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Volatility Modeling of Asset Returns

Actuarial and Financial Engineering

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Abstract

The research investigates financial market volatility modeling through an analysis of daily stock price data from Tallink Grupp together with OMX Baltic Index data spanning from 01 February 2007 until 10 October 2023. Financial econometric theory guides the analysis through the combination of Autoregressive Moving Average (ARMA) models with Generalized Autoregressive Conditional Heteroskedasticity (GARCH) frameworks to properly model heteroskedasticity. This research evaluates asymmetric GARCH extensions including TGARCH, EGARCH and other asymmetric variants to account for the leverage effect and examine how market shocks affect volatility differently based on their positive or negative nature.

The 'rugarch' package in R serves as a tool and provides a robust and flexible framework for specifying, fitting, and comparing various volatility models. The research further investigates heteroskedasticity and asymmetry characteristics in the volatility dynamics of Tallink Grupp stock prices and OMX Baltic Index data across three economic periods: the global financial crisis (2007–2010), the stable market phase (2011–2019), and the COVID-19 pandemic (2020–2023).

The research provides both theoretical and practical value by advancing knowledge about risk-trade-off and the relationship between expected returns and associated risk. Advanced GARCH and asymmetric models which further support the performance of the models in capturing market shocks and volatility effects.

Keywords: Volatility modeling, rugarch, GARCH, asymmetric GARCH, financial markets, Tallink Grupp, OMX Baltic index.

CERCS code and name: P160 Statistics, operations research, programming, actuarial mathematics.

Varade tootluse volatiilsuse modelleerimine

Magistritöös

Alex Chiwete Michael

Lühikokkuvõte

Käesolev uurimus käsitleb volatiilsuse modelleerimist finantsturgudel, analüüsides Tallink Grupi aktsia ja OMX Balti indeksi päevaseid hindu ajavahemikus 1. veebruar 2007 kuni 10. oktoober 2023. Tuginedes finantsökonomeetria teooriale, ühendab analüüs autoregressiivsete liikuvate keskmiste (ARMA) mudelid üldistatud autoregressiivsete tingliku heteroskedastilisuse (GARCH) raamistikuga, et tõhusalt modelleerida hindade päevaseid tulususi. Volatiilsuse asümmeetrilisele reageerimisele erinevatele turušokkidele hinnatakse mitmeid GARCH-mudelite asümmeetrilisi laiendusi, sealhulgas TGARCH, EGARCH ja teisi mudeleid, mis arvestavad nn leverage-efektiga, mille puhul negatiivsed hinnašokid suurendavad volatiilsust rohkem kui positiivsed samas ulatuses hinnašokid.

Mudelite hindamine ja diagnostiline testimine viiakse läbi R-i paketi rugarch abil, mis pakub paindlikku ja usaldusväärset platvormi tingliku autoregressiivse heteroskedastilisuse modelleerimiseks.

Märksõnad: volatiilsuse modelleerimine, rugarch pakett, GARCH, asümmeetriline GARCH, finantsturud, Tallink Grupp, OMX Balti indeks.

CERCS kood ja nimetus: P160 Statistika, operatsioonianalüüs, programmeerimine, kindlustusmatemaatika.

Contents

Introduction

In order to understand and predict market volatility in modern financial data, models with the ability to capture the complexity of asset price movements have become necessary and relevant over time. Conditional variance, a key factor that reflects how asset prices change over time, requires accurate estimation, giving room for investors as well as policymakers to make informed decisions and anticipate market fluctuations. Autoregressive conditional heteroskedasticity (ARCH) and Generalized autoregressive conditional heteroskedasticity (GARCH) models have significant contributions in the analysis of conditional variance. However, different markets pose different challenges. In my research, when analyzing the Baltic Market, new challenges were encountered requiring more advanced approach and modeling.

The Baltic market, which is relatively small in size in comparison to larger markets, comprises regional dependencies and its sensitivity to shocks presents features that lead to volatility clustering and conditional heteroskedasticity. These features that only traditional models may fail to capture introduce advanced GARCH models and asymmetric models to account for the different effects of shocks, i.e, positive and negative shocks, and further give a better analysis of the volatility effects.

In this study, statistical model approaches are applied in examining the conditional variances of the daily returns of Tallink Grupp's stock prices and OMX Baltic index prices. GARCH models introduced by Engle [6] and Bollerslev in 1986 [7] offer a framework for modeling volatility clustering and heteroskedasticity. Furthermore, advanced and asymmetric variants as discussed by Hentschel [12] and applied using the 'rugarch' package in R (Ghalanos, 2022 [24]), are well suited in capturing volatility trends in these markets.

The recent studies on stock returns and exchange rates confirm the flexibility of GARCH models and their effectiveness in different applications. Hence, the application of these models in analyzing different volatility trends and providing a comprehensive interpretation for the volatility dynamics in these markets.

My research is structured into three chapters that focus on the different details of the volatility modeling. The first chapter provides the theoretical background for modeling. It introduces the behavior of asset returns, ARMA models, and the concepts of ARCH AND GARCH models. ARMA models, introduced for handling autocorrelation in the returns, and furthermore explain how the GARCH model builds on the ARCH framework through the introduction of past shocks and lagged conditional variances.

Building on the foundations, advanced GARCH extensions and asymmetric variants designed to capture leverage effects and shocks were also explored, along with important diagnostic tools used in analyzing conditional heteroskedasticity and model performance. Furthermore, reviews of significant studies that highlight the success of GARCH models in volatility modeling in financial markets. In summary, this chapter gives the framework for the application of model techniques in the analysis of Tallink Grupp stock prices and the OMX Baltic Index prices.

The second chapter moves to the practical application of the theory of ARCH and GARCH, further demonstrating how these models are applied using the 'rugarch' package in R. It follows through the model estimation and fitting process, selection criteria, and diagnostic tests used to evaluate performance. In addition, the practical application of GARCH variants and asymmetric variants that account for leverage effects.

The third chapter gives the analysis and results of daily stock prices from Tallink Grupp and the OMX Baltic Index. Step-by-step details of how the models are applied to investigate conditional heteroskedasticity and how the volatility responds to changing market conditions.

The analysis is further divided into significant periods to reflect some major economic phases. The complete period (2007-2023), the post global financial crisis period (2007–2010), a relatively stable phase (2011–2019), the pre-COVID period (2007-2019) and the post-COVID period (2020–2023), to uncover structural breaks and changes in volatility tied to economic events. Additionally, the analysis highlights how different models perform under varying conditions, drawing insights that guide future research and model selection.

1 Behavior of Asset Returns

Historically, the logarithmic prices of financial assets $\ln(P_t)$ have been modeled using the random walk process. This process, as explained by Fama [1], assumes that asset returns are independent, identically distributed (i.i.d.) random variables with constant variance.

Mathematically, the random walk process can be expressed as:

$$\ln(P_t) = \ln(P_{t-1}) + \mu + \varepsilon_t,$$

or in terms of continuous returns:

$$\Delta \ln(P_t) = \mu + \varepsilon_t.$$

where $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$ represents a normal, independent, and identically distributed random process.

Under this model, asset returns are assumed to be uncorrelated. This means that the past returns do not predict the future returns, supporting the assumption of the market being weakly efficient.

However, evidence from financial data suggests deviations from the assumptions with empirical observations such as autocorrelation in residuals ε_t, volatility clustering and periods where volatility stays persistently high or low indicate that returns are not truly i.i.d. Furthermore, returns may have heavy tails and asymmetry, deviating from the normal distribution.

Conditional heteroskedasticity is a feature in time series analysis where the conditional variance is not constant but changes over time, and this variation is often influenced by past events. This phenomenon is prominent in financial time series, where periods of high volatility tend to be followed by further high volatility, and periods of low volatility tend to be followed by low volatility.

Another feature observed is the leverage effect first introduced by Black [2]; this refers to the tendency for bad news (e.g., a negative return) typically leads to a larger increase in volatility than good news of the same size. One explanation is that a decline in a firm's stock price increases its financial leverage, leading to greater risk and higher volatility.

Studies have demonstrated that conditional variance reacts in response to the volatility of previous periods. And when time series exhibit heteroskedasticity, traditional methods may become unreliable. Hence, the modeling of conditional variance using approaches such as ARCH, GARCH, and other advanced models.

Another factor is the non-trading period effect. During weekends and holidays, markets are closed; however, the accumulation of information during market closures continues. When the markets reopen, prices may adjust, which sometimes leads to differences in variance return compared to trading periods. Ignoring these effects may lead to an underestimation of volatility during these non-traded periods.

Here, log prices can be adjusted using interpolation techniques as described by Dacorogna [3], which assumes that each non-trading day contributes a fixed fraction of a trading day's variance to the overall market variance.

1.1 ARMA Models

Referenced from - Gebhard Kirchg¨assner and J¨urgen Wolters. [4].

In time series analysis, second-order weak stationarity (also known as covariance stationarity) is a fundamental property required for modeling the conditional mean using ARMA models. A stochastic process r_t is said to be second-order weakly stationary if; the mean of the series is constant: $E(r_t) = \mu$, the variance is constant and finite: $D(r_t) = \sigma^2$, and the covariance between any two values r_t and r_{t-k} depends only on the lag k, and not on the specific time t: $Cov(r_t, r_{t-k}) = \gamma(k)$. We say that a time series is weakly stationary if the corresponding stochastic process is weakly stationary.

In financial time series, asset returns sometimes exhibit serial correlation, which means that past values influence future values. To capture these dependencies and the persistence in returns, the Autoregressive Moving Average (ARMA) models are applied, providing a flexible framework for representing stationary time series with both autoregressive (AR) and moving average (MA) components.

AR(p) Models

An AR(p) process models a time series where each value depends on its previous p values and a stochastic error term. The p-th order autoregressive process AR(p) is called the second order weakly stationary process in the form:

$$r_t = c + \sum_{i=1}^{p} \phi_i r_{t-i} + \varepsilon_t, \tag{1}$$

where ε_t represents the white noise error term with zero mean and constant variance, and ϕ_i represents the coefficients. The mean value, $\mu \;\; = E(r_t)$ of the process has the form:

$$\mu = \frac{c}{1 - \sum_{i=1}^{p} \phi_i}.$$

Using lag operator B, the AR(P) process can be presented as:

$$\phi(B)\tilde{r}_t = \varepsilon_t, \quad \phi(x) = 1 - \sum_{i=1}^{p} \phi_i x^i. \tag{2}$$

where:

$$\tilde{r}_t = r_t - \mu.$$

and lag operator B:

$$B^k r_t = r_{t-k}, \quad \text{for } k = 1, 2, \ldots$$

AR(p) models are particularly useful for identifying the appropriate lag order of a time series, typically through the analysis of the ACF and PACF functions.

An AR(p) process is weakly stationary of second order if and only if the moduli of the roots x_i of the characteristic polynomial, $\phi(x)$, and satisfy the condition $|x_i| > 1$ for all $i = 1, 2, \ldots, p$.

The autocorrelation coefficients $\rho(k)$ of an AR(p) process expressed as a linear function of its past values is given by:

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p),$$

On the other hand, the partial autocorrelation coefficients $\pi(k)$ of an AR(p) process exhibit a cut-off property and it follows that if $\tilde{r}_t$ is a second-order weakly stationary AR(p) process, then the partial autocorrelations become zero beyond lag p, that is:

$$\pi(k) = 0, \quad \text{for all } k \geq p + 1,$$

This truncation property of the PACF, alongside the gradual decay of the ACF, serves as a practical tool for identifying the appropriate order p of an AR process in empirical analysis.

Moving Average (MA) Models

An MA(q) process models a time series as a linear combination of a white noise error term and its past values. It is expressed as:

$$r_t = c + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}, \tag{3}$$

where θ_i represents the Moving average coefficients.

Using lag operator B, the MA(q) process can be presented as:

$$\tilde{r}_t = \theta(B)\varepsilon_t, \quad \theta(x) = 1 - \sum_{i=1}^{q} \theta_i x^i. \tag{4}$$

The Moving Average process is a stationary process by construction.

The autocorrelation function (ACF) of an MA(q) process exhibits the following behavior:

$$\rho(k) = \begin{cases} \dfrac{-\theta_k + \sum_{i=1}^{q-k} \theta_i \theta_{i+k}}{1 + \sum_{i=1}^{q} \theta_i^2}, & \text{for } k = 1, 2, \ldots, q, \\ 0, & \text{for } k > q. \end{cases}$$

This truncation property is a defining characteristic of MA models and is especially useful in model identification. When the ACF cuts off sharply after lag q, it is an indicator that the process may follow an MA(q) structure.

In contrast, the partial autocorrelation function (PACF) of an MA(q) process does not truncate, but instead decays gradually, reflecting the diminishing marginal contribution of higher-order lags to the current value.

ARMA(p,q)

The ARMA(p, q) process is second-order weakly stationary and can be presented in the following forms:

$$\tilde{r}_t = \sum_{i=1}^{p} \phi_i \tilde{r}_{t-i} + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

Using lag operator B, the ARMA(p, q) process can be represented as:

$$\phi(B)\tilde{r}_t = \theta(B)\varepsilon_t,$$

where:

$$\phi(x) = 1 - \sum_{i=1}^{p} \phi_i x^i, \quad \theta(x) = 1 - \sum_{i=1}^{q} \theta_i x^i.$$

The $\phi(B)$ represents the autoregressive (AR) component, and $\theta(B)$ representing the moving average (MA) component,

The ARMA(p, q) process is second-order weakly stationary, if and only if the moduli of the roots x_i of the autoregressive polynomial $\phi(x)$ lie outside the unit circle.

The autocorrelation function (ACF) of an ARMA(p, q) process displays specific behavior depending on the values of p and q. For lags $k > q$, the autocorrelation coefficients $\rho(k)$ of the ARMA(p, q) process satisfy the following recursive relation:

$$\rho(k) = \sum_{i=1}^{p} \phi_i \rho(k-i), \quad k > q.$$

In the special case where $p = 1$, this simplifies to:

$$\rho(k) = \phi^{k-q} \rho(q), \quad k > q.$$

This result implies that for an ARMA($1, q$) process, the absolute values of the autocorrelation coefficients decay exponentially after lag q.

In an ARMA(p, q) process, the absolute values of the autocorrelation coefficients begin to decay after lag q. This behavior occurs because, beyond lag q, the influence of the moving average (MA) component fades, and the autocorrelation structure becomes primarily driven by the autoregressive (AR) part of the model. In the case of the stationary process roots of the AR characteristic polynomial lie outside the unit circle and the autocorrelations will gradually decline towards zero.

In contrast, the partial autocorrelation function (PACF) of an ARMA(p, q) process shows a different pattern, where the absolute values of the partial autocorrelation coefficients begin to decay slowly after lag p, reflecting the diminishing impact of the AR terms beyond this point.

ARIMA (p, d ,q) process

In time series analysis, the ARIMA(p, d, q) process, known as the Autoregressive Integrated Moving Average model, provides a powerful framework for modeling non-stationary time series data. The key feature of the ARIMA model lies in its ability to transform a non-stationary series into a stationary one through differencing.

Specifically, a time series r_t is said to follow an ARIMA(p, d, q) process if its d-th order differenced series $W_t = (1 - B)^d r_t$ becomes a stationary ARMA(p, q) process. In this context, the differencing operator $(1 - B)^d$ eliminates trends or other forms of non-stationarity present in the original series.

Mathematically, the ARIMA(p, d, q) model is expressed as:

$$\phi(B)(1 - B)^d r_t = \theta(B)\varepsilon_t,$$

This general form highlights that the ARIMA model integrates differencing to handle non-stationarity while maintaining the ARMA structure to capture serial dependence and shocks in the stationary transformed series.

Building upon the ARIMA framework, advanced models such as ARCH and GARCH have become particularly relevant for modeling conditional heteroskedasticity. While ARIMA models effectively capture the dynamics of the conditional mean, ARCH and GARCH models extend this framework by modeling the conditional variance, allowing for a more comprehensive representation of financial time series characterized by volatility clustering and time-varying risk.

1.2 ARCH and GARCH Models

Referenced from - Gebhard Kirchg¨assner and J¨urgen Wolters. [4].

1.2.1 ARCH Model

The ARCH model, introduced by Engle [6] in 1982, captures volatility clustering by linking current variance to past squared errors.

Let ε_t denote the return on a financial asset at time t. We assume that:

$$\varepsilon_t = \sigma_t z_t.$$

where $z_t \sim N(0,1)$, and in the case of ARCH (1) model, the σ_t^2 has the form:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2, \quad \omega > 0, \quad \alpha_1 \geq 0. \tag{5}$$

where ω is the baseline variance, and α_1 captures the impact of past squared residuals.

The ARCH(1) process has several properties for effective volatility modeling. In this context, $I_{t-1} = \{\varepsilon_{t-1}, \varepsilon_{t-2}, \dots\}$ denotes the set of previous returns, which captures historical shocks that shape the current conditional variance.

1. The ARCH(1) process has a zero mean. The conditional mean is given by:

$$E(\varepsilon_t \mid I_{t-1}) = \sigma_t E(z_t \mid I_{t-1}) = 0.$$

as $E(\varepsilon_t \mid I_{t-1}) = 0$. Using the law of iterated expectations (LIE), the unconditional mean is also zero:

$$E(\varepsilon_t) = E(E(\varepsilon_t \mid I_{t-1})) = 0.$$

2. The ARCH(1) process is serially uncorrelated as:

$$E(\varepsilon_t \varepsilon_{t-1}) = E(E(\varepsilon_t \varepsilon_{t-1} \mid I_{t-1})) = E(\varepsilon_{t-1} E(\varepsilon_t \mid I_{t-1})).$$

it follows that:

$$\mathrm{Cov}(\varepsilon_t, \varepsilon_{t-1}) = E(\varepsilon_t \varepsilon_{t-1}) = 0.$$

3. Conditional Variance of the ARCH (1) process.
The conditional variance of ε_t given the information set I_{t-1} (which includes all past residuals up to time $t-1$) is mathematically presented as:

$$D(\varepsilon_t \mid I_{t-1}) = E(\varepsilon_t^2 \mid I_{t-1}) = E(\sigma_t^2 z_t^2 \mid I_{t-1}) = \sigma_t^2 E(z_t^2 \mid I_{t-1}) = \sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2.$$

This equation shows that today's volatility σ_t^2 depends on a constant term ω and the magnitude of the previous period's shock ε_{t-1}^2. Larger shocks in the past lead to higher expected volatility today, which explains the persistence of volatility over time.

To analyze the properties of the variance equation, we define a disturbance term v_t as the deviation of the squared residual from its conditional expectation:

$$v_t = \varepsilon_t^2 - \sigma_t^2.$$

Taking the conditional expectation:

$$E(v_t \mid I_{t-1}) = E(\varepsilon_t^2 \mid I_{t-1}) - E(\sigma_t^2 \mid I_{t-1}).$$

Since $\sigma_t^2 = E(\varepsilon_t^2 \mid I_{t-1})$ it follows that:

$$E(v_t \mid I_{t-1}) = \sigma_t^2 - \sigma_t^2 = 0.$$

This implies that ω and α_1 can be estimated by regressing ε_t^2 onto an intercept and ε_{t-1}^2. Thus, ε_t^2 follows an AR(1) process in the form:

$$\varepsilon_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + v_t.$$

with:

$$E(v_t) = 0.$$

4. Unconditional Variance of the ARCH (1) Process.
While the conditional variance captures how volatility evolves based on past information, the unconditional variance represents the long-run average variance of the process, independent of time. It reflects the steady-state level of volatility that the process reverts to over time. To derive the unconditional variance of the ARCH(1) model, we take the expectation of both sides of the conditional variance equation.

$$D(\varepsilon_t) = E(\varepsilon_t^2) = E[E(\varepsilon_t^2 \mid I_{t-1})] = E[\omega + \alpha_1 \varepsilon_{t-1}^2] = \omega + \alpha_1 E(\varepsilon_t^2).$$

From this, it follows that if $0 < \alpha_1 < 1$, then:

$$D(\varepsilon_t) = E(\varepsilon_t^2) = \frac{\omega}{1 - \alpha_1}.$$

ARCH(p) Model

The ARCH(p) model generalizes the ARCH(1) model by including p-lagged squared error terms. Mathematically, the conditional variance σ_t^2 in the ARCH(p) process is expressed as:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2. \tag{6}$$

Similar to the ARCH(1), it follows that:

$$\varepsilon_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + v_t. \tag{7}$$

This equation indicates that the ARCH(p) process is equivalent to modeling ε_t^2 as an autoregressive process of order p.

The ARCH model addressed limitations of constant variance assumptions in traditional time series models, but it has its own limitations, such as complexity of estimating a large number of parameters, inefficiency in capturing long-term volatility, sensitivity to outliers, difficulty in modeling asymmetries, and potential issues with model specification. These drawbacks have led to the development of more sophisticated models like GARCH, which address many of the limitations of ARCH models.

1.2.2 GARCH Model

The Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model enhances the ARCH model by including lagged conditional variances, making it more flexible for modeling financial time series. The most commonly used GARCH(1,1) model is defined as:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{8}$$

where $\omega, \alpha_1, \beta_1$ are parameters. The ARCH term $\alpha_1 \varepsilon_{t-1}^2$ captures the effect of past shocks on current volatility, while the GARCH term $\beta_1 \sigma_{t-1}^2$ reflects the persistence of past volatility. For the process to be second order weak stationarity, the parameters must satisfy $0 < \alpha_1 + \beta_1 < 1$, and the unconditional variance is given by:

$$\mathrm{Var}(\varepsilon_t) = \frac{\omega}{1 - \alpha_1 - \beta_1}.$$

The model can be generalized to GARCH(p,q) which incorporates p lagged squared residuals and q lagged conditional variances to model the time-varying variance:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2. \tag{9}$$

The variance equation of the GARCH(p, q) model closely mirrors the structure of an ARMA$(\max(p, q), q)$ process applied to the squared residuals ε_t^2.

In the GARCH framework, the lagged squared residual terms ε_{t-i}^2 form the autoregressive (AR) component, capturing the persistence of past shocks, while the lagged conditional variance terms σ_{t-j}^2 constitute the moving average (MA) component, reflecting the influence of past forecast errors. This ARMA-like structure in the variance equation allows the GARCH model to effectively capture time-varying volatility, particularly the serial correlation and volatility clustering observed in financial time series. As such, the combined ARMA-GARCH approach provides a comprehensive framework for modeling both the conditional mean and conditional variance dynamics in financial return data.

GARCH-in-Mean Model

In the GARCH-in-Mean (GARCH-M) model, the relationship between risk and return is explicitly incorporated, reflecting the expectation that risk-averse investors demand higher returns (risk premium) for riskier assets. The model is formulated as:

$$r_t = d + \delta\sigma_t^2 + u_t, \tag{10}$$

where δ represents the risk premium parameter, typically positive, indicating that higher conditional variance (σ_t^2) leads to higher expected returns. The residual term is given by:

$$u_t = \sigma_t \varepsilon_t,$$

The conditional variance equation follows a standard GARCH(1,1) process:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

By introducing σ_t^2 into the mean equation, the GARCH-M model effectively captures the trade-off between risk and return, where greater volatility is associated with higher expected returns. The coefficient δ measures this relationship, reinforcing the theoretical foundation that investors require additional compensation for bearing higher levels of risk.

1.3 Advanced GARCH Models (Asymmetric GARCH)

Referenced from - Degiannakis, S. and Xekalaki. [5].
The GARCH model effectively captures important features of financial time series, however, a limitation of the standard GARCH model is its symmetry. The conditional variance σ_t^2 depends only on the magnitude of the innovations ε_t and not their sign. This conflicts with observed market behavior, where the leverage effect suggests that negative shocks (bad news) have more impact on future volatility than positive shocks (good news) of the same magnitude.

To address this limitation, asymmetric GARCH models were developed, incorporating terms to account for the leverage effect and market shocks. The simplest asymmetric GARCH model, as introduced by Glosten-Jagannathan-Runkle [10], modifies the traditional model to include a term for the leverage effect:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \gamma \varepsilon_{t-1}^2 Z_{t-1}, \tag{11}$$

Here, the indicator function Z_{t-1} is defined as:

$$Z_{t-1} = \begin{cases} 1, & \text{if } \varepsilon_{t-1} < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Where $\omega > 0$ is the baseline variance, $\alpha_1 \geq 0$ captures the impact of past squared innovations, $\beta_1 \geq 0$ measures the persistence of past volatility, and $\gamma \geq 0$ represents the asymmetric effect of negative innovations.

When the innovation ε_{t-1} is positive, its impact on σ_t^2 is determined by α_1. For negative ε_{t-1}, the impact increases by $\alpha_1 + \gamma$, capturing the leverage effect.

GJRGARCH (p,q)

The GJRGARCH (p,q) model further generalizes the basic model by allowing the effects of shocks on volatility across several lags. While in the simple model, only one lag of past squared innovations and volatility is included, the (p,q) model allows for p lags of both symmetric and asymmetric shocks and q lags of past volatility. The indicator function Z_{t-i} is set to detect negative shocks, which have a stronger impact on volatility than positive ones, improving the model's ability to analyze the time series data. The mathematical equation of the conditional variance is given by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} (\alpha_i \varepsilon_{t-i}^2 + \gamma_i \varepsilon_{t-i}^2 Z_{t-i}) + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{12}$$

TGARCH (p,q) (Threshold GARCH)

The Threshold GARCH (TGARCH) model, introduced by Zakoian (1994) [11], was designed to differentiate between the effects of positive and negative shocks on volatility which is achieved through a mechanism, that captures the asymmetry. Typically, negative shocks like a sharp decline in asset prices, result in a leverage effect.

Mathematically, the TGARCH(p, q) model can be represented as:

$$\sigma_t = \omega + \sum_{i=1}^{p} (\alpha_i \varepsilon_{t-i}^+) - \sum_{i=1}^{p} (\gamma_i \varepsilon_{t-i}^-) + \sum_{j=1}^{q} \beta_j \sigma_{t-j}, \tag{13}$$

Where the positive shocks (good news) contribute via $\alpha_i \varepsilon_{t-i}^+$ and the negative shocks (bad news) via $-\gamma_i\varepsilon_{t-i}^{-}$. If $\gamma_i > \alpha_i$, this implies that bad news has a stronger effect on future volatility than good news of the same magnitude, thereby capturing the leverage effect.

APARCH (p,q) (Asymmetric Power ARCH)

The Asymmetric Power ARCH (APARCH) model, developed by Ding et al. (1993) [20], incorporates both asymmetry and non-linearity in modeling volatility. Unlike standard GARCH models, which rely on a fixed quadratic relationship between past shocks and volatility, APARCH uses a power transformation enhancing its ability to account for volatility clustering and long-memory effects. This flexibility means that APARCH is not limited to a single structure, and encompasses various ARCH and GARCH specifications depending on the values assigned to its parameters.

The APARCH(p, q) model is formulated as:

$$\sigma_t^\delta = \omega + \sum_{i=1}^{p} \alpha_i \left(|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i}\right)^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta, \tag{14}$$

where $\delta > 0$ represents the Box-Cox power parameter, governing the transformation of the conditional standard deviation.

AVGARCH (p,q)

Taylor (1986) [9] introduced the Absolute Value GARCH (AVGARCH) model to capture the relationship between shocks and conditional volatility by using absolute deviations instead of squared residuals. This modification allows the model to better reflect both the magnitude and asymmetry of shocks in volatility dynamics. Schwert (1989) [13] further developed this approach by incorporating absolute value transformations into financial volatility models, emphasizing their ability to capture volatility clustering and nonlinear market behavior.

The AVGARCH(p, q) model is defined as:

$$\sigma_t = \omega + \sum_{i=1}^{p} \alpha_i \left(\left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \gamma_{2i} \right| - \gamma_{1i} \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \gamma_{2i} \right) \right) + \sum_{j=1}^{q} \beta_j \sigma_{t-j}, \tag{15}$$

AVGARCH models σ_t, allowing for a better representation of volatility, ensuring that the model captures the size of shocks without squaring them, making it more sensitive to large deviations in asset returns. The model introduces two asymmetry parameters for each lag i = 1, 2, ...: $\gamma_1 i$ which measures the leverage effect by allowing the model to differentiate between the effects of positive and negative shocks, while $\gamma_2 i$ acting as a threshold adjustment or shift parameter, adjusting the reference point of the asymmetry.

By responding directly to absolute shocks, the model provides a more flexible framework for periods of uncertainty and asymmetric market responses.

NGARCH (p,q)

The Nonlinear GARCH (NGARCH) model enhances traditional volatility modeling by incorporating a power parameter λ to allow for more flexible modeling of volatility dynamics. NGARCH allows for a non-linear transformation, which enables the model to capture asymmetric volatility responses and nonlinear behavior.

Mathematically, the NGARCH(p, q) model is given by:

$$\sigma_t^{\lambda} = \omega + \sum_{i=1}^{p} \alpha_i |\varepsilon_{t-i}|^{\lambda} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^{\lambda}. \tag{16}$$

This formulation introduces the power parameter λ, governing the transformation of volatility, and by adjusting the parameter, the model accommodates persistence and sensitivity to past shocks.

Higgins and Bera (1992) tested the NGARCH model on financial datasets, and the model outperformed standard GARCH models in capturing these asymmetries, making them more reflective of real market fluctuations.

NAGARCH (p,q)

The Nonlinear Asymmetric GARCH (NAGARCH) model, focuses on how volatility responds differently to positive and negative shocks by introducing a leverage term which adjusts past shocks before squaring them to reflect a more realistic representation of financial time series.

The NAGARCH model works by incorporating an adjustment term $\gamma\sigma_{t-i}$ which modifies past residuals before computing their squared impact on conditional variance.

The NAGARCH(p, q) model can be mathematically represented as:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \left(\varepsilon_{t-i} - \gamma \varepsilon_{t-i}\right)^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2. \tag{17}$$

EGARCH (p,q) (Exponential GARCH)

The Exponential GARCH (EGARCH) model, proposed by Nelson [8], offers a major improvement over traditional GARCH models. EGARCH models the logarithm of variance, which ensures a strictly positive variance without requiring non-negativity constraints on model parameters.

$$\log(\sigma_t^2) = \omega + \sum_{i=1}^{p}(\alpha_i \frac{\varepsilon_{t-i}}{\sigma_{t-i}} + \gamma_i \left(\left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| - E \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| \right)) + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j}^2), \tag{18}$$

EGARCH has the ability to account for asymmetries and volatility clustering in financial data. This is achieved by incorporating past standardized shocks and their expected values, enabling the model to capture both the magnitude and direction of past returns, differentiating between unexpected upward and downward movements.

CSGARCH (p,q)

The Component GARCH (CSGARCH) model, developed by Engle and Lee (1999) [16], splits volatility into two components: a permanent (long-term) component q_t and a transitory (short-term) component $\sigma_t^2 - q_t$. This separation allows distinguishing between persistent volatility trends and temporary fluctuations caused by market shocks.

Mathematically, the CSGARCH(p, q) model is represented as:

$$\sigma_t^2 = q_t + \sum_{i=1}^{p} \alpha_i \left(\varepsilon_{t-i}^2 - q_{t-i} \right) + \sum_{j=1}^{q} \beta_j \left(\sigma_{t-j}^2 - q_{t-j} \right). \tag{19}$$

Here, q_t represents the long-term volatility component, which evolves as an autoregressive process:

$$q_t = \omega + \rho q_{t-1} + \phi \left(\varepsilon_{t-1}^2 - q_{t-1} \right).$$

This formulation ensures that short-term volatility reverts to long-term volatility over time. The term $\varepsilon_{t-1}^2 - q_{t-1}$ determines how unexpected past shocks influence the long-term variance.

In contrast to the GARCH model which assumes that the mean returns to the unconditional variance is constant for all time, the CSGARCH allows the mean returns to a time varying level q_t. $\sigma_t^2 - q_t$, that is the difference between the conditional variance and its trend, represents the short run component or the transitory component of the conditional variance.

In periods of financial distress, the transitory component $\sigma_t^2 - q_t$ exhibits higher fluctuations, while the long-term component q_t ensures that volatility remains elevated over extended periods, making the model useful for modeling financial markets, where volatility spikes tend to persist due to structural changes in market behavior.

ALLGARCH

The ALLGARCH model is an extension of standard GARCH frameworks, and it is a hybrid structured model that allows it to account for a wide range of volatility behaviors. The model brings together features from several GARCH models into a single flexible framework by allowing for transformations and asymmetry parameters to improve its ability to describe volatility.

Mathematically, the ALLGARCH model can be expressed as:

$$\sigma_t^{\lambda} = \omega + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^{\lambda} \left(\left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \gamma_{2i} \right| - \gamma_{1i} \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \gamma_{2i} \right) \right)^{\lambda} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^{\lambda}, \tag{20}$$

FGARCH Model

The Family GARCH (FGARCH) model, introduced by Hentschel (1995) [12], offers a versatile framework by combining non-linearity, asymmetry, and leverage effects. FGARCH introduces a power transformation that allows for varying degrees of persistence and sensitivity to shocks, giving the model the flexibility to adapt to various market conditions.

Mathematically, the FGARCH(p, q) model is expressed as:

$$\sigma_t^{\lambda} = \omega + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^{\lambda} \left(\left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \gamma_{2i} \right| - \gamma_{1i} \left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \gamma_{2i} \right) \right)^{\delta} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^{\lambda}, \tag{21}$$

This formulation introduces the power parameter λ, the leverage effect parameter γ_i accounts for asymmetry, and the exponent δ, which further enhances flexibility by allowing for non-linear relationships between past shocks and future volatility.

FGARCH can simulate different GARCH models through parameter choices, making it highly adaptable. Each of these models is defined by distinct parameter configurations, allowing for tailored representations of volatility dynamics as outlined below:

- **GARCH Model:** $\lambda = \delta = 2, \gamma_{1i} = \gamma_{2i} = 0$ (see Equation 8).
  Captures symmetric volatility where past squared residuals and conditional variances explain future volatility with no asymmetry or shift.

- **Absolute Value GARCH (AVGARCH) Model:** $\lambda = \delta = 1, |\gamma_{1i}| \leq 1, \gamma_{2i} \leq 1$ (see Equation 15).
  Models volatility clustering using absolute residuals; allows asymmetry via γ_{1i} and shift adjustment via γ_{2i}.

- **GJR-GARCH Model:** $\lambda = \delta = 2, \gamma_{1i} \neq 0, \gamma_{2i} = 0$ (see Equation 12).
  Introduces leverage effect by allowing negative shocks to affect volatility more than positive ones.

- **Threshold GARCH (TGARCH):** $\lambda = \delta = 1, \gamma_{1i} \neq 0, \gamma_{2i} = 0$ (see Equation 13).
  Differentiates impact of positive vs. negative shocks through a threshold mechanism with asymmetry.

- **Nonlinear ARCH (NGARCH) Model:** $\delta = \lambda, \gamma_{1i} = \gamma_{2i} = 0$ (see Equation 16). Introduces nonlinear effects using a power parameter, but does not account for asymmetry or shifting.

- **Nonlinear Asymmetric GARCH (NAGARCH) Model:** $\delta = \lambda = 2, \gamma_{1i} \neq 0, \gamma_{2i} = 0$ (see Equation 17).
  Adds asymmetry through a leverage term but without a centering shift.

- **Asymmetric Power ARCH (APARCH) Model:** $\delta = \lambda, \gamma_{1i} \neq 0, \gamma_{2i} = 0$ (see Equation 14).
  Combines asymmetry and nonlinearity; γ_{1i} controls the sign effect, but there's no shift component.

- **ALLGARCH Model:** $\delta = \lambda, \gamma_{1i} \neq 0, \gamma_{2i} \neq 0$ (see Equation 20).
  Integrates all effects asymmetry, persistence, shift, and nonlinear transformations for comprehensive modeling of volatility behavior.

1.4 Testing for Conditional Heteroskedasticity

In testing for conditional heteroskedasticity in time series models, it is necessary to determine if any ARCH (Autoregressive Conditional Heteroskedasticity) or GARCH effects are left in the residuals of the fitted models. One possibility involves analyzing the squared residuals ($\hat{\epsilon}_t^2$) and starts by calculating the sample variance of these residuals as:

$$\hat{\sigma}^2 = \sum_{t=1}^{T} \frac{\hat{\varepsilon}_t^2}{T},$$

Next, calculate and plot the sample autocorrelations of the squared residuals using the formula:

$$\rho_i = \frac{\sum_{t=i+1}^{T} (\hat{\varepsilon}_t^2 - \hat{\sigma}^2)(\hat{\varepsilon}_{t-i}^2 - \hat{\sigma}^2)}{\sum_{t=1}^{T} (\hat{\varepsilon}_t^2 - \hat{\sigma}^2)^2},$$

For larger samples, the standard deviation of ρ_i can be approximated by $1/\sqrt{T}$. Significant deviations of ρ_i from zero suggest the presence of ARCH-GARCH effects. To test for these effects, we use the Ljung-Box test, which evaluates whether groups of autocorrelations are significantly different from zero.

Another possibility is the Lagrange Multiplier test (LM Test) which involves estimating :

$$\hat{\varepsilon}_t^2 = w + \alpha_1 \hat{\varepsilon}_{t-1}^2 + \alpha_2 \hat{\varepsilon}_{t-2}^2 + \cdots + \alpha_p \hat{\varepsilon}_{t-p}^2,$$

Under the null hypothesis that there are no ARCH or GARCH effects, the coefficients

α_i should all be zero. The test statistic, $T \times R^2$, where R^2 is the coefficient of determination from the regression, follows a chi-square distribution with p degrees of freedom.

The Sign Bias Test checks if positive and negative shocks have different effects on future volatility. Specifically, it tests whether the model's conditional variance responds differently to positive and negative lagged residuals.

The test can be implemented by estimating the following auxiliary regression:

$$\varepsilon_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^{-} \varepsilon_{t-1} + \varepsilon_t,$$

where: Z_{t-1} is a dummy variable that equals 1 if $\varepsilon_{t-1} < 0$ (indicating a negative shock) and 0 otherwise, γ_0 is the intercept term andγ_1 is the coefficient of the lagged residual associated with the sign of the shock.

The Null Hypothesis (H_0): $\gamma_1 = 0$ implies that there is no sign bias—positive and negative shocks have symmetric effects on volatility.

And Alternative Hypothesis (H_A): $\gamma_1 \neq 0$ implies that there is sign bias negative and positive shocks have different effects on volatility.

If γ_1 is significantly different from zero, it suggests that the model exhibits sign bias, indicating that the model does not fully capture the asymmetry in the volatility response to positive and negative shocks.

Model selection

In conditional heteroskedasticity modeling, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are pivotal for model selection as they strike a balance between model complexity and fit as discussed in Claeskens and Hjort [17].

Akaike Information Criterion (AIC)

The AIC evaluates the relative quality of statistical models, combining data fit and model simplicity. It is useful for comparing different conditional heteroskedasticity models and can be represented as:

$$\text{AIC} = 2k - 2\ln(L), \tag{22}$$

where k denotes the number of parameters, and L represents the maximum likelihood of the model. A lower AIC value indicates a preferred model.

Bayesian Information Criterion (BIC)

Similar to AIC but with a stronger penalty for models with more parameters, BIC is particularly effective with large datasets and helps to avoid overfitting in complex models. Mathematically:

$$\text{BIC} = \ln(n)k - 2\ln(L), \tag{23}$$

where k is the number of parameters, n is the number of observations, and L is the maximum likelihood of the model. A lower BIC value indicates a preferable model.

Both AIC and BIC are instrumental in selecting a well-fitted model for analyzing volatility, ensuring that the chosen model explains the data without unnecessary complexity hence, the model with the lowest AIC or BIC value is generally preferred.

1.5 Volatility modeling of stock prices, Literature Review

Empirical studies consistently show how powerful ARCH and GARCH models are in capturing the complex nature of volatility in financial markets. The work by Sohail, Shahid, and Imran (2012)[15] applied GARCH models to share price data from a Muslim commercial bank in Pakistan, and found them effective in modeling persistent volatility and heavy tails. These features which commonly seen in financial time series. The study emphasized the importance of using distributions like the Student's t-distribution, to better capture the extreme values, thereby providing more robust estimates of volatility. Similarly, Atoi in 2014 [14] studied the Nigerian stock market, where the TGARCH model proved particularly good at detecting leverage effects, highlighting the influence of negative shocks on volatility compared to positive shocks of similar magnitude. These findings underscore the importance of selecting models that are suited to the specific characteristics of financial data, especially in emerging markets.

More recent studies have expanded the application of GARCH models to broader and more interconnected markets. For instance, the Asian Economic Financial Review (2023) [23] examined regional indices like the FTSE, Bursa Malaysia KLCI, Indonesia's LQ45, and Thailand's SET, using daily historical price data. The study used univariate GARCH and ARMA-GARCH models, and found EGARCH to be particularly strong in capturing asymmetric effects and other nonlinear behaviors in returns. Furthermore, the results indicated significant correlations and bidirectional causality among the indices, highlighting the need for advanced coordination and predictive tools to enhance market efficiency and the reliance of sophisticated GARCH variants for addressing the complexities of interconnected markets.

In developed markets, advanced econometric techniques have also proven effective in addressing volatility clustering and asymmetry in developed markets. Engle and Lee [16] introduced the Component GARCH (CSGARCH) model, which separates volatility into permanent and transitory components. This decomposition is relevant for markets with cyclical behaviors or enduring shocks, as seen in studies of the US and European stock markets during the global financial crisis.

While GARCH models are robust, their application to smaller, regionally sensitive markets like the Baltic region encounters challenges due to external vulnerabilities. In such cases, advanced and asymmetric GARCH models have proven more suitable, as they can better reflect how volatility reacts to different kinds of market shocks. For example, Zhang et al. (2021)[18] demonstrated that such models were especially effective in capturing the heightened volatility and uncertainty triggered by the COVID-19 pandemic on global stock markets. Their findings are particularly relevant for markets like the Baltic region, which experienced significant disruptions due to the pandemic.

2 Rugarch package in R

The `rugarch` package in R documented by Alexios Ghalanos [24], provides a robust and efficient framework for specifying and estimating GARCH models, facilitating the modeling of conditional heteroskedasticity in financial time series data. Its core functions, `ugarchspec` and `ugarchfit`, streamline the process of defining and estimating both mean and variance equations while accommodating flexible error distribution specifications. The short specifications of the ugarchspec and ugarch fit functions are as follows:

```
ugarchspec(variance.model = list(model = "sGARCH", garchOrder = c(1,1),
                                 submodel = "-", external.regressors = NULL,
                                 variance.targeting = FALSE),
           mean.model = list(armaOrder = c(0,0)),include.mean = TRUE,
                                     archm = TRUE, archpow = 2),
                   distribution.model = "std")

ugarchfit(data = returns, spec = spec, solver = "hybrid").
```

The `ugarchspec` function serves as the foundation for defining the structure of a GARCH model, enabling users to specify the variance model, mean model, and residual error distribution. The variance model includes the parameter `garchOrder = c(p, q)`, which captures lagged squared residuals (ARCH terms) and conditional variances (GARCH terms), essential for modeling volatility clustering.

The `model` parameter determines the specific GARCH variant to be estimated, offering the following options:

- **GARCH**: The standard GARCH model, which assumes symmetric effects of shocks on volatility.

- **EGARCH**: Exponential GARCH, which captures asymmetries in volatility responses, effectively modeling the leverage effect where negative shocks disproportionately impact volatility (see Equation 18).

- **CSGARCH**: Component GARCH decomposes volatility into permanent and transitory components, making it ideal for distinguishing long-term trends from short-term fluctuations (see Equation 19).

- **FGARCH**: The Family GARCH (FGARCH) model (see Equation 21), introduced by Hentschel [12], provides a unified framework for modeling diverse volatility dynamics, including symmetry, asymmetry, leverage effects, and non-linearity. Common GARCH variants emerge as special cases, defined using the `submodel` parameter. These include the GARCH model (symmetric dynamics), AVGARCH (absolute residuals for asymmetry), GJR-GARCH (leverage effects), TGARCH (threshold effects), NGARCH (non-linearity with symmetry), NAGARCH (asymmetry via residual shifts), APARCH (combined asymmetry and non-linearity) and ALLGARCH, are all implemented through the `submodel` parameter in the `ugarchspec` function.

The mean model, defined by `armaOrder = c(p, q)`, incorporates autoregressive (AR) and moving average (MA) terms to capture temporal dependencies in the time series, improving the accuracy of volatility modeling. The `distribution.model` parameter allows users to select error distributions, such as the Normal distribution, Student's t-distribution, or Generalized Error Distribution (GED), enhancing the model's ability to handle heavy tails and extreme values.

The GARCH-M model can also be estimated using the `ugarchspec` function in R. In this setup, the argument `archpow = 2` specifies the power applied to the conditional standard deviation, while `archm = TRUE` activates the GARCH-in-Mean component. The resulting output includes parameter estimates including the risk premium parameter δ (see Equation 10).

Once the model structure is specified, the `ugarchfit` function estimates the parameters through Maximum Likelihood Estimation (MLE), optimizing both the mean and variance equations to align with the data. For example, in a GARCH(1,1) model, the estimation includes one ARCH term to account for past squared residuals and one GARCH term to address lagged conditional variances. The fitting of the model using the `"hybrid"` solver enhances the likelihood of convergence.

The output from `ugarchfit` includes parameter estimates, diagnostic statistics, and information criteria such as Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), which help evaluate model performance and fit. Diagnostic tests like the Ljung-Box test for residuals and squared residuals ensure that the model effectively captures autocorrelation and conditional heteroskedasticity. These rigorous diagnostics validate that the chosen GARCH specification aligns with the underlying dynamics of the data.

In the 'rugarch' package, parameters are designed to keep things clear and consistent when setting up GARCH models. The mean equation, μ represents the expected return or the drift term. The Variance Equation Parameters $(\omega, \alpha_i, \beta_j)$, capture Variance Equation Parameters $(\omega, \alpha_i, \beta_j)$ how shocks influence current volatility. For asymmetry, there are two types: 'eta11', 'eta12', etc.(in our equations 'gammaa11', 'gamma12'), handle rotational asymmetry (Asymmetry1), while 'eta21', 'eta22' (in our equations 'gammaa21', 'gamma22'), and so on address shift asymmetry (Asymmetry2). However, in the case of CSGARCH, the parameters, 'gammaa11' and 'gamma21' represent ρ and ϕ which account for the persistence of the long-term component and response of the long-term component to shocks respectively. Parameters like 'delta' (asymmetry power) and 'lambda' (conditional sigma power) add flexibility for modeling more complex volatility behaviors.

Tests from `rugarch`

The `rugarch` package in R includes several diagnostic tests that help verify whether a fitted GARCH model behaves as expected. One important test is the Weighted Ljung-Box Test, which checks for autocorrelation in the standardized residuals. Ideally, after a GARCH model is properly specified, it should eliminate autocorrelation, leaving residuals that behave like white noise. A high p-value here suggests the model has successfully removed autocorrelation from the residuals.

Another version of this test focuses on Squared Residuals, examining if the model has adequately captured the volatility dynamics. Again, the squared residuals should not display any autocorrelation and a high p-value indicates that the GARCH model has adequately accounted for these dynamics, with no remaining volatility patterns in the residuals.

The ARCH LM test is used to detect if their are any remaining ARCH effects in the residuals after fitting the model. High p-values indicate that the model is adequate and captures conditional heteroskedasticity, however, significant results suggest the need for a more complex model or additional terms to capture the volatility structure.

The sign bias test is relevant for assessing asymmetry in GARCH models. Asymmetric It helps determine whether positive and negative shocks have different impacts on future volatility, that is, it checks for leverage effect. If significant sign bias is detected, it means that a standard GARCH model may need to be extended to an asymmetric variant to capture the volatility dynamics.

Conditional Distributions

The `rugarch` package allows users to specify for various univariate distributions, such as the Normal ('norm') and Student's t ('std') distributions, which can be selected using the `distribution.model` option within the `ugarchspec` function.

The Normal distribution assumes returns are symmetrically distributed, defined by its mean (μ) and variance (σ^2), and is characterized by zero skewness and no excess kurtosis.. The probability density function for a normally distributed variable x is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

In the context of GARCH models, after removing the mean and scaling by conditional variance, the residuals ϵ, when normalized by σ, follow a standard normal distribution:

$$f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} f(z) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2},$$

The conditional likelihood at each time step, denoted by LL_t, uses the conditional standard deviation σ_t as a scaling factor:

$$LL_t(z_t;\sigma_t) = \frac{1}{\sigma_t} f(z_t),$$

The Student's t distribution, offers an alternative to the Normal distribution for modeling standardized innovations. This distribution adds a shape parameter ν, with the probability density function expressed as:

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\beta\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{(x-\alpha)^2}{\beta\nu}\right)^{-\frac{\nu+1}{2}},$$

Here, α represents the location parameter, β the scale parameter, with Γ representing the Gamma function. The variance for this distribution is:

$$\mathrm{Var}(x) = \frac{\beta\nu}{\nu-2},$$

For standardization, the variance is set to 1 by selecting:

$$\beta = \frac{\nu-2}{\nu},$$

Substituting this value into the density function results in the standardized Student's t distribution:

$$f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} f(z) = \frac{1}{\sigma} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{(\nu-2)\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{z^2}{\nu-2}\right)^{-\frac{\nu+1}{2}},$$

In R, this distribution is implemented through the `dt` function, adjusted for the standard deviation:

$$dt\left(\frac{\epsilon_t}{\sigma\sqrt{\frac{\nu-2}{\nu}}}, \nu\right) \times \frac{1}{\sigma\sqrt{\frac{\nu-2}{\nu}}},$$

The Student's t distribution with its zero skewness and excess kurtosis of $\frac{6}{\nu-4}$ for $\nu > 4$ is particularly effective for data exhibiting heavy tails..

3 Modeling Financial Time series of Prices

3.1 Tallink Grupp Stock Prices

To begin modeling, we sourced from Nasdaq.com the stock price data of Tallink Group, covering the period from February 1, 2007, to October 10, 2023. Figure 1 illustrates the time series of raw daily closing stock prices, showing clear signs of non-stationarity, making the raw prices inaccurate for time series modeling.

To resolve this, a differenced logarithmic transformation of prices was applied:

$$\Delta \log P_t = \log P_t - \log P_{t-1},$$

where P_t represents the daily closing prices. This transformation allowed the time series to meet the stationarity requirement, as illustrated in Figure 2, with a stabilized mean. However, Figure 3 and 4 identified specific lags over the confidence limit, indicating that there may be autocorrelation. These findings highlight that simple time series models may not fully capture the data's intricate dynamics.

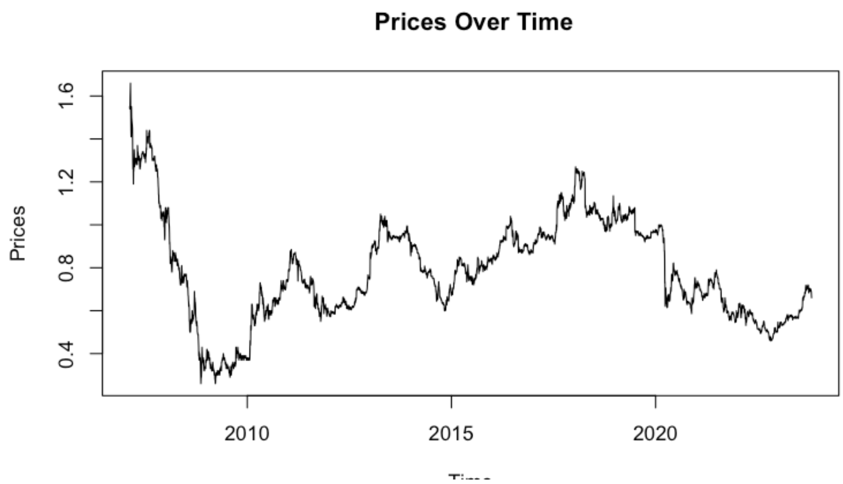

Figure 1: Tallink Group Stock Data

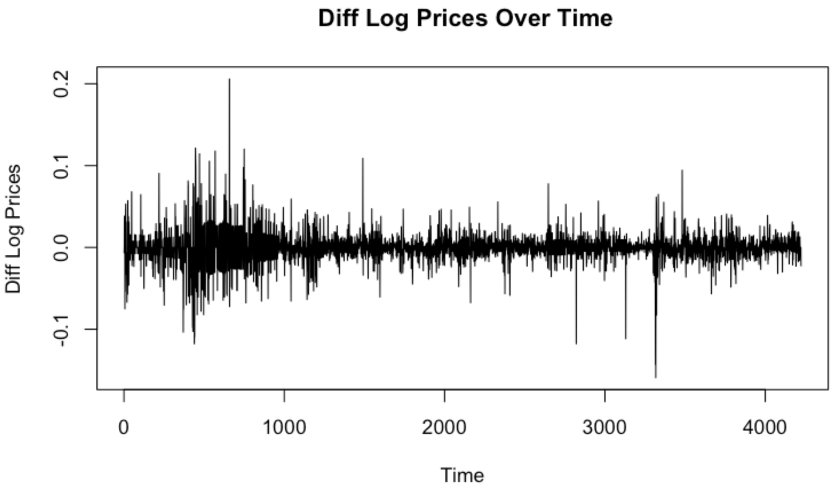

Figure 2: Differenced-Log Prices

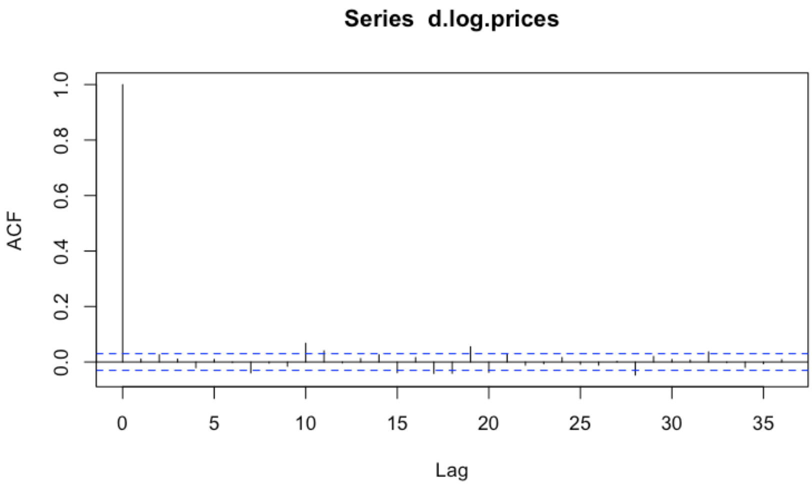

Figure 3: ACF

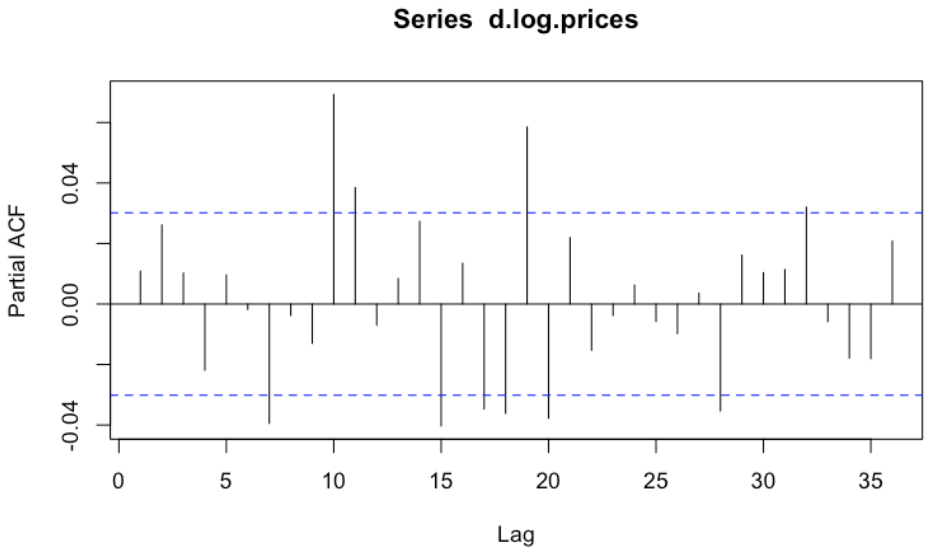

Figure 4: PACF

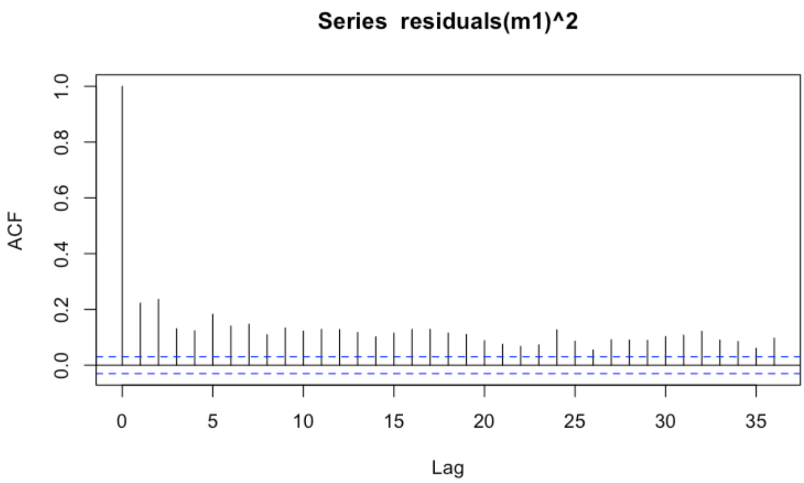

Figure 5: ACF of squared residuals

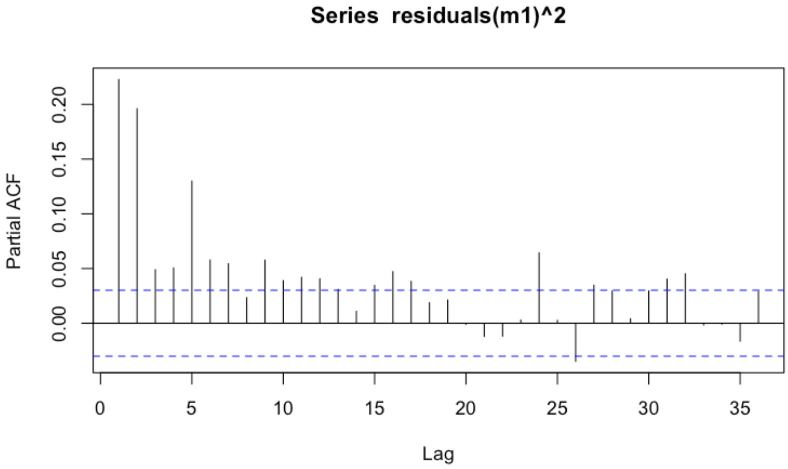

Figure 6: PACF of squared residuals

To model possible autocorrelations, ARIMA (Autoregressive Integrated Moving Average) models were tested with the logarithmic prices $logP_t$ by incorporating autoregressive (AR) and moving average (MA) components. Among the tested models, ARIMA(0,1,0) with an AIC of -21654.79, BIC -21648.44 and ARIMA(3,1,8) with an AIC of -21665.06, BIC -21588.88 demonstrated good performance. However, ARIMA(0,1,0) was preferred for its simplicity, fewer parameters, and lesser BIC value. The ARIMA(0,1,0) model represents the simplest case within the ARIMA framework with no autoregressive (AR) or moving average (MA) components, and applies first-order differencing to the series. Here, the current value of the series depends solely on the previous value plus a random shock, with no influence from past residuals. Despite this, squared residual analysis revealed persistent autocorrelation with signs of heteroskedasticity, as evident from the ACF and PACF plots on Figure 5 and 6, prompting the use of more advanced volatility modeling approaches, such as ARCH and GARCH.

To address heteroskedasticity, the simplest model ARCH (Autoregressive Conditional Heteroskedasticity) model was introduced with ARMA(0,0) serving as the baseline model. In the normal distribution, ARCH(7) performed well with an AIC of -5.5296 and BIC of -5.5161, however, ARCH(8) yielded superior results under the Student's t-distribution with an AIC of -5.7862 and BIC -5.7697, as evidenced by lower AIC and BIC values.

ARCH(8):

$$\begin{aligned}\sigma_t^2 &= 0.000059 + 0.423101\epsilon_{t-1}^2 + 0.161184\epsilon_{t-2}^2 + 0.083330\epsilon_{t-3}^2 \\ &\quad + 0.047410\epsilon_{t-4}^2 + 0.048554\epsilon_{t-5}^2 + 0.086040\epsilon_{t-6}^2 \\ &\quad + 0.085288\epsilon_{t-7}^2 + 0.064092\epsilon_{t-8}^2.\end{aligned}$$

To further confirm our choice of distribution, a visualization using the qqplot was done by fitting a normal distribution and a Student-t distribution to the differenced log returns.

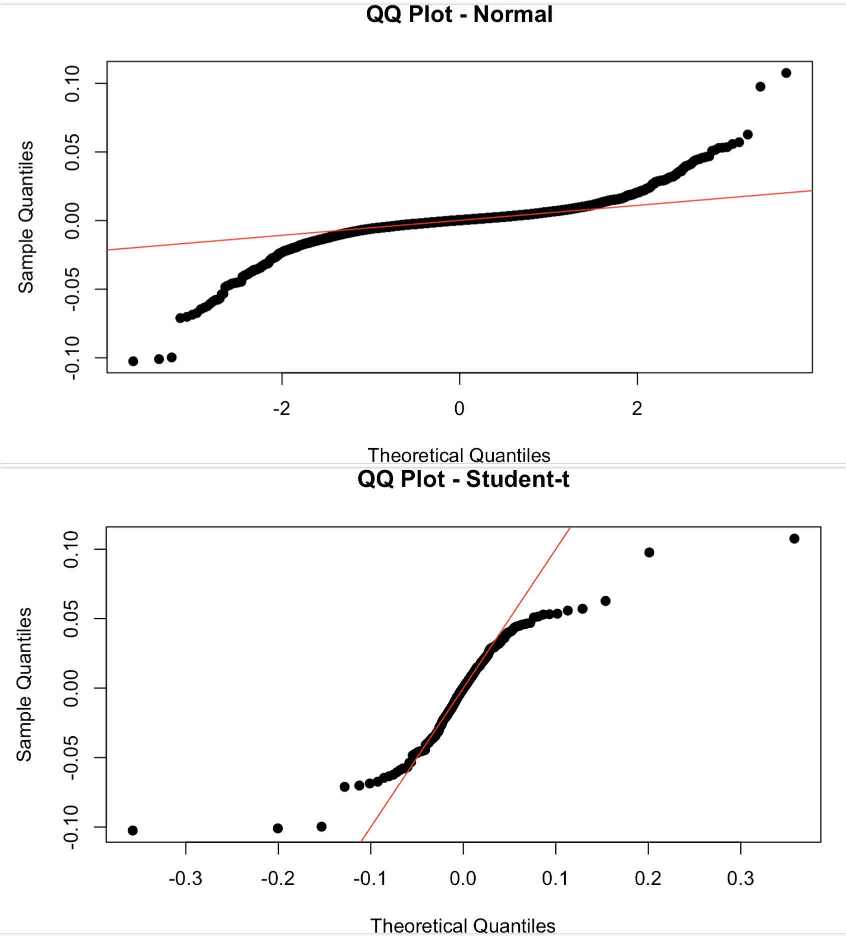

Figure 7: Comparison of Normal and Student-t distribution fits for differenced log returns.

The QQ plot comparison showed that the returns deviate significantly from the normal distribution, indicating heavier tail behavior. While the Student-t distribution provided a much better fit, aligning more closely with the empirical quantiles, particularly in the center of the distribution, suggesting the Student-t distribution is the more appropriate distribution for modeling the returns.

The ARCH model passed the ARCH-LM tests (p-values greater than 0.05) indicating that the ARCH effects were fully captured, the Ljung-Box tests on the residuals (p-values greater than 0.05) confirming that there were no autocorrelation left in the residuals, and the sign test with p-values greater than 0.05, confirming that there were no leverage effects (See Table 1). Due to the large number of parameters presented by the ARCH model, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model was introduced.

Building on the ARCH model, GARCH(1,1) model was applied with the ARMA(0,0) serving as the baseline model. This model effectively resolved autocorrelation and conditional heteroskedasticity with fewer parameters, providing a more parsimonious representation of volatility dynamics.

GARCH(1,1):

$$\sigma_t^2 = 0.000009 + 0.195755\epsilon_{t-1}^2 + 0.803245\sigma_{t-1}^2.$$

A further concern arose regarding the impact of non-traded days on model performance. Here, estimated returns for non-traded days were incorporated into the dataset. The interpolation formula used was:

$$\Delta \log \tilde{P}_i = (1 - d \cdot N_t) \cdot \Delta \log P_i \tag{24}$$

where $\Delta \log \tilde{P}_i$ represents the adjusted price on day i, N_t: Number of non-trading days between D_{i-1} and D_i, d; the discount factor per non-trading day set to $d = 0.2$ to represent the adjustment of 20%. This formula implies that if there's a gap between trading days, the next price is discounted by 20%.

After including these estimates, the parameters from the results of the adjusted dataset closely aligned with those based solely on trading days, indicating that the non-trading days had minimal impact on model performance. (see Table A5 in Appendix).

The GARCH-in-Mean (GARCH-M) model was applied to assess whether increased volatility, interpreted as financial risk, is associated with higher expected returns. The parameter of primary interest, `archm`, reflects the risk premium coefficient (δ) incorporated into the mean equation. Although the parameter estimate for `archm` is positive, suggesting that higher volatility could lead to higher returns, the estimation results show that it is not statistically significant (p-value > 0.05), indicating no strong evidence of a risk-return trade-off within the data. This suggests that investors in this market do not systematically demand higher returns in exchange for bearing greater volatility. (See Table A6).

Advanced GARCH models were also tested, and among these, the Component GARCH (CSGARCH(1,1)) model demonstrated superior performance. The parameter (γ_{11}) shows the persistence of the long-term component q_t and (γ_{21}), the response of long-term component to shocks. The p-values below 0.05 indicates that they are statistically significant. And like in the case of the GARCH (1,1) model, the ARCH-LM tests indicated that the ARCH effects were fully captured, the Ljung-Box tests on the residuals confirmed that there was no autocorrelation left in the residuals, and the sign test indicated no leverage effects. Empirical results demonstrated that the CSGARCH(1,1) model outperformed both the ARCH(8) and GARCH(1,1) models, and offered the best overall fit to the data with the lowest AIC and BIC values.

CSGARCH:

$$\sigma_t^2 = q_t + 0.276242(\varepsilon_{t-1}^2 - q_{t-1}) + 0.323941(\sigma_{t-1}^2 - q_{t-1}).$$

And

$$q_t = 0.000002 + 0.998750 q_{t-1} + 0.084898(\varepsilon_{t-1}^2 - q_{t-1})$$

Table 1 summarizes the performance metrics of the tested models, including ARCH(8), GARCH(1,1), and the best-performing CSGARCH(1,1), providing a comprehensive comparison of their effectiveness.

The asterisks (***, **, *) denote significance levels at 1%, 5%, and 10%, respectively.

Table 1: **Model Results and Diagnostic Tests for ARCH(8), GARCH(1,1), and CSGARCH(1,1)**

Parameters / Tests	**arch(8)**		**garch(1,1)**		**csgarch(1,1)**	
Parameter Estimates						
mu	-0.000319	*	-0.000284	*	-0.000298	**
omega	0.000059	***	0.000009	**	0.000002	-
alpha1	0.423101	***	0.195755	***	0.276242	***
alpha2	0.161184	***	-	-	-	-
alpha3	0.083330	**	-	-	-	-
alpha4	0.047410	**	-	-	-	-
alpha5	0.048554	**	-	-	-	-
alpha6	0.086040	**	-	-	-	-
alpha7	0.085288	***	-	-	-	-
alpha8	0.064092	**	-	-	-	-
beta1	-	-	0.803245	***	0.323941	***
gamma11	-	-	-	-	0.998750	-
gamma21	-	-	-	-	0.084898	***
Shape	3.393750	***	3.412024	***	3.589824	***
Information Criteria						
Akaike	-5.7862	-	-5.7897	-	-5.8062	-
Bayes	-5.7697	-	-5.7821	-	-5.7957	-
Ljung-Box Standardized Residuals (p-value)						
Lag[1]	0.1647	-	0.1911	-	0.14873	-
Lag[2*(p+q)+(p+q)-1][2]	0.1299	-	0.1355	-	0.09953	-
Lag[4*(p+q)+(p+q)-1][5]	0.1533	-	-	-	0.10144	-
Ljung-Box Squared Residuals (p-value)						
Lag[1]	0.3423	-	0.9045	-	0.4561	-
Lag[2*(p+q)+(p+q)-1][5]	0.9746	-	0.6778	-	0.7048	-
Lag[4*(p+q)+(p+q)-1][9]	0.9644	-	0.7579	-	0.8974	-
ARCH LM Tests (p-value)						
ARCH Lag[3]	0.9549	-	0.4491	-	0.6660	-
ARCH Lag[5]	0.9384	-	0.4607	-	0.8463	-
ARCH Lag[7]	0.9804	-	0.5717	-	0.9394	-
Sign Bias Test (p-value)						
Negative Sign Bias	0.7838	-	0.7285	-	0.6439	-
Positive Sign Bias	0.0630	-	0.6625	-	0.1941	-
Joint Effect	0.1295	-	0.7941	-	0.4894	-

To comprehensively assess the performance and adaptability of the selected volatility models, the dataset was divided into four distinct time periods: Period 1 (01 February 2007–30 December 2010), characterized by post-global financial crisis adjustments; Period 2 (03 January 2011–30 December 2019), reflecting a relatively stable economic phase; Period 3 (Pre-COVID: 01 February 2007–30 December 2019), combining the first two periods for a broader pre-pandemic analysis; and Period 4 (Post-COVID: 02 January 2020–10 October 2023), defined by the economic volatility and uncertainty brought on by the COVID-19 pandemic. This segmentation enabled a detailed analysis of the models' performance under varying market conditions.

The results revealed that different models performed best across the various periods. During Periods 1 and 3, both the AVGARCH(1,1) and ALLGARCH(1,1) models demonstrated strong performance. However, AVGARCH(1,1) was the preferred model based on its lower BIC values and the reliability of its parameter estimates. In Period 1, the asymmetry parameter was positive, suggesting the presence of a leverage effect. While in Period 3, the asymmetry parameter was negative, suggesting that positive shocks increased volatility more than negative shocks. These outcomes are consistent with findings by Ding et al. (1993) [20] and Brooks and Persand (2001) [22], who talked about the asymmetric GARCH-type models in capturing changing volatility behaviors under different market conditions. In Period 2, the CSGARCH(1,1) model outperformed others, as indicated by both AIC and BIC, with positive significant parameters reflecting how the model captures volatility persistence and short-term volatility shocks.

In Period 4, representing the post-COVID era, the GARCH(1,1) model outperformed all other models. The significant market uncertainties and disruptions caused by the pandemic created an environment where simpler symmetric models like GARCH(1,1) excelled. As noted by Bollerslev (1986)[7], GARCH(1,1) is highly effective at adapting to rapidly changing variance dynamics, capturing the core features of conditional variance without relying on extensive asymmetry modeling. Its performance during this period underscores its suitability for scenarios involving extreme volatility shocks and rapidly evolving market behaviors.

The variation in best-performing models across different time periods arises from the distinct market dynamics and volatility patterns defining each phase. In shorter periods, such as post-crisis adjustments or stable economic phases, models like AVGARCH and GARCH(1,1) excel due to their ability to capture localized asymmetric effects or simpler symmetric volatility dynamics. However, for the full dataset spanning 01 February 2007 to 10 October 2023, the CSGARCH model outperforms others as it effectively handles the complexity of long-term and short-term volatility components across diverse economic conditions, including the Global Financial Crisis and the COVID-19 pandemic.

Table 2 presents the best-performing models across various time periods. Period 1(AVGARCH(1,1)), Period 2(CSGARCH(1,1)), Period 3(AVGARCH(1,1)), Period 4(GARCH(1,1))

Table 2: **Best Model Results Across Periods**

	Period 1 avgarch(1,1)		**Period 2 csgarch(1,1)**		**Period 3 avgarch(1,1)**		**Period 4 garch(1,1)**	
Parameter								
mu	-0.001030	**	-0.000160	-	-0.000275	**	-0.000686	-
omega	0.000176	-	0.000007	***	0.000352	***	0.000020	**
alpha1	0.133305	***	0.360836	***	0.195693	***	0.258588	***
beta1	0.912539	***	0.128966	-	0.849456	***	0.722315	***
gamma11	0.490685	***	0.995271	***	-0.096393	**	-	-
gamma21	-0.330207	***	0.128962	***	0.280728	***	-	-
shape	4.349569	***	3.047357	***	3.170596	***	3.161174	***
Information Criteria								
Akaike	-4.5891		-6.2589		-5.7421		-6.0119	
Bayes	-4.5545		-6.2412		-5.7290		-5.9866	
Ljung-Box Standardized Residuals (p-value)								
Lag[1]	0.6934		0.5341		0.4954		2.234e-06	
Lag[2]	0.5430		0.7359		0.5833		4.376e-07	
Lag[5]	0.6398		0.7787		0.5260		8.001e-07	
Ljung-Box Squared Residuals (p-value)								
Lag[1]	0.1196		0.5376		0.8467		0.6352	
Lag[5]	0.4289		0.8629		0.8207		0.9059	
Lag[9]	0.6389		0.9723		0.9052		0.9351	
ARCH LM Tests (p-value)								
ARCH Lag[3]	0.7802		0.6754		0.5115		0.8493	
ARCH Lag[5]	0.7690		0.9416		0.6518		0.7997	
ARCH Lag[7]	0.8669		0.9791		0.7613		0.8473	
Sign Bias Test (p-value)								
Negative Sign Bias	0.1470		0.8083		0.9345		0.4044	
Positive Sign Bias	0.1255		0.1665		0.8357		0.8714	
Joint Effect	0.2033		0.4314		0.4632		0.8322	

3.2 OMX Baltic

Similar to the analysis of Tallink Group stock prices, we obtained daily closing index prices from Nasdaq.com to explore volatility and model performance. Figure 7 displays the raw data, showcasing trends and variations typical of financial time series. As with the Tallink dataset, initial inspections of the autocorrelation function (ACF) and partial autocorrelation function (PACF) of the log-differenced data indicated that there may be autocorrelation.

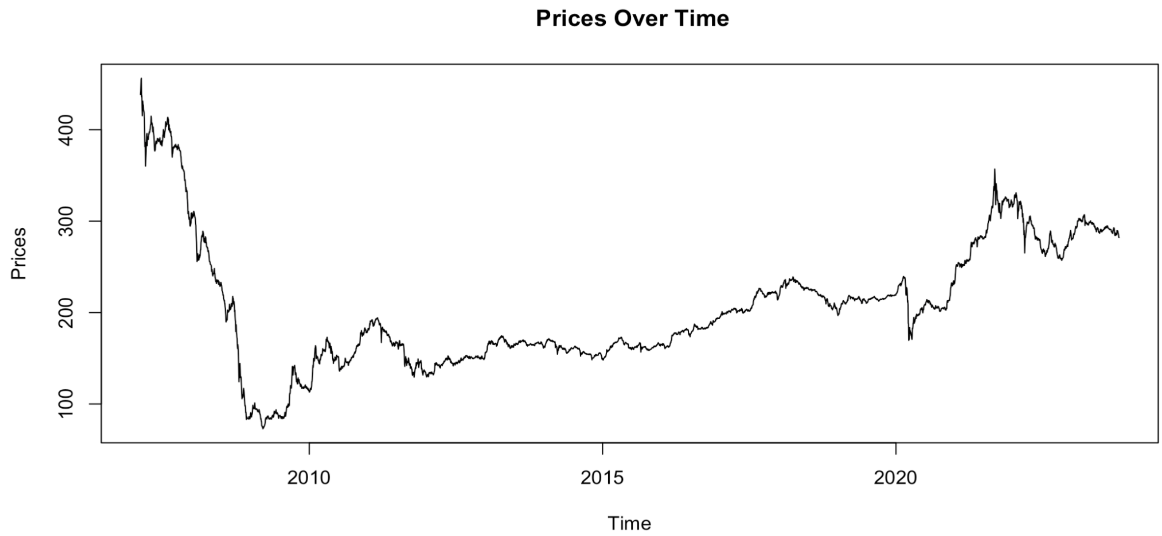

Figure 8: OMX index prices Data

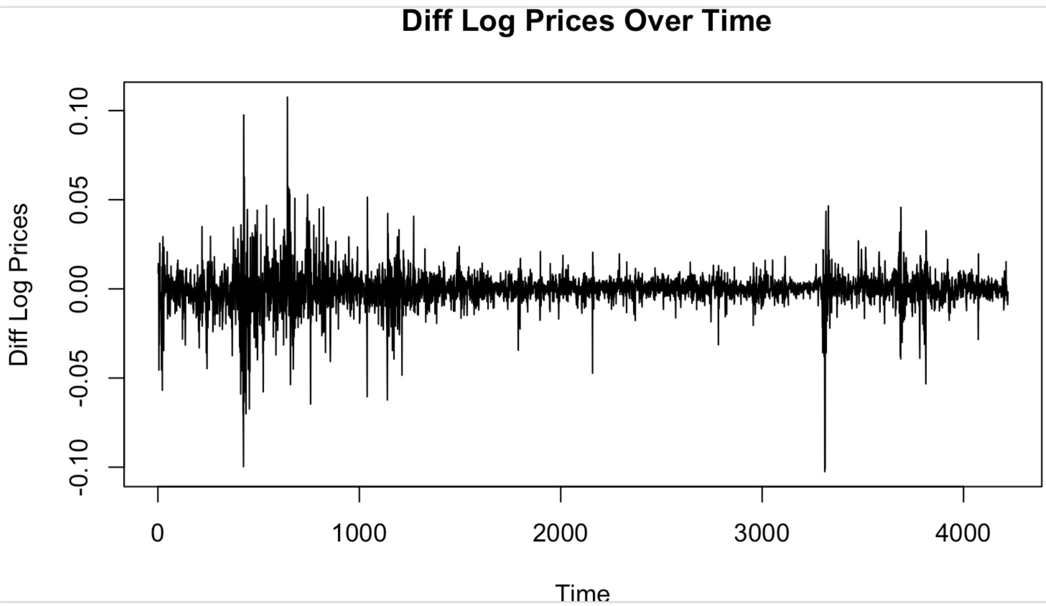

Figure 9: Differenced-Log Prices

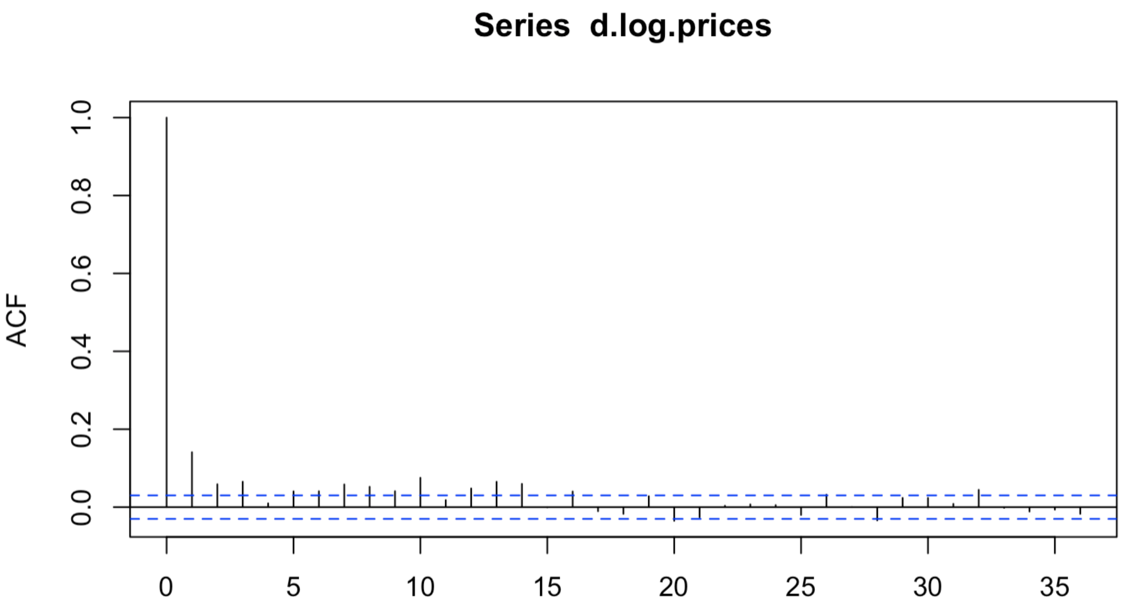

Figure 10: ACF

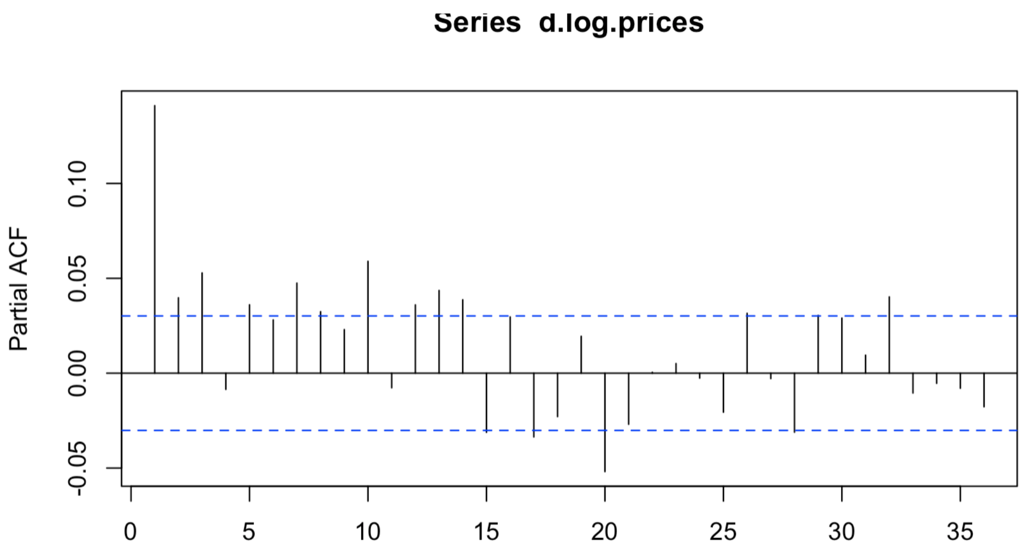

Figure 11: PACF

In this analysis, various ARMA models were combined with the GARCH framework to account for dependencies between current and past values. Although several AIRMA specifications were tested, including ARIMA(0,1,0) with AIC -26303.3 and BIC-26296.95, ARIMA (1,1,1) with AIC 26405.05 and BIC -26386 and ARIMA(10, 1, 5) with AIC -26454.4 and BIC -26352.97. Although the ARIMA(10,1,5) persisted in capturing autocorrelation, the ARIMA(1,1,1,) model was chosen as our preferred model because it had the lowest BIC value and fewer parameters, hence less complexity in modeling. However, persistent autocorrelation and distinct patterns remained in the squared residuals, as evidenced by the residual plots in Figures 12 and 13.

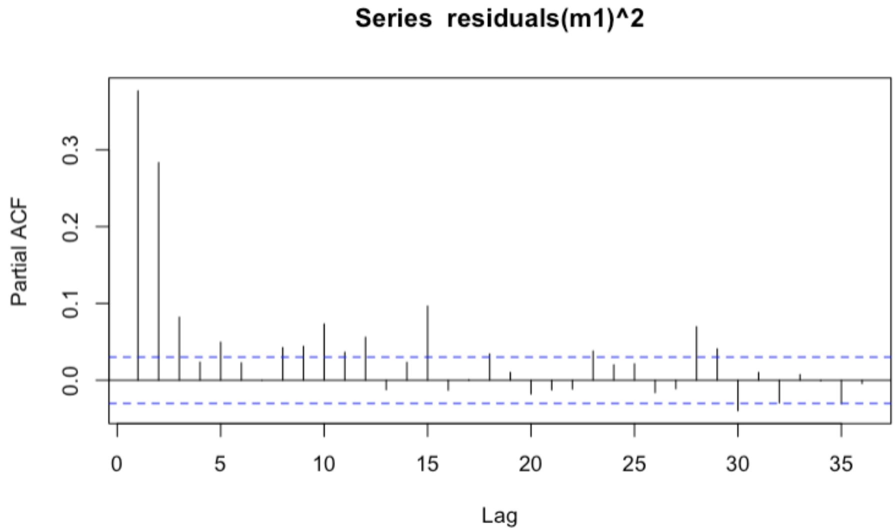

Figure 12: ACF squared residuals

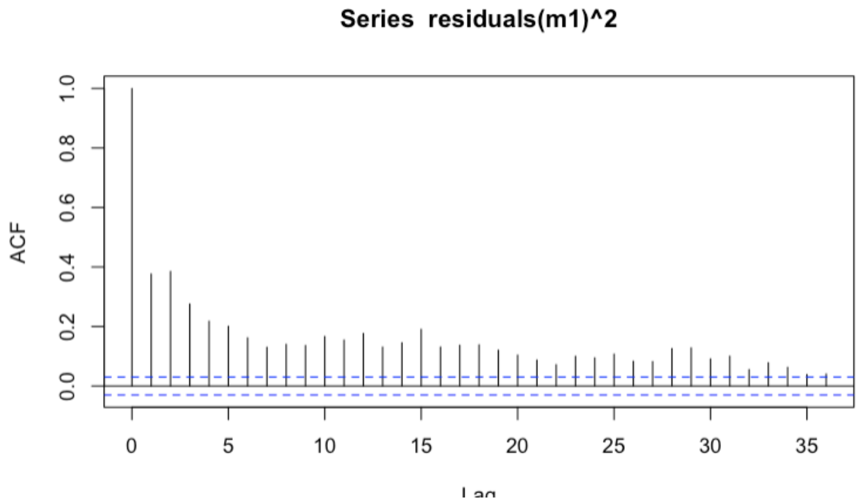

Figure 13: PACF squared residuals

Among the ARCH models evaluated, the ARCH(4) model with a Student's t-distribution emerged as the most effective, capturing heteroskedasticity and providing accurate parameter estimates. While it satisfied most diagnostic tests, the Ljung-Box Test on Standardized Residuals yielded p-values below 0.05, indicating potential shortcomings in addressing all autocorrelation patterns.

ARCH(4):

$$\begin{aligned}\sigma_t^2 &= 0.000016 + 0.434496\epsilon_{t-1}^2 + 0.249888\epsilon_{t-2}^2 \\ &\quad + 0.164029\epsilon_{t-3}^2 + 0.150586\epsilon_{t-4}^2.\end{aligned}$$

More advanced models, including the GARCH(1,3) framework and asymmetric GARCH variants, were explored to overcome these limitations. Additionally, to account for the impact of non-trading days for example holidays between traded days, in the verge of trying to capture the observed autocorrelation, another interpolation formula was introduced. Here, we assume that if between days t_1 and t_2, there was a holiday, then for the non-trading days $t_1 + i$, we find the daily prices according to the formula:

$$P_{t_1+i} = P_{t_1} + i \cdot \frac{P_{t_2} - P_{t_1}}{k}, \quad \text{for } i = 1, 2, \dots, k-1,$$

where $k-1$ represents the number of non-traded days between prices P_{t_1} and P_{t_2}. The results from this adjusted dataset closely mirrored those based solely on trading days, similar to Tallink stock prices that non-trading days had minimal influence on model performance. (see Table A8)

The GARCH-in-Mean (GARCH-M) model was employed to test whether higher volatility is linked to higher expected returns. However, the risk premium coefficient (δ), represented by `archm` parameter estimate is positive, similar to Tallink stock prices, was not statistically significant (p-value > 0.05), indicating no strong evidence of a trade-off between risk and return. (see Table A9).

GARCH(1,3):

$$\begin{aligned}\sigma_t^2 &= 0.000002 + 0.259580\epsilon_{t-1}^2 + 0.500692\sigma_{t-1}^2 \\ &\quad + 0.000002\sigma_{t-2}^2 + 0.238725\sigma_{t-3}^2.\end{aligned}$$

Extensive testing identified the CSGARCH(1,1) model as the top performer, achieving the lowest AIC and BIC values. This model effectively captured volatility dynamics and asymmetries. However, despite its strong diagnostic performance, the ljung-Box Test on standardized residuals still produced low p-values, pointing to unresolved residual autocorrelation and missing exogenous regressors.

CSGARCH:

$$\begin{aligned}\sigma_t^2 &= q_t + 0.245263\left(\varepsilon_{t-1}^2 - q_{t-1}\right)\\ &\quad + 0.675897\left(\sigma_{t-1}^2 - q_{t-1}\right).\end{aligned}$$

And

$$q_t = 0.999714\, q_{t-1} + 0.245263\left(\varepsilon_{t-1}^2 - q_{t-1}\right)$$

To assess model adaptability, the dataset was divided into four distinct periods: Period 1 (01 February 2007–30 December 2010), which captures post-financial crisis adjustments; Period 2 (03 January 2011–30 December 2019), which represents a stable market phase; Period 3 (01 February 2007–30 December 2019), which encompasses pre-COVID data; and Period 4 (01 January 2020–10 October 2023), reflecting pandemic-induced volatility.

The analysis revealed that the AVGARCH(1,1) model performed best in Period 1, effectively addressing post-crisis volatility asymmetries with a negative, suggesting that positive shocks in- creased volatility more than negative shocks. For Periods 2 and 3, characterized by stability, the CSGARCH(1,1) model demonstrated superior performance, capturing both long-term volatility trends and asymmetries. However, during Period 4, marked by pandemic-induced disruptions, the simpler GARCH(1,1) model outperformed, proving effective in adapting to rapidly shifting variance dynamics.

The persistent low p-values in the ljung-Box Test on standardized residuals across all models indicate possible unaccounted nonlinear dependencies or higher-order autocorrelations. This underscores the need for Future research to make use of advanced techniques, such as hybrid models or machine learning approaches, to address these residual issues.

Table 3: **Model Results and Diagnostic Tests for ARCH(4), GARCH(1,3), and CSGARCH(1,3)**

Parameters	**arch(4)**		**garch(1,3)**		**csgarch(1,1)**	
Parameter						
mu	0.000241	**	0.000240	**	0.000250	**
ar1	-	***	0.852808	***	0.839532	***
ma1	-	***	-0.807193	***	-0.790410	***
omega	0.000016	***	0.000002	-	0.000000	-
alpha1	0.434496	***	0.259580	***	0.245263	***
alpha2	0.249888	***	-	-	-	-
alpha3	0.164029	***	-	-	-	-
alpha4	0.150586	***	-	-	-	-
beta1	-	-	0.500692	***	0.675897	***
beta2	-	-	0.000002	-	-	-
beta3	-	-	0.238725	**	-	-
gamma11	-	-	-	-	0.999714	***
gamma21	-	-	-	-	0.020215	***
shape	4.017947	***	4.714468	***	4.797420	***
Information Criteria						
Akaike	-7.0776	-	-7.1182	-	-7.1319	-
Bayes	-7.0640	-	-7.1047	-	-7.1184	-
Ljung-Box Standardized Residuals (p-value)						
Lag[1]	1.078e-06	-	1.639e-07	-	3.798e-07	-
Lag[2*(p+q)+(p+q)-1][5]	0.000e+00	-	0.000e+00	-	0.000e+00	-
Lag[4*(p+q)+(p+q)-1][9]	1.272e-12	-	3.331e-16	-	1.288e-14	-
Ljung-Box Squared Residuals (p-value)						
Lag[1]	0.40209	-	0.2659	-	0.1205	-
Lag[2*(p+q)+(p+q)-1][5]	0.20522	-	0.2861	-	0.2563	-
Lag[4*(p+q)+(p+q)-1][9]	0.03994	*	0.2972	-	0.2725	-
ARCH LM Tests (p-value)						
ARCH Lag[5]	0.4507	-	0.1123	-	0.3973	-
ARCH Lag[7]	0.6619	-	0.1136	-	0.3275	-
ARCH Lag[9]	0.1505	-	0.1909	-	0.3395	-
Sign Bias Test (p-value)						
Negative Sign Bias	0.9775	-	0.1322	-	0.0623	*
Positive Sign Bias	0.4140	-	0.4722	-	0.6034	-
Joint Effect	0.7172	-	0.1735	-	0.0703	*

Table 4: **Best Model Results and Diagnostic Tests Across Periods**

	Period 1 avgarch(1,1)		**Period 2 csgarch(1,1)**		**Period 3 csgarch(1,1)**		**Period 4 garch(1,1)**	
mu	-0.000496	-	0.000256	**	0.000211	**	0.000412	*
ar1	0.860107	***	-0.467161	-	0.866040	***	0.180141	***
ma1	-0.788502	***	0.466935	-	-0.833095	***	-	-
omega	0.000515	**	0.000000	-	0.000000	-	0.000005	*
alpha1	0.172260	***	0.297948	***	0.236986	***	0.290607	***
beta1	0.839380	***	0.344310	***	0.579055	***	0.642344	***
gamma11	-0.085423	-	0.998832	***	0.999553	***	-	-
gamma21	0.219608	***	0.031725	***	0.027722	***	-	-
shape	4.841245	***	4.906876	***	4.844175	***	5.299556	***
Information Criteria								
Akaike	-5.7715	-	-7.6909	-	-7.1061	-	-7.2343	-
Bayes	-5.7269	-	-7.6683	-	-7.0893	-	-7.2038	-
Standardized Residuals (p-value)								
Lag[1]	0.0106	*	0.0009	***	0.00005	***	0.0756	-
Lag[5]	1.008e-6	***	0.0000	***	0.0000	***	0.0018	*
Lag[9]	0.0420	*	1.539e-7	***	1.435e-9	***	0.0057	*
Squared Residuals (p-value)								
Lag[1]	0.5951	-	0.1474	-	0.2463	-	0.8608	-
Lag[5]	0.9617	-	0.4326	-	0.4373	-	0.5480	-
Lag[9]	0.9430	-	0.2912	-	0.4663	-	0.5959	-
ARCH LM Tests (p-value)								
ARCH Lag[3]	0.6663	-	0.6904	-	0.3022	-	0.1816	-
ARCH Lag[5]	0.9214	-	0.5599	-	0.5175	-	0.2061	-
ARCH Lag[7]	0.8437	-	0.4376	-	0.5625	-	0.3668	-
Sign Bias Test (p-value)								
Negative Sign Bias	0.3445	-	0.0957	*	0.1794	-	0.3029	-
Positive Sign Bias	0.2959	-	0.2212	-	0.5048	-	0.6411	-
Joint Effect	0.2070	-	0.0176	**	0.1599	-	0.6220	-

Conclusion

This study assessed the effectiveness of GARCH-type models in capturing volatility dynamics in the Tallink Group stock and the OMX Baltic Index. For the Tallink Group stock, the CSGARCH(1,1) model performed consistently well over the full period (2007–2023), effectively modeling both short and long-term volatility components. Significant estimates for α_1 and β_1 confirmed volatility persistence. The persistence of the long-term component γ_{11} and response of the long-term component to shocks γ_{21} appeared statistically significant under standard errors, indicating that both the permanent component and its response to past shocks play a meaningful role in volatility dynamics and the Sign Bias tests did not reveal a significant leverage effect. This implies that positive and negative shocks had a similar impact on volatility, suggesting symmetric market reactions.

Period specific model comparisons revealed nuanced differences. The AVGARCH(1,1) model was best suited for Periods 1 and 3, effectively capturing asymmetric shocks and persistent variance during times of crisis and recovery. In Period 2, the CSGARCH(1,1) model outperformed others due to its strength in modeling both long and short term variance components. Period 4, marked by post-COVID shocks, was best modeled by the GARCH(1,1) model, because of its simple, and symmetric structure fitting well with sharp, and short-term volatility.

For the OMX Baltic Index, the CSGARCH(1,1) model stood out by effectively capturing long-term volatility trends and asymmetry. α_1 and β_1 are high and significant, indicating strong short-term and long-term volatility persistence. Again, the long-term component γ_{11} and the response of the component to shocks γ_{21} were statistically significant, while the results of the sign bias test indicate there is some weak evidence of asymmetry. Though asymmetry is not strongly significant, the Sign Bias Tests hint at a possible leverage effect.

The CSGARCH(1,1) model was the preferred model choice, in Periods 2 and 3, where it captured both persistent and asymmetric volatility. The significance of the sign bias tests in Period 2 suggest the presence of a leverage effect. In Period 1, the AVGARCH(1,1) model captured volatility linked to the global financial crisis, while in Period 4, the simpler GARCH(1,1) model sufficed, reflecting symmetric market responses to pandemic-era shocks. Diagnostic tests across all models in the squared residuals and ARCH LM tests confirmed good specification.

The lack of significant asymmetry in Tallink stock returns may be attributed to the relatively small and less liquid nature of the Baltic market and to specific investor behavior. Compared to larger markets like in the U.S., where information flows are more reactive and efficient, Baltic markets may show weaker responses to bad news. In contrast, the OMX Baltic Index, being a broader composite of assets, exhibited greater sensitivity to systemic shocks, explaining the noticeable leverage effects.

In conclusion, while GARCH-type models proved highly effective in modeling volatility dynamics in both Tallink and the OMX Baltic Index, their performance varied across time and assets. These findings highlight the importance of selecting appropriate models for different market conditions. Future research should consider hybrid models or time varying parameter GARCH approaches to further enhance forecasting accuracy and account for structural breaks, ensuring continued relevance in increasingly complex and evolving financial environments.

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A Appendix

Table A5: **Talink stock prices Untransformed and Transformed data**

	garch(1,1)		**Transformed garch(1,1)**		**csgarch(1,1)**		**Transformed csgarch(1,1)**	
mu	-0.000284	*	-0.000260	*	-0.000298	**	-0.000272	**
omega	0.000009	***	0.000008	**	0.000002	-	0.000002	**
alpha1	0.195755	***	0.190233	***	0.276242	***	0.284493	***
beta1	0.803245	***	0.808767	***	0.323941	***	0.298909	***
gamma11	-	-	-	-	0.998750	***	0.998787	***
gamma21	-	-	-	-	0.264685	***	0.080231	***
shape	3.412024	***	3.358006	***	3.589824	***	3.542229	***
Information Criteria								
Akaike	-5.7897	-	-5.8810	-	-5.8062	-	-5.8987	-
Bayes	-5.7821	-	-5.8735	-	-5.7957	-	-5.8882	-
Standardized Residuals (p-value)								
Lag[1]	0.1911	-	0.19707	-	0.14873	-	0.14919	
Lag[2]	0.1355	-	0.11118	-	0.09953	-	0.08078	
Lag[5]	0.1291	-	0.09169	-	0.10144	-	0.07105	
Squared Residuals (p-value)								
Lag[1]	0.9045	-	0.9722	-	0.4561	-	0.4218	
Lag[5]	0.6778	-	0.5641	-	0.7048	-	0.6126	
Lag[9]	0.7579	-	0.6261	-	0.8974	-	0.8434	
ARCH LM Tests (p-value)								
ARCH Lag[3]	0.4491	-	0.3851	-	0.6660	-	0.6449	
ARCH Lag[5]	0.4607	-	0.3549	-	0.8463	-	0.8489	
ARCH Lag[7]	0.5717	-	0.4444	-	0.9394	-	0.9429	
Sign Bias Test (p-value)								
Negative Sign Bias	0.7285	-	0.6985	-	0.6439	-	0.5913	
Positive Sign Bias	0.6625	-	0.8377	-	0.1941	-	0.2179	
Joint Effect	0.7941	-	0.8191	-	0.4894	-	0.5029	

Table A6: **Tallink stock prices All models Untransformed data I**

	tgarch(1,1)		gjr-garch(1,1)		avgarch(1,1)		garch-M	
mu	-0.000430	***	-0.000330	**	-0.000369	**	-0.000363	*
archm	-	-	-	-	-	-	0.497869	-
omega	0.000346	***	0.000009	***	0.000444	***	0.000009	**
alpha1	0.172209	***	0.194691	***	0.201653	***	0.196791	***
beta1	0.873199	***	0.803211	***	0.840525	***	0.802206	***
gamma11	0.128050	**	0.075070	*	-0.088072	-	-	-
gamma21	-	-	-	-	0.264685	***	-	-
shape	3.123302	***	3.437607	***	3.151735	***	3.412003	***
Information Criteria								
Akaike	-5.7971	-	-5.7902	-	-5.8016	-	-5.7893	-
Bayes	-5.7881	-	-5.7812	-	-5.7911	-	-5.7803	-
Standardized Residuals (p-value)								
Lag[1]	0.1564	-	0.1903	-	0.1366	-	0.1917	-
Lag[5]	0.1155	-	0.1303	-	0.0932	-	0.1299	-
Squared Residuals (p-value)								
Lag[1]	0.5688	-	0.9351	-	0.6786	-	0.9005	-
Lag[5]	0.8447	-	0.6582	-	0.8233	-	0.6772	-
ARCH LM Tests (p-value)								
ARCH Lag[3]	0.6442	-	0.4410	-	0.5764	-	0.4470	-
ARCH Lag[5]	0.7046	-	0.4570	-	0.6000	-	0.4578	-
ARCH Lag[7]	0.7986	-	0.5700	-	0.6883	-	0.5685	-
Sign Bias Tests (p-value)								
Negative Sign Bias	0.7899	-	0.8778	-	0.7663	-	0.7127	-
Positive Sign Bias	0.8214	-	0.8295	-	0.9336	-	0.595	-
Joint Effect	0.5606	-	0.7234	-	0.4045	-	0.6869	-

Table A7: **Tallink stock prices All models Untransformed data II**

Parameters	**nagarch**		**aparch**		**egarch**		**csgarch**	
mu	-0.000357	***	-0.000370	***	-0.000365	***	-0.000298	*
omega	0.000009	***	0.000142	*	-0.131992	***	0.000002	-
alpha1	0.194127	***	0.192474	***	-0.015897	-	0.276242	***
beta1	0.800553	***	0.856591	***	0.983549	***	0.323941	***
gamma21	0.149178	*	0.106958	**	-	-	0.084898	***
lambda	-	-	1.249837	***	-	-	-	-
shape	3.437693	***	3.146985	***	3.125230	***	3.589824	***
Information Criteria								
Akaike	-5.7907	-	-5.7983	-	-5.7901	-	-5.8062	-
Bayes	-5.7816	-	-5.7878	-	-5.7811	-	-5.7957	-
Standardized Residuals (p-value)								
Lag[1]	0.1876	-	0.1621	-	0.1884	-	0.1487	-
Lag[2*(p+q)+(p+q)-1][2]	0.1321	-	0.1230	-	0.1251	-	0.0995	-
Lag[4*(p+q)+(p+q)-1][5]	0.1257	-	0.1158	-	0.1215	-	0.1014	-
Squared Residuals (p-value)								
Lag[1]	0.9442	-	0.7663	-	0.4547	-	0.4561	-
Lag[2*(p+q)+(p+q)-1][5]	0.6567	-	0.8007	-	0.8387	-	0.7048	-
Lag[4*(p+q)+(p+q)-1][9]	0.7408	-	0.8753	-	0.9311	-	0.8974	-
ARCH LM Tests (p-value)								
ARCH Lag[3]	0.4382	-	0.5369	-	0.6692	-	0.6660	-
ARCH Lag[5]	0.4488	-	0.5904	-	0.7278	-	0.8463	-
ARCH Lag[7]	0.5614	-	0.6950	-	0.8273	-	0.9394	-
Sign Bias Tests (p-value)								
Sign Bias	0.2330	-	0.2807	-	0.3477	-	0.4366	-
Negative Sign Bias	0.8643	-	0.8992	-	0.4099	-	0.6439	-
Positive Sign Bias	0.8477	-	0.9889	-	0.4830	-	0.1941	-
Joint Effect	0.6108	-	0.6431	-	0.5492	-	0.4894	-

OMX Baltic Index prices

Table A8: **OMX Baltic index prices Untransformed and Transformed(Second Transformation) data**

	garch(1,3)		Transformed garch(1,3)		csgarch(1,1)		Transformed csgarch(1,1)	
mu	0.000240	**	0.000266	**	0.000250	**	0.000285	***
ar1	0.852808	***	0.847115	***	0.839532	***	0.830547	***
ma1	-0.807193	***	-0.797096	***	-0.790410	***	-0.776206	***
omega	0.000002	-	0.000002	**	0.000000	-	0.000000	-
alpha1	0.259580	***	0.284286	***	0.245263	***	0.290757	***
beta1	0.500692	***	0.481541	***	0.675897	***	0.609034	***
beta2	0.000002	-	0.000003	-	-	-	-	-
beta3	0.238725	**	0.233169	***	-	-	-	-
gamma11	-	-	-	-	0.999714	***	0.999707	***
gamma21	-	-	-	-	0.020215	***	0.018678	***
shape	4.714468	***	4.372891	***	4.706719	***	4.448667	***
Information Criteria								
Akaike	-7.1182	-	-7.1791	-	-7.1319	-	-7.1982	-
Bayes	-7.1047	-	-7.1659	-	-7.1184	-	-7.1850	-
Standardized Residuals (p-value)								
Lag[1]	1.639e-07	***	5.539e-08	***	3.798e-07	***	1.768e-07	***
Lag[5]	0.0000	***	0.0000	***	0.0000	***	0.0000	***
Lag[9]	3.331e-16	***	0000	***	1.288e-14	***	2.665e-15	***
Squared Residuals (p-value)								
Lag[1]	0.2659	-	0.4377	-	0.1205	-	0.3359	-
Lag[11]	0.2861	-	0.3295	-	0.2563	-	0.5053	-
Lag[19]	0.2972	-	0.3353	-	0.2725	-	0.5384	-
ARCH LM Tests (p-value)								
ARCH Lag[5]	0.1123	-	0.2198	-	0.3973	-	0.3661	-
ARCH Lag[7]	0.1136	-	0.1667	-	0.3275	-	0.4674	-
ARCH Lag[9]	0.1909	-	0.2434	-	0.3395	-	0.6062	-
Sign Bias Test (p-value)								
Negative Sign Bias	0.1322	-	0.1428	-	0.0623	-	0.4237	-
Positive Sign Bias	0.4722	-	0.4481	-	0.6034	-	0.3282	-
Joint Effect	0.1734	-	0.2423	-	0.0703	-	0.6602	-

Table A9: **OMX Baltic index prices All models Untransformed data I**

	tgarch(1,3)		gjrgarch(1,3)		avgarch(1,3)		garch-M	
mu	0.000200	**	0.000198	*	0.000173	-	0.000140	-
ar1	0.864830	***	0.855694	***	0.863130	***	0.879198	***
ma1	-0.820856	***	-0.808153	***	-0.822409	***	-0.834692	***
archm	-	-	-	-	-	-	3.168697	-
omega	0.000248	***	0.000002	-	0.000247	***	0.000002	-
alpha1	0.250102	***	0.259711	***	0.249859	***	0.261426	***
beta1	0.541397	***	0.501803	***	0.555922	***	0.503157	***
beta2	0.000006	-	0.000001	-	0.000003	-	0.000002	-
beta3	0.246336	***	0.236280	***	0.230603	*	0.234415	**
gamma11	0.120386	***	0.068114	**	0.016216	-	-	-
gamma21	-	-	-	-	0.134863	**	-	-
shape	4.678665	***	4.718380	***	4.675558	***	4.698643	***
Information Criteria								
Akaike	-7.1200	-	-7.1188	-	-7.1201	-	-7.1184	-
Bayes	-7.1049	-	-7.1038	-	-7.1036	-	-7.1034	-
Standardized Residuals (p-value)								
Lag[1]	8.082e-08	***	2.291e-07	***	4.784e-08	***	7.354e-08	***
Lag[5]	0.0000	***	0.0000	***	0.0000	***	0.000e+00	***
Lag[9]	1.110e-16	***	2.331e-15	***	0.0000	***	0.000e+00	***
Squared Residuals (p-value)								
Lag[1]	0.0494	-	0.3861	-	0.0422	-	0.3121	-
Lag[11]	0.0099	-	0.3930	-	0.0093	-	0.3129	-
Lag[19]	0.0170	-	0.3619	-	0.0187	-	0.3207	-
ARCH LM Tests (p-value)								
ARCH Lag[5]	0.1703	-	0.1264	-	0.1680	-	0.1200	-
ARCH Lag[7]	0.2225	-	0.1293	-	0.2091	-	0.1187	-
ARCH Lag[9]	0.3526	-	0.2262	-	0.3507	-	0.2001	-
Sign Bias Tests (p-value)								
Negative Sign Bias	0.1523	-	0.2950	-	0.0668	-	0.2081	-
Positive Sign Bias	0.7229	-	0.6311	-	0.4929	-	0.5167	-
Joint Effect	0.5270	-	0.6133	-	0.2393	-	0.2406	-

Table A10: **OMX Baltic index prices All models Untransformed data II**

Parameters	nagarch		aparch		egarch		csgarch	
mu	0.000175	*	0.000197	*	0.000197	-	0.000250	**
ar1	0.861772	***	0.859429	***	0.855573	***	0.839532	***
ma1	-0.815067	***	-0.814293	***	-0.807162	***	-0.790410	***
omega	0.000002	-	0.000040	-	-0.300541	***	0.000000	-
alpha1	0.255707	***	0.264630	***	-0.049088	***	0.245263	***
beta1	0.508195	***	0.525728	***	0.769495	***	0.675897	***
beta2	0.000000	-	0.000000	-	-0.100435	**	-	-
beta3	0.229116	***	0.244655	***	0.300530	***	-	-
gamma11	-	-	0.094472	**	-	-	0.999714	***
gamma21	0.152959	***	-	-	-	-	0.020215	***
eta1	-	-	-	-	0.441544	***	-	-
lambda	-	-	1.370823	***	-	-	-	-
shape	3.275898	***	2.901267	***	2.898315	***	4.706719	***
Information Criteria								
Akaike	-7.1199	-	-7.1211	-	-7.1239	-	-7.1319	-
Bayes	-7.1048	-	-7.1046	-	-7.1088	-	-7.1184	-
Standardized Residuals(p-value)								
Lag[1]	1.987e-07	***	1.308e-07	***	3.767e-07	***	3.798e-07	***
Lag[5]	0.0000	***	0.0000	***	0.0000	***	0.0000	***
Lag[9]	1.221e-15	***	3.331e-16	***	2.154e-14	***	1.288e-14	***
Squared Residuals(p-value)								
Lag[1]	0.3166	-	0.1755	-	0.1440	-	0.1205	-
Lag[11]	0.3960	-	0.1553	-	0.1339	-	0.2563	-
Lag[19]	0.3624	-	0.1718	-	0.1483	-	0.2725	-
ARCH LM Tests (p-value)								
ARCH Lag[5]	0.1416	-	0.1249	-	0.1144	-	0.3973	-
ARCH Lag[7]	0.1528	-	0.1386	-	0.1574	-	0.3275	-
ARCH Lag[9]	0.2655	-	0.2373	-	0.2545	-	0.3395	-
Sign Bias Tests (p-value)								
Negative Sign Bias	0.2845	-	0.2472	-	0.0804	*	0.0623	-
Positive Sign Bias	0.6803	-	0.6536	-	0.9718	-	0.6034	-
Joint Effect	0.7253	-	0.6211	-	0.3381	-	0.0703	-

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