University of Tartu
Faculty of Science and Technology
Institute of Mathematics and Statistics

Marcus Lõo

Slicely Countably Determined Points in Banach Spaces

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Supervisors:
Johann Langemets, Associate Professor, University of Tartu
Miguel Martín, Professor, University of Granada

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Masters's thesis

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Abstract. In 2010, A. Avilés, V. Kadets, M. Martín, J. Merí and V. Shepelska introduced the concept of slicely countably determined Banach spaces in order to generalize separable Banach spaces which have the Asplund or the Radon–Nikodým property. The aim of the thesis is to extend the concept of the slicely countably determined sets to the non-separable setting, by introducing a pointwise version of the property.

CERCS research specialisation: P140 Series, Fourier analysis, functional analysis.

Keywords: slices, Radon–Nikodým property, Daugavet property.

Loenduva arvu viiludega määratud punktid Banachi ruumides

Magistritöö

Marcus Lõo

Lühikokkuvõte. Aastal 2010 tõid A. Avilés, V. Kadets, M. Martín, J. Merí ja V. Shepelska sisse loenduva arvu viiludega määratud Banachi ruumide mõiste, et üldistada Asplundi ja Radon–Nikodými omadustega separaableid Banachi ruume. Magistritöös laiendatakse loenduva arvu viiludega määratud hulga mõistet uurimaks nimetatud omadust mitte-separaablites Banachi ruumides. Selleks tuuakse sisse loenduva arvu viiludega määratud punkti mõiste.

CERCS teaduseriala: P140 Jadad, Fourier analüüs, funktsionaalanalüüs.

Märksõnad: viilud, Radon–Nikodými omadus, Daugaveti omadus.

Contents

1 Introduction

In [2], A. Avilés, V. Kadets, M. Martín, J. Merí and V. Shepelska introduced slicely countably determined sets and spaces. One of the main motivations was the generalization of separable Banach spaces with the Radon–Nikodým property or the Asplund property.

An open question on this topic was whether this concept can be extended to the non-separable case. The main goal of this thesis is to solve this open question. We will provide a pointwise description of the slicely countably determined property, which enables us to investigate this subject in non-separable Banach spaces, and moreover, allows us to prove some notable results in the context of slicely countably determined sets and spaces aswell.

The thesis consists of four sections. The first section focuses initially on the preliminaries, composed by fundamental definitions and results of Banach space geometry and the definitions of widely known classes of Banach spaces, such as spaces with the Daugavet property or the Radon–Nikodým property. We also recall the localized versions of these classes of Banach spaces. Furthermore, we recollect the definitions of slicely countably determined sets and spaces with examples.

In the second section, we introduce the main definition of our thesis, conveyed with many properties and examples. Additionally, we prove a general theorem (Theorem 3.16), which for instance, enables us to describe slicely countably determined points in the unit balls of L_1-preduals.

The third section is dedicated to the research of slicely countably determined points in the unit ball of the direct sum of Banach spaces. In this section we show, in particular, that the unit ball of the direct sum $\bigoplus_{n=1}^{\infty} X_n$ of non-trivial Banach spaces, endowed with the p-norm, where $1 < p < \infty$, always contains a slicely countably determined point.

In the last section we study slicely countably determined points in projective tensor products of Banach spaces. By studying when elementary tensors are slicely countably determined points (Theorem 5.6), we can deduce that for Banach spaces X and Y, if B_X is slicely countably determined and B_Y is dentable, then the unit ball of the projective tensor product of X and Y must also be slicely countably determined.

In this thesis, we consider Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, if we do not state otherwise. Given a Banach space X over $\mathbb{K}$, we write B_X for its closed unit ball and S_X for its unit sphere. The dual space of X is denoted by X^*. Let A be a subset of X. We write $\operatorname{span}(A)$ and $\overline{\operatorname{span}}(A)$ for the linear hull and the closed linear hull of A respectively, $\operatorname{conv}(A)$ and $\overline{\operatorname{conv}}(A)$ for the convex hull and the closed convex hull of A respectively. The quotient space of a Banach space X with respect to a subspace $Y \subset X$ is denoted by X/Y.

The diameter of A is denoted by $\operatorname{diam}(A)$ and the set of extreme points of A by $\operatorname{ext}(A)$. For a set $B \subset X$ and $x \in X$, we write $d(x, B) = \inf\{\|x - b\| : b \in B\}$. The identity operator on Banach space X is denoted by Id. Given a set I and a function $f\colon I \to \mathbb{K}$, we denote $\operatorname{supp}(f) = \{i \in I\colon f(i) \neq 0\}$ as the support of f.

2 Definitions and preliminaries

In this section our goal is to recall some definitions and results necessary for introducing slicely countably determined points. To this end, we give the essential definitions such as the slice of a convex set and the determining sequence of subsets for a set, as well as the definitions of some widely known classes of Banach spaces, in particular, the ones having the Daugavet property or the Radon–Nikodým property, including the pointwise characterization of these properties. In addition to the preliminaries, we recall the definitions of slicely countably determined sets and spaces.

Throughout this section, let X be a Banach space.

2.1 Various geometric notions in Banach spaces

Definition 2.1. Suppose that $A \subset X$ is bounded and convex. A *slice of* A is the non-empty intersection of A with an open half-space.

It is well known that every slice of A can be written in the form

$$S(A, x^*, \alpha) = \{x \in A\colon \operatorname{Re} x^*(x) > \sup_{b \in A} \operatorname{Re} x^*(b) - \alpha\},$$

where $x^* \in X^* \setminus \{0\}$ and $\alpha > 0$. Note that a slice is non-empty, relatively weakly open and convex.

Definition 2.2. Let $S_1, \ldots, S_m$, where $m \in \mathbb{N}$, be slices of a convex and bounded set $A \subset X$. A *convex combination of slices of* A is a set of the form

$$\sum_{n=1}^{m} \lambda_n S_n,$$

where $m \in \mathbb{N}$, $\lambda_n \in [0, 1]$ for every $n \in \{1, \ldots, m\}$ and $\sum_{n=1}^{m} \lambda_n = 1$.

J. Bourgain was the first to observe that convex combination of slices play a special role in non-empty relatively weakly open subsets.

Lemma 2.3 (Bourgain's Lemma, [2, Lemma 2.16]). *Let $K \subset X$ be bounded, convex and closed. Then every non-empty relatively weakly open subset of K contains a convex combination of slices of K.*

Remark 2.4. As per [2, Remark 2.17], the assumption of K being closed can be omitted.

For convenience, we also state some fundamental theorems of functional analysis.

Theorem 2.5 (Mazur's Theorem, [7, Theorem 3.45]). *Let X be a Banach space and $C \subset X$ a convex set. Then the weak closure and the norm closure of C coincide.*

Theorem 2.6 (Tukey–Klee Separation Theorem, [14, Theorem 2.2.28]). *Let X be a Banach space and assume that K and C are disjoint non-empty convex subsets of X such that K is compact and C is closed. Then there exists $x^* \in S_{X^*}$ such that*

$$\min\{\operatorname{Re} x^*(x) \colon x \in K\} > \sup\{\operatorname{Re} x^*(x) \colon x \in C\}.$$

Theorem 2.7 (Krein–Milman Theorem, [14, Corollary 2.10.9]). *If X is a Banach space, then B_{X^*} is the weak* closed convex hull of its set of extreme points.*

Next we recall some well-established geometrical properties of Banach spaces.

Definition 2.8 ([8, §III]). Let $A \subset X$ be bounded and convex. A point $a \in A$ is called an *SCS point of A* (SCS standing for small combinations of slices), if for every $\varepsilon > 0$ there exists a convex combination of slices $\sum_{n=1}^{m} \lambda_n S_n$, $m \in \mathbb{N}$, such that $a \in \sum_{n=1}^{m} \lambda_n S_n$ and $\operatorname{diam}\big(\sum_{n=1}^{m} \lambda_n S_n\big) < \varepsilon$.

One particular subset of SCS points leads to the definition of the Radon–Nikodým property.

Definition 2.9. Let $A \subset X$ be bounded, convex and closed. A point $a \in A$ is a *denting point of A*, if for every $\varepsilon > 0$ there exists a slice S of set A such that $a \in S$ and $\operatorname{diam}(S) < \varepsilon$. The set of all denting points of A is denoted by $\operatorname{dent}(A)$.

Definition 2.10 ([8, §III]). A bounded, convex and closed set $A \subset X$ is called *dentable*, if A is the closed convex hull of its denting points, i.e. $A = \overline{\operatorname{conv}}(\operatorname{dent}(A))$.

Definition 2.11. A Banach space X has the *Radon–Nikodým property* (shortly, X has the *RNP*), if every bounded, convex and closed set $A \subset X$ is dentable.

Example 2.12. The Banach spaces ℓ_p, where $1 < p < \infty$, are reflexive and reflexive spaces are known to have the RNP. Moreover, the Banach space ℓ_1 is a separable dual space, which means that ℓ_1 also has the RNP (see [6, Page 218] for both claims).

With the use of the Radon–Nikodým property, another class of Banach spaces is defined.

Definition 2.13. A Banach space X has the *Asplund property*, if the dual X^* has the Radon–Nikodým property.

Example 2.14. The Banach space c_0 has the Asplund property, since its dual space is ℓ_1, which by Example 2.12 has the RNP.

We now recall the definition of a Banach space having the Daugavet property. These Banach spaces are known to be remarkably different from Banach spaces having the RNP or the Asplund property, regarding their geometry. Specifically speaking, it is known that every slice of the unit ball of a Banach space with the Daugavet property has diamater two, whereas Banach spaces with the RNP have arbitrarily small diameter slices in the unit ball.

Definition 2.15. We say that a Banach space X *has the Daugavet property*, if

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

for every $T\colon X \to X$, which is linear, bounded and rank-one.

Example 2.16. The Banach spaces $C[0,1]$ and $L_1[0,1]$ have the Daugavet property (see [19, page 77]).

Lastly, we recall the localized version of the Daugavet property.

Definition 2.17 ([1])**.** A point $x \in S_X$ is called a *Daugavet point*, if

$$B_X = \overline{\mathrm{conv}}(\{y \in B_X \colon \ \|x - y\| \geq 2 - \varepsilon\})$$

for every $\varepsilon > 0$.

It is well known that a Banach space X has the Daugavet property if and only if every $x \in S_X$ is a Daugavet point [1, Proposition 1.2].

2.2 Slicely countably determined sets and spaces

The aim of this subsection is to recall the notion of slicely countably determined sets and spaces (SCD sets and spaces in short) and give some related examples.

We first recollect the definition of determining sequence of subsets for a convex bounded set.

Definition 2.18 ([2, Definiton 2.1])**.** Let $A \subset X$ be bounded and convex. A sequence $\{V_n : n \in \mathbb{N}\}$ of subsets of A is *determining for* A, if $A \subset \overline{\mathrm{conv}}(B)$ for every $B \subset A$ intersecting all the sets V_n.

Using the former definition, we can recall the definitions of slicely countably determined sets and spaces.

Definition 2.19 ([2, Definition 2.5])**.** A bounded convex subset A of Banach space X is called *slicely countably determined* (an *SCD set* in short), if there exists a determining sequence of slices of A.

Definition 2.20 ([2, Definition 3.1])**.** A separable Banach space X is called *slicely countably determined* (an *SCD space* in short), if every bounded convex subset of X is an SCD set.

Example 2.21. Every bounded, convex and closed set $A \subset X$, which is separable and dentable, is an SCD set (see [2, Proposition 2.8]). On the other hand, the unit ball of a separable Banach space with the Daugavet property is not an SCD set (see [2, Example 2.13]).

Example 2.22. Every separable Banach space having the Radon–Nikodým or the Asplund property is an SCD space (see [2, Examples 3.2]). In particular, reflexive separable Banach spaces are SCD spaces. However, if X is a separable Banach space which admits an equivalent renorming with the Daugavet property, X is not an SCD space (see [2, Examples 3.3]).

3 Slicely countably determined points

The main goal of our work is to generalize the slicely countably determined property to the non-separable setting. A natural way to proceed is to carry this property from sets to points. In this section, we introduce the main definition of the thesis with characterizations. For instance, we show that it is possible to replace slices in the determining sequence with non-empty relatively weakly open sets or with convex combinations of slices. We also give several examples of slicely countably determined points. Furthermore, we prove a sufficient condition for a Banach space to admit no slicely countably determined points in the unit ball. With the use of this condition, we give a description of SCD points in the unit ball of L_1-preduals and spaces having the Daugavet property.

3.1 Definitions and characterizations

Let X be a Banach space. First we naturally extend the concept of a determining sequence of subsets for a point.

Definition 3.1. Let $A \subset X$ be bounded, convex and closed and $a \in A$. We say that a sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A is *determining for a*, if $a \in \overline{\text{conv}}(B)$ for every $B \subset A$ intersecting all the sets V_n.

Remark 3.2. We can assume the set B in the Definition 3.1 to be convex. Indeed, assume that we have $\{V_n \colon n \in \mathbb{N}\}$ such that for every convex subset $C \subset A$, satisfying $C \cap V_n \neq \emptyset$ for every $n \in \mathbb{N}$, we have $a \in \overline{\text{conv}}(C) = \overline{C}$. Let $B \subset A$ be a set satisfying $B \cap V_n \neq \emptyset$ for every $n \in \mathbb{N}$. Then $\text{conv}(B) \subset A$ and $\text{conv}(B)$ also intersects every V_n, hence $a \in \overline{\text{conv}}(\text{conv}(B)) = \overline{\text{conv}}(B)$, meaning that the original condition also holds. The converse implication is evident.

Moreover, we can also assume closedness of set B in Definition 3.1, by using similar proof as with convexity.

First we give some equivalent conditions for a sequence of subsets to be determining for a point. These characterizations will be used throughout the thesis.

Proposition 3.3. *Let $A \subset X$ be bounded, convex and closed and $a \in A$. For a sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A, the following conditions are equivalent:*

(i) $\{V_n : n \in \mathbb{N}\}$ *is determining for* a;

(ii) *for every slice* S *of* A *with* $a \in S$, *there is* $m \in \mathbb{N}$ *such that* $V_m \subset S$;

(iii) *if* $x_n \in V_n$ *for every* $n \in \mathbb{N}$, *then* $a \in \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$.

Proof. (i) $\Rightarrow$ (ii). Let S be a slice of A with $a \in S$. Assume on the contrary that $V_n \not\subset S$ for every $n \in \mathbb{N}$. Therefore, $(A \setminus S) \cap V_n \neq \emptyset$ for all $n \in \mathbb{N}$. By (i) $a \in \overline{\text{conv}}(A \setminus S) = A \setminus S$, a contradiction.

(ii) $\Rightarrow$ (iii). Let us prove the contrapositive. Assume that for every $n \in \mathbb{N}$ there exists $x_n \in V_n$ such that $a \notin C$, where $C := \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$. Since $\{a\}$ is compact, set C closed and convex and $\{a\} \cap C = \emptyset$, there exists a functional $x^* \in S_{X^*}$ such that

$$\operatorname{Re} x^*(a) > \sup_{c \in C} \operatorname{Re} x^*(c),$$

by Theorem 2.6. Let $\alpha := \sup_{x \in A} \operatorname{Re} x^*(x) - \sup_{c \in C} \operatorname{Re} x^*(c) > 0$. Then $a \in S(A, x^*, \alpha)$, but for arbitrary $n \in \mathbb{N}$ we get $V_n \not\subset S(A, x^*, \alpha)$, because $x_n \in V_n$ and

$$\operatorname{Re} x^*(x_n) \leq \sup_{c \in C} \operatorname{Re} x^*(c) = \sup_{x \in A} \operatorname{Re} x^*(x) - \alpha,$$

hence $x_n \notin S(A, x^*, \alpha)$.

(iii) $\Rightarrow$ (i). It is clear, because if $B \cap V_n \neq \emptyset$, then there are $b_n \in V_n \cap B$ for every $n \in \mathbb{N}$ and by (iii) we have $a \in \overline{\text{conv}}(\{b_n : n \in \mathbb{N}\}) \subset \overline{\text{conv}}(B)$. $\square$

We now give the main definition of the thesis.

Definition 3.4. Assume that $A \subset X$ is bounded, convex and closed and $a \in A$. Point $a \in A$ is called a *slicely countably determined point of* A (an *SCD point of* A in short), if there exists a determining sequence of slices of A for the point a. We denote the set of all SCD points of A by $\text{SCD}(A)$.

Remark 3.5. Evidently we can assume that the functionals defining the slices, which determine a point, to be norm one.

Lemma 3.6. *Let* $A \subset X$ *be bounded, convex and closed. The set* $\text{SCD}(A)$ *is convex and weakly closed.*

Proof. Fix $x_1, x_2 \in \mathrm{SCD}(A)$, $x_1 \neq x_2$ and $\lambda \in [0,1]$. It suffices to show that $\lambda x_1 + (1-\lambda)x_2 \in \mathrm{SCD}(A)$. By assumption there exists two determining sequence of slices, one for x_1 and the other for x_2. Consider the union of these sequences. If we have $B \subset A$ such that B intersects all slices in the union, we get $a \in \overline{\mathrm{conv}}(B)$ and $b \in \overline{\mathrm{conv}}(B)$. It is clear that in this case $\lambda a + (1-\lambda)b \in \overline{\mathrm{conv}}(B)$.

It is easy to check that $\mathrm{SCD}(A)$ is closed in the norm topology. Indeed, fix a sequence $a_n \to a$, where $a_n \in \mathrm{SCD}(A)$ for all $n \in \mathbb{N}$. Assume that for each a_n the determining sequence of slices is $\{S_n^m : m \in \mathbb{N}\}$. We show that the sequence $\{S_n^m : n, m \in \mathbb{N}\}$ is determining for point a, using Proposition 3.3 condition (ii). Let S be a slice containing a. Since a_n converge to a, we can find an index $k \in \mathbb{N}$ such that $a_k \in S$. Since the sequence of slices $\{S_k^m : m \in \mathbb{N}\}$ determines a_k, there exists $S_k^m \subset S$.

To conclude, we use Theorem 2.5 to see that

$$\mathrm{SCD}(A) = \overline{\mathrm{SCD}(A)} = \overline{\mathrm{SCD}(A)}^{w}. \qquad \square$$

To show that Definition 3.4 is indeed reasonable, we prove the following connection between SCD sets and SCD points.

Lemma 3.7. *For a bounded, convex and closed subset $A \subset X$, the following statements hold:*

(a) If A is an SCD *set, then every $a \in A$ is an* SCD *point.*

(b) If every $a \in A$ is an SCD *point and A is separable, then A is an* SCD *set.*

Proof. (a). By Proposition 3.3 condition (ii) and [2, Proposition 2.2] it is easy to see that the sequence of slices $\{S_n : n \in \mathbb{N}\}$ determining set A also determines every point $a \in A$.

(b). We need to find a determining sequence of slices for A. By assumption we have a countable set $\{x_n : n \in \mathbb{N}\}$, which is dense in A. Moreover, for every $n \in \mathbb{N}$ we have $x_n \in \mathrm{SCD}(A)$. Hence for each x_n there exists a determining sequence of slices $\{S_n^m : m \in \mathbb{N}\}$. Let us now consider sequence of slices $\{S_n^m : n, m \in \mathbb{N}\}$. To show that this sequence is determining for A we use once again Proposition 3.3 condition (ii). Fix an arbitrary slice S of A. Since the set $\{x_n : n \in \mathbb{N}\}$ is dense, there exists $n \in \mathbb{N}$ such that $x_n \in S$ and then, by assumption, there also exists $m \in \mathbb{N}$ so that $S_n^m \subset S$. $\square$

The upcoming proposition provides us with an effective tool in our analysis.

Proposition 3.8. *Let $A \subset X$ be bounded, convex and closed. The following conditions are equivalent:*

(i) $a \in \mathrm{SCD}(A)$*;*

(ii) *there is a sequence of non-empty relatively weakly open subsets $\{W_n : n \in \mathbb{N}\}$ of A, which is determining for a;*

(iii) *there exists a sequence of convex combinations of slices $\{C_n : n \in \mathbb{N}\}$ of A, which is determining for a.*

Proof. Firstly, note that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i). Assume that there exists a sequence of convex combinations of slices $\{C_n : n \in \mathbb{N}\}$, which is determining for a. For every $n \in \mathbb{N}$, the set C_n is defined as follows:

$$C_n = \sum_{i=1}^{k_n} \lambda_i^n S_i^n, \ \sum_{i=1}^{k_n} \lambda_i^n = 1, \ k_n \in \mathbb{N}, \tag{3.1}$$

where S_i^n, $i \in \{1, \dots, k_n\}$ are slices of A. Let us show that the sequence of slices $\{S_i^n : n \in \mathbb{N}, \ i \in \{1, \dots, k_n\}\}$ is determining for a. For that, let $B \subset A$ be convex and $B \cap S_i^n \neq \emptyset$ for every $n \in \mathbb{N}$ and every $i \in \{1, \dots, k_n\}$. Hence there exists $b_i^n \in B \cap S_i^n$, from which we can construct another set

$$\hat{B} := \mathrm{conv}(\{b_i^n : n \in \mathbb{N}, \ i \in \{1, \dots, k_n\}\}) \subset B \subset A.$$

Note that $\hat{B} \cap C_n \neq \emptyset$ for every $n \in \mathbb{N}$. Indeed, by fixing $n \in \mathbb{N}$ arbitrarily, we have C_n defined by condition (3.1). Now using the fact that $b_i^n \in S_i^n$, we get

$$\sum_{i=1}^{k_n} \lambda_i^n b_i^n \in \sum_{i=1}^{k_n} \lambda_i^n S_i^n = C_n.$$

Moreover, since $\hat{B}$ is convex, we have $\sum_{i=1}^{k_n} \lambda_i^n b_i^n \in \hat{B}$. Therefore, by (iii), we have

$$a \in \overline{\mathrm{conv}}(\hat{B}) \subset \overline{\mathrm{conv}}(B).$$

(ii) $\Rightarrow$ (iii). Assume that we have a determining sequence of non-empty relatively weakly open sets $\{W_n : n \in \mathbb{N}\}$ for a. By Lemma 2.3, for each $n \in \mathbb{N}$ there exists C_n such that $C_n \subset W_n$, where C_n is defined by condition (3.1). Suppose $B \cap C_n \neq \emptyset$ for each $n \in \mathbb{N}$. Then $B \cap W_n \neq \emptyset$ for all $n \in \mathbb{N}$ and since the sequence $\{W_n : n \in \mathbb{N}\}$ is determining for a, we have $a \in \overline{\text{conv}}(B)$. This in turn shows that $\{C_n : n \in \mathbb{N}\}$ is determining for a. □

Using Proposition 3.8, we get the following example of SCD points.

Example 3.9. Let $A \subset X$ be bounded, convex and closed. If a is an SCS point of A, then a is an SCD point of A.

Proof. Assume that a is an SCS point. According to Definition 2.8, we can generate convex combinations of slices C_n as follows:

$$C_n = \sum_{i=1}^{k_n} \lambda_i^n S_i^n, \ a \in C_n, \ \text{diam}(C_n) < \frac{1}{n},$$

where $k_n \in \mathbb{N}$. By Proposition 3.8 condition (iii), it suffices to show that the sequence $\{C_n : n \in \mathbb{N}\}$ is determining for a. Let $B \subset A$ and $B \cap C_n \neq \emptyset$ for every $n \in \mathbb{N}$, hence there exists $b_n \in B \cap C_n$. We need to show that $a \in \overline{\text{conv}}(B)$. Pick an arbitrary $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $1/m < \varepsilon$. Then

$$\|a - b_m\| \leq \text{diam}(C_m) < \frac{1}{m} < \varepsilon,$$

therefore $a \in \overline{\text{conv}}(B)$. □

Corollary 3.10. *Every denting point of A is an* SCD *point of A.*

In order to finish this subsection, we bring out an example, where each member of the determining sequence of slices is far from the determined point.

Example 3.11. Assume that X has the RNP and let $x_0 \in S_X$ be a Daugavet point (a space like this exists by [18]). Then for every $\varepsilon > 0$ there exists a sequence of slices $\{S_n : n \in \mathbb{N}\} \subset B_X$ determining for x_0 such that $d(x_0, S_n) > 2 - \varepsilon$ for every $n \in \mathbb{N}$.

Proof. Fix $\varepsilon > 0$. We have, by assumption, a determining sequence of slices $\{T_n \colon n \in \mathbb{N}\}$ for x_0. Since $B_X = \overline{\text{conv}}(\text{dent}(B_X))$, we have $T_n \cap \text{dent}(B_X) \neq \emptyset$ for every $n \in \mathbb{N}$, meaning we have an element $x_n \in T_n$ for each $n \in \mathbb{N}$, which is also a denting point of B_X. This enables us to find a slice S_n such that $x_n \in S_n \subset T_n$ and $\text{diam}(S_n) < \varepsilon/2$ for every $n \in \mathbb{N}$. Observe that in this case the slices $\{S_n \colon n \in \mathbb{N}\}$ also determine the point x_0 (this is obvious by condition (ii) in Proposition 3.3), and since x_0 is a Daugavet point, we have $\|x_0 - x_n\| = 2$ for every $n \in \mathbb{N}$ [11, Proposition 3.1]. In conclusion

$$d(x_0, S_n) \geq \|x_0 - x_n\| - \text{diam}(S_n) \geq 2 - \frac{\varepsilon}{2} > 2 - \varepsilon. \qquad \square$$

3.2 SCD points in the unit ball of a Banach space

In this subsection, we prove a sufficient condition for the unit ball of a Banach space to have no SCD points. This condition will be used to describe SCD points in the unit ball of L_1-preduals and in Banach spaces having the Daugavet property. For this purpose, we introduce a class of Banach spaces failing the (-1)-ball covering property. We encourage the reader to consult [9] and [5] for further reading on the (-1)-ball covering property.

We start by proving some useful properties of SCD points in the unit ball of a Banach space. To this end, let X be a Banach space.

Proposition 3.12. *Suppose that $a \in B_X$. Then $a \in \text{SCD}(B_X)$ if and only if $-a \in \text{SCD}(B_X)$.*

Proof. Necessity. Assume that $a \in \text{SCD}(B_X)$, meaning that we have $\{S_n \colon n \in \mathbb{N}\}$, where $S_n = S(B_X, x_n^*, \alpha_n)$ such that for every $S = S(B_X, x^*, \alpha) \ni a$, there exists $m \in \mathbb{N}$ such that $S_m \subset S$. For every $n \in \mathbb{N}$, write $\hat{S}_n = S(B_X, -x_n^*, \alpha_n)$ and fix an arbitrary slice $S = S(B_X, x^*, \alpha) \ni -a$. Then

$$\text{Re}\big(-x^*(a)\big) = \text{Re}\, x^*(-a) > 1 - \alpha,$$

meaning that $a \in S(B_X, -x^*, \alpha)$. By assumption there exists $m \in \mathbb{N}$ such that $S(B_X, x_m^*, \alpha_m) \subset S(B_X, -x^*, \alpha)$. Now we see that $\hat{S}_m \subset S$. Indeed, if $x \in \hat{S}_m$, then

$$\operatorname{Re}\big(-x_m^*(x)\big) > 1-\alpha_m \iff \operatorname{Re} x_m^*(-x) > 1-\alpha_m.$$

Since $S(B_X, x_m^*, \alpha_m) \subset S(B_X, -x^*, \alpha)$, we get

$$\operatorname{Re} x^*(x) = \operatorname{Re}\big(-x^*(-x)\big) > 1-\alpha.$$

Sufficiency can be deduced from the earlier proof by taking the element $-a$ as the initial SCD point. □

The last proposition gives us a convenient way to check whether the unit ball has any SCD points at all.

Corollary 3.13. $\operatorname{SCD}(B_X) = \emptyset$ *if and only if* $0 \notin \operatorname{SCD}(B_X)$.

Proof. Sufficiency. Assume that there exists $a \in \operatorname{SCD}(B_X)$. By Proposition 3.12 we have $-a \in \operatorname{SCD}(B_X)$. Since the set of SCD points is convex, we have

$$0 = \frac{1}{2}a + \frac{1}{2}(-a) \in \operatorname{SCD}(B_X).$$

Necessity is clear. □

Definition 3.14 ([5, Proposition 2.2])**.** A Banach space X is said fail the (-1)-*ball covering property* ((-1)-*BCP* in short) if for any separable subspace $Y \subset X$ there exists $x \in S_X$ such that the equality

$$\|y + \lambda x\| = \|y\| + |\lambda| \tag{3.2}$$

holds for every $y \in Y$ and $\lambda \in \mathbb{R}$.

Example 3.15. The spaces $\ell_1(I)$, where I is an uncountable set, ℓ_∞/c_0 and $L_\infty[0,1]$ all fail the (-1)-BCP (see [5]).

Theorem 3.16. *Let X be a Banach space and assume that X^* fails the* (-1)-BCP*. Then* $\operatorname{SCD}(B_X) = \emptyset$*.*

Proof. By Corollary 3.13 it suffices to prove that $0 \notin \operatorname{SCD}(B_X)$. Pick an arbitrary sequence of slices $\{S_n \colon n \in \mathbb{N}\}$, defined as $S_n = S(B_X, x_n^*, \alpha_n)$, where $x_n^* \in S_{X^*}$

for each $n \in \mathbb{N}$ and let us find a sequence $(x_n) \subset B_X$ such that $x_n \in S_n$ for every $n \in \mathbb{N}$ and $0 \notin \overline{\mathrm{conv}}(\{x_n : n \in \mathbb{N}\})$. Since X^* fails the (-1)-BCP, we can find $x^* \in S_{X^*}$ such that

$$\|x_n^* + x^*\| = \|x_n^*\| + 1 = 2,$$

By this condition, we can find for each $n \in \mathbb{N}$ elements $x_n \in S_X$ such that

$$\mathrm{Re}(x_n^* + x^*)(x_n) > 2 - \min\Big\{\alpha_n, \frac{1}{2}\Big\}.$$

From this we infer that $x_n \in S_n$ for every $n \in \mathbb{N}$, because

$$1 + \mathrm{Re}\, x_n^*(x_n) \geq \mathrm{Re}\, x^*(x_n) + \mathrm{Re}\, x_n^*(x_n) > 2 - \min\Big\{\alpha_n, \frac{1}{2}\Big\} \geq 2 - \alpha_n.$$

Similarly we see that $\mathrm{Re}\, x^*(x_n) > 1/2$ for every $n \in \mathbb{N}$. To conclude the proof, notice that

$$\{x_n : n \in \mathbb{N}\} \subset \Big\{y \in B_X :\ \mathrm{Re}\, x^*(y) \geq \frac{1}{2}\Big\}.$$

Now making use of Theorem 2.5, we get

$$\begin{aligned}\overline{\mathrm{conv}}(\{x_n : n \in \mathbb{N}\}) &\subset \overline{\mathrm{conv}}\Big(\Big\{y \in B_X :\ \mathrm{Re}\, x^*(y) \geq \frac{1}{2}\Big\}\Big)\\ &= \overline{\mathrm{conv}}^w\Big(\Big\{y \in B_X :\ \mathrm{Re}\, x^*(y) \geq \frac{1}{2}\Big\}\Big)\\ &= \Big\{y \in B_X :\ \mathrm{Re}\, x^*(y) \geq \frac{1}{2}\Big\},\end{aligned}$$

but $\mathrm{Re}\, x^*(0) = 0$, therefore $0 \notin \overline{\mathrm{conv}}(\{x_n : n \in \mathbb{N}\})$. □

By Example 2.21, we know that if X has the Daugavet property, B_X is not an SCD set. With Theorem 3.16 we obtain something stronger.

Corollary 3.17. *Let X be a Banach space with the Daugavet property. Then* $\mathrm{SCD}(B_X) = \emptyset$.

Proof. Let us show first that X^* fails the (-1)-BCP. For this, fix a separable subspace $Y \subset X^*$. Additionally pick $x \in S_X$ and $\varepsilon > 0$ arbitrarily. We use [13, Lemma 2.12] to find $x^* \in S_{X^*}$ such that

$$\mathrm{Re}\, x^*(x) \geq 1 - \varepsilon \text{ and } \|x^* + y^*\| = 1 + \|y^*\|$$

for every $y^* \in Y$. Fix now $y^* \in Y$ and $\lambda \in \mathbb{R}$. Clearly, if $\lambda = 0$, equation (3.2) holds. Considering the case where $\lambda \neq 0$, we see that

$$\|y^* + \lambda x^*\| = |\lambda| \left\| \frac{y^*}{\lambda} + x^* \right\| = |\lambda| \left(1 + \left\| \frac{y^*}{\lambda} \right\| \right) = \|y^*\| + |\lambda|,$$

therefore equation (3.2) still holds. Since we have shown that X^* fails the (-1)-BCP, we can apply Theorem 3.16 to obtain that $\mathrm{SCD}(B_X) = \emptyset$. □

So far the only known separable Banach spaces failing to be SCD spaces were the Banach spaces having the Daugavet property.

Example 3.18. There are separable Banach spaces without the Daugavet property and which have no SCD points in the unit ball. Indeed, let X be a separable Banach space with the Daugavet property. Then there is an absolute normalized norm N on $X \times X$ such that $X \oplus_N X$ fails the Daugavet property and $(X \oplus_N X)^*$ fails the (-1)-BCP (see [4, Remark 6.9]), the latter meaning that $\mathrm{SCD}(B_{X \oplus_N X}) = \emptyset$, by application of Theorem 3.16.

Next we study SCD points in the unit ball of L_1-preduals.

Definition 3.19. We say that a Banach space X is an L_1-*predual*, if

$$X^* = L_1(S, \Sigma, \mu)$$

for some measure space (S, Σ, μ).

Definition 3.20. Given a measure space (S, Σ, μ), we say that the set $A \in \Sigma$ is an *atom*, if $\mu(A) > 0$ and for any measurable subset $B \subset A$, if $\mu(B) < \mu(A)$, then the set B has measure zero.

For a measure space (S, Σ, μ), we have $\mathrm{ext}(B_{L_1(S,\Sigma,\mu)}) = \emptyset$, if μ has no atoms (see [3, Theorem 4.8]). However, if X is an L_1-predual, Theorem 2.7 implies that X^* has extreme points in the unit ball. This means that if X is an L_1-predual, the unit ball of the dual space X^* must have at least one atom.

For these spaces, we use the decomposition

$$L_1(S, \Sigma, \mu) = L_1(S, \Sigma, \nu) \oplus_1 \ell_1(\{A \in \Sigma \colon A \text{ is an atom of } \mu\}), \tag{3.3}$$

where ν is a measure, which has no atoms (see [10, Theorem 2.1]). In particular, if μ only consists of atoms, we have $L_1(S, \Sigma, \nu) = \{0\}$. Let us first prove a useful lemma.

Lemma 3.21. *Suppose that $(x, y) \in \text{ext}(B_{X\oplus_1 Y})$, where $x \neq y$. Then either $x = 0$ and $y \in \text{ext}(B_Y)$ or $y = 0$ and $x \in \text{ext}(B_X)$.*

Proof. Assume that $(x, y) \in \text{ext}(B_{X\oplus_1 Y})$, but $x \neq 0$ and $y \neq 0$. Then we can write

$$(x, y) = \|x\| \Big(\frac{x}{\|x\|}, 0\Big) + \|y\| \Big(0, \frac{y}{\|y\|}\Big),$$

where $\|(x, y)\|_1 = \|x\| + \|y\| = 1$. This is a contradiction, since we have represented the extreme point (x, y) as a convex combination of two different elements of $B_{X\oplus_1 Y}$, hence $x = 0$ or $y = 0$.

Assume now that $x = 0$. Then we have $(0, y) \in \text{ext}(B_{X\oplus_1 Y})$. It is clear from the definition of extreme points that in this case $y \in \text{ext}(B_Y)$. We arrive to the same conclusion if $y = 0$. $\square$

We provide a description of SCD points in the unit ball of an L_1-predual by the following theorem.

Theorem 3.22. *Let X be an L_1-predual. Then the following statements hold:*

(a) if $\text{ext}(B_{X^*})$ is finite, then X is an SCD space. In particular, $\text{SCD}(B_X) = B_X$;

(b) if $\text{ext}(B_{X^*})$ is infinite and countable, then X is an SCD space. In particular, $\text{SCD}(B_X) = B_X$;

(c) if $\text{ext}(B_{X^*})$ is uncountable, then $\text{SCD}(B_X) = \emptyset$.

Proof. (a). First, note that Theorem 2.7 implies that there always exists extreme points in the unit ball of a dual space. If the amount of extreme points is finite, we have a finite-dimensional dual space, which implies that X is finite-dimensional. In this case, since every finite-dimensional Banach space is reflexive, we get by Example 2.12 that X has the RNP, which in turn means that X is an SCD space (and in particular $\text{SCD}(B_X) = B_X$) by Example 2.22.

(b). If $\operatorname{ext}(B_{X^*})$ is infinite and countable, then by [3, Theorem 3.21] the dual space X^* is separable. According to Example 2.12 separable dual spaces have the RNP, therefore X has the Asplund property, which in turn implies, by Example 2.22, that X is an SCD space and in particular, $\mathrm{SCD}(B_X) = B_X$.

(c). Assume that $\operatorname{ext}(B_{X^*})$ is uncountable and consider the decomposition (3.3). Recall that the Banach space $L_1(S, \Sigma, \nu)$ does not have any extreme points in the unit ball, if ν is a measure with no atoms. By Lemma 3.21 we see that

$$\operatorname{ext}(B_{X^*}) = \{0\} \times \operatorname{ext}(B_{\ell_1(I)}). \tag{3.4}$$

It is also well known that $\operatorname{ext}(B_{\ell_1(I)}) = \{\theta e_i \colon i \in I,\, \theta \in \mathbb{K},\, |\theta| = 1\}$, where e_i are the standard basis vectors. Now since the set of extreme points of B_{X^*} is uncountable, we deduce that I is uncountable (considering (3.4)). In this case the Banach space $\ell_1(I)$ fails the (-1)-BCP [9, Corollary 25] and finally, we apply [9, Proposition 8] to obtain that the direct sum $X^* = L_1(S, \Sigma, \nu) \oplus_1 \ell_1(I)$ also fails the (-1)-BCP. Now by Theorem 3.16 we have $\mathrm{SCD}(B_X) = \emptyset$. □

Following the proof of Theorem 3.22, we get the following.

Corollary 3.23. *If X is an L_1-predual, then either X is separable and has the Asplund property (hence an SCD space, which means that* $\mathrm{SCD}(B_X) = B_X$*) or* $\mathrm{SCD}(B_X) = \emptyset$.

4 SCD points in direct sums of Banach spaces

Now we consider how does the concept of SCD points relates to the direct sum of Banach spaces. For instance, we are interested whether being an SCD point in either components unit ball of the direct sum of Banach spaces is again an SCD point in the unit ball of the direct sum. In addition, we find an example of a separable Banach space in which the only SCD point of the unit ball is the zero element.

Let X and Y be Banach spaces. Note that throughout the section, we will make use of identifications

$$(X \oplus_p Y)^* = \begin{cases} X^* \oplus_q Y^*, \text{ if } 1 < p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1, \\ X^* \oplus_\infty Y^*, \text{ if } p = 1, \\ X^* \oplus_1 Y^*, \text{ if } p = \infty. \end{cases}$$

4.1 SCD points in the direct sum $X \oplus_\infty Y$

Let X and Y be Banach spaces. Our goal in this subsection is to research SCD points in the unit ball of the Banach space $X \oplus_\infty Y$. Observe that in this case $B_{X\oplus_\infty Y} = B_X \times B_Y$. This observation makes our investigation more straightforward.

For convenience, we denote $Z := X \oplus_\infty Y$ throughout this subsection. Let us first prove some useful properties. Note that similar statements were proven in the context of SCD sets in [12, Lemmata 2.3 and 2.4].

Lemma 4.1. *For any $(x^*, y^*) \in Z^*$ and $\alpha, \beta, \gamma > 0$, the following conditions hold:*

(a) $S(B_X, x^*, \alpha) \times S(B_Y, y^*, \beta) \subset S(B_Z, (x^*, y^*), \alpha + \beta)$.

(b) If $a \in B_X$, $b \in B_Y$ satisfy $(a, b) \in S(B_Z, (x^*, y^*), \gamma)$, then $a \in S(B_X, x^*, \gamma)$ and $b \in S(B_Y, y^*, \gamma)$.

Proof. (*a*). Pick $(x, y) \in S(B_X, x^*, \alpha) \times S(B_Y, y^*, \beta)$. Then

$$\begin{aligned}
&\operatorname{Re} x^*(x) > \|x^*\| - \alpha,\ \operatorname{Re} y^*(y) > \|y^*\| - \beta \\
&\implies \operatorname{Re}(x^*, y^*)(x, y) > \|x^*\| + \|y^*\| - (\alpha + \beta) \\
&\iff \operatorname{Re}(x^*, y^*)(x, y) > \|(x^*, y^*)\|_1 - (\alpha + \beta),
\end{aligned}$$

hence $(x, y) \in S(B_Z, (x^*, y^*), \alpha + \beta)$.

(*b*). Let $a \in B_X$, $b \in B_Y$ be such that $(a, b) \in S(B_Z, (x^*, y^*), \gamma)$. This means that

$$\operatorname{Re} x^*(a) + \|y^*\| \geq \operatorname{Re} x^*(a) + \operatorname{Re} y^*(b) > \|(x^*, y^*)\|_1 - \gamma = \|x^*\| + \|y^*\| - \gamma,$$

from which we can conclude that $\operatorname{Re} x^*(a) > \|x^*\| - \gamma$, therefore $a \in S(B_X, x^*, \gamma)$. The proof for $b \in S(B_Y, y^*, \gamma)$ is analogous. □

Lemma 4.2. *Assume that* $(a, b) \in \operatorname{SCD}(B_Z)$. *Then there exists a sequence* $((x_n^*, y_n^*), \alpha_n) \in Z^* \times (0, \infty)$ *such that for every* $(x^*, y^*) \in Z^*$, $\alpha > 0$ *satisfying* $(a, b) \in S(B_Z, (x^*, y^*), \alpha)$, *there exists* $m \in \mathbb{N}$ *such that*

$$S(B_X, x_m^*, \alpha_m) \subset S(B_X, x^*, \alpha) \ \text{ and } \ S(B_Y, y_m^*, \alpha_m) \subset S(B_Y, y^*, \alpha). \tag{4.1}$$

Proof. Let $S_n = S(B_Z, (x_n^*, y_n^*), 2\alpha_n)$ be the determining sequence of slices for $(a, b) \in B_Z$. We show that the desired sequence is $((x_n^*, y_n^*), \alpha_n)$, where $n \in \mathbb{N}$. Pick $(x^*, y^*) \in Z^*$, $\alpha > 0$ so that $(a, b) \in S(B_Z, (x^*, y^*), \alpha)$. Since the sequence $\{S_n : n \in \mathbb{N}\}$ is determining for (a, b), we can find $m \in \mathbb{N}$ such that

$$S(B_Z, (x_m^*, y_m^*), 2\alpha_m) \subset S(B_Z, (x^*, y^*), \alpha).$$

An application of Lemma 4.1 (*a*) gives us

$$S(B_X, x_m^*, \alpha_m) \times S(B_Y, y_m^*, \alpha_m) \subset S(B_Z, (x^*, y^*), \alpha).$$

Observe that inclusions (4.1) are satisfied. Indeed, making use of Lemma 4.1 (*b*):

$$\begin{aligned}(x,y)\in S(B_X,x_m^*,\alpha_m)\times S(B_Y,y_m^*,\alpha_m) &\implies (x,y)\in S(B_Z,(x^*,y^*),\alpha)\\ &\implies x\in S(B_X,x^*,\alpha),\ y\in S(B_Y,y^*,\alpha),\end{aligned}$$

which implies that $(x,y)\in S(B_X,x^*,\alpha)\times S(B_Y,y^*,\alpha)$. □

With the use of the presented lemmata, we get the following theorem.

Theorem 4.3. *An element* $(a,b)\in \mathrm{SCD}(B_Z)$ *if and only if* $a\in\mathrm{SCD}(B_X)$ *and* $b\in\mathrm{SCD}(B_Y)$.

Proof. Necessity. Assume that $(a,b)\in\mathrm{SCD}(B_Z)$ and let

$$((x_n^*,y_n^*),\alpha_n)\in Z^*\times(0,\infty)$$

be the sequence from Lemma 4.2. Let us show that slices $S_n=S(B_X,x_n^*,\alpha_n)$ are determining for a. Fix a slice $S(B_X,x^*,\alpha)\ni a$. Observe that $(x^*,0)\in Z^*$ and $(a,b)\in S(B_Z,(x^*,0),\alpha)$, hence, by Lemma 4.2, there exists $m\in\mathbb{N}$ such that

$$S_m=S(B_X,x_m^*,\alpha_m)\subset S(B_X,x^*,\alpha),$$

so $a\in\mathrm{SCD}(B_X)$. Analogously we deduce that $b\in\mathrm{SCD}(B_Y)$.

Sufficiency. Now presume that $a\in\mathrm{SCD}(B_X)$ and $b\in\mathrm{SCD}(B_Y)$ and let the sequence $\{S_n^a\colon n\in\mathbb{N}\}$ determine a and the sequence $\{S_n^b\colon n\in\mathbb{N}\}$ determine b. We show, by using Proposition 3.8 condition (ii), that the sequence of non-empty relatively weakly open subsets $\{S_n^a\times S_k^b\colon n,k\in\mathbb{N}\}$ determines point (a,b). Fix a slice $S=S(B_Z,(x^*,y^*),\alpha)$ containing (a,b), where $\|(x^*,y^*)\|_1=1$.

Now fix $\gamma>0$ arbitrarily. Observe that

$$\operatorname{Re}x^*(a)>\|x^*\|-(\|x^*\|-\operatorname{Re}x^*(a)-\gamma),\ \operatorname{Re}y^*(b)>\|y^*\|-(\|y^*\|-\operatorname{Re}y^*(b)-\gamma).$$

From this we infer that

$$a\in S_1:=S(B_X,x^*,\|x^*\|-\operatorname{Re}x^*(a)-\gamma),\ b\in S_2:=S(B_Y,y^*,\|y^*\|-\operatorname{Re}y^*(b)-\gamma),$$

meaning it is possible to find $i, j \in \mathbb{N}$ such that $S_i^a \subset S_1$ and $S_j^b \subset S_2$. Now making use of Lemma 4.1 (a), we get $S_1 \times S_2 \subset S$, because

$$\begin{aligned}(\|x^*\| - \operatorname{Re} x^*(a) - \gamma) + (\|y^*\| - \operatorname{Re} y^*(b) - \gamma) \\ &= \|(x^*, y^*)\|_1 - \operatorname{Re} x^*(a) - \operatorname{Re} y^*(b) - 2\gamma \\ &= 1 - \operatorname{Re}(x^*, y^*)(a, b) - 2\gamma < \alpha - 2\gamma < \alpha.\end{aligned}$$

In conclusion, we have $S_i^a \times S_j^b \subset S_1 \times S_2 \subset S$, which means that the sequence of non-empty relatively weakly open subsets $\{S_n^a \times S_k^b \colon n, k \in \mathbb{N}\}$ determines point (a, b). $\square$

4.2 SCD points in the direct sum $X \oplus_1 Y$

Let X and Y be Banach spaces. This subsection is dedicated to the study of SCD points in the unit ball of the Banach space $X \oplus_1 Y$.

Proposition 4.4. *Suppose that $a \in S_X$. Then $(a, 0) \in \operatorname{SCD}(B_{X \oplus_1 Y})$ if and only if $a \in \operatorname{SCD}(B_X)$.*

Proof. Necessity. Let the sequence of slices $\{S_n \colon n \in \mathbb{N}\}$, where for every $n \in \mathbb{N}$

$$S_n = S(B_{X \oplus_1 Y}, (x_n^*, y_n^*), \alpha_n),$$

be determining for point $(a, 0)$. Here we assume that

$$\|(x_n^*, y_n^*)\|_\infty = \max\{\|x_n^*\|, \|y_n^*\|\} = 1$$

for all $n \in \mathbb{N}$. Let us show that the sequence of slices $\{T_n \colon n \in \mathbb{N}\}$, where $T_n = S(B_X, x_n^*, \alpha_n)$, determines point a. Pick for every $n \in \mathbb{N}$ an element $x_n \in T_n$. Our goal is to show that $a \in \overline{\operatorname{conv}}(\{x_n \colon n \in \mathbb{N}\})$. To do so, denote

$$I := \{n \in \mathbb{N} \colon \|x_n^*\| = 1\}, \; J := \mathbb{N} \setminus I,$$

and define elements $z_n \in B_{X\oplus_1 Y}$ as follows:

$$z_n = \begin{cases} (x_n, 0), \ n \in I; \\ (0, y_n), \ n \in J, \end{cases}$$

where for every $n \in \mathbb{N}$ the element y_n is picked such that $y_n \in S(B_Y, y_n^*, \alpha_n)$. It is easy to see that every z_n belongs to the slice S_n and since the sequence $\{S_n : n \in \mathbb{N}\}$ is determining for point $(a, 0)$, we have $(a, 0) \in \overline{\text{conv}}(\{z_n : n \in \mathbb{N}\})$.

Fix $\varepsilon > 0$ and find $\lambda_n \in [0, 1]$ for every $n \in \mathbb{N}$, where finitely many of λ_n are non-zero and $\sum_{n=1}^{\infty} \lambda_n = 1$, such that

$$\left\| (a, 0) - \sum_{n=1}^{\infty} \lambda_n z_n \right\| < \frac{\varepsilon}{2}. \tag{4.2}$$

Then we have

$$\begin{aligned} \left\| (a, 0) - \sum_{n=1}^{\infty} \lambda_n z_n \right\|_1 &= \left\| (a, 0) - \Big(\sum_{n\in I} \lambda_n x_n, 0 \Big) - \Big(0, \sum_{n\in J} \lambda_n y_n \Big) \right\|_1 \\ &= \left\| \Big(a - \sum_{n\in I} \lambda_n x_n, - \sum_{n\in J} \lambda_n y_n \Big) \right\|_1 \\ &= \left\| a - \sum_{n\in I} \lambda_n x_n \right\| + \left\| \sum_{n\in J} \lambda_n y_n \right\|, \end{aligned}$$

and using the estimation (4.2), we get

$$\left\| a - \sum_{n\in I} \lambda_n x_n \right\| < \frac{\varepsilon}{2}.$$

Notice that using the reverse triangle inequality

$$\frac{\varepsilon}{2} > \left\| a - \sum_{n\in I} \lambda_n x_n \right\| \geq \left| \|a\| - \left\| \sum_{n\in I} \lambda_n x_n \right\| \right| = \left| 1 - \left\| \sum_{n\in I} \lambda_n x_n \right\| \right|,$$

therefore

$$1-\frac{\varepsilon}{2}<\left\|\sum_{n\in I}\lambda_n x_n\right\|\leq\sum_{n\in I}\lambda_n\leq 1.$$

Denote $\Lambda := \sum_{n\in I}\lambda_n$ and see that $1-\varepsilon/2<\Lambda\leq 1$. Moreover, notice that

$$\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n\in\operatorname{conv}(\{x_n\colon n\in\mathbb{N}\})$$

and

$$\begin{aligned}\frac{\varepsilon}{2}&>\left\|a-\sum_{n\in I}\lambda_n x_n\right\|=\left\|a-\sum_{n\in I}\lambda_n x_n-\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n+\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n\right\|\\&\geq\left\|a-\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n\right\|-\left\|\sum_{n\in I}\Big(\lambda_n-\frac{\lambda_n}{\Lambda}\Big)x_n\right\|.\end{aligned}$$

To conclude, we see that

$$\begin{aligned}\left\|\sum_{n\in I}\Big(\lambda_n-\frac{\lambda_n}{\Lambda}\Big)x_n\right\|&\leq\sum_{n\in I}\left|\lambda_n-\frac{\lambda_n}{\Lambda}\right|\|x_n\|\leq\sum_{n\in I}\left|\lambda_n-\frac{\lambda_n}{\Lambda}\right|\\&=\sum_{n\in I}\lambda_n\left|\frac{\Lambda-1}{\Lambda}\right|=\frac{|\Lambda-1|}{\Lambda}\cdot\sum_{n\in I}\lambda_n\\&=|\Lambda-1|=1-\Lambda<\frac{\varepsilon}{2},\end{aligned}$$

therefore

$$\left\|a-\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n\right\|<\frac{\varepsilon}{2}+\left\|\sum_{n\in I}\Big(\lambda_n-\frac{\lambda_n}{\Lambda}\Big)x_n\right\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence $a\in\overline{\operatorname{conv}}(\{x_n\colon n\in I\})\subset\overline{\operatorname{conv}}(\{x_n\colon n\in\mathbb{N}\})$.

Sufficiency. Suppose that $a\in\operatorname{SCD}(B_X)$, with the determining sequence of slices $\{S_n\colon n\in\mathbb{N}\}$, where $S_n=S(B_X,x_n^*,\alpha_n)$, $x_n^*\in S_{X^*}$ for every $n\in\mathbb{N}$. Without loss of generality we can assume that $\alpha_n<1$ for every $n\in\mathbb{N}$.

Define for every $n,k\in\mathbb{N}$ a slice $S_n^k=S(B_{X\oplus_1 Y},(x_n^*,0),\alpha_n/k)$. We show that the sequence of slices $\{S_n^k\colon n,k\in\mathbb{N}\}$ is determining for point $(a,0)$. Fix elements $z_n^k=(x_n^k,y_n^k)\in S_n^k$ for every $n,k\in\mathbb{N}$. It is easy to see that $x_n^k\in S_n$ and since the sequence $\{S_n \colon n \in \mathbb{N}\}$ determines point a, we have $a \in \overline{\operatorname{conv}}(\{x_n^k \colon n \in \mathbb{N}\})$ for every $k \in \mathbb{N}$. Furthermore, we see that

$$1 \geq \left\|(x_n^k, y_n^k)\right\|_1 = \left\|x_n^k\right\| + \left\|y_n^k\right\| > 1 - \frac{\alpha_n}{k} + \left\|y_n^k\right\|,$$

from which

$$\left\|y_n^k\right\| < \frac{\alpha_n}{k} < \frac{1}{k}.$$

Pick an arbitrary $\varepsilon > 0$ and find $K \in \mathbb{N}$ such that $1/K < \varepsilon/2$. By the argument presented before, we can find $z \in \operatorname{conv}(\{x_n^K \colon n \in \mathbb{N}\})$ such that $\|z - a\| < \varepsilon/2$. Let $z = \sum_{n=1}^{\infty} \lambda_n x_n^K$, where $\sum_{n=1}^{\infty} \lambda_n = 1$, $\lambda_n \in [0,1]$ and the number of non-zero elements λ_n is finite. Now

$$\begin{aligned}
\left\|(a,0) - \sum_{n=1}^{\infty} \lambda_n (x_n^K, y_n^K)\right\|_1 &= \left\|\left(a - \sum_{n=1}^{\infty} \lambda_n x_n^K, -\sum_{n=1}^{\infty} \lambda_n y_n^K\right)\right\|_1 \\
&= \left\|a - \sum_{n=1}^{\infty} \lambda_n x_n^K\right\| + \left\|\sum_{n=1}^{\infty} \lambda_n y_n^K\right\| \\
&< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \lambda_n \left\|y_n^K\right\| < \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \lambda_n \cdot \frac{1}{K} \\
&= \frac{\varepsilon}{2} + \frac{1}{K} \sum_{n=1}^{\infty} \lambda_n \\
&\leq \frac{\varepsilon}{2} + \frac{1}{K} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence $(a,0) \in \overline{\operatorname{conv}}(\{z_n^K \colon n \in \mathbb{N}\}) \subset \overline{\operatorname{conv}}(\{z_n^k \colon n, k \in \mathbb{N}\})$. □

By the proposition presented above, we obtain the following result.

Theorem 4.5. *Let* $(a,b) \in S_{X \oplus_1 Y}$*, where* $a \in X \setminus \{0\}$ *and* $b \in Y \setminus \{0\}$*. Then* $(a,b) \in \operatorname{SCD}(B_{X \oplus_1 Y})$ *if and only if* $\frac{a}{\|a\|} \in \operatorname{SCD}(B_X)$ *and* $\frac{b}{\|b\|} \in \operatorname{SCD}(B_Y)$*.*

Proof. Necessity. Let the slices $S_n = S(B_{X \oplus_1 Y}, (x_n^*, y_n^*), \alpha_n)$, $n \in \mathbb{N}$, determine point (a,b). We prove that the slices $\{T_n \colon n \in \mathbb{N}\}$, where $T_n = S(B_X, x_n^*, \alpha_n)$, is determining for point $\frac{a}{\|a\|}$. Pick for each $n \in \mathbb{N}$ an element $x_n \in T_n$. We assume that $\|(x_n^*, y_n^*)\|_\infty = \max\{\|x_n^*\|, \|y_n^*\|\}$=1, and using this fact, we construct the sets

$$I := \{n \in \mathbb{N} \colon \|x_n^*\| = 1\},\ J := \mathbb{N} \setminus I.$$

Now we define elements $z_n \in B_{X \oplus_1 Y}$:

$$z_n = \begin{cases} (x_n, 0),\ n \in I; \\ (0, y_n),\ n \in J, \end{cases}$$

where y_n belongs to slice $S(B_Y, y_n^*, \alpha_n)$ for every $n \in \mathbb{N}$. Evidently $z_n \in S_n$ for each $n \in \mathbb{N}$ and consequently, since the sequence $\{S_n \colon n \in \mathbb{N}\}$ determines point (a, b), we have $(a, b) \in \overline{\operatorname{conv}}(\{z_n \colon n \in \mathbb{N}\})$.

Let $\varepsilon > 0$. We can find $\delta \in (0, 1)$ such that

$$\frac{2\delta}{\|a\|^2 - \delta \|a\|} < \varepsilon$$

and $\sum_{n=1}^\infty \lambda_n z_n \in \operatorname{conv}(\{z_n \colon n \in \mathbb{N}\})$, where $\lambda_n \in [0, 1]$ and only finitely many λ_n are non-zero, such that

$$\left\| (a, b) - \sum_{n=1}^\infty \lambda_n z_n \right\|_1 < \delta.$$

Then

$$\begin{aligned} \delta > \left\| (a, b) - \sum_{n=1}^\infty \lambda_n z_n \right\|_1 &= \left\| (a, b) - \sum_{n \in I} \lambda_n (x_n, 0) - \sum_{n \in J} \lambda_n (0, y_n) \right\|_1 \\ = \left\| \left(a - \sum_{n \in I} \lambda_n x_n, b - \sum_{n \in J} \lambda_n y_n \right) \right\|_1 &= \left\| a - \sum_{n \in I} \lambda_n x_n \right\| + \left\| b - \sum_{n \in J} \lambda_n y_n \right\|, \end{aligned}$$

therefore

$$\delta > \left\| a - \sum_{n \in I} \lambda_n x_n \right\| \geq \|a\| - \left\| \sum_{n \in I} \lambda_n x_n \right\| \geq \|a\| - \sum_{n \in I} \lambda_n,$$

meaning that $\sum_{n \in I} \lambda_n > \|a\| - \delta$. Similarly we can see that $\sum_{n \in J} \lambda_n > \|b\| - \delta$.

On the other hand

$$\sum_{n\in I}\lambda_n = 1-\sum_{n\in J}\lambda_n < 1-\|b\|+\delta = \|a\|+\|b\|-\|b\|+\delta = \|a\|+\delta,$$

so in conclusion we have $\left|\|a\|-\sum_{n\in I}\lambda_n\right| < \delta$. Now by denoting $\Lambda := \sum_{n\in I}\lambda_n$, we see that

$$\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n \in \operatorname{conv}(\{x_n : n\in\mathbb{N}\})$$

and

$$\begin{aligned}
\left\|\frac{a}{\|a\|}-\sum_{n\in I}\frac{\lambda_n}{\Lambda}x_n\right\| &= \left\|\frac{a}{\|a\|}-\frac{\sum_{n\in I}\lambda_n x_n}{\Lambda}\right\| = \frac{\left\|\Lambda a-\|a\|\sum_{n\in I}\lambda_n x_n\right\|}{\Lambda\|a\|} \\
&= \frac{\left\|\Lambda a-\|a\|\sum_{n\in I}\lambda_n x_n+\|a\|a-\|a\|a\right\|}{\Lambda\|a\|} \\
&= \frac{\left\|a(\Lambda-\|a\|)+\|a\|(a-\sum_{n\in I}\lambda_n x_n)\right\|}{\Lambda\|a\|} \\
&\le \frac{\|a(\Lambda-\|a\|)\|+\left\|\|a\|(a-\sum_{n\in I}\lambda_n x_n)\right\|}{\Lambda\|a\|} \\
&\le \frac{\|a\|\,|\Lambda-\|a\||+\|a\|\left\|a-\sum_{n\in I}\lambda_n x_n\right\|}{\Lambda\|a\|} < \frac{\delta+\delta}{\Lambda\|a\|} \\
&= \frac{2\delta}{\Lambda\|a\|} \le \frac{2\delta}{(\|a\|-\delta)\|a\|} = \frac{2\delta}{\|a\|^2-\delta\|a\|} < \varepsilon,
\end{aligned}$$

Therefore $\frac{a}{\|a\|}\in\overline{\operatorname{conv}}(\{x_n : n\in\mathbb{N}\})$. The proof for the point $\frac{b}{\|b\|}$ is analogous.

Sufficiency. Assume $\frac{a}{\|a\|}\in\operatorname{SCD}(B_X)$ and $\frac{b}{\|b\|}\in\operatorname{SCD}(B_Y)$ and observe that

$$(a,b) = \|a\|\left(\frac{a}{\|a\|},0\right)+\|b\|\left(0,\frac{b}{\|b\|}\right),$$

where $\|(a,b)\|_1 = \|a\|+\|b\| = 1$. By Proposition 4.4 we have

$$\left(\frac{a}{\|a\|},0\right)\in\operatorname{SCD}(B_{X\oplus_1 Y})\ ,\ \left(0,\frac{b}{\|b\|}\right)\in\operatorname{SCD}(B_{X\oplus_1 Y})$$

and since the set of SCD points is convex, we have $(a,b)\in\operatorname{SCD}(B_{X\oplus_1 Y})$. □

4.3 SCD points in the direct sum $(\bigoplus_{n=1}^{\infty} X_n)_p$

In this subsection, let $X_n \neq \{0\}$, $n \in \mathbb{N}$, be a sequence of Banach spaces. We now investigate the Banach space $\big(\bigoplus_{n=1}^{\infty} X_n\big)_p$ endowed with the norm

$$\|x\| = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p},$$

where $x = (x_n)_{n=1}^{\infty}$, $x_n \in X_n$ for every $n \in \mathbb{N}$ and $1 < p < \infty$. The study of this space yields interesting results, in particular, by taking a countable sequence of Banach spaces, which contain no SCD points in the unit ball, the unit ball of the direct sum of these spaces does indeed contain only one SCD point. One of our main results on this topic is the following theorem.

Theorem 4.6. *Consider the Banach space* $X := \big(\bigoplus_{n=1}^{\infty} X_n\big)_p$, *where* $1 < p < \infty$. *The element* $0 \in B_X$ *is an* SCD *point of* B_X.

Proof. Firstly, for each $n \in \mathbb{N}$ select an element $x_n^* \in S_{X_n^*}$ and for every $k \in \mathbb{N}$ define a slice

$$S_n^k = S\Big(B_X, (\underbrace{0, 0, \dots, x_n^*}_{n \text{ components}}, 0, 0, \dots), \frac{1}{k}\Big).$$

Let us prove that the sequence of slices $\{S_n^k \colon n, k \in \mathbb{N}\}$ is determining for point $0 = (0, 0, \dots) \in B_X$. To this end, pick $x_n^k \in S_n^k$ for every $n, k \in \mathbb{N}$ and fix $\varepsilon > 0$. It suffices to find $x \in \overline{\mathrm{conv}}(\{x_n^k \colon n, k \in \mathbb{N}\})$ such that $\|x\| < \varepsilon$.

Pick $K \in \mathbb{N}$, such that $K \neq 1$ and $(p/K)^{1/p} < \varepsilon/2$. Now for given $n \in \mathbb{N}$,

$$x_n^K = (w_{n,1}^K, w_{n,2}^K, \dots, w_{n,n}^K, w_{n,n+1}^K, \dots),$$

so we have

$$\begin{aligned}1 - \frac{1}{K} &< (0, 0, \dots, x_n^*, 0, 0, \dots)(w_{n,1}^K, w_{n,2}^K, \dots, w_{n,n}^K, w_{n,n+1}^K, \dots)\\ &= x_n^*(w_{n,n}^K) \leq \|x_n^*\| \left\|w_{n,n}^K\right\| = \left\|w_{n,n}^K\right\|,\end{aligned}$$

hence $\left\|w_{n,n}^K\right\| > 1 - 1/K$. Moreover, we have $x_n^K \in B_X$, which means that

$$\|x_n^K\|^p = \sum_{i=1}^{\infty} \|w_{n,i}^K\|^p \leq 1.$$

Recall also the generalization of the Bernoulli inequality, which states that

$$(1+s)^r > 1 + rs$$

for every $r > 1$ and $s \neq 0$ satisfying $s > -1$ (see [15, Page 34]). Now we can derive another estimation:

$$\sum_{i\neq n}^{\infty} \|w_{n,i}^K\|^p = \Big(\sum_{i=1}^{\infty} \|w_{n,i}^K\|^p\Big) - \|w_{n,n}^K\|^p < 1 - \Big(1 - \frac{1}{K}\Big)^p \leq 1 - \Big(1 - \frac{p}{K}\Big) < \frac{p}{K}.$$

Define for each $n \in \mathbb{N}$ an element

$$\hat{x}_n^K = (0, 0, \dots, w_{n,n}^K, 0, 0, \dots).$$

Using the estimation above we get

$$\left\|x_n^K - \hat{x}_n^K\right\| = \left\|(w_{n,1}^K, \dots, w_{n,n-1}^K, 0, w_{n,n+1}^K, \dots)\right\| = \Big(\sum_{i\neq n}^{\infty} \|w_{n,i}^K\|^p\Big)^{1/p} < \Big(\frac{p}{K}\Big)^{1/p}.$$

Now consider the inclusion operator $I\colon \ell_1 \to \ell_p$. Note that $\|I\| \leq 1$, since for any $y \in B_{\ell_1}$

$$\|Iy\|_p = \|y\|_p \leq \|y\|_1 \leq 1,$$

where the inequality $\|y\|_p \leq \|y\|_1$ is a consequence of the Hölder inequality [7, Exercise 1.15]. Since the Banach spaces ℓ_1 and ℓ_p are not isomorphic [7, Proposition 4.49], I cannot be an isomorphism, which means that there cannot exist constants $\alpha, \beta > 0$ such that

$$\alpha \|y\|_1 \leq \|Iy\|_p = \|y\|_p \leq \beta \|y\|_1$$

for every $y \in S_{\ell_1}$. Since $\beta = 1$ satisfies the right-hand side of the inequalities, it must hold that for any $\alpha > 0$ there exists $y \in S_{\ell_1}$ such that

$$\|Iy\|_p < \alpha \|y\|_1 \leq \alpha.$$

Using this fact, we can find a sequence $(\lambda_n) \in S_{\ell_1}$, $\lambda_n \in [0,1]$ for every $n \in \mathbb{N}$, satisfying

$$\|I(\lambda_n)\|_p = \|(\lambda_n)\|_p = \Big(\sum_{n=1}^{\infty} \lambda_n^p\Big)^{1/p} < \frac{\varepsilon}{2}.$$

Now consider the element

$$x = \sum_{n=1}^{\infty} \lambda_n x_n^K$$

and observe that $x \in \overline{\text{conv}}(\{x_n^k \colon n, k \in \mathbb{N}\})$. Indeed, fix $\delta > 0$ arbitrarily. Because $\sum_{n=1}^{\infty} \lambda_n = 1$ we can find an index $N \in \mathbb{N}$ such that

$$r := \sum_{n=1}^{N} \lambda_n > 1 - \frac{\delta}{2}.$$

Now we can define another sequence

$$(\mu_n) = (\lambda_1, \lambda_2, \dots, \lambda_N, 1 - r, 0, \dots),$$

which obviously satisfies $\mu_n \in [0,1]$ for every $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \mu_n = 1$ and because there are finitely many μ_n being non-zero, we have

$$\sum_{n=1}^{\infty} \mu_n x_n^K \in \text{conv}(\{x_n^k \colon n, k \in \mathbb{N}\}). \tag{4.3}$$

Now

$$\begin{aligned}
\left\|x - \sum_{n=1}^{\infty} \mu_n x_n^K\right\| &= \left\|\sum_{n=1}^{\infty} \lambda_n x_n^K - \sum_{n=1}^{\infty} \mu_n x_n^K\right\| = \left\|\sum_{n=1}^{\infty} (\lambda_n - \mu_n) x_n^K\right\| \\
&\leq \sum_{n=1}^{\infty} |\lambda_n - \mu_n| \left\|x_n^K\right\| \leq \sum_{n=1}^{\infty} |\lambda_n - \mu_n| \\
&= \sum_{n=N+1}^{\infty} |\lambda_n - \mu_n| = |\lambda_{N+1} - (1-r)| + \sum_{n=N+2}^{\infty} \lambda_n \\
&= (1-r) - \lambda_{N+1} + \sum_{n=N+2}^{\infty} \lambda_n < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\end{aligned}$$

which means that $x \in \overline{\operatorname{conv}}(\{x_n^k \colon n, k \in \mathbb{N}\})$. To conclude, we apply the estimations derived above and see that

$$\begin{aligned}
\|z\| &= \left\|\sum_{n=1}^{\infty} \lambda_n x_n^K + \sum_{n=1}^{\infty} \lambda_n \hat{x}_n^K - \sum_{n=1}^{\infty} \lambda_n \hat{x}_n^K\right\| \\
&\leq \left\|\sum_{n=1}^{\infty} \lambda_n x_n^K - \sum_{n=1}^{\infty} \lambda_n \hat{x}_n^K\right\| + \left\|\sum_{n=1}^{\infty} \lambda_n \hat{x}_n^K\right\| \\
&= \left\|\sum_{n=1}^{\infty} \lambda_n (x_n^K - \hat{x}_n^K)\right\| + \left\|\sum_{n=1}^{\infty} \lambda_n \hat{x}_n^K\right\| \\
&\leq \sum_{n=1}^{\infty} \lambda_n \left\|x_n^K - \hat{x}_n^K\right\| + \Big(\sum_{n=1}^{\infty} \lambda_n^p \left\|\hat{x}_n^K\right\|^p\Big)^{1/p} \\
&\leq \Big(\frac{p}{K}\Big)^{1/p} \sum_{n=1}^{\infty} \lambda_n + \Big(\sum_{n=1}^{\infty} \lambda_n^p\Big)^{1/p} \\
&= \Big(\frac{p}{K}\Big)^{1/p} + \|(\lambda_n)\|_p < \Big(\frac{p}{K}\Big)^{1/p} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

□

For the upcoming discussion, let I be an uncountable set. Recall that the Banach space $c_0(I)$ consists of all bounded functions $f \colon I \to \mathbb{K}$ such that the set

$$\{i \in I \colon |f(i)| \geq \varepsilon\}$$

is finite for every $\varepsilon > 0$. The norm on $c_0(I)$ is defined as

$$\|f\| = \sup_{i \in I} |f(i)| .$$

Note that the Banach space $c_0(I)$ has the Asplund property, since $(c_0(I))^* = \ell_1(I)$, which has the RNP (see [6, Page 218]). On the other hand, we can easily deduce that $\mathrm{SCD}(B_{c_0(I)}) = \emptyset$. Indeed, By Example 3.15 the Banach space $\ell_1(I)$ fails the (-1)-BCP. An application of Theorem 3.16 results in $\mathrm{SCD}(B_{c_0(I)}) = \emptyset$.

Proposition 4.7. *Consider the Banach space $X := c_0(I) \oplus_p Y$, where Y is an arbitrary Banach space and $1 < p < \infty$. If $(a, b) \in \mathrm{SCD}(B_X)$, then $a = 0$.*

Proof. We prove that if $a \neq 0$ then $(a, b) \notin \mathrm{SCD}(B_X)$. Let $a \neq 0$ and select an arbitrary sequence of slices $\{S_n : n \in \mathbb{N}\}$, where

$$S_n = S(B_X, (a_n^*, b_n^*), \alpha_n).$$

Note that here $(a_n^*, b_n^*) \in (c_0(I) \oplus_p Y)^* = \ell_1(I) \oplus_q Y^*$, where q is the Hölder conjugate of p. Pick for every $n \in \mathbb{N}$ elements $x_n = (a_n, b_n) \in S_n$ and consider sets

$$A := \mathrm{supp}(a),\ A_n := \mathrm{supp}(a_n),\ B_n := \mathrm{supp}(a_n^*),\ n \in \mathbb{N}.$$

These sets are countable, since every element in $c_0(I)$ or $\ell_1(I)$ has countable support (see [7, Page 9]). Therefore we can define another countable set

$$J := A \cup \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n.$$

Since I is uncountable, we can find $i \in I$ such that $i \notin J$. Observe that for every $n \in \mathbb{N}$

$$(a_n + \|a_n\| e_i, b_n) \in S_n,$$

where $e_i = (\delta_{ik})_{k \in I}$. Indeed, fix $n \in \mathbb{N}$. Since $i \notin A_n$, it is easy to see that

$\|a_n + \|a_n\| e_i\| = \|a_n\|$. By using the fact that $e_i \notin B_n$

$$\begin{aligned}(a_n^*, b_n^*)(a_n + \|a_n\| e_i, b_n) &= a_n^*(a_n) + \|a_n\| a_n^*(e_i) + b_n^*(b_n) \\ &= a_n^*(a_n) + b_n^*(b_n) = (a_n^*, b_n^*)(a_n, b_n) > 1 - \alpha_n.\end{aligned}$$

To conclude the proof, it suffices to show that

$$(a, b) \notin \overline{\text{conv}}(\{(a_n + \|a_n\| e_i, b_n) \colon n \in \mathbb{N}\}). \tag{4.4}$$

Assume that we have for all $n \in \mathbb{N}$ elements $\lambda_n \in [0, 1]$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$ and only finitely many of λ_n are non-zero. We now divide our discussion into two possibilities.

(a) Suppose that the following estimation holds:

$$\sum_{n=1}^{\infty} \lambda_n \|a_n\| < \frac{\|a\|}{2}.$$

In this case, select $k \in I$ such that and $|a(k)| = \|a\|$. Since $\|a\| > 0$, we have $k \in A$, which means that $k \neq i$ and in turn $e_i(k) = 0$. Now

$$\begin{aligned}&\left\|(a, b) - \sum_{n=1}^{\infty} \lambda_n (a_n + \|a_n\| e_i, b_n)\right\|_p \\ &= \left\|\Big(a - \sum_{n=1}^{\infty} \lambda_n (a_n + \|a_n\| e_i), b - \sum_{n=1}^{\infty} \lambda_n b_n\Big)\right\|_p \\ &\geq \left\|a - \sum_{n=1}^{\infty} \lambda_n (a_n + \|a_n\| e_i)\right\| \\ &\geq \left|a(k) - \sum_{n=1}^{\infty} \lambda_n \big(a_n(k) + \|a_n\| e_i(k)\big)\right| \\ &\geq |a(k)| - \sum_{n=1}^{\infty} |a_n(k)| \geq \|a\| - \sum_{n=1}^{\infty} \lambda_n \|a_n\| \\ &> \|a\| - \frac{\|a\|}{2} = \frac{\|a\|}{2},\end{aligned}$$

which means that (4.4) is satisfied.

(b) Assume now that

$$\sum_{n=1}^{\infty} \lambda_n \|a_n\| \geq \frac{\|a\|}{2}.$$

Using similar approach and the fact that $i \notin J$, we obtain

$$\begin{aligned}
\left\| (a,b) - \sum_{n=1}^{\infty} \lambda_n (a_n + \|a_n\| e_i, b_n) \right\|_p &\geq \left\| a - \sum_{n=1}^{\infty} \lambda_n (a_n + \|a_n\| e_i) \right\| \\
&\geq \left| a(i) - \sum_{n=1}^{\infty} \lambda_n \big(a_n(i) + \|a_n\| e_i(i)\big) \right| \\
&= \left| a(i) - \sum_{n=1}^{\infty} \lambda_n a_n(i) - \sum_{n=1}^{\infty} \lambda_n \|a_n\| \right| \\
&= \left| - \sum_{n=1}^{\infty} \lambda_n \|a_n\| \right| \geq \frac{\|a\|}{2},
\end{aligned}$$

hence (4.4) holds once again.

□

The results above lead to an interesting conclusion.

Theorem 4.8. *Consider the Banach space* $X := \big(\bigoplus_{n=1}^{\infty} c_0(I)\big)_p$ *with* $1 < p < \infty$. *Then* $\mathrm{SCD}(B_X) = \{0\}$.

Proof. First, we know that $0 \in \mathrm{SCD}(B_X)$ by Theorem 4.6. Assume now that $(a_n) \in \mathrm{SCD}(B_X)$. We show that $a_n = 0$ for all $n \in \mathbb{N}$, starting with a_1. To this end, consider the mapping

$$\Big(\bigoplus_{n=1}^{\infty} c_0(I)\Big)_p \ni (x_1, x_2, \dots) \mapsto (x_1, (x_2, x_3, \dots)) \in c_0(I) \oplus_p \Big(\bigoplus_{n=2}^{\infty} c_0(I)\Big)_p.$$

It is easy to see that this mapping is an isometry, consequently $(a_1, a_2, \dots)$ is an SCD point in the unit ball of X if and only if $(a_1, (a_2, a_3, \dots))$ is SCD point in the unit ball of $c_0(I) \oplus_p \big(\bigoplus_{n=2}^{\infty} c_0(I)\big)_p$. Now by Proposition 4.7 we have that $a_1 = 0$. The rest of the coordinates work similarly. □

We now prove an analogous result to Proposition 4.7 for spaces with the Daugavet property. To this end, we use the following lemma.

Lemma 4.9. *Let X be a Banach space with the Daugavet property. Then for every sequence of slices $\{S_n \colon n \in \mathbb{N}\} \subset B_X$ and every $x \in S_X$, there is a sequence (x_n) with $x_n \in S_n$ for each $n \in \mathbb{N}$, such that $x \notin \overline{\operatorname{span}}(\{x_n \colon n \in \mathbb{N}\})$.*

Proof. For the proof of this lemma, check the proof of [2, Example 2.13] □

Proposition 4.10. *Consider the Banach space $X := E \oplus_p Y$, where E has the Daugavet property, Y is arbitrary and $1 < p < \infty$. If $(a, b) \in \mathrm{SCD}(B_X)$, then $a = 0$.*

Proof. We again prove the contrapositive. Let $a \neq 0$ and fix an arbitrary sequence of slices $\{S_n \colon n \in \mathbb{N}\}$, where $S_n = S(B_X, (a_n^*, b_n^*), \alpha_n)$. Moreover, we assume that $\|(a_n^*, b_n^*)\|_q = 1$, where q is the Hölder conjugate of p. We find a sequence $(x_n) \subset B_X$ such that $x_n \in S_n$ for each $n \in \mathbb{N}$ and $(a, b) \notin \overline{\operatorname{conv}}\{x_n \colon n \in \mathbb{N}\}$. Define

$$A := \{n \in \mathbb{N} \colon a_n^* \neq 0\},\ B = \mathbb{N} \setminus A = \{n \in \mathbb{N} \colon a_n^* = 0\}.$$

Using these sets, let us define the desired sequence. First, if $n \in B$, we will pick $x_n = (0, v_n) \in S_n$, where the first coordinate can be taken zero due to the fact that $a_n^* = 0$. Now, for every $n \in A$ define a slice

$$T_n = S\Big(B_E, a_n^*, \frac{\alpha_n}{4}\Big),$$

and consider the sequence of slices $\{T_n \colon n \in A\}$. By Lemma 4.9 we can find a sequence $(u_n)_{n \in A} \subset B_E$ so that $u_n \in T_n$ for all $n \in A$ and

$$a \notin \overline{\operatorname{span}}(\{u_n \colon n \in A\}).$$

The latter means that there exists $\varepsilon > 0$ such that $\|a - u\| \geq \varepsilon$ for every element $u \in \operatorname{span}(\{u_n \colon n \in A\})$. Consider now the following claim.

Claim:

$$\forall n \in A\ \exists r_n, s_n \in \mathbb{R}\ \exists q_n \in B_Y \colon u_n \in T_n \implies (r_n u_n, s_n q_n) \in S_n. \tag{4.5}$$

Proof of Claim. Fix $n \in A$ and observe that we can find $(e_n, y_n) \in S_n$ such that

$$(a_n^*, b_n^*)(e_n, y_n) = a_n^*(e_n) + b_n^*(y_n) > 1 - \frac{\alpha}{2}.$$

Take $r_n := \|e_n\|$ and $s_n := \|y_n\|$. In addition, find $q_n \in B_Y$ such that

$$b_n^*(q_n) > \|b_n^*\| - \frac{\alpha}{4}.$$

By assumption $u_n \in T_n$, which means that

$$a_n^*(u_n) > \|a_n^*\| - \frac{\alpha}{4}.$$

To conclude, first notice that

$$(r_n u_n, s_n q_n) = (\|e_n\|\, u_n, \|y_n\|\, q_n) \in B_X.$$

Indeed, we have

$$\begin{aligned} \|(r_n u_n, s_n q_n)\|_p^p &= \|(\|e_n\|\, u_n, \|y_n\|\, q_n)\|_p^p \\ &= \|e_n\|^p \|u_n\|^p + \|y_n\|^p \|q_n\|^p \\ &\leq \|e_n\|^p + \|y_n\|^p \leq 1, \end{aligned}$$

and finally, using all the estimations derived above

$$\begin{aligned} (a_n^*, b_n^*)(r_n u_n, s_n q_n) &= a_n^*(u_n)\, \|e_n\| + b_n^*(q_n)\, \|y_n\| \\ &> \Big(\|a_n^*\| - \frac{\alpha_n}{4} \Big) \|e_n\| + \Big(\|b_n^*\| - \frac{\alpha_n}{4} \Big) \|y_n\| \\ &= \|a_n^*\|\, \|e_n\| + \|b_n^*\|\, \|y_n\| - \frac{\alpha_n}{4} \Big(\|e_n\| + \|y_n\| \Big) \\ &\geq a_n^*(e_n) + b_n^*(y_n) - \frac{\alpha_n}{2} > 1 - \frac{\alpha_n}{2} - \frac{\alpha_n}{2} = 1 - \alpha_n, \end{aligned}$$

which means that $(r_n u_n, s_n q_n) \in S_n$, hence claim (4.5) holds. $\square$

So we are able to pick, by using this claim, for each $n \in A$

$$x_n = (r_n u_n, s_n q_n) \in S_n.$$

Pick arbitrary $\lambda_n \in [0,1]$, finitely many being non-zero and let $\sum_{n=1}^{\infty} \lambda_n = 1$. We have

$$\begin{aligned}
\left\| (a,b) - \sum_{n=1}^{\infty} \lambda_n x_n \right\|_p &= \left\| (a,b) - \sum_{n\in A} \lambda_n (r_n u_n, s_n q_n) - \sum_{n\in B} \lambda_n (0, v_n) \right\|_p \\
&= \left\| \Big(a - \sum_{n\in A} \lambda_n r_n u_n, b - \sum_{n\in B} \lambda_n (s_n q_n - v_n) \Big) \right\|_p \\
&= \left(\left\| a - \sum_{n\in A} \lambda_n r_n u_n \right\|^p + \left\| b - \sum_{n\in B} \lambda_n (s_n q_n - v_n) \right\|^p \right)^{1/p} \\
&\geq \left\| a - \sum_{n\in A} \lambda_n r_n u_n \right\| \geq \varepsilon,
\end{aligned}$$

which means that $(a,b) \notin \overline{\operatorname{conv}}(\{x_n \colon n \in \mathbb{N}\})$. $\square$

By following the proof of Theorem 4.8, we see that the following theorem also holds.

Theorem 4.11. *Consider the Banach space* $X := \big(\bigoplus_{n=1}^{\infty} E_n \big)_p$, *where* $1 < p < \infty$ *and* E_n *are spaces with the Daugavet property. Then* $\operatorname{SCD}(B_X) = \{0\}$.

Theorem 4.11 has important consequences. Namely, we can find a separable Banach space with only one SCD point in the unit ball. Indeed, consider the Banach space $X = \big(\bigoplus_{n=1}^{\infty} C[0,1] \big)_p$, where $1 < p < \infty$. Then by Theorem 4.11 we have $\operatorname{SCD}(B_X) = \{0\}$. It is also worth noting that by Corollary 3.17 we know that $\operatorname{SCD}(C[0,1]) = \emptyset$. This means that we can generate SCD points (in the unit ball of the direct sum) even if the components themselves have no SCD points (in the unit ball of the component).

To finish the section, we generalize Theorems 4.8 and 4.11. The proof of this theorem is obvious by the proof of Theorem 4.8.

Theorem 4.12. *Consider the Banach space* $X := \big(\bigoplus_{n=1}^{\infty} Y_n \big)_p$, *where* $1 < p < \infty$ *and for every* $n \in \mathbb{N}$, *the Banach space* Y_n *is either* $c_0(I)$ *or has the Daugavet property. Then* $\operatorname{SCD}(B_X) = \{0\}$.

5 SCD points in projective tensor products

We now turn our attention to SCD points in projective tensor products. We start by giving some preliminary definitions about projective tensor products. For further reading on tensor products, we advise the reader to consult [17].

Let X and Y be Banach spaces, both over $\mathbb{R}$.

Definition 5.1. A mapping $B\colon X \times Y \to \mathbb{R}$ is called a *bilinear form*, if it satisfies the following conditions:

(a) $B(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y)$;

(b) $B(x, \mu_1 y_1 + \mu_2 y_2) = \mu_1 B(x, y_1) + \mu_2 B(x, y_2)$,

for all $x_1, x_2, x \in X$, $y_1, y_2, y \in Y$ and all $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

All bilinear forms on $X \times Y$ form a vector space, denoted by $B(X \times Y)$. We denote its algebraic dual by $B(X \times Y)^{\#}$. The *elementary tensor of $x \in X$ and $y \in Y$* is a linear functional on the space $B(X \times Y)$, denoted by $x \otimes y$ and defined by the equation

$$(x \otimes y)(B) = B(x, y), \; B \in B(X \times Y).$$

Definition 5.2. The *algebraic tensor product* $X \otimes Y$ is the subspace of $B(X \times Y)^{\#}$, which is defined as

$$X \otimes Y = \operatorname{span}\{x \otimes y \colon x \in X, \; y \in Y\}.$$

We observe one natural way to define a norm on the algebraic tensor product.

Definition 5.3. The *projective norm* $\|\cdot\|_\pi$ on the algebraic tensor product is given as

$$\|z\|_\pi = \inf\Big\{ \sum_{n=1}^{m} \|x_n\| \, \|y_n\| : z = \sum_{n=1}^{m} x_n \otimes y_n, \; m \in \mathbb{N} \Big\}.$$

Hence, we obtain the normed space

$$X \otimes_\pi Y := (X \otimes Y, \|\cdot\|_\pi).$$

As generally this space is not a Banach space (unless X and Y are both finite-dimensional), we work in the completion of this space.

Definition 5.4. The *projective tensor product* $X\hat{\otimes}_\pi Y$ is defined as the completion of the normed space $X \otimes_\pi Y$.

Note that throughout the discussion above, we will make use of the isometry $B(X \times Y) = (X\hat{\otimes}_\pi Y)^*$ (see [17, Page 22]).

We will now present the main result of this section, by first introducing an useful lemma.

Lemma 5.5 ([16, Lemma 3.4])**.** *Suppose that we have a norm one bilinear form* $B \in (X\hat{\otimes}_\pi Y)^*$ *and* $\varepsilon > 0$*. Then*

$$S(B_{X\hat{\otimes}_\pi Y}, B, \varepsilon^2) \subset \overline{\text{conv}}(\{x \otimes y \colon x \in B_X,\, y \in B_Y,\, B(x, y) > 1 - \varepsilon\}) + 4\varepsilon B_{X\hat{\otimes}_\pi Y}.$$

Theorem 5.6. *Assume that we have* $a \in \text{dent}(B_X)$ *and* $b \in \text{SCD}(B_Y) \setminus \{0\}$*. Then* $a \otimes b \in \text{SCD}(B_{X\hat{\otimes}_\pi Y})$*.*

Proof. First, notice that since a is a denting point of B_X, we can find for each $n \in \mathbb{N}$ a slice $S(B_X, x_n^*, \alpha_n)$, where $\|x_n^*\| = 1$ and

$$a \in S(B_X, x_n^*, \alpha_n) \subset B\Big(a, \frac{1}{n}\Big). \tag{5.1}$$

On the other hand we can find a sequence of slices

$$\{S(B_Y, y_n^*, \beta_n) \colon n \in \mathbb{N},\ \|y_n^*\| = 1\}, \tag{5.2}$$

which is determining for point b. Let us now define for each $n, m, k \in \mathbb{N}$ the following slices:

$$S_{n,m}^k = S\Big(B_{X\hat{\otimes}_\pi Y}, x_n^* \otimes y_m^*, \frac{1}{k}\Big),$$

where

$$(x_n^* \otimes y_m^*)(x \otimes y) = x_n^*(x) y_m^*(y)$$

for every $x \in X$ and $y \in Y$. Note here that the function $x_n^* \otimes y_m^*$ is bilinear and bounded, therefore $(x_n^* \otimes y_m^*) \in (X\hat{\otimes}_\pi Y)^*$. Our goal is to prove that the sequence of slices $\{S_{n,m}^k \colon n, m, k \in \mathbb{N}\}$ is determining for the elementary tensor $a \otimes b$. To this end, we will use Proposition 3.3 condition (ii). Let $S = S(B_{X\hat{\otimes}_\pi Y}, B, \alpha)$, where

B is a norm one bounded bilinear form and $a \otimes b \in S$. It suffices to find a member of the sequence of slices $\{S^k_{n,m} : n, m, k \in \mathbb{N}\}$, which is contained in S.

First, since $a \otimes b \in S$, we have

$$B(a, b) > 1 - \alpha \implies \exists \gamma > 0 \colon \; B(a, b) > 1 - \alpha + \gamma.$$

This in turn means that

$$a \in \{x \in B_X : B(x, b) > 1 - \alpha + \gamma\},$$

where the set above is actually a slice of B_X, since for the mapping $x \mapsto B(x, b)$ is clearly linear and continuous.

Take $n \in \mathbb{N}$ such that $1/n < \gamma/32$. Then by (5.1)

$$a \in S(B_X, x^*_n, \alpha_n) \subset B\Big(a, \frac{1}{n}\Big) \subset B\Big(a, \frac{\gamma}{32}\Big) \tag{5.3}$$

Using the fact that $B(a, b) > 1 - \alpha + \gamma$, we see that similarly to the last case

$$b \in \{y \in B_Y : B(a, y) > 1 - \alpha + \gamma\},$$

where the set is again a slice of B_Y. Since the sequence of slices in (5.2) determines b, we can find $m \in \mathbb{N}$ such that

$$S(B_Y, y^*_m, \beta_m) \subset \{y \in B_Y : B(a, y) > 1 - \alpha + \gamma\}. \tag{5.4}$$

Consider now the following set:

$$S^{\otimes} := \{u \otimes v \colon u \in S(B_X, x^*_n, \alpha_n), \; v \in S(B_Y, y^*_m, \beta_m)\}.$$

We claim that

$$S^{\otimes} \subset \Big\{z \in B_{X \hat{\otimes}_\pi Y} : B(z) > 1 - \alpha + \frac{31\gamma}{32}\Big\}. \tag{5.5}$$

Assume that we have $u \in S(B_X, x^*_n, \alpha_n)$ and $v \in S(B_Y, y^*_m, \beta_m)$. By (5.4) we get

$B(a,v) > 1-\alpha+\gamma$ and using (5.3), we obtain

$$\begin{aligned} B(u\otimes v) = B(u,v) &= B(a,v) - B(a-u,v) \\ &> 1-\alpha+\gamma - \|B\|\,\|v\|\,\|a-u\| > 1-\alpha+\gamma-\frac{1}{n} \\ &> 1-\alpha+\gamma-\frac{\gamma}{32} > 1-\alpha+\frac{31\gamma}{32}. \end{aligned}$$

Hence (5.5) holds.

To proceed, pick for each $n,m\in\mathbb{N}$ another $k\in\mathbb{N}$ satisfying $1/k < \min\{\alpha_n,\beta_m\}$. Now we claim that

$$S\Big(B_X\otimes B_Y, x_n^*\otimes y_m^*, \frac{1}{k}\Big) \subset S^{\otimes}, \tag{5.6}$$

where $B_X\otimes B_Y = \{x\otimes y\colon x\in B_X,\ y\in B_Y\}$.

Fix $x\otimes y\in S(B_X\otimes B_Y, x_n^*\otimes y_m^*, 1/k)$, i.e.

$$(x_n^*\otimes y_m^*)(x\otimes y) = x_n^*(x)y_m^*(y) > 1-\frac{1}{k}.$$

Since $y\in B_Y$, we have

$$x_n^*(x) \ge x_n^*(x)y_m^*(y) > 1-\frac{1}{k} > 1-\alpha_n,$$

which means that $x\in S(B_X, x_n^*, \alpha_n)$. It is easy to see that analogous discussion leads to $y\in S(B_Y, y_m^*, \beta_m)$. In conclusion

$$x\otimes y\in S^{\otimes}.$$

Let $k\in\mathbb{N}$ be such that $4/k<\gamma/32$. In order to finish the proof, we show that

$$S\Big(B_{X\hat{\otimes}_\pi Y}, x_n^*\otimes y_m^*, \frac{1}{k^2}\Big) = S_{n,m}^{k^2} \subset S = S(B_{X\hat{\otimes}_\pi Y}, B, \alpha). \tag{5.7}$$

To show that (5.7) holds, we use Lemma 5.5, which states in particular that

$$S\Big(B_{X\hat{\otimes}_\pi Y}, x_n^*\otimes y_m^*, \frac{1}{k^2}\Big) \subset \overline{\operatorname{conv}}\bigg(S\Big(B_X\otimes B_Y, x_n^*\otimes y_m^*, \frac{1}{k}\Big)\bigg) + \frac{4}{k}B_{X\hat{\otimes}_\pi Y}. \tag{5.8}$$

Pick an element $z \in S(B_{X\hat{\otimes}_\pi Y}, x_n^* \otimes y_m^*, 1/k^2)$. By (5.8) by can write

$$z = a + \frac{4}{k}h,\ a \in \overline{\text{conv}}\left(S\Big(B_X \otimes B_Y, x_n^* \otimes y_m^*, \frac{1}{k}\Big)\right),\ h \in B_{X\hat{\otimes}_\pi Y}.$$

This means that we can find another element $\hat{a} \in \text{conv}(S(B_X \otimes B_Y, x_n^* \otimes y_m^*, 1/k))$ such that $\|a - \hat{a}\| < \gamma/32$. By defining $\hat{z} = \hat{a} + (4/k)h$, it is obvious that

$$\|z - \hat{z}\| = \|a - \hat{a}\| < \frac{\gamma}{32}.$$

In addition, by (5.6) we have

$$\text{conv}(S(B_X \otimes B_Y, x_n^* \otimes y_m^*, 1/k)) \subset \Big\{z \in B_{X\hat{\otimes}_\pi Y} \colon B(z) > 1 - \alpha + \frac{31\gamma}{32}\Big\},$$

because a slice is a convex set. With this, we obtain

$$B(\hat{z}) = B(\hat{a}) + \frac{4}{k}B(h) > 1 - \alpha + \frac{31\gamma}{32} - \frac{4}{k}.$$

Now using the above estimations as well as (5.5) and (5.6), we get

$$\begin{aligned} B(z) &= B(z) - B(\hat{z}) + B(\hat{z}) = B(\hat{z}) - B(\hat{z} - z) \\ &> 1 - \alpha + \frac{31\gamma}{32} - \frac{4}{k} - \|B\|\,\|\hat{z} - z\| \\ &> 1 - \alpha + \frac{31\gamma}{32} - \frac{4}{k} - \frac{\gamma}{32} \\ &= 1 - \alpha + \frac{30\gamma}{32} - \frac{4}{k} > 1 - \alpha + \frac{\gamma}{32} - \frac{4}{k} > 1 - \alpha. \end{aligned}$$ □

It is unknown whether $X\hat{\otimes}_\pi Y$ is an SCD space, whenever X and Y are SCD spaces. The following can be considered as partial progress towards this open question.

Corollary 5.7. *Let B_X be dentable and* $\text{SCD}(B_Y) = B_Y$*. Then* $\text{SCD}(B_{X\hat{\otimes}_\pi Y}) = B_{X\hat{\otimes}_\pi Y}$*. If additionally B_X is separable and B_Y is an SCD set, then $B_{X\hat{\otimes}_\pi Y}$ is an SCD set.*

Proof. Notice that using elementary properties of projective tensor products together with the assumption, we can write

$$B_{X\hat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y) = \overline{\text{conv}}\big(\overline{\text{conv}}(\text{dent}(B_X)) \otimes \text{SCD}(B_Y)\big)$$
$$= \overline{\text{conv}}(\text{dent}(B_X) \otimes \text{SCD}(B_Y)).$$

By Theorem 5.6

$$\text{dent}(B_X) \otimes (\text{SCD}(B_Y) \setminus \{0\}) \subset \text{SCD}(B_{X\hat{\otimes}_\pi Y}). \tag{5.9}$$

This in turn implies that

$$\text{dent}(B_X) \otimes \text{SCD}(B_Y) \subset \text{SCD}(B_{X\hat{\otimes}_\pi Y}). \tag{5.10}$$

Indeed, pick $x \otimes y \in \text{dent}(B_X) \otimes \text{SCD}(B_Y)$, where $y \neq 0$ (such elements clearly do exist). Then by (5.9) we directly obtain that $x \otimes y \in \text{SCD}(B_{X\hat{\otimes}_\pi Y})$. Using Corollary 3.13 we have $0 \in \text{SCD}(B_{X\hat{\otimes}_\pi Y})$. Now picking $x \otimes 0 \in \text{dent}(B_X) \otimes \text{SCD}(B_Y)$, we have

$$x \otimes 0 = 0 \in \text{SCD}(B_{X\hat{\otimes}_\pi Y})$$

and consequently (5.10) holds.

Also note that the set of SCD points is closed and convex, meaning

$$\overline{\text{conv}}(\text{dent}(B_X) \otimes \text{SCD}(B_Y)) \subset \text{SCD}(B_{X\hat{\otimes}_\pi Y}).$$

In conclusion

$$B_{X\hat{\otimes}_\pi Y} = \overline{\text{conv}}(\text{dent}(B_X) \otimes \text{SCD}(B_Y)) \subset \text{SCD}(B_{X\hat{\otimes}_\pi Y}) \subset B_{X\hat{\otimes}_\pi Y}.$$

If additionally B_X is separable and B_Y is an SCD set, the set $B_{X\hat{\otimes}_\pi Y}$ must also be separable, hence by Lemma 3.7 we have that $B_{X\hat{\otimes}_\pi Y}$ is an SCD set. □

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6 Summary

In [2], A. Avilés, V. Kadets, M. Martín, J. Merí and V. Shepelska introduced slicely countably determined sets and spaces. One of the main motivations was the generalization of separable spaces with the Radon–Nikodým or the Asplund property. The goal of this thesis is to answer to an open question whether the slicely countably determined property can be naturally extended to non-separable Banach spaces. To this end, we introduce the slicely countably determined points, together with numerous examples and properties. We study these points in widely known classes of Banach spaces, and moreover, we investigate the stability of these points in direct sums and projective tensor products of Banach spaces. With the help of this pointwise approach, we are able to prove new results for slicely countably determined sets and spaces aswell, making the research on this topic appealing.

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