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Back-testing the *VaR* risk measure: an empirical study

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ABSTRACT

This thesis verifies the worst case losses (Value-at-Risk) of financial returns over a specified time period with a certain level of confidence. The measurement of VaR hinges on the distribution of investment returns. In order to test whether or not the VaR model accurately represents reality, back-testing is carried out for one day horizon for a yearly rolling window. The standard VaR parametric model which is based on normal distribution of returns is tested on real data. Findings are that this model is better for historical VaR estimation for bigger exceedance probabilities such as 5%, 1%, 2% etc, while the Student's t-distribution seems to be better for smaller exceedance probabilities such as 0.5%, 0.1% etc.

Keywords: Value-at-Risk, parametric methods, return distribution.

CERCS: P160 Statistics, Operation research, programming, actuarial mathematics

VaR riskimõõdu empiiriline testimine

Magistritöö

Ibraheem Olanrewaju Ola-Adua

Lühikokkuvõte

Magistritöös testitakse Value-at-Risk (VaR) metoodika kasutatavust tegelikel andmetel. VaR on riskimõõt, mis näitab suurimat tõenäolist kahju, mis võib investeeringut tabada etteantud ajahorisondi lõpuks. VaR arvutamine põhineb investeeringu tulususe tõenäosusjaotusel. Töö eesmärgiks on testida empiiriliselt, kas tulususte normaaljaotusel põhinev VaR-metoodika annab teooriaga kooskõlalisi tulemusi. New-Yorgi börsi andmete analüüs näitas, et mõõdukate usaldustõenäosuste (95%, 98%, 99%) korral on normaaljaotusel põhinev VaR õigustatud, kuid suuremate tõenäosuste puhul (99,9% jne) tuleks kasutada t-jaotust, mille sabad on raskemad.

Võtmesõnad: riski all olev väärtus, parameetriline meetod, tulususe jaotus

CERCS: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika

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Ibraheem O. Ola-Adua

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CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND

Financial institutions need some capital (reserve) large enough to cater for future unexpected losses. The unexpected losses could be viewed as risks in financial terms. Risk is simply the possibility of an unfavourable outcome and its negative effect. Oxford dictionary defines risk with a modifier as "A person or thing regarded as a threat or likely source of danger".

A resolution was reached 1998 in Basel, Switzerland which turned into a recommendation for banking regulations with regard to credit, market and operational risks. According to Abad et al (2014), this resolution or agreement was called Basel I also known as the Basel Accord by the Basel Committee on Bank Supervision (BCBS) in a meeting which involves chairmen of various central banks across Europe and the United States of America. The purpose of the resolution is to ensure that financial institutions hold enough capital on account to meet obligations and absorb unexpected loss. However, financial risk cannot be measured practically in actual sense but can only be inferred from behaviours of observed market prices using some distribution tests as mathematical principles.

Risk measure takes place when these mathematical principles are applied to the computation of risk. The statistic obtained during risk measure is referred to as risk measurement which tells us the extent to which a damage is done and how severe is the negative effect of risk on an investment. Therefore, we measure risk in order to have the idea of how big a quantity, the unexpected loss would seem. The measurement of risks in financial institutions becomes crucial with the development of some instruments such as Mean Variance Portfolio Theory, volatility and Value-at-Risk (VaR) among others. The main focus of risk measure in this study is the VaR also called Riskmetrics which was believed to have been invented or introduced by JP Morgan in the late 80's (Moscoso, 2012). VaR has become an essential tool for risk measure in many financial institutions and this has sprung up an increase in academic literature over the last decade on the study of VaR with so many modifications especially in the aspect of finding a different distribution for returns other that the normal distribution as postulated in Riskmetrics.

VaR is a universal concept which summarizes in a single number all the risks of a portfolio including interest rate risk, foreign exchange risk and so on. It combines loss (quantile) and probability, it facilitates comparison between different asset classes. It is a significant step forward with respect to traditional measures such as the greek and gamma which measures the sensitivities of options to underlying risk factors (Danielsson, 2011).

VaR as a risk measure is the scope of this thesis which aims to study and verify the interesting fact if returns are normally distributed or follow some other distributions such as the Student's t-distribution. Value-at-Risk (VaR) is defined as a quantitative tool for determining the maximum potential loss in the return of investment over a given period of time at a specified confidence level. More precisely, VaR is the α-quantile of the profit (loss) distribution of the investment. Mathematically, $P(X \leq VaR) = \alpha,$ where X is the profit (loss) of the investment over the given time horizon. By this definition, VaR is usually a negative number. It also refers to the far-left tail of the unconditional return distributions. There is going to be difficulty in estimating VaR of financial returns when the distribution is unknown. The traditional method of estimating VaR is to assume normality but return distribution can be fitted in actual sense for correctness. The empirical calculations involve the estimation of the lower-order quantile, for example 1% or 5% quantiles of the return distribution. It is noteworthy that VaR seems easy but its accurate measurement is a very challenging statistical problem. Under normal assumption, VaR either underestimates, that is, the number of risky returns is greater than the expected number when smaller quantile is specified or overestimates when bigger quantile is specified.

Doric and Doric (2011), noted that risk analysis of VaR can be done in two stages; first, by expressing profit and loss in terms of returns and secondly, by modelling the returns statistically and estimate the VaR returns by computing appropriate quantiles.

The empirical distribution function of the sample returns is an approximation of the true distribution of returns which usually is reasonably accurate in the centre of distribution. However, in order to estimate extreme quantile such as VaR, a reasonable estimate is needed not just in the centre but in the extreme tail as well.

1.2 STATEMENT OF THE HYPOTHESIS

Value-at-Risk (VaR) is a measure of worst-case losses over a specified time period with a certain level of confidence. The measurement of VaR hinges on the distribution of investment returns. In order to test whether or not the model accurately represents reality, back-testing can be carried out. A failed back-test means that the VaR model must be re-evaluated. In this master thesis, we will analyse market data and apply parametric method for calculation of VaR. The models obtained will be back-tested against real data. It will be interesting to see whether the normal distribution fits the return distribution sufficiently well, or an alternative distribution (e.g. t-distribution) should be used.

1.3 SIGNIFICANCE OF THE STUDY

Recent articles have pointed out that the return distributions of a real market data are leptokurtic, that is, fat or heavy tailed as against the widely known standard method of VaR construction which assumes that financial returns are independently and identically distributed and having a normal distribution. It has been proven that distribution of investment returns have three stylized facts, first, the presence of volatility clustering, indicated by high autocorrelation of absolute and squared returns, secondly, excess kurtosis (fat tails) and thirdly, skewness in the density of the unconditional return distributions that returns are negatively skewed. This research work verifies this fact through empirical assessment of Value at Risk models by back-testing procedures.

1.4 DELIMITATIONS

The scope of this study is focused on the use of VaR as a financial instrument or model for risk measure by applying it on real financial market data as there are well known other methods of risk measures such as the Expected Shortfall "which is not applicable in real sense (in practice) (Danielsson, 2010, p. 160)" and conditional VaR which is applicable to time series data. These two methods could perhaps enhance the efficiency of VaR. However, it is not in the scope of this research work to dwell on the best model to apply but to analyse market data through unconditional parametric calculation of VaR and to verify, if the normal distribution assumption gives unbiased results in back-testing, or other distributions (e.g. t-distribution) should be preferred.

CHAPTER TWO

LITERATURE REVIEW

This chapter focuses on some articles that were reviewed in the course of finding a good approach to writing and conducting this research study. No gainsaying that no research is novel!

2.1 EMPIRICAL ASSESSMENT OF VaR

Abad et al (2014) categorised the methodologies initially developed for calculating a portfolio VaR into three. These are; 1) the variance-covariance approach (the parametric method), 2) the historical simulation (non-parametric method) and 3) the Monte-Carlo simulation (semi-parametric method). He noted further that these standard models have numerous shortcomings that have led to the development of new proposals. Among the parametric approaches the first model for VaR estimation is Riskmetrics and a major drawback of the model is the normal distribution assumption for investment returns which is against the empirical evidence that investment returns do not necessarily follow the normal distribution; secondly, this relates to the model used to estimate conditional volatility; thirdly, this involves the assumption that investment returns are independently identically distributed. He mentioned that parametric methods have moved in several directions to counter this drawbacks in the estimation of VaR. Hence, better return distributions other than the normal distribution should be fitted to real market data or investment returns.

In his article, in the context of non-parametric method, he noted that several non-parametric density estimations have been implemented with improvement on the results by historical simulation. However, in the semi-parametric models, new approaches have been proposed, some of which are; the Filtered Historical Simulation proposed by Barone-Adesi et al (1999), this method is applied by Sommacampagna (2003) where she used the Kalman filter for estimating VaR, the CAViaR (Conditional Autoregressive VaR) method proposed by Engle and Manganelli (2004) and the conditional and unconditional approaches based on the Extreme Value Theory.

Abad et al (2014) emphasised that the performance of the parametric approach in estimating VaR depends on the assumed investment return, when asymmetric and fat tail distributions are considered, the VaR estimate improves considerably, under a normal distribution the VaR estimate is not very accurate.

Doric and Doric (2011) used several alternative models on return distribution and compare predictive ability of VaR estimates based on them. They used the means of back-testing for the whole sample and did not discover the asymmetric behaviour of returns in the case for many stock indexes. However, it was proved that based on VaR estimation, Student's t-distribution and Normal Inverse Gaussian (NIG) distribution are considered good for all α-values.

