show that under the assumption of common local units, our definition of strong Morita equivalence between two posemigroups S and T is necessary and sufficient for the Pos-equivalence of the Pos-categories of closed right S-posets and closed right T-posets. This result is an analogue of the original definition of Morita equivalence from ring theory, since Theorem 5.1 generalizes Banaschewski's theorem (Theorem 1.5) and shows that we cannot have a Pos-equivalence between the full Pos-categories of Pos_S and Pos_T without S and T being isomorphic.

5.1 Closed S-posets

We shall first show that we cannot define two posemigroups to be Morita equivalent iff the Pos-categories Pos_S and Pos_T are Pos-equivalent. We also remark on an issue that may seem to arise from this fact.

Recall from [26] that two pomonoids S and T are called Morita equivalent if the full Pos-categories of monoid posets Pos_S and Pos_T are Pos-equivalent. We now extend Banaschewski's theorem (Theorem 1.5) to posemigroups.

Theorem 5.1 *Let S and T be arbitrary posemigroups. If the Pos-categories Pos_S and Pos_T are Pos-equivalent, then S and T are isomorphic posemigroups.*

Proof. We merely repeat Banaschweski's original argument. Let S^1 and T^1 be the pomonoids obtained by adjoining external identities to S and T. The Pos-category of pomonoid S^1-posets, Pos_{S^1}, is Pos-isomorphic to the Pos-category of S-posets Pos_S (we just add or remove the external identity and its identity action to an S-poset). Note that the morphisms of those Pos-categories are exactly the same, i.e. S-poset morphisms. If Pos_S and Pos_T are Pos-equivalent, then so are Pos_{S^1} and Pos_{T^1}, i.e. S^1 and T^1 are Morita equivalent as pomonoids. Theorem 1.11 now provides $e \in E(S^1)$ and $l, l' \in S^1$ such that $el = l$, $l'l = 1$ and $eS^1e \cong T^1$ as pomonoids. Since $l'l = 1$, we have $l = l' = 1$, so $e = e1 = el = l = 1$. Consequently $S^1 = 1S^11 = eS^1e \cong T^1$ as pomonoids and thus $S \cong T$ as posemigroups. □

Therefore we need to restrict ourselves to some Pos-subcategory of Pos_S to get a useful notion of Morita equivalence. A suitable candidate is the Pos-category of closed right S-posets FPos_S. But there is the following slightly alarming observation, the consequences of which are dealt with in Remark 5.5.

Lemma 5.2 *Let S be a posemigroup with common weak local units. Then all unitary right S-posets are closed.*

Proof. Let X be a unitary right S-poset. We have to verify that $\mu_X : X \otimes_S S \to X$ reflects order. Take $x, x' \in X$, $s, s' \in S$ and let $xs \leq x's'$ in X. Then there exists $e \in E(S)$ such that $s = se$ and $s' = s'e$. Thus

$$x \otimes s = x \otimes se = xs \otimes e \leq x's' \otimes e = x' \otimes s'e = x' \otimes s'. \qquad \square$$

Proposition 5.3 *Let S and T be posemigroups with common local units. Then the following are equivalent:*

(1) *the full Pos-subcategories of unitary S- and T-posets UPos_S and UPos_T are Pos-equivalent;*

(2) *FPos_S and FPos_T are Pos-equivalent;*

(3) *$C(S)$ and $C(T)$ are Pos-equivalent.*

Proof. Lemma 5.2 provides (1)⇔(2). The equivalence (2)⇔(3) will be proved for posemigroups with ordered local units in Theorem 5.14 and Corollary 5.18. □

Remark 5.4 Both Talwar ([44], [44]) and Lawson ([29]) consider the category of unitary acts over a semigroup, although only as an intermediate step to closed (Lawson) or fixed action (Talwar) acts. Chen and Shum (see [16]) investigated the equivalences between categories of unitary acts as an independent point of interest.

Remark 5.5 While Lemma 5.2 shows that all unitary right S-posets are closed, there are always non-unitary S-posets. For example, any poset with at least two elements and with S-actions that all map every element to a single fixed element are non-unitary. Such posets are, of course, not pomonoid S-posets if S happens to be a pomonoid. Therefore we do not have the situation where if S and T are posemigroups with common local units, then all right S-posets are closed and so $S \cong T$ by Theorem 5.1. This would, for example, trivialize a number of our results on Morita invariants.

5.2 From closed S-posets to Cauchy completions

We now modify Lawson's proof (see Theorem 1.9) that Morita equivalence implies the equivalence of Cauchy completions and show the same for posemigroups.

Lemma 5.6 *Let S be a posemigroup with ordered weak local units. Then the right S-poset S_S is closed.*

Proof. Proving that the map $s \otimes s' \mapsto ss'$ is a right S-poset morphism is easy if one keeps in mind Lemma 3.20. It is surjective because S is factorizable. Finally, take $ss' \leq tt'$ for some $s, s', t, t' \in S$. Then there exist $e, e', g, g' \in S$ such that $ess' = ss'$, $e'tt' = tt'$, $e \leq e'$, $s'g = s'$ and $t'g' = t'$. So

$$s \otimes s' = ess' \otimes g = e \otimes ss' \leq e' \otimes tt' = e'tt' \otimes g' = t \otimes t'. \qquad \square$$

Lemma 5.7 *Let S be a posemigroup with ordered weak local units and let X be a right S-poset. Then $X \otimes_S S$ is a closed right S-poset.*

Proof. Because S_S is closed by Lemma 5.6, we have a right S-poset isomorphism $(X \otimes_S S) \otimes_S S \to X \otimes_S (S \otimes_S S) \to X \otimes_S S$, where $(x \otimes s) \otimes s' \mapsto x \otimes (s \otimes s') \mapsto x \otimes (ss')$. But this is exactly $\mu_{X \otimes_S S}$, since $\mu_{X \otimes_S S}((x \otimes s) \otimes s') = (x \otimes s)s' = x \otimes (ss')$. Therefore $X \otimes_S S$ is closed. $\square$

Lemma 5.8 (cf. Prop 2.3 of [29]) *Let S be a posemigroup with ordered weak local units and let X be a unitary right S-poset. Then the following are equivalent*

1) *X is closed;*

2) *$X \otimes_S S \cong X$ for some isomorphism in Pos_S.*

Proof. The implication 1) ⇒ 2) is trivial. For the converse, let $\varphi : X \otimes_S S \to X$ be a right S-poset isomorphism. Then $\varphi \otimes_S S : (X \otimes_S S) \otimes_S S \to X \otimes_S S$ must also be an isomorphism in Pos_S because Pos-functors preserve isomorphisms. Observe that $\mu_X : X \otimes_S S \to X$ is natural in X, since for any $f : X \to X'$ and $x \in X$, $s \in S$ we have

$$(\mu_{X'} \circ (f \otimes_S S))(x \otimes s) = f(x)s = f(xs) = (f \circ \mu_X)(x \otimes s).$$

Therefore we have a commutative diagram in Pos_S:

$$\begin{tikzcd}
(X \otimes_S S) \otimes_S S \arrow[r, "\mu_{X \otimes_S S}"] \arrow[d, "\varphi \otimes_S S"'] & X \otimes_S S \arrow[d, "\varphi"] \\
X \otimes_S S \arrow[r, "\mu_X"'] & X
\end{tikzcd}.$$

Now, $X \otimes_S S$ is closed by Lemma 5.7 and therefore $\mu_{X \otimes_S S}$ is an isomorphism. Thus μ_X must also be an isomorphism. $\square$

The previous lemma allows us to ignore checking that an isomorphism $X \otimes_S S \cong X$ is actually μ_X if we need to show that a unitary right S-poset X is closed.

Lemma 5.9 *Coproducts in* Pos_S *and* FPos_S *are constructed as disjoint unions with componentwise order and action.*

Proof. For Pos_S, this is essentially proved in Section 2 of [14]. Observe that the Pos-functor $- \otimes_S S : \mathsf{Pos}_S \to \mathsf{Pos}_S$ as a left adjoint preserves colimits. Thus if we take $X_i \in \mathsf{FPos}$, $i \in I$, we get the following isomorphisms in Pos_S:

$$(\coprod_i X_i) \otimes_S S \cong \coprod_i (X_i \otimes_S S) \cong \coprod_i X_i.$$

So a Pos_S-coproduct of closed right S-posets is closed by Lemma 5.8. Because FPos_S is a full Pos-subcategory, the FPos_S-coproduct of closed right S-posets is the same as in Pos_S. □

The amalgamated coproduct of a right S-poset B with itself over an S-subposet $B' \subseteq B$ is the pushout of the embedding $B' \hookrightarrow B$ along itself. Let us recall from Section 2 of [14] that it can be realized as the set $(\{1,2\} \times (B \setminus B')) \cup B'$ with the action

$$(i,b)s = \begin{cases} (i,bs) & \text{if } bs \in B \setminus B' \\ bs & \text{if } bs \in B' \end{cases}, i = 1,2, b \in B \setminus B', s \in S.$$

The order relation for $b_1, b_2 \in B \setminus B'$, $i, j \in \{1,2\}$ is

$$(i,b_1) \leq (j,b_2) \iff (i = j \wedge b_1 \leq b_2) \vee (i \neq j \wedge b_1 \leq b'' \leq b_2 \text{ for some } b'' \in B').$$

For $b' \in B'$, $b \in B \setminus B'$, $i \in \{1,2\}$, it is

$$b' \leq (i,b) \iff b' \leq b \quad \text{and} \quad (i,b) \leq b' \iff b \leq b'.$$

The amalgamated coproduct of B with itself over B' is denoted by $B \coprod_{B'} B$. The pushout morphisms $i_1 : B \to B \coprod_{B'} B$ and $i_2 : B \to B \coprod_{B'} B$ are defined by

$$i_j(b) = \begin{cases} (j,b) & \text{if } b \in B \setminus B', \\ b & \text{if } b \in B'. \end{cases}$$

Lemma 5.10 (cf. Prop 2.4 of [29]) *Let S be a posemigroup with ordered weak local units. Then all epimorphisms in* FPos_S *are surjective.*

Proof. Let $f : A \to B$ be an epimorphism in FPos_S. Lemma 5.7 shows that $A \otimes_S S$ and $B \otimes_S S$ are closed. Since the same lemma also demonstrated that μ_X is natural in X, we have the following commutative diagram in FPos_S:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
A \otimes S \arrow[u, "\mu_A"] \arrow[r, "f \otimes S"'] & B \otimes_S S \arrow[u, "\mu_B"']
\end{tikzcd}$$

Since μ_A is an epimorphism, $f \otimes_S S \in \mathsf{FPos}_S$ as a product of epimorphisms must also be an epimorphism. Assume that f is not surjective, put $B' = \mathrm{Im} f$, take the embedding $\iota : B' \hookrightarrow B$ and the amalgamated coproduct $C := B \coprod_{B'} B$ in Pos_S as constructed above and let $i_1, i_2 : B \to C$ be the pushout maps. Then obviously $i_1 f = i_2 f$. Yet for all $b \in B \setminus B'$ we get that $i_1(b) = (1, b) \neq (2, b) = i_2(b)$. Observe that due to Lemma 5.7 we have the following commutative diagram in FPos_S:

$$\begin{tikzcd}
A \otimes_S S \arrow[r, "f \otimes_S S"] & B \otimes_S S \arrow[r, shift left, "i_1 \otimes_S S"] \arrow[r, shift right, "i_2 \otimes_S S"'] & C \otimes_S S.
\end{tikzcd}$$

Take $b \in B \setminus B' \neq \emptyset$ and $s \in S$ such that $bs = b$ (this can be done as B is unitary). If $(i_1 \otimes_S S)(b \otimes s) = (i_2 \otimes_S S)(b \otimes s)$ then $i_1(b) \otimes s = i_2(b) \otimes s$. Therefore $\mu_C(i_1(b) \otimes s) = \mu_C(i_2(b) \otimes s)$, so $i_1(b) = i_1(bs) = i_1(b)s = i_2(b)s = i_2(bs) = i_2(b)$, a contradiction. So f must be surjective. □

We say that an object A from some category $\mathcal{A}$ of either S-acts or S-posets is *indecomposable* if there do not exist non-initial objects $A_1, A_2 \in \mathcal{A}_0$ such that $A \cong A_1 \coprod A_2$. In particular, we use this in the cases $\mathcal{A} = \mathsf{Act}_S$, $\mathcal{A} = \mathsf{Pos}_S$ and $\mathcal{A} = \mathsf{FPos}_S$.

Lemma 5.11 (cf. Lemma 3.1 of [29]) *Let S be a posemigroup with ordered weak local units. Then the right S-posets eS, $e \in E(S)$, are indecomposable and projective in* FPos_S.

Proof. First, the right S-posets eS are clearly unitary because $es = (ee)s$ for any $s \in S$. If we take $s, s', t, t' \in S$ and let $est \leq es't'$, then

$$es \otimes t = ees \otimes t = e \otimes est \leq e \otimes es't' = ees' \otimes t' = es' \otimes t'$$

in $eS \otimes_S S$. So eS are also closed.

Since the underlying S-acts eS are indecomposable (into any coproduct of S-acts), eS are also indecomposable as closed right S-posets. Now, let $f : A \to B$ be an epimorphism in FPos_S and $g : eS \to B$ an arbitrary morphism in FPos_S. Then f is surjective by Lemma 5.10 and there exists $a \in A$ so that $f(a) = g(e)$.

$$\begin{tikzcd}
 & eS \arrow[ld, "\lambda_a"'] \arrow[d, "g"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Therefore the left translation $\lambda_a : eS \to A$ is still the required unique right S-poset morphism for which $f\lambda_a = g$. □

Lemma 5.12 (cf. Lemma 3.2 of [29]) *Let S be a posemigroup with ordered local units. If $A \in \mathsf{FPos}_S$ then there exists a projective $P \in \mathsf{FPos}_S$ and an epimorphism $\pi : P \to A$ in FPos_S.*

Proof. Take $A \in \mathsf{FPos}_S$. Since A is unitary and S has local units, for every $a \in A$ there exists $e_a \in E(S)$ such that $ae_a = a$. Form the coproduct $\coprod_{a\in A} e_aS$. Because e_aS are closed and the left adjoint $- \otimes_S S : \mathsf{Pos}_S \to \mathsf{Pos}_S$ preserves coproducts, $\coprod_{a\in A} e_aS$ is closed by Lemma 5.8. Since e_aS are also projective and coproducts of projectives are projective, $\coprod_{a\in A} e_aS$ is a projective in FPos_S. As the S-poset morphism $\pi : \coprod_{a\in A} e_aS \to A$, defined by $\pi(e_as) = as$, is an epimorphism in Pos_S, it is also an epimorphism in FPos_S. □

Proposition 5.13 (cf. Prop 3.3 of [29]) *Take a posemigroup S with ordered local units. Then a closed right S-poset is indecomposable and projective in FPos_S if and only if it is isomorphic to eS for some $e \in E(S)$.*

Proof. Sufficiency is proved in Lemma 5.11. Take an indecomposable projective P in FPos_S. Lemma 5.12 provides an epimorphism $\pi : \coprod_{p\in P} e_pS \to P$, with $e_p \in E(S)$. Since P is projective, there is a coretraction $\iota : P \to \coprod_{p\in P} e_pS$ so that $\pi\iota = 1_P$. Therefore $\iota(P) = \mathrm{Im}\iota \cong P$ is also indecomposable, so $\iota(P) \subseteq e_pS$ for some $p \in P$.

$$\begin{tikzcd} & P \arrow[ld, "\iota"'] \arrow[d, "1_P"] \\ \coprod_{p\in P} e_pS \arrow[r, "\pi"'] & P \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[ld, "\varphi"'] \arrow[d, "1_P"] \\ e_pS \arrow[r, "\lambda_x"'] & P \end{tikzcd}$$

Because $\pi\iota = 1_P$, we get that $P = \pi(e_pS) = \pi(e_p)S$. Denoting $x = \pi(e_p) \in P$, we thus have $P = xS$ and $xe_p = \pi(e_p)e_p = \pi(e_p^2) = \pi(e_p) = x$. We can now consider the epimorphism $\lambda_x : e_pS \to P$ in FPos_S and use the projectivity of P to find a coretraction $\varphi : P = xS \to e_pS$ such that $\lambda_x\varphi = 1_P$. If we put $e = \varphi(x)$ then

$$e^2 = \varphi(x)e = \varphi(xe) = \varphi(\lambda_x(e)) = \varphi(\lambda_x(\varphi(x))) = \varphi((\lambda_x\varphi)(x)) = \varphi(x) = e,$$

so $e \in E(S)$. Because the coretraction φ is a regular monomorphism and $e = \varphi(x)$, $P \cong eS$, as required. □

Theorem 5.14 (cf. Thm 3.4 of [29]) *Let S and T be posemigroups with ordered local units. If FPos_S and FPos_T are Pos-equivalent, then $C(S)$ and $C(T)$ are also Pos-equivalent.*

Proof. Let the Pos-functors $F : \mathsf{FPos}_S \to \mathsf{FPos}_T$ and $G : \mathsf{FPos}_T \to \mathsf{FPos}_S$ form a Pos-equivalence. Since F and G map indecomposable projectives to indecomposable projectives, the full Pos-subcategories FIP_S and FIP_T generated by indecomposable projectives are also Pos-equivalent. Due to Proposition 5.13, every indecomposable projective in FPos_S is isomorphic to eS for some $e \in E(S)$, so those poset isomorphisms provide a Pos-equivalence between FIP_S and the full Pos-subcategory of FPos_S generated by objects eS, $e \in E(S)$, which we denoted by $\mathcal{IP}_S$. Similarly, FIP_T is Pos-equivalent to $\mathcal{IP}_T$. Proposition 2.32 shows that the Pos-categories $\mathcal{IP}_S$ and $C(S)$ (and $\mathcal{IP}_T$ and $C(T)$) are Pos-equivalent and therefore $C(S)$ and $C(T)$ are Pos-equivalent. □

5.3 From Cauchy completions to closed S-posets

We now prove the converse of what we showed in the previous section, namely that the Pos-equivalence of Cauchy completions $C(S)$ and $C(T)$ implies the Pos-equivalence of the Pos-categories FPos_S and FPos_T.

Lemma 5.15 (cf. Prop 3.14 of [29]) *Let S and T be posemigroups with local units that have a joint enlargement R such that the biposets $P = SRT \in {}_S\mathsf{Pos}_T$ and $Q = TRS \in {}_T\mathsf{Pos}_S$ are Pos-unitary. Then the biposets P and Q (constructed for the Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ in Proposition 3.21) are closed as right T- and S-posets .*

Proof. Since $P = SRT$ and $Q = TRS$ are Pos-unitary, they are also unitary. For them to be closed as well it is sufficient to prove that the mappings μ_P and μ_Q are order-reflecting. We will only show this for $\mu_P : P \otimes_T T \to P$, since the case for Q is essentially the same. Let $pt \le p't'$ in P. Because $P = SRT$ and T has weak local units, we can write

$$p = s_1 r_1 t_1, p' = s'_1 r'_1 t'_1, t = tu_1, t' = t'u_2$$

for $s_1, s'_1 \in S$, $t_1, t'_1, u_1, u_2 \in T$, $r_1, r'_1 \in R$. As P is Pos-unitary, $S \subseteq R = RTR$ and T has weak local units, we can find $s_2, s'_2 \in E(S)$, $r_3, r_4 \in R$, $t_2, u \in T$ so that

$$s_2 pt = pt, s'_2 p't' = p't', \quad s_2 \le s'_2, s'_2 = r_3 t_2 r_4 \quad \text{and} \quad t_2 = ut_2.$$

Then $t_2 r_4 p = t_2 r_4 s_1 r_1 t_1 \in TRSRT = TRT = T$, $t_2 r_4 p' = t_2 r_4 s'_1 r'_1 t'_1 \in T$ and $s'_2 r_3 u \in SRT = P$. We can now calculate in $P \otimes_T T$ that

$$\begin{aligned}
p \otimes t &= p \otimes tu_1 = pt \otimes u_1 = s_2 pt \otimes u_1 \le s'_2 pt \otimes u_1 = (s'_2)^2 pt \otimes u_1 \\
&= s'_2 (r_3 t_2 r_4) pt \otimes u_1 = s'_2 r_3 (ut_2) r_4 pt \otimes u_1 = s'_2 r_3 ut_2 r_4 p \otimes tu_1 \\
&= s'_2 r_3 ut_2 r_4 p \otimes t = (s'_2 r_3 u)(t_2 r_4 p) \otimes t = s'_2 r_3 u \otimes (t_2 r_4 p)t \\
&= s'_2 r_3 u \otimes (t_2 r_4)(pt) \le s'_2 r_3 u \otimes (t_2 r_4)(p't') = s'_2 r_3 u \otimes (t_2 r_4 p')t' \\
&= s'_2 r_3 u \otimes (t_2 r_4 p')(t'u_2) = s'_2 r_3 u \otimes ((t_2 r_4 p')t')u_2 \\
&= (s'_2 r_3 u)(t_2 r_4 p')t' \otimes u_2 = s'_2 r_3 (ut_2) r_4 p't' \otimes u_2 \\
&= s'_2 (r_3 t_2 r_4) p't' \otimes u_2 = (s'_2)^2 (p't') \otimes u_2 = s'_2 (p't') \otimes u_2 \\
&= p't' \otimes u_2 = p' \otimes t'u_2 = p' \otimes t'.
\end{aligned}$$

□

Corollary 5.16 *Let S and T be posemigroups with ordered local units that have a joint enlargement R, with* Pos*-unitary ${}_SSRT_T \in {}_S\mathsf{Pos}_T$ and ${}_TTRS_S \in {}_T\mathsf{Pos}_S$. Then there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps where P and Q are closed right S- and T-posets.*

Proof. Due to Proposition 3.21, there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps. Since $P = SRT$ and $Q = TRS$ are Pos-unitary, we can apply Lemma 5.15 and get that P and Q are closed right S- and T-posets. □

Theorem 5.17 (cf. Prop 3.16 of [29], Thm 1.1) *Let S and T be posemigroups with ordered local units. If $(S, T, P, Q, \langle -, - \rangle, [-, -])$ is a unitary Morita context with surjective maps and P and Q are closed, then the* Pos*-categories* FPos_S *and* FPos_T *are* Pos*-equivalent via* Pos*-functors*

$$- \otimes_Q P : \mathsf{FPos}_S \to \mathsf{FPos}_T \quad \text{and} \quad - \otimes_T Q : \mathsf{FPos}_T \to \mathsf{FPos}_S .$$

Proof. The proof closely mirrors Proposition 3.16 of [29]. Let $A \in \mathsf{FPos}_S$ be a closed right S-poset. Then $(A \otimes P) \otimes T \cong A \otimes (P \otimes T) \cong A \otimes P$ and the first Pos-functor is well-defined. The same applies to the second Pos-functor. Fix a closed right S-poset A. Then we have right S-poset isomorphisms

$$(A \otimes P) \otimes Q \cong A \otimes (P \otimes Q) \cong A \otimes S \cong A$$

by Corollary 2.37, mapping

$$(a \otimes p) \otimes q \mapsto a \otimes (p \otimes q) \mapsto a \otimes \langle p, q \rangle \mapsto a\langle p, q \rangle,$$

so their composite $\alpha_A : (A \otimes P) \otimes Q \to A$ maps $\alpha_A((a \otimes p) \otimes q) = a\langle p, q \rangle$. Now we merely need to demonstrate the naturality of α in A. Take $f : A \to A'$ in FPos_S and $a \in A$, $p \in P$, $q \in Q$. Then

$$\begin{aligned}
(f\alpha_A)((a \otimes p) \otimes q) &= f(a\langle p, q \rangle) = f(a)\langle p, q \rangle = \alpha_{A'}((f(a) \otimes p) \otimes q) \\
&= (\alpha_{A'}((f \otimes P) \otimes Q))((a \otimes p) \otimes q).
\end{aligned}$$

$$\begin{tikzcd}
(A \otimes P) \otimes Q \arrow[r, "\alpha_A"] \arrow[d, "(f\otimes P)\otimes Q"'] & A \arrow[d, "f"] \\
(A' \otimes P) \otimes Q \arrow[r, "\alpha_{A'}"'] & A'
\end{tikzcd}$$

Similarly, there exists a natural isomorphism $\beta : (- \otimes_T Q) \otimes_S P \to 1_{\mathsf{FPos}_T}$. □

Corollary 5.18 *Let S and T be posemigroups with common local units. If the Cauchy completions $C(S)$ and $C(T)$ are* Pos*-equivalent, then the* Pos*-categories* FPos_S *and* FPos_T *are* Pos*-equivalent.*

Proof. Theorem 3.17 and Corollary 3.18 show that S and T have a joint enlargement R such that $P = SRT \in {}_S\mathsf{Pos}_T$ and $Q = TRS \in {}_T\mathsf{Pos}_S$ are Pos-unitary. By Corollary 5.16, there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps where P and Q are closed right S- and T-posets. Therefore we can apply Theorem 5.17 to get that the Pos-categories FPos_S and FPos_T are Pos-equivalent. □

Example 3.19 shows that our assumption of the existence of common local units is an artefact of the proof used in Theorem 3.17 and it is not necessary for satisfying the Pos-unitarity requirement of Corollary 5.16. Thus we have the following open question.

