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STUDIES OF PRE-BIG BANG AND BRANEWORLD COSMOLOGY

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Contents

List of original publications

This thesis consists of an introductory review part, followed by three research publications [I–III]. These are listed below and reprinted after the review part.

Papers indexed in *Current Contents*:

I P. Kuusk and M. Saal, *"Hamilton-Jacobi approach to pre-big bang cosmology and the problem of initial conditions"*, Gen. Rel. Grav. **34**, 3, pp. 353–364, 2002, [gr-qc/9910093].

II P. Kuusk and M. Saal, *"Long-wavelength approximation for string cosmology with barotropic perfect fluid"*, Gen. Rel. Grav. **34**, 12, pp. 2135–2148, 2002, [gr-qc/0205091].

III P. Kuusk and M. Saal, *"A cosmological model of holographic brane gravity"*, Gen. Rel. Grav. **36**, 5, pp. 1001–1014, 2004, [gr-qc/0309084].

Other related publications:

IV P. Kuusk and M. Saal, *"WKB approximation in String Cosmology"*, in Proceedings of the Fourth Alexander Friedmann International Seminar on Gravitation and Cosmology, pp. 421–431, Eds. Yu.N. Gnedin, A.A. Grib, V.M. Mostepanenko, A.W. Rodriges Jr, Campinas, SP: Unicamp/Imecc, Brazil, 1999 .

V M. Saal, *"Stringikosmoloogia"*, Eesti Füüsika Seltsi aastaraamat 1999, lk 88-89, Tartu 2000 .

VI P. Kuusk and M. Saal, *"Long-wavelength approximation for string cosmology"*, 21th Texas Symposium on Relativistic Astrophysics, Book of Abstracts, p. 18, Florence, December 9-13, 2002 .

VII P. Kuusk and M. Saal, *"A cosmological model for a two-brane world"*, 17th International Conference on General Relativity and Gravitation, Book of Abstracts, pp. 153-154, Dublin, July 19-23, 2004 .

VIII P. Kuusk and M. Saal, *"Advances in string and brane cosmology"* in Biannual Report 2002/2003, pp. 44–47, Institute of Physics, Tartu, 2004 .

Chapter 1

Introduction

Until recently cosmology [1] - [4] was mainly a theoretical science since the observations were not precise enough to confirm or to confute the predictions made by theory. Now the situation has changed and cosmological parameters are measured to the accuracy of a few percent. Today cosmology is an observational and an exact science.

According to the standard theory, space and time sprung into being about 14 billion years ago in an event known as a *big bang*. The emergence of the Universe lies in the "Planck epoch" were all physical parameters were nearby at their limiting values. The existence and possible structure of this initial state, known as a *cosmological singularity*, is still unknown. At that very instant the Universe went under rapid expansion called *inflation* [5] - [7] which made the Universe extremely flat and homogeneous. Quantum fluctuations of the scalar field which drove the inflation were the seeds of the large-scale structure we observe today.

Today it seems that we must supplement this standard paradigm with a surprising discovery made recently. Namely, the observations of Type Ia supernovae indicate [8], [9] that the expansion of our Universe is accelerating at least at present epoch. Before that discovery, it was commonly believed that the expansion is decelerating and the question needing to be answered is that whether the Universe is open and expands forever or is it closed and starts to contract at some future moment? The geometry of the Universe is dictated by the energy density of the Universe. If the energy density is greater than the critical value $\rho_{cr} \sim 10^{-29}$ g cm^{-3} the expansion will stop and the Universe will begin to contract. If the energy density is less than the critical value the Universe will expand forever but the expansion rate will slow down as time goes on. Recent observations [10], [11] indicate that the energy density of the Universe ρ is very close to the critical value ρ_{cr}: $\Omega \equiv \rho/\rho_{cr} = 1$. The dimensionless quantity Ω is known as the *density parameter*. The question about the fate of the Universe still remains open. It is important to remark, that matter contents (both visible and *dark matter*) make up less than one third of the critical density and therefore most of the energy density is in an unknown form, called *dark energy*. This dark energy drives the accelerating expansion. Since the expansion is accelerating, it is a temptation to conclude that the Universe is open and expands forever. However, we don't know what is the dark energy which causes the present day acceleration. It is not excluded that the acceleration may be a temporal epoch in the evolution of the Universe and at some moment it starts decelerating again. There are two leading candidates for the dark energy. The first candidate is the famous *cosmological constant* [12] - [14] which has been one of the greatest challenges for theoretical physics since 1916, when Einstein introduced it for making the Universe static. The second is a "dynamical cosmological constant" known as *quintessence* [15] - [18]. In most of the models this is nothing but the *scalar field* Q with a specific self-interaction potential $V(Q)$. Both the cosmological constant and the quintessence lead to a specific equation of state for cosmological fluid which drives the present day accelerated evolution.

A good cosmological theory should give a unique mechanism which predicts, starting from fundamental principles (whatever they exactly are), generically the Universe like ours. In this context the rivalry of different scenarios, making different predictions, is the only way to improve our understanding about the Universe we live in. In this thesis we describe some extensions and alternatives to the standard cosmological scenario. We focus on the *pre-big bang scenario* and to a cosmological scenario following from the *braneworld conception*. Both, the pre-big bang scenario and the braneworld cosmology are based, more or less, on *string theory* [19], [20]. String theory is considered to be the leading candidate for description of physics at the Planck scale.

Until recently the most successful string theory inspired cosmological model was the pre-big bang scenario introduced by Gasperini and Veneziano [21], [22]. This scenario introduces the generic properties of string theory, namely *duality symmetries*, into cosmology. A particular case of T*-duality*, known as the *scale-factor duality*, is applied to solutions of the field equations, derived from the low energy effective action of string theory. The scale-factor duality maps a Friedmann-Robertson-Walker (FRW) cosmology evolving from a singularity in the past, into a pre-big bang cosmology going towards a singularity in the future. The main problem of the scenario, known as *graceful exit*, is the impossibility to describe the smooth (non-singular) transition from the pre-big bang phase into the post-big bang phase. Another problem is related to the initial state of the pre-big bang Universe.

Superstring theory is formulated in a 10-dimensional spacetime, while their extension *M-theory* needs an 11-dimensional spacetime. The observable Universe, on the other hand, is described as a 4-dimensional spacetime. The main motivation to consider the extra dimensions comes thus from string theory, where at least six or seven extra dimensions are required to be compactified. In the conventional approach they are assumed to be very small (at the level of the Planck scale). The first attempt to include an extra dimension into the theory was made by Kaluza [23] and Klein [24] in the early of twentieth century in a bit different context. They tried to unify gravity and electromagnetism proposing a 5-dimensional theory where an extra dimension was compactified on a circle. Recent developments of string theory have given a new motivation and a deeper physical basis to such ideas, but the central procedure which originates from the Kaluza-Klein approach is also applicable in the context of string theory. It is assumed, that the geometry of a D-dimensional spacetime, which is a solution of the D-dimensional Einstein equations, can be represented as a direct product $\mathcal{M}^4 \times X^{D-4}$. Here $\mathcal{M}^4$ is our 4-dimensional spacetime manifold and X^{D-4} is an internal compact manifold of extra dimensions. If the compactification scale L_{ed} is small enough then the effects originating from extra dimensions are too small to measure. However, at scales less than L_{ed} the existence of extra dimensions becomes important. Since there is no evidence for the existence of the extra dimensions, it is assumed that their characteristic scale is in the order of the Planck scale: $L_{ed} \sim L_{_{Pl}} \sim 10^{-33}$ cm.

Recently it was realized, that it is reasonable to consider the models where the extra dimensions (at least one of them) are not compactified and should not be very small. Moreover, they may be infinite. The basic idea, in the context of string theory, which leads to this kind of scenario was proposed in the works of Hořava and Witten [25], [26] and developed by Lukas et al [27]. Hořava and Witten showed, in the framework of 11-dimensional M-theory, that gauge fields may consistently appear on the 10-dimensional boundary of a Z_2-symmetric 11-dimensional spacetime. In their scenario, as shown later by Witten [28], one extra dimension is bigger than the others. According to an earlier proposal by Antoniadis [29], string theory is consistent even if the size of the extra dimensions is as large as the electroweak scale $L_{ed} \sim L_{ew} \sim 10^{-17}$ cm. The braneworld conception, which states that D-dimensional spacetime (*bulk*) contains two (or one) $(D-1)$-dimensional boundaries (branes), was born.

The next important step was done by Arkani-Hamed et al [30], [31] who considered the *hierarchy problem*. Together with the *cosmological constant problem* [12] it has been the most important challenge to theoretical physics. The hierarchy problem is that there exists a huge gap between the 4-dimensional Planck scale $M_{Pl} = 10^{19}$ GeV and the electroweak scale $M_{ew} \sim 10^2$ GeV. In other words, there is no explanation why the characteristic scales of particle physics and gravity are so different. In their proposal [30], [31] the 4-dimensional Planck scale M_{Pl} is not a fundamental scale for gravity but only a 4-dimensional effective value. The fundamental Planck scale M is much closer to the electroweak scale and this is achieved by the presence of *large extra dimensions*. The scales are comparable $M \sim M_{ew}$ if the extra dimensions are large enough. This leads to a requirement to modify the gravitational interaction at distances shorter than 0.1 mm. At the scale $10^{-3} - 10^{28}$ cm the Newton gravitational law works well for non-relativistic gravitational interactions but we don't know how gravity behaves at distances shorter than 10^{-3} cm and at distances larger than 10^{28} cm. At large scales the Newton law is tested accurately and no deviations from the r^{-2} behaviour have been discovered. On the other hand, the Newton gravitational law must be modified at small scales, if extra dimension(s) exist. Recent experiments [32], [33] lower the limit down to 0.1 mm but below this scale the deviations are still possible.

These ideas were developed further by Randall and Sundrum [34] who argued that the large hierarchy is due to the highly curved background spacetime generated by the appropriate inter-brane distance which must be stabilized at a certain value. One possible stabilization mechanism was proposed by Goldberger and Wise [35] and requires the inclusion of bulk scalar fields. Randall and Sundrum introduced an elegant classical construction and presented two proposals known as RSI [34] and RSII [36]. They demonstrated [36] that highly curved bulk constrains gravity near the brane and Newton's law with small corrections can be recovered on the brane even if the extra dimension is infinite.

Let us summarize the situation described above. The basic idea of the braneworld scenario is that the observable Universe could be a $(3+1)$-hypersurface, called a *3-brane*, which is embedded in a $(3+1+d)$-dimensional spacetime, called the *bulk*. An important ingredient of the scenario is that all matter and gauge fields are bound by some mechanism to the brane and only gravity and other gravitational sector fields (dilaton fields for example) can propagate in the direction perpendicular to the brane, i.e., into extra dimensions. Since the standard model fields are confined to the brane and are insensitive at low energies to the presence of extra dimensions they do not put very strong constraints to the size of extra dimensions. Especially interesting is the 5-dimensional case where a 3-brane is embedded in a 5-dimensional bulk spacetime.

The braneworld cosmology was developed independently by Binétruy et al [37], [38] and significant differences from the standard cosmological scenario have been discovered. Most important, it was realized that the Friedmann equation is modified: at high energies the Hubble parameter is proportional to energy density $H \propto \rho$, while in the standard cosmology $H \propto \sqrt{\rho}$. However, the standard case is obtained at low energies and hence the constraints coming from the nucleosynthesis can be satisfied. Since the papers of Randall and Sundrum [34], [36] a lot of work has been done on the cosmological aspects of the model and we discuss the basic results in Ch. 5.

If braneworld conception is realistic it can be addressed to answer the question of the origin of the brane Universe. The deep connection with string theory is essential and a step toward this direction was the *ekpyrotic scenario* [39]. In this scenario branes move in bulk spacetime and can collide. This collision can be interpreted as a big bang. The ekpyrotic scenario has inspired the *cyclic scenario* [40], [41] where big bangs and *big crunches* repeat oneself. After collision the branes are moving apart from each other, reach the maximum and then start to move closer to each other again. Finally, the new collision gives birth to a new braneworld and all the cycle starts again.

A special model, known as *holographic brane gravity* was developed by Kanno and Soda in [42], [43] using the low energy expansion scheme. Related cosmological model, known as *born-again braneworld* [44] conjectures that two branes collide and emerge as reborn branes. During the collision the brane tensions change the signs. It is possible to make a conformal transformation to the frame where the born-again scenario resembles the pre-big bang scenario. In our paper III [45] we presented another cosmological model which is based on the holographic brane gravity.

Until recently, there was no evidence from gravity experiments, to confirm the existence of the extra dimensions. Therefore the higher-dimensional effects must be suppressed or the whole conception is wrong.

On the other side, recent advances in observational cosmology have given a lot of precise data and a couple of experiments are starting soon in the nearest future or are planned. This enables one to test and to constrain the cosmological models developed on theoretical grounds. In this thesis we briefly review these ideas and basic results, and pay a special attention to the cosmology which follows if the braneworld conception is applied to describe our Universe.

The introductory review part of the thesis is organized as follows. In Ch. 2 we review the standard cosmological model, including a brief discussion of the inflation and quintessence. Some aspects of string theory, which are important in the following chapters, are also presented. Ch. 3 deals with the pre-big bang cosmology. First, field equations are derived from the low energy effective action and scale-factor duality is applied to extend the cosmological model to negative time values. The initial and final stages of dual cosmologies are investigated. Then the initial conditions and the graceful exit problems are outlined. Ch. 4 introduces the main aspects of the braneworld conception. Finally, Ch. 5 is devoted to the braneworld cosmology. First, the cosmology of a single brane Universe is outlined. Three different approaches to derive the Friedmann equation on the brane in the case of empty bulk are presented and compared to each other. The bulk scalar field and inflation on the brane are briefly discussed in Subsecs. 5.2.4 and 5.2.5. Finally, the cosmology of a two-brane system is discussed focusing to the model of holographic brane gravity (Subsec. 5.3.4). However, moduli space approximation (Subsec. 5.3.2), ekpyrotic/cyclic model (Subsec. 5.3.3) and born-again braneworld (Subsec. 5.3.5) are also discussed in brief.

Chapter 2

General Framework

2.1 Cosmology

In this section we review the standard cosmological model and *cosmological inflation* introduced in order to solve the standard model problems (see below pp 22-24). The standard cosmology is based on three assumptions. First, it postulates the *cosmological principle* which states that our observable Universe is homogeneous and isotropic on large scales. Mathematically this means that the geometry of our Universe is described by the Friedmann-Robertson-Walker (FRW) line element. Secondly, cosmological particles are moving along timelike geodesics that do not intersect except at a singular point in the finite or infinite past (or future). This means that the matter content of our Universe may be taken to be an *ideal fluid*. Finally, the underlying theory that governs the dynamics of our Universe is *general relativity*. If the geometry is described by FRW line element and the matter is taken to be an ideal fluid, the general Einstein equations reduce to ordinary differential equations, known as the *Friedmann equations*. At present it is commonly believed that an early accelerated expansion or some alternative mechanism is needed in order to solve the standard model problems.

The most successful model of *inflation* is based on a slowly rolling *scalar field* and is later referred to as the standard (or *slow-roll*) inflation. For that reason, in what follows we refer the scenario where early inflationary epoch is included as the *standard cosmological model*. After a short description of this standard scenario in Subsecs. 2.1.1 and 2.1.2 we briefly review the simplest extension of the Einstein general relativity. This is known as the *Brans-Dicke theory* of gravity and in this theory the gravitational constant becomes time dependent. The time dependent gravitational constant requires a new degree of freedom, a scalar field, which appears in the action of the theory. Since almost all new extensions of the standard cosmological model are based or apply more or less string (or M-) theory as the most fundamental theory we briefly mention the most important aspects of string theory in Sec. 2.2.

2.1.1 Standard cosmological model

An important cornerstone of the standard cosmological model is the assumption that the Universe is homogeneous and isotropic on the very large scale. This requirement determines the metric up to an arbitrary function $a(t)$ and a discrete parameter k

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \\ &= -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] . \end{aligned} \tag{2.1}$$

Here $a(t)$ is the scale factor which measures the time evolution of the Universe and $k = -1, 0, 1$ determines the spatial geometry of the Universe, corresponding to hyperbolic, Euclidean and spherical spatial section, respectively. The metric (2.1) is known as the Friedmann-Robertson-Walker (FRW) metric and it is assumed to describe the *local* geometry of our 4-dimensional Universe. Sometimes it is more suitable to use *conformal time* defined as $d\tau = \frac{dt}{a(t)}$, then the FRW line element can be written as follows

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] , \tag{2.2}$$

where $a(\tau) \equiv a(t(\tau))$.

The energy-momentum tensor $T_{\mu\nu}$ of cosmological matter is assumed to be in a form of ideal (perfect) fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} , \tag{2.3}$$

where u^μ is the fluid 4-velocity, ρ is the energy density and p is the pressure of the ideal fluid. The energy density ρ and the pressure p are measured in the rest frame of the fluid: in comoving coordinates with respect to which the fluid is at rest, the 4-velocity is $u^\mu = (1, 0, 0, 0)$ and $T^\mu_{\ \nu} = diag(-\rho, p, p, p)$. The energy density ρ and the pressure p are functions of time and throughout the present thesis we assume barotropic equation of state

$$p = (\Gamma - 1)\rho \equiv w\rho\,, \tag{2.4}$$

where Γ and w are constants. The most important particular cases are: radiation ($w = 1/3$), dust ($w = 0$), stiff fluid ($w = 1$), "phenomenological" cosmological constant ($w = -1$).

The standard cosmology uses the Einstein general relativity as an underlying theory. The total action of the theory consists of the Einstein-Hilbert action I_{EH} and the matter action I_m

$$I = I_{EH} + I_m\,, \tag{2.5}$$

where

$$I_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,(R - 2\Lambda)\;, \qquad I_m = \int d^4x\sqrt{-g}\mathcal{L}_{matter}\,. \tag{2.6}$$

Here $\kappa^2 = 8\pi G_N$, R is the Ricci scalar and Λ is the cosmological constant. We have introduced the cosmological constant Λ explicitly in the action (2.6) but an equivalent way is to do that through the energy-momentum tensor of the ideal fluid taking $p_\Lambda = -\rho_\Lambda = const. = -\Lambda/8\pi G_N$. The Einstein equations derived from the action (2.5) are

$$G_{\mu\nu} \equiv \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}\;, \tag{2.7}$$

where the energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I_m}{\delta g^{\mu\nu}}\,. \tag{2.8}$$

The Bianchi identities $\nabla_\mu G^\mu{}_\nu = 0$ (where ∇_μ is the covariant derivative with respect to metric $g_{\mu\nu}$) require that $\nabla_\mu T^\mu{}_\nu = 0$ which is the conservation law for the energy-momentum tensor. If the energy-momentum is assumed to be that of the ideal fluid (2.3) then in the FRW universe (2.1) the conservation law can be written as follows

$$\dot\rho + 3\frac{\dot a}{a}(\rho + p) = 0\,, \qquad \Longrightarrow \qquad \rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(w+1)}\,. \tag{2.9}$$

Here dot means the derivative with respect to the FRW time t and subscript zero means, through this chapter, the present day value of corresponding quantity.

The Einstein equations (2.7) can be written, using (2.1), as follows

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \qquad (2.10)$$

$$\dot{H} + H^2 \equiv \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda}{3}\,, \qquad (2.11)$$

where H is the *Hubble parameter.* Its present day value is measured with high accuracy: $H_0 = H(t_0) = 100h$ km s^{-1} Mpc^{-1}, where $h = 0.71^{+0.04}_{-0.03}$ [10], [11].

The set of equations (2.9) - (2.11) are known as the *Friedmann equations* and they govern the dynamics of the FRW cosmology. Sometimes only equation (2.10) is called the Friedmann equation and equation (2.11) is called the *Raychaudhuri equation.* Due to the Bianchi identities, equations (2.9) - (2.11) are not independent, namely, equation (2.11) can be derived from equations (2.9) and (2.10). In general, equation (2.11) is the dynamical equation for the scale factor a and equation (2.10) is a constraint equation and can be used to fix the integration constant.

Dividing equation (2.10) by H^2 we can write

$$1 = \Omega - \frac{k}{H^2a^2} = \Omega_m + \Omega_\Lambda - \frac{k}{H^2a^2}\,, \qquad (2.12)$$

where the *density parameter* Ω is defined as

$$\Omega = \frac{\rho_{tot}}{\rho_{cr}} = \frac{\rho_m + \rho_\Lambda}{\rho_{cr}} = \Omega_m + \Omega_\Lambda\,. \qquad (2.13)$$

Here ρ_m is the energy density of matter (we added the subscript m), $\rho_\Lambda = \frac{\Lambda}{8\pi G_N}$ is the energy density of the cosmological constant and the *critical energy density* ρ_{cr} is defined as follows

$$\rho_{cr} \equiv \frac{3H^2}{8\pi G_N}\,. \qquad (2.14)$$

The present value of the critical density is $\rho_{cr} = 1.88 \times 10^{-29}h^2$ g cm^{-3}. If $\rho_{tot} = \rho_{cr}$, then $\Omega = 1$ and the spatial geometry should be flat, $k = 0$. Actual local geometry is determined by the density parameter Ω and can be summarized as follows:

- $\Omega < 1, \quad \Longleftrightarrow \quad k = -1, \quad \Longleftrightarrow$ the Universe is open ,
- $\Omega = 1, \quad \Longleftrightarrow \quad k = \;\; 0, \quad \Longleftrightarrow$ the Universe is flat ,

- $\Omega > 1, \quad \Longleftrightarrow \quad k = \;\; 1, \quad \Longleftrightarrow \quad$ the Universe is closed .

The Friedmann equations (2.9) - (2.11) can be easily solved assuming the flat Universe ($k = 0$) without the cosmological constant ($\Lambda = 0$)

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^{\frac{2}{3(w+1)}} , \tag{2.15}$$

where a_0 and t_0 are the present day values of the scale factor and the time parameter. Another simple example is the flat ($k = 0$), empty ($\rho = 0$) Universe with non-vanishing cosmological constant ($\Lambda \neq 0$). The corresponding solution of equation (2.10) is

$$a(t) \sim \begin{cases} \sinh\left(\sqrt{\frac{\Lambda}{3}} t\right) , & k = -1 \\ e^{\sqrt{\frac{\Lambda}{3}} t} , & k = 0 \\ \cosh\left(\sqrt{\frac{\Lambda}{3}} t\right) , & k = 1 \end{cases} \tag{2.16}$$

and it is known as the *de Sitter solution.*

Horizons

The wavelength of a photon, propagating in the spacetime, increases as the spacetime expands $\lambda \propto a(t)$ and the *redshift* z is given by the ratio of the scale factor today to the scale factor at the time when the photon was emitted:

$$1 + z = \frac{a(t_0)}{a(t_{em})} . \tag{2.17}$$

Physical distance L_{pd} between two particles in an expanding Universe can be written

$$L_{pd} = a(t) \times L_{cd} , \tag{2.18}$$

where the comoving distance L_{cd} is measured in *comoving coordinates* and is constant in time in the case of freely moving particles.

In the framework of standard cosmology the Universe has a finite age and the photons have been propagating a finite distance. This introduces the concept of a *horizon.* The horizon delimits the separation of the particles which can be causally connected.

The *particle horizon* d_{ph} determines the size of causally connected region what an observer can see in principle at a given time t:

$$d_{ph}(t) = a(t) \int_0^t \frac{dt'}{a(t')} \,. \tag{2.19}$$

If the distance between the particles is larger than horizon $d_{ph} < L_{pd}$ they are not causally connected. For example, if $a \sim t^q$, $0 < q < 1$ then the particle horizon is $d_{ph} = \frac{q}{1-q} H^{-1}$.

The *event horizon* delimits the part of the Universe from which we can ever (up to t_{max}) receive information about events taking place at the moment t

$$d_{eh}(t) = a(t) \int_t^{t_{max}} \frac{dt'}{a(t')} \,. \tag{2.20}$$

For example, if $a \sim t^q$, $0 < q < 1$ there is no event horizon $d_{eh} \to \infty$ since $t_{max} \to \infty$. If $a \sim e^{Ht}$, $H = const.$, the event horizon is a constant $d_{eh} = H^{-1}$ and an observer can receive information no farther away than H^{-1}.

Confirmation from observations

The standard big bang model based on the FRW spacetime is widely acknowledged and has many observational confirmations.

- *The observed expansion of the Universe.*
  The galaxies are separating from each other with the rate which is proportional to the distance d (*Hubble law*): $v = Hd$. This relation is verified with high precision. The factor of proportionality is the Hubble parameter $H(t)$.

- *The synthesis of light elements during the nucleosynthesis.*
  The theory of nucleosynthesis predicts the relative abundance of the elements with approximately 75% hydrogen, 24% helium, and a small fraction of percent for light elements such as deuterium and helium-4. These theoretical predictions are confirmed by observations at high accuracy.

- *The Cosmic Microwave Background Radiation (CMBR).*
  The discovery of the CMBR (corresponding to 2.72 K black body

radiation [10]) was the most important test of the standard big bang model. First, the isotropy of the CMBR on large scales confirms the assumption made for the local geometry, i.e., the FRW metric. Secondly, the CMBR brings us the information about the state of the Universe at the time of *recombination* ($z \approx 1100$) and supports, alongside the theory of nucleosynthesis, that the early Universe was very hot and has been cooled down as the Universe expands: $T = T_0(1+z)$. Density fluctuations in the hot plasma give rise to temperature fluctuations in the CMBR and measurements of the temperature anisotropy on the small scales allows us to estimate the density contrast at the time of recombination. Since the discovery of CMBR anisotropy at the level of one part in 10^5 by Cosmic Background Explorer (COBE) [46], the precise measurements of the CMBR have become one of the finest method to estimate cosmological parameters, peaking with recent results from Wilkinson Microwave Anisotropy Probe (WMAP) [10], [11].

Cosmological constant

Estimates of the density parameter of the matter Ω_m at present are constrained as follows: $0.1 \leq \Omega_m \leq 0.4$. The WMAP data suggests $\Omega_m = 0.27 \pm 0.04$ [10]. Here "matter" does not mean only the baryonic matter, density parameter Ω_b of which is measured to be $\Omega_b = 0.044 \pm 0.004$ [10], but also all kinds of exotic particles, including *dark matter*. On the other hand, precise measurements of the CMBR lead to the conclusion that the total density parameter is very close to one: $\Omega = 1.02 \pm 0.02$ [10], supporting the flat Universe ($k = 0$). This means that the matter is not dominating in the Universe and some form of *dark energy* is needed. *Cosmological constant* Λ with density parameter $\Omega_{_\Lambda} \simeq 0.7$ and with energy density $\rho_{_\Lambda} = 1.3 \times 10^{-29}$ g cm^{-3} $= (10^{-3}\ \text{eV})^4$ fits well. This conclusion is in agreement with earlier observations made independently by two groups, High-Z Supernova Team [8] and Supernova Cosmology Project [9]. They used Type Ia supernovas as standard candles and concluded that the cosmological model with $\Omega_{_\Lambda} \sim 0.7$ fits the data better than the matter dominated model.

Here we face the famous cosmological constant problem [12]. If it is assumed that the cosmological constant (or vacuum energy) is the energy of quantum fluctuations of the vacuum then methods of quantum field theory lead to the energy density of fluctuations $\rho_{\Lambda}^{the} \sim M_{Pl}^4 = (10^{27}\ \mathrm{eV})^4$ which differs from the observed result ρ_{Λ} by a fraction of 10^{120}

$$\rho_{\Lambda}^{the} = 10^{120}\ \rho_{\Lambda}\,. \tag{2.21}$$

Lots of efforts have been undertaken to explain such a huge gap or to "hide" it but a satisfactory explanation is still unknown. For a recent review, see [13], [14].

Another interesting observation related to the vacuum energy is the *coincidence* problem. The observed density parameters of the vacuum energy and the matter are of the same order of magnitude, $\Omega_m \simeq \Omega_{\Lambda}$, at the present moment but the ratio is changing continuously as follows

$$\frac{\Omega_{\Lambda}}{\Omega_m} = \frac{\rho_{\Lambda}}{\rho_m} = \begin{cases} const. \times a^4\ , & \text{radiation} \\ const. \times a^3\ , & \text{dust}\ . \end{cases} \tag{2.22}$$

The transition from the matter domination to the vacuum energy domination occurs during a short period in the history of the Universe and it is remarkable that we observe the Universe at that moment. In other words, why should we live in a moment when dark energy only slightly dominates over the other forms of matter? Among others, an anthropic arguments are used to explain that coincidence, see [14].

Quintessence

If the vacuum energy dominates in the present day Universe it inevitably results in the accelerated expansion of the Universe. So far we have assumed that the dark energy should be the cosmological constant ($w = -1$) and its energy density does not vary in time. From equation (2.11) it follows that the acceleration $\ddot{a} > 0$ results also in the case if cosmological constant is absent but if the following condition is fulfilled for so-called dark energy:

$$\frac{\ddot{a}}{a} \propto -(\rho + 3p) \qquad \Longrightarrow \qquad p < -\frac{1}{3}\rho\,. \tag{2.23}$$

In accordance with equation (2.11) it follows that the acceleration is possible in the case when the energy density redshifts away slower than $\rho \sim a^{-2}$ as the Universe expands.

Recent data from WMAP [10] suggest the dark energy equation of state to be $w < -0.78$ with the assumption that $w \geq -1$. Thus, the observations allow also a dynamical component for the missing energy with slowly varying energy density which is mimicking a nonzero cosmological constant but possibly with $w \neq -1$. This choice is assumed to be more suitable to explain the coincidence problem and also offering a possibility to consider a cosmological model where the acceleration epoch is temporal and at some future moment the Universe may enter into the decelerating phase again. A realization of that idea which uses a time dependent, spatially inhomogeneous scalar field $Q(t, x^\mu)$ with a suitable potential $V(Q)$ is known as *quintessence* [16], [17]. In this case the energy density of quintessence field Ω_Q replaces the energy density of cosmological constant Ω_Λ and gives the main contribution to the energy density of the Universe: $\Omega_Q \approx 0.7$. The barotropic index w will vary with time in this case and can be tuned to fit the observed value since

$$w = \frac{\frac{1}{2}\dot{Q}^2 - V(Q)}{\frac{1}{2}\dot{Q}^2 + V(Q)} \qquad \Longrightarrow \qquad -1 < w < 1\,. \tag{2.24}$$

The driving field of quintessence Q has much in common with the scalar field ϕ to be used to generate early accelerating phase known as cosmological *inflation* (see the next subsection). The common idea in both cases is that the field must evolve slowly ($\dot{Q}$ is small) and the leading contribution to the energy density comes from the potential $V(Q)$ with a rather flat slope. In this case the slowly rolling Q-field can be in the rôle of the dark energy and its potential gives rise to an almost constant energy density of the dark energy $\rho_Q \approx V(Q) \approx const.$ Since the potential determines the dynamics of the model, different proposals are presented, for a review see [18]. If we suppose that at some future moment the accelerated evolution stops we should look for potentials that asymptotically go to zero. Typical potentials with this property are $V(Q) \sim e^{-\alpha(w)Q}$ or $V(Q) \sim Q^{-n}$.

Problems of the standard model

The standard cosmological model describes rather well the observable Universe. However, despite its success there are some open questions and some unsolved problems. Some of them are believed to be explainable but the others require modification or complementation of the model. Let us mention the most important problems:

- *The horizon problem.*
  Let us consider the observed uniformity of CMBR at large scales. The

radiation decoupled from the rest of the matter at the temperature $T_{rec} = 3000$ K and this corresponds to the surface of the *last scattering* at the redshift of $z \approx 1100$.

The observable Universe is proportional to the inverse of the Hubble parameter, called the *Hubble radius*, $d_{ph} \sim H^{-1}$. The length scale corresponding to the size of present day particle horizon was at the time of recombination as follows

$$L(t_{rec}) = d_{ph}(t_0)\left(\frac{a(t_{rec})}{a(t_0)}\right) = d_{ph}(t_0)\left(\frac{T_0}{T_{rec}}\right) . \qquad (2.25)$$

Here we assumed that the expansion was adiabatic $aT \approx const.$ On the other hand, particle horizon in a matter dominated Universe evolves differently

$$d_{ph}(t_{rec}) = d_{ph}(t_0)\left(\frac{H(t_0)}{H(t_{rec})}\right) = d_{ph}(t_0)\left(\frac{T_0}{T_{rec}}\right)^{\frac{3}{2}} . \qquad (2.26)$$

We see that the length scale corresponding to the size of observable Universe was bigger than the particle horizon, i.e., the size of causally connected domains. The volume elements of corresponding scales are

$$[L(t_{rec})]^3 = \left(\frac{T_0}{T_{rec}}\right)^{-\frac{3}{2}} [d_{ph}(t_{rec})]^3 \approx 10^5\ [d_{ph}(t_{rec})]^3 . \qquad (2.27)$$

This means that the present Hubble volume contains 10^5 domains which were causally disconnected at the time of recombination and they can't be "correlated" through causal processes. Widely separated points on the last scattering surface are outside of each other's horizon and are expected to be "uncorrelated" with each other. However, the CMBR does not show this "uncorrelation" but is almost uniform across the sky. A mechanism to establish this uniformity would need transmittance of information at about 100 c, where c is the speed of light.