However, they noted that since the unconditional parametric models assume that investment returns are independently and identically distributed (iid), the density given as:

$$f_x(x) = \frac{1}{\sigma} f_{r*}\left(\frac{x-\mu}{\sigma}\right),$$

where f_r is the density function of the distribution of r_t and f_{r*} is the density function of the standardized distribution of r_t. The parameters μ and σ are the mean value (trend) and standard deviation (volatility) of r_t. The VaR for return $r_t$for long trading positions is given by

$$VaR_{long} = \mu + r_\alpha^* \sigma$$

For short trading positions, VaR is equal to,

$$VaR_{short} = \mu + r_{1-\alpha}^* \sigma,$$

where r_α^* is the α-quantile of f_r^* and $\alpha = 0.05$.

According to Danielsson (2011), there is no intrinsic reason for VaR to be positive, that is VaR might end up on the negative side if the mean of the density of profit/loss distribution is sufficiently large, the probability quantile corresponding to VaR might end up using the negative side especially for long holding periods. He summarized some issues which might arise in applying VaR. These include, VaR is only a quantile on the return distribution, it is a coherent risk measure for some special cases and lastly, it is easy to manipulate. However, VaR has provided the best estimate among other available risk measures, it has underpinned most practical risk models.

2.2 RETURN DISTRIBUTIONS

This section focuses on the normal distribution, student's t-distribution, normal inverse Gaussian distribution and the stable distribution (Doric and Doric, 2011).

2.2.1 NORMAL DISTRIBUTION

The normal distribution also known as the Gaussian distribution is defined with two parameters μ (trend) and σ (volatility). VaR under normal distribution is calculated using these two parameters which is the most technical and widely applied model for assets returns until about ten years ago when findings are that assets returns are not necessarily normally distributed.

The normal density function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

The fitting of the normal distribution uses the maximum likelihood estimate (MLE) for μ and σ.

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} r_i \qquad \hat{\sigma} = \sqrt{\frac{\sum(r_i - \widehat{\mu}_r)^2}{n-1}}$$

where n is the number of observations in the return series. The normal distribution is said to have a zero skewness and kurtosis of 3. However a different method of fitting the distribution is employed in this research study using R-program.

2.2.2 STUDENT'S T-DISTRIBUTION

Doric and Doric (2011) noted that the Student's t-distribution has become an appropriate distribution in developing a model for asset return as seen in many empirical distribution. This is because it has a fat tail and its skewness is not zero as in the case of the normal distribution.

The density function of t-distribution is defined by;

$$f(x) = \frac{\Gamma\left(\frac{v}{2} + 1/2\right)}{\Gamma\left(\frac{v}{2}\sqrt{\pi v b}\right)} \left(1 + \frac{(x-\mu)^2}{bv}\right)^{-(v+1)/2},$$

where $v > 2$ is the degrees of freedom and $b > 0$ is the scale parameter.

Rozga and Arneric (2009) stated that the standard t-distribution has heavier tail with degrees of freedom in the interval, $4 \leq \text{df} \leq 30$. This becomes a fact after fitting the t-distribution in this research work. They stated that degrees of freedom (df) can also be estimated by using the formula;

$$\widehat{K} = \frac{6}{\widehat{df} - 4},$$

where $\widehat{K}$is the kurtosis of the investment returns and $\widehat{\mathrm{df}}$ is the degrees of freedom. This is the method of moments for estimation of degrees of freedom.

2.2.3 NORMAL INVERSE GAUSSIAN DISTRIBUTION

The NIG distribution is characterised by fours parameters α, β, σ and μ.

The density function is defined by;

$$f_{NIG}(x) = \frac{\alpha\sigma}{\pi} \frac{k_1\left(\alpha\sqrt{\sigma^2 + (x-\mu)^2}\right)}{\sqrt{\sigma^2 + (x-\mu)^2}}\ e^{\sigma\sqrt{\alpha^2 - \beta^2 + \beta(x-\mu)}},$$

where k_1 denotes the modified Bessel function of the third kind of order 1, μ and σ denote the scale and location parameters respectively. The conditions for parameters are $|\beta| \leq \alpha$ and $\delta > 0$. The parameters α and β refer to the flatness of the density function and the skewness of the distribution respectively. The greater the α, the greater the concentration of the probability mass around μ and a negative β means heavier left tail while a positive β means heavier right tail. The value β = 0 means the symmetric distribution around μ (Doric and Doric, 2011).

Aas and Haff (2006) argued that NIG is one of the most promising distributions for financial returns among other distributions because it is analytically tractable among other attractive theoretical properties.

2.2.4 HYPERBOLIC DISTRIBUTION

The hyperbolic distribution had been used in various fields before it was applied by Eberlein and Keller (1995). The hyperbolic distribution permits heavier tail than the normal distribution because its log-density is a hyperbola instead of a parabola as in the normal distribution (Doric and Doric, 2011).

The density function is defined by;

$$f_H(x) = \frac{\sigma^2 - \beta^2}{2\alpha\sigma k_1\left(\sigma\sqrt{\sigma^2 - \beta^2}\right)}\ e^{-\alpha\sqrt{\sigma^2 + (x-\mu)^2} + \beta(x-\mu)},$$

where $k_1$is the modified Bessel function of the third kind with index 1. Parameters α and β determine the shape of the density while σ and μ determine the scale and location respectively.

2.2.5 STABLE DISTRIBUTION

The linear combination of two independent random samples is to be stable if it has the same distribution for both the location and the scale parameters. They are sometimes referred to as the Levy alpha-stable distribution.

Doric and Doric (2011) related that the use of the stable distribution to model stock prices was first proposed by Mandelbrot (1963), when he used it for modelling stock and commodity prices and also regarded it as a better description than the normal distribution. He noted that although, most stable distributions and their probability densities cannot be described in closed mathematical form but their characteristic functions can be expressed in closed form. He stated that stable distributions are characterized by four parameters α, β, σ, and μ and the characteristic function of the general stable function is given by;

$$E\left(e^{i\theta x}\right) = \begin{cases} e^{-\sigma^{\alpha}|\theta|^{\alpha}\left(1-i\beta\tan\frac{\pi\alpha}{2}\sin\theta\right)+i\mu\theta} \; ; \; \alpha \neq 1 \\ e^{-\sigma|\theta|\left(1+i\beta\frac{2}{\pi}\ln|\theta|\sin\theta\right)+i\mu\theta} \; ; \; \alpha = 1 \end{cases}$$

He explained the parameters of this characteristic function as follows. The characteristics of the exponent or index α lies in the half-open interval (0, 2] and measures the rate at which the tails of the density function decline to zero. The skewness parameter β lies in the closed interval [-1, 1] and is a measure of asymmetric of the distribution. Stable distribution can be skewed to the left or right depending on the sign β. The scale parameter, σ > 0 measures the spread of the distribution and location parameter, μ is a rough measure of the midpoint of the distribution. The stable distribution with these parameters is denoted as $S_{\alpha}(\beta, \sigma. \mu)$.

2.2.6 BINOMIAL DISTRIBUTION

From the basic idea of probability, Andersen and Frederiksen (2010) noted that the binomial distribution is useful when dealing with random variable with two possible outcomes, success and failure. They noted that this idea is also useful in risk management because of the continuous interest in the evaluation of the risk models.

Say one is interested in evaluating whether the loss in a given portfolio is below (success) or above (failure) some arbitrary threshold. Given a sample of n-trial observations and X, a random variable that equals the number of successes in these $n-$ trials. If p is the probability of success and $1-p$ is the probability of a failure and the four conditions below are met, then the series of random variable can be defined as binomial distributed. These four conditions are:

i. only two outcomes are possible for every trial,
ii. each trial's outcome has the same probability of success,
iii. each outcome does not depend on previous outcomes, that is, independence between outcomes,
iv. there is a fixed number of trials.

When these conditions are met, it is possible to calculate the number of successes with the combinatorial approach below:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \tag{2.1}$$

and the related probability $p^x(1-p)^{n-x}$. Summing up, the probability of x successes within a series of $n-$ trials can be calculated as:

$$P(x) = \binom{n}{x}p^x(1-p)^{n-x},\ where\ x = 0,1,2,\dots,n \tag{2.2}$$

The mean of the binomial distribution equals the proportion of the rate of success of the trials, that is, $\mu = np$ and the standard deviation is $\sigma = \sqrt{np(1-p)}$.

2.3 BACK-TESTING

Abad et al (2014) noted that among the standard tests for the accuracy of VaR models there is back-testing criterion. Others include, the unconditional and conditional coverage tests and the dynamic quantile test. They noted that to implement all these tests an exception indicator must be defined as follows:

$$I_{t+1} = \begin{cases} 1 & if\ \ r_{t+1} < VaR(\alpha) \\ 0 & if\ \ r_{t+1} > VaR(\alpha)' \end{cases}$$

where r_{t+1} is the return after day $t = 252$.

Kupiec showed that if the probability of an exception is constant, then the number of exceptions, $x = \sum I_{t+1}$ follows a binomial distribution, $\mathrm{B}(\mathrm{N}, \alpha)$. For back-testing criterion, the test for the significance of the departure of $\hat{\alpha}$ from α is carried out using the z-statistic which follows an asymptotic normal distribution:

$$z = \frac{(N\hat{\alpha} - N\alpha)}{\sqrt{N\alpha(1-\alpha)}}$$

(Kupiec, 1995 as cited in Abad et al 2014).

Danielsson (2011) noted that there is no best model for forecasting risk as individual models can be checked for parametric significance or analysing residuals. According to him, back-testing evaluates VaR forecasts by checking how a VaR forecast model performs over a period. It is a procedure used to compare various risk models. It aims to take an ex ante VaR forecasts from a particular model and compare them with ex post realized investment returns (historical observations), whenever losses exceed VaR, a VaR violation is said to have occurred. He noted that back-testing can be useful in identifying the weakness of risk forecasting models and providing an improvement. It prevents underestimation of VaR and which ensures that a financial institution carries significantly high capital, it as well reduces overestimating VaR which can lead to excessive conservatism. Danielsson (2011) emphasized that the violation ratio is the actual number of VaR violation compared with expected value. This he said is the main tool in back-testing.

His idea is as below:

$$\text{Violation Ratio, VR} = \frac{observed\ number\ of\ violations}{expected\ number\ of\ violations} = \frac{v_1}{p\ x\ W_T}$$

$$v_1 = \sum I_{t+1}$$

$$v_0 = W_T - v_1,$$

where W_T is the testing window (the difference between the number of returns, N and number of trading days in a year, 252), v_1 is the count of the indicator I_{t+1} = 1 and v_0 is the count of I_{t+1} = 0.

He emphasized that if VR > 1 then the VaR model underforecasts risk and if VR < 1 then the VaR model overforecasts risk. However, if $VR \in [0.8,\ 1.2]$, it is a good forecast and if $VR < 0.5\ or\ VR > 1.5$, the model is imprecise for $\alpha = 0.01$.

CHAPTER THREE

METHODOLOGY OF BACK-TESTING

This chapter focuses on the procedures involved in the parametric evaluation of VaR using the standard model, the normal distribution. However, after the discovery of some inadequacies the Student's t-distribution was applied.

3.1 DATA COLLECTION

A 3-year historical trading data from 14/02/2014 to 14/02/2017 (756 days) gotten from www.finance.yahoo.com, the adjusted closing price for 12 companies form NASDAQ stock market was used in analysing the VaR at 90%, 95%, 98%, 99%, 99.5% and 99.9% confidence probabilities for both the normal and Student's t-distributions. The companies are listed in the table below with the following label for identification purpose.

COMPANY	**STOCK NUMBER**
APPLE	1
INTEL	2
MICROSOFT	3
MICRON	4
SIRIUS	5
POPEYES	6
CISCO	7
FACEBOOK	8
ON	9
FRONTIER	10
SPARTAN	11
GILEAD	12

3.2 INSTRUMENTATION

3.2.1 ESTIMATING INVESTMENT RETURNS

Let P_t be the price of an investment asset on day t and $P_{t-1}$the price the day before, that is, $t-1$. The investment return is given and calculated by;

$$r_t = \frac{P_t}{P_{t-1}} - 1 \quad (3.1)$$

3.2.2PARAMETRIC MODEL FOR VaR FOR NORMAL DISTRIBUTION

The calculation of VaR for normal distribution was based on the standard formula,

$$VaR_\alpha = \mu - qnorm_\alpha \sigma, \tag{3.2}$$

where α is the confidence probability (90%, 95% etc), $qnorm_\alpha$ is the standard normal α-quantile, μ is the mean and σ is the standard deviation of the return.

The expected return, μ is estimated by the sample mean

$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} r_i,$

where $r_i$is the return on the day i, n is the number of trading days in a year.

The standard deviation σ is estimated by sample standard deviation

$$\hat{\sigma} = \sqrt{\frac{\sum(r_i - \hat{r})^2}{n-1}}$$

The region for the profit/loss distribution is specified with the diagram below.

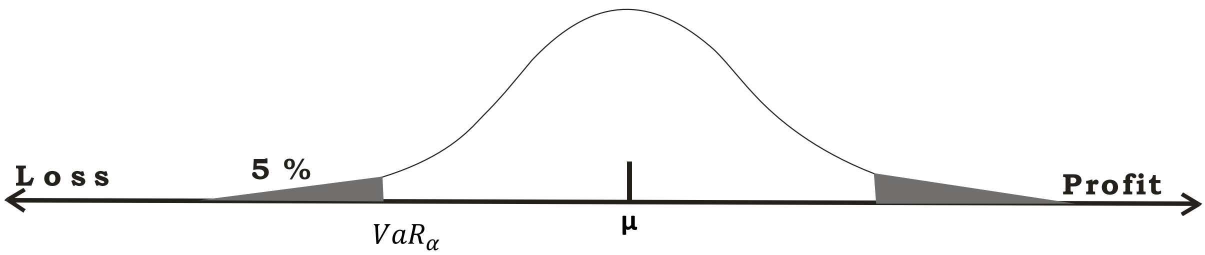


Figure 3.1. Graphical representation of VaR (own drawing).

The VaR formula stated above is used because our focus is on the loss i.e. on the negative side of the return distribution.

3.2.3 PARAMETRIC MODEL FOR VaR FOR STUDENT'S T-DISTRIBUTION

The normal distribution does not fit return investment in some cases, whereas, the Student's t-distribution fits properly in these failed cases. Hence, the calculation of VaR for Student's t-distribution was based on the formula,

$$VaR_\alpha = \mu - qt_\alpha^v \sigma, \tag{3.3}$$

where α is the confidence probability (90%, 95% etc), v is the same as degrees of freedom (df), qt_{α}^{v} is the α-quantile for the t-distribution, μ is the mean and σ is the standard deviation of the return.

The expected return, μ is estimated by the sample mean,

$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} r_i$,

where r_i is the return on day i, n is the number of trading days in a year.

The standard deviation is estimated by the sample standard deviation

$$\hat{\sigma} = \sqrt{\frac{\sum(r_i - \hat{r})^2}{n-1}}$$

3.2.4 GENERAL SCHEME OF THE BACK-TESTING PROCEDURE

General scheme of our back-testing procedure is the following:

1. Get adjusted closing price for 3 trading years of stock data for any company, say Apple Incorporation.
2. Proceed as follows:
   - i. Calculate VaR_{α} on the basis of 1 year data (252 days)
   - ii. Compare VaR_{α} with actual loss of the next day
   - iii. Repeat the steps i-ii 503 times, each time with a new window shifted by 1 day forward
   - iv. Count how many times (out of 503) VaR_{α} did not work (number of violations).

In an ideal case, the relative frequency of violations is close to the probability α.

The whole idea of the back testing procedure is explained by the diagram below.

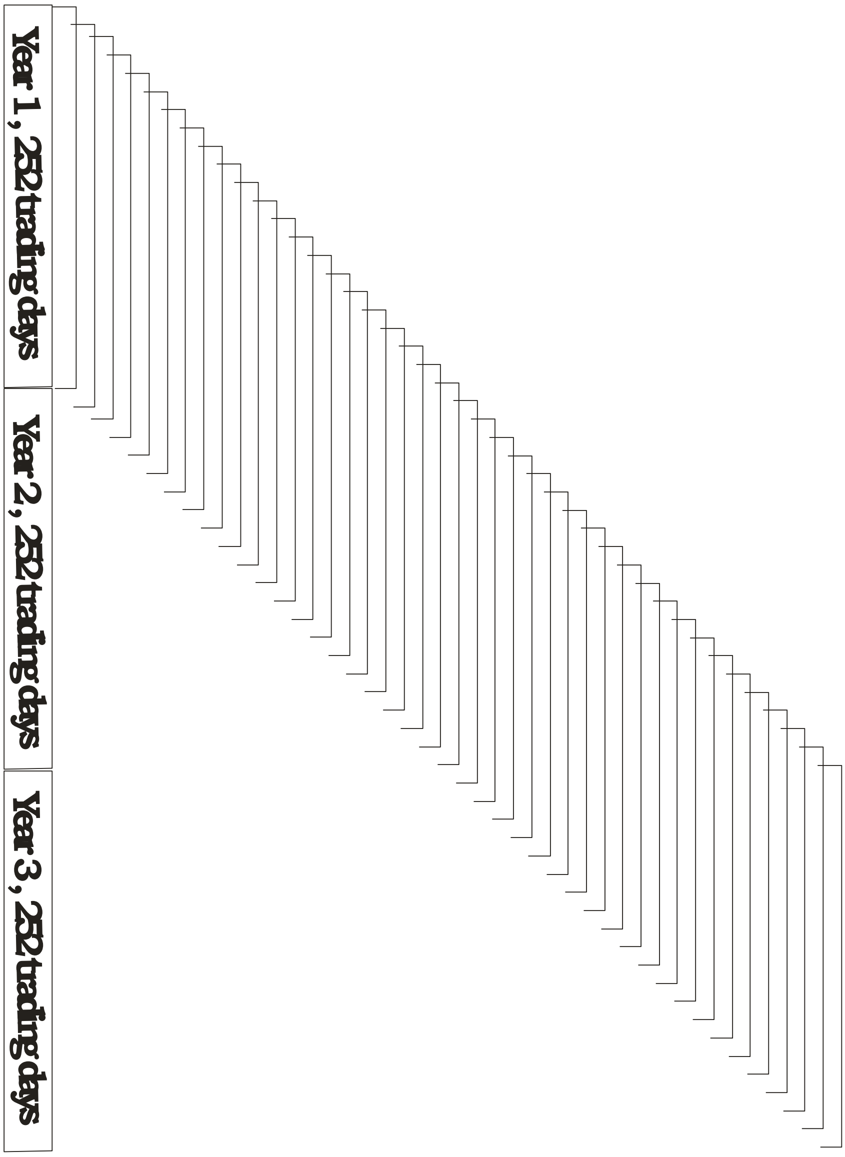


Figure 3.2 showing the back-testing procedure for a 756 trading days starting with a 252-days and shifting the test window each time by one-day (own drawing).

The idea of back-testing as it applies to failure rate gives a quantitative measure of the accuracy of the model. Since the loss values are on the negative side of the profit/loss distribution, the failure rates are expected to be 1% for 99% confidence level, 5% for 95% confidence level etc. Howbeit, statistical tests are needed to verify if the failure rate is too high or low compared to the expected failure rate. For if the number of violations is too low, the model is too conservative leading to an inefficient allocation of capital. Hence, the company applying the model will not act in their owner's (shareholder's etc) best interest. This failure rate could also be called the violation (Danielsson, 2011) or the Indicator (Abad et al, 2014).

3.2.5 BINOMIAL DISTRIBUTION AND THE BACK-TESTING FAILURE RATE

From the binomial distribution discussed in the preceding chapter, when testing for failure rate, it is known that there are two possible outcomes at each point in time, that is VaR can either be violated or not. Because of this, each daily outcome can be treated as a Bernoulli trial with a binomial distribution.

Letting n be the total number of trials, p the assumed probability of failure (the probability of violating VaR) and n_A the number of failures in the series of $n-$trials, the failure rate can be estimated by $\hat{p} = \frac{n_A}{n}$.

We test the proportion hypothesis as follows to know whether to reject the model or accept it.

$$H_0 : p = p_0, where\ p_0 = \alpha$$

Because the proportion test is two-sided, it gives the spread within which the sample failure rate will be in line with the population failure rate. If this hypothesis is rejected, the model is also rejected.

3.2.6 CONFIDENCE INTERVAL AND THE BACK-TESTING FAILURE RATE

The confidence interval is a good choice to check if the test for failure rate is accurate with a specific model by checking the interval for which the test values should fall within.

The confidence interval for the failure rate p is given by;

$$\hat{p}_{LCL}, \hat{p}_{UCL} = \hat{p} \pm Z_{0.975}\, \hat{\sigma}, \tag{3.4}$$

where $\hat{p}$ denotes the proportion of failure rate $\hat{p} = \frac{n_A}{n}$, n_A denotes the number of failure of investment returns, n denotes the total number of the values of VaR, $\hat{p}_{LCL}$ denotes the lower confidence level, $\hat{p}_{UCL}$ denotes the upper confidence level, $Z_{0.975}$ is the standard 0.975-quantile of the standard normal distribution and $\hat{\sigma}$ is the standard deviation of the failure rate proportion given by

$$\hat{\sigma} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

The value of the failure rate p is within the interval, $\hat{p}_{LCL} < p < \hat{p}_{UCL}$, with probability 0.95. the confidence limit above are based on the assumption of independence of $n-$ trials.

3.2.7 LOG-LIKELIHOOD TEST AND THE BACK-TESTING FAILURE RATE

Christoffersen (1998) used a log-likelihood test to compare empirical failure rate $\widehat{p}$ with $p_0 = \alpha$ (called the unconditional coverage test).

His setup for the test is as follows.

$$H_0 : p = p_0 \; vs \; H_1 : p \neq p_0$$

Combined with equation 2.1 which states that the probability of seeing n_A violations is $(1 - p_0)^{n_0} {p_0}^{n_A}$ where $n_0 = n - n_A$ is the total number of non-violating observations, it is possible to test the likelihood of this hypothesis against the observed probability of n_A; $(1 - \hat{p})^{n_0} \hat{p}^{n_A}$ where π is an estimate of the true failure rate, $\hat{p} = \frac{n_A}{n}$. This gives the log-likelihood ratio (LR) test simplified by (Andersen and Frederiksen, 2010) as below

$$LR_{uc} = -2 \log \left[\frac{(1-p_0)^{n_0} {p_0}^{n_A}}{(1-\hat{p})^{n_0} \hat{p}^{n_A}} \right] \sim x^2(1) \qquad (3.5)$$

or

$$LR_{uc} = -2[n_0 \log(1 - p_0) + n_A \log(p_0) - n_0 \log(1 - \hat{p}) - n_A \log(\hat{p}) \;], \qquad (3.6)$$

where $x^2(1)$ is the chi-square distribution with 1 degree of freedom (df = 1).

In practice, equation 3.6 is applied as some mathematical software programs have problem calculating (3.5) because when $p \rightarrow 0$ and $n \rightarrow \infty$ the denominator in 3.5 tends to zero and when n and p combined pass some threshold, some software programs fail to work.

The test for unconditional coverage explains the goodness-of-fit of the failure rate compared to the proposed failure rate under H_0. Therefore, H_0 is accepted when $LR_{uc} < {x^2}_{1-\alpha}(1)$ which means that the overall observed failure rate is in line with the expected failure rate, otherwise, $H_0$is rejected and $H_1$is accepted.

3.2.8 LOG-LIKELIHOOD TEST FOR INDEPENDENCE OF BACK-TESTING VIOLATIONS.

The unconditional coverage test does not test for clustering and thereby it does not reveal if there is a tendency for large violations to come in clusters. Hence, the test for independence can be carried out to reveal this fact.

Andersen and Frederiksen (2010) stated that a more scientific test which will make it possible to accept or reject the model based on the failure rate, is a likelihood test. This method tests to verify whether violations are independently and identically distributed (iid) as postulated by Christoffersen (1998)

When violations are not independent, the probability of a violation tomorrow, given there has been a violation today, is no longer equal to p. Because of this, it is necessary to set up a test which will reveal such tendencies.

By defining

- n_{0A} as the number of observations where a non-violation is followed by a violation
- n_{AA} when a violation is followed by another violation
- n_{A0} when a violation is followed by a non-violation
- n_{00} when a non-violation is followed by another non-violation

Define $\mathbb{\Pi}_1$ as:

$$\mathbb{\Pi}_1 = \begin{pmatrix} \pi_{0A} \\ \pi_{AA} \end{pmatrix} = \begin{pmatrix} \dfrac{n_{0A}}{n_{00} + n_{0A}} \\ \dfrac{n_{AA}}{n_{A0} + n_{AA}} \end{pmatrix}$$

The log-likelihood ratio test for independence thereby becomes a test for the null hypothesis,

$$H_0 : \mathbb{\Pi}_1 = \mathbb{\Pi}_2 \quad vs \quad H_1 : \mathbb{\Pi}_1 \neq \mathbb{\Pi}_2,$$

where $\mathbb{\Pi}_2 = \binom{\hat{p}}{\hat{p}}$ and $\hat{p} = \frac{n_{0A} + n_{AA}}{n} = \frac{n_A}{n}$. The test statistic to test H_0 is

$$LR_{in} = -2\log\left[\frac{(1-\pi_{0A})^{n_{00}}\pi_{0A}{}^{n_{0A}}(1-\pi_{AA})^{n_{A0}}\ \pi_{AA}{}^{n_{AA}}}{(1-\hat{p})^{(n_{00}+n_{A0})}(\hat{p})^{(n_{0A}+n_{AA})}}\right] \sim x^2(1), \qquad (3.9)$$

The hypothesis test above is to test if the general failure rate is the same as the likelihood of observing a violation following a violation. If this is true, then the series of violations do not cluster. This test completely ignores p_0 and only tests if it is comfortably probable that the probability of n_{AA} and n_{0A} is the same as the general probability of observing a violation. If this test's p-value is too high compared to the chi-squared value, the test of independence is rejected with the consequence of being that the violations are not independently and identically distributed (iid) (Andersen and Frederiksen, 2010).

Worthy of note is that if LR_{uc} is rejected but LR_{in} is accepted, then this is a special case, where $p_0 \neq p$ but where the violations are iid. On the other hand, if the test for unconditional coverage and independence is accepted then it means that the observed failure rate is close to the confidence level and that the probability of observing subsequent violations is also close to this confidence level.

3.2.9 LJUNG-BOX TEST

One of the assumptions for the application of Value-at-Risk formula (3.2) is the assumption of independence of returns (r_i). More exactly, the independence of returns is necessary in order for $\hat{\sigma}$ to be an unbiased estimator of σ. Therefore, we also need to test autocorrelations in return data.

According to Danielsson (2011), the Ljung-Box test verifies for the correlation of investment returns, it verifies if the correlations of return are zero. If the autocorrelations of return are not different from zero then the expected value of the distribution would be the best guess for tomorrow's portfolio return. It tests the overall randomness based on a number of lags, instead of testing randomness at each lag. However, the Ljung-Box test is a test of the general independently and identically distributed (iid) assumption of returns, where the LR_{in} is only tested for independence between violations. Hence, positive deviation must be followed by negative deviation on the average.

H_0: Returns are independently distributed

The test statistic is given by

$$Q_{LB} = n(n+2)\sum_{j=1}^{h}\frac{\hat{\rho}(j)^2}{n-j},$$

where $\hat{\rho}(j)^2$ is the correlation factor between the jth observation and the initial observation, n is the sample size, and h is the number of lags being tested. The Ljung-Box test is thereby a test of the predictability between observations. We reject H_0 if $Q_{LB} > \ x^2{}_{1-\alpha}(h)$, where $x^2{}_{1-\alpha}(h)$is the α-quantile distribution with h degrees of freedom.

3.3 JARQUE-BERA TEST

Among the assumptions for the application of Riskmetrics, Value-at-Risk formula (3.2) is to assume that returns are normally distributed. We need to ascertain this fact by carrying the normality test.

According to Danielsson (2011), the Jarque-Bera test is a goodness-of-fit test which can be used to test if the return follows a normal distribution based on the observations of skewness and kurtosis of the empirical distribution. The test statistic is defined as

$$JB = \frac{n-k}{6}\left(S^2 + \frac{1}{4}(K-3)^2\right),$$

where n is the number of observations, k is the number of explanatory variables if the data come from the residuals of a linear regression, otherwise, $k = 0$, S is the asymmetry coefficient of the sample tested and K is the kurtosis of the sample tested.

Mathematically, S and K are defined by;

$S = \frac{\hat{\mu}_3}{\hat{\sigma}^3}$ is the empirical distribution's skewness and $K = \frac{\hat{\mu}_4}{\hat{\sigma}^4}$, where $\frac{\hat{\mu}_4}{\hat{\sigma}^4}$ is the kurtosis, $\hat{\mu}_3$and $\hat{\mu}_4$ are the third and fourth moment estimators respectively and $\hat{\sigma}^3$ and $\hat{\sigma}^4$ can be estimated from the variance as below,

$$\hat{\mu}_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3, \quad \hat{\sigma}^3 = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^{3/2}, \ \hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^4, \ \hat{\sigma}^3 = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2,$$

where $\bar{x}$ is the average of the sample.

This test follows a chi-square distribution with degrees of freedom (df) = 2.

These tests are implemented in R-program.

CHAPTER FOUR

DATA ANALYSIS AND INTERPRETATION

This chapter presents the analysis of data used in conducting this research study. The interpretation of statistical findings based on the drawn hypotheses in Chapter one using the data gathering and analysis instrument described in Chapter three.

4.1 DESCRIPTIVE STATISTICAL ANALYSIS OF INVESTMENT RETURNS OF 12 STOCKS

Stock	Min Return	Max Return	Median	Mean	Standard deviation	Skewness	Kurtosis
1	-0.066	0.081	0.001	0.001	0.015	0.148	6.667
2	-0.091	0.093	0.001	0.001	0.014	0.063	8.852
3	-0.093	0.105	0.000	0.001	0.015	0.469	12.505
4	-0.182	0.127	0.000	0.000	0.029	-0.009	6.825
5	-0.057	0.048	0.000	0.000	0.012	-0.230	4.235
6	-0.093	0.144	0.000	0.001	0.017	0.953	13.026
7	-0.058	0.096	0.000	0.001	0.013	0.805	12.017
8	-0.069	0.115	0.001	0.001	0.018	0.668	11.482
9	-0.106	0.109	0.001	0.001	0.022	-0.186	5.756
10	-0.137	0.143	0.000	0.000	0.023	0.238	9.524
11	-0.130	0.153	0.001	0.001	0.020	0.927	14.256
12	-0.143	0.059	0.001	0.000	0.019	-1.149	10.053

4.2 GRAPHICAL REPRESENTATIONS OF INVESTMENT RETURNS

4.2.1 QQ PLOTS FOR APPLE RETURNS

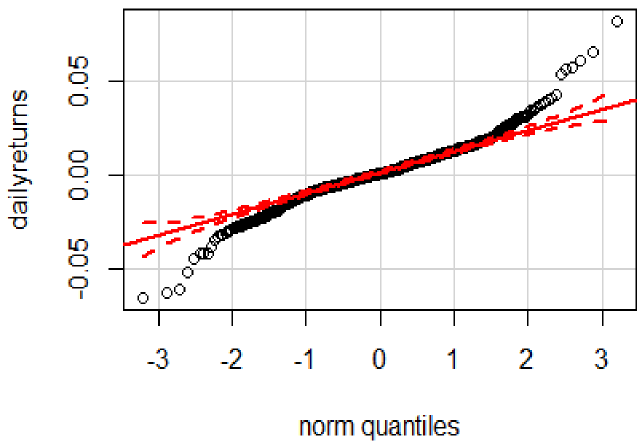


Figure 4.2.2(a)

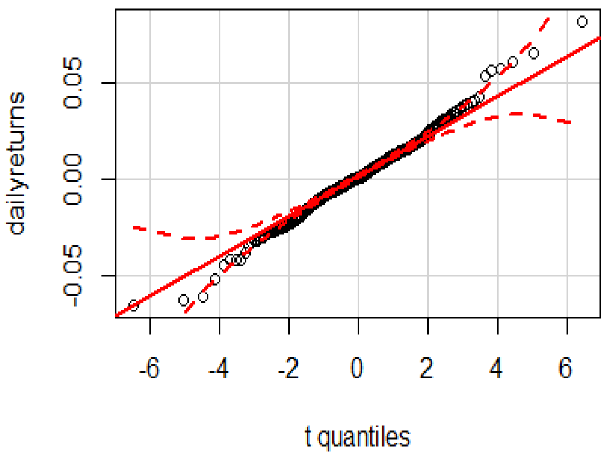
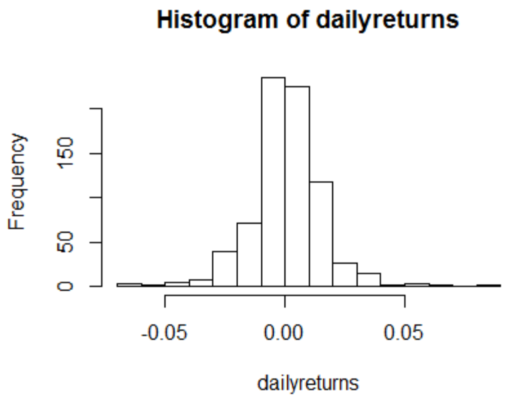
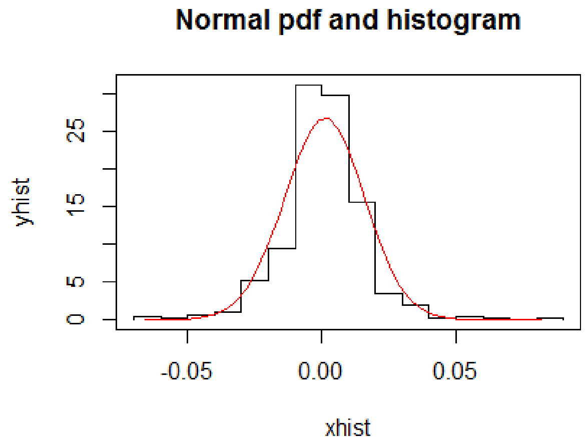


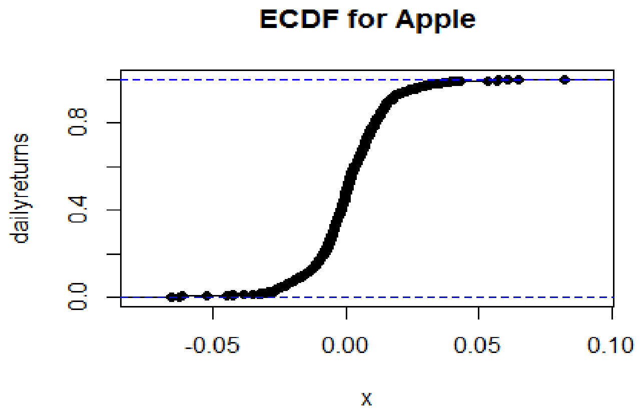
Figure 4.2.2(b)

Figure 4.2.2(a) shows the QQ plot (norm quantiles) for normal distribution. It is obvious that this plot did not contain returns at the extreme ends (light tail) with the red lines. However, the QQ plot (t quantiles) for Student’s t-distribution is a better plot for its heavy/fat tail property.

4.2.2 NORMAL PROBABILITY DENSITY FUNCTION AND HISTOGRAM FOR APPLE RETURNS







From these charts above we see that, qualitatively, the distribution of Apple returns is close to normal.

4.3 DISTRIBUTION TESTS FOR APPLE RETURNS

Now we perform statistical tests of normality of Apple returns

4.3.1 Result table of Test for normality for Apple returns.

Test	$p - value$
Jarque-Bera	2.2e-16
Shapiro-Wilk	9.34e-15

The above table shows the test for normality based on the investment returns. The $p - values$ for each test is less than 0.05 which means we reject that the distribution of return is normal.

4.3.2 Result table of test for Student's t-distribution for Apple returns.

Test	$p - value$
t.test	0.0894

The $p - value > 0.05$ for the t-test above. We accept H_0 and conclude that the distribution of return of the Apple equity could actually be from the t-distribution.

4.4 TEST FOR AUTOCORRELATION OF RETURNS OF APPLE

4.4.1 LJUNG-BOX TEST

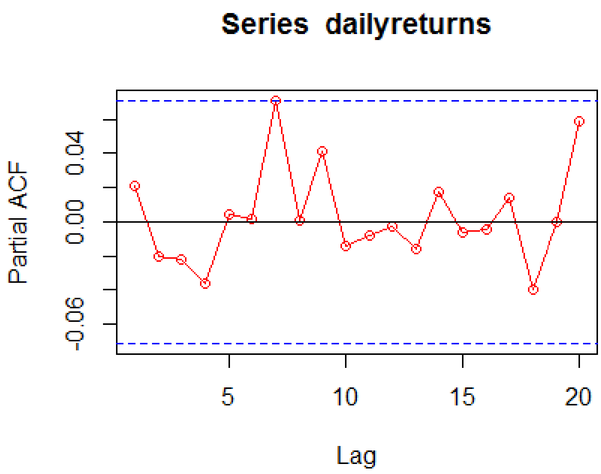
A 3-year historical trading data from 14/02/2014 to 14/02/2017 (756 days) for Apple equity was used for this test.

Test	$p - value$	h
Ljung-Box	0.5606	1

From the result in the table above the $p - value > 0.05$. Hence, we accept H_0 and conclude that investment returns of Apple can be regarded as independent.

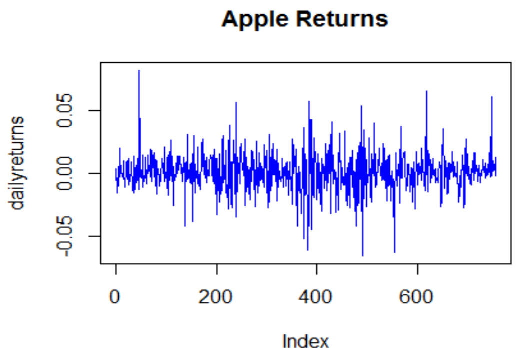
4.4.2 PARTIAL AUTOCORRELATION FUNCTION PLOT OF APPLE RETURNS

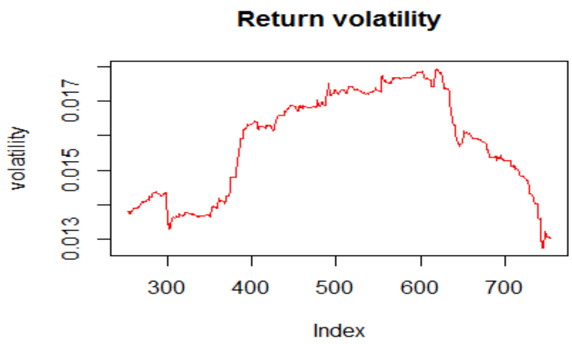
A 3-year historical trading data from 14/02/2014 to 14/02/2017 (756 days) for Apple equity was used for this test.

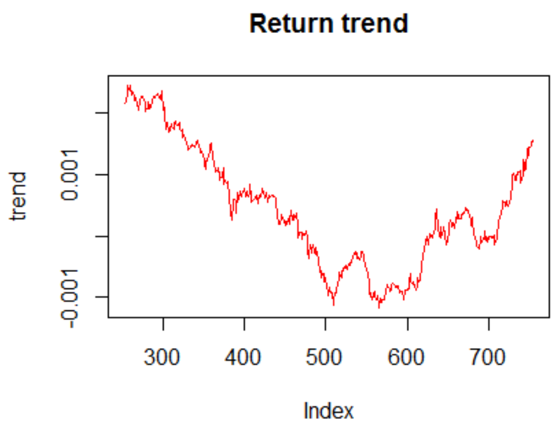


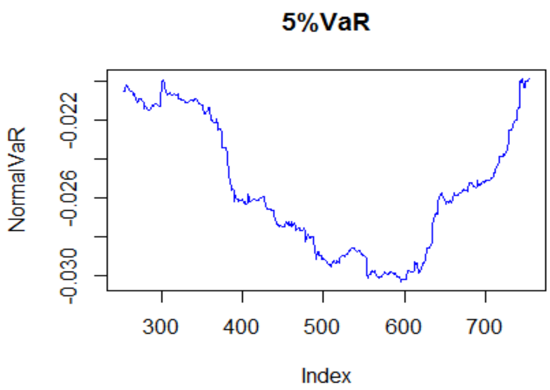
4.5 SOME EMPIRICAL RESULTS PLOTS FOR APPLE DATA

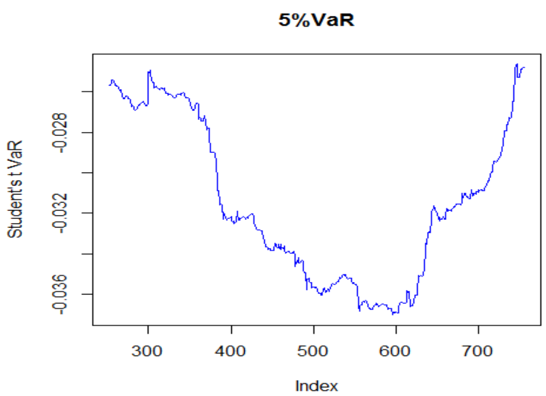
Next we present some graphics depicting the returns of Apple stock, its volatility, trend and values of $VaR_{0.05}$ that are calculated by two methods (normal distribution based & t-distribution based).











Return volatility is calculated on the basis of 252 previous days. The same is true for trend and 5% VaR based on normal and t-distribution.

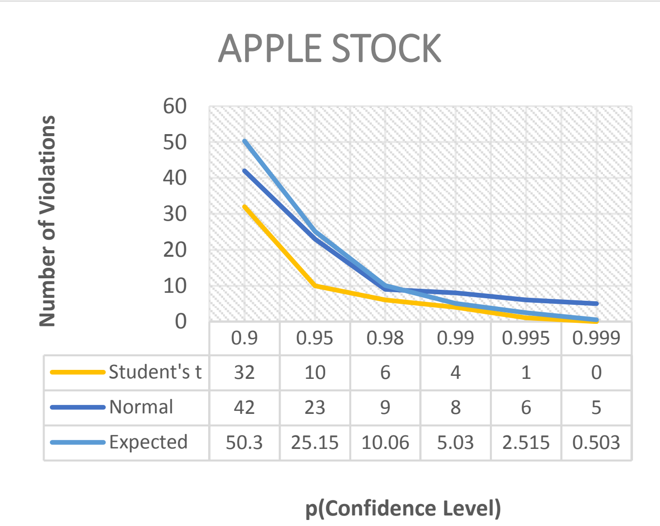
4.6 CORRELATION MATRIX OF INVESTMENT RETURNS FOR 12 STOCKS

CORRELATION MATRIX

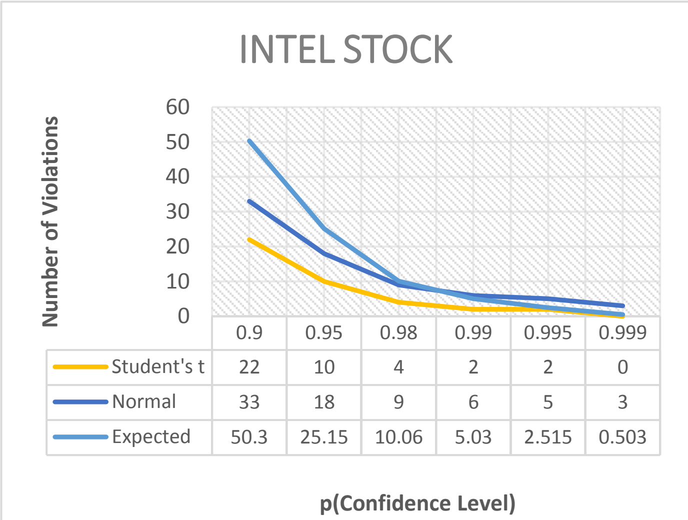
	MICRON	*TWITTER*	*SPARTAN*	*SIRIUS*	*POPEYES*	*MICROSOFT*	*GILEAD*	*FRONTIER*	*FACEBOOK*	*APPLE*	*CISCO*	*INTEL*
MICRON	1.000											
TWITTER	0.246	1.000										
SPARTAN	0.203	0.115	1.000									
SIRIUS	0.364	0.222	0.274	1.000								
POPEYES	0.185	0.147	0.209	0.236	1.000							
MICROSOFT	0.363	0.181	0.219	0.406	0.237	1.000						
GILEAD	0.278	0.152	0.108	0.264	0.160	0.277	1.000					
FRONTIER	0.239	0.164	0.164	0.267	0.083	0.239	0.191	1.000				
FACEBOOK	0.280	0.241	0.217	0.376	0.245	0.427	0.303	0.160	1.000			
APPLE	0.082	-0.020	-0.021	0.066	0.037	0.078	0.117	0.074	0.035	1.000		
CISCO	0.392	0.182	0.220	0.399	0.225	0.517	0.309	0.272	0.329	0.069	1.000	
INTEL	0.486	0.167	0.177	0.385	0.188	0.559	0.263	0.264	0.340	0.081	0.538	1.000

4.7 BACK-TESTING RESULTS TABLES FOR VaR ESTIMATES

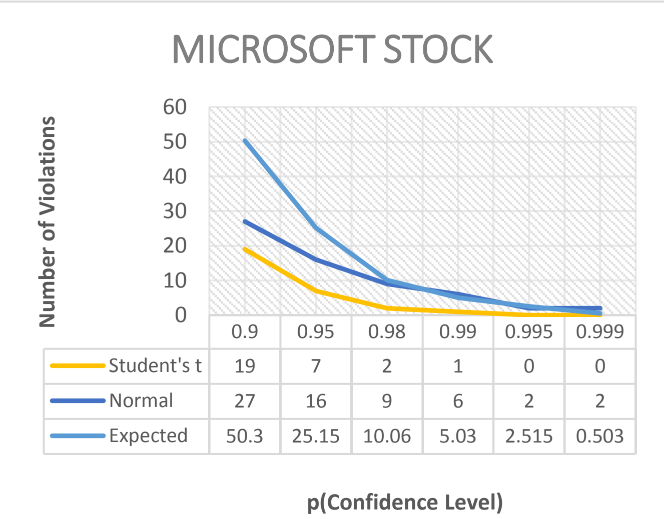
In the graph below the number of violations is plotted as graph against the confidence probabilities. The expected number of violations and the empirical number of violations from the normal distribution and the t-distribution is shown with the table of values at confidence probabilities 90%, 95%, 98%, 99%, 99.5% and 99.9%.



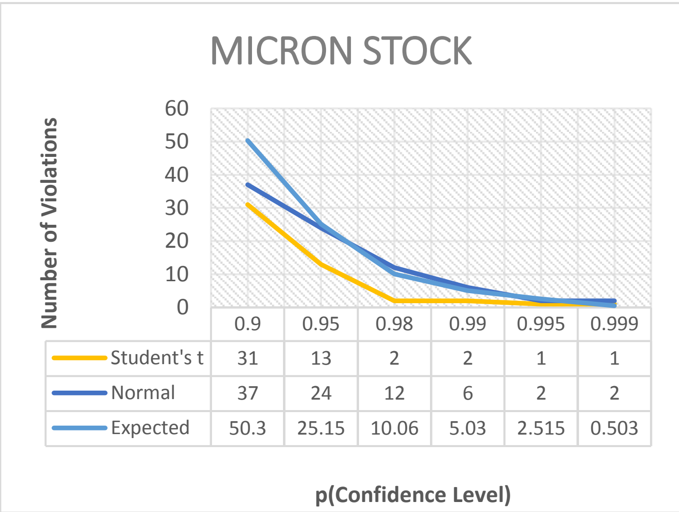
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	32	10	6	4	1	0
Normal	42	23	9	8	6	5
Expected	50.3	25.15	10.06	5.03	2.515	0.503



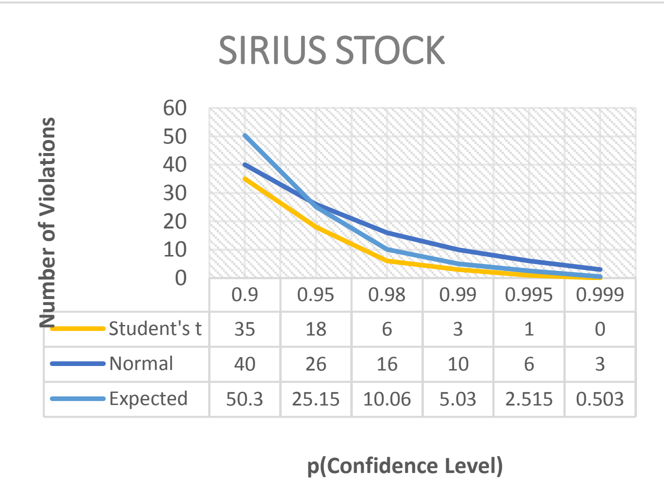
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	22	10	4	2	2	0
Normal	33	18	9	6	5	3
Expected	50.3	25.15	10.06	5.03	2.515	0.503



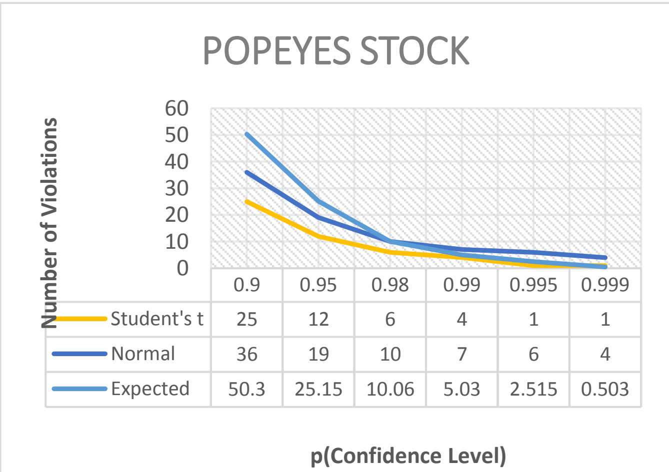
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	19	7	2	1	0	0
Normal	27	16	9	6	2	2
Expected	50.3	25.15	10.06	5.03	2.515	0.503



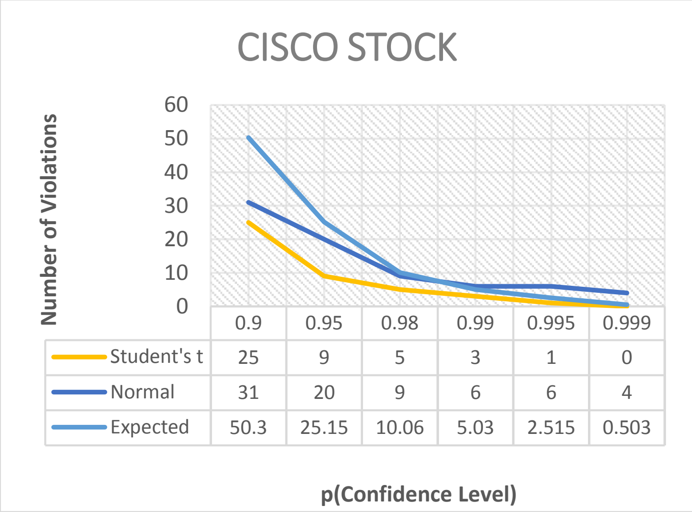
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	31	13	2	2	1	1
Normal	37	24	12	6	2	2
Expected	50.3	25.15	10.06	5.03	2.515	0.503



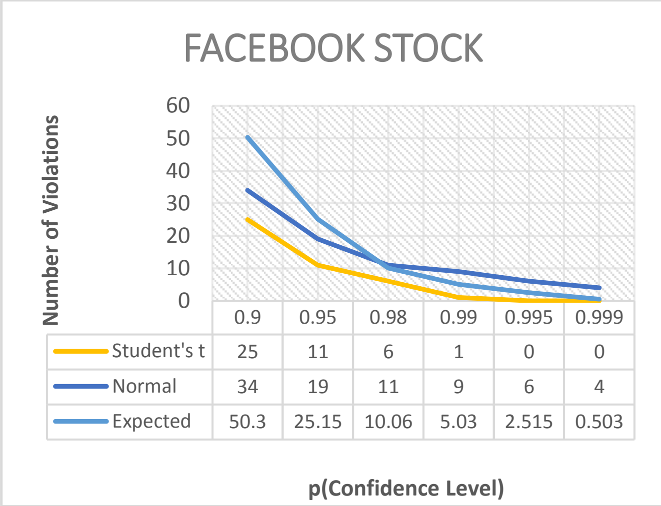
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	35	18	6	3	1	0
Normal	40	26	16	10	6	3
Expected	50.3	25.15	10.06	5.03	2.515	0.503



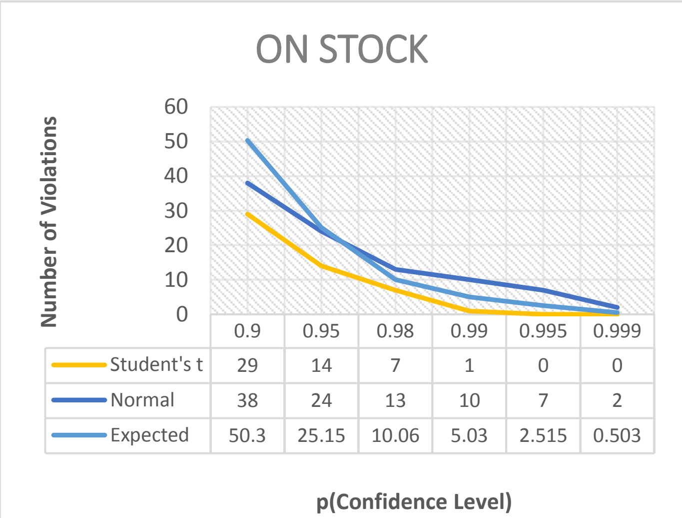
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	25	12	6	4	1	1
Normal	36	19	10	7	6	4
Expected	50.3	25.15	10.06	5.03	2.515	0.503



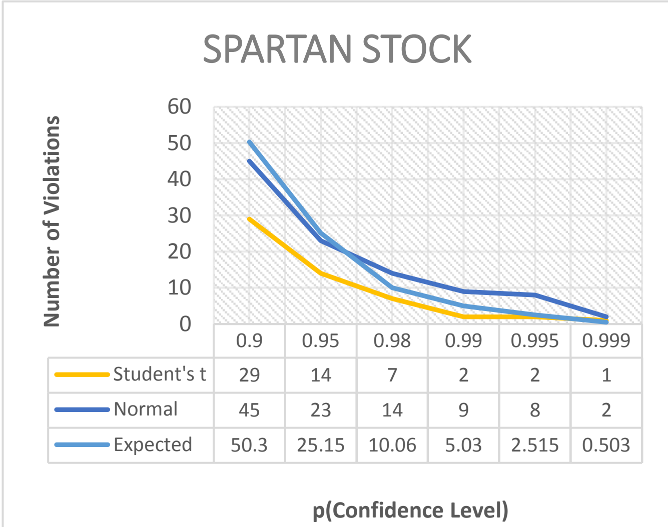
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	25	9	5	3	1	0
Normal	31	20	9	6	6	4
Expected	50.3	25.15	10.06	5.03	2.515	0.503



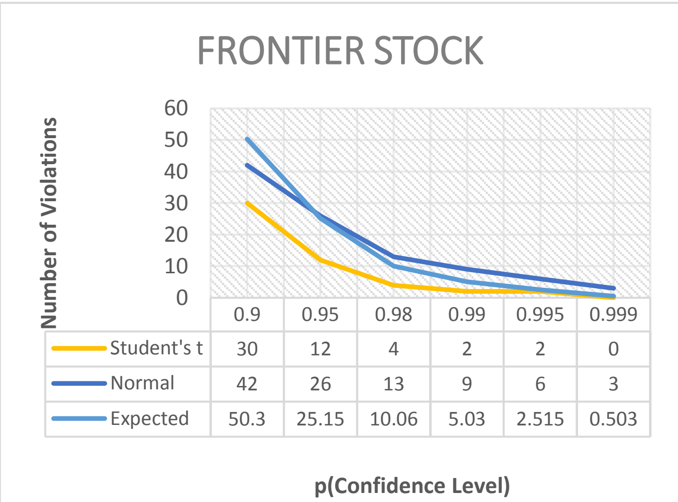
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	25	11	6	1	0	0
Normal	34	19	11	9	6	4
Expected	50.3	25.15	10.06	5.03	2.515	0.503



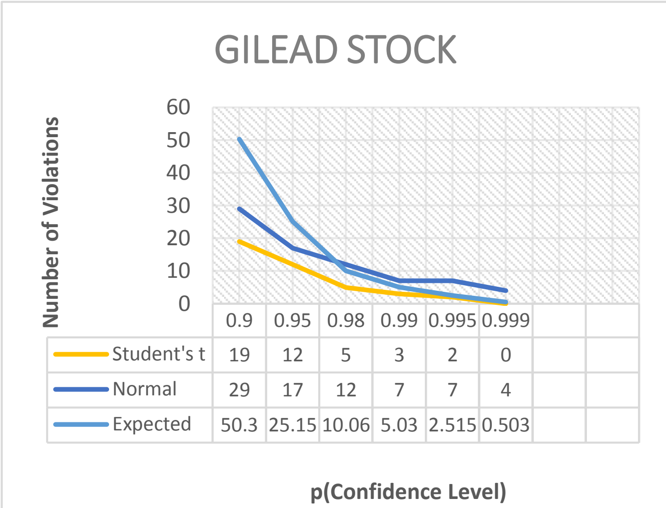
	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	29	14	7	1	0	0
Normal	38	24	13	10	7	2
Expected	50.3	25.15	10.06	5.03	2.515	0.503



	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	29	14	7	2	2	1
Normal	45	23	14	9	8	2
Expected	50.3	25.15	10.06	5.03	2.515	0.503



	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	30	12	4	2	2	0
Normal	42	26	13	9	6	3
Expected	50.3	25.15	10.06	5.03	2.515	0.503



	0.9	0.95	0.98	0.99	0.995	0.999
Student's t	19	12	5	3	2	0
Normal	29	17	12	7	7	4
Expected	50.3	25.15	10.06	5.03	2.515	0.503

From the above charts with table of values, from the empirical estimation of VaR, the normal distribution produces the better results for $\alpha = 5\%$, the same applies to $\alpha = 10\%$ where-as for other values like $\alpha = 1\%$, $\alpha = 0.5\%$ *and* $\alpha = 0.1\%$, the Student's t-distribution seems to be a better model. For $\alpha = 2\%$, the normal distribution seems good for some stocks, while the Student's t-distribution seems good for some other stocks.

4.7.1 Results table for failure rate $\hat{p}$.

Here we present the names of data in numerical form.

Empirical failure rate $\hat{p}$													
Model		Normal						Student's t					
Stock	$\boldsymbol{\alpha}$	**0.1**	**0.05**	**0.02**	**0.01**	**0.005**	**0.001**	**0.1**	**0.05**	**0.02**	**0.01**	**0.005**	**0.001**
1		0.08	0.046	0.018	0.016	0.0119	0.0099	0.064	0.019	0.012	0.008	0.0019	0.000
2		0.07	0.036	0.018	0.012	0.0099	0.0059	0.043	0.019	0.008	0.004	0.0039	0.000
3		0.05	0.032	0.018	0.012	0.0039	0.0039	0.037	0.014	0.004	0.002	0.00	0.000
4		0.07	0.048	0.024	0.012	0.0039	0.0039	0.061	0.026	0.004	0.004	0.0019	0.002
5		0.08	0.052	0.032	0.019	0.0119	0.0059	0.069	0.036	0.012	0.006	0.0019	0.000
6		0.07	0.038	0.019	0.014	0.0119	0.0079	0.049	0.024	0.012	0.008	0.0059	0.002
7		0.06	0.039	0.018	0.012	0.0119	0.0079	0.049	0.018	0.009	0.006	0.0019	0.00
8		0.07	0.038	0.022	0.018	0.0119	0.0079	0.049	0.022	0.012	0.002	0.00	0.00
9		0.08	0.048	0.026	0.019	0.0139	0.0039	0.058	0.028	0.014	0.002	0.00	0.00
10		0.08	0.052	0.026	0.018	0.0119	0.0059	0.059	0.024	0.008	0.004	0.0039	0.00
11		0.09	0.046	0.029	0.018	0.0159	0.0039	0.058	0.028	0.014	0.004	0.0039	0.002
12		0.06	0.034	0.024	0.014	0.0139	0.0079	0.038	0.024	0.009	0.004	0.0059	0.00

Next we need to test whether the empirical failure rates in the table are in accordance with respective theoretical values of α (shown at the top of each column of the table).

4.7.2 Results table for Log-likelihood test.

Kupiec Coverage Test													
Model		Normal						Student's t					
Stock	**α**	**0.1**	**0.05**	**0.02**	**0.01**	**0.005**	**0.001**	**0.1**	**0.05**	**0.02**	**0.01**	**0.005**	**0.001**
1	p	0.21	0.66	0.73	0.22	0.062	0.0002	0.004	0.0004	0.16	0.63	0.28	0.000
2	-	0.006	0.12	0.73	0.67	0.17	0.02	0.00	0.0004	0.03	0.12	0.74	0.000
3	v	0.00	0.05	0.73	0.67	0.74	0.11	0.00	0.00	0.002	0.03	0.00	0.000
4	a	0.05	0.81	0.55	0.67	0.74	0.11	0.002	0.01	0.002	0.12	0.28	0.54
5	l	0.11	0.86	0.08	0.05	0.06	0.02	0.02	0.12	0.16	0.33	0.28	0.000
6	u	0.03	0.19	0.98	0.41	0.06	0.002	0.00	0.003	0.16	0.63	0.77	0.54
7	e	0.002	0.28	0.73	0.67	0.06	0.002	0.00	0.00	0.07	0.33	0.28	0.000
8	s	0.01	0.19	0.77	0.11	0.06	0.002	0.00	0.001	0.16	0.03	0.00	0.000
9		0.06	0.81	0.37	0.05	0.02	0.11	0.00	0.01	0.30	0.03	0.00	0.000
10		0.21	0.86	0.37	0.11	0.06	0.02	0.00	0.003	0.03	0.12	0.74	0.000
11		0.42	0.66	0.24	0.11	0.01	0.11	0.00	0.01	0.30	0.12	0.74	0.54
12		0.00	0.07	0.55	0.40	0.02	0.002	0.00	0.003	0.07	0.33	0.76	0.000

The p-values in the table are to be used in the following way: if a p-value is smaller than 0.05, then we reject the null hypothesis i.e. the empirical failure rate is different from its respective theoretical failure rate α. However, if the p-value is larger than 0.05, then the empirical failure rate does not differ significantly from its respective α. The smaller p-values are highlighted in colour grey.

Finally, the Christofferssen log-likelihood independence test of violations fails showing very small p-values. Hence, we reject H_0 and conclude that there are dependencies in our back-testing sequences. However, this fact does not have serious consequences in practice since it does not affect the overall number of violations.

CHAPTER FIVE

CONCLUSION

It has been seen from the empirical assessment of VaR carried out in this research study that investment returns do not necessarily follow the normal distribution especially at extreme tails. The empirical distribution for 0.05 and other bigger quantiles seem to be a truism in most cases for the normal distribution while the Student's t distribution is in fact a good model for smaller quantiles. More so, investment returns are not independently identically distributed.

The effectiveness of any VaR model depends on the specified confidence level, the trend and volatility of market for every financial trading as being considered for a yearly rolling window in this research through back-testing procedure. As it was investigated and proved that the VaR violations for normal distribution using a bigger quantile produced a good result but the Student's t-distribution overestimates risk using a bigger quantile leading to conservatism, howbeit, the Student's t-distribution produced a good result when a smaller quantile is specified.

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