Problem 5.19 Let S and T be posemigroups with ordered local units. If the Cauchy completions $C(S)$ and $C(T)$ are Pos-equivalent, are the Pos-categories FPos_S and FPos_T also Pos-equivalent?

Lemma 5.20 (cf. Lemma 3.17 of [29]) *Let S and T be two posemigroups with ordered local units. If $(S, T, P, Q, \langle -, - \rangle, [-, -])$ is a unitary Morita context with surjective maps and P and Q are closed as right posets then they are also closed as left posets.*

Proof. By Corollary 2.37, we have the following (S, T)-biact isomorphisms:

$$S \otimes P \cong (P \otimes Q) \otimes P \cong P \otimes (Q \otimes P) \cong P \otimes T \cong P$$

and

$$T \otimes Q \cong (Q \otimes P) \otimes Q \cong Q \otimes (P \otimes Q) \cong Q \otimes S \cong Q.$$ □

Proposition 5.21 (cf. Thm 3.18 of [29]) *Let S and T be two posemigroups with common local units. If the Pos-categories FPos_S and FPos_T are Pos-equivalent then the Pos-categories ${}_S\mathsf{FPos}$ and ${}_T\mathsf{FPos}$ are also Pos-equivalent.*

Proof. By Theorem 3.1, Theorem 5.14 and Corollary 5.18, the Pos-categories FPos_S and FPos_T are Pos-equivalent if and only if there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ with surjective maps and P and Q closed as right posets. Dualizing this result to the Pos-categories ${}_S\mathsf{FPos}$ and ${}_T\mathsf{FPos}$ yields a similar condition with P and Q closed as left posets. Now we only need to apply Lemma 5.20. □

6 Properties of Morita equivalence

We now turn to the applications of Theorem 3.1. As expected, strong Morita equivalence preserves a great deal of algebraic structure, although occasionally we need to assume the existence of stronger local units. Somewhat surprisingly, the order structure is very much variable. Some of it can still be recovered if we require the existence of even stronger local units. In total, we prove a series of necessary and sufficient conditions for when a posemigroup is strongly Morita equivalent to another posemigroup from a given well-known class of posemigroups. Furthermore, we establish a number of Morita invariants such as lattices of ideals, the validity of inequalities and greatest commutative images. The chapter is based on [47] and [50], with a small section from [49].

6.1 Strong Morita equivalence classes and Morita invariants in general

We first provide two corollaries of Theorem 3.1, showing that while the algebraic properties of strongly Morita equivalent posemigroups are still quite close, their order-related features may be entirely different. Moreover, it turns out that if two posemigroups are strongly Morita equivalent, then they are also strongly Morita equivalent as semigroups.

Proposition 6.1 (cf. Prop 5.1 of [29]) *Let S and T be posemigroups with local units which are strongly Morita equivalent. Then*

(1) *each local subpomonoid of S is isomorphic to a local subpomonoid of T and vice versa;*

(2) *S is regular if and only if T is regular;*

(3) *the cardinalities of the sets of regular $\mathcal{D}$-classes in S and T are the same;*

(4) *the lattices of two-sided ideals in S and T are order-isomorphic;*

(5) *the posets of principal two-sided ideals in S and T are order-isomorphic.*

Proof. Strong Morita equivalence holds if and only if $C(S)$ and $C(T)$ are Pos-equivalent by Theorem 3.1. But Pos-equivalence implies usual equivalence of categories. Therefore $C(S)$ is equivalent to $C(T)$ and our proposition holds by Theorem 1.1 and Proposition 5.1 of [29]. □

Proposition 6.2 *The one-element posemigroup* **1** *is strongly Morita equivalent to any poset considered as a left zero semigroup.*

Proof. It is clear that such left zero semigroups are posemigroups. Let S be one such. Then it is easy to verify that $C(S)_0 = S$ and

$$C(S)(t, s) = \{(s, s, t)\}$$

for all $s, t \in S$. So $C(S)$ is a groupoid in the category theoretical sense. Thus $C(S)$ is Pos-equivalent to $C(\mathbf{1})$ via the unique Pos-functor $C(S) \to C(\mathbf{1})$. □

The above example is also a consequence of Proposition 6.10.

We need the following lemma to show that strong Morita equivalence of posemigroups implies Morita equivalence of semigroups. To state it precisely, let us introduce a notation of convenience. Let S and T be posemigroups, let P be an (S, T)-biposet and Q a (T, S)-biposet. Then we have two tensor products: that of the biposets P and Q over the posemigroup T, which we have denoted by $P \otimes_T Q$, and that of biacts P and Q over the semigroup T, which we will denote by $P\bar{\otimes}_T Q$. The first is naturally an (S, S)-biposet and the second an (S, S)-biact. We will write $p\bar{\otimes}q$ for the elements of $P\bar{\otimes}_T Q$.

Lemma 6.3 *The mapping $\pi : P\bar{\otimes}_T Q \to P \otimes_T Q$, defined by $p\bar{\otimes}q \mapsto p \otimes q$, is a surjective (S, S)-biact morphism.*

Proof. First, $p\bar{\otimes}q = p'\bar{\otimes}q'$ iff there is a scheme

$$\begin{array}{rclcrcl}
p & = & p_1u_1 & & & & \\
p_1v_1 & = & p_2u_2 & & u_1q & = & v_1q_2 \\
 & \vdots & & & & \vdots & \\
p_nv_n & = & p' & & u_nq_n & = & v_nq'
\end{array}$$

for some $u_1, \ldots, u_n, v_1, \ldots, v_n \in T^1$. Second, $p \otimes q = p' \otimes q'$ iff $p \otimes q \leq p' \otimes q'$ and $p \otimes q \geq p' \otimes q'$, and $p \otimes q \leq p' \otimes q'$ iff there is a scheme

$$\begin{array}{rclcrcl}
p & \leq & p_1u_1 & & & & \\
p_1v_1 & \leq & p_2u_2 & & u_1q & \leq & v_1q_2 \\
 & \vdots & & & & \vdots & \\
p_nv_n & \leq & p' & & u_nq_n & \leq & v_nq'
\end{array}$$

for some $u_1, \ldots, u_n, v_1, \ldots, v_n \in T^1$. Obviously $p\bar{\otimes}q = p'\bar{\otimes}q'$ implies $p\otimes q \leq p'\otimes q'$ and $p \otimes q \geq p' \otimes q'$ and therefore π is a well-defined mapping. It is clearly both surjective and an (S, S)-biact morphism. □

Note that since π is a proper mapping, equalities (M3) (i) and (ii) will still hold if we precompose $\langle -, - \rangle$ and $[-, -]$ with it. So $(S, T, P, Q, \langle -, - \rangle \circ \pi, [-, -] \circ \pi)$ is a Morita context of semigroups with surjective mappings and we have proved the following result.

Proposition 6.4 *If two posemigroups S and T are strongly Morita equivalent as posemigroups, then they are Morita equivalent as semigroups.*

It is easy to see that the converse of Proposition 6.4 does not hold in general (see Example 6.9 for a slightly non-trivial example).

6.2 Characterizations of some strong Morita equivalence classes

We now verify a series of necessary and sufficient conditions for a given posemigroup to be strongly Morita equivalent to a posemigroup from one of a number of well-known classes of (po)semigroups, such as groups and monoids. Because of Proposition 6.4, these are extensions of similar results that hold for unordered semigroups (see [29] or [25], for instance).

Proposition 6.5 (cf. Thm 8 of [25]) *Let S be a posemigroup. Then the following are equivalent:*

(1) *S is strongly Morita equivalent to a posemigroup with local units;*

(2) *S is strongly Morita equivalent to a sandwich posemigroup;*

(3) *S is strongly Morita equivalent to the posemigroup $E(S)SE(S)$;*

(4) *S is a sandwich posemigroup.*

Proof. We have (1) ⇒ (2) because having local units implies being a sandwich posemigroup, (2) ⇒ (4) because it holds for unordered semigroups. (4) ⇒ (3) is similar to Theorem 8 of [25] and we can take $T = \bigcup_{e,e' \in E(S)} eSe' = E(S)SE(S)$, $P = \bigcup_{e \in E(S)} Se = SE(S)$ and $Q = \bigcup_{e \in E(S)} eS = E(S)S$ as sub(po)semigroups of S and define the mappings $\langle -, - \rangle : P \otimes_T Q \to S$ and $[-, -] : Q \otimes_S P \to T$ by

$$\langle se, e's' \rangle = see's' \quad \text{and} \quad [es, s'e'] = ess'e'.$$

The mapping $\langle -, - \rangle$ is surjective because S is a sandwich semigroup and $[-, -]$ is surjective because S is factorizable. By Lemma 3.20, these mappings are monotone, so we do have a unitary posemigroup Morita context with surjective mappings. Finally, (3) ⇒ (1) is trivial. □

Proposition 6.6 (cf. Thm 9 of [25]) *Let S be a posemigroup. Then the following are equivalent:*

(1) *S is strongly Morita equivalent to a pomonoid;*

(2) *S is strongly Morita equivalent to eSe for some $e \in E(S)$;*

(3) *$S = SeS$ for some $e \in E(S)$.*

Proof. The same as before. (1) ⇒ (3) because it holds in the unordered case (note that Theorem 9 of [25] shows that we can take $e = (\langle p, q\rangle)^2$, where $[q, p] = 1$ is the identity of the pomonoid) and (2) ⇒ (1) trivially. (3) ⇒ (2) can be done as in Theorem 9 of [25]. We take $T = eSe$, $P = Se$, $Q = eS$ and define the context maps by

$$\langle se, es'\rangle = ses' \quad \text{and} \quad [es, s'e] = ess'e.$$

Lemma 3.20 again shows that such maps are monotone. □

A *pogroup* is a group that has a partial order compatible with its multiplication. We say that a partially ordered Rees matrix semigroup $S = \mathcal{M}(G, I, J, M)$ over a pogroup G is *normally ordered* if $eSe \cong G$ as posemigroups for all $e \in E(S)$. Note that we always have $eSe \cong G$ as semigroups but not necessarily as posemigroups. Therefore not every partially ordered Rees matrix semigroup is normally ordered. A simple example of such is obtained by taking a non-trivially ordered pogroup G, a poset I, a constant matrix map M and ordering the Rees matrix semigroup $\mathcal{M}(G, I, I, M)$ by

$$(i, g, j) \leq (i', g', j') \Leftrightarrow (i \leq i') \wedge (j \leq j') \wedge (g = g').$$

Theorem 6.7 (cf. Thm 12 of [25]) *Let S be a posemigroup. Then S is strongly Morita equivalent to a pogroup G if and only if S is isomorphic to a normally ordered Rees matrix semigroup over G.*

Proof. Suppose that S is strongly Morita equivalent to a pogroup G. Then it is isomorphic (as a semigroup) to a Rees matrix semigroup over the group G by Proposition 6.4 and Theorem 12 of [25]. Let $\varphi : S \to \mathcal{M}(G, U, V, M)$ be that isomorphism. Then we can define a partial order on the Rees matrix semigroup by taking $\varphi(s) \leq \varphi(s')$ iff $s \leq s'$. In this way, φ will also be a posemigroup isomorphism between S and the partially ordered Rees matrix semigroup. Due to Proposition 3.22, $C(S)$ and $C(G)$ are Pos-equivalent Pos-categories, whence $eSe \cong C(S)(e, e) \cong C(G)(1, 1) \cong G$ as posemigroups for all $e \in E(S)$.

If S is isomorphic to a normally ordered Rees matrix semigroup, then it is a completely simple semigroup and hence there exists an idempotent e such that eSe is a group. In our case, it is also a pogroup and $eSe \cong G$ since the partially ordered Rees matrix semigroup is normally ordered. Now we merely need to take $T = eSe$ and complete the proof exactly as it was done in Proposition 6.6. □

Corollary 6.8 (cf. Cor 13 of [25]) *Two normally ordered Rees matrix semigroups* $S = \mathcal{M}(G, I, J, M)$ *and* $S' = \mathcal{M}(G', I', J', M')$ *are strongly Morita equivalent if and only if* $G \cong G'$ *as pogroups.*

Proof. By Corollary 3.6, every local subpomonoid eSe of S is isomorphic to some local subpomonoid $fS'f$ of S', $e \in E(S)$, $f \in E(S')$. Therefore normality implies that $G \cong eSe \cong fS'f \cong G'$ as posemigroups for some $e \in E(S)$, $f \in E(S')$. Hence $G \cong G'$ as pogroups. For the converse, $G \cong G'$ implies strong Morita equivalence of G and G' due to Theorem 3.1 (2), S is strongly Morita equivalent to G and S' is strongly Morita equivalent to G' by Theorem 6.7, and the relation of strong Morita equivalence is symmetric and transitive. □

Example 6.9 A non-trivially ordered group G is Morita equivalent to any Rees matrix semigroup over its underlying (trivially ordered) group $\underline{G}$ by Corollary 13 of [25], yet it is not strongly Morita equivalent to the corresponding Rees matrix posemigroup over $\underline{G}$ by Corollary 6.8.

Proposition 6.10 (cf. Thm 16 of [25]) *Let* S *be any posemigroup. Then* S *is strongly Morita equivalent to a one-element posemigroup if and only if* S *is a rectangular poband.*

Proof. The same as in Proposition 6.5, using Theorem 16 of [25]. □

Corollary 6.11 *All rectangular bands with any compatible order are strongly Morita equivalent.*

Note that the compatible orders on rectangular bands are exactly the product orders.

Example 6.12 The one-element pomonoid is strongly Morita equivalent to any non-trivial rectangular poband, which is not a monoid.

Proposition 6.13 (cf. Thm 5.5 (4), (6), (7) of [29]) *Let* S *be a posemigroup with weak local units. Then* S *is strongly Morita equivalent to an orthodox posemigroup (*E*-solid posemigroup; union of groups) if and only if it is regular and locally orthodox (regular and locally* E*-solid; regular, locally a union of groups and* $S/\mathcal{J}$ *is a lower semilattice under subset inclusion).*

Proof. Necessity holds due to Proposition 2 of [27], Proposition 3.22, an argument similar to that used in proving Corollary 3.6, and Proposition 5.1 (5) of [29], since regularity implies the existence of local units.

For sufficiency, let S be regular and either locally orthodox, locally E-solid or locally a union of groups and such that $S/\mathcal{J}$ is a lower semilattice under subset inclusion. In Sections 3 and 5 of [33], McAlister constructs a consolidation q on S such that $C(S)^q$ is regular and respectively orthodox, E-solid or a union of groups. Since the identity $1_{C(S)^q}$ is certainly a strict local isomorphism, S is strongly Morita equivalent to $C(S)^q$ by Theorem 3.1. □

A regular posemigroup S is called *$\mathcal{L}$-unipotent* if each $\mathcal{L}$-class of S contains exactly one idempotent.

Proposition 6.14 (cf. Prop 1.2 and Prop 1.6 of [33]) *Let S be a regular posemigroup the idempotents of which satisfy the identity $uef = ufef$ and let ρ be a posemigroup congruence on S defined by*

$$s\,\rho\,t \Leftrightarrow xsy = xty \quad \forall x, y \in S.$$

Then the projection homomorphism $\pi : S \to S/\rho$ is a surjective strict local isomorphism to an $\mathcal{L}$-unipotent posemigroup.

Proof. It is easy to see that ρ is compatible with multiplication. We first show that $s \leq_\rho t$ implies $xsy \leq xty$ for all $x, y \in S$. Fix $x, y \in S$. If there exist $n \in \mathbb{N}$ and $s_i, t_i \in S$, $i = 1, \ldots, n$ such that

$$s \leq s_1 \rho t_1 \leq s_2 \ldots s_n \rho t_n \leq t,$$

then

$$xsy \leq xs_1y = xt_1y \leq xs_2y = \ldots \leq xs_nt = xt_ny \leq xty.$$

The relation ρ satisfies the closed chains condition, i.e. it is a posemigroup congruence, because if $s \leq_\rho t \leq_\rho s$ and $x, y \in S$, then $xsy \leq xty \leq xsy$, implying $xsy = xty$. Now, π satisfies (LI2) due to Lallement's Lemma because its domain is regular. (LI3) is a consequence of the surjectivity of π. By Lemma 2.42, we can replace condition (LI1) with (LI1'). So take $s, t \in eSf$ for some $e, f \in E(S)$. Assume that $\pi(s) \leq \pi(t)$, i.e. $s \leq_\rho t$, whence $xsy \leq xty$ for all $x, y \in S$. Then $s = esf \leq etf = t$ and consequently π is a strict local isomorphism.

Now we need to verify that S/ρ is $\mathcal{L}$-unipotent. Take $a, b \in E(S/\rho)$ such that $a\mathcal{L}b$. Again, Lallement's Lemma allows us to lift idempotents and therefore we can find $e, f \in E(S)$ such that $a = \pi(e)$ and $b = \pi(f)$. Then $a = ab$ and $b = ba$, so $e\,\rho\,ef$ and $f\,\rho\,fe$. Thus we get that $uev = (uef)v = (ufef)v = (uff)v = ufv$ for all $u, v \in E(S)$. Because the posemigroup S is regular, any element $x \in S$ can be written as $x = xx'x \in E(S)S \cap SE(S)$, so $xey = xfy$ for all $x, y \in S$. Hence $e\,\rho\,f$ and $a = b$, making S/ρ $\mathcal{L}$-unipotent. □

Theorem 6.15 (cf. Thm 5.5 (5) of [29]) *Let S be a posemigroup with weak local units. Then S is strongly Morita equivalent to an $\mathcal{L}$-unipotent posemigroup if and only if it is regular and locally $\mathcal{L}$-unipotent.*

Proof. Necessity again follows from Proposition 2 of [27], Proposition 3.22 and an argument similar to the one used in proving Corollary 3.6.

For sufficiency, let S be regular and locally $\mathcal{L}$-unipotent. In Section 4 of [33], McAlister constructs a consolidation q on $C(S)$ such that the idempotents of $C(S)^q$ satisfy the identity $uef = ufef$. By Proposition 6.14, there is a strict local isomorphism from $C(S)^q$ to an $\mathcal{L}$-unipotent posemigroup T. Therefore Theorem 3.1 applies and S is strongly Morita equivalent to the $\mathcal{L}$-unipotent T. □

Finally, we remark that the necessary and sufficient conditions beginning with Proposition 6.5 provide us with a wealth of strongly Morita invariant properties. Most of those characterizations admit a corollary of the following type.

Corollary 6.16 *Being a rectangular poband is an invariant of strong Morita equivalence.*

Proof. Let T be a posemigroup that is strongly Morita equivalent to a rectangular poband S. By Proposition 6.10, S is strongly Morita equivalent to a one-element posemigroup. Therefore T is also strongly Morita equivalent to the same one-element posemigroup and is a rectangular poband by Proposition 6.10. □

6.3 Strong Morita equivalence classes of regular posemigroups

We can now make use of McAlister's extensive work on regular Rees matrix covers (in particular, [35]) and link his results with some of ours from Section 4.3. This provides us with several additional sufficient conditions for the strong Morita equivalence of two regular posemigroups.

Recall that a partially ordered semigroup is *naturally ordered* if the partial order extends the natural partial order on the set of idempotents. Furthermore, the greatest idempotent of a posemigroup is the greatest element in its subposet of idempotents.

Corollary 6.17 *Let S be a naturally ordered regular posemigroup with a greatest idempotent u. Then S is strongly Morita equivalent to the regular pomonoid uSu.*

Proof. Take a naturally ordered regular posemigroup S with a greatest idempotent u. Then $S = SuS$ by Corollary 1.6 of [35] and therefore S is strongly Morita equivalent to the regular pomonoid uSu by Proposition 6.6. □

Corollary 6.18 *Let S be a regular posemigroup with a greatest element u. Then S is strongly Morita equivalent to the posemilattice uSu. In particular, S is locally a semilattice and $S/\mathcal{J}$ is a semilattice under subset inclusion.*

Proof. Lemma 1.7 and Corollary 1.6 of [35] establish that $u^2 = u$, $S = SuS$ and uSu is a semilattice. Proposition 6.6 implies that S is strongly Morita equivalent to uSu and Theorem 5.5 (3) of [29] shows that S is locally a semilattice and $S/\mathcal{J}$ is a semilattice under subset inclusion. □

Actually, a similar lemma from [36] can be somewhat improved as follows.

Lemma 6.19 *Let S be a naturally ordered sandwich posemigroup with a greatest idempotent u. Then $S = SuS$.*

Proof. Take two idempotents $e, f \in E(S)$ such that $e \leq f$. Then

$$(ef)^2 \leq ef^3 = ef = e^3 f \leq (ef)^2,$$

so ef and similarly fe are also idempotents. Furthermore, $e \leq ef$, so efe is an idempotent as well. Now suppose that u is the greatest idempotent of S. Take $e \in E(S)$, so $e \leq u$. Then eue is an idempotent and $e \leq eue$. Yet $eeue = euee = eue$ and, as S is naturally ordered, $eue \leq e$, whence $eue = e$.

Fix an $s \in S$ and take $e \in E(S)$, $s_1, s_2 \in S$ such that $s = s_1 e s_2$. Then $s = s_1 e s_2 = s_1 eue s_2 \in SuS$. □

Corollary 6.20 *Let S be a naturally ordered sandwich posemigroup that has a greatest idempotent u. Then S is strongly Morita equivalent to the pomonoid uSu.*

We need the following result to connect our work with that of McAlister. Recall that Rees matrix posemigroups in [35] have cartesian order and are frequently unital, i.e. they are considered over a monoid and the identity of that monoid is an entry in the sandwich matrix.

Proposition 6.21 *Let S and T be regular posemigroups. Assume that there exist a cartesian regular Rees matrix posemigroup $\mathcal{R}_c$ and a surjective local isomorphism $\tau : \mathcal{R}_c \to T$ in the sense of McAlister (see [35]). Let $\mathcal{R}_m$ be the regular Rees matrix posemigroup over S. Then $\tau : \mathcal{R}_m \to T$ is a surjective strict local isomorphism.*

Proof. It is easy to see that in the case of regular posemigroups, our notion of surjective strict local isomorphism coincides with that of local isomorphism in [35]. Condition (LI3) is trivial, (LI2) is covered by Lallement's Lemma and (LI1) can be reduced to the usual definition via Lemma 2.42 and Lemma 1.3 of [32].

Now, the only difference between the two kinds of local isomorphism is that $\mathcal{R}_m$ is a weakly ordered version of $\mathcal{R}_c$. As before, (LI2) and (LI3) are immediate. Since τ preserves the stronger order, it is also monotone as a mapping $\mathcal{R}_m \to T$. The only remaining question is whether it reflects order on local subpomonoids of $\mathcal{R}_m$. But since the local subpomonoids of $\mathcal{R}_m$ and $\mathcal{R}_c$ are of the form

$$(u, s, v)\mathcal{R}(u, s, v) = \{(u, z, v) | \, z = sM(v, u')z'M(v', u)s \text{ for some } (u', z', v') \in \mathcal{R}\},$$

this is equivalent to τ reflecting order on the local subpomonoids of $\mathcal{R}_c$. □

Proposition 6.22 *Let S be a locally inverse regular posemigroup for which there exists an idempotent e such that $S = SeS$. Then S is strongly Morita equivalent to the inverse pomonoid eSe. In particular, there exists a surjective strict local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ from a unital cartesian regular Rees matrix posemigroup.*

Proof. Corollary 2.5 of [35] shows that there exists a surjective local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ from a unital cartesian regular Rees matrix posemigroup. Due to Proposition 6.21, $\tau : \mathcal{R}_m(eSe, U, V, M) \to S$ is also a surjective strict local isomorphism from a regular Rees matrix posemigroup. It satisfies the condition $eSe\mathrm{Im}(M)eSe = eSe$ because the sandwich matrix is unital and so Theorem 4.13 implies that S is strongly Morita equivalent to eSe. □

This allows us to provide a description of semigroups that are strongly Morita equivalent to inverse pomonoids.

Theorem 6.23 (cf. Thm 5.5 (2) of [29]) *Let S be a posemigroup with weak local units. Then the following are equivalent:*

(1) S is strongly Morita equivalent to an inverse pomonoid T;

(2) S is regular, locally inverse and $S = SeS$ for some $e \in E(S)$;

(3) there exist an inverse pomonoid T, a regular Rees matrix posemigroup $\mathcal{R}_m = \mathcal{R}(T, U, V, M)$ for which $T\mathrm{Im}(M)T = T$ and a surjective strict local isomorphism $\tau : \mathcal{R}_m \to S$;

(4) S has local units, there exist an inverse pomonoid T, a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(T, U, V, M)$ with $T\mathrm{Im}(M)T = T$ and a surjective strict local isomorphism $\tau : \mathcal{M} \to S$.

Proof. (1)⇒(2) Let S be strongly Morita equivalent to an inverse pomonoid T. Then S is regular by Proposition 2 of [27] and hence S has local units. Due to Proposition 3.22, $C(S)$ and $C(T)$ are Pos-equivalent. Any local subpomonoid eSe for $e \in E(S)$ is isomorphic to $C(S)(e, e)$, which by Pos-equivalence is isomorphic to $C(T)(i, i) \cong iTi$ for some $i \in E(T)$. Since T is inverse, all its local subpomonoids are also inverse and therefore S must be locally inverse. Finally, $S = SeS$ for some $e \in E(S)$ by Proposition 6.6.

(2)⇒(3) This is the second claim of Proposition 6.22, seen in the light of Proposition 6.21.

(1)⇔(3) Holds because of Theorem 4.13, since the image of a regular (po)semigroup is regular.

(1)⇔(4) Follows from Theorem 4.8. □

Proposition 6.24 *Let S be a simple (bisimple) regular posemigroup and let e be an idempotent of S. Then S is strongly Morita equivalent to the simple (bisimple) pomonoid eSe. In particular, there exists a surjective strict local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ from a unital cartesian regular Rees matrix posemigroup.*

Proof. Again, the result follows from Corollary 2.6 of [35], Proposition 6.21 and Theorem 4.13. □

Once more, we are able to derive a necessary and sufficient condition.

Theorem 6.25 *Let S be a posemigroup with weak local units. Then the following are equivalent:*

(1) S is strongly Morita equivalent to a simple (bisimple) regular pomonoid T;

(2) S is a simple (bisimple) regular posemigroup;

(3) there exist a simple (bisimple) regular pomonoid T, a regular Rees matrix posemigroup $\mathcal{R}_m = \mathcal{R}(T, U, V, M)$ with $T\mathrm{Im}(M)T = T$ and a surjective strict local isomorphism $\tau : \mathcal{R}_m \to S$;

(4) S has local units, there exist a simple (bisimple) regular pomonoid T, a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(T, U, V, M)$ with $T\mathrm{Im}(M)T = T$ and a surjective strict local isomorphism $\tau : \mathcal{M} \to S$.

Proof. (1)⇒(2). Let S be strongly Morita equivalent to a simple (bisimple) regular pomonoid T. Then S is regular by Proposition 2 of [27]. Due to Proposition 6.4 and Proposition 5.1 (4) of [29] (or Corollary 9.12 of [44]), S is simple (bisimple).

(2)⇒(3) This is the second claim of Proposition 6.24 (and Proposition 6.21).

(1)⇔(3) and (1)⇔(4) again follow from Theorem 4.13 and Theorem 4.8. □

Now, we can refine the previous result for locally inverse regular posemigroups even further (recall from [38] that a regular semigroup has a compatible natural order if and only if it is locally inverse).

Proposition 6.26 *Let S be a naturally ordered regular posemigroup which has a greatest idempotent e. Then S is strongly Morita equivalent to the naturally ordered inverse pomonoid eSe. In particular, there exists a surjective strict local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ from a naturally ordered cartesian regular Rees matrix posemigroup that also has a greatest idempotent, with U and V possessing greatest elements, denoted by 1, $M(1, 1) = e$ and $\mathrm{Im}(M) \subseteq E(T)$.*

Proof. Theorem 3.3 of [35] shows that there exists a surjective local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ as required above, with eSe inverse. Because of Proposition 6.21, the same mapping is also a surjective strict local isomorphism $\tau : \mathcal{R}_m(eSe, U, V, M) \to S$. Since $\mathcal{R}_c(eSe, U, V, M)$ is unital, we are able to apply Theorem 4.13 and get that S is strongly Morita equivalent to eSe, which is a naturally ordered pomonoid because S is naturally ordered. □

Proposition 6.27 *Let S be a naturally ordered regular posemigroup which has a greatest element e. Then S is strongly Morita equivalent to the naturally ordered posemilattice eSe. In particular, there exists a surjective strict local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ from a naturally ordered cartesian regular Rees matrix posemigroup that also has a greatest element, with U and V also possessing greatest elements, denoted by 1, and $M(1, 1) = e$.*

Proof. This is derived from Corollary 3.5 of [35] as above. □

Recall that a *Dubreil-Jacotin semigroup* is a posemigroup S for which there exists a posemigroup homomorphism $h : S \to G$ onto a pogroup G such that the poset $\{s \in S \,|\, h(s^2) \leq h(s)\}$ has a greatest element.

Proposition 6.28 *Let S be a naturally ordered regular Dubreil-Jacotin semigroup. Then S has a greatest idempotent e and S is strongly Morita equivalent to the naturally ordered Dubreil-Jacotin inverse pomonoid eSe. In particular, there is a surjective strict local isomorphism $\tau : \mathcal{R}_c(eSe, U, V, M) \to S$ from a naturally ordered Dubreil-Jacotin cartesian regular Rees matrix posemigroup, with posets U and V also having greatest elements, denoted by 1, $M(1, 1) = e$ and $\mathrm{Im}(M) \subseteq E(T)$.*

Proof. Again, we use the same results as before and Theorem 3.7 of [35]. □

While McAlister's results are generally both necessary and sufficient and so is Theorem 4.13, we don't have a suitable converse for Proposition 6.21. Thus the question of whether the above sufficient conditions are also necessary is presently open.

6.4 Morita invariants

In this section we investigate which general properties of posemigroups remain invariant under strong Morita equivalence.

Due to Proposition 6.10, we can immediately conclude that

Corollary 6.29 *The invariants of strong Morita equivalence do not include any purely order-related property that does not model the entire* Pos. *Moreover, the congruence lattices of strongly Morita equivalent posemigroups are not necessarily isomorphic.*

Proof. Either the one-element posemigroup satisfies an order-related property or it does not. By our assumption, there exists a poset that does not satisfy (or does satisfy, as the case may be) the same property. We turn that poset into a left zero posemigroup, which is a rectangular poband and therefore strongly Morita equivalent to the one-element posemigroup by Proposition 6.10.

Similarly, the congruence lattice of the one-element posemigroup is singleton, while nontrivial rectangular bands have at least two congruences. □

Still, if we assume sufficiently “good“ local units, we can prove that some properties are Morita invariants for such classes of posemigroups. The following use of common local units to establish the existence of Morita invariants is a very useful technique, and will be employed to prove Proposition 6.33 as well.

Proposition 6.30 *Let S and T be strongly Morita equivalent posemigroups with common local units. If the order on S is either total, discrete, directed or a semiorder, then the order on T is also total, discrete, directed or a semiorder.*

Proof. We will prove only the claim about chains, since the others can be proved in the same way. Let S be a chain and take $t, t' \in T$. As T has common local units, there exist $i, j \in E(T)$ such that $t = it = tj$ and $t' = it' = t'j$, so $t, t' \in iTj$. By extending Corollary 3.6, $iTj \cong eSf$ for some $e, f \in E(S)$. But since S is a chain, eSf and therefore iTj are also chains. So either $t \leq t'$ or $t' \leq t$, as required. □

We note that common local units carry over to finite subsets.

Lemma 6.31 *If a posemigroup S has common (two-sided, weak) local units, then any finite subset of S also has common (two-sided, weak) local units.*

Proof. We only need to verify this for a three-element subset $\{s_1, s_2, s_3\} \subseteq S$. Take $e \in S$ such that $es_1 = s_1$ and $es_2 = s_2$. Now take $f \in S$ so that $fe = e$ and $fs_3 = s_3$. Then $fs_1 = fes_1 = es_1 = s_1$ and similarly $fs_2 = s_2$, as required. □

We use the term “inequality“ in the sense of [9], where it is introduced for ordered universal algebras. For example, a posemigroup S satisfies an inequality $x^2 \leq x$ if $s^2 \leq s$ for every $S \in S$.

Theorem 6.32 (cf. Thm 5 of [27]) *Let S and T be strongly Morita equivalent posemigroups with common two-sided weak local units. If S satisfies an inequality, then T also satisfies the same inequality.*

Proof. The proof is a minor modification of Theorem 5 of [27]. Take two terms $w(x_1, \ldots, x_n) = x_{i_1} \ldots x_{i_k}$ and $w'(x_1, \ldots, x_n) = x_{j_1} \ldots x_{j_l}$. Suppose that S satisfies the inequality $w \leq w'$. Take any $t_1, \ldots, t_n \in T$ and fix $e = [q, p] \in T$ such that $t_i = et_i = t_ie$ for all $i = 1, \ldots, n$.

Then for $m_1, \ldots, m_a \in \{1, \ldots, n\}$ we have

$$\begin{aligned} t_{m_1} \ldots t_{m_a} &= et_{m_1}e \ldots et_{m_{a-1}}et_{m_a}e = [q, p][t_{m_1}q, p] \ldots [t_{m_{a-1}}q, p][t_{m_a}q, p] \\ &= [q, \langle p, t_{m_1}q \rangle \ldots \langle p, t_{m_{a-1}}q \rangle \langle p, t_{m_a}q \rangle p]. \end{aligned}$$

So

$$\begin{aligned} w(t_1, \ldots, t_n) &= [q, \langle p, t_{i_1}q \rangle \ldots \langle p, t_{i_{k-1}}q \rangle \langle p, t_{i_k}q \rangle p] \\ &= [q, w(\langle p, t_1q \rangle), \ldots, \langle p, t_nq \rangle)p] \\ &\leq [q, w'(\langle p, t_1q \rangle), \ldots, \langle p, t_nq \rangle)p] \\ &= [q, \langle p, t_{j_1}q \rangle \ldots \langle p, t_{j_{l-1}}q \rangle \langle p, t_{j_l}q \rangle p] = w'(t_1, \ldots, t_n). \end{aligned}$$ □

There is also an alternative proof under slightly different assumptions.

Proposition 6.33 *Let S and T be strongly Morita equivalent posemigroups with common local units. If S satisfies an inequality, then T also satisfies the same inequality.*

Proof. Suppose that S and T are strongly Morita equivalent posemigroups with common local units. Morover, let S satisfy an inequality $w \leq w'$, where the terms are $w(x_1, \dots, x_n) = x_{i_1} \dots x_{i_k}$ and $w'(x_1, \dots, x_n) = x_{j_1} \dots x_{j_l}$. By an extension of Corollary 3.6, the posemigroup T satisfies $w \leq w'$ locally. Take $t_1, \dots, , t_n \in T$ and $e, f \in E(T)$ such that $t_i = et_if$ for all $i = 1, \dots, n$. Then $t_i \in eTf$ and thus

$$w(t_1, \dots, t_n) \leq w'(t_1, \dots, t_n). \qquad \square$$

Remark 6.34 Since an identity is equivalent to two inequalities, the above results hold for identities as well.

Corollary 6.35 *Commutativity and being a band or a semilattice are invariant properties of Morita equivalent posemigroups with common (two-sided weak) local units.*

Actually, since idempotence involves an identity with only one variable, we can easily derive the following version of Theorem 6.32.

Proposition 6.36 *Let S and T be strongly Morita equivalent posemigroups with (two-sided weak) local units. If S is a band, then so is T.*

The following result can be seen as a generalization of the result e.g. from [24] that commutative Morita equivalent monoids are always isomorphic.

Theorem 6.37 (cf. Thm 4 of [27]) *Let S and T be two strongly Morita equivalent posemigroups with common two-sided weak local units. Then their greatest commutative images are isomorphic posemigroups.*

Proof. For any posemigroup S, let α_S^1 be the binary relation

$$\alpha_S^1 = \{(cabd, cbad) | \, a, b \in S, c, d \in S^1\} \subseteq S \times S$$

and let $\alpha_S = \theta(\alpha_S^1) = \leq_{\alpha_S^1} \cap \geq_{\alpha_S^1}$. Then α_S is a posemigroup congruence and S/α_S is the greatest commutative image of S. Let $\pi_S : S \to S/\alpha_S$ be the canonical projection. Now, suppose that S and T are strongly Morita equivalent posemigroups with common two-sided weak local units. Define two mappings $f : S/\alpha_S \to T/\alpha_T$ and $g : T/\alpha_T \to S/\alpha_S$ by

$$f(\pi_S(s)) = \pi_T([q, sp]), \text{where } s = us = su \text{ and } u = \langle p, q \rangle,$$
$$g(\pi_T(t)) = \pi_S(\langle p, tq \rangle), \text{where } t = vt = tv \text{ and } v = [q, p].$$

We show that f is a posemigroup homomorphism. Then g is also a homomorphism by symmetry. First, we check that the choice of u does not influence the definition of f. For this suppose that $s = us = su = u's = su'$ for $u = \langle p, q\rangle$ and $u' = \langle p', q'\rangle$. Then $su = su' = us$ yields $\langle sp, q\rangle = \langle sp', q'\rangle = \langle p, qs\rangle$. Therefore

$$\begin{aligned} [q, sp] &= [q, su'p] = [q, \langle sp', q'\rangle p] = [q, sp'][q', p] \\ &\alpha_T \; [q', p][q, sp'] = [q', \langle p, q\rangle sp'] = [q', usp'] = [q', sp'] \end{aligned}$$

and thus $\pi_T([q, sp]) = \pi_T([q', sp'])$, as required.

Moreover, take $(s, s') = (cabd, cbad) \in \alpha_S^1$. We only consider the case when $c, d \in S$, the others can be proved similarly. So let $u = \langle p, q\rangle \in S$ be such that $a = ua = au$, $b = ub = bu$, $c = uc = cu$ and $d = ud = du$. Then

$$\begin{aligned} f(\pi_S(s)) &= \pi_T([q, cabdp]) = \pi_T([q, cuaubudp]) \\ &= \pi_T([q, c\langle p, q\rangle\langle ap, q\rangle\langle bp, q\rangle dp]) \\ &= \pi_T([q, cp][qa, p][qb, p][qd, p]) \\ &= \pi_T([q, cp][qb, p][qa, p][qd, p]) \\ &= \pi_T([q, c\langle p, q\rangle\langle bp, q\rangle\langle ap, q\rangle dp]) \\ &= \pi_T([q, cubuaudp]) = \pi_T([q, cbadp]) = f(\pi_S(s')). \end{aligned}$$

Additionally, if $s \leq s'$, then we can again take $u = \langle p, q\rangle \in S$ such that $s = su = us$ and $s' = s'u = us'$ and get $f(\pi_S(s)) = \pi_T([q, sp]) \leq \pi_T([q, s'p]) = f(\pi_S(s'))$.

To verify that f is monotone, we need to show that if $s \leq_{\alpha_S^1} s'$ then also $f(\pi_S(s)) \leq f(\pi_S(s'))$. For $s \leq_{\alpha_S^1} s'$, we have $n \in \mathbb{N}$, $s_i, t_i \in S$, $1 \leq i \leq n$ such that $s \leq s_1 \, \alpha_S^1 \, t_1 \leq s_2 \, \alpha_S^1 \, t_2 \leq \ldots \leq s_n \, \alpha_S^1 \, t_n \leq s'$. Then

$$\begin{aligned} f(\pi_S(s)) &\leq f(\pi_S(s_1)) = f(\pi_S(t_1)) \leq f(\pi_S(s_2)) = f(\pi_S(t_2)) \\ &\leq \ldots \leq f(\pi_S(s_n)) = f(\pi_S(t_n)) \leq f(\pi_S(s')). \end{aligned}$$

So f is monotone and consequently well-defined.

To see that f is a homomorphism, take $s, s' \in S$ and let $u = \langle p, q\rangle \in S$ be such that $s = su = us$ and $s' = s'u = us'$. Then $ss' = uss' = ss'u$ and

$$\begin{aligned} f(\pi_S(ss')) &= \pi_T([q, ss'p]) = \pi_T([q, sus'p]) = \pi_T([q, s\langle p, q\rangle s'p]) \\ &= \pi_T([q, sp[q, s'p]]) = \pi_T([q, sp][q, s'p]) \\ &= \pi_T([q, sp])\pi_T([q, s'p]) = f(\pi_S(s))f(\pi_S(s')). \end{aligned}$$

Finally, take $s \in S$ and $u = \langle p, q\rangle \in S$ such that $s = su = us$. Then

$$[q, sp] = [q, sup] = [q, s\langle p, q\rangle p] = [q, sp][q, p],$$

similarly $[q, sp] = [q, p][q, sp]$ and therefore

$$\begin{aligned} (gf)(\pi_S(s)) &= g(\pi_T([q, sp])) = \pi_S(\langle p, [q, sp]q\rangle) \\ &= \pi_S(\langle p, q\rangle\langle sp, q\rangle) = \pi_S(usu) = \pi_S(s). \end{aligned}$$

So $gf = 1_{S/\alpha_S}$ and similarly $fg = 1_{T/\alpha_T}$. □

Corollary 6.38 (cf. Cor 1 of [27], Cor 6.2 (1) of [24]) *Let S and T be strongly Morita equivalent commutative semigroups with common two-sided weak local units. Then S and T are isomorphic.*

We now show that the lattices of various kinds of ideals of strongly Morita equivalent posemigroups are isomorphic. This allows us to make several further observations on strong Morita equivalence. Note that we allow empty ideals so that the posets of ideals are proper lattices.

By $\Downarrow I$, $\Uparrow I$ and $\mathrm{Conv}(I)$ we will denote the down-set, up-set and convex poset generated by a fixed subposet $I \subseteq S$, i.e.

$$\Downarrow I = \{x \in S \mid \exists y \in I \text{ such that } x \leq y\},$$

$$\Uparrow I = \{x \in S \mid \exists y \in I \text{ such that } y \leq x\},$$

$$\mathrm{Conv}(I) = \{x \in S \mid \exists y, z \in I \text{ such that } y \leq x \leq z\}.$$

We say that a two-sided ideal I of a posemigroup S is *downwards closed* (*upwards closed*, *convex*) if $\Downarrow I = I$ ($\Uparrow I = I$, $\mathrm{Conv}(I) = I$) and use the notation $\mathrm{Id}(S)$, $\mathrm{DId}(S)$, $\mathrm{UId}(S)$ and $\mathrm{CId}(S)$ for the lattices of (two-sided) ideals, downwards closed ideals, upwards closed ideals and convex ideals of S. The lattice operations on the ideal posets with respect to inclusion are the usual intersection and union operations, except for convex ideals where the join of two convex ideals is the convex ideal generated by their union.

Assume that there is a Morita context $(S, T, P, Q, \langle -,- \rangle, [-,-])$ with posemigroups S and T. We can define the following mappings

$$\Phi : \mathrm{Id}(S) \to \mathrm{Id}(T), \quad \Theta : \mathrm{Id}(T) \to \mathrm{Id}(S)$$

$$\Phi_{\downarrow} : \mathrm{DId}(S) \to \mathrm{DId}(T), \quad \Phi_{\uparrow} : \mathrm{UId}(S) \to \mathrm{UId}(T), \quad \Phi_{\updownarrow} : \mathrm{CId}(S) \to \mathrm{CId}(T)$$

$$\Theta_{\downarrow} : \mathrm{DId}(T) \to \mathrm{DId}(S), \quad \Theta_{\uparrow} : \mathrm{UId}(T) \to \mathrm{UId}(S), \quad \Theta_{\updownarrow} : \mathrm{CId}(T) \to \mathrm{CId}(S)$$

by

$$\Phi(I) = [QI, P] = \{[qi, p] \mid p \in P, q \in Q, i \in I\},$$

$$\Theta(J) = \langle PJ, Q \rangle = \{\langle pj, q \rangle \mid p \in P, q \in Q, j \in J\},$$

$$\Phi_{\downarrow}(I) = \Downarrow [QI, P], \quad \Phi_{\uparrow}(I) = \Uparrow [QI, P], \quad \Phi_{\updownarrow}(I) = \mathrm{Conv}[QI, P],$$

$$\Theta_{\downarrow}(I) = \Downarrow \langle PJ, Q \rangle, \quad \Theta_{\uparrow}(I) = \Uparrow \langle PJ, Q \rangle, \quad \Theta_{\updownarrow}(I) = \mathrm{Conv}\langle PJ, Q \rangle.$$

Lemma 6.39 *Let $(S, T, P, Q, \langle -,- \rangle, [-,-])$ be a Morita context. Then*

$$\mathrm{Conv}\langle P\mathrm{Conv}X, Q \rangle = \mathrm{Conv}\langle PX, Q \rangle,$$

$$\Downarrow \langle P \Downarrow X, Q \rangle = \Downarrow \langle PX, Q \rangle, \quad \Uparrow \langle P \Uparrow X, Q \rangle = \Uparrow \langle PX, Q \rangle,$$

$$\mathrm{Conv}[Q\mathrm{Conv}Y, P] = \mathrm{Conv}[QY, P],$$

$$\Downarrow [Q \Downarrow Y, P] = \Downarrow [QY, P] \quad \text{and} \quad \Uparrow [Q \Uparrow Y, P] = \Uparrow [QY, P]$$

for any subsets $X \subseteq T$, $Y \subseteq S$.

Proof. We will prove only the first equality. Take $x \in \mathrm{Conv}\langle P\mathrm{Conv}X, Q\rangle$, i.e. $\langle py, q\rangle \leq x \leq \langle p'z, q'\rangle$ and $y_1 \leq y \leq y_2$, $z_1 \leq z \leq z_2$ for some $p, p' \in P$, $q, q' \in Q$, $y, z \in T$, $y_1, y_2, z_1, z_2 \in X$. Then $\langle py_1, q\rangle \leq x \leq \langle p'z_2, q'\rangle$, i.e. $x \in \mathrm{Conv}\langle PX, Q\rangle$. The converse is obvious since $X \subseteq \mathrm{Conv}X$. □

The following theorem is a generalization of Theorem 3 of [27], which in turn ultimately arises from a similar ring-theoretic result (e.g. Proposition 21.11 of [4]).

Theorem 6.40 (cf. Prop 5.1 (4), (5) of [29]) *Let S and T be two strongly Morita equivalent posemigroups with weak local units. Then*

(1) $\Phi : \mathrm{Id}(S) \to \mathrm{Id}(T)$ *is a lattice isomorphism that takes finitely generated ideals to finitely generated ideals and principal ideals to principal ideals;*

(2) $\Phi_\downarrow : \mathrm{DId}(S) \to \mathrm{DId}(T)$ *is a lattice isomorphism that takes down-sets of finitely generated ideals to down-sets of finitely generated ideals and down-sets of principal ideals to down-sets of principal ideals;*

(3) $\Phi_\uparrow : \mathrm{UId}(S) \to \mathrm{UId}(T)$ *is a lattice isomorphism that takes up-sets of finitely generated ideals to up-sets of finitely generated ideals and up-sets of principal ideals to up-sets of principal ideals;*

(4) $\Phi_\updownarrow : \mathrm{CId}(S) \to \mathrm{CId}(T)$ *is a lattice isomorphism that takes convex subsets generated by finitely generated ideals to convex subsets generated by finitely generated ideals and convex subsets generated by principal ideals to convex subsets generated by principal ideals.*

Proof. (1): This is implied by Theorem 3 of [27] and Proposition 6.4. The mappings Φ and Θ clearly preserve subset inclusion, i.e. are monotone. Since $\langle -, -\rangle$ and $[-, -]$ are biposet morphisms, Φ and Θ take any subset to a two-sided ideal. If $I \in \mathrm{Id}(S)$, then

$$\begin{aligned}\Theta(\Phi(I)) &= \langle P[QI, P], Q\rangle = \langle\langle P, QI\rangle P, Q\rangle \\ &= \langle P, Q\rangle I\langle P, Q\rangle = SIS = I.\end{aligned}$$

Similarly $\Phi(\Theta(J)) = J$ for all $J \in \mathrm{Id}(T)$, so Φ is a lattice isomorphism.

Now, take a finitely generated $I \in \mathrm{Id}(S)$, i.e. $I = \bigcup_{i=1}^{n} Ss_iS$, and $s \in I$. Since $\langle -, -\rangle$ is surjective, there exist $i \in \{1, \ldots, n\}$ and $p, p' \in P$, $q, q' \in Q$ such that $s = \langle p, q\rangle s_i\langle p', q'\rangle$. Because S has weak local units, we can put $s_i = \langle p_i, q_i\rangle s_i\langle p_i', q_i'\rangle$ for some $p_i, p_i' \in P$, $q_i, q_i' \in Q$. So for any $p'' \in P$ and $q'' \in Q$ we get that

$$\begin{aligned}
[q''s, p''] &= [q''\langle p, q\rangle\langle p_i, q_i\rangle s_i\langle p'_i, q'_i\rangle\langle p', q'\rangle, p''] \\
&= [q'', p][q, p_i][q_i s_i, \langle p'_i, q'_i\rangle\langle p', q'\rangle p''] \\
&= [q'', p][q, p_i][q_i s_i, p'_i][q'_i, p'][q', p''] \\
&\in T[q_i s_i, p'_i]T \subseteq \bigcup_{j=1}^{n} T[q_j s_j, p'_j]T.
\end{aligned}$$

Conversely, $t[q_j s_j, p'_j]t' = [(tq_j)s_j, p'_j t'] \in [QI, P] = \Phi(I)$ and therefore we have $\Phi(\bigcup_{i=1}^{n} S s_i S) = \bigcup_{j=1}^{n} T[q_j s_j, p'_j]T$.

(2): First of all, $\mathrm{DId}(S)$ and $\mathrm{DId}(T)$ are indeed both sublattices of $\mathrm{Id}(S)$ and $\mathrm{Id}(T)$, respectively, since intersections and unions of down-sets are again down-sets. Second, the mappings Φ and Θ clearly preserve subset inclusion, i.e. are monotone. Moreover, take $I \in \mathrm{DId}(S)$, $x \in \Phi_\downarrow(I)$ and $t, t' \in T$. Because $x \leq [qi, p]$ for some $p \in P$, $q \in Q$, $i \in I$, we also have $txt' \leq t[qi, p]t' = [(tq)i, pt']$, so $txt' \in \downarrow[QI, P] = \Phi_\downarrow(I)$. Therefore $\Phi_\downarrow(I) \in \mathrm{DId}(T)$. By Lemma 6.39, we get for any $I \in \mathrm{DId}(S)$ that

$$\begin{aligned}
\Theta_\downarrow(\Phi_\downarrow(I)) &= \downarrow\langle P \downarrow[QI, P], Q\rangle = \downarrow\langle P[QI, P], Q\rangle \\
&= \downarrow\langle\langle P, QI\rangle P, Q\rangle = \downarrow(\langle P, Q\rangle I\langle P, Q\rangle) = \downarrow(SIS) = \downarrow I = I.
\end{aligned}$$

Symmetrically $\Phi_\downarrow(\Theta_\downarrow(J)) = J$ for all $J \in \mathrm{DId}(T)$ and thus $\Theta_\downarrow$ is also a lattice isomorphism.

Fix a finitely generated ideal $I = \bigcup_{i=1}^{n} S s_i S \in \mathrm{Id}(S)$. Then $\downarrow I \in \mathrm{DId}(S)$, so we can consider $\Phi_\downarrow(\downarrow I)$. Take any $s \in I$. Since $\langle -, -\rangle$ is surjective, there exist $i \in \{1, \ldots, n\}$ and $p, p' \in P$, $q, q' \in Q$ such that $s = \langle p, q\rangle s_i\langle p', q'\rangle$. Because S has weak local units, we can put $s_i = \langle p_i, q_i\rangle s_i\langle p'_i, q'_i\rangle$ for some $p_i, p'_i \in P$, $q_i, q'_i \in Q$. So $s = \langle p, q\rangle\langle p_i, q_i\rangle s_i\langle p'_i, q'_i\rangle\langle p', q'\rangle$. Thus we get for any $p'' \in P$, $q'' \in Q$ and $x \leq [q''y, p'']$, $y \in \downarrow s$ that

$$x \leq [q''s, p''] = [q'', p][q, p_i][q_i s_i, p'_i][q'_i, p'][q', p''],$$

so

$$x \in \downarrow(T[q_i s_i, p'_i]T), \text{ i.e. } \downarrow[Q \downarrow I, P] \subseteq \bigcup_{j=1}^{n} \downarrow(T[q_j s_j, p'_j]T).$$

For the converse, if $y \leq t[q_j s_j, p'_j]t'$, then also $y \leq [(tq_j)s_j, p'_j t'] \in [Q \downarrow I, P]$, so $y \in \downarrow[Q \downarrow I, P] = \Phi(\downarrow I)$. Combining the two preceding inclusions therefore yields $\Phi(\downarrow\bigcup_{i=1}^{n} S s_i S) = \bigcup_{j=1}^{n} \downarrow(T[q_j s_j, p'_j]T) = \downarrow\bigcup_{j=1}^{n} (T[q_j s_j, p'_j]T)$, as required.

(3): Dual to (2).

(4): $\mathrm{CId}(S)$ and $\mathrm{CId}(T)$ are lattices, but not necessarily sublattices of $\mathrm{Id}(S)$ and $\mathrm{Id}(T)$. The mappings $\Theta_{\updownarrow}$ and $\Phi_{\updownarrow}$ are again monotone and yield two-sided ideals. Once more, $\Theta_{\updownarrow}$ is a poset isomorphism, because Lemma 6.39 implies for any $I \in \mathrm{CId}(S)$ that

$$\begin{aligned}\Theta_{\updownarrow}(\Phi_{\updownarrow}(I)) &= \mathrm{Conv}\langle P\mathrm{Conv}[QI, P], Q\rangle = \mathrm{Conv}\langle P[QI, P], Q\rangle \\ &= \mathrm{Conv}\langle\langle P, QI\rangle P, Q\rangle = \mathrm{Conv}(\langle P, Q\rangle I\langle P, Q\rangle) \\ &= \mathrm{Conv}(SIS) = \mathrm{Conv}(I) = I\end{aligned}$$

and likewise $\Phi_{\updownarrow}(\Theta_{\updownarrow}(J)) = J$ for all $J \in \mathrm{CId}(T)$.

Take $I = \bigcup_{i=1}^{n} S s_i S \in \mathrm{Id}(S)$. Clearly $\mathrm{Conv}(I) \in \mathrm{CId}(S)$, so we can consider $\Phi_{\updownarrow}(\mathrm{Conv}(I))$. Fix any $x \in \Phi_{\updownarrow}(\mathrm{Conv}(I))$. Then $[qs, p] \leq x \leq [q's', p']$ for some $s, s' \in \mathrm{Conv}(I)$. We can again write $s \geq \langle p_1, q_1\rangle\langle p_{i_1}, q_{i_1}\rangle s_{i_1}\langle p'_{i_1}, q'_{i_1}\rangle\langle p'_1, q'_1\rangle$ and $s' \leq \langle p_2, q_2\rangle\langle p_{i_2}, q_{i_2}\rangle s_{i_2}\langle p'_{i_2}, q'_{i_2}\rangle\langle p'_2, q'_2\rangle$. Thus

$$\begin{aligned}&[q, p_1][q_1, p_{i_1}][q_{i_1} s_{i_1}, p'_{i_1}][q'_{i_1}, p'_1][q'_1, p] \leq x, \\ &x \leq [q', p_2][q_2, p_{i_2}][q_{i_2} s_{i_2}, p'_{i_2}][q'_{i_2}, p'_2][q'_2, p'],\end{aligned}$$

implying that

$$x \in \mathrm{Conv}\left(T[q_{i_1} s_{i_1}, p'_{i_1}]T \cup T[q_{i_2} s_{i_2}, p'_{i_2}]T\right) \subseteq \mathrm{Conv} \bigcup_{j=1}^{n} T[q_j s_j, p'_j]T.$$

Finally, if $t_1[q_{j_1} s_{j_1}, p'_{j_1}]t'_1 \leq y \leq t_2[q_{j_2} s_{j_2}, p'_{j_2}]t'_2$, then

$$[(t_1 q_{j_1}) s_{j_1}, p'_{j_1} t'] \leq y \leq [(t_2 q_{j_2}) s_{j_2}, p'_{j_2} t'],$$

i.e. $y \in \mathrm{Conv}[QI, P] \subseteq \mathrm{Conv}[Q\mathrm{Conv}(I), P] = \Phi_{\updownarrow}(\mathrm{Conv}(I))$, whence

$$\Phi_{\updownarrow}(\mathrm{Conv}(\bigcup_{i=1}^{n} S s_i S)) = \mathrm{Conv} \bigcup_{j=1}^{n} (T[q_j s_j, p'_j]T). \qquad \square$$

Every commutative band (semilattice) has a compatible *natural order* ω defined by $a\omega b \Leftrightarrow ab = a$. Yet there can be other compatible partial orders on that semilattice as well. We call a commutatice band with any kind of compatible partial order a *posemilattice*.

Proposition 6.41 *Two posemilattices with common weak local units are strongly Morita equivalent if and only if they are isomorphic.*

Proof. Sufficiency is obvious. Let $(S, T, P, Q, \langle -,- \rangle, [-,-])$ be the Morita context. Define an additional order $\preceq$ on $\mathrm{PrId}(S)$, the poset of principal ideals of S, by $SsS \preceq StS$ iff $s \le t$. Since S is a semilattice, every principal ideal SsS has a unique generator $s \in S$, so this order is well-defined. Consider the posemilattices $(S, \cdot, \le)$ and $(\mathrm{PrId}(S), \cap, \preceq)$. First, $S \cong \mathrm{PrId}(S)$ as semigroups. To see this, we observe that $SsS \cap StS = SstS$ for all $s, t \in S$. Obviously $SsS \cap StS \supseteq SstS$. For the converse, take $z = s_1 s s_2 = t_1 t t_2 \in SsS \cap StS$ for some $s_1, s_2, t_1, t_2 \in S$. Then

$$z = z^2 = (s_1 s s_2)(t_1 t t_2) = s_1(st)(s_2 t_1 t_2) \in SstS,$$

because S is a commutative band. It is clear that now $(S, \cdot, \le) \cong (\mathrm{PrId}(S), \cap, \preceq)$ as posemigroups. The lattice isomorphism Φ from Theorem 6.40 is a semigroup isomorphism between the posets of principal ideals. To conclude the proof, it now suffices to show that Φ preserves and reflects the above order $\preceq$ on $\mathrm{PrId}(S)$ and $\mathrm{PrId}(T)$. We do this via generators, so take $s, s' \in S$ and fix $e, f \in S$ such that $es = es'$ and $sf = s'f$. Since $[-,-]$ is surjective, there exist $p, p' \in P$ and $q, q' \in Q$ such that $e = \langle p, q \rangle$, $f = \langle p', q' \rangle$, so $s = \langle p, q \rangle s \langle p', q' \rangle$ and $s' = \langle p, q \rangle s' \langle p', q' \rangle$. If $s \le s'$, then clearly $[qs, p'] \le [qs', p']$. Conversely, if $[qs, p'] \le [qs', p']$, then

$$s = \langle p, q \rangle s \langle p', q' \rangle = \langle p, [qs, p'] q' \rangle \le \langle p, [qs', p'] q' \rangle = \langle p, q \rangle s' \langle p', q' \rangle = s'. \quad \square$$

A *polattice* means a posemilattice $(S, \cdot, \le)$ where the natural order ω is a lattice order. A *directed posemilattice* is a (lower) posemilattice $(S, \cdot, \le)$ where the natural order ω is an (upwards) directed order.

The preceding result can thus be reformulated as

Corollary 6.42 *Two directed posemilattices are strongly Morita equivalent if and only if they are isomorphic.*

Corollary 6.43 *Two polattices are strongly Morita equivalent if and only if they are isomorphic.*

The next claim holds for unordered semigroups (see Proposition 15 of [25]), but is an open question for posemigroups.

Problem 6.44 *Are two posemilattices which have weak local units strongly Morita equivalent if and only if they are isomorphic?*

A summary of Morita invariants is given in the table below. If not yet proved, showing that they are invariants is similar to Corollary 6.16. Note that conditions (1) and (3) of Proposition 6.1 do not require the existence of local units, since strong Morita equivalence implies the Pos-equivalence of Cauchy completions even for factorizable posemigroups (see Proposition 3.22) and the proofs of (1) and (3) in Proposition 5.1 of [29] only use basic properties of Cauchy completions without referencing local units.

Semigroup	Factorizable
Invariants	structure of local subpomonoids eSe, $e \in E(S)$ cardinalities of the sets of regular $\mathcal{D}$-classes sandwich posemigroup $S = SeS$ for some $e \in E(S)$ rectangular poband

Semigroup	Weak local units
Invariants	regular regular, locally inverse, $S = SeS$ for some $e \in E(S)$ regular, simple, $S = SeS$ for some $e \in E(S)$ regular, bisimple, $S = SeS$ for some $e \in E(S)$ regular and locally orthodox regular and locally E-solid regular, locally a union of groups, $S/\mathcal{J}$ a semilattice regular and locally $\mathcal{L}$-unipotent lattice of ideals lattice of finitely generated ideals poset of principal ideals lattice of upwards closed ideals lattice of downwards closed ideals lattice of convex ideals

Semigroup	Common local units	Semigroup	Common two-sided weak local units
Invariants	inequalities identities commutativity poband chain antichain directed partial order semiorder	Invariants	inequalities identities commutativity poband greatest comm. images

Conclusion

We have proved that by making a few natural modifications in the relevant definitions, most of the equivalent conditions for (strong) Morita equivalence introduced in [44], [29] and [28] for semigroups are still equivalent (see Theorem 3.1). One difference is that we were able to prove that condition (6) from Remark 3.3 is equivalent to the rest only by assuming the existence of slightly stronger local units. Since a semigroup with trivial order has ordered (weak) local units if and only if it has (weak) local units, this is not an unreasonable restriction.

Our fifth chapter establishes that if both partially ordered semigroups have common local units, then the conditions of Theorem 3.1 are equivalent to the existence of an equivalence between the subcategories of closed partially ordered acts. This result is an analogue of the equivalence of strong Morita equivalence and Morita equivalence between semigroups with local units. Since Talwar has shown in [45] that strong Morita equivalence implies Morita equivalence for all factorizable semigroups, it would be interesting to see whether the same holds for factorizable partially ordered semigroups. Furthermore, Example 3.19 shows that the restriction to common local units is not necessary. Since most of the relevant results from Section 5.3 use ordered local units instead of common local units, it is possible that the same result could be proved by only assuming the existence of ordered local units. This open question has been stated as Problem 5.19.

A related issue that we have mostly ignored in the thesis is the subcategory of unitary partially ordered acts. For semigroups, Chen and Shum have made an investigation of the equivalences between such subcategories in [16]. It would be interesting to see whether their results can also be transferred to partially ordered semigroups.

The seven equivalent conditions form a solid basis for further research into (strong) Morita equivalence of partially ordered semigroups. One can use any of our conditions that proves convenient to either establish that two partially ordered semigroups are strongly Morita equivalent, to construct new posemigroups that are strongly Morita equivalent to a given posemigroup or to investigate the properties of two strongly Morita equivalent partially ordered semigroups. There are quite a few problems available for investigation. One such has already been referred to in Problem 6.44. We can study what happens to one of the partially ordered semigroups when we assume that the other has some interesting properties. Some examples worth considering are commutative monoids, posemilattices, inverse posemigroups, lattice-ordered groups, lattice-ordered posemigroups, posemigroups with zero, posemigroups with a greatest idempotent and Dubreil-Jacotin semigroups. While Proposition 6.2 shows that some of those properties are not Morita invariants in general, they may be if we assume the existence of “better“ local units. One can also study other Morita invariants such as centers or Picard groups.

Another similar line of research can be seen in condition (5) of Theorem 3.1. It is of interest to see whether assuming that one partially ordered semigroup has some property is reflected in the structure of the Rees matrix posemigroup, specifically in the matrix map. One example of such a result is present in the thesis as Theorem 4.10.

We have also spent some effort on studying the strong Morita equivalence of regular semigroups, proving that we can restrict the Rees matrix posemigroups in condition (5) of Theorem 3.1 to regular Rees matrix posemigroups. This provides a one-way link to D. McAlister’s series of papers ([35], [32], [33], [34], [31]) on Rees matrix covers of (partially ordered) semigroups. An open problem is to investigate whether this link can be reversed under some conditions, i.e. can we use one of our equivalent conditions to construct McAlister’s special Rees matrix semigroups. If true, that would lead to characterization theorems similar to our Theorem 6.22 and Theorem 6.25.

One more general point of interest is to find a way around the anomaly of Proposition 6.2. It is generally not very desirable that two strongly Morita equivalent partially ordered semigroups can have very different partial orders. One way to try to get around this would be to strengthen the definition of strong Morita equivalence by adding some kind of further order-related requirements for the Morita context. On the other hand, one can still work with the current definition and simply try to find out whether assuming stronger local units can compensate, somewhat in the style of Proposition 6.30.

In the end, this thesis is a starting point for further study into (strong) Morita equivalence of partially ordered semigroups. It provides a characterization theorem for strong Morita equivalence and a first list of Morita invariants. Those are largely up to date with the results for Morita equivalence of semigroups. Further progress in either field will likely be also at least somewhat transferrable.

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Kokkuvõte (*Summary in Estonian*)

Osaliselt järjestatud poolrühmade Morita ekvivalentsus

Käesolevas väitekirjas uuritakse, kas ja millisel määral on võimalik üldistada Morita ekvivalentsuse mõistet ja teooriat poolrühmadelt osaliselt järjestatud poolrühmadele. Kuna üldjuhul ei ole see võimalik, tuleb mõlemal juhul sisse tuua kitsendus ja nõuda nn lokaalsete ühikelementide olemasolu.

Morita ekvivalentsuse mõiste pärineb ringiteooriast ja on inspireeritud klassikalisest Artini-Wedderburni teoreemist. Seda on ringiteooriast edasi arendatud nii ühikuta ringide, monoidide, osaliselt järjestatud monoidide, poolrühmade kui ka kategooriate jaoks. Kaks ringi (monoidi, poolrühma) on Morita-ekvivalentsed, kui nende moodulite (polügoonide) kategooriad on ekvivalentsed.

Morita teooria põhiidee on taandada ühe struktuuri uurimine teise, temaga Morita-ekvivalentse ja loodetavasti lihtsama struktuuri uurimisele. Selle jaoks sisaldab iga konkreetse struktuuri Morita teooria endas mitmesuguseid vahendeid ja tulemusi.

Doktoritöös defineeritakse osaliselt järjestatud poolrühmade tugev Morita ekvivalentsus nn Morita kontekstide abil. Kahte osaliselt järjestatud poolrühma S ja T nimetatakse tugevalt Morita-ekvivalentseteks siis, kui leidub sürjektiivsete kujutustega unitaarne Morita kontekst $(S, T, P, Q, \langle -, - \rangle, [-, -])$. Seejärel tuuakse sisse rida konstruktsioone ja näidatakse, et kui poolrühmadelt S ja T eeldada teatud lokaalsete ühikute olemasolu, siis on tugev Morita ekvivalentsus samaväärne nende konstruktsioonide abil saadud seostega. Olgu S ja T lokaalsete ühikutega osaliselt järjestatud poolrühmad. Siis on sellisteks seosteks

- kategooriateoreetiline Morita ekvivalentsus ehk nn Cauchy täieldite $C(S)$ ja $C(T)$ ekvivalentsus,
- järjestatud poolrühmade S ja T ühise laiendi olemasolu,
- range lokaalse isomorfismi $C(S)^q \to T$ olemasolu, kus $C(S)^q$ on Cauchy täieldil $C(S)$ põhinev osaliselt järjestatud poolrühm,
- range lokaalse isomorfismi $\mathcal{M}(S, U, V, M) \to T$ olemasolu, kus $\mathcal{M}(S, U, V, M)$ on spetsiifiline Reesi maatrikspoolrühm üle S.

Juhul kui S ja T on järjestatud lokaalsete ühikutega osaliselt järjestatud poolrühmad, on eelnevate tingimustega samaväärne ka

- range lokaalse isomorfismi $Q \otimes_S P \to T$ olemasolu, kus $Q \otimes_S P$ on teatud liiki Morita poolrühm üle S.

Kui S ja T on ühiste lokaalsete ühikutega osaliselt järjestatud poolrühmad, on tugev Morita ekvivalentsus samaväärne teatud Morita ekvivalentsuse vormiga, milleks on

- kinniste osaliselt järjestatud polügoonide kategooriate FPos_S ja FPos_T ekvivalentsus.

Juhul kui S ja T on mõlemad regulaarsed osaliselt järjestatud poolrühmad, võib osaliselt järjestatud Reesi maatrikspoolrühma $\mathcal{M}(S, U, V, M)$ asendada vaid $\mathcal{M}(S, U, V, M)$ regulaarsetest elementidest moodustatud osaliselt järjestatud alampoolrühmaga.

Peale eelmainitud tingimuste samaväärsuse uuritakse väitekirjas ka nn Morita invariante ehk omadusi, mis kanduvad osaliselt järjestatud poolrühmalt üle temaga tugevalt Morita-ekvivalentsetele osaliselt järjestatud poolrühmadele. See uurimissuund jaotub omakorda kaheks:

- leitakse tarvilikud ja piisavad tingimused selleks, et osaliselt järjestatud poolrühm oleks tugevalt Morita-ekvivalentne osaliselt järjestatud poolrühmaga mõnest tuntud klassist;
- uuritakse omadusi ja struktuure, mis on samad (isomorfismi täpsuseni) kõigil tugevalt Morita-ekvivalentsetel osaliselt järjestatud poolrühmadel.

Doktoritöös vaadeldakse tarvilikke ja piisavaid tingimusi mitmete osaliselt järjestatud poolrühmade klasside jaoks. Kriteeriumid on õnnestunud leida muuhulgas järgnevate klasside jaoks: osaliselt järjestatud monoidid, osaliselt järjestatud inverssed monoidid, osaliselt järjestatud ristkülikpoolrühmad, lineaarselt ja diskreetselt järjestatud poolrühmad, osaliselt järjestatud (bi)lihtsad monoidid, täielikult regulaarsed osaliselt järjestatud poolrühmad, suunatud järjestusega poolrühmad ja osaliselt järjestatud sändvitšpoolrühmad.

Lisaks tehakse kindlaks, et mõnevõrra tugevamaid lokaalseid ühikuid nõudes on kahel tugevalt Morita-ekvivalentsel osaliselt järjestatud poolrühmal isomorfsed suurimad kommutatiivsed kujutised, ideaalide võred, alla- ja ülespoole kinniste ideaalide võred ning kumerate ideaalide võred. Peale selle rahuldavad need poolrühmad samaaegselt kõikvõimalikke samasusi ja võrratusi.

Viimasena esitatakse rida lahtisi küsimusi ja võimalikke uurimissuundi edaspidiseks tööks.

Töö kolmas peatükk on suunatud avaldamiseks artiklina [49] (kõik viited on esitatud doktoritöö allikanimekirja järgi). Neljanda ja viienda peatüki tulemusi sisaldavate artiklite eelversioonid on [48] ja [51]. Kuues peatükk jaotub kahe artikli esialgsete versioonide ([47] ja [50]) vahel.

Curriculum vitae

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