- *The flatness problem.*
  Let us explain this problem with the help of the Friedmann equation (for different formulations, see [3]). Equation (2.12) can be written as follows

$$|\Omega - 1| = |k|\,\dot{a}^{-2} \propto \begin{cases} const. \times t\ , & \text{radiation} \\ const. \times t^{2/3}\ , & \text{dust}\,. \end{cases} \qquad (2.28)$$

If $\Omega = 1$, it remains as such for all time (unstable fixed point, see [47]) but if $\Omega \neq 1$ initially, then the value of Ω moves away from unity as the Universe expands. WMAP data indicate that at the present moment $t_0 \sim 13.7$ Gyr $= 4.3 \times 10^{17}$ s the density parameter is close to unity $\Omega = 1.02 \pm 0.02$. In this case at the time of nucleosynthesis, $t_{ns} \sim 1$ s, the density of the Universe should be critical at the level of accuracy 10^{-17}

$$|\Omega - 1| < 10^{-17}\,, \qquad \Longrightarrow \qquad \rho(t_{ns}) = (1 \pm 10^{-17})\rho_{cr} \tag{2.29}$$

and at the Planck time $t_{_{Pl}} \sim 10^{-43}s$ the accuracy should be huge

$$|\Omega - 1| < 10^{-60}\,, \qquad \Longrightarrow \qquad \rho(t_{_{Pl}}) = (1 \pm 10^{-60})\rho_{cr}\,. \tag{2.30}$$

This enormous fine-tuning is known as the *flatness* problem.

- *The initial singularity problem.*
  The FRW models predict an *initial singularity* which is known as a big bang. The solutions (2.9) and (2.15) indicate that the energy density goes to infinity as $t \to 0$ and the solutions cannot be continued to $t < 0$.

2.1.2 Inflation

The horizon problem and the flatness problem can be solved by an assumption that in the early Universe there was a period during which the physical scale L_{pd} evolves faster than the horizon scale $d_{ph} \sim H^{-1}$

$$\frac{d}{dt}\left(\frac{L_{pd}}{H^{-1}}\right) > 0\,. \tag{2.31}$$

The scales which were outside of the horizon ($L_{pd} > H^{-1}$) at the time of recombination, had a possibility to be within the horizon ($L_{pd} < H^{-1}$) at some earlier epoch and are therefore causally connected. Sometimes it is more convenient to define the inflation as the decrease of the comoving Hubble length L_{cd}

$$\dot{L}_{cd} \equiv \frac{d}{dt}\left(\frac{H^{-1}}{a}\right) = \left(\frac{-\ddot{a}}{\dot{a}^2}\right) < 0\,. \tag{2.32}$$

In brief, inflation is an epoch of the Universe during which the scale factor a is accelerating $\ddot{a} > 0$. From the equation (2.11) with $\Lambda = 0$ follows the condition for inflation:

$$\ddot{a} > 0 \quad \Longleftrightarrow \quad (\rho + 3p) < 0 \quad \Longrightarrow \quad p < -\frac{1}{3}\rho \,. \tag{2.33}$$

In general, inflation is supposed to begin at $t_{inf} \sim 10^{-36}$ s after the big bang within a small patch of spacetime of Planck size M_{Pl}^{-1}. Since our spacetime has four macroscopic dimensions it is assumed that during the inflation at least three spatial dimensions expanded very fast. According to string theory the spacetime manifold should be 10-dimensional. It is believed, that during the inflation at least six dimensions are contracting or growing to some finite size and then stabilized due to some unknown mechanism. Why only three spatial dimensions expanded, is unknown. The possibility of the existence of large extra dimensions and their influence to cosmology will be discussed in Chs. 4, 5.

Dynamics of the scalar field

The generic feature of inflation is the introduction of a minimally coupled scalar field ϕ, called the *inflaton*. The action for a minimally coupled real scalar field is given by

$$I_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \, \nabla_\nu \phi - V(\phi) \right] , \tag{2.34}$$

which leads to the energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu \phi \, \nabla_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \, \nabla_\alpha \phi \nabla_\beta \phi + V(\phi) \right] . \tag{2.35}$$

Assuming that the scalar field is homogeneous and isotropic $\phi(t)$, i.e., the gradient terms $\nabla_i \phi = 0$ are negligible, then the equation of the scalar field in the case of the FRW spacetime is as follows

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \,. \tag{2.36}$$

This coincides with the conservation equation (2.9), since from the energy-momentum tensor we get for the energy density and pressure:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V \,, \qquad\qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V \,. \tag{2.37}$$

Taking into account these expressions, the condition for inflation (2.33) can be written $\dot{\phi}^2 < V(\phi)$, i.e., the potential energy dominates over the kinetic energy. This requires the potential to be positive and very flat to keep the kinetic energy of the scalar field under control. The scalar field is slowly rolling down its potential during the inflation. The corresponding equation of state $p_\phi \simeq -\rho_\phi$ is approximately the same as in the case of cosmological constant $p_{_\Lambda} = -\rho_{_\Lambda}$. The simplest inflationary dynamics follows immediately. From the conservation law (2.9), written for ρ_ϕ and p_ϕ, it follows that $\rho_\phi = \rho_i \approx const.$ and from equation (2.10) we get ($\Lambda = 0$, $k = 0$)

$$H^2 = \frac{8\pi G_{_N}}{3}\rho_i \qquad \Longrightarrow \qquad a = a_i\, e^{\sqrt{\frac{8\pi G_N \rho_i}{3}}\,(t-t_i)} = a_i\, e^{H_i(t-t_i)}\,, \quad (2.38)$$

where a_i denotes the value of the scale factor and t_i denotes the time when inflation starts. The exponential growth of the scale factor is typical for standard inflation but this is not necessary to fulfil the conditions (2.31) or (2.32).

We have set $k = 0$ in equation (2.38) but since we want to solve the flatness problem, it is inconsistent to assume it. Still, obviously the curvature term becomes rapidly negligible (see equation (2.12)) once inflation is switched on

$$|\Omega - 1| = \frac{1}{H_i^2}\,\frac{|k|}{a_i^2}\, e^{-2H_i(t-t_i)}\,, \qquad \Longrightarrow \qquad \Omega = 1\,. \qquad (2.39)$$

At the end of the inflation the Universe is extremely flat $\Omega = 1$ and this can be taken as a suitable initial condition for the radiation dominated epoch.

Slow-roll approximation

The conditions that $\ddot{\phi} \approx 0$, $\dot{\phi}^2 \ll V(\phi)$ are known as the *slow-roll approximation* and in this case the Friedmann equation (2.10) and the Klein-Gordon equation (2.36) can be written as follows

$$H^2 \simeq \frac{8\pi G_{_N}}{3}\, V(\phi)\,, \qquad (2.40)$$

$$-3H\dot{\phi} \simeq \frac{dV}{d\phi}\,. \qquad (2.41)$$

Here we assumed that the energy density of the cosmological constant is negligible $\rho_{_\Lambda} \approx 0$ in the early Universe and that the Universe is flat, $k = 0$, which is supported by inflation. Necessary (not sufficient) conditions for the slow-roll approximation to hold can be expressed through the *slow-roll parameters* ϵ and η defined as follows

$$\epsilon(\phi) = \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \approx -\frac{\dot{H}}{H^2}, \tag{2.42}$$

$$\eta(\phi) = \frac{M_{Pl}}{8\pi}\frac{V''}{V} \approx \frac{V''}{3H^2}, \tag{2.43}$$

where $V' = \frac{dV}{d\phi}$. The slow-roll approximation works well until $\epsilon \ll 1$ and $|\eta| \ll 1$.

During the inflation, the scalar fields roll down from the potential from the initial value ϕ_i at time t_i to the value ϕ_f at time t_f when inflation stops (slow-roll conditions (2.42) and 2.43 break down). The *number of e-foldings*

$$N = \ln\frac{a(t_f)}{a(t_i)} = \int_{t_i}^{t_f} H dt\,, \tag{2.44}$$

measures the amount of expansion during the inflation and is an important characteristic. We describe the solution of the horizon problem in the context of a simple exponential solution (2.38), detailed discussion can be found in [52]. During the inflation the particle horizon stays almost constant $d_{ph} \sim H^{-1} \approx const.$ but all length scales L_{pd} are stretched exponentially $L_{pd} \sim a$. Even the scales which were within the horizon (and are therefore causally related) before inflation, are stretched outside the horizon during the inflation and seem to be "uncorrelated". In order to solve the horizon problem it is required that during the inflation the scale factor of the Universe grows at least e^{70} times:

$$\frac{a(t_f)}{a(t_i)} \geq e^{70}\,, \qquad \Longrightarrow \qquad N \geq 70\,. \tag{2.45}$$

In the slow-roll approximation the number of e-folds can be expressed

$$N \simeq -\frac{8\pi}{M_{Pl}^2}\int_{\phi_i}^{\phi_f} \frac{V}{V'}\,d\phi\,. \tag{2.46}$$

If we use the alternative definition of the inflation (2.32), the amount of inflation can be characterized by the ratio of the initial comoving Hubble length to the final one [4]

$$Z = \ln\left(\frac{L_{cd}^i}{L_{cd}^f}\right) = \ln\frac{a(t_f)\;H(t_f)}{a(t_i)\;H(t_i)}\,. \tag{2.47}$$

The difference between Z and N is not big, because the scale factor varies much faster than the Hubble parameter during the inflation.

Chaotic inflation

Detailed dynamics of the inflation in the slow-roll approximation is determined by the potential $V(\phi)$. One of the most popular scenarios is the *chaotic inflation* [48], in which the Universe emerges from the Planck epoch with the scalar field taking different values in different parts of the Universe. We are located in the part of the Universe were the inflation took place. This model is described by a massive non-interacting scalar field ϕ with effective potential

$$V(\phi) = \frac{1}{2}m^2\phi^2\,, \tag{2.48}$$

where m is the mass of the scalar field ϕ. It is assumed that the initial energy density is of the order of the Planck density: $\rho_i \simeq M_{Pl}^4$. Equations (2.40) and (2.41) can be easily solved and we get a linearly evolving dilaton

$$\phi(t) = \phi_i - \frac{m}{\sqrt{12\pi G_N}}t\,, \tag{2.49}$$

and an approximately exponentially evolving scale factor

$$a(t) = a_i e^{\left[2\pi G_N\left(\phi_i^2-\phi^2\right)\right]}\,. \tag{2.50}$$

The inflation proceeds until the slow-roll condition $\epsilon \ll 1$ will be violated $\epsilon(\phi_f) = 1$ and this constraint implies that the inflation ends no later than $\phi_f = \frac{M_{Pl}}{\sqrt{4\pi}}$. The maximum value of ϕ_i can be calculated from the constraint that initial potential energy equals M_{Pl}^4 and this gives that $\phi_i = \frac{\sqrt{2}M_{Pl}^2}{m}$. The total number of e-foldings for the potential (2.48) can be calculated from relation (2.46)

$$N = -\frac{8\pi}{M_{Pl}^2}\left(\phi_f^2 - \phi_i^2\right) = 4\pi\frac{M_{Pl}^2}{m^2} - \frac{1}{2}\,. \tag{2.51}$$

From the requirement that the density fluctuations are not too large, the constraint $m < 10^{-5}M_{Pl}$ arises [3] and in this case at least 10^{11} e-foldings are possible.

After the end of inflation $\phi < \phi_f$ the scalar field oscillates near the bottom of the potential and starts to interact with other fields. The energy of the scalar field is transformed into the energy of radiation which equilibrates rapidly at a temperature known as the *reheat temperature* T_{rt}, and radiation dominated era begins. The thermalization does not occur simultaneously everywhere in the Universe. Along the regions which are evolving like a radiation dominated Universe there exist regions which are still inflating. This is the eternally existing *self-reproducing chaotic inflationary Universe* [49]. A detailed discussion of the chaotic inflation can be found in [3] and detailed discussion of reheating in [4].

Another important potential discussed in the literature (see for example [4]) is the exponential potential

$$V(\phi) = V_0 \, e^{\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{Pl}}\right)}, \tag{2.52}$$

where V_0 and p are constants. In this case it is possible to find an exact solution to the field equations and we get for the scale factor: $a = a_i \, t^p, \; p > 1$. This type of solution is called the *power-law inflation*.

In a recent paper [50] Borde et al showed that even if the *weak energy condition* ($\rho \geq 0$ and $\rho + p \geq 0$) is violated, the inflation can not be past eternal and the singularity problem remains unsolved. There is also a concern about the suitable *initial conditions* for inflation. If once initiated, the slow-roll inflation effectively solves the problems unanswered by the standard cosmology but the initial state is quite subtle in respect of the initial inhomogeneities (remember that we omitted the gradient term $\nabla_i\phi$) and in respect of the initial curvature. A detailed discussion of the initial conditions of the standard inflation is summarized by Goldwirth and Piran in [51]. However, most predictions made by inflation are confirmed by observations and are in good agreement with recent data. This gives a strong evidence that inflation is a suitable paradigm for the early Universe.

By the term "inflation" we understand a general paradigm which contains a wide class of models with different potentials and with different initial conditions. More detailed reviews of the inflation can be found in [2], [3], [4]. As we pointed out already in the Introduction, we don't consider the generation and evolution of perturbations here, for a review see [53], [52].

So far the Einstein general relativity has been considered as the fundamental theory of gravity. One possible extension of general relativity is the *Brans-Dicke theory* [55] (or more generally a *scalar-tensor theory*) of gravity where the scalar field is non-minimally coupled to gravity.

The importance of the scalar-tensor theory is twofold. First, it is an extension of the standard gravitational theory with an additional degree of freedom which can be interpreted as a variable gravitational constant. Secondly, the scalar-tensor gravity is a model theory for a low energy limit of string theory, where the scalar degree of freedom, called *dilaton*, is a necessary partner of the graviton. The dilaton appears in the low energy effective theory due to the Kaluza-Klein reduction, according to which higher-dimensional theory is reduced to lower dimensional theory.

The scalar-tensor theory was conceived originally by Jordan and later developed by Brans and Dicke [55] who proposed the following action

$$I_{BD} = \int d^4x\sqrt{-g}\left(\varphi R - \omega\frac{1}{\varphi}\, g^{\mu\nu}\nabla_\mu\varphi\,\nabla_\nu\varphi + L_{matter}(\Psi)\right)\,, \qquad (2.53)$$

where ω is a dimensionless constant and Ψ represents matter fields. Observationally it is required that $\omega > 3600$ [54]. Note, that the matter Lagrangian does not contain the scalar field φ. In the case of general scalar-tensor theories ω is not a constant but is a scalar field dependent function: $\omega = \omega(\varphi)$. By comparing the first terms in the Einstein-Hilbert action (2.6) and in the Brans-Dicke action (2.53) it is easy to deduce that the Brans-Dicke theory contains an *effective gravitational constant* G_{eff}, defined by

$$\varphi = \frac{1}{16\pi G_{eff}}\,. \qquad (2.54)$$

The action (2.53) is presented in the *Jordan frame* where the scalar field is non-minimally coupled to the metric. It is possible to transform the action (2.53) into another frame, called the *Einstein frame*, where the metric and the scalar field are minimally coupled. In the case of scalar-tensor theories of gravity, there has been long debate, dating back to the works of Brans and Dicke [55], [56] to answer the question, which frame represents the physical one. We don't discuss this question here but make some comments about the frame in Sec. 3.1 . A comprehensive discussion of the Brans-Dicke gravity was presented recently by Fujii and Maeda [54].

The scalar field is important in cosmology. In the standard inflation the potential of the scalar field drives the slow-roll evolution and the scalar field is also one possible candidate for the dark energy, which drives the observed late time acceleration. In the framework of the Einstein general relativity the action of the scalar field is added by hand into the total action. In the scalar-tensor theory, the scalar field is a part of the mathematical description of gravity.

The field equations derived from action (2.53) (in the case of general coupling, $\omega(\varphi)$) are analyzed extensively. The FRW cosmology with a perfect fluid sources is studied by Barrow et al in [57] and by Navarro et al in [58]. The convergence of scalar-tensor theories towards general relativity in the cosmological context is analyzed by Serna et al [59].

2.2 String/M- theory

It is well known, that up to now all attempts to develop a consistent theory of quantum gravity have not been successful. String theory [19], [20] has been a subject of extensive studies as the most promising approach to unify gravity with other interactions and provide us with the theory of quantum gravity. Despite great progress in theoretical aspects, direct evidence supporting string theory is still lacking.

Since string theory is intended to describe the spacetime also at large curvature scales and the matter at high energies, it is legitimate to apply it for the early Universe. Special hope is that it can be applied to solve the initial singularity problem, which remained unsolved by inflation.

String theory is certainly not the subject of this thesis and therefore we only mention some characteristic aspects of it.

- *String theory is a quantum theory which contains gravity.*

The closed string has a zero mode described by a symmetric second rank tensor which behaves in the low energy limit as the spacetime metric. The requirement of supersymmetry and absence of conformal anomalies [19], [20] restrict the number of consistent string theories down to five: Type IIA, Type IIB (both having $N = 2$ supersymmetry), Type I, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic (having $N = 1$ supersymmetry). Each theory has a consistent perturbative expansion. It has been argued that the $E_8 \times E_8$ heterotic string theory contains the standard model described by gauge group $SU(3)\times SU(2)\times U(1)$ and is therefore a serious candidate to describe the real world.

• *String theories are related by duality symmetries.*
Duality symmetries between different theories lead to a conjecture that they all represent different low energy realizations of a more fundamental theory, called *M-theory.* Duality symmetries indicate that there exists one more low energy realization of the M-theory which corresponds to 11-dimensional supergravity [99]. The M-theory is a covering theory, formulated in 11-dimensional spacetime, which reduces, in the case of certain conditions, to the five superstring theories or to the quantum completion of 11-dimensional supergravity. *T-duality* states that there is no difference between a compactification on a circle of radius R and a compactification on a circle of radius $1/R$: $R \leftrightarrow 1/R$. This means that the length of the compactification radius is not an invariant concept in string theory. *S-duality* relates a strongly coupled theory to an other theory which is weakly coupled: $f_A(g) = f_B(1/g)$, where f_A (f_B) is a physical observable of the theory A (theory B) and g is the coupling constant.

• *String theory is a higher-dimensional theory.*
An important feature of string theory is that it can be consistently formulated in more than 4-dimensions. M-theory is a 11-dimensional theory. However, the present day Universe, at low energies looks 4-dimensional and therefore the theory requires six extra spatial dimensions beyond the four that we observe. A conventional approach assumes that the extra dimensions are compactified into tiny compact space of the Planck scale. However, recent progress in string theory has led to a possibility that the extra dimensions (at least one of them) should not be of the Planck scale. This possibility has lead to the braneworld conception, introduced at the fundamental level in [26], [27] and at the phenomenological level in [30], [34].

• *String theory contains objects of more than one dimension.*
In addition to 1-dimensional strings, the theory contains p-dimensional extended objects, called *p-branes* [20].

• *String theory contains a dilaton field.*
The compactification of extra dimensions induces a variety of *moduli fields*, which characterize the size and the shape of extra dimensions. All string theories contain a massless scalar field ϕ, called the *dilaton*, which belongs to the same multiplet as the graviton and which determines the string coupling constant $g_s^2 = e^{\phi}$.

2.3 Summary

The observations show that the standard cosmological model supplemented with the early inflationary epoch, may indeed be a realistic model of our observable Universe and we refer to it as the standard paradigm.

The observed features allow us to believe that the late Universe we see today is well described by the Friedmann-Robertson-Walker (FRW) spacetime. This leads to a conclusion that any modification of the standard cosmological model must give rise to the Universe which expands, at least in some epoch, in a way essentially similar to the FRW Universe. This allows us to use the FRW cosmology as a constraint. At the same time we are quite free to modify the theory for the early Universe (the epoch before nucleosynthesis) and to study the imprints of these modifications in the evolution of the late Universe.

The inflationary epoch is the earliest period in the history of the Universe whose existence and basic predictions are at least partially confirmed by recent observations of the Cosmic Microwave Background Radiation (CMBR) and the large-scale structure. If we study some alternative cosmological model or an extension of the standard inflation, the modifications can't be very rude because they must imply the same imprints for the CMBR and large-scale structure. If we are looking for an extension, for example in string theory context, the question which should be asked is weather the theory admits an inflationary solution or some other mechanism with a similar outcome. If we remember the definition of inflation (2.32), it follows that besides the accelerated expansion it contains the possibility to consider the accelerated contraction, $\ddot{a} > 0$, $\dot{a} < 0$. Moreover, it has not been proven that the slow-roll inflation is the only way to solve the standard cosmological problems.

Chapter 3

Pre-big bang scenario

According to the prevailing paradigm, the inflation or a similar phenomenon is needed to solve the problems of the standard cosmology. On the other hand, string theory as a fundamental theory should be applied to describe the early Universe. In general, it has been difficult to derive potentials from string theory which are suitable for slow-roll inflation.

The *pre-big bang scenario*, developed by Gasperini and Veneziano in [21], [22] is a string theory based model of cosmology where the inflation is realized differently compared with the standard chaotic inflation (see Subsec. 2.1.2). Namely, the accelerated expansion of the scale factor is driven by the kinetic energy of dilaton and not by the potential of the scalar field. This model employs a string duality within a cosmological setting and thus provides an interesting opportunity to consider a non-standard cosmology in the context of string theory. The field equations derived from the low energy effective action of string theory have a symmetry property, called the *scale-factor duality.* This duality symmetry relates the FRW Universe and the *superinflationary* pre-big bang Universe. The low energy effective action and the duality symmetry of the solutions are discussed in Secs. 3.1 and 3.2. Unfortunately, the pre-big bang scenario suffers from the problem of singularity and the problem of initial conditions (see Secs. 3.3 and 3.4). It was hoped that the singularity problem can be cured with the help of non-perturbative stringy effects but the complete solution to this problem has not yet been found.

Since the comprehensive reviews by Lidsey et al [60] and by Gasperini et al [61] are available and an up-to-date web page about pre-big bang cosmology is maintained [62], we present here only a short review of the pre-big bang cosmology. A lot of topics, including generation and evolution of quantum fluctuations during the pre-big bang inflation are discussed thoroughly in [60] and [61].

3.1 Effective low energy action

Effective bosonic actions of five different string theories contain several terms, which are common to all of them, see [60] for a review. In all cases the gravitational sector of the effective action contains graviton g_{MN}, dilaton Φ and antisymmetric 2-form potential B_{MN}.

In the Jordan (string) frame, in the case of a 10-dimensional spacetime, the effective action of the gravitational sector is

$$\begin{aligned} I_{eff} &= \frac{1}{2\lambda_s^8}\int d^{10}x\sqrt{-g}e^{-\Phi}\left[{}^{10}R + g^{MN}\nabla_M\Phi\nabla_N\Phi - \frac{1}{12}H_{MNR}H^{MNR}\right] \\ &+ [\text{ higher order terms in } \lambda_{\mathrm{s}}^{\mathrm{d}-1}\cdot\nabla^2] \\ &+ [\text{ higher order terms in } \mathrm{e}^{\Phi}]\,, \end{aligned} \tag{3.1}$$

where $H_{MNR} = \nabla_{[M}B_{NR]}$ represents the antisymmetric tensor field B_{MN} through its field strength, ${}^{10}R$ is 10-dimensional scalar curvature, Φ is the dilaton field determining the strength of the string coupling through $g_s^2 \equiv e^{\Phi}$ and $\lambda_s \sim \sqrt{\alpha'}$ is the string length scale defined through the inverse of string tension α'. The string length λ_s is replacing the Planck length l_{Pl} in string theory (see 3.4) and is the minimal observable scale in the theory.

Since all five superstring theories are formulated in a 10-dimensional spacetime, the compactification down to 4-dimensions should be the first step. We omit it here and refer for the discussion of that question to the review by Lidsey et al [60]. Upon compactification, in the lowest order in the inverse string tension and coupling, the action (3.1) can be written as follows

$$I_{eff} = \frac{1}{2\lambda_s^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right]. \tag{3.2}$$

In four dimensions the antisymmetric tensor field $B_{\mu\nu}$ is equivalent to a pseudo-scalar, usually called the *axion* A, through relation

$$H^{\mu\nu\rho} \equiv e^{\phi}\epsilon^{\mu\nu\rho\lambda}\nabla_\lambda A\,, \quad \Longrightarrow \quad H^{\mu\nu\rho}H_{\mu\nu\rho} \to \frac{1}{2}e^{2\phi}\nabla^\lambda A\nabla_\lambda A\,, \tag{3.3}$$

where $\epsilon^{\mu\nu\rho\lambda}$ is the covariantly constant four-form. We don't consider *moduli fields*, arising from the compactification, and assume that the internal dimensions are stabilized. Comparing the action (3.2) with the Einstein-Hilbert action (2.6), we can deduce the following relations:

$$\sqrt{8\pi G_N} \equiv l_{Pl} = \lambda_s e^{\frac{\phi}{2}} \qquad \Longrightarrow \qquad g_s^2 \equiv e^{\phi} = \left(\frac{l_{Pl}}{\lambda_s}\right)^2 . \tag{3.4}$$

The weak coupling region coincides with $\phi \ll -1$. The string length λ_s is assumed to provide a cut-off scale of the theory and physical quantities are bound by appropriate powers of λ_s: $R \sim H^2 \sim G_N \rho < \lambda_s^{-2}$.

String frame vs. Einstein frame

Actions (3.1) and (3.2) are written in the so-called string or the Jordan frame. In the string frame the dilaton is non-minimally coupled to the metric and it is possible to go to an other frame, called the Einstein frame, where the dilaton is minimally coupled. In general, these two frames are related by a *conformal transformation*

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \Omega^2(x)\; g_{\mu\nu} \,, \tag{3.5}$$

where $\Omega(x)$ is conformal factor[1].

In the case of action (3.2) the conformal transformation from the string frame to the Einstein frame can be done by a dilaton-dependent rescaling

$$\begin{aligned} \bar{g}_{\mu\nu} &= e^{\left(-\frac{2\phi}{d-1}\right)}\; g_{\mu\nu} \,, \\ \bar{\phi} &= \sqrt{\frac{2}{d-1}}\; \phi \,, \end{aligned} \tag{3.6}$$

where $d = 3$ in our Universe. In the Einstein frame, $\bar{g}_{\mu\nu}$, the gravitational part of the action (3.2), takes the standard Einstein-Hilbert form. In the σ-model approach [63], the string frame metric $g_{\mu\nu}$ coincides with the metric to which test strings are directly coupled. Thus, motions of a free string follow geodesic surfaces with respect to the string frame metric.

In the string frame, the string length λ_s is taken to be fixed, while in the Einstein frame the string length is dilaton dependent. In contrast, in the Einstein frame, the Planck length l_{Pl} is considered to be constant and

[1]The conformal factor $\Omega^2(x)$ is not related to the density parameter Ω, defined in Subsec. 2.1.1.

in the string frame it evolves as $l_{Pl} = e^{\phi/2}\,\lambda_s,$, i.e., it is dilaton (time) dependent.

In some cases it is technically simpler to find solutions and study perturbations in the Einstein frame, but it is believed that the string frame provides the physical interpretation. All physical quantities must be, of course, independent of the frame. In asymptotic states, where the dilaton approaches a constant value, the Einstein and string frames become practically equivalent. For a more detailed analysis of frame-independence in string cosmology see, e.g., [64]. In what follows we shall work in the string frame.

3.2 Field equations and scale-factor duality

By varying the effective action (3.2) with respect to metric $g_{\mu\nu}$, dilaton ϕ and antisymmetric field $B_{\mu\nu}$, respectively, we get the corresponding equations of motion

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\alpha\beta}\, H_\nu{}^{\alpha\beta} = \lambda_s^2 e^{\phi} T_{\mu\nu}\,, \tag{3.7}$$

$$R + 2\nabla^\mu \nabla_\mu \phi - g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{12} H_{\mu\nu\rho}\, H^{\mu\nu\rho} = 0\,, \tag{3.8}$$

$$\nabla_\nu \left(e^{-\phi}\, H^{\nu\rho\mu}\right) = 0\,. \tag{3.9}$$

In the case when the dilaton and ordinary matter are minimally coupled the usual conservation equation holds

$$\nabla^\mu\, T_{\mu\nu} = 0\,. \tag{3.10}$$

A possible contribution of the matter sources is represented by the energy-momentum tensor of perfect fluid (2.3) $T_{\mu\nu}$. In what follows we assume a spatially flat FRW background (2.1), homogeneous dilaton $\phi = \phi(t)$ and trivial axion field $\nabla_\mu A = 0$. We solve the field equations (3.7) - (3.10) in the case of pure dilaton gravity and in the case of dilaton gravity with perfect barotropic fluid matter. We will consider a particular class of solutions with a power law evolution of the scale factor and a logarithmically evolving dilaton

$$a = a_0\, \tau^{\alpha(\Gamma)}, \qquad \phi = \phi_0 - \beta(\Gamma) \ln \tau, \tag{3.11}$$

where τ is a time parameter.

The Friedmann equations (2.9) - (2.11), derived from the Einstein-Hilbert action (2.6), are invariant under the time reversal $t \to -t$ transformation. If $a(t)$ is a solution of equations (2.9) - (2.11), then $a(-t)$ is also a solution. However, the Hubble parameter H changes its sign under the time reversal: $H \to -H$. To any standard solution $H(t)$, describing decelerating expansion $H > 0$, corresponds a reflected solution $H(-t)$, describing decelerating contraction $H < 0$.

In the case of the FRW metric with $k = 0$, equations of motion (3.7) - (3.9) derived from the low energy string effective action have an additional symmetry property. Namely, the equations are invariant under the inversion of the scale factor $a \to \bar{a} = a^{-1}$, which is accompanied by an appropriate transformation of the dilaton $\phi \to \bar{\phi} = \phi - 6 \ln a$. Thus, if $\{a,\ \phi\}$ is a solution, then $\{\bar{a},\ \bar{\phi}\}$ is also a solution of the field equations. This invariance is called the *scale-factor duality* and it is a particular case of *T-duality*, mentioned in Sec. 2.2 in the context of string theory. The duality transformation is independent of the time reversal, and in general cosmological solutions can be divided into two branches, I and II:

$$\bullet I(+).\{a(t):\ \dot{a}(t) > 0,\ \ddot{a}(t) < 0,\ H(t) > 0,\ \dot{H}(t) < 0;\ t > 0\},$$

$$\bullet II(+).\{\bar{a}(t):\ \dot{\bar{a}}(t) < 0,\ \ddot{\bar{a}}(t) > 0,\ \bar{H}(t) < 0,\ \dot{\bar{H}}(t) < 0;\ t > 0\},$$

$$\bullet I(-).\{a(-t):\ \dot{a}(-t) < 0,\ \ddot{a}(-t) < 0,\ H(-t) < 0,\ \dot{H}(-t) < 0;\ t < 0\},$$

$$\bullet II(-).\{\bar{a}(-t):\ \dot{\bar{a}}(-t) > 0,\ \ddot{\bar{a}}(-t) > 0,\ \bar{H}(-t) > 0,\ \dot{\bar{H}}(-t) > 0;\ t < 0\}.$$

A particularly interesting case is the transformation from branch $I(+)$ to branch $II(-)$: $I(+) \to II(-)$, which combines the scale-factor duality with the time reversal:

$$a(t) \to \bar{a}(t) = a^{-1}(-t), \qquad \phi(t) \to \bar{\phi}(t) = \phi(-t) - 6 \ln a(-t). \tag{3.12}$$

This transformation maps a solution with decelerated expansion ($H(t) > 0$) and decreasing curvature ($\dot{H}(t) < 0$) to a dual solution with accelerated expansion ($\bar{H}(-t) > 0$) and increasing curvature ($\dot{\bar{H}}(-t) > 0$). The dual solution describes the *pre-big bang* scenario, it has no analogue in the context of the standard cosmology, described in Subsec. 2.1.1. The pre-big bang scenario is characterized by a long period of dilaton-driven accelerated expansion with growing curvature and this is usually called *superinflation.*

In what follows, we present some results from our paper II [66], were we have analyzed the field equations (3.7) - (3.9) in particular cases.

Pure dilaton gravity

The field equations for the Hubble parameter $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ and the dilaton $\phi(t)$ can be written as follows

$$\dot{H} + 3H^2 - H\dot{\phi} = 0, \tag{3.13}$$

$$6H^2 - 6H\dot{\phi} + \dot{\phi}^2 = 0, \tag{3.14}$$

$$\ddot{\phi} + 3H\dot{\phi} - \dot{\phi}^2 = 0. \tag{3.15}$$

The system is not overdetermined, since only two equations are independent.

The most general isotropic solution reads

$$a = a_0(At + t_0)^{\frac{B}{\sqrt{3}}}, \tag{3.16}$$

$$\phi = \phi_0 + (B\sqrt{3} - 1)\ln(At + t_0). \tag{3.17}$$

Here $B = \pm 1$ denote two branches of solutions. The range of variation of the time coordinate t must be chosen such that $At + t_0 \geq 0$, so the time inversion amounts to a change of constant $A \to -A$. The dilaton gravity analogue of a post-big bang model corresponds to $A > 0, B = 1$, a standard pre-big bang model corresponds to $A < 0, B = -1$. Integration constants a_0, ϕ_0 are equal to values of $a(t), \phi(t)$ at an initial moment $t_i = \frac{1-t_0}{A}$ and A determines initial values for $\dot{a}(t)$, $\dot{\phi}(t)$.

An one-to-one correspondence between post-big bang solutions and pre-big bang solutions can be arranged by duality transformations (3.12). In terms of the general solution (3.16), (3.17), the correspondence between a particular post-big bang solution ($B = 1$) with a_0, ϕ_0 and a pre-big bang solution ($B = -1$) is encoded in the integration constants $\bar{a}_0, \bar{\phi}_0, A$ that have undergone the same duality transformation (3.12):

$$\bar{a}(t) = a_0^{-1}(t_0 - At)^{-\frac{1}{\sqrt{3}}}, \tag{3.18}$$

$$\bar{\phi}(t) = \phi_0 - 6\ln a_0 - (\sqrt{3} + 1)\ln(t_0 - At). \tag{3.19}$$

After some manipulations the field equations become

$$\dot{H} + 3H^2 - H\dot{\phi} = \frac{1}{2}e^{\phi}\rho(\Gamma - 1)\,, \tag{3.20}$$

$$6H^2 - 6H\dot{\phi} + \dot{\phi}^2 = e^{\phi}\rho\,, \tag{3.21}$$

$$\ddot{\phi} + 3H\dot{\phi} - \dot{\phi}^2 = \frac{1}{2}e^{\phi}\rho(3\Gamma - 4)\,, \tag{3.22}$$

$$\dot{\rho} + 3H\Gamma\rho = 0\,, \tag{3.23}$$

where the last equation is the conservation law of matter. Loosely speaking, equations (3.20), (3.22) and (3.23) describe the evolution of H, ϕ and ρ, the remaining equation (3.21) imposes a constraint.

A quasi-isotropic solution of the field equations can be written as follows

$$a = a_0(t_0 \pm |\chi|t)^{\frac{2(\Gamma-1)}{\Delta}}\,, \tag{3.24}$$

$$\phi = \phi_0 - \frac{2(4-3\Gamma)}{\Delta}\ln(t_0 \pm |\chi|t)\,, \tag{3.25}$$

$$\rho = \rho_0(t_0 \pm |\chi|t)^{-\frac{6\Gamma(\Gamma-1)}{\Delta}}\,, \tag{3.26}$$

where

$$\chi \equiv \pm\sqrt{\frac{e^{\phi_0}\rho_0\;(\Delta)^2}{-4\;(\Delta-2)}}\,, \tag{3.27}$$

$$\Delta \equiv 3(\Gamma-1)^2 + 1 > 0\,, \tag{3.28}$$

and a_0, ϕ_0, ρ_0 are integration constants which are equal to the values of functions $a(t), \phi(t), \rho(t)$ at an initial moment $t_i = \frac{1-t_0}{\pm|\chi|}$. The solution is not the most general one, since initial values of $\dot{a}(t)$ and $\dot{\phi}(t)$ are both uniquely determined by barotropic index Γ and the initial values of $\phi(t)$ and $\rho(t)$; in the most general case it must contain an additional arbitrary integration constant. As a result, solution (3.24) - (3.28) turns out to be singular at $\Gamma_s = 1 \pm \frac{1}{\sqrt{3}}$, where $\Delta - 2 = 0$, $\chi \to \infty$. In what follows we don't consider barotropic indices belonging to the neighborhood of Γ_s. Note that in the case of unusual matter with $(\Gamma-1)^2 > 1/3$ we must take $\rho_0 < 0$.

The range of variation of the time coordinate t must be chosen such that $t_0 \pm |\chi|t \geq 0$; t_0 is a numerical constant corresponding to the freedom of constant shift in time. In comparison with the solutions of the pure dilaton gravity (3.16)–(3.17), the integration constant A is replaced by the constant $\chi = \chi(\rho_0, \phi_0, \Gamma)$ and the choice of the branch, $B = \pm 1$, is imitated by the choice of the barotropic index Γ, or more exactly, by the sign of the expression $\Gamma - 1 \in [-1, +1]$.

Solution (3.24) for $a(t)$ determines the Hubble parameter as

$$H = \frac{\pm|\chi|}{(t_0 \pm |\chi| t)} \frac{2}{\Delta} (\Gamma - 1), \qquad \dot{H} = \frac{-\chi^2}{(t_0 \pm |\chi| t)^2} \frac{2}{\Delta} (\Gamma - 1). \tag{3.29}$$

We see that the constant χ is proportional to the initial value of the Hubble parameter at the initial moment t_i, $t_0 \pm |\chi| t_i = 1$.

If we take $\Gamma > 1$ and the temporal argument $(t_0 + |\chi| t)$ we get a decelerating post-big bang Universe. For example for $\Gamma = \frac{4}{3}$ (radiation dominated stage) we have $|\chi| = \sqrt{\frac{2}{3} e^{\phi_0} \rho_0}$ and $a(t) = a_0 (t_0 + |\chi| t)^{\frac{1}{2}}$, matter density is decreasing, $\rho = \rho_0 (t_0 + |\chi| t)^{-2}$, while the dilaton is frozen to a constant, $\phi = \phi_0$. The integration constants a_0 and ρ_0 are the (initial) values of the corresponding variables at the moment $t_i = \frac{1 - t_0}{|\chi|}$. A singularity ($a \to 0$, $\rho \to \infty$) is reached at the moment $t_s = -\frac{t_0}{|\chi|}$. The domain of the time variable is $t \in (t_s, +\infty)$. Taking into account our choice of the initial moment t_i we see that $t_i = t_s + \frac{1}{|\chi|}$, i.e., $\frac{1}{|\chi|}$ is the time interval from the singularity to the initial moment. If $t_0 > 0$ we have $t_s \leq t_i \leq 0$ and the Universe is regular at the moment $t = 0$. If the source matter is exotic ($\Gamma < 1$), solution (3.24) - (3.26) represents a contracting Universe beginning from a singularity at t_s and going through t_i.

The pre-big bang branch of solutions corresponds to the minus sign in front of $|\chi| t$ and exotic matter $\Gamma - 1 < 0$. The domain of the time coordinate is $t \in (-\infty, \frac{t_0}{|\chi|} \equiv t_\infty)$, i.e., $t_0 - |\chi| t > 0$, and the behaviour of the scale factor is superinflationary ($\dot{a} > 0$, $\ddot{a} > 0$, $H > 0$, $\dot{H} > 0$). Integration constants a_0, ϕ_0, and ρ_0 are the values of corresponding variables at the moment $t_i \equiv \frac{1 - t_0}{-|\chi|}$ and here $t_i = t_s - \frac{1}{|\chi|}$, i.e., $\frac{1}{|\chi|}$ is the time interval from the initial moment to the singularity. A contracting solution with a future singularity is obtained if $\Gamma > 1$.

Expressions (3.25) for ϕ and (3.29) for H imply

$$H = -(\Gamma - 1)\dot{\bar{\phi}}, \qquad \dot{\bar{\phi}} \equiv \dot{\phi} - 3H \tag{3.30}$$

and the behaviour of solutions depending on barotropic index Γ can be summarized on a phase diagram analogous to the corresponding one familiar from the pre-big bang scenario of the pure dilaton gravity, see our paper II [66] (Figure 1).

As in the case of pure dilaton gravity, solution (3.24) - (3.26) contains both the post-big bang and the pre-big bang branch. The latter one can be related to the former one by duality transformations (3.12), but in order to satisfy the field equations, here we need an additional transformation for barotropic index Γ and for energy density ρ:

$$\frac{p}{\rho} \to \frac{\bar{p}}{\bar{\rho}} = -\frac{p}{\rho} \quad \Longrightarrow \bar{\Gamma} \to 2 - \Gamma\,, \qquad \rho \to \rho a^6(-t)\,. \tag{3.31}$$

The corresponding pre-big bang solutions (i.e., with the time dependence in the form $(t_0 - |\chi|t)$) now read

$$\bar{a} = a_0^{-1}\,(t_0 - |\chi|t)^{\frac{2(\bar{\Gamma}-1)}{\Delta}}\,, \tag{3.32}$$

$$\bar{\phi} = \phi_0 - 6\ln a_0 - \frac{2(4-3\bar{\Gamma})}{\Delta}\ \ln(t_0 - |\chi|t)\,, \tag{3.33}$$

$$\bar{\rho} = \rho_0 a_0^6\,(t_0 - |\chi|t)^{-\frac{6\bar{\Gamma}(\bar{\Gamma}-1)}{\Delta}}\,. \tag{3.34}$$

Note that $\rho_0 e^{\phi_0} = \bar{\rho}_0 e^{\bar{\phi}_0}$, $\Delta = \bar{\Delta}$ and t_0 don't change under duality transformations (3.12), (3.31).

3.3 Initial conditions and fine-tuning

Solutions (3.18) and (3.19) (and solutions (3.32) - (3.34)) may be continued to the past infinity in time ($t \to -\infty$). The basic postulate of the pre-big bang scenario [22] is that the Universe emerged an infinite time ago ($t \to \infty$) in a state of low curvature ($H \to 0$) and weak coupling ($\phi \to -\infty$), corresponding to string perturbative vacuum. To characterize the initial state of the pre-big bang the principle of *asymptotic past triviality* (APT) is proposed [65]. The APT describes the asymptotic past through the classical field equations of the low energy tree level effective action of string theory, i.e., with the action (3.2). In contrast to the standard cosmology, the initial state is cold, flat and empty and is not hidden beyond the Planckian, quantum gravity regime. The problem of initial state is separated from the problem of initial singularity and this is one attractive feature of the pre-big bang scenario. However, despite the differences compared with the standard cosmology, the initial state of pre-big bang is not well understood.

It is commonly believed that the present homogeneity was achieved from a *chaotic* (inhomogeneous) initial state by inflation. However, at the same time a strong inhomogeneity and initial spatial curvature will suppress the onset and duration of inflation, see for a review [51]. In the pre-big bang scenario, the homogeneity and flatness can be obtained by a superinflation and like in the standard case the initial inhomogeneities and spatial curvature will influence the onset of superinflation.

The *generic* initial state should be simple according to APT but chaotic in a sense that spatial gradients are assumed to be in a same order of magnitude as time derivatives. This inhomogeneous state should evolve into a state where spatial gradients are negligible in comparison with time derivatives and which is asymptotically flat. It was argued by Veneziano [67] that on a constant time hypersurface, defined in the synchronous gauge, sufficiently isotropic initial patches with $\nabla g \sim \dot{g}$ and with a growing dilaton would inevitably expand and evolve towards homogeneity where $\nabla g \ll \dot{g}$. At the time when strong coupling is reached the Universe is already sufficiently homogeneous. An essentially same conclusion is reached by Buonanno et al in [68] where the equations are extended to an arbitrary number of dimensions and the two-form field $B_{\mu\nu}$ is included.

The evolution of initial inhomogeneities was also discussed in our papers I [69] and II [66]. In the paper I [69] the Hamilton-Jacobi equation, derived from the low energy tree level effective action, was solved using the *gradient expansion method* developed by Salopek et al [70], [71]. In the second order of approximation the effect of spatial gradients was included and it was found that metric corrections die off during the superinflation. This means that initial inhomogeneities, which originate from spatial gradients of the seed metric, are smoothed out and spacetime becomes homogeneous. However, dilaton corrections are growing and this means that dilaton inhomogeneities are not decaying during the superinflation. In the paper II [66] the Euler-Lagrange equations derived from the low energy string action were solved iteratively using the gradient expansion, developed in [72], [73] (in the context of Lagrange formalism the method is also called a *long-wavelength approximation*). The second order corrections, which include the effect of spatial gradients, coincide with the solution of the Hamilton-Jacobi equation in the second order of approximation, presented in I [69]. In addition, the second order corrections are found if the barotropic perfect fluid matter is included. We concluded that in the case of an exotic matter with barotropic index $\Gamma = 0$ and $\Gamma = 2$ it is possible to achieve the decay of all second order correction terms at the end of the pre-big bang stage.

Numerical calculations presented by Maharana et al [74] and Chiba [75] contain controversial results concerning the decay of initial inhomogeneities.

Initial state and fine-tuning

It has been argued that pre-big bang initial conditions have to be fine-tuned in order to give expected results [76]. Turner and Weinberg [77] concluded that curvature terms postpone the onset of inflation and can prevent getting sufficient amount of inflation before higher-order loop and string corrections become important. Kaloper, Linde and Bousso [78] have argued that horizon and flatness problems will be solved if the Universe at the onset of inflation is exponentially large and homogeneous. Counterarguments for justifying the pre-big bang inflationary model have been given in [79], [80].

Let us consider the horizon problem in the context of the pre-big bang scenario. In this case the conventional slow-roll inflation is replaced by pre-big bang superinflation which should provide a solution to the horizon (and the flatness) problem. In what follows we assume that radiation dominated hot big bang starts shortly after the moment t_f when superinflation ends. In the case when a smooth, non-singular transition into post-big bang occurs then the end of superinflation is well-defined. Namely, superinflation ends when the curvature and coupling reaches the string scale and at that time the size of the horizon is of the order of string length: $H_f^{-1} \sim \lambda_s \sim l_{Pl}$. In order to solve the horizon problem the comoving Hubble length

$$L_{cd} = \left(\frac{H^{-1}}{a}\right) = \dot{a}^{-1} \tag{3.35}$$

must decrease by a factor at least 10^{30} during the superinflation. The condition can be written in the case of pure dilaton gravity (3.16) as follows

$$\frac{L^f_{cm}}{L^i_{cm}} = \left(\frac{(t_0 - t_f)}{(t_0 - t_i)}\right)^{1+\frac{1}{\sqrt{3}}} \leq 10^{-30} \ , \tag{3.36}$$

where t_i and t_f mark the initial and final moment of the superinflation, respectively. The condition (3.36) can be written as follows

$$(t_0 - t_f) \approx 10^{-19}\,(t_0 - t_i) \qquad \Longrightarrow \qquad H_i^{-1} \approx 10^{19}\,\lambda_s\,. \tag{3.37}$$

This means that the initial horizon and thus the initial homogeneous patch, is very large in string units. This has been criticized by Kaloper et al in [78] since in the case of conventional inflation the size of initial homogeneous patch is of order l_{Pl} and the size of the horizon is of the same order. Here we stress the important kinematical difference between the power law $a \sim t^q$, $q > 1$ and superinflationary $a \sim (-t)^q$, $q < 0$ backgrounds. Namely, the event horizon (2.20), $d_{eh} \sim H^{-1}$, is growing in the power law case and is shrinking in the superinflationary case. The question, which is the natural scale for the initial homogeneous patch, the Planck (string) scale or the horizon scale, is still under debate [80].

In our paper I [69] we adopted the procedure presented by Nambu and Taruya [81] in the context of the de Sitter inflation to the pre-big bang cosmology to estimate the effect of the initial curvature. We found that positive spatial curvature reduces the horizon size compared with the flat case and therefore favors inflation. The negative spatial curvature, on the other hand, increases the horizon size and through that it is unfavorable for inflation. The situation is contrary to the case of the de Sitter inflation [81]. We estimated the initial curvature radius and got that it must be huge $C^i_{curv} \sim 10^{16}\ \lambda_s$. This means that the initial patch which can be inflated should be very flat.

3.4 Graceful exit problem in string cosmology

The standard FRW cosmology has a singularity in the past but the dual pre-big bang has a singularity in the future. The transition from the pre-big bang branch to the post-big bang branch is a necessary part of the scenario. There exist *no-go theorems* [82], [83] which state that neither a dilaton potential nor the matter sources in a form of a perfect fluid, added to the action (3.2), can accommodate the transition from pre-big bang to the post big bang. It has been argued [22], [82] that the transition from pre- to post-big bang will take place in the regime where the curvature is about the string scale $R \sim \lambda_s^{-2}$ and the coupling is strong $e^\phi \sim 1$. The lowest order approximation breaks down before either limit is reached and correction terms in the action (3.2) must be taken into account. Near the singularity, both the α'-corrections, controlling the finite-size effects in string theory, and higher order expansion in the string coupling e^ϕ, controlling the loop corrections, are important.

In the lowest order, evolution equations do not contain α' (or λ_s^2) and are singular since there is no scale at which the growth of the curvature is stopped. To the first order in α', the simplest effective action can be written as [84]

$$I_{eff} = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g} e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{\bar{\kappa}\alpha'}{4} R^2_{\mu\nu\alpha\beta} \right], \tag{3.38}$$

where $\bar{\kappa} = 1$ for the bosonic, $\bar{\kappa} = \frac{1}{2}$ for the heterotic and $\bar{\kappa} = 0$ for the Type II strings[2]. It is more convenient to treat the field equations that do not contain higher than the second order derivatives. It is possible to perform a field redefinition [85] and in this case the action (3.38) becomes [86]

$$I_{eff} = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g} e^{-\phi} \left[R + (\nabla\phi)^2 - \frac{\kappa\alpha'}{4} \left(R^2_{GB} - (\nabla\phi)^4 \right) \right], \tag{3.39}$$

here

$$R^2_{GB} = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2 \tag{3.40}$$

is the Gauss-Bonnet invariant. In this case, as we can see from (3.39), the α'-corrections are dilaton dependent.

In [79], [86] it was assumed that α'-corrections become important before the loop corrections. This means that the Universe reaches the high-curvature phase, i.e., $R \sim \lambda_s^{-2}$, but the coupling still remains small, i.e., $g_s^2 \ll 1$. In this case action (3.39) and respective field equations are in force. In a spatially flat case ($k = 0$, during superinflation the term $\frac{k}{a(t)^2}$ approaches zero) the superinflation approaches asymptotically a stage of exponential inflation with linearly growing dilaton (called de Sitter or string phase):

$$a(t) = -\frac{1}{H_s} e^{H_s(t-t_s)}, \qquad \phi(t) = \phi_s + c(t - t_s), \qquad c = const. \tag{3.41}$$

Here H_s, ϕ_s, and t_s are constants denoting the values of the Hubble parameter, dilaton and time at the transition between the superinflationary and the string phases. This solution acts as a late time attractor of the superinflationary solution obtained from the action (3.2). If so, the curvature

[2] In the latter case the the first corrections appear in the second order in α', i.e., $\sim R^4_{\mu\nu\alpha\beta}$

approaches a constant value $R \sim H_s^2 \sim \lambda_s^{-2}$, but the dilaton keeps growing. This indicates the importance of loop corrections to perform a complete transition from the string phase to the radiation dominated phase with a constant dilaton. In [86] it was found, after numerical integration of the equations of motion, that the higher derivative terms prevent the growth of the curvature even in the general case (i.e., $k = +1$ or $k = -1$).

The effect of antisymmetric tensor field $B_{\mu\nu}$ on the large curvature phase was investigated by Foffa and Maggiore [87] and they concluded that a non-vanishing value of $B_{\mu\nu}$ leads to an anisotropic expansion.

Loop corrections

The loop corrections are purely quantum phenomena and the power of g_s^2 counts the number of loops in the string worldsheet topology. The corresponding models in four spacetime dimensions are of rather complicated nature, but one can investigate the 2-dimensional models, which still capture the underlying essential physics of four dimensions. Up to one-loop quantum corrections the 2-dimensional effective action is given by [88]

$$\begin{aligned} I_{eff} &\equiv I_{tree} + I_{1-loop} \\ &= \int \frac{d^2x}{2\pi}\sqrt{-g}\left[e^{-2\phi}\left(R + 4(\nabla\phi)^2\right) - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right] \\ &\quad -\frac{\bar{\kappa}}{2}\int \frac{d^2x}{2\pi}\sqrt{-g}\left[R\,\Box^{-1}R + 2\phi R\right]. \end{aligned} \tag{3.42}$$

Here R is 2-dimensional scalar curvature, ϕ denotes the dilaton, f_i are N massless scalar fields and $\bar{\kappa} = \frac{(N-24)}{24}$. Here the central charge deficit is taken to be zero: $\Lambda_2 = 0$. The trace-anomaly term $R\,\Box^{-1}R$ has been supplemented by a local, covariant counter-term in order to preserve a useful classical symmetry [88].

The tree level part of the action (3.42) represents the 2-dimensional analogy of the scenario derived from the action (3.2). Again two branches occur, related to each other by scale-factor duality and separated by curvature and coupling singularity. Thus a smooth transition does not appear in this two-dimensional classical theory. The second term in action (3.42) represents the one-loop correction. Due to this correction, the scale factor and the dilaton interpolate the superinflationary and FRW branches [88]. For example, the scalar curvature evolves as

$$R = \frac{16\, e^{2\phi}}{\left(1 + \frac{|\bar{\kappa}| e^{2\phi}}{2}\right)^3} , \tag{3.43}$$

which vanishes at the past and future infinity at which $\phi \to \pm\infty$ [88]. This indicates that back-reaction is sufficiently strong to wash out the classical divergences and a smooth transition between the branches becomes possible. Whether the same is true for the quantum corrected string vacuum in four dimensions is still under investigation, see for a review [61].

3.5 Summary

The pre-big bang scenario is a string theory motivated cosmological model where duality symmetry are applied in the cosmological context. In this scenario the beginning of our Universe is not hidden into the initial cosmological singularity and the history of the Universe is supposed to reach over the time interval $-\infty < t < \infty$. The initial state is assumed to be flat, cold and empty, in contrast to a highly curved, hot and dense initial state of the standard big bang scenario. The initial state corresponds to a string perturbative vacuum where some instability was growing due to superinflation. Thus the pre-big bang scenario is the first cosmological model where the problem of initial conditions is separated from the problem of the classical singularity. In any case the pre-big bang superinflation should play the same rôle as the conventional inflation in the standard cosmological model. For example, it must solve the standard cosmological problems. During the superinflation quantum fluctuations are generated and amplified to form seeds for the large-scale structure we observe today. However, the perturbation spectrum of these fluctuations (both scalar and tensor) differs from the spectrum generated in the slow-roll inflation (see for a review [61]). After all, the conventional inflation and superinflation have some remarkable differences. First, in the case of superinflation the event horizon is shrinking in contrast to standard inflation where the event horizon is constant (or increasing). Secondly, the superinflation is driven by kinetic energy of the dilaton and there is no need for a scalar potential to get the inflationary behaviour.

Despite the string theory background, the simple pre-big bang model is not free from problems. The inflationary and the FRW branches are separated from each other by a curvature and coupling singularity and the inclusion of the higher order corrections, both $\alpha'-$ and loop corrections, into the action is necessary to improve the situation. However, until now it has not been rigorously proven that such a transition occurs and a standard cosmological evolution, without the initial singularity, follows.

Chapter 4

Braneworlds

In this chapter we give a short review of the braneworld conception and describe the main attempts to reconcile cosmology with ideas that are motivated by string theory. As we saw in the previous chapter the pre-big bang approach was an attempt in a modest way to apply the string theory results to the early Universe, but unfortunately a lot of problems remain unsolved. This chapter is a necessary preparation for the next chapter where the braneworld cosmology is discussed.

In this chapter we concentrate on the development of ideas that lead to the conception that our 4-dimensional Universe is embedded in a higher-dimensional spacetime and at least one extra dimension is large compared with the Planck scale. It is argued that this extra dimension may be *non-compact* in contrast to the usual Kaluza-Klein approach where it was assumed that the extra dimensions are *compact*.

The hypothesis that the world may be a hypersurface in a higher-dimensional spacetime is rather old and originates from Rubakov and Shaposhnikov [98]. Recent interest in the idea comes from the works of Hořava and Witten [25], [26] and is strongly motivated by string/M-theory. The essential structure of a braneworld is represented by two phenomenological models. First is the Arkani-Hamed-Dimopoulos-Dvali (ADD) approach [30], [105] which was motivated by the hierarchy problem. Second is the Randall-Sundrum model with two different versions, known as RSI [34] and RSII [36]. The ADD-type model is constructed on the basis of particle phenomenology but the RS-type models are constructed in the framework of general relativity. It is important to stress, that Arkani-Hamed-Dimopoulos-Dvali approach and Randall-Sundrum approach are phenomenological in the sense that they are motivated and supported by string theory but not rigorously derived from it. The Hořava-Witten model and the following cosmology, on the other hand, is a direct realization of string/M-theory. In what comes next we will describe a bit closer three main approaches that lead to the braneworld conception.

Conventions and Notations

The capital Latin indices from the beginning of the alphabet $I, J, K, ... = 0, ..., 9, 10$ are used to parametrize the 11-dimensional spacetime while barred Latin indices $\bar{I}, \bar{J}, \bar{K}, ... = 0, ..., 9$ are used for the 10-dimensional hypersurface orthogonal to the eleventh coordinate. Indices with hats $\hat{I}, \hat{J}, \hat{K}, ... = 4, ..., 9$ are used for the internal Calabi-Yau space. Capital indices from the middle of the alphabet take the values $M, N, R, ... = 0, ..., 3, 10(5)$ and are used for 5-dimensional coordinates. Greek letters $\mu, \nu, \rho, ... = 0, ..., 3$ are used for four spacetime coordinates and small Latin letters $m, n, l, ... = 1, 2, 3$ for three space coordinates.

4.1 Hořava-Witten theory

The Hořava-Witten theory is a M-theory realization of the braneworld conception. As already mentioned in Sec. 2.2, the 11-dimensional supergravity is conjectured to be a low energy realization of M-theory. On the other hand, different compactifications of the M-theory give different string theories.

The possibility of the existence of a non-Planckian extra dimension arose first in the works of Hořava and Witten [25], [26] who investigated the strong coupling behaviour of $E_8 \times E_8$ heterotic string theory in 10 dimensions and conjectured that it is a specific realization of a more fundamental 11-dimensional theory, called M-theory. They concluded that a dual low energy theory is given by 11-dimensional supergravity compactified on an orbifold with Z_2 symmetry. The eleventh dimension is compact and periodic: $x^{11} \in [-\pi\rho, \pi\rho]$, where the endpoints are identified. The Z_2 symmetry requires that $x^{11} = -x^{11}$. The orbifold fixed points $x^{11} = 0$ and $x^{11} = \pi\rho$ are 10-dimensional boundaries of spacetime and gauge fields consistently appear on these boundaries. Gauge fields do not propagate in the bulk and two boundaries influence each other only through gravity. Compared with the basic idea of string theory that all extra dimensions are compactified and the size of extra dimension is Planckian $L \sim M_{Pl}^{-1}$, it was argued by Witten [28] that the size between the boundary branes, i.e., the size of the orbifold $L_{of} = \pi\rho$ may be much larger than the size of other six extra dimensions $L_{CY} \sim M_{Pl}^{-1}$, compactified on the Calabi-Yau manifold : $L_{CY} < L_{of}$. Since $L_{of} > M_{Pl}^{-1}$ it indicates that the spacetime is effectively 5-dimensional. The 5-dimensional, low energy realization of the model of Hořava-Witten was discussed by Lukas et al [27] and soon cosmological solutions were found [100].

In the Hořava-Witten model the 11-dimensional bulk is described by a graviton supermultiplet and by a 10-dimensional Yang-Mills supermultiplet living on each boundary. The bosonic part of the resulting action is [27]

$$\begin{aligned}
I_{HW}^{11} &= I_{SUGRA} + I_{YM} \\
&= -\frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}_{11}} d^{11}x\,\sqrt{-g_{11}}\left\{R + \frac{1}{24}G_{IJKL}G^{IJKL}\right. \\
&+ \left.\frac{\sqrt{2}}{1728}\epsilon^{I_1\cdots I_{11}}C_{I_1I_2I_3}G_{I_4\cdots I_7}G_{I_8\cdots I_{11}}\right\} \\
&- \frac{1}{8\pi\kappa_{11}^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{M}_{10}^{(1)}} d^{10}x\sqrt{-g_{10}}\left\{tr(F^{(1)})^2 - \frac{1}{2}trR^2\right\} \\
&- \frac{1}{8\pi\kappa_{11}^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\int_{\mathcal{M}_{10}^{(2)}} d^{10}x\,\sqrt{-g_{10}}\left\{tr(F^{(2)})^2 - \frac{1}{2}trR^2\right\} \quad (4.1)
\end{aligned}$$

Here $F_{\bar{I}\bar{J}}^{(i)}$ are the E_8 gauge field strengths and C_{IJK} is the 3-form with field strength given by $G_{IJKL} = 24\partial_{[I}C_{JKL]}$. The term trR^2 was absent in paper [26] and was added by Lukas et al in [27] in order to have well-defined supersymmetry.

The reduction of the 11-dimensional theory to the 5-dimensional starts with the metric ansatz

$$ds_{11}^2 = \mathcal{V}^{-\frac{2}{3}}\; g_{MN}(x^M)\; dx^M dx^N + \mathcal{V}^{\frac{1}{3}}\; \Omega_{\hat{M}\hat{N}}(z^{\hat{M}})\; dz^{\hat{M}}\; dz^{\hat{N}} \qquad (4.2)$$

where $\mathcal{V} \equiv e^{\phi}$ is a modulus field measuring the volume of the Calabi-Yau space, x^M are the coordinates of the 5-dimensional spacetime, $z^{\hat{M}}$ are the coordinates of the Calabi-Yau space, and $\Omega_{\hat{M}\hat{N}}$ is the Calabi-Yau metric. The detailed reduction process of the action I_{HW}^{11} down to 5-dimensional effective action I_{HW}^{5} is described in [27]. The resulting theory is 5-dimensional $N = 1$ supergravity coupled to 4-dimensional gauge supermultiplets. Symbolically this can be written

$$\mathcal{M}_{11} = \mathcal{M}_{10} \times S^1/Z_2 \quad \longrightarrow \quad \mathcal{M}_{11} = \mathcal{M}_4 \times X \times S^1/Z_2 \qquad (4.3)$$

where X is smooth Calabi-Yau threefold, $\mathcal{M}_{10}$ and $\mathcal{M}_4$ are 10-dimensional and 4-dimensional Minkowski spacetime, respectively.

The simplest 5-dimensional effective action derived from the 11-dimensional action (4.1) can be written as

$$
\begin{aligned}
I^5_{HW} &= -\frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left\{R + \frac{1}{2}(\nabla\phi)^2 + 2e^{-\phi}\nabla_M\xi\nabla^M\bar{\xi} + \frac{1}{3}\alpha^2 e^{-2\phi}\right\} \\
&+ \frac{\sqrt{2}}{\kappa^2}\int d^4x\sqrt{-h}\;\alpha e^{-\phi} - \frac{\sqrt{2}}{\kappa^2}\int d^4x\sqrt{-\tilde{h}}\;\alpha e^{-\phi} \qquad (4.4)
\end{aligned}
$$

where α is a constant, $h_{\mu\nu}$, $\tilde{h}_{\mu\nu}$ are the induced metrics on the boundaries, ξ is a complex scalar field $\xi = e^{\rho+i\theta}$, $\theta = const.$ In action (4.4) several terms are assumed to vanish in the context we will proceed.

For the 5-dimensional metric the following ansatz was made [102]

$$ds^2 = -e^{2U(t,y)}d\tau^2 + e^{2A(t,y)}ds_3^2 + e^{2B(t,y)}dy^2 \qquad (4.5)$$

with

$$\phi = \phi(t,y), \qquad \rho = \rho(t,y)\ . \qquad (4.6)$$

Here ds_3^2 is the metric of the 3-dimensional space and we have taken $x^{11} = y$. A separation of variables $U(t,y) = U_1(t) + U_2(y)$ etc. was assumed. The field equations derived from the action (4.4) are presented and solved under certain restrictions in [100], [102]. The solutions depending only on y are as follows [1]

$$
\begin{aligned}
e^{A_2} &= e^{U_2} = a_0 H^{\frac{1}{2}}\ , \qquad & e^{B_2} = b_0 H^2\ , \qquad (4.7)\\
e^{\phi_2} &= b_0 H^3\ , & \xi = e^{i\theta}(d_0 H^4 + \xi_0)\ , \qquad (4.8)
\end{aligned}
$$

where H is

$$H(y) = \frac{\sqrt{2}}{3}\alpha|y| + c_0 \qquad (4.9)$$

and a_0, b_0, c_0, d_0, ξ_0 are all constants. The field equations for time dependent components can be solved if it is assumed that $e^{B_1} = e^{\phi_1}$ and gauge is fixed as $e^{U_1} = const.$ The solutions for the brane scale factor $a(t)$ and

[1] Here a_0 is not the scale factor on the brane and H is not the Hubble parameter, although we use a_0 in Ch. 5 to mark the scale factor on the brane and H to mark the Hubble parameter.

for the scalar field e^{ϕ_1} in a comoving time $dt = e^{U_1} d\tau$ can be expressed as follows [100], [101]

$$e^{A_1} \equiv a(t) = P|t-t_0|^p , \qquad p = \frac{3}{11}\left(1 \mp \frac{4}{3\sqrt{3}}\right) ; \qquad (4.10)$$

$$e^{B_1} \equiv e^{\phi_1(t)} = Q|t-t_0|^q , \qquad q = \frac{2}{11}\left(1 \pm 2\sqrt{3}\right) . \qquad (4.11)$$

where P, Q and t_0 are constants. These are the simplest cosmological solutions found in the framework of Hořava-Witten theory. These solutions are generalized to spatially curved FRW models [102] and for different exponential potentials [103]. The scale factor e^{A_1} determines the size of brane worldvolume and the scalar fields e^{B_1} determine the size of the orbifold. Since the size of the orbifold changes the distance between the branes changes also. The question of stabilization of the moduli fields is discussed by Huey et al [104].

4.2 Arkani-Hamed, Dvali, Dimopoulos approach

This is the first phenomenological model that realizes the braneworld conception, introduces macroscopically large extra dimensions, and solves the hierarchy problem. An important motivation for the work [30] was the assumption that the electroweak scale M_{ew} is the only fundamental scale in nature and therefore the fundamental Planck mass should be of the same order $M \sim M_{ew}$. The scenario combines the braneworld conception with the Kaluza-Klein compactification. In [30] it was assumed that there exist n *compact* extra spatial dimensions with a characteristic size L.

Experiments rule out the possibility that standard model particles can propagate in extra dimensions that are larger than the electroweak scale M_{ew}^{-1} and it was assumed that they are trapped to a 4-dimensional hypersurface. For gravity this experimental constraint is much milder since until recently it has not been tested how gravity behaves at distances shorter than 1 mm. The fact that gravitons can propagate through the $(4+n)$-dimensional bulk spacetime offers a verifiable possibility to solve the hierarchy problem.

The action that realizes the ideas presented above can be written

$$I_{ADD} = \frac{M^{2+n}}{2} \int d^4x \int_0^L d^n y \sqrt{-G}\, R(G) + \int d^4x \sqrt{-h} L_{SM}(\Psi) \qquad (4.12)$$

where G_{MN} is the $(4+n)$-dimensional metric, $h_{\mu\nu}$ is 4-dimensional induced metric on the hypersurface and $L_{SM}(\Psi)$ is the standard model Lagrangian.

The gravitational part of the action (4.12) can be easily reduced via the Kaluza-Klein procedure to 4-dimensions

$$\frac{M^{2+n}}{2}\int d^4x \int_0^L d^n y\sqrt{-G}\ R(G) \quad \rightarrow \quad \frac{M^{2+n}L^n}{2}\int d^4x\ \sqrt{-h}\ R(h)\,, \tag{4.13}$$

giving a relation between the 4-dimensional Planck mass M_{Pl} and the $(4+n)$-dimensional Planck mass M

$$M_{Pl}^2 = L^n\ M^{2+n}\ . \tag{4.14}$$

If $M = M_{ew}$, then the characteristic scale for extra dimensions is given by

$$L = L_{ew}\left(\frac{M_{Pl}}{M_{ew}}\right)^{\frac{2}{n}} \sim 10^{\frac{30}{n}-17}\ cm\ . \tag{4.15}$$

Here it is assumed that all extra dimensions have the same characteristic size L. The case of one extra dimension $n = 1$ is excluded because this gives $L \sim 10^{13}$ cm. In this range the Newton law is tested well and no evidence for an extra dimension is found. The case $n = 2$ gives $L \sim 10^{-2}$ cm which is acceptable from the point of view of gravity experiments. In this case at distances larger than L the influence of the extra dimensions should be very small and the dynamics is effectively 4-dimensional. For example, gravitational potential energy between two static masses m_1 and m_2 at distances larger than the extra dimension $r \gg L$, remains unchanged compared with the standard case

$$V = -G_N\ \frac{m_1\ m_2}{r}, \qquad r \gg L\ . \tag{4.16}$$

At distances shorter than the size of the extra dimension $r \ll L$ the potential energy feels the extra dimensions and gets modified as follows

$$V = -M^{-(2+n)}\ \frac{m_1\ m_2}{r^{1+n}}, \qquad r \ll L\ . \tag{4.17}$$

Testing gravity at short distances is crucial for this approach [33]. Some observable effects for the models in astrophysics, cosmology, and accelerator physics are discussed in [30]. An important conclusion arrived at in [30] was that the theory of nucleosynthesis remains unchanged in this framework because the typical energy scale at the time of nuclesynthesis was $\sim$ MeV which is too low to feel the extra dimensions.

4.3 Randall-Sundrum model

This is the most popular version of the braneworld scenario and it is based on the conjecture that our world is a hypersurface (codimension 1 brane) in a 5-dimensional spacetime with Z_2 symmetry along the extra dimension. It is another low energy realization of the Hořava-Witten construction in a 5-dimensional spacetime with compact or noncompact 5th dimension and a negative cosmological constant in the bulk. This corresponds to the anti-de Sitter space (AdS) which is a maximally symmetric solution of the Einstein equation with a negative cosmological constant $\Lambda < 0$.

The key issue of the model is the assumption that the background geometry is *nonfactorizable*. In the usual Kaluza-Klein approach it is assumed that the geometry is factorizable which means that the 4-dimensional metric is independent of the position in the extra dimensions. In the case of nonfactorizable metric the 4-dimensional part of the metric is multiplied by a *warp factor* which is rapidly changing in the direction of extra dimensions.

Randall and Sundrum proposed two different models, known as RSI [34] and RSII [36]. The aim of the RSI was to solve the hierarchy problem introduced in Sec. 1. The model assumes that there are two branes at the orbifold fixed points and it preserves the requirement that the extra dimension must be *compact*. Moreover, the size of the extra dimension should be suitably small to remove the large hierarchy. In RSII, on the other hand, there is only one brane and it can be considered as an "uncompactified" variant of RSI where the second brane has been moved to infinity. This model concentrates on the recovering of the Newtonian gravity on the brane at low energies. A remarkable conclusion is that this can be done even if the extra dimension is *noncompact*.

Randall-Sundrum model I

The extra dimension is compactified on an orbifold S^1/Z_2 and is parametrized with the variable φ ranging from $-\pi \leq \varphi \leq \pi$. This means that the coordinate is periodic: $\varphi = \varphi + 2\pi$. The orbifold symmetry requires that $\varphi \to -\varphi$. The 3-branes are located at the orbifold fixed points $\varphi|_{hb} = 0$ (hidden brane) and $\varphi|_{vb} = \pi$ (visible brane). The metric ansatz for the warped 5-dimensional spacetime that preserves the 4-dimensional

Poincaré invariance can be taken to be

$$ds^2 = e^{-2A(\varphi)}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\varphi^2 \ . \qquad (4.18)$$

The coordinate φ can be rescaled to hide the coefficient r_c which characterizes the compactification radius of the extra dimension prior to orbifolding. After orbifolding the size of the extra dimension is $r_c\pi$.

The action of the model consists of the 5-dimensional Einstein-Hilbert action with a cosmological constant I_{EH}, visible brane (called also the standard model brane) action I_{vb} and hidden brane (called also the Planck brane) action I_{hb} [2]

$$I_{RS} = \frac{1}{2\kappa^2}\int d^4x \int_{-\pi}^{\pi} d\varphi\sqrt{-g}\left(R - 2\kappa^2\Lambda\right)$$

$$+\int_{y_{vb}} d^4x\sqrt{-h_{vb}}\left(L_{vb} - \sigma_{vb}\right) + \int_{y_{hb}} d^4x\sqrt{-h_{hb}}\left(L_{hb} - \sigma_{hb}\right) \ , \qquad (4.19)$$

Here σ_{vb}, σ_{hb} are the tensions of the visible and the hidden brane and L_{vb}, L_{hb} represent the Lagrangian densities of the standard model fields on the respective branes. The induced metric on the visible brane is $h^{vb}_{\mu\nu} = \delta^M_\mu \delta^N_\nu g_{MN}(\varphi = \pi)$ while the induced metric on the hidden brane is $h^{hb}_{\mu\nu} = \delta^M_\mu \delta^N_\nu g_{MN}(\varphi = 0)$. Here δ^M_μ and δ^N_ν are projection operators.

With the metric (4.18) the 5-dimensional Einstein equations become ordinary differential equations for the warp factor A

$$A'^{\,2} = -\frac{\kappa^2}{6} r_c^2 \,\Lambda \ , \qquad (4.20)$$

$$A'' = -\frac{\kappa^2}{3} r_c \left[\sigma_{vb}\ \delta(\varphi) + \sigma_{hb}\ \delta(\varphi - \pi)\right] \ . \qquad (4.21)$$

Here the prime denotes a differentiation with respect to φ. The solution of equation (4.20) is

$$A(y) = r_c|\varphi|\sqrt{-\frac{\kappa^2}{6}\Lambda} \equiv r_c|\varphi|k \ , \qquad \Rightarrow \qquad \Lambda < 0 \ , \qquad (4.22)$$

where φ is in the interval $-\pi < \varphi < \pi$ and

$$|\varphi| = \begin{cases} -\varphi \ , & -\pi < \varphi < 0 \\ \varphi \ , & 0 < \varphi < \pi \ . \end{cases} \qquad (4.23)$$

[2] The Randall-Sundrum original [34] conventions were different: $2M^3 \equiv \frac{1}{2\kappa^2}$, $\sigma_{vb} \equiv V_{vis}$, $\sigma_{hb} \equiv V_{hid}$.

The requirement $\Lambda < 0$ implies that the spacetime between the branes is a slice of AdS_5. The metric (4.18) is free from coordinate singularities but contains δ-function singularities at the boundaries $(0,\ r_c\pi)$. Computing the second derivative of the solution (4.22)

$$A'' = 2r_c k[\delta(\varphi) - \delta(\varphi - \pi)] \tag{4.24}$$

and comparing it with equation (4.21) leads to expression for the brane tensions

$$\sigma_{hb} = -\sigma_{vb} = \frac{6k}{\kappa^2} \ . \tag{4.25}$$

The hidden brane has therefore positive tension and the visible brane has negative tension. The bulk cosmological constant Λ and the brane tensions σ_{hb}, σ_{vb} are fine-tuned as follows

$$\sigma_{hb}^2 = \sigma_{vb}^2 = -\frac{6}{\kappa^2}\Lambda \ . \tag{4.26}$$

This condition preserves the 4-dimensional Poincaré invariance which was required by the ansaz (4.18).

The main motivation of the RSI model was the hierarchy problem and for that reason we need to present low energy effective quantities in terms of the 5-dimensional fundamental scales. Inserting the perturbed 4-dimensional Minkowski metric

$$\bar{h}_{\mu\nu}(x^\mu) \equiv \eta_{\mu\nu} + f_{\mu\nu}(x^\mu) \tag{4.27}$$

into action (4.19), we can derive the 4-dimensional effective action [34]

$$I_{eff} \propto \int d^4x \int_{-\pi}^{\pi} 2d\varphi M^3 r_c e^{-2kr_c|\varphi|}\sqrt{-\bar{h}}R(\bar{h}) \ . \tag{4.28}$$

Here $f_{\mu\nu}$ represents the physical graviton in 4-dimensional theory which is the massless mode of the Kaluza-Klein decomposition of the 5-dimensional metric g_{MN} [34]. Using the effective action (4.28) a relation between the 4-dimensional Planck mass M_{Pl} and fundamental 5-dimensional Planck mass M can be established

$$M_{Pl}^2 = M^3\ r_c \int_{-\pi}^{\pi} d\varphi e^{-2kr_c\varphi} = \frac{M^3}{k}\left[1 - e^{-2kr_c\pi}\right] \ . \tag{4.29}$$

The Planck mass M_{Pl} depends only weakly on the compactification scale r_c due to exponential damping. In a similar way it is possible to show that any mass parameter m_{eff} on the visible brane is related to the fundamental mass m_0 via

$$m_{eff} \equiv e^{-kr_c\pi} m_0 \, , \tag{4.30}$$

measured with respect to the effective metric $\bar{h}_{\mu\nu}(x^\mu)$. If $kr_c \sim 12$ the electroweak scale physical masses M_{ew} are produced from the fundamental Planck scale M_{Pl} without introducing large numbers. It seems that the warp factor makes the gauge hierarchy much milder without introducing supersymmetry.

Despite solving of the hierarchy problem, the RSI scenario suffers from issues which remain unsolved or are in conflict with the observations. The main problems of the RSI scenario are the following:

- The standard model brane has negative tension $\sigma_{vb} < 0$ and this leads to difficulties when cosmological expansion is considered [106], [107]. The non-static brane is achieved by perturbing the RS solution by introducing additional energy density on the brane without compensating it with the bulk cosmological constant. In the case of negative tension the matter source term in the modified Friedmann equation has opposite sign in comparison with ordinary matter. This change leads to a collapsing Universe for any kind of matter when the barotropic index obeys $\Gamma < \frac{4}{3}$. This conclusion rules out the RSI-type theories where the standard model brane has a negative tension or there is only one extra dimension. However, a possible solution can also be provided by introducing additional fields in the bulk.

- There is fine-tuning between the bulk cosmological constant and the brane tension expressed by (4.26). These conditions can be interpreted also as fine-tuning conditions for the 4-dimensional effective cosmological constant. Only this case leads to a Ricci flat brane in the absence of ordinary matter. Therefore the cosmological constant problem remains unsolved in this simple setup.

- In general, the compactification radius r_c is the vacuum expectation value of the *modulus field* $T(x)$ (called *radion*): $\langle 0|T|0\rangle = r_c$. The modulus field T also fluctuates and in the effective 4-dimensional theory these fluctuations will be represented as a massless scalar. A mechanism is needed to make it massive and freeze it at a classical value r_c^2. One such mechanism was presented by Goldberger and Wise [35] where the inclusion of the bulk scalar fields was essential. They proved that in the case of a suitable choice of parameters, a potential for the distance of the two branes $V(r_c)$ has a stable minimum.

In the case of a single brane the general metric that preserves the Z_2 symmetry, i.e., $y \to -y$ and 4-dimensional Poincaré invariance can be written as

$$ds^2 = e^{2A(y)}(-dt^2 + dx^{i\ 2}) + dy^2 \equiv e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2\ . \tag{4.31}$$

The extra dimension is not compact any more and therefore it is more convenient to parametrize the extra dimension with y instead of φ. The hidden brane action I_{hb} can be removed from (4.19) and the 5-dimensional Einstein equations are

$$A'^{\ 2} = -\frac{\kappa^2}{6}\Lambda\ , \tag{4.32}$$

$$A'' = -\frac{\kappa^2}{3}\sigma_{vb}\ \delta(y)\,. \tag{4.33}$$

The general solution of equation (4.32) is

$$A(y) = \begin{cases} -\sqrt{-\frac{\kappa^2}{6}\Lambda}y + A_0 = -ky + A_0 & y > 0\ , \\ \\ \sqrt{-\frac{\kappa^2}{6}\Lambda}y + A_0 = \quad ky + A_0 & y < 0\ . \end{cases} \tag{4.34}$$

There exist coordinate singularities at infinities $y \to \pm\infty$ and a delta function singularity at the location of the brane $y = 0$. The integration constant A_0 can be removed by transforming the coordinates on the brane. In this scenario the visible brane has positive tension

$$\sigma_{vb} = \frac{6k}{\kappa^2} \tag{4.35}$$

(as opposed to RSI were the visible brane has negative tension). Perturbing the Minkowski metric in a similar way as in the case of RSI it is possible [36] to get an analogous relation for the 4-dimensional Planck mass as in RSI (4.29)

$$M_{Pl}^2 = 2M^3 \int_0^{y_\star} dy e^{-2ky} = \frac{M^3}{k}\left[1 - e^{-2ky_\star}\right]\ , \tag{4.36}$$

where $y_\star$ is the characteristic scale of the extra dimension. The 4-dimensional Planck mass remains finite even if $y_\star \to \infty$. This indicates that massless graviton is bounded to the brane and 5-dimensional gravity looks effectively 4-dimensional for the observer on the brane. Small corrections come from massive gravity modes. To prove that, Randall and Sundrum performed the Kaluza-Klein reduction down to 4-dimensions and solved the wave equation

$$\left[\frac{1}{2}\left(e^{2k|y|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta + \partial_y^2\right) + 2k\delta(y) - 2k^2\right] f_{\mu\nu}(x^\mu, y) = 0 \tag{4.37}$$

for a small gravitational fluctuation $f_{\mu\nu}(x^\mu, y)$ around the background (4.31). Zero energy (massless) ground state of the graviton equation has the solution that is localized around the brane and can be interpreted as 4-dimensional gravitation on the brane. In addition, a continuum of massive states gives rise to correction terms in Newtonian potential and the resulting potential energy between two point masses on the brane is

$$V(r) \approx G_N \frac{m}{r}\left(1 + \frac{\ell^2}{r^2}\right) . \tag{4.38}$$

Here ℓ is AdS curvature radius and $k = \ell^{-1}$. If the warping is strong, i.e., the constant k is large, the massive modes are suppressed near the brane and usual Newton's law is recovered on the brane. On the other hand, if warping is very weak, the potential remains 5-dimensional. A more careful analysis of equation (4.37) was done by Garriga and Tanaka [108], who found a different prefactor (2/3 instead of 1) for the corrective term in the Newtonian potential (4.38). Newton's law in $(5+n)$-dimensional spacetime, where n extra dimensions on the brane are compactified with a characteristic size M_{Pl}^{-1} was found by Ito [109]. So far Newton's law is tested in sub-millimeter scales and no evidence of an extra dimension is found [32], [33].

4.4 Summary

In this chapter we reviewed three basic models which lead to or employ the conception that our Universe may be a hypersurface in a higher-dimensional spacetime. One of them, the Hořava-Witten construction is based on the M-theory, a conjectured fundamental theory which relates all string theories (and thus provides a quantum description of gravity and unifies all interactions at high energies). The other two approaches, Arkani-Hamed, Dvali, Dimopoulos approach and Randall-Sundrum approach, are phenomenological. In the Randall-Sundrum model the bulk spacetime is AdS_5 and it is highly warped. This warping is essential to localize the standard model particles on the brane. An additional scalar field is needed in the bulk to stabilize the distance between the branes in the case when two branes are involved. In the Hořava-Witten model the bulk is less warped in comparison with the Randall-Sundrum model and the moduli fields arising from the compactification will serve as bulk scalar fields.

Chapter 5

Braneworld cosmology

In this chapter we describe how the ideas presented in the previous chapter can be realized in cosmology and how the unconventional setup changes the dynamics on the brane which is identified with our 4-dimensional Universe.

The presence of the brane(s) changes also the dynamics of the bulk. The *timelike brane* ($\mathcal{M}_{D-1} \equiv \Sigma$, $h_{\mu\nu}$) in a bulk spacetime ($\mathcal{M}_D$, g_{MN}) splits the bulk into two parts, $\mathcal{M}_D^{\pm}$. The split corresponds to a δ-function singularity in the n-dimensional energy-momentum tensor of the brane. In this case we get additional constraints to the gravity and to the matter fields from *junction conditions*. The junction conditions follow from the fact that the Einstein equations are second order differential equations which cannot contain discontinuities in solutions to describe a well-defined geometry. The junction conditions, in the form used in this review, where first derived by Israel [110] to describe surface layers in general relativity. Later, it became the standard framework for consistency of the Einstein equations in the presence of *domain walls* [111], for a review see [112]. More recently, a class of domain walls were considered as a possible candidate of braneworlds [113], [114]. But the main setup and motivation comes from the Randall-Sundrum model. In what follows we also adopt their definition of a *3-brane*: it is a 4-dimensional timelike subspace within the (4+d)-dimensional spacetime [34].

In Sec. 5.1 we give the underlying definitions, basic equations, and general junction conditions for future reference. In Secs. 5.2 and 5.3 we consider a 5-dimensional bulk spacetime containing one or two timelike branes as boundaries of spacetime. The braneworld cosmology has been extensively investigated and reviewed by different authors [89] – [96]. We don't include branes of higher codimension although they have been investigated [97]. As typical for this review, we don't discuss the generation and evolution of perturbations but we refer to the original works [115], [116], [117] about braneworld perturbations. The braneworld setup has influence to the CMBR and it was reviewed by Maartens in [126].

5.1 Basic equations and junction conditions

We start with the 5-dimensional action

$$I = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\,(R-2\Lambda) + \int d^5x\sqrt{-g}\,L_{mat} \ , \tag{5.1}$$

where Λ is the cosmological constant of the bulk, L_{mat} is matter Lagrangian and the 5-dimensional gravitational coupling constant κ^2 is related to the reduced 5-dimensional Planck mass M as follows

$$\kappa^2 \equiv 8\pi G_5 \equiv M^{-3} \ . \tag{5.2}$$

The 5-dimensional Einstein equations derived from action (5.1) are

$$G_{MN} \equiv R_{MN} - \frac{1}{2}g_{MN}R = -\Lambda g_{MN} + \kappa^2\, T^{tot}_{MN} \ . \tag{5.3}$$

The total energy-momentum tensor T^{tot}_{MN} on the right hand side of the Einstein equations (5.3) can be decomposed into the bulk and the brane contribution

$$T^{tot}_{MN} \quad \rightarrow \quad T_{MN}|_{bulk} + T_{MN}|_{brane} = T_{MN} + S_{MN}\delta(y) \ . \tag{5.4}$$

Here T_{MN} represents any 5-dimensional energy-momentum of the gravitational sector (e.g. bulk scalar fields) and the *surface energy-momentum tensor* S_{MN} is the energy-momentum tensor of matter confined to the brane. It can be defined [1] through the integral

$$S^{M}_{\ N} = \lim_{\epsilon\rightarrow 0}\left[\int_{-\epsilon}^{+\epsilon} T^{M\ tot}_{\ N}\, dn\right] \ , \tag{5.5}$$

where the integration takes place with respect to the proper distance which is measured perpendicularly through the brane located at $y=0$. The distributional nature of the brane energy-momentum tensor comes from the fact that the brane is considered as an infinitely thin wall. The brane with non-vanishing thickness Δ was considered by Kanti et al in [118]. They concluded that the 4-dimensional Planck mass M^2_{Pl} is proportional to the thickness of the brane Δ: $M^2_{Pl} = M^3(2b\Delta)$, where $b = const.$

To describe a hypersurface $\mathcal{M}_{D-1}$ in a higher-dimensional spacetime $\mathcal{M}_D$ let us introduce the concept of the *extrinsic curvature*, which characterizes how the brane is bending in the 5-dimensional spacetime

$$K_{MN} = \nabla_M n_N = h^A_{\ M}\, h^B_{\ N}\, \nabla_A n_B\,, \tag{5.6}$$

where $h^A_{\ M}$ is the *induced metric* on the hypersurface, ∇_A is the covariant derivative with respect to the higher-dimensional metric g_{MN} and n_M is the unit spacelike vector field ($g_{MN} n^M n^N = 1$) normal to the hypersurface ($h_{MN} n^M = 0$). The induced metric h_{MN} on the brane can be expressed in terms of the bulk metric g_{MN} and the normal vector n_M of the hypersurface as follows

$$h_{MN} = g_{MN} - n_M n_N\,. \tag{5.7}$$

This definition represents the induced metric h_{MN} as a 5-dimensional tensor which is *degenerate* since it is nontrivial only on the hypersurface. The induced metric acts as a projection operator which allows to project out components tangential to the hypersurface.

The 5-dimensional Einstein equations (5.3) contain a δ-function singularity on the brane due to the distributional nature of the brane matter. Since the metric must be continuous also across the brane and the Einstein equations contain the metric up to the second derivatives, the δ-function term in the distribution of matter introduces a discontinuity in the first derivative of the metric with respect to the extra coordinate y. Formally we can write

$$g'' = g''_c + [g']\,\delta(y) \tag{5.8}$$

where g''_c is a continuous function and $[g'] = g'(y=0+) - g'(y=0-)$ denotes the jump of the first derivative of the metric across the brane $y = 0$.

By expressing the Einstein equations (5.3) in terms of extrinsic curvature and by matching the terms that contain the δ-function singularity a relation between the brane energy-momentum tensor S_{MN} and extrinsic curvature K_{MN} can be obtained [1], [110]

$$\Delta K_{MN} \equiv K^+_{MN} - K^-_{MN} = -\kappa^2 \left(S_{MN} - \frac{1}{3}\, S\, h_{MN} \right)\,. \tag{5.9}$$

This expression is known as the Israel-Darmois junction condition and it indicates that the geometries on two sides of the hypersurface which carries energy-momentum, may be different. Here ΔK_{MN} characterizes the jump in the components of extrinsic curvature and if there is no distributional matter the jump is zero, $\Delta K_{MN} = 0$.

Let us assume a mirror or Z_2-symmetry, which means that if we approach the brane from one side and go through it, we arrive into bulk which looks the same, only the normal is reversed: $n^M \to -n^M$. In this case $K^+_{MN} = -K^-_{MN}$ and we can write the junction conditions (5.9) as follows (omitting the $\pm$ for brevity)

$$K_{MN} = -\frac{\kappa^2}{2}\left(S_{MN} - \frac{1}{3}\,S\,h_{MN}\right). \tag{5.10}$$

As we pointed out earlier, the metric should be continuous across the brane and we can add the following conditions for the induced metric

$$h^+_{MN} - h^-_{MN} = 0\,. \tag{5.11}$$

The junction condition (5.9) was derived in more general way by Chamblin and Reall [114] who considered the domain wall motion in a higher-dimensional spacetime. They included a Gibbons-Hawking boundary term

$$I_{GH} = -\frac{1}{\kappa^2}\int_{\Sigma_\pm} d^4x\,\sqrt{-h}\,K \tag{5.12}$$

and domain wall action I_{DW} into the Einstein-Hilbert action (5.3). Upon varying the total action $I = I_{EH} + I_{GH} + I_{DW}$ the Israel conditions (5.9) follow.

5.2 The cosmology of a single brane Universe

In this section we study the cosmology of a 3-brane Universe which is moving in a 5-dimensional bulk spacetime. First we review the simplest version of the brane cosmology developed by Binétruy et al [37] slightly before the RSII model gave an additional motivation to consider this kind of cosmologies. There are two approaches to consider the simplest brane cosmology: the brane based point of view and the bulk based point of view, reviewed in Subsecs. 5.2.1, 5.2.2, respectively. In the brane based approach we are dealing with a static brane in a dynamical bulk while in a bulk based approach we consider a dynamic brane in a static bulk. Both approaches lead to the same cosmology on the brane. In Subsec. 5.2.4 we describe some scenarios where a bulk scalar field is included and in Subsec. 5.2.5 we briefly review how the brane effects influence the dynamics of inflation on the brane.

5.2.1 Brane based point of view

Let us consider an infinitely thin 3-brane of a constant spatial curvature embedded in a 5-dimensional spacetime and look for a FRW-type homogeneous and isotropic cosmology on the brane described by the metric

$$ds^2_{brane} = -dt^2 + a_0(t)^2 \gamma_{ij} dx^i dx^j \ . \tag{5.13}$$

Here γ_{ij} is a maximally symmetric 3-dimensional metric, $a_0(t)$ is the scale factor on the brane and t is the proper time on the brane. Near the brane it is convenient to represent the 5-dimensional metric in a Gaussian normal coordinate system where the brane is located at the origin $y = 0$

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N = g_{\mu\nu}\, dx^\mu dx^\nu + dy^2 \\ &= -n(t,y)^2 dt^2 + a(t,y)^2 \gamma_{ij}\, dx^i dx^j + dy^2 \ . \end{aligned} \tag{5.14}$$

The metric (5.14) is consistent with the homogeneity and isotropy requirement of our brane world: $a(t,0) = a_0(t)$ is the brane scale factor and $n(t,0) = 1$ if t is the proper time on the brane. To avoid coordinate singularities far off the brane, other coordinate systems must be used as discussed in Subsec. 5.2.2.

In the coordinate system (5.14), the components of the 5-dimensional Einstein equation (5.3) can be written [37], [38]

$$G_{00} \equiv 3\frac{\dot{a}^2}{a^2} - 3n^2\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) + 3n^2\frac{k}{a^2} = \kappa^2 T^{tot}_{00} + n^2\Lambda, \tag{5.15}$$

$$\begin{aligned} G_{ij} &\equiv a^2\gamma_{ij}\left(2\frac{a''}{a} + \frac{n''}{n} + \frac{a'^2}{a^2} + 2\frac{a'n'}{an}\right) \\ &+ \frac{a^2}{n^2}\gamma_{ij}\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}\dot{n}}{an}\right) - k\gamma_{ij} = \kappa^2 T^{tot}_{ij} - \Lambda\gamma_{ij} \ , \end{aligned} \tag{5.16}$$

$$G_{0y} \equiv 3\left(\frac{n'}{n}\frac{\dot{a}}{a} - \frac{\dot{a}'}{a}\right) = \kappa^2 T^{tot}_{0y} \ , \tag{5.17}$$

$$\begin{aligned} G_{yy} &\equiv 3\left(\frac{a'^2}{a^2} + \frac{a'n'}{an}\right) - \frac{3}{n^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a}\frac{\dot{n}}{n}\right) \\ &- 3\frac{k}{a^2} = \kappa^2 T^{tot}_{yy} - \Lambda \ , \end{aligned} \tag{5.18}$$

where a dot denotes a derivative with respect to the proper time t and a prime denotes a derivative with respect to the fifth coordinate, y.

Since we are interested in cosmological solutions on the brane we must take under consideration the junction conditions. Using the notation of (5.8) and matching the components of the Einstein equations (5.15), (5.16) that contain a δ-function singularity (a'', n''), we obtain the following relations

$$\left.\frac{[a']}{a}\right|_{y=0} = \frac{\kappa^2}{3} S^0_{\ 0} \ , \qquad \left.\frac{[n']}{n}\right|_{y=0} = \frac{\kappa^2}{3}\left(S^i_{\ i} - 2S^0_{\ 0}\right) \ . \tag{5.19}$$

These relations can be derived directly from general junction conditions (5.9) if we take into account that the extrinsic curvature in the case of the Gaussian normal coordinate system can be written

$$K_{MN} = \frac{1}{2}\frac{\partial g_{MN}}{\partial y} \ . \tag{5.20}$$

If we use Z_2 symmetry

$$\begin{aligned} a(y) &= -a(-y), & a'(y) = -a'(-y), \\ n(y) &= -n(-y), & n'(y) = -n'(-y), \end{aligned} \tag{5.21}$$

and assume that the brane matter is ideal fluid $S^M_{\ N} = diag(-\rho_b, p, p, p, 0)$, we can write the junction conditions (5.19) as follows

$$\left.\left(\frac{a'}{a}\right)\right|_{y=0} = -\frac{\kappa^2}{6}\rho_b \ , \qquad \left.\left(\frac{n'}{n}\right)\right|_{y=0} = \frac{\kappa^2}{6}(3p + 2\rho_b) \ . \tag{5.22}$$

In this case, inserting the junction conditions (5.22) into the $(0y)$-component of the Einstein equation (5.17) we get the usual conservation law for the brane ideal fluid

$$\dot{\rho_b} + 3\frac{\dot{a}_0}{a_0}(p + \rho_b) = 0 \ . \tag{5.23}$$

Here we assumed that bulk contains only the cosmological constant Λ, i.e., $T_{MN} = 0$ and there is no energy flow along the fifth dimension which implies that $T_{0y} = 0$. The cosmology in the presence of energy influx from the bulk was considered by Tetradis in [119]. The influx can lead to accelerated expansion on the brane, depending on the equations of state of the bulk and brane matter. The conservation law (5.23) can be derived also directly from the Bianchi identity $\nabla^N G^M_{\ N} = 0$.

As shown by Langlois et al in [38], if the bulk contains only the cosmological constant Λ, any set of functions a and n satisfying equations

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{a'}{a} + \frac{\Lambda}{6} - \frac{k}{a^2} + \frac{\mathcal{C}}{a^4} , \tag{5.24}$$

$$\frac{n'}{n}\frac{\dot{a}}{a} = \frac{\dot{a}'}{a} \tag{5.25}$$

are also local solutions in the bulk of all Einstein equations (5.15) - (5.18). Here $\mathcal{C}$ is an integration constant which is time independent if the bulk cosmological constant is constant in time. Using the junction conditions (5.22) in equation (5.24) we get an equation for the scale factor on the brane

$$H_0^2 \equiv \frac{\dot{a}_0^2}{a_0^2} = \frac{\kappa^4}{36}\rho_b^2 + \frac{\Lambda}{6} - \frac{k}{a_0^2} + \frac{\mathcal{C}}{a_0^4} . \tag{5.26}$$

This is a modified Friedmann equation and this equation together with conservation law (5.23) rule the dynamics of the brane Universe. Sometimes it is more convenient to use the dynamical equation for the Hubble parameter H_0 and in this case the Friedmann equation (5.26) can be seen as a constraint equation. Inserting the junction conditions (5.22) into the yy-component of the Einstein equations (5.18) we get

$$\dot{H}_0 + 2H_0^2 = -\frac{\kappa^4}{36}\rho_b(\rho_b + 3p) + \frac{\Lambda}{3} . \tag{5.27}$$

Another way of deriving equation (5.26) is given in [118].

If we compare the generalized Friedmann equation (5.26) with the standard Friedmann equation (2.10) we see the following remarkable differences:

- The energy density of the brane enters quadratically $H^2 \propto \rho_b^2$ in contrast with the standard 4-dimensional Friedmann equation (2.10) where the energy density enters linearly $H^2 \propto \rho$.
- The effective 4-dimensional gravitational constant does not appear in this equation and all dynamics depends only on the 5-dimensional gravitational constant κ.
- Equation (5.26) contains an additional integration constant $\mathcal{C}$, whose influence can be shown to correspond to an effective radiation term and it is therefore called *dark radiation*.

If we assume that brane matter obeys a barotropic equation of state $p = (\Gamma - 1)\rho_b$ and that the bulk cosmological constant and the dark radiation term vanish ($\Lambda = 0$ and $\mathcal{C} = 0$), we can easily solve equations (5.23) and (5.26). The time dependence of the scale factor is as follows

$$a_0 \propto t^{\frac{1}{3\Gamma}} \tag{5.28}$$

and it is slower than in the standard case $a \propto t^{\frac{2}{3\Gamma}}$ (see Sec. 2.1.1), which is also approved by the nucleosynthesis. A drastic change in the expansion rate modifies the prediction for the abundances of light elements and is therefore in contradiction with the standard cosmological scenario [37]. The conclusion is that a brane embedded in an empty Minkowski bulk spacetime is not confirmed by standard observations.

Let us now follow the procedure presented by Langlois et al [38]. To get a more suitable evolution law for the brane scale factor compared with the expression (5.28) it is necessary to consider non-vanishing dark radiation and to separate the brane cosmological constant (or brane tension in the language of Randall-Sundrum) and the rest of the brane matter ρ as follows

$$\rho_b = \sigma + \rho \,, \qquad \sigma = const. \tag{5.29}$$

Substituting expression (5.29) into the modified Friedmann equation (5.26) we get

$$H_0^2 = \frac{\Lambda}{6} + \frac{\kappa^4}{36}\sigma^2 + \frac{\kappa^4}{18}\sigma\rho + \frac{\kappa^4}{36}\rho^2 - \frac{k}{a_0^2} + \frac{\mathcal{C}}{a_0^4} \ . \tag{5.30}$$

If we choose the brane tension σ such that it completely compensates the bulk cosmological constant Λ

$$\Lambda_4 \equiv \frac{\Lambda}{6} + \frac{\kappa^2}{36}\sigma^2 = 0 \ , \tag{5.31}$$

and make the identification

$$\kappa_4^2 \equiv 8\pi G_N = \frac{\kappa^4}{6}\sigma \tag{5.32}$$

we can write the Friedmann equation (5.30) as follows

$$H_0^2 = \frac{8\pi G_N}{3}\rho\left(1 + \frac{\rho}{2\sigma}\right) - \frac{k}{a_0^2} + \frac{\mathcal{C}}{a_0^4} \ . \tag{5.33}$$

The first term in the right hand side is exactly the linear term of the standard Friedmann equation and it dominates at low energies (at late times) when the brane tension σ is much greater that the usual energy density ρ on the brane, i.e., $\sigma \gg \rho$. On the contrary, at high energies when $\sigma \ll \rho$ the quadratic term dominates and changes the conventional dynamics. The last term, which characterizes the influence of the bulk, decreases quickly and becomes unimportant at late times. The condition (5.31) is known as the *criticality condition* and the corresponding braneworld as *critical.* This is exactly the case discussed by Randall and Sundrum [36] and it can be interpreted as vanishing of an effective 4-dimensional cosmological constant Λ_4 on the brane. The case $\Lambda_4 < 0$ (brane tension σ is greater than bulk cosmological constant Λ) corresponds to *subcritical* (anti-de Sitter) brane and $\Lambda_4 > 0$ (brane tension σ is smaller than the bulk cosmological constant Λ) corresponds to *supercritical* (de Sitter) brane. If we assume that the matter on the brane obeys the barotropic equation of state the solution of equation (5.33) in the case of the critical brane $\Lambda_4 = 0$ can be written [38], [89]

$$a_0(t) = \left[\frac{\kappa^2\rho_0\Gamma}{2}\left(\frac{\kappa^2\rho_0\Gamma}{4}\sigma t^2 + t\right)\right]^{\frac{1}{3\Gamma}} . \tag{5.34}$$

Here ρ_0 is the initial value of energy density on brane. For supercritical brane ($\Lambda_4 > 0$) the evolution of the scale factor is [38], [89]

$$a_0(t) = \left[\frac{\kappa^2\rho_0}{6\sqrt{\alpha}}\sinh(3\Gamma\sqrt{\alpha}t) + \frac{\kappa^4\sigma\rho_0}{36\alpha}\left[\cosh(3\Gamma\sqrt{\alpha}t - 1)\right]\right]^{\frac{1}{3\Gamma}} , \tag{5.35}$$

and for subcritical brane ($\Lambda_4 < 0$) [89]

$$a_0(t) = \left[\frac{\kappa^2\rho_0}{6\sqrt{-\alpha}}\sin(3\Gamma\sqrt{\alpha}t) + \frac{\kappa^4\sigma\rho_0}{36\alpha}\left[\cos(3\Gamma\sqrt{\alpha}t - 1)\right]\right]^{\frac{1}{3\Gamma}} . \tag{5.36}$$

As we can see, at late times the usual behaviour of the scale factor appears and the constraints imposed by the theory of nucleosynthesis are satisfied quite well.

5.2.2 Bulk based point of view

In the brane based approach we used the Gaussian normal coordinate system, Z_2 symmetry and solved the junction condition to get the cosmology on the brane. In the bulk based approach we assume an appropriate 5-dimensional bulk metric and then find how the brane is located in this metric. In this chapter we follow the works of Kraus [121], Ida [122] and Bowcock et al [120]. For a review see [89], a generalization to arbitrary dimensions can be found in [95].

First, let us assume that the bulk is empty, $T_{MN} = 0$. In this case the bulk spacetime is the Einstein space, i.e., $R_{MN} = \frac{2}{3}\Lambda g_{MN}$. Its maximally symmetric solution in the case of $\Lambda < 0$ is the anti-de Sitter spacetime (AdS_5) and in the case of $\Lambda < 0$ the de Sitter spacetime (dS_5). The generalization of AdS_5 that preserves the 4-spatial isotropy, has 3-spatial homogeneity and solves the 5-dimensional vacuum Einstein equation, is called *the Schwarzschild-AdS*$_5$ *solution.* It follows that the FRW braneworld can be a foliation of this bulk spacetime and it is therefore a cosmological generalization of the RS braneworld.

Starting with the general 5-dimensional metric which has planar (spherical/hyperboloidal) symmetry in three of its spatial directions and which has one brane with a δ-function source on it, Bowcock et al showed [120] that the resulting metric can be written as

$$ds^2 = -f(R)d\tau^2 + \frac{dR^2}{f(R)} + R^2\gamma_{ij}dx^i dx^j \ , \tag{5.37}$$

where

$$f(R) \equiv k - \frac{\Lambda}{6}R^2 - \frac{\mathcal{C}}{R^2} \ . \tag{5.38}$$

This is the Schwarzschild-(A)dS_5 metric. The term containing $\mathcal{C}$ comes from the "electric" part of the Weyl tensor (see Subsec. 5.2.3) and it is the 5-dimensional analog of the Schwarzschild mass (the 5-dimensional gravitational potential behaves as R^{-2} instead of R^{-1}). It can be interpreted as a bulk black hole and this black hole gives rise to dark radiation on the brane. If $\mathcal{C} = 0$, the metric (5.37) reduces to the anti-de Sitter metric and this is exactly the case discussed by Randall and Sundrum. On the other hand, if $\Lambda = 0$ the metric reduces to a 5-dimensional analog of the Schwarzschild metric. Metric coefficients are time independent and metric (5.37) is static. This is a generalization of the Birkhoff theorem which states that spherically symmetric solution of the Einstein vacuum equation is static and has the structure of the Schwarzschild geometry. A more detailed discussion of the generalized Birkhoff theorem can be found in [95]. Since the bulk spacetime is static the brane is no longer a fixed boundary of the spacetime but moves along a timelike trajectory.

Let us use parametrization $X^A = (x^a, \tau(t), R(t))$ to characterize the 3-brane as a boundary of the 5-dimensional bulk. Here t is proper time for an observer comoving with the brane and it must be continuous at the brane while τ may be discontinuous at the brane. The metric coefficient $R(t)$ describes the trajectory of the brane in the 5-dimensional spacetime along the extra dimension and it is possible to interpret the expansion of the Universe as a motion of the brane through the static bulk. The components of the tangent vector (brane velocity) of the brane can be written as

$$u^{A} = \Big(\mathbf{0},\, \dot{\tau}(t),\, \dot{R}(t)\Big) \tag{5.39}$$

and here dot is the derivative with respect to the brane proper time t. The normalization condition

$$g_{MN} u^{M} u^{N} = -1 \tag{5.40}$$

leads to the relation

$$f\dot{\tau}^2 - f^{-1}\dot{R}^2 = 1 \ . \tag{5.41}$$

Inserting this expression into metric (5.37) we get the usual FRW metric on the brane

$$ds^2_{brane} = -dt^2 + R^2(t)\gamma_{ij}\, dx^i dx^j \ , \tag{5.42}$$

so we can identify the brane scale factor $a_0(t)$ and the metric coefficient $R(t)$. Taking into account the expressions

$$n_{M}\, u^{M} = 0, \qquad\qquad n_{M}\, n^{M} = 1 \tag{5.43}$$

we can find the unit normal vector for the brane (up to the sign ambiguity)

$$n_{A} \;=\; \Big(\mathbf{0},\, n^{\tau},\, n^{R}\Big) = \left(\mathbf{0},\, \dot{R},\, -\frac{1}{f}\sqrt{f+\dot{R}^2}\right) \ . \tag{5.44}$$

The spatial components of the extrinsic curvature can be computed using the definition (5.6)

$$K^{\pm}_{ij} = -\Gamma^{\mu}_{ij}\, n_{\mu} = \pm\sqrt{f^{\pm}+\dot{R}^2}\, R\, \gamma_{ij} \ . \tag{5.45}$$

Junction conditions (5.9) lead to relation

$$\frac{\left(\sqrt{f^{+}+\dot{R}^2}+\sqrt{f^{-}+\dot{R}^2}\right)}{R}\, g_{ij} = -\kappa^2\left(S_{ij} - \frac{1}{3}\, S g_{ij}\right) \ , \tag{5.46}$$

where $g_{ij} = R^2(t)\gamma_{ij}$. If we assume that the brane matter is an ideal fluid (2.3) and substitute expression (5.38) into equation (5.46), we get finally

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\kappa^4}{36}\rho^2 - \frac{k}{R^2} + \frac{\Lambda}{6} - \frac{(\mathcal{C}^{+}+\mathcal{C}^{-})}{2R^4} + \frac{9}{4}\frac{(\mathcal{C}^{+}-\mathcal{C}^{-})}{\kappa^4\rho^2 R^8} \ . \tag{5.47}$$

If we assume now the Z_2 symmetry, i.e., $\mathcal{C}^+ = \mathcal{C}^-$ and take into account that $a_0(t) \equiv R(t)$, we get the Friedmann equation (5.26) as in Subsec. 5.2.1 when we considered the brane based approach.

We can conclude, that both descriptions lead to the same cosmology and they are therefore physically equivalent. In some cases it is more convenient to use the bulk based approach, and in some cases the brane based approach. The transformation functions between the Gaussian normal coordinate system (5.14) and the coordinate system (5.37) were found by Mukohyama et al in [123].

5.2.3 Covariant curvature formulation

Shiromizu et al [124] derived the effective 4-dimensional Einstein equations for the 4-dimensional metric by projecting the 5-dimensional metric onto the brane. The procedure is essentially similar to the 3 + 1 split of 4-dimensional spacetime used in the treatment of the initial value problem of general relativity [1]. This approach gives a deeper understanding about a relation between the 4-dimensional and 5-dimensional dynamics.

The starting point is the *Gauss equation* which relates the 4-dimensional curvature tensor ${}^4R_{ABCD}$, constructed from the induced metric h_{MN} (5.7), to the 5-dimensional one R_{ABCD}, constructed from g_{MN}, and to the extrinsic curvature K_{AB}

$$ {}^4R_{ABCD} = R_{KLMN}\ h^K_{\ A}\ h^L_{\ B}\ h^M_{\ C}\ h^N_{\ D} + K_{AC}K_{BD} - K_{AD}K_{BC}\ . \qquad (5.48) $$

The 5-dimensional curvature tensor can be decomposed into the Weyl tensor C_{ABCD}, the Ricci tensor R_{AB} and the scalar curvature R as follows

$$ \begin{aligned} R_{ABCD} = C_{ABCD} \quad &+ \quad \frac{2}{3}\Big[\Big(g_{AC}R_{BD} - g_{AD}R_{BC}\Big) - \Big(g_{BC}R_{AD} - g_{BD}R_{AC}\Big)\Big] \\ &- \quad \frac{1}{6}R\left(g_{AC}g_{BD} - g_{AD}g_{BC}\right)\ . \end{aligned} \qquad (5.49) $$

Using this equation together with the 5-dimensional Einstein equations (5.3) and with the Gauss equation (5.48), the 4-dimensional Einstein equations read as

$$ \begin{aligned} {}^4G_{MN} \quad = \quad & \frac{2}{3}\kappa^2\left[T_{RS}\ h^R_{\ M}\ h^S_{\ N} + \left(T_{RS}n^R n^S - \frac{1}{4}T^R_{\ R}\right)h_{MN}\right] - E_{MN} \\ & + K\ K_{MN} - K^S_{\ M}\ K_{NS} - \frac{1}{2}h_{MN}\left(K^2 - K^{AB}K_{AB}\right)\ , \end{aligned} \qquad (5.50) $$

where

$$ E_{MN} \equiv C^A_{\ BRS}\, n_A\, n^R\ h^B_{\ M}\, h^S_{\ N}\ , \qquad E^R_{\ R} = 0 \qquad (5.51) $$

is called the "electric" part of the Weyl tensor. It is the projection of the component $C^{N}{}_{BNS}$ of the Weyl tensor orthogonal to normal n^{A} and carries information about the gravitational field outside the brane [124].

The surface energy-momentum tensor on the brane can be decomposed as in the Subsec. 5.2.1 and its covariant form is as follows

$$S_{MN} = \tau_{MN} - \sigma h_{MN}, \qquad \tau_{MN} n^{M} = 0 \tag{5.52}$$

where τ_{MN} is the energy-momentum tensor of ordinary matter confined to the brane and σ is the brane tension measured by a 5-dimensional observer.

Now Z_2 symmetry can be assumed. Substituting the brane energy-momentum tensor (5.52) into junction conditions (5.9) and the result into equation (5.50) we finally arrive at the 4-dimensional Einstein equations on the brane

$$^{4}G_{MN} = -\Lambda_4\, h_{MN} + \kappa_4^2\, \tau_{MN} + \kappa^4\, \pi_{MN} - E_{MN} + \frac{2}{3}\kappa^2\, F_{MN}\ , \tag{5.53}$$

where we have made the identifications

$$\begin{aligned}
\kappa_4^2 &= 8\pi G_N = \frac{1}{6}\sigma\kappa^2\ , && (5.54)\\
\Lambda_4 &= \frac{1}{2}\left(\Lambda + \frac{1}{6}\kappa^4\sigma^2\right)\ , && (5.55)\\
\pi_{MN} &= \frac{1}{12}\tau\,\tau_{MN} - \frac{1}{4}\tau_{MA}\,\tau^{A}{}_{N} + \frac{1}{24}h_{MN}\,\tau_{AB}\,\tau^{AB} - \frac{1}{24}h_{MN}\,\tau^2\ , && (5.56)\\
F_{MN} &= T_{AB}\, g^{A}{}_{M}\, g^{B}{}_{N} + \left[T_{AB}\, n^{A}\, n^{B} - \frac{1}{4}T\right]\, g_{MN}\ . && (5.57)
\end{aligned}$$

These equations were first derived by Shiromizu et al [124], generalized to include bulk contribution F_{MN} by Maartens [126] and to arbitrary dimensions by Padilla [95]. If we assume that bulk spacetime contains only the cosmological constant Λ it follows that $F_{MN} = 0$. The cosmology of non-empty bulk was discussed in [127], [128] and we give a short review in Subsec. 5.2.4. In the Randall-Sundrum case $\Lambda_4 = 0$ due to the fine-tuning between the brane tension σ and the bulk cosmological constant Λ. The $\pi_{\mu\nu}$ term is quadratic in brane energy-momentum $\tau_{\mu\nu}$ and therefore recovers the behavior discussed in the case of modified Friedmann equation (5.26).

If we choose the Gaussian normal coordinate system (5.14) near the brane so that $n_{A}\, dx^{A} = dy$, $n_{M} = (0,0,0,0,1)$ we can replace the 5-dimensional indices $M, N, ...$ in (5.53) with 4-dimensional indices $\mu, \nu, ...$ and drop the superscript 4.

Now let us consider the *Codazzi equation* which relates the 5-dimensional Ricci tensor R_{AB} to the extrinsic curvature K_{AB}

$$\nabla_B K^B_{\ A} - \nabla_A K = R_{BC}\, h^B_{\ A}\, n^C = G_{BC}\, h^B_{\ A}\, n^C \ . \qquad (5.58)$$

If we use the 5-dimensional Einstein equation (5.3) and the junction condition (5.9), we obtain from the Codazzi equation the general conservation law for the brane matter

$$\nabla^D \tau_{AD} = -2\, T_{BC}\, h^B_{\ A}\, n^C \ . \qquad (5.59)$$

As we pointed out in the previous section, there can be exchange of energy-momentum between the bulk and the brane. If we assume that bulk contains only the cosmological constant Λ, i.e., $T_{MN} = 0$ and use the Gaussian normal coordinate system we arrive at the usual conservation law for the brane matter $\nabla^\nu \tau_{\mu\nu} = 0$. This means that the brane and the bulk energy exchange can be only gravitational and it occurs through the projection of the Weyl tensor $E_{\mu\nu}$ which characterizes the gravitational field outside the brane. If we compute the covariant derivative of the Einstein equation (5.53) and take into account the Bianchi identities $\nabla^\nu G_{\mu\nu} = 0$ we get the following relation for $E_{\mu\nu}$

$$\nabla^\nu E_{\mu\nu} = \kappa^4\, \nabla^\nu \pi_{\mu\nu} \ . \qquad (5.60)$$

As pointed out in [124], [126], the effective field equations (5.53) do not form a closed system and must therefore be supplemented with equation (5.60). The questions associated with the term coming from the projection of the Weyl tensor are studied in [124], [125], [126]. Here we only mention that in the case of a purely anti-de Sitter bulk the term $E_{\mu\nu}$ vanish. If we assume the FRW metric on the brane and take the brane matter to be an ideal fluid (2.3), the 4-dimensional Einstein equations (5.53) lead to the modified Friedmann equation (5.26) without the dark radiation term which comes from the projection of the bulk Weyl tensor.

5.2.4 Bulk scalar field

In the Randall-Sundrum model and in the simplest cosmology inspired by this proposal, the 5-dimensional Einstein gravity was supplemented by the 5-dimensional cosmological constant. Thus the gravity is the only field propagating in the bulk and this situation seems unlikely. The moduli fields and the associated gravitational sector scalar fields (dilaton) arising in the models of compactified M-theory are also filling the bulk and in a more realistic model it is necessary to take this into account. Inclusion of the scalar field is also motivated by the Hořava-Witten model (see Subsec. 4.1) and by the extensions of the Randall-Sundrum model.

In this subsection we first review the works of Maeda et al [127] and Mennim et al [128] where the covariant curvature formulation described in Sec. 5.2.3 was generalized to the case when bulk scalar field is included. As in the case of empty bulk, the field equations on the brane do not form a closed system. There are two parameters which carry the influence of the bulk:

- the projected component of the Weyl tensor $E_{\mu\nu}$ and the corresponding dark radiation term in the case of homogeneous and isotropic cosmology on the brane,
- the *loss parameter* $\Delta\Phi_2$, defined below, which is completely undetermined by the 4-dimensional dynamics. In some special cases the loss parameter vanishes but in general the determination of it requires the complete description of the full 5-dimensional spacetime.

To summarize, the aim of this section is to extend the coordinate independent formalism presented in Subsec. 5.2.3 to the case where there is a scalar field in the bulk and to introduce other attempts to treat the dynamics of the brane and the bulk influenced by the presence of the scalar fields.

Scalar field on the brane

The 5-dimensional action supplemented by the scalar field is

$$\begin{aligned} I &= \int d^5x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\nabla_{M}\Phi\nabla^{M}\Phi - \Lambda(\Phi)\right] \\ &+ \int d^4x\sqrt{-h}\left[\frac{1}{\kappa^2}K^{\pm} + L_{matter} - \sigma(\Phi)\right], \end{aligned} \tag{5.61}$$

where $h_{\mu\nu}$ is induced metric on the brane (5.7), $K^{\pm}$ is the extrinsic curvature of the brane and we have replaced the cosmological constant Λ, included in the action (5.1) by the scalar field dependent "cosmological constant" in the following way: $\Lambda \rightarrow \kappa^2\Lambda(\Phi)$. The Φ dependent cosmological constant is actually nothing but the self-interaction potential energy of the bulk scalar field and is sometimes denoted by $V(\Phi)$.

The bulk equations of motion are [127]

$$G_{AB} = \kappa^2 T_{AB} \equiv \kappa^2 \left[\nabla_A \Phi \nabla_B \Phi - g_{AB}\left(\frac{1}{2}\nabla_M \Phi \nabla^M \Phi + \Lambda\right)\right] \ , \tag{5.62}$$

$$\nabla_M \nabla^M \Phi - \frac{d\Lambda}{d\Phi} - \frac{d\lambda}{d\Phi}\delta(y) = \frac{1}{\Omega}\frac{d\Omega}{d\Phi}\,\tau\,\delta(y) \ , \tag{5.63}$$

where τ is the trace of the energy-momentum tensor of the brane matter. The right hand side of the dilaton equation arises from the fact, that the matter on the brane may be minimally coupled with respect to conformally related metric $\tilde{g}_{\mu\nu} = \Omega^2(\Phi) g_{\mu\nu}$. This term disappears if the matter Lagrangian L_{matter} is independent of the scalar field Φ.

It is suitable to choose a coordinate y such that $n_A dx^A = dy$ and the brane is located at $y = 0$. In this case it is possible to use the Gaussian normal coordinate system (5.14) in the neighborhood of the brane. The dilaton field can be expanded near the brane as follows

$$\Phi = \phi(x) + \Phi_1(x)|y| + \frac{1}{2}\Phi_2(x)y^2 + \mathcal{O}(y^3), \tag{5.64}$$

this leads to an additional junction conditions for the dilaton field

$$\Phi_1 = \frac{1}{2}\left(\frac{d\sigma}{d\phi} + \frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right) . \tag{5.65}$$

From the junction conditions (5.9), (5.65), equations (5.62) and (5.53) we can write the 4-dimensional Einstein equation as follows [127]

$$\begin{aligned} G_{\mu\nu} &= \frac{2\kappa^2}{3}\left(\nabla_\mu \phi \nabla_\nu \phi - \frac{5}{8} h_{\mu\nu} \nabla_\rho \phi \nabla^\rho \phi\right) \\ &+ \left[-\Lambda_4 + \frac{\kappa^2}{16}\left(2\frac{d\sigma}{d\phi} + \frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right)\frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right] h_{\mu\nu} \\ &+ \kappa_4^2(\phi)\tau_{\mu\nu} + \kappa^4\,\pi_{\mu\nu} - E_{\mu\nu} \ , \end{aligned} \tag{5.66}$$

where $\pi_{\mu\nu}$ is given by expression (5.56). The effective Newton constant on the brane G_N and effective 4-dimensional cosmological constant Λ_4 become scalar field dependent and are given by the following expressions

$$8\pi G_N = \frac{\kappa^4}{6}\sigma(\phi) \ , \tag{5.67}$$

$$\Lambda_4 = \frac{1}{2}\kappa^2 \left[\Lambda + \frac{1}{6}\kappa^2\,\sigma^2 - \frac{1}{8}\left(\frac{d\sigma}{d\phi}\right)^2\right] \ . \tag{5.68}$$

The 4-dimensional dilaton equation can be written

$$\nabla_\mu \nabla^\mu \phi = \frac{d}{d\phi}\left(\Lambda + \frac{\kappa^2}{6}\sigma^2\right) - \frac{\kappa^2}{12}\left[\frac{d\lambda}{d\phi} - (4\sigma - \tau)\frac{1}{\Omega}\frac{d\Omega}{d\phi}\right]\tau - \Phi_2 \,. \quad (5.69)$$

The term Φ_2 is analogous to the term which is coming from the bulk Weyl tensor in equation (5.66). It represents the effect of the bulk and the 5-dimensional nature of the scalar field. It cannot be determined locally from the field equation on the brane. As we saw in Sec. 5.2.3, it was possible using the Bianchi identity to connect the "bulk term" $E_{\mu\nu}$ with "brane term" $\pi_{\mu\nu}$ via relation (5.60). Now the situation is even worse. The quantity

$$\Delta\Phi_2 \equiv \Phi_2 - \frac{1}{4}\frac{d\sigma}{d\phi}\frac{d^2\sigma}{d\phi^2} \quad (5.70)$$

is known as the *loss parameter* and it measures the scalar energy-momentum loss from the brane to the bulk. In the Randall-Sundrum case the dilaton field is constant in the bulk and hence $\Phi_1 = 0$ and $\Phi_2 = 0$.

Finally, the Codazzi equation (5.58) gives the conservation law

$$\nabla_\nu \tau_\mu{}^\nu = -\frac{1}{\Omega}\frac{d\Omega}{d\phi}\,\tau\,\nabla_\mu \phi \,, \quad (5.71)$$

which with conformally rescaled metric $\tilde{g}_{\mu\nu}$ gives the usual conservation law for brane matter

$$\tilde{\nabla}_\nu \tilde{\tau}_\mu^\nu = 0 \,, \qquad \tilde{\tau}_{\mu\nu} = \frac{1}{\Omega^2}\,\tau_{\mu\nu} \,. \quad (5.72)$$

In what follows we review the work of Maeda et al [127] and present some particular solutions on the FRW brane. Assuming that there is no energy transfer from the brane to the bulk and ignoring the brane matter ($\tau_{\mu\nu} = 0$) it is possible to write the dilaton equation and its solution in a simple form

$$\ddot{\phi} + 3H\dot{\phi} = 0 \qquad \Longrightarrow \qquad \dot{\phi} = \frac{C_\phi}{a^3} \,, \quad (5.73)$$

where C_ϕ is an integration constant. The modified Friedmann type equation for the scale factor can be written as [1]

$$H^2 = \frac{\kappa^2 C_\phi^2}{6a^6} - \frac{k}{a^2} + \frac{\mathcal{C}}{a^4} \,. \quad (5.74)$$

[1]In what follows we don't use the subscript zero to mark the quantities on the brane, as distinct from notation used in Subsec. 5.2.1. Here the subscript zero is used to mark the initial values of the corresponding quantity.

The first term in the r.h.s. corresponds to stiff fluid matter ($\Gamma = 2$) and the last term corresponds to dark radiation. The general solution with an initial singularity $a = 0$ is found in [127]. All initial and asymptotic solutions with different parameter ranges (C_ϕ, $\mathcal{C}$, k) are analyzed in [127]. Here we pay attention to two important remarks.

- In the case of spatially flat models ($k = 0$) with $\mathcal{C} > 0$ the standard radiation-dominated Universe is recovered at late time

$$a(t) = (4\mathcal{C})^{\frac{1}{4}}\ t^{1/2}\ . \tag{5.75}$$

  On the other hand, if $\mathcal{C} < 0$ the Universe will recollapse.

- There exists a non-singular solution for $k = -1$ when $\mathcal{C} \le -\sqrt{\frac{2}{3}}\kappa|C_\phi|$ [127]

$$a^2 = \frac{{a_0}^2}{(1-\tau^2)}\left[1 - \left(\frac{\tau}{\tau_*}\right)^2\right]\ , \tag{5.76}$$

  where

$$\begin{aligned} \tau &= \tanh|\eta|\ , \quad \eta = \int \frac{dt}{a}\ , \\ a_0^2 &= \frac{1}{2}\left[|\mathcal{C}| + \sqrt{\mathcal{C}^2 - \frac{2}{3}\kappa^2 C_\phi^2}\right]\ , \\ \tau_* &= \frac{\sqrt{3}}{\sqrt{2}\kappa|C_\phi|}\left[|\mathcal{C}| + \sqrt{\mathcal{C}^2 - \frac{2}{3}\kappa^2 C_\phi^2}\right], \quad \tau_* \ge 1\ . \end{aligned} \tag{5.77}$$

  This solution describes an infinitely large Universe contracting from the past infinity $t \to -\infty$, bouncing at $t = 0$, $a = a_0$ and then expanding to infinity at $t \to \infty$.

A remark is in order. If we compare the action (5.61) used in this section with the action (4.4) used by Lukas et al [100] to derive the cosmological model of the Hořava-Witten theory we see remarkable similarities. After dropping the second brane part from the action (4.4) and making the identifications

$$\Phi = \frac{1}{\sqrt{2}\kappa}\bar{\phi}, \tag{5.78}$$

$$\Lambda(\Phi) = \frac{\alpha^2}{6\kappa^2}e^{-2\sqrt{2}\kappa\bar{\phi}}, \tag{5.79}$$

$$\sigma(\Phi) = \frac{\sqrt{2}\alpha}{\kappa^2}e^{-\sqrt{2}\kappa\bar{\phi}}, \tag{5.80}$$

we recover the low energy Hořava-Witten theory. Here $\bar{\phi}$ denotes the scalar field used in Sec. 4.1. If the bulk cosmological constant Λ and brane tension σ are given by expressions (5.79) and (5.80) the 4-dimensional cosmological constant vanishes ($\Lambda_4 = 0$) despite the value of α (see Sec. 4.1) as can be seen from conditions (5.68). Note that the 5-dimensional cosmological constant is positive here and this contradicts to the RSI model. We can conclude that the fundamental theory allows an exponential potential for the bulk scalar field.

On the other hand, as shown by Kachru et al [129], it is possible to obtain Minkowski spacetime on the brane with vanishing 4-dimensional cosmological constant $\Lambda_4 = 0$ if the 5-dimensional cosmological constant is zero $\Lambda = 0$. From the expressions (5.68) it follows that in this case the brane tension is

$$\sigma = \sigma_0 e^{\frac{2}{\sqrt{3}}\kappa^2\phi} , \tag{5.81}$$

where σ_0 is an integration constant.

A general Friedmann equation for a brane coupled to the bulk scalar field ϕ is presented in [130], [131] [2] and can be written as follows

$$\begin{aligned} H^2 &= \frac{\kappa^4}{36}\rho^2 + \frac{\kappa^4}{18}\sigma\rho + \frac{1}{3a^4}\int dt \frac{da^4}{dt}\left(\Lambda_4 - \frac{1}{4}\kappa^2\dot{\phi}^2\right) \\ &- \frac{\kappa^2}{18a^4}\int dt a^4 \frac{d\sigma}{dt}\rho + \frac{\mathcal{C}}{a^4} . \end{aligned} \tag{5.82}$$

The 5-dimensional Klein-Gordon equation (5.63) is given by

$$\ddot{\phi} + 4H\dot{\phi} + \frac{1}{4}\left(\frac{4}{3} - \Gamma\right)\kappa^2\rho\frac{d\sigma}{d\phi} = -\frac{2}{\kappa^2}\frac{d\Lambda_4}{d\phi} + \Delta\Phi_2 . \tag{5.83}$$

Here $\Lambda_4(\phi)$ is in the role of the effective 4-dimensional potential. If the brane tension σ and 5-dimensional cosmological constant Λ are constants the Randall-Sundrum case is recovered with $\Lambda_4 = 0$ and $G_N = const$. The equations (5.82), (5.83) were analyzed at different evolution epochs of the Universe, including the radiation ($\Gamma = \frac{4}{3}$) and matter ($\Gamma = 1$) epochs, in [130], [131] and the corresponding approximate solutions were found. In all cases it was assumed that the loss parameter vanishes $\Delta\Phi_2 = 0$. First

[2] We present the following equations using the notations used above and they are different from the ones used in [130], [131]. However, instead of using ϕ as a brane scalar field, we use it here as a bulk scalar field.

they considered the case when the 4-dimensional potential possesses a self-tuned minimum, which means that for each value of the brane tension there exists a value of the scalar field that represents the vacuum state $V(\phi_0)$. The main conclusion was that in the late Universe the Hubble parameter oscillates with a period $t_p = 400$ s and it rules out this class of models [130].

The second class of models discussed in [130], [131] are based on the model of supergravity in singular spaces [132] where in addition to the 5-dimensional supergravity the boundaries are taken into account. Then the dynamics is specified by the superpotential W, related to the brane and bulk potentials by

$$\sigma(\phi) = \frac{3}{2\kappa^2} W \,, \tag{5.84}$$

$$\Lambda(\phi) = \frac{9}{32}\frac{1}{\kappa^4}\left(\frac{dW}{d\phi}\right)^2 - \frac{3}{8}\frac{1}{\kappa^2} W^2 \tag{5.85}$$

In this case the 4-dimensional cosmological constant vanishes automatically, $\Lambda_4 = 0$. Supergravity determines the superpotential to be of the exponential type [133]

$$W = \xi e^{\frac{2\kappa}{\sqrt{3}}\alpha\phi} \,, \qquad \alpha = \frac{1}{\sqrt{3}}, \, \frac{1}{\sqrt{12}} \,, \tag{5.86}$$

where ξ is a characteristic scale related to the brane tension. The solutions for the scale factor and the scalar field without matter on the brane are as follows

$$a = \left(1 - \alpha^2 \xi \, |y|\right)^{\frac{1}{4\alpha^2}} \,, \tag{5.87}$$

$$\phi = \frac{1}{\alpha} \ln\left(1 - \alpha^2 \xi \, |y|\right) \,. \tag{5.88}$$

If $\alpha \to 0$ the RS model is recovered with $a(y) = e^{-\frac{\xi}{4}y}$. Since the Ricci scalar is proportional to the superpotential [134] there is a singularity in the bulk ($y = \frac{1}{\xi\alpha^2}$), i.e., $R \to \infty$ if $\phi \to \infty$, for details see [134].

In the real world supersymmetry must be broken and the brane tension in the case of broken supersymmetry can be written as

$$\bar{\sigma} = T\sigma, \qquad T > 1 \,, \tag{5.89}$$

where T is a real number. If $T > 1$ we have the de Sitter and if $T < 1$ the anti-de Sitter brane. Since supersymmetry is broken the relation between the brane tension and the superpotential is modified. Therefore, in general, the 4-dimensional cosmological constant is non-vanishing

$$\Lambda_4 = \frac{(1-T^2)}{2}\kappa^2\Lambda\,. \tag{5.90}$$

At late times the scale factor (with negligible matter contribution) evolves as [130]

$$a(t) \sim t^{\left(\frac{1}{3}+\frac{1}{6\alpha^2}\right)}\,. \tag{5.91}$$

This solution corresponds to a 4-dimensional FRW solution with barotropic index Γ

$$\Gamma = \frac{4\alpha^2}{1+2\alpha^2} \tag{5.92}$$

and if $\alpha = \frac{1}{\sqrt{12}}$ it follows that $\Gamma = 2/7$. Since $\Gamma < 1$ this leads to an accelerated expansion of the scale factor and can be interpreted as a model of *brane quintessence* [130], [133]. It is possible to fine-tune the accelerating epoch to correspond to the present day Universe. The supergravity cosmological solutions including the brane matter were discussed by Brax et al [130] and the bulk scalar fields in context of supergravity were discussed also by Davis in [135], [136].

Solutions of the bulk equations

In the previous subsection we reviewed attempts to deal with the solutions on the brane without trying to solve the full system of equations (5.62) and (5.63) in the bulk. To understand the effects of bulk scalar field, especially the loss parameter $\Delta\Phi_2$, we need also general bulk solutions. In order to solve the bulk equations of motion some restrictions can be made and the general solution to the resulting equations was found by Chamblin et al [114] and by Langlois et al [137]. In [137], two different strategies were employed to find global solutions.

First strategy introduces a relation between the scalar field and the bulk scale factor

$$\phi = \frac{\sqrt{3}}{\kappa}\lambda \ln a \tag{5.93}$$

and leads to the result that the bulk potential for the scalar field is necessarily of the exponential form

$$V = V_0\, e^{-\frac{2}{\sqrt{3}}\kappa\lambda\phi}\,. \tag{5.94}$$

Here λ is a parameter related to the equation of state of the brane matter. It was argued by Langlois et al [137] that the assumption (5.93) leads essentially to the same type static metric discussed already in Subsec. 5.2.2

$$ds^2 = -h(R)d\tau^2 + \frac{dR^2}{g(R)} + R^2\delta_{ij}dx^i dx^j \ , \tag{5.95}$$

where

$$h(R) = -\frac{\kappa^2 V_0}{6\left(1-\frac{\lambda^2}{4}\right)} R^{2-2\lambda^2} - \mathcal{C}R^{\lambda^2-2} \ , \tag{5.96}$$

$$g(h) = -\frac{\kappa^2 V_0}{6\left(1-\frac{\lambda^2}{4}\right)} R^{2-2\lambda^2} - \mathcal{C}R^{-2-\lambda^2} \ . \tag{5.97}$$

Note that in the case $\lambda = 0$ the results of Subsec. 5.2.2 are recovered. The interpretation is also the same.

Alongside with the static solution (5.95) there exist non-static solutions [137] due to the bulk scalar fields. As shown by Langlois et al [137], it is possible to get a power law expansion on the brane (in the Einstein frame) in the case of barotropic perfect fluid ($\Gamma \neq 0$)

$$a \sim t^p, \qquad p = \frac{1}{3\Gamma}\left[1 + \frac{(3\Gamma-1)}{\lambda^2}\right], \qquad p \in \left[1, \ \frac{1}{3\Gamma}\right] \tag{5.98}$$

and a logarithmically evolving dilaton

$$\phi = \frac{\sqrt{27}}{\kappa}\frac{\lambda(3\Gamma-1)}{\lambda^2 + (3\Gamma-1)} \ \ln a \ . \tag{5.99}$$

In the case of this particular solution the energy density on the brane is proportional to the Hubble parameter on the brane $H \sim \rho$. If $\Gamma = 0$ the situation corresponds to the case of cosmological constant on the brane and the power law evolution (5.98) is replaced by exponential evolution [137]

$$a \sim e^{\sqrt{-\frac{2\kappa^2 V_0}{9}(\lambda^2-1)} \ t} \ . \tag{5.100}$$

In general, the method described in Subsec. 5.2.2 is hard to fit with the case of a bulk scalar field because the bulk is not static anymore. For a general discussion of generalized Birkhoff's theorem see [95], [94].

5.2.5 Inflation on the brane

As we discussed in Subsec. 2.1.1, the inflationary behaviour in the early Universe is necessary to solve some standard puzzles and to generate an appropriate scale-invariant perturbation spectrum. Inflation, or alternative mechanism which can solve the same problems, must take place also in the brane Universe. In the case of the brane Universe it is possible that inflation occurs on the brane and/or also in the bulk. This possibility was emphasized by Kaloper and Linde in [138]. They performed a general analysis of the inflation in the presence of large internal dimensions and concluded that the inflaton field must be extremely light and it is necessary to have a stage of inflation in the bulk before the compactification of internal dimensions [138]. In this short review we concentrate our attention to the inflation on the brane. There are a lot of possibilities to get the inflationary behaviour on the brane.

Slow-roll inflation on the brane

The first and simplest case to obtain inflation on the brane was discussed by Maartens et al [139]. They considered a RSII-type model with empty bulk. In this case inflation on the brane must be driven by a 4-dimensional brane scalar field ϕ with a self-interaction potential $V(\phi)$. The scalar field is *ad hoc* like in the case of usual inflation. We just assume, that there exists a scalar field alongside with ordinary matter or without ordinary matter. This is completely analogous to the case of usual inflation and the difference comes from modified Friedmann equation (5.26).

In what follows we use the modified Friedmann equation (5.33) and the Klein-Gordon equation (2.36) as the basic equations which govern the dynamics of the inflationary brane Universe in this approach. As the inflation is switched on, the curvature term and the dark radiation term will be rapidly diluted in the modified Friedmann equation (5.26) and can be neglected. In the slow-roll approximation (see Subsec. 2.1.2) the modified Friedmann equation (5.26) and the Klein-Gordon equation (2.36) can be written as

$$H^2 \simeq \frac{8\pi G_N}{3} V\left(1+\frac{V}{2\sigma}\right) , \tag{5.101}$$

$$3H\dot{\phi} \simeq \frac{dV}{d\phi} . \tag{5.102}$$

At low energies, $\sigma \gg V$, the standard behaviour of the Hubble parameter is recovered and the usual inflationary behaviour occurs. In contrary, at high energies $\sigma \ll V$, the extra term is important and it enhances the value of the Hubble parameter at the time of the same energy density compared with the usual case (2.40). Therefore the correction term relaxes the conditions of slow-roll inflation and allows more steep potentials because there is more friction for the rolling scalar field compared with the standard case [139], [140]. The slow-roll parameters (2.42) can be written as [139]

$$\epsilon(\phi) = \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2\left[\frac{2\sigma(2\sigma+2V)}{(2\sigma+V)^2}\right] , \tag{5.103}$$

$$\eta(\phi) = \frac{M_{Pl}}{8\pi}\frac{V''}{V}\left[\frac{2\sigma}{2\sigma+V}\right] , \tag{5.104}$$

and again the standard values (2.42) are recovered at low energies $V \ll \sigma$ but extra terms are important at high energies $V \gg \sigma$. The number of e-foldings (2.46) is given by

$$N \simeq -\frac{8\pi}{M_{pl}^2}\int_{\phi_i}^{\phi_f}\frac{V}{V'}\left(1+\frac{V}{2\sigma}\right)d\phi . \tag{5.105}$$

The number of e-folds is greater compared with the case of standard inflation because the value of the Hubble parameter at the time of the same energy density is greater or we can get the same number of e-folds for a smaller initial value of the scalar field ϕ_i [139]. The power spectrum of perturbations was computed in [139] and concluded that it is exactly scale-invariant as expected but the amplitude of the scalar perturbation is enhanced over that of the standard inflationary scenario.

The dynamics of the scalar field on the brane with different potentials (a power-law, an inverse-power-law and an exponential) was discussed by Mizuno et al [141]. In the case of a power-law potential ($V \sim \phi^2$, $V \sim \phi^4$) there exists an inflationary solution in a stage when the quadratic term in the Friedmann equation dominates. Since the expansion is faster, a shorter time is needed for the same amount of expansion (the same number of e-folds) compared with the conventional inflation. The inverse-power-law potential $V \sim \phi^{-\alpha}$ with $2 < \alpha < 6$ was discussed as a possible candidate for the brane quintessence scenario [141], [142]. The stability analysis of different potentials against the linear perturbations was also done in [141].

An interesting proposal was made by Himemoto and Sasaki [143] where inflation was realized without inflaton on the brane. They considered a minimally coupled massive scalar field ϕ in the bulk and solved the bulk scalar field equations perturbatively in the AdS background. In this case the Friedmann equation can be written as [143]

$$3H^2 = \frac{\kappa^4}{12}\rho_m^2 + \kappa_4^2(\rho_m + \rho_s) - \Lambda_4 - 3\frac{k}{a^2} - E_{tt} \ . \tag{5.106}$$

Here ρ_m is the energy density of the usual matter on the brane, ρ_s is the energy density of the scalar field

$$\rho_s = \frac{3}{\kappa^2\sigma}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = \frac{1}{2}\dot{\Phi}^2 + \bar{V}(\Phi)\,, \tag{5.107}$$

where last equality is nothing but a field redefinition. The last term in equation (5.106) can be expressed as follows

$$E_{tt} = \frac{\kappa^2}{2a^4}\int a^4\,\dot{\phi}\left(\partial_y^2\phi + \frac{\dot{a}}{a}\dot{\phi}\right)dt \ . \tag{5.108}$$

As discussed in Sec. 5.2.3, the Friedmann equation is not closed on the brane because the contribution of ρ_s and E_{tt} carry the influence of the bulk dynamics.

In the absence of matter $\rho_m \simeq 0$ and in the case of fine-tuned cosmological constant $\Lambda_4 \equiv 0$ (i.e. $\sigma = \sqrt{\frac{-6\Lambda}{\kappa^2}} = \frac{6}{\kappa^2\ell}$), an important contribution to the Friedmann equation comes from E_{tt} term which carries the information of the bulk gravitational field. The integration constant in expression (5.108) gives rise to dark radiation term proportional to a^{-4}. The term E_{tt} can be neglected in equation (5.106) and the inflation appears in a similar way as in the usual 4-dimensional case (see Subsec. 2.1.2) if both $\dot{\phi}$ and $\partial_y^2\phi$ are small. The condition for inflation to occur on the brane is that ϕ is a slowly varying function with respect to t and y near the brane, i.e.,

$$\dot{\phi}^2 \ll V(\phi)\ , \qquad\qquad |\partial_y^2\phi| \ll \frac{\dot{a}}{a}|\dot{\phi}|\ . \tag{5.109}$$

The potential of the bulk scalar field is modeled by

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2, \tag{5.110}$$

where m is the mass of the scalar field and in the approach presented in [143] it was required that $m^2 < 0$. At lowest order the region where $\frac{1}{2}|m^2|\phi^2 \ll V_0$ was considered. In this case the bulk spacetime is effectively AdS_5 with the curvature radius of the background spacetime $\bar{\ell}$ which is bigger than the Randall-Sundrum value ℓ. In [143] a perturbative spherically symmetric solution to the 5-dimensional field equation for ϕ was found that satisfies the conditions (5.109). However, the additional assumption that the scalar field is factorizable, $\phi(t, y) = \psi(t)\, u(y)$, was made. The conclusion is that at the lowest order of perturbation the term E_{tt} can be neglected in equation (5.106) and slow-roll inflation is realized on the brane if $|m^2| \gg H^2$. Hence the bulk scalar field *mimics* the 4-dimensional inflation dynamics and the system is described by an effective 4-dimensional scalar field Φ. A generalization of the model was made in [144], [145] by arguing that there is no need to assume the factorizable form of the scalar field and the negative mass square $m^2 < 0$. The 5-dimensional field equation

$$(-\Box_5 + m^2)\phi = 0 \tag{5.111}$$

for the bulk scalar field in the AdS_5 with the de Sitter boundary brane was solved as an initial value problem by using the properties of the retarded Green function. After a long period of the de Sitter inflation the bulk scalar field evaluated on the brane behaves as an effective 4-dimensional scalar field with $m_{eff}^2 = \frac{m^2}{2}$ [144]. Evolution of quantum fluctuations in this scenario was discussed by Sago et al [146].

However, in [130] it was argued that in the slow-roll approximation where the time variation of ϕ is slower compared to the time variation of a, the Friedmann equation can be written as (neglecting the dark radiation term)

$$H^2 = -\frac{1}{8}\left(\frac{1}{2}\dot{\phi}^2 - V\right). \tag{5.112}$$

This does not mimic the 4-dimensional inflation dynamics (see (2.40)) as in the r.h.s. is the effective pressure.

We end up with a remark about the bulk scalar field. In the Subsec. 5.2.4, the one-brane model with a non-empty bulk, with additional scalar field, was reviewed. As we saw, it was possible to get “inflation-type” solutions on the brane. The underlying field, that causes the inflationary behaviour in this case is the bulk scalar field through its projection on the brane. In this approach the scalar field on the brane comes from a fundamental theory and is not *ad hoc* like in the conventional inflation.

5.2.6 Summary

In this section we reviewed the cosmology of a single brane Universe which was inspired by the second Randall-Sundrum model (RSII). We reviewed the cosmology starting from the 5-dimensional Einstein-Hilbert action and ignored higher order curvature terms appearing in the low energy effective action from the dimensional reduction of M-theory. As illustrated in section 5.2 there are various setups to study the braneworld cosmology and several classes of solutions that describe the dynamics on the brane. The big bang nucleosynthesis is sensitive with respect to the change of the evolution of the scale factor and a strongly modified evolution of the scale factor gives a modified abundance of the light elements. Using the Randall-Sundrum idea that the tension of the brane will be compensated by a bulk cosmological constant it is possible to recover the standard cosmological evolution at late times when the energy density of radiation (matter) is less than the brane tension.

5.3 The cosmology of a two-brane Universe

In the previous section we gave a review of the cosmology on a single brane which was supposed to correspond to our Universe. Strong motivation to consider this kind of braneworld originates from the second Randall-Sundrum model. According to the fundamental Hořava-Witten theory presented in Sec. 4.1 there are two branes which are boundaries in a higher-dimensional spacetime. The first Randall-Sundrum model, which was supposed to offer a solution to the hierarchy problem, describes also a two-brane Universe but it contains various problems discussed in Sec. 4.3. In this section we give a review of other models which contain two branes, one of which is referred to as a visible brane (A-brane) and the other as a hidden brane (B-brane). The visible brane is identified with our 4-dimensional Universe.

Certainly the dynamics of a two-brane Universe is more complicated compared with the case if only one brane is considered. The distance between the branes can affect the dynamics on the branes and the dynamics of the whole system. One possibility is to assume, that the interbrane distance is fixed due to a stabilization mechanism, proposed in [35]. The cosmological model with a stabilized extra dimension was considered in [147] with the conclusion that the effective 4-dimensional theory leads to the standard cosmology at low energies. In general, we assume that the interbrane distance is not fixed. In this case an additional scalar degree of freedom appears and the effective theory looks like a *scalar-tensor theory of gravity*. The additional scalar field is called the *radion* and it describes the relative distance between the branes. The radion can be given also as a metric component along the extra dimension.

The cosmology of a two-brane system is investigated using various approximations which are valid in the low energy regime. Here we describe two approaches to derive the low energy effective action which governs the cosmological dynamics at low energies. First, *moduli space approximation* developed by Brax et al [153] assumes that the brane motion is slow and proposes that it is possible to replace the static bulk configuration with a spacetime dependent one. Secondly, the *gradient expansion*, developed by Kanno and Soda in the context of one brane [42] and generalized to the case of two branes [43], assumes that the brane tension is much bigger than the energy density on the brane. Then the bulk metric $g_{MN}(x^\mu, y)$ is expanded as perturbative series starting from the induced metric on the brane $h_{\mu\nu}(x^\mu)$. The resulting 4-dimensional effective theory allows to construct the 5-dimensional bulk geometry and is therefore called a *holographic brane gravity*. A spin-off cosmological model, called the *born-again braneworld* is briefly considered in Sec. 5.3.5. Cosmological dynamics on the visible brane (A-brane) is investigated in our paper III [45] reprinted after the review part of the thesis. Summary of the paper is presented in Ch. 7. The *ekpyrotic* and *cyclic* models of the brane Universe are briefly described in Subsec. 5.3.3.

5.3.1 Basic setup

It was pointed out already in [108] that in the case of two branes the effective 4-dimensional theory on the brane is in a form of a Brans-Dicke theory (see Subsec. 2.1.1). It can be demonstrated that the *modulus field*, denoted in the case of RSI by $T(x^\mu)$, arising from the compactification down to four dimensions can be identified with the Brans-Dicke scalar field. The Brans-Dicke scalar field in this case measures the distance between the branes and is called the *radion*.

Replacing the compactification radius r_c in (4.22) with the modulus field $T(x^\mu)$ and the Minkowski metric $\eta_{\mu\nu}$ with a general metric $\bar{g}_{\mu\nu}$, the RSI line element (4.18) can be written

$$ds^2 = e^{-2kT(x)|\varphi|}\,\bar{g}_{\mu\nu}dx^\mu dx^\nu + T(x)^2 d\varphi^2\,. \tag{5.113}$$

Following Chiba [148], and performing the integration over the extra dimension in the action (4.19) (omitting the matter Lagrangian) we get the

4-dimensional action on the positive tension brane[3]

$$I^{(+)}_{RS} = \int d^4x\sqrt{-h^{(+)}}\left[\frac{\left(1-e^{-2k\pi T}\right)}{2k\kappa^2}R(h^{(+)}) - \frac{3k\pi^2}{\kappa^2}e^{-2k\pi T}(\nabla T)^2\right], \tag{5.114}$$

where we have used the conditions (4.25) and (4.26). Here $h^{(+)}_{\mu\nu} \equiv \bar{g}_{\mu\nu}$ is the induced metric on the positive tension brane and the derivative ∇ is with respect to the metric $h^{(+)}_{\mu\nu}$. The 4-dimensional effective action on the brane with negative tension $I^{(-)}_{RS}$ is obtained from the action (5.114) with the help of a conformal transformation

$$h^{(-)}_{\mu\nu} = e^{-2k\pi T}\bar{g}_{\mu\nu} = e^{-2k\pi T}h^{(+)}_{\mu\nu} \tag{5.115}$$

and it reads

$$I^{(-)}_{RS} = \int d^4x\sqrt{-h^{(-)}}\left[\frac{\left(e^{-2k\pi T}-1\right)}{2k\kappa^2}R(h^{(-)}) + \frac{3k\pi^2}{\kappa^2}e^{2k\pi T}(\nabla T)^2\right], \tag{5.116}$$

where the derivative ∇ is with respect to the metric $h^{(-)}_{\mu\nu}$. Comparing the actions $I^{(+)}_{RS}$ and $I^{(-)}_{RS}$ with the Brans-Dicke action (2.53), the Brans-Dicke scalar field can be recovered[4]

$$\phi^{\pm}_{BD} = \pm\frac{1}{2k\kappa^2}\left(1-e^{\mp 2k\pi T}\right) = \pm\frac{1}{k\kappa^2}\, e^{\mp k\pi T}\ \sinh(k\pi T) \tag{5.117}$$

and for the Brans-Dicke coupling we get

$$\omega_{\pm}(T) = \frac{3}{2}\left(e^{\pm 2k\pi T}-1\right) = \pm 3e^{\pm k\pi T}\ \sinh(k\pi T)\ . \tag{5.118}$$

Expressions (5.117) and (5.118) agree with the results obtained in [108] if the distance between the branes is replaced by $\pi T(x)$. For the positive tension brane the coupling function satisfies $0 < \omega < \infty$ (remember that $\omega < 3600$ [54]) and for the negative tension brane $-\frac{3}{2} < \omega < 0$. To summarize, gravity on the brane at low energies is described by a scalar-tensor type theory where the scalar field is in the role of relative distance between the branes. If an additional scalar field is included into the bulk the effective theory on the brane is a *bi-scalar-tensor* theory.

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[3] Here we use a different parametrization of the extra dimension compared with the work of Chiba [148].

[4] Actually these expressions are typical for a general scalar-tensor theory where the Brans-Dicke parameter is a function of the scalar field and not a constant.

Let us summarize the basic picture presented in this section. Two spatially homogeneous and isotropic 3-branes are assumed to live in a 5-dimensional spacetime. They are moving along the time coordinate and, in general, 3-branes are in motion also along the extra dimension (the extra coordinate is denoted by y). It is convenient to choose the coordinate system where one brane, called in this context the reference brane, is at rest in the extra dimension, $y = 0$. If so, it is possible to introduce the Gaussian normal coordinate system (5.14) near the reference brane. The second brane is moving along the extra dimension and its position at any time will be characterized by a homogeneous (ignoring the spatial dependence) cosmological radion $y = \mathcal{R}(t)$. This is a nonperturbative definition of the radion used by Binétruy et al [149]. Since the radion mode corresponds to displacement of the brane, it introduces an additional physical degree of freedom if two branes are present. The relative motion of the branes leads to a modification of gravity on the brane. As stated above, without stabilization of the radion mode a scalar-tensor type theory appears as an effective 4-dimensional theory of gravity on the brane.

Radion dynamics is studied extensively in the braneworld context and usually the perturbative approach [150], [151] is applied but in this case the nonlinear behaviour disappears in the lowest order. An interesting approach is given by Binétruy et al [149], who derived nonlinear equations of motion for the homogeneous radion directly from junction conditions. The same equations of motion can be obtained from the 4-dimensional effective action for the radion, derived explicitly without any approximation in [152]. The nonlinear evolution of the radion was considered in the case if matter on the branes behaves like a cosmological constant [149].

5.3.2 Moduli space approximation

In this section we review the approximation method developed by Brax et al [153]. The approximation assumes that the motion of the brane is slow and allows to construct the effective 4-dimensional theory starting from a general 5-dimensional action.

The starting point is the 5-dimensional action which contains gravity g_{MN}, the bulk scalar field ψ and the bulk potential $U(\psi)$ (for details, see [153])

$$I^5_{bulk} = I(R, \psi, U) \ . \tag{5.119}$$

The setup also contains two branes, one has a positive tension and the other has a negative tension. Accordingly two terms are added to the total action

$$I_1 = I(g_{MN}, -U_B\delta(y_1)) \,, \qquad\qquad I_2 = I(g_{MN}, U_B\delta(y_2)). \tag{5.120}$$

Here U_B is in the rôle of brane tension which is proportional to superpotential W, see Subsec. 5.2.4. Finally, the Gibbons-Hawking term (5.12) is also included into the total action.

As we discussed before, the brane positions may be arbitrary and can be described by coordinate dependent scalar fields. The position of the first brane is $y_1 = \phi(x^\mu)$ and the position of the second brane is $y_2 = \sigma(x^\mu)$. These scalar fields play the rôle of the scalar degrees of freedom in the gravitational sector of the effective 4-dimensional theory and can be therefore interpreted as *moduli fields*. The number of moduli fields is independent of the coordinate system [153]. Since the *moduli space approximation* requires that the brane motion is slow it is equivalent with the statement that the time-variation of the moduli fields must be small. It is believed that the motion of the branes is slow when the relative distance between the branes is large and this corresponds to late time cosmology.

In order to include the graviton zero mode the Minkowsky metric $\eta_{\mu\nu}$ should be replaced by a coordinate dependent metric $g_{\mu\nu}(x^\mu)$. In this case the 5-dimensional metric can be chosen as follows

$$ds^2 = a(y)^2 g_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dy^2 \,, \tag{5.121}$$

Substituting the ansatz (5.121) into 5-dimensional action (5.119) and using the solutions of a static configuration, i.e., for the static metric $\eta_{\mu\nu}$ [133] [5]

$$a(y) = (1 - 4k\alpha^2 y)^{\frac{1}{4\alpha^2}} \,, \tag{5.122}$$

$$\psi = -\frac{1}{\alpha} \ln\left(1 - 4k\alpha^2 y\right) , \tag{5.123}$$

$$\alpha = \left\{ \frac{1}{\sqrt{3}}, \;\; -\frac{1}{\sqrt{12}} \right\} , \tag{5.124}$$

the 4-dimensional action (in the case of moduli space approximation) can be written after lengthy calculations as [153]

$$I_{msa} = \int d^4x \sqrt{-g} \Big[f(\phi, \sigma) R(g) + \frac{3}{4} a^2(\phi) \frac{U_B(\phi)}{\kappa^2} (\nabla\phi)^2 -$$

[5] These are exactly the same solutions as discussed in a single brane case, see equations (5.87) and (5.88), where $\xi = 4k$.

$$-\frac{3}{4}a^2(\sigma)\frac{U_B(\sigma)}{\kappa^2}(\nabla\sigma)^2\Bigg] . \tag{5.125}$$

Here $g_{\mu\nu}$ is a 4-dimensional metric and the effective gravitational constant is

$$f(\phi,\sigma) = \frac{1}{\kappa^2}\int_{\phi}^{\sigma} dya^2(y) . \tag{5.126}$$

Finally, the moduli fields

$$\tilde{\phi} = (1-4k\alpha^2\phi)^{2\beta} , \qquad \tilde{\sigma} = (1-4k\alpha^2\sigma)^{2\beta}, \tag{5.127}$$

$$\tag{5.128}$$

can be redefined as

$$\tilde{\phi} = Q\cosh\mathcal{R} , \qquad \tilde{\sigma} = Q\sinh\mathcal{R} , \tag{5.129}$$

where $\beta = \frac{2\alpha^2+1}{4\alpha^2}$. Introducing the Einstein frame, $\tilde{g}_{\mu\nu} = Q^2 g_{\mu\nu}$, the gravitational sector of the action (5.125) can be written as (omitting tilde right now) [153]

$$I_{EF} = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left[R(g) - \frac{12\alpha^2}{1+2\alpha^2}\frac{(\nabla Q)^2}{Q^2} - \frac{6}{2\alpha^2+1}(\nabla\mathcal{R})^2\right] . \tag{5.130}$$

In the Einstein frame the Newton constant is independent of the moduli

$$\kappa_4^2 = 8\pi G_N = k\kappa^2(1+2\alpha^2) = const . \tag{5.131}$$

The action (5.130) describes 4-dimensional gravity with two massless scalar fields Q and $\mathcal{R}$ and is in a form of bi-scalar-tensor theory.

Note that values (5.124) of the parameter α were obtained in a theory with supergravity in singular spaces [133] but if we treat it as a free parameter the Randall-Sundrum case is recovered if $\alpha \to 0$: $a(y) = e^{-ky}$. This solution corresponds to strongly warped bulk and helps us to give interpretation to these fields. If $\alpha \to 0$ the Q-field decouples (in the RS model there is no scalar field in the bulk) and the other scalar field $\mathcal{R}$ must therefore describe the relative distance between the branes:

$$\tanh\mathcal{R} = e^{-k(\sigma-\phi)} . \tag{5.132}$$

This means that in the case $\alpha \to 0$ the field $\mathcal{R}$ can be interpreted as the radion.

Taking into account the transformations (5.129) it is possible to give an interpretation to special points in moduli space. If $\mathcal{R} = 0$, the negative tension brane hits the bulk singularity at the position $\sigma = y_{bs} = \frac{1}{4k\alpha^2}$. The positive tension brane is at the same time at the position $\phi = y_{bs}(1 - Q^{\frac{1}{\beta}})$. There are at least two possibilities which can happen if the brane hits the singularity:

- The brane will be destroyed at the singularity and the remaining positive tension brane will evolve as a single one.
- The brane will bounce from the singularity. It was argued in [153] that it is hard to find a suitable potential that will drive the brane away from the singularity because the radion will grow again in this case and this behaviour destroys the decaying nature of the radion (see below).

If $Q = 0$, then distance between the branes is zero and it can be interpreted as the brane collision.

By converting the moduli part of the action (5.130) into a non-linear sigma model, it was argued by Brax et al [153] that observations give the upper bound for $\alpha \leq 10^{-2}$ and $\mathcal{R} \leq 0.2$ today. The smallness of $\mathcal{R}$ requires that distance between the branes must be sufficiently large.

Since the matter on the brane interacts with the induced metric $h_{\mu\nu}$ the matter part of the action can be written as

$$I_1^{matter} = I_1(\Psi_{(1)}, h^{(1)}_{\mu\nu}), \qquad I_2^{matter} = I_2(\Psi_{(2)}, h^{(2)}_{\mu\nu}), \tag{5.133}$$

where $\Psi_{(i)}$ are the matter fields on each brane. In the Einstein frame the matter part of the action is

$$I_1^{matter} = I_1\left[\Psi_{(1)}, A^2(Q,\mathcal{R}) g^{(1)}_{\mu\nu}\right], \tag{5.134}$$

$$I_2^{matter} = I_2\left[\Psi_{(2)}, B^2(Q,\mathcal{R}) g^{(2)}_{\mu\nu}\right], \tag{5.135}$$

where the conformal factors are

$$A = Q^{-\frac{2\alpha^2}{(1+2\alpha^2)}} (\cosh \mathcal{R})^{\frac{1}{1+2\alpha^2}}, \qquad B = Q^{-\frac{2\alpha^2}{(1+2\alpha^2)}} (\sinh \mathcal{R})^{\frac{1}{1+2\alpha^2}}. \tag{5.136}$$

In a similar way it can be shown, that if a potential term V is added on the brane, then in the Einstein frame it can be written

$$V_{eff}(Q,\mathcal{R}) = Q^{-\frac{8\alpha^2}{(1+2\alpha^2)}} (\cosh \mathcal{R})^{\frac{4}{1+2\alpha^2}} V, \tag{5.137}$$

$$W_{eff}(Q,\mathcal{R}) = Q^{-\frac{8\alpha^2}{(1+2\alpha^2)}} (\sinh \mathcal{R})^{\frac{4}{1+2\alpha^2}} W. \tag{5.138}$$

Defining a new variable $\bar{\phi}$ as $Q = e^{\bar{\phi}}$, the Friedmann equation derived from the action (5.130), (5.134), after straightforward but lengthy calculations can be written as

$$H^2 = \frac{8\pi G}{3}\left(\rho_1 + \rho_2 + V_{\rm eff} + W_{\rm eff}\right) + \frac{2\alpha^2}{1+2\alpha^2}\dot{\bar{\phi}}^2 + \frac{1}{1+2\alpha^2}\dot{\mathcal{R}}^2 \,. \quad (5.139)$$

The equations for $\mathcal{R}$ and $\bar{\phi}$ are as follows

$$\ddot{\mathcal{R}} + 3H\dot{\mathcal{R}} = \quad (5.140)$$

$$-\frac{8\pi G_N}{6}(1+2\alpha^2)\left[\frac{\partial V_{\rm eff}}{\partial \mathcal{R}} + \frac{\partial W_{\rm eff}}{\partial \mathcal{R}} + \alpha_{\mathcal{R}}^{(1)}(\rho_1 - 3p_1) + \alpha_{\mathcal{R}}^{(2)}(\rho_2 - 3p_2)\right] ,$$

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} = \quad (5.141)$$

$$-8\pi G_N \left(\frac{1+2\alpha^2}{12\alpha^2}\right)\left[\frac{\partial V_{\rm eff}}{\partial \bar{\phi}} + \frac{\partial W_{\rm eff}}{\partial \bar{\phi}} + \alpha_{\bar{\phi}}^{(1)}(\rho_1 - 3p_1) + \alpha_{\bar{\phi}}^{(2)}(\rho_2 - 3p_2)\right] ,$$

and the conservation law is

$$\dot{\rho}_{(i)} + 3H(\rho_{(i)} + p_{(i)}) = -\alpha_{\bar{\phi}}^{(i)}\,\dot{\bar{\phi}}\,T_{(i)} - \alpha_{\mathcal{R}}^{(i)}\,\dot{\mathcal{R}}\,T_{(i)} \,. \quad (5.142)$$

Here $T_{(i)}$ is the trace of energy-momentum tensor for brane matter and the coupling functions are

$$\alpha_{\bar{\phi}}^{(1)} = -\frac{2\alpha^2}{1+2\alpha^2}\,, \qquad \alpha_{\bar{\phi}}^{(2)} = -\frac{2\alpha^2}{1+2\alpha^2}\,, \quad (5.143)$$

$$\alpha_{\mathcal{R}}^{(1)} = \frac{\tanh\mathcal{R}}{1+2\alpha^2}\,, \qquad \alpha_{\mathcal{R}}^{(2)} = \frac{(\tanh\mathcal{R})^{-1}}{1+2\alpha^2}\,. \quad (5.144)$$

These equation were analyzed numerically [153] in the case of vanishing potentials and also in the case when the potential on the positive tension brane is taken into account. The matter is assumed to be on the positive tension brane. An important conclusion, true in both cases, made in [153], is that there is a *cosmological attractor* driving the radion $\mathcal{R}$ towards small values and the efficiency of the attractor depends on the matter content on the negative tension brane. This means that branes are far apart from each other and general relativity is a late time attractor. If exponential potential is assumed then at late times (small $\mathcal{R}$) the potential supports the accelerated expansion and the field $\bar{\phi}$ is playing the rôle of a quintessence field.

Approximate (assuming that the potentials V_{eff}, W_{eff} are negligible and α, $\mathcal{R}$ are small) analytical solutions to equations (5.139) - (5.142), on the positive tension brane, in the matter-domination era ($p_{(i)} = 0, \rho_2 = 0$), were also found [153] and are as follows

$$\rho_1 = \rho_0 \left(\frac{a}{a_0}\right)^{-3-\frac{2}{3}\alpha^2}, \tag{5.145}$$

$$a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}-\frac{4}{27}\alpha^2}, \tag{5.146}$$

$$\bar{\phi} = \bar{\phi}_0 + \frac{1}{3}\ln\frac{a}{a_0}, \tag{5.147}$$

$$\mathcal{R} = \mathcal{R}_1 \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} + \mathcal{R}_2 \left(\frac{t}{t_0}\right)^{-\frac{2}{3}}. \tag{5.148}$$

Small corrections $\mathcal{O}(\alpha^2)$, compared with the usual FRW cosmology at the time of matter domination, are acceptable. Despite the decay of the radion $\mathcal{R}$ as $t \gg t_0$, it still must be quite small at the time of nucleosynthesis.

5.3.3 Ekpyrotic and cyclic scenario

Involving two branes in previous subsection, it was possible to construct a scenario were the branes can collide with each other and the collision with the bulk singularity is also possible. The structure of the bulk singularity is still unknown but it is widely accepted that the singularity plays a crucial rôle in a 5-dimensional picture.

An interesting scenario, called *ekpyrotic Universe*, was presented by Khoury et al [39], where it was emphasized that the underlying fundamental theory is the 5-dimensional heterotic M-theory reviewed briefly in Sec. 4.1. Besides the approach presented by Lukas et al [27] the ekpyrotic scenario is another realization of ideas originating from the work of Hořava and Witten [26]. Thus the original version of the ekpyrotic scenario is based on a fundamental theory. This is not true for the *cyclic model* of the Universe [40]. The cyclic model is a scenario which takes the main interpretation from the ekpyrotic scenario but the initial set-up is usual general relativity with an additional phenomenological term.

It was argued in [39] that the ekpyrotic scenario can resolve standard cosmological problems outlined in Subsec. 2.1.1 without slow-roll inflation and may offer a solution to the initial singularity problem.

The initial singularity is one of the most important motivation to apply string theory to cosmology. In what follows we describe briefly the main ideas that lead to ekpyrotic/cyclic scenario and draw some conclusions and concerns related to this scenario. A more exhaustive review of ekpyrotic scenario is presented in [96].

Old ekpyrotic scenario

The spacetime of heterotic M-theory is a 5-dimensional manifold $\mathcal{M}_5 = \mathcal{M}_4 \times S_1/Z_2$ with two boundary branes in orbifold fixed points, $y_1 = 0$ (visible brane) and $y_2 = R$ (hidden brane). The branes are assumed to be flat, parallel and empty at the beginning. This initial state corresponds to Bogomolny-Prasad-Sommerfield (BPS) state in the language of supersymmetry. It is believed that some attractor mechanism drives the Universe toward the BPS state starting from more general (unknown) initial condition [39]. Between the two boundary branes there is a bulk brane, $y_3 = Y$ ($0 < Y < R$), which is emitted from the hidden brane ("small instanton") or created by some unknown mechanism in the bulk. The BPS state requires the bulk brane to be nearly stationary but nonperturbative effects introduce a potential V which attracts the bulk brane towards the visible brane. The attractive force is stronger near the visible brane and tends to zero near the hidden brane. This requirement determines the shape of the potential to be exponential $V \sim -e^{-Y}$. Finally, the bulk brane will dissolve into the visible brane after the collision, called *ekpyrosis*. The visible brane is heated up during ekpyrosis and standard big bang with slightly modified initial conditions occurs.

The action of the model consists of three parts:

$$I = I_{het} + I_{bi} + I_{matter} \tag{5.149}$$

where I_{het} is the action of heterotic M-theory, I_{bi} is the action of brane interaction, and I_{matter} is the action of usual matter. The heterotic M-theory part of the action I_{het} is described briefly in Sec. 4.1, but in the context of ekpyrotic scenario an additional term is included (see criticism below). The brane interaction part of the action I_{bi} should be the basis of effective potential, responsible for attracting the visible brane and the bulk brane. The matter part of the action I_{matter} should be included after the collision because the matter is assumed to be created from the kinetic energy of the bulk brane during ekpyrosis. The hidden brane remains empty in this case.

The starting point is the static (BPS) solution for the 5-dimensional spacetime [101], [39]

$$\begin{aligned} ds^2 &= D(y)(-N^2dt^2 + A^2\,\gamma_{ij}dx^i dx^j) + B^2 D^4(y) dy^2\,, && (5.150)\\ e^{\phi} &= BD^3(y)\,, && (5.151) \end{aligned}$$

where[6]

$$D(y) = \begin{cases} \alpha y + C\,, & y < Y \\ (\alpha - \beta)y + \beta Y + C\,, & y > Y \end{cases}\,, \qquad (5.152)$$

N, A, B, C are dimensionless constants and Y is constant but has the dimension of length. In the original version of the ekpyrotic scenario [39] the tension of the visible brane $(-\alpha)$ is negative, the tension of the bulk brane β is positive and the tension of the hidden brane $(\alpha - \beta)$ is positive. It was required that $C > 0$ and this means that the curvature singularity $D = 0$ does not fall between the boundary branes.

An important property of the original model is the fact that the warp factor is growing if we move away from the visible brane $D(0) < D(R)$. This was claimed to lead to a blue tilt of the perturbation spectrum [39], [154], although the matter is still under debate [156], [158], [159].

The 4-dimensional effective theory is obtained using the moduli space approximation introduced in Subsec. 5.3.2. Note, that the moduli space approximation is valid when the motion of the branes is assumed to be slow. According to the moduli space approximation the constants N, A, B, C, Y of static solutions (5.150), (5.152) become moduli fields. If additionally homogeneity and isotropy are assumed they can be treated as functions of time: $N, A, B, C, Y \to N(t), A(t), B(t), C(t), Y(t)$. Substituting the resulting modified solution into the 5-dimensional action (5.149) and integrating over the fifth dimension, i.e., over y, we get the 4-dimensional *moduli space action* I_{msa} [39] (assuming additionally that $B = const.$ and $C = const.$)

$$I^{(4)}_{msa} \approx 3M^3 \int d^4x \tilde{n}\tilde{a} \left(-\frac{1}{\tilde{n}^2}\frac{\dot{\tilde{a}}^2}{\tilde{a}^2} + \frac{\beta}{I_3}\left[\frac{1}{2}\frac{1}{\tilde{n}^2} D(Y)\dot{Y}^2 - \frac{1}{BI_3M} V(Y)\right]\right)\,, \qquad (5.153)$$

[6] In Sec. 4.1 we used the quantity $H(y)$ instead of $D(y)$ and the α used in Sec. 4.1 is equal to $\frac{3}{\sqrt{2}}\alpha$ used here.

where $I_3(t)$ is a positive function which is constant to zeroth order in $\frac{\beta}{\alpha}$ [96] and $\tilde{n}$ and $\tilde{a}$ are given by

$$\tilde{n} \equiv N\sqrt{BI_3M}, \qquad \tilde{a} \equiv A\sqrt{BI_3M}. \tag{5.154}$$

The 4-dimensional effective action (5.153) is assumed to describe the dynamics of the system if the branes are moving slowly.

Two remarks are in order. First, it is important to note that $\tilde{n}$ and $\tilde{a}$ in (5.153) are not the lapse function and the scale factor measured on the visible brane. The lapse function $n(t,0)$ and the scale factor $a(t,0)$ on the visible brane can be found from (5.150) and are

$$n(t,0) \simeq \tilde{n}(t)\sqrt{\frac{C}{BI_3(0)M}}, \quad a(t,0) \simeq \tilde{a}(t)\sqrt{\frac{C}{BI_3(0)M}}. \tag{5.155}$$

Now the Friedmann equation on the visible brane reads [96]

$$H^2 \equiv \frac{\dot{a}(t,0)^2}{a(t,0)^2} \approx \frac{\beta}{CI_3}\left(\frac{1}{2}a(t,0)^2 D(Y)^2\dot{Y}^2 + V(Y)\right) \tag{5.156}$$

and $n(t,0) = 1$ if the time derivative is computed with respect to proper time on the visible brane.

Secondly, the potential $V(Y)$ is added by hand into the action (5.153) and must be quite specific

$$V(Y) = -F(Y)\, V_0 e^{-cY}, \tag{5.157}$$

where $V_0 = const.$ and $c = const.$ At small values of Y (near collision) the potential must quickly become zero and this is ensured by a suitable choice of function $F(Y)$. Conversely to the standard inflation the potential $V(Y)$ must be negative and steep. The potential causes serious troubles since the metric ansatz (5.150) with potential (5.157) does not satisfy the field equations derived from the action (5.149). The Friedmann equation (5.157) must be compared with the Friedmann equation derived directly from the action (5.149) under the assumption that the brane interaction has only a δ-function support at the bulk brane. In this case the Friedmann equation reads [155], [96]

$$\frac{\dot{a}(t,0)^2}{a(t,0)^2} = \frac{1}{36M^6}\rho_m^2 - \frac{1}{6M^3}\alpha e^{-\phi_1}\rho_r - \frac{1}{6M^3}\alpha e^{-\phi_0}\rho_d + \frac{\mathcal{C}}{a(t,0)^4}, \tag{5.158}$$

where ϕ_0 is a constant value of the moduli field ϕ at the position of the visible brane and $\phi_1 = const.$, $\mathcal{C} = const.$ Here the matter energy density

ρ_m is divided into the radiation energy density ρ_r and into other brane matter energy density ρ_d: $\rho_m = \rho_r + \rho_d$. One of the conditions to recover approximately the standard Friedmann equation (2.10) is

$$G_{_N} = \frac{(-\alpha)}{16\pi M^3} e^{-\phi_0} \tag{5.159}$$

and it requires that the tension of the visible brane must be positive $(-\alpha) > 0$. It is evident that it is quite impossible to fit together the equation (5.156) and (5.158). In [96] it was argued that the problem may arise from the integration over the fifth dimension and it is more appropriate to consider the induced equations on the brane.

The old ekpyrotic scenario has been criticized by Kallosh et al [156]. First, they argued that the original action derived in [27] was complemented with additional 4-form gauge field $\mathcal{A}_{_{ABCD}}$ in [39] which was also introduced in [132] but in a different context. The other important remark was that in the Hořava-Witten model the tension of the visible brane is positive. The same conclusion was also reached by Enqvist et al [155] who considered gravity as seen by an observer on the visible brane. In this case the coupling constant on the positive tension brane is small. Since the volume of the internal Calabi-Yau manifold is proportional to the inverse of the coupling constant this leads to conclusion that the Calabi-Yau volume on the positive tension brane is bigger than on the negative tension brane.

In the light of these arguments the *pyrotechnic scenario* was introduced in [156]. In this scenario the tension of the visible brane is positive and the tension of the hidden brane is negative. The warp factor is replaced

$$D(y) \rightarrow \tilde{D}(y) = C - |\alpha| y\,, \tag{5.160}$$

and decreases if one moves away from visible brane, i.e., $\tilde{D}(0) < \tilde{D}(R)$. In this case the perturbation spectrum has a red tilt [156].

New ekpyrotic scenario

Among the criticism presented above, an important aspect for presenting a new version of the ekpyrotic scenario [154], [157] was the fact that the fifth dimension may not be stabilized but should collapse. This conclusion was revised by Räsänen [160] who argued that contraction will take place if the tension of the visible brane is positive (remember that in old ekpyrotic scenario the tension of the visible brane was negative). In this case the boundary branes reach each other and at the moment when fifth dimension vanishes, they collide. There is no need for the bulk brane and in the new ekpyrotic scenario there are only two boundary branes. In the new ekpyrotic scenario the hidden brane should also contain matter since it is reasonable to assume that the collision energy will be distributed equally between the branes.

During the collision the branes are not absorbed into each other but bounce apart from each other and the scale factor on the brane should start expanding. To reverse contraction into expansion requires the violation of the *null energy condition* ($\rho + p \geq 0$) or passage through a singularity where the scale factor passes through zero and this process is hoped to be non-singular. The last possibility was discussed by Khoury et al [157]. They argued that it is possible to find variables which remain finite at the collision and it is possible to match the contracting and the expanding solutions at the bounce. They emphasized the importance of string theory at the moment of bounce and argued that 5-dimensional spacetime looks like a *Milne universe* (flat Universe) near the collision. However, it was argued in [160] that energy density on the brane curves the the spacetime and leads to deviations from the Milne metric. The metric was derived under the assumption that it remains non-singular near the collision which requires that the scale factor approaches a finite value (to keep the Riemann tensor and energy-momentum tensor bounded) as the branes approach each other. This in turn confines the brane matter to have negative energy density after the collision and additional fields are needed to cure the situation [160]. The moduli space approximation and conclusions following from it, were also criticized [160], especially near the collision when the approximation breaks down anyway.

Cyclic model of the Universe

The cyclic scenario, presented and advocated by Steinhardt and Turok [40], [41] is inspired by the new ekpyrotic scenario but unlike the latter one it is phenomenological. The braneworld conception (new ekpyrotic scenario) helps to give an interpretation to some results and provides the possibility to connect the scenario with the underlying string theory. Even if the brane collision takes place as described by ekpyrotic scenario, there remains the question about initial conditions since it is unknown what happens before the branes reach the BPS state and start moving towards each other. The new ekpyrotic scenario assumes that there is only one collision and after that the attractive potential, which drives the brane movement, becomes zero.

The cyclic scenario states that the evolution of the Universe is cyclic and even the collision of branes, which in some sense mimics the big bang, is not a beginning of time but rather a transition event. By definition, there is no beginning or end of time. In this scenario, the space and time have infinite past and future as the Universe undergoes an endless sequence of cycles in which the contraction and expansion phases are followed by each other. Each cycle lasts trillions of years. The expansion phase includes a period of radiation-, matter-, and dark energy/quintessence-dominations. Accelerated expansion is followed by a period of decelerating expansion which turns to contraction and ends with a big crunch. Since there is no beginning of time, the singularity problem changes and in some sense it is milder than in the case of usual big bang. In the cyclic model the singularity is a transition epoch from the contracting phase to the expanding phase and at that moment only the fifth dimension, separating the two branes, collapses. Since three spatial dimensions remain large, it was argued in [41] that time should continue smoothly. A smooth transitions from a big crunch to a big bang is crucial part of the model and string theory must be taken into account near the collision, as pointed out already in the context of the new ekpyrotic scenario [157]. In brief, to repeat the cycle, the Universe must pass smoothly from big crunch to big bang.

An important ingredient of the scenario is an interbrane potential which ensures cyclic behaviour. The potential is the same before and after collision and the reversal from expansion to contraction is caused by negative potential not by spatial curvature. To provide the appropriate sequence of different epochs, the potential, like in the ekpyrotic scenario, must be specific and it is argued in [40] that it is suitable to choose it as [7]

$$V(\phi) = F(\phi)\, V_0 \left(1 - e^{-c\phi}\right) . \tag{5.161}$$

Here ϕ is a scalar field that parametrizes the interbrane distance, $F(\phi)$ is a function that ensures that the potential goes to zero $V(\phi) \to 0$ at the moment of collision when $\phi \to -\infty$, $V_0 > 0$ corresponds to dark energy density observed at the present Universe and $c \geq 10$ is a constant. A detailed description of the requirements for the potential can be found in [41].

[7] Here $8\pi G_N = 1$ and hence ϕ is dimensionless.

It is proposed that the 4-dimensional effective theory consists of the Einstein gravity plus a scalar field but interpreted in the context of ekpyrotic set-up. It is supposed that the effective 4-dimensional theory can be derived from a more fundamental theory but at present it is *ad hoc*. The Einstein frame action of the model is [40]

$$I = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_{_N}}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \beta^4(\phi)(\rho_m + \rho_r)\right)\,, \tag{5.162}$$

where $\beta^4(\phi)$ is the coupling between the scalar field ϕ and the energy density of matter ρ_m and radiation ρ_r. In the case of a flat and homogeneous Universe, the Friedmann equation derived from the action (5.162) is

$$H^2 = \frac{8\pi G_{_N}}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \beta^4(\rho_r + \rho_m)\right)\,. \tag{5.163}$$

An important statement of the cyclic scenario is that matter and radiation couple to the metric $\beta^2(\phi)\, g_{\mu\nu}$ rather than to the Einstein frame metric $g_{\mu\nu}$. This means that the scale factor seen by an observer on the visible brane is $\hat{a} = \beta(\phi)a$ rather than a. The size and warp of the extra dimension is also determined by a coupling function $\beta(\phi)$ and the scale factor $\hat{a}$ remains finite ($\beta a \to$ *const.*) even if $a \to 0$ since $\beta \to \infty$. Since the Einstein frame scale factor a is singular at the transition from big crunch to big bang, it is more appropriate to consider new non-singular variables a_0, a_1 defined as follows [157], [41]

$$a_0 = 2a\cosh\bar{\phi}\,, \qquad \Longrightarrow \qquad \beta = 2\cosh\bar{\phi}\,, \tag{5.164}$$

$$a_1 = -2a\sinh\bar{\phi}\,, \qquad \Longrightarrow \qquad \beta = -2\sinh\bar{\phi}\,, \tag{5.165}$$

where

$$\bar{\phi} = \frac{\phi - \phi_\infty}{\sqrt{6}}\,. \tag{5.166}$$

Here ϕ_∞ is a constant shift and $\phi < \phi_\infty$ which ensures that a_1 is positive. The scalar field ϕ in this context regularizes the Einstein frame singularity. These new variables are simply related to the Einstein frame scale factor $a = \frac{1}{2}\sqrt{a_0^2 - a_1^2}$. If the branes are in a bulk AdS$_5$, a_0 can be interpreted as the scale factor of the positive tension brane and a_1 as the scale factor of the negative tension brane [157], [41].

Using these new variables and taking into account that a_1 is interpreted as the scale factor on the negative tension brane (this brane is considered as our Universe), it is possible to write the Friedmann equation (5.163) on the brane as follows

$$\begin{aligned}\left(\frac{\dot{a}_1}{a_1}\right)^2 &= \frac{8\pi G_N}{3}\left[\beta^4(\rho_r+\rho_m)+V(\bar{\phi})\right]+\dot{\bar{\phi}}^2\left(1+\coth^2\bar{\phi}\right)\\ &+ 2\dot{\bar{\phi}}\coth\bar{\phi}\sqrt{\dot{\bar{\phi}}^2+\frac{8\pi G_N}{3}\left[\beta^4(\rho_r+\rho_m)+V(\bar{\phi})\right]}. \end{aligned} \quad (5.167)$$

The equation of the scalar field, derived from the action (5.162) is [40]

$$\ddot{\phi}+3H\dot{\phi}=-V_{,\phi}-\beta_{,\phi}\beta^3\rho \quad (5.168)$$

and the continuity equation reads [40]

$$\hat{a}\frac{d\rho}{d\hat{a}}\equiv a\frac{\partial\rho_i}{\partial a}+\frac{\beta}{\beta_{,\phi}}\frac{\partial\rho_i}{\partial_i}=-3(\rho_i+p_i) \quad (5.169)$$

where $V_{,\phi}=\frac{dV}{d\phi}$, $\beta_{,\phi}=\frac{d\beta}{d\phi}$ and $i=m,r$ refers to matter and radiation energy densities. These three equations (5.167) - (5.169) are assumed to determine the dynamics on the visible brane. They are analyzed before and after the bounce, and discussed how the incoming and outgoing states are connected to each other [41]. Approximate solutions are presented for the radiation-, matter-, and dark energy-dominated epochs [41]. With a suitable choice of parameters the solutions fit well into the model and are also in agreement with the observational constraints. As an alternative to inflation, the cyclic model should provide a long enough period of cosmic acceleration to solve the standard model puzzles, discussed in Subsec. 2.1.1. The late time accelerated expansion we observe today fits well and since the expansion is driven by the potential energy (5.161) the number of e-foldings can be computed by formula (2.46)

$$N=\int d\phi\,\frac{V}{V_{,\phi}}\approx\frac{e^{c\phi_C}}{c^2}\,, \quad (5.170)$$

where the potential (5.161) is applied. Here ϕ_C is the rest value of the scalar field and the required number of e-folds, $N>60$, is easily achieved when ϕ_C is of order of unity in Planck units. In the cyclic model the acceleration is very slow compared with the usual inflation but it lasts much longer and therefore the number of e-folds is large enough. It is emphasized, that unlike inflation, in the cyclic model there is only one epoch of accelerated evolution and this will take place in the late Universe just before the expansion will be replaced by the contraction. Strong criticism against the cyclic model was presented in [161].

5.3.4 Holographic brane gravity

In this subsection we reproduce the key results from the paper by Kanno and Soda [43] (and do not refer to it steadily during the presentation), since the results of their work were applied in our paper III [45] to develop a particular cosmological model. Kanno and Soda derived the 4-dimensional low energy effective theory to describe gravity on the branes. They considered a two-brane system in the Randall-Sundrum type setup.

Quite generally, RSII argued (see Subsec. 5.3.1) that if two branes are involved the effective 4-dimensional theory behaves like the Brans-Dicke theory (scalar-tensor theory, in general) and the Brans-Dicke scalar field acts as a radion. Essentially the same conclusion was reached in [43], under the assumption that the brane tensions σ_A and σ_B are much bigger than the corresponding energy densities ρ_A and ρ_B, called the *low energy expansion scheme*. The bulk equations of motion are solved using the low energy expansion scheme developed in [42] but in the context of a single brane. If the second brane is involved, an additional degree of freedom, the radion, should be taken into account and the expansion scheme modifies.

Basic set-up

The starting point is the 5-dimensional Randall-Sundrum action I_{RS} (4.19) but here we are making the identification $-2\Lambda\kappa^2 = 12\ell^{-2}$, where ℓ is the AdS_5 curvature radius. The 5-dimensional metric for the two brane system can be chosen in a form where the radion appears explicitly

$$ds^2 = e^{\phi(y,x^\mu)}dy^2 + g_{\mu\nu}(x^\mu, y)dx^\mu dx^\nu \ . \tag{5.171}$$

The brane locations are chosen to be: $y = 0$ (A-brane) and $y = \ell$ (B-brane). In this case the radion $\phi(x^\mu, y)$ measures the proper distance between branes

$$d(x) = \int_0^\ell e^{\phi(y,x^\mu)}dy \ . \tag{5.172}$$

In what follows we use the argument presented by Kanno et al in [43] and assume that the radion is independent of the fifth dimension: $\phi(x^\mu, y) = \phi(x^\mu)$.

In the coordinate system (5.171) the field equations derived from the action (4.19) are as follows

$$R^\mu{}_\nu + e^{-\phi}(e^{-\phi}K^\mu{}_\nu)_{,y} - (e^{-\phi}K)(e^{-\phi}K^\mu{}_\nu) - \nabla^\mu\nabla_\nu\phi - \nabla^\mu\phi\nabla_\nu\phi =$$

$$= -\frac{4}{\ell^2}\delta^\mu_\nu + \kappa^2 \left(\frac{1}{3}\delta^\mu_\nu \sigma_A + T^{A\mu}{}_\nu - \frac{1}{3}\delta^\mu_\nu T^A\right) e^{-\phi}\delta(y)$$
$$+\kappa^2 \left(\frac{1}{3}\delta^\mu_\nu \sigma_B + \tilde{T}^{B\mu}{}_\nu - \frac{1}{3}\delta^\mu_\nu \tilde{T}^B\right) e^{-\phi}\delta(y-\ell)\,, \quad (5.173)$$

$$e^{-\phi}(e^{-\phi}K)_{,y} - (e^{-\phi}K^{\alpha\beta})(e^{-\phi}K_{\alpha\beta}) - \nabla^\alpha\nabla_\alpha\phi - \nabla^\alpha\phi\nabla_\alpha\phi$$
$$= -\frac{4}{\ell^2} - \frac{\kappa^2}{3}\left(-4\sigma_A + T^A\right) e^{-\phi}\delta(y)$$
$$-\frac{\kappa^2}{3}\left(-4\sigma_B + \tilde{T}^B\right) e^{-\phi}\delta(y-\ell)\,, \quad (5.174)$$

$$\nabla_\nu(e^{-\phi}K_\mu{}^\nu) - \nabla_\mu(e^{-\phi}K) = 0\,. \quad (5.175)$$

Here the extrinsic curvature $K_{\mu\nu}$ is defined as follows

$$K_{\mu\nu} = -\frac{1}{2}\frac{\partial}{\partial y}g_{\mu\nu} \equiv -\frac{1}{2}g_{\mu\nu},_y\ . \quad (5.176)$$

Sub- and superscripts A and B refer to corresponding branes. The junction conditions are obtained by collecting together the terms in field equations which contain a δ-function and can be written as

$$e^{-\phi}\left[K^\mu{}_\nu - \delta^\mu_\nu K\right]|_{y=0} \;=\; \frac{\kappa^2}{2}\left(-\sigma_A\delta^\mu_\nu + T^{A\mu}{}_\nu\right)\ , \quad (5.177)$$
$$e^{-\phi}\left[K^\mu{}_\nu - \delta^\mu_\nu K\right]|_{y=\ell} \;=\; -\frac{\kappa^2}{2}\left(-\sigma_B\delta^\mu_\nu + \tilde{T}^{B\mu}{}_\nu\right)\ . \quad (5.178)$$

Let us decompose the extrinsic curvature into the traceless and trace part

$$e^{-\phi}K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4}g_{\mu\nu}Q\,, \qquad Q = -e^{-\phi}\frac{\partial}{\partial y}\ln\sqrt{-g} \quad (5.179)$$

which allows us to write the field equations (5.173) - (5.175) in the bulk as follows

$$e^{-\phi}\Sigma^\mu{}_{\nu,y} - Q\Sigma^\mu{}_\nu = -\left[R^\mu{}_\nu - \frac{1}{4}\delta^\mu_\nu R - \nabla^\mu\nabla_\nu\phi - \nabla^\mu\phi\nabla_\nu\phi\right.$$
$$\left.+\frac{1}{4}\delta^\mu_\nu\left(\nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi\right)\right]\,, \quad (5.180)$$
$$\frac{3}{4}Q^2 - \Sigma^\alpha{}_\beta\Sigma^\beta{}_\alpha = R + \frac{12}{\ell^2}\ , \quad (5.181)$$
$$e^{-\phi}Q_{,y} - \frac{1}{4}Q^2 - \Sigma^{\alpha\beta}\Sigma_{\alpha\beta} = \nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi - \frac{4}{\ell^2}\ , \quad (5.182)$$
$$\nabla_\lambda\Sigma^\lambda{}_\mu - \frac{3}{4}\nabla_\mu Q = 0\ . \quad (5.183)$$

The junction conditions (5.177) and (5.178) reduce to

$$\left[\Sigma^{\mu}{}_{\nu} - \frac{3}{4}\delta^{\mu}_{\nu}Q\right]\Bigg|_{y=0} = \frac{\kappa^2}{2}(-\sigma_A\delta^{\mu}_{\nu} + T^{A\mu}{}_{\nu}) \,, \tag{5.184}$$

$$\left[\Sigma^{\mu}{}_{\nu} - \frac{3}{4}\delta^{\mu}_{\nu}Q\right]\Bigg|_{y=\ell} = -\frac{\kappa^2}{2}(-\sigma_B\delta^{\mu}_{\nu} + \tilde{T}^{B\mu}{}_{\nu}) \,. \tag{5.185}$$

The junction conditions determine the dynamics of the induced metric and thus provide the effective theory of gravity on the branes. The bulk equations (5.180) - (5.183) should be solved using the Dirichlet boundary conditions on the A-brane: $g_{\mu\nu}(y = 0, x^{\mu}) = h_{\mu\nu}(x^{\mu})$. In what follows we describe the approximation scheme developed in [42].

Low energy expansion scheme

The basis of the approximation scheme is the assumption that the energy density of matter ρ on the brane is smaller than the brane tension σ. Equivalently, the bulk curvature scale ℓ is much smaller than the characteristic length scale of the curvature L on the brane, $\ell \ll L$. This means that the curvature on the brane can be neglected compared with the extrinsic curvature. This can be summarized as follows

$$\frac{\rho_i}{\sigma_i} \sim \left(\frac{\ell}{L}\right)^2 \ll 1. \tag{5.186}$$

This allows us to expand the metric in perturbative series starting from the induced metric on the A-brane $h_{\mu\nu}$ as the first term

$$g_{\mu\nu}(y, x^{\mu}) = a^2(y, x^{\mu})\left[h_{\mu\nu}(x^{\mu}) + g^{(1)}_{\mu\nu}(y, x^{\mu}) + g^{(2)}_{\mu\nu}(y, x^{\mu}) + \cdots\right] , \tag{5.187}$$

where

$$g_{\mu\nu}(y = 0, x^{\mu}) = h_{\mu\nu}(x^{\mu}) \tag{5.188}$$

is the induced metric on the A-brane and therefore

$$g^{(n)}_{\mu\nu}(y = 0, x^{\mu}) = 0 \,, \qquad n = 1, 2, 3, \cdots \,. \tag{5.189}$$

Accordingly the extrinsic curvature can be expanded as

$$K^{\mu}{}_{\nu} = K^{(0)\mu}{}_{\nu} + K^{(1)\mu}{}_{\nu} + K^{(2)\mu}{}_{\nu} + \cdots \,. \tag{5.190}$$

The approximation (5.186) was assumed also by Shiromizu et al [162], [163] in adopting the covariant curvature formalism to the case of two branes. Their approximation scheme leads to same conclusion as the approximation scheme developed by Kanno et al in [42].

Zeroth and first order approximation

At the zeroth order, the gradient terms and matter on the branes can be ignored. In this case the field equations (5.180) - (5.183) reduce to the system

$$e^{-\phi}\Sigma^{(0)\mu}{}_{\nu,y} - Q^{(0)}\Sigma^{(0)\mu}{}_{\nu} = 0\ , \tag{5.191}$$

$$\frac{3}{4}Q^{(0)2} - \Sigma^{(0)\alpha}{}_{\beta}\Sigma^{(0)\beta}{}_{\alpha} = \frac{12}{\ell^2}\ , \tag{5.192}$$

$$e^{-\phi}Q^{(0)}_{,y} - \frac{1}{4}Q^{(0)2} - \Sigma^{(0)\alpha\beta}\Sigma^{(0)}_{\alpha\beta} = -\frac{4}{\ell^2}\ , \tag{5.193}$$

$$\nabla_\lambda\Sigma^{(0)\lambda}{}_{\mu} - \frac{3}{4}\nabla_\mu Q^{(0)} = 0 \tag{5.194}$$

and junction conditions (5.184), (5.185) reduce as follows

$$\left[\Sigma^{(0)\mu}{}_{\nu} - \frac{3}{4}\delta^\mu_\nu Q^{(0)}\right]\Bigg|_{y=0} = -\frac{\kappa^2}{2}\sigma_A\delta^\mu_\nu\ , \tag{5.195}$$

$$\left[\Sigma^{(0)\mu}{}_{\nu} - \frac{3}{4}\delta^\mu_\nu Q^{(0)}\right]\Bigg|_{y=\ell} = \frac{\kappa^2}{2}\sigma_B\delta^\mu_\nu\ . \tag{5.196}$$

Taking into account the definition (5.179), equation (5.191) can be integrated

$$\Sigma^{(0)\mu}{}_{\nu} = \frac{C^\mu{}_\nu(x^\mu)}{\sqrt{-g}}\,, \qquad C^\mu{}_\mu = 0\,, \tag{5.197}$$

where $C^\mu{}_\nu$ are integration constants, but the junction conditions (5.195) and (5.196) require that $\Sigma^{(0)\mu}{}_{\nu} = 0$. From the equation (5.192) it follows that

$$Q^{(0)} = \frac{4}{\ell} \tag{5.198}$$

and the remaining equations (5.193), (5.194) are trivially satisfied. Substituting this solution into relation (5.179) and taking into account the definition (5.176) we get the equation for the zeroth order metric

$$-\frac{1}{2}\frac{\partial}{\partial y}g^{(0)}_{\mu\nu} = \frac{1}{\ell}e^{\phi}g^{(0)}_{\mu\nu}\ . \tag{5.199}$$

This equation can be integrated and it gives (here $g^{(0)}_{\mu\nu}(y,x^\mu) = a^2(y,x^\mu)\, h_{\mu\nu}$)

$$a(y,x^\mu) = e^{-\frac{ye^\phi}{\ell}} . \tag{5.200}$$

The junction conditions (5.195) and (5.196) reduce to simple constraints $\kappa^2\sigma_A = \frac{6}{\ell}$, $\kappa^2\sigma_B = -\frac{6}{\ell}$, known already from Sec. 4.3.

In the next order of approximation the matter and curvature terms can't be ignored any more and the field equations (5.180) - (5.183) can be written as follows

$$e^{-\phi}\Sigma^{(1)\mu}{}_{\nu,y} - \frac{4}{\ell}\Sigma^{(1)\mu}{}_\nu = -\left[R^\mu{}_\nu - \nabla^\mu\nabla_\nu\phi - \nabla^\mu\phi\nabla_\nu\phi\right]^{(1)}_{trless} , \tag{5.201}$$

$$\frac{6}{\ell}Q^{(1)} = [R]^{(1)} , \tag{5.202}$$

$$e^{-\phi}Q^{(1)}_{,y} - \frac{2}{\ell}Q^{(1)} = \left[\nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi\right]^{(1)} , \tag{5.203}$$

$$\nabla_\lambda\Sigma^{(1)\lambda}{}_\mu - \frac{3}{4}\nabla_\mu Q^{(1)} = 0 , \tag{5.204}$$

and the junction conditions (5.184), (5.185) are

$$\left[\Sigma^{(1)\mu}{}_\nu - \frac{3}{4}\delta^\mu_\nu Q^{(1)}\right]\Bigg|_{y=0} = \frac{\kappa^2}{2}T^{A\mu}{}_\nu , \tag{5.205}$$

$$\left[\Sigma^{(1)\mu}{}_\nu - \frac{3}{4}\delta^\mu_\nu Q^{(1)}\right]\Bigg|_{y=\ell} = -\frac{\kappa^2}{2}\tilde{T}^{B\mu}{}_\nu . \tag{5.206}$$

It is convenient to express the Ricci tensor $R_{\mu\nu}(g^{(0)})$ in terms of $R^\mu_\nu(h)$

$$\begin{aligned}[R^\mu{}_\nu(g)]^{(1)} = \frac{1}{a^2}\Bigg[& R^\mu{}_\nu(h) + 2\frac{y}{\ell}e^\phi\left(\phi^{|\mu}{}_{|\nu} + \phi^{|\mu}\phi_{|\nu}\right) + \delta^\mu_\nu\frac{y}{\ell}e^\phi\left(\phi^{|\alpha}{}_{|\alpha} + \phi^{|\alpha}\phi_{|\alpha}\right) \\ &+2\frac{y^2}{\ell^2}e^{2\phi}\phi^{|\mu}\phi_{|\nu} - 2\delta^\mu_\nu\frac{y^2}{\ell^2}e^{2\phi}\phi^{|\alpha}\phi_{|\alpha}\Bigg] ,\end{aligned} \tag{5.207}$$

where vertical bar | denotes the covariant derivative with respect to the induced metric $h_{\mu\nu}(x^\mu)$. The second derivative of scalar field with respect to metric $g_{\mu\nu}$ can be expressed through the derivatives with respect to metric $h_{\mu\nu}$ as follows

$$[\nabla^\mu\nabla_\nu\phi]^{(1)} = \frac{1}{a^2}\left[\phi^{|\mu}{}_{|\nu} + 2\frac{y}{\ell}e^\phi\phi^{|\mu}\phi_{|\nu} - \frac{y}{\ell}e^\phi\delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha}\right] . \tag{5.208}$$

Substituting the trace $R(g)$ (computed from (5.207)) into equation (5.202) we get the expression for $Q^{(1)}$

$$Q^{(1)} = \frac{\ell}{a^2}\left[\frac{1}{6}R(h) + \frac{ye^{\phi}}{\ell}\left(\phi^{|\alpha}{}_{|\alpha} + \phi^{|\alpha}\phi_{|\alpha}\right) - \frac{y^2e^{2\phi}}{\ell^2}\phi^{|\alpha}\phi_{|\alpha}\right] . \quad (5.209)$$

Substituting equations (5.207) and (5.208) into equation (5.201) we get the equation for $\Sigma^{(1)\mu}{}_{\nu}$

$$\Sigma^{(1)\mu}{}_{\nu,y} - \frac{4}{\ell}e^{\phi}\Sigma^{(1)\mu}{}_{\nu} = -\frac{e^{\phi}}{a^2}\left[C_0 + 2\frac{y}{\ell}e^{\phi}C_1 + 2\frac{y^2}{\ell^2}e^{2\phi}C_2\right] , \quad (5.210)$$

where C_0, C_1 and C_2 are y-independent functions:

$$C_0(x^\mu) = R^\mu_\nu(h) - \phi^{|\mu}{}_{|\nu} - \phi^{|\mu}\phi_{|\nu} , \quad (5.211)$$

$$C_1(x^\mu) = \phi^{|\mu}{}_{|\nu} + \delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha} + \frac{1}{2}\delta^\mu_\nu\phi^{|\alpha}{}_{|\alpha} , \quad (5.212)$$

$$C_2(x^\mu) = \phi^{|\mu}\phi_{|\nu} - \delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha} . \quad (5.213)$$

Integrating the equation (5.210) and taking into account that the solution should be traceless we get for $\Sigma^{(1)\mu}{}_{\nu}$

$$\begin{aligned}\Sigma^{(1)\mu}{}_{\nu} &= \frac{\ell}{a^2}\left[\frac{1}{2}\left(R^\mu{}_\nu - \frac{1}{4}\delta^\mu_\nu R\right) + \frac{ye^{\phi}}{\ell}\left(\phi^{|\mu}{}_{|\nu} - \frac{1}{4}\delta^\mu_\nu\phi^{|\alpha}{}_{|\alpha}\right)\right. \\ &\left. + \left(\frac{y^2e^{2\phi}}{\ell^2} + \frac{ye^{\phi}}{\ell}\right)\left(\phi^{|\mu}\phi_{|\nu} - \frac{1}{4}\delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha}\right)\right] + \frac{\chi^\mu{}_\nu(x^\mu)}{a^4} . \quad (5.214)\end{aligned}$$

The equation (5.203) is trivially satisfied and equation (5.204) requires that $\chi^\mu{}_{\nu|\mu} = 0$. From the definition (5.179) we get the equation for $g^{(1)}_{\mu\nu}$

$$\frac{1}{2}e^{-\phi}g^{(0)\alpha\mu}\frac{\partial}{\partial y}g^{(1)}_{\alpha\nu} = \Sigma^{(1)\mu}_{\nu} + \frac{1}{4}\delta^\mu_\nu Q^{(1)} \quad (5.215)$$

which can be integrated in a similar way like equation (5.210). The first order correction to the bulk metric is as follows

$$\begin{aligned}g^{(1)}_{\mu\nu} &= -\frac{\ell^2}{2}\left(\frac{1}{a^2} - 1\right)\left[R_{\mu\nu} - \frac{1}{6}h_{\mu\nu}R\right] - \frac{y^2e^{2\phi}}{a^2}\left(\phi_{|\mu}\phi_{|\nu} - \frac{1}{2}h_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\right) \\ &+\frac{\ell^2}{2}\left(\frac{1}{a^2} - 1 - \frac{2ye^{\phi}}{\ell}\frac{1}{a^2}\right)\left(\phi_{|\mu\nu} + \frac{1}{2}h_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\right) \\ &-\frac{\ell}{2}\left(\frac{1}{a^4} - 1\right)\chi_{\mu\nu}(x^\mu) , \quad (5.216)\end{aligned}$$

where the boundary condition $g^{(1)}_{\mu\nu}(y=0,x^\mu)=0$ is taken into account. The relation between $\chi_{\mu\nu}(x^\mu)$ and the components of Weyl tensor $C_{y\mu y\nu}$ were found in [43]

$$C_{y\mu y\nu} = \frac{2}{\ell a^4}\,\chi_{\mu\nu}(x^\mu)\,. \tag{5.217}$$

Effective 4-dimensional field equations

Substituting expressions (5.209) and (5.214) into junction conditions (5.205) leads to the effective Einstein equations on the A-brane

$$G^\mu{}_\nu(h) + \frac{2}{\ell}\,\chi^\mu{}_\nu = \frac{\kappa^2}{\ell} T^{A\mu}{}_\nu\,. \tag{5.218}$$

This equation is non-local due to the generalized dark radiation term $\chi^\mu{}_\nu$ which is still undetermined. Using the notation $\Omega \equiv a(y=\ell,x^\mu) = e^{-e^\phi}$ the junction conditions (5.206) on the B-brane can be written as

$$\begin{aligned}\frac{\ell}{2\Omega^2}G^\mu{}_\nu(h) &+ \frac{\ell e^\phi}{\Omega^2}\left(\phi^{|\mu}{}_{|\nu} - \delta^\mu_\nu\phi^{|\alpha}{}_{|\alpha} + \phi^{|\mu}\phi_{|\nu} - \delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha}\right)\\ &+\frac{\ell e^{2\phi}}{\Omega^2}\left(\phi^{|\mu}\phi_{|\nu} + \frac{1}{2}\delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha}\right) + \frac{\chi^\mu{}_\nu}{\Omega^4} = -\frac{\kappa^2}{2\Omega^2}T^{B\mu}{}_\nu\,,\end{aligned} \tag{5.219}$$

where $T^{B\mu}{}_\nu = \Omega^2\,\tilde{T}^{B\mu}{}_\nu$ is the energy-momentum tensor on the B-brane with the index raised by the A-brane metric $h_{\mu\nu}$. Using the induced metric on the B-brane

$$g^B_{\mu\nu}\Big|_{y=\ell} = \Omega^2(h_{\mu\nu} + g^{(1)}_{\mu\nu}) \equiv f_{\mu\nu} + \Omega^2 g^{(1)}_{\mu\nu} \tag{5.220}$$

the equation (5.219) can be written

$$G^\mu{}_\nu(f) + \frac{2}{\ell\,\Omega^4}\,\chi^\mu{}_\nu = -\frac{\kappa^2}{\ell}\tilde{T}^{B\mu}{}_\nu\,. \tag{5.221}$$

In the case of two branes it is possible to eliminate the non-local quantity $\chi^\mu{}_\nu$ from the equations (5.218) with the help of (5.219) and this leads to the local Einstein equations on the A-brane

$$\begin{aligned}G^\mu{}_\nu(h) = \frac{\kappa^2}{\ell\Psi}T^{A\mu}{}_\nu &+ \frac{\kappa^2(1-\Psi)}{\ell\Psi}T^{B\mu}{}_\nu + \frac{1}{\Psi}\left(\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu\Psi^{|\alpha}{}_{|\alpha}\right)\\ &+\frac{\omega(\Psi)}{\Psi^2}\left(\Psi^{|\mu}\Psi_{|\nu} - \frac{1}{2}\delta^\mu_\nu\Psi^{|\alpha}\Psi_{|\alpha}\right)\,,\end{aligned} \tag{5.222}$$

where the new scalar field Ψ is defined as $\Psi \equiv 1-\Omega^2$, where $\Psi \in [0,1]$ and the coupling function $\omega(\Psi)$ has a definite form

$$\omega(\Psi) \equiv \frac{3}{2}\frac{\Psi}{1-\Psi}\,. \tag{5.223}$$

On the other hand, we can eliminate also the Einstein tensor from the equations (5.218) and (5.219) and this gives the expression for ${\chi^\mu}_\nu$

$$\chi^\mu{}_\nu = -\frac{\kappa^2(1-\Psi)}{2\Psi}\left(T^{A\mu}{}_\nu + T^{B\mu}{}_\nu\right)$$

$$-\frac{\ell}{2\Psi}\left[\left(\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha}\right) + \frac{\omega(\Psi)}{\Psi}\left(\Psi^{|\mu}\Psi_{|\nu} - \frac{1}{2}\delta^\mu_\nu \Psi^{|\alpha}\Psi_{|\alpha}\right)\right]\,. \tag{5.224}$$

Note, that $\chi^\mu{}_\nu$ is expressed through the quantities defined on the branes. Since χ^μ_ν is traceless $\chi^\mu_\nu = 0$ equation (5.224) leads to an equation for the scalar field Ψ

$$\Psi^{|\mu}{}_{|\mu} = \frac{\kappa^2}{\ell}\frac{T^A + T^B}{2\omega+3} - \frac{1}{2\omega+3}\frac{d\omega}{d\Psi}\Psi^{|\mu}\Psi_{|\mu}\,, \tag{5.225}$$

where definition (5.223) is taken into account. The conservation laws for the A-brane and B-brane matter with respect to A-brane metric $h_{\mu\nu}$ are as follows

$$T^{A\mu}{}_{\nu|\mu} = 0\,, \qquad T^{B\mu}{}_{\nu|\mu} = \frac{\Psi_{|\mu}}{1-\Psi}T^{B\mu}{}_\nu - \frac{1}{2}\frac{\Psi_{|\nu}}{1-\Psi}T^B\,. \tag{5.226}$$

Equations (5.222) and (5.225) represent a scalar-tensor gravity with a coupling function $\omega(\Psi)$ on the A-brane and they can be derived from the action

$$\begin{aligned} I^A &= \frac{\ell}{2\kappa^2}\int d^4x\sqrt{-h}\left[\Psi R(h) - \frac{\omega(\Psi)}{\Psi}\Psi^{|\alpha}\Psi_{|\alpha}\right] + \int d^4x\sqrt{-h}\mathcal{L}^A \\ &+ \int d^4x\sqrt{-h}\,(1-\Psi)^2\,\mathcal{L}^B\,. \end{aligned} \tag{5.227}$$

The effective equations for B-brane metric $f_{\mu\nu}$ and for the scalar field defined as $\Phi = \Omega^{-2} - 1$ are derived alike, see [43]. The dynamics on both branes are not independent and the transformation rules

$$\Phi = \frac{\Psi}{1-\Psi}\,, \tag{5.228}$$

$$g^B_{\mu\nu} = (1-\Phi)\left[h_{\mu\nu} + g^{(1)}_{\mu\nu}\left(y=\ell, h_{\mu\nu}, \Psi, T^A_{\mu\nu}, T^B_{\mu\nu}\right)\right] \tag{5.229}$$

allow us to compute the gravity on the B-brane if the gravity on the A-brane is known. The corresponding B-brane action is

$$I^B = \frac{\ell}{2\kappa^2}\int d^4x\sqrt{-f}\left[\Phi R(f)+\frac{3}{2(1+\Phi)}\Phi^{;\alpha}\Phi_{;\alpha}\right]+\int d^4x\sqrt{-f}\mathcal{L}^B$$

$$+ \int d^4x\sqrt{-f}\,(1+\Phi)^2\,\mathcal{L}^A\,, \tag{5.230}$$

where semicolon marks the covariant derivate with respect to the metric $f_{\mu\nu}$ and we have not defined the coupling function (analogue of the $\omega(\Psi)$).

The gravity couples to the A-brane and B-brane matter with different coupling functions and the theory is called by Kanno et al as *quasi-scalar-tensor gravity.* The bulk metric (5.187) is determined if we know the energy-momentum tensors on both branes $T^A_{\mu\nu}$, $T^B_{\mu\nu}$, induced metric on the A-brane $h_{\mu\nu}$ and the scalar field Ψ. Since the 4-dimensional fields allow us to construct the 5-dimensional bulk geometry, the quasi-scalar-tensor theory works as a *holographic brane gravity* at low energy. A generalization of the low energy expansion scheme to dilatonic braneworld where the bulk scalar field φ and nontrivial potential $V(\varphi)$ are included was also investigated in [164]. In this case the additional scalar field $\eta(x^\mu)$ and the effective potential $V_{eff}=V_{eff}(\Psi,\sigma,V)$ appear in the 4-dimensional action and therefore this approach represents a more realistic model and can be compared with the moduli space approximation. Note, that the moduli space approximation (see Subsec. 5.3.2) is equivalent to the approximation scheme discussed here if one makes the identification

$$\Psi = \frac{1}{\cosh^2\mathcal{R}}\,, \tag{5.231}$$

where $\mathcal{R}$ is defined by expressions (5.129). In order to satisfy the phenomenological constraints for $\mathcal{R}$ (see Subsec. 5.3.2) it follows that at present Ψ should be close to one and this means that the distance between the branes is large.

5.3.5 Born-again braneworld

Born-again braneworld [44] is a cosmological scenario based on the effective theory represented by actions (5.227) and (5.230). In this scenario the branes can collide and this collision changes the signs of the brane tensions. The scenario does not contain a cyclic behaviour, the collision can happen only once.

The matter Lagrangians in action (5.227) are assumed to be $\mathcal{L}^A = -\delta\sigma^A$ and $\mathcal{L}^B = -\delta\sigma^B$, where $\delta\sigma^A$ and $\delta\sigma^B$ are the detuned brane tensions. This means that there are two de Sitter or anti-de Sitter branes in the AdS_5 bulk. Transforming the action (5.227) into the Einstein frame ($\bar{h}_{\mu\nu} = \Psi h_{\mu\nu}$) a nontrivial potential appears into the action (see details [44]) and this potential can drive the branes into the collision, where the fifth dimension collapses, $\Psi \to 0$.

In this model it is assumed that before the collision our brane had negative tension which changes to positive tension during the collision. This is realized with the assumption that Ψ smoothly becomes negative after collision. A replacement $\Psi \to -\tilde{\Psi}$ transforms action (5.227) into the action

$$
\begin{aligned}
-I^A &= \frac{\ell}{2\kappa^2}\int d^4x\sqrt{-h}\left[\tilde{\Psi}R + \frac{3}{2}\frac{1}{1+\tilde{\Psi}}\tilde{\Psi}^{|\alpha}\tilde{\Psi}_{|\alpha}\right] + \int d^4x\sqrt{-h}(-\mathcal{L}^A) \\
&+ \int d^4x\sqrt{-h}\left(1-\tilde{\Psi}\right)^2(-\mathcal{L}^B)
\end{aligned}
\tag{5.232}
$$

which is the B-brane action (5.230) (here the coupling function $\omega(\Psi)$ is presented explicitly) with the overall change of sign and changes of signs in matter Lagrangians. It can be interpreted that the positive tension brane becomes the negative tension brane and vice versa during the collision.

In the born-again braneworld scenario it is assumed that our brane was initially with negative tension and it changed to positive at the collision. Cosmological evolution on the negative tension brane ($\delta\sigma_B < 0$), described by action (5.230) is investigated [44] in the original brane (Jordan) frame where the Friedmann equation is

$$
H^2 = -\frac{\kappa^2}{3\ell}\delta\sigma^B - \frac{k}{a^2} + \frac{\mathcal{C}}{a^4}
\tag{5.233}
$$

and the equation of the scalar field Φ is

$$
\ddot{\Phi} + 3H\dot{\Phi} = \frac{4\kappa^2}{3\ell}(1+\Phi)\left[\delta\sigma^B + (1+\Phi)\delta\sigma^A\right] + \frac{1}{2}\frac{1}{1+\Phi}\dot{\Phi}^2 .
\tag{5.234}
$$

Equation (5.234) was integrated numerically and it was concluded, that Φ passes smoothly through zero at the moment $t_c \neq 0$. The Friedmann equation (5.233) does not "feel" the collision [44]. At the collision Φ changes the sign and finally approaches asymptotically to -1. This is interpreted in

[44] as the separation of reborn branes to infinity. Near the collision $t = t_c$ the field Φ is regular and behaves as

$$\Phi = -2(1-\sqrt{\gamma})H_c\,(t-t_c)\,, \qquad (5.235)$$

where

$$\gamma = 1 - \frac{H_\star^2}{H_c^2}\left(1+\frac{1}{\beta}\right), \qquad H_\star^2 = \frac{\kappa^2}{3\ell}(-\delta\sigma_B)\,, \qquad \beta = \frac{\delta\sigma^B}{\delta\sigma^A}\,, \qquad (5.236)$$

and H_c is Hubble parameter at the time of collision $t = t_c$.

On the other hand, at present ($\Phi \to -1$) our Universe is described by the Einstein frame and corresponding transformations to the Einstein frame are

$$a_{_E} \equiv b = \sqrt{\Phi}\;a\,, \qquad dt_{_E} = \sqrt{\Phi}\;dt_{_J}\,. \qquad (5.237)$$

The Hubble parameter in the Einstein frame is given by

$$H_{_E} \equiv \frac{\dot b(t_{_E})}{b(t_{_E})} = \frac{1}{3t_{_E}} + \frac{H_c}{\left(3(1-\sqrt{\gamma})H_c|t_{_E}|\right)^{1/3}}\,, \qquad (5.238)$$

where we have used the relation $t_{_E} = \frac{2}{3}\sqrt{\Phi}(t_{_J} - t_c)$. The collision time in the Einstein frame is $t_{_E} = 0$ and therefore the Hubble parameter diverges at the collision: $H_{_E} \to -\infty$. The authors of the paper [44] interpret the fact that the newborn Universe starts with a large Hubble parameter in the Einstein frame as a possibility to connect the contracting (pre-big bang) phase and expanding (post-big bang) phase since there is no divergence (singularity) in the Jordan frame. In this case the born-again scenario includes a non-singular realization of the pre-big bang scenario in the braneworld context [44]. However, the curvature perturbation diverges logarithmically at the collision both in the Einstein and in the Jordan frame, see for example [159]. The pre-big bang scenario on the brane was investigated also by Foffa in [165], [166].

5.3.6 Summary

In this section we reviewed the cosmology inspired by the first Randall-Sundrum model or by a more fundamental Hořava-Witten theory. In both cases two branes are involved and an additional degree of freedom, called the radion, is needed to describe the interbrane distance. We concentrated to the case when radion is not fixed, i.e., the relative distance between the branes changes. We reviewed two approximation schemes to reduce the 5-dimensional theory down to 4-dimensional and then considered cosmology provided by the effective theory. Different models are possible, and all of them are hoped to offer a solution to the initial singularity problem. However, all of them are suffering some drawbacks. As already a tradition, it is believed that the inclusion of string and quantum corrections will solve the problem.

Despite their different nature, the graceful exit problem in the context of pre-big bang cosmology and the collision of branes have some common failures. In both cases the transition from the contraction to the expansion must occur and in both cases this requires the violation of the null energy conditions or, as pointed out in [157], an unknown mechanism is needed to pass through the singularity. In both cases the string and quantum corrections are assumed to be important near the transition. An important difference which may give some advantages to the concept of the braneworlds (brane collision) is that the transition in some cases occurs in the weak coupling regime. In the pre-big bang scenario the transition occurs in the strongly coupled regime and is therefore hardly tractable as pointed out in [157].

However, the ideas developed in [153], [39], [157], [40], [44], [45] are different and hardly comparable. It seems that a comprehensive analysis, where all models will be compared to each other, may give some hints for future work.

Chapter 6

Summary of review part

Since we have summarized the basic ideas and results at the end of each chapter we add here only a very brief resume. The review part of the thesis deals with three models which have been proposed to describe our Universe:

- Standard cosmological model
- Pre-big bang scenario
- Braneworld cosmology

The standard model, based on the cosmological principle and the assumption that general relativity is the underlying theory which governs the dynamics of the Universe, is a cornerstone of modern cosmology. If this model is supplemented by the inflationary epoch taking place very early in the history of the Universe this model fits well with the data received from recent observations. During the inflation the physical scale L_{pd} evolves faster than the horizon scale d_{pd} and quantum fluctuations are amplified to primeval density inhomogeneities of order $\frac{\delta\rho}{\rho} \sim 10^{-5}$. Fluctuations in the temperature of the Cosmic Microwave Background Radiation are of a similar order. Once the Universe becomes matter dominated these inhomogeneities are amplified by gravity and grow into the structure we see today.

In the framework of the standard cosmological model the Universe has a finite age and solutions describing the evolution of the Universe are singular at that initial moment. Since the initial singularity problem is believed to be conceptual in the framework of the standard model we should look for extensions. An important reason for developing a cosmological model where the underlying theory is not the general relativity but string theory is the fact that string theory is believed to be more fundamental and thus applicable also at very high energies, i.e, for the early Universe.

Until recently the pre-big bang scenario was the most successful realization of string theory in the cosmological context. The cosmological solutions found from the low energy effective action of string theory are related by a duality transformation. In the pre-big bang scenario the evolution of the Universe starts from the string perturbative vacuum and evolves inflationarily until the transition to the Friedmann-Robertson-Walker universe occurs. The decay of the initial vacuum state and the transition to Friedmann-Robertson-Walker universe are not completely understood. At present the scenario is developed up to the level where some observational evidence is needed for future work. Since the pre-big bang scenario predicts very specific gravitational wave background, it is believed, that planned gravitational wave detectors will provide the crucial experiment.

Braneworld cosmology is a developing approach which, as time goes on, links more and more string theory and cosmology with each other. String theory can offer new conceptions to develop cosmology and vice versa, cosmology is hoped to be a laboratory for string theory. The most remarkable prediction of the braneworld idea is that at least some of the extra dimensions could be quite large. An experimental limitation on the scale of extra dimension arises from the accuracy with which gravity obeys the inverse square law. According to the latest experiments the r^{-2} force law persists down to distances of the order of 0.1 mm.

However, both the pre-big bang scenario and the braneworld cosmology (in all realizations) have some unsolved problems, partially discussed in this review. So far we have no reliable theory which can answer the question how and why our Universe sprung into being. Maybe the answer to this question is out of the physics.

Chapter 7

Summary of original papers

Motivated by superstring theory, several new approaches have emerged recently in the mathematical cosmology. The first stringy cosmological model was the pre-big bang scenario proposed in 1991 by G. Veneziano, which added an infinitely long epoch preceding the big bang singularity to the history of our universe. It is described by solutions of the modified Einstein equations for 4-metric $g_{\mu\nu}$ and dilaton field ϕ which follow from the low energy effective action of the heterotic superstring theory.

In our paper I [69], a pre-big bang universe with vanishing matter tensor is considered and the evolution of cosmological perturbations is investigated with the aim to clarify the role of initial inhomogeneities. The corresponding Hamilton-Jacobi equation is solved using the long-wavelength approximation (the gradient expansion). The zero order solution is taken to be the standard pre-big bang model with a cosmological singularity at t_0. From the second order solution it follows that metric corrections die off during the superinflation as $t \to t_0$, but dilaton corrections are growing. This means that initial classical inhomogeneities, which originate from spatial gradients of the seed metric, are smoothed out during the superinflation, but not dilaton inhomogeneities. Going backwards in time dilaton corrections become negligible, but influence of initial classical inhomogeneities of the seed metric and the initial curvature are growing. Thus the adequacy of the gradient expansion for investigating the pre-big bang superinflation decreases in the direction of the past as well as of the future (singularity). However, we can estimate the characteristic size of initial inhomogeneities and conclude that the inflating domain must be large in string units but smaller than the initial horizon.

The Lagrangian field equations for the pre-big bang cosmology are solved in our paper II [66] using the long-wavelength (gradient) approximation in the case of a matter tensor describing a perfect fluid with a barotropic equation of state $p = (\Gamma - 1)\rho$, $0 \leq \Gamma \leq 2$, where p is the pressure and ρ is the density of cosmological matter. In the zero order, a quasi-isotropic solution is found and compared with the solution with a vanishing matter tensor [69]. The second order corrections include the effect of spatial gradients and are symmetric in respect of the singularity: terms which are growing in the pre-big bang phase are decaying in the post-big bang phase and vice versa. In the case of an exotic matter with barotropic index $\Gamma = 0$ or $\Gamma = 2$ the decay of all second order correction terms at the end of the pre-big bang stage can be achieved. However, corrections for the post-big bang stage always include terms which are growing in time. This indicates that a simple model without any dilaton potential or cosmological constant leads to a result which can hardly fit our present understandings.

A totally different string-inspired cosmology was proposed in 1999 by L. Randall and R. Sundrum. To solve the hierarchy problem of the standard model they considered our spacetime as a 4-hypersurface (a world volume of a 3-brane) in a 5-dimensional bulk spacetime with Z_2 symmetry along the extra dimension. They proposed two cosmological scenarios, with one brane and with two branes moving in a 5-dimensional bulk spacetime. In our paper III [45], a model with two branes is considered using the gradient expansion method with respect to the coordinate normal to a brane. The low energy effective theory gives a scalar-tensor type gravitation on both branes with a specific coupling function. The matter is described by a barotropic perfect fluid on the A-brane and by a phenomenological time dependent "cosmological constant" on the B-brane. Some special solutions are found for the scale factor of the three-metric on the A-brane and for the scalar radion field which determines the proper distance between the branes. For all values of the barotropic index Γ, at late time the dynamics on the A-brane is well described by the Einstein general relativity. In the case of a phenomenological cosmological constant on the A-brane ($\Gamma = 0, p = -\rho$) the de Sitter type evolution at late time follows. This feature seems to be typical also in other braneworld scenarios discussed recently and fits well with the experimental evidence of late time acceleration. Compared with the phenomenological theory (quintessence) the braneword model gives a more motivated theoretical ground to this result. In the case $\Gamma \neq 0$ the power law evolution for the metric of the A-brane can be assumed. This type of solution lacks one integration constant which encodes the influence of the bulk on the brane (dark radiation) and in this case the cosmological evolution of our universe on the A-brane coincides with the familiar Friedmann-Robertson-Walker model. This means that it shares all its observational evidences and contains no additional hints for approving the braneworld model. Such hints could be found in the explicit solutions with the dark radiation term presented by us in special cases of barotropic index $\Gamma = 0,\ 4/3,\ 2/3,\ 1$. The dynamics of the radion is discussed in detail. We conclude that the collision of the branes can take place at a distinct moment determined by matter tensors on the branes. The evolution of the scale factor and the radion field is regular at the moment of collision.

Errata

Equations (36), (49) and (52) of the paper III [45] should read as follows:

$$\dot{\chi} + H\chi \mp \sqrt{k\lambda_0^B}\,\chi^2 = 0\,,$$

$$a^2 = 2\sqrt{k\rho_0 + C}(t - t_0), \qquad H = \frac{1}{2(t - t_0)}\,,$$

$$\dot{\chi} + H\chi \mp \sqrt{k\lambda_0^B\,\chi^4 + \frac{C}{a^4}} = 0\,.$$

We apologize for a misprint in the name of Dr. A. Serna in the paper III.

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Summary in Estonian

Uurimusi Suure Paugu eelse Universumi ja braanimaailma kohta

Superstringiteooria on inspireerinud esitama mitmeid uusi matemaatilise kosmoloogia mudeleid. Esimese stringiteoorial baseeruva kosmoloogilise mudeli konstrueeris G. Veneziano 1991. aastal. Selle mudeli järgi eelneb Suure Paugu singulaarsusele lõpmatult pikk Universumi evolutsioonietapp, mida kirjeldavad modifitseeritud Einsteini võrrandite lahendid 4-meetrika $g_{\mu\nu}$ ja dilatoni ϕ jaoks. Võrrandid ise on tuletatud heterootilise superstringiteooria madala energia lähendi mõjufunktsionaalist.

Publikatsioonis [I] on käsitletud häirituste evolutsiooni Suure Paugu eelses Universumis juhul, kui mateeriatensor puudub, ja eesmärgiks on selgitada algsete mittehomogeensuste mõju mudelile. Vastav Hamiltoni-Jacobi võrrand on lahendatud kasutades pikalainelist lähendust (gradientarendust). Nullindat järku lahend kirjeldab standardset Suure Paugu eelset Universumit ja sisaldab kosmoloogilist singulaarsust hetkel t_0. Teist järku lahendist järeldub, et meetrika parandused kahanevad superinflatsioonis, kui $t \to t_0$, kuid dilatoni parandused kasvavad. See tähendab, et algsed klassikalised meetrika mittehomogeensused kaovad superinflatsioonis, aga dilatoni mittehomogeensused säilivad. Ajas tagasi liikudes dilatoni parandused seevastu kaovad, meetrika mittehomogeensused ja algne kõverus aga kasvavad. Seega gradientarenduse täpsus, uurimaks Suure Paugu eelset superinflatsiooni, kahaneb nii singulaarsuse läheduses kui ka liikudes asümptootilise mineviku poole. On võimalik hinnata algsete mittehomogeensuste iseloomulikku mõõdet ja järeldada, et inflatsioneeruv domeen peab olema väga suur, mõõdetuna stringi pikkusskaalas, samas on see domeen väiksem kui algne horisont.

Publikatsioonis [II] on pikalainelises lähendis lahendatud Lagrange'i võrrandid, mis kirjeldavad Suure Paugu eelset kosmoloogiat. Mateeria on kirjeldatud ideaalse vedelikuna ja on eeldatud, et ta rahuldab barotroopset olekuvõrrandit: $p = (\Gamma - 1)\rho$, $0 < \Gamma < 2$, kus p on rõhk ja ρ on energiatihedus. Nullindat järku kvaasi-isotroopset lahendit on võrreldud lahendiga, mis on publikatsioonis [I] leitud juhul, kui mateeria puudub. Teist järku parandused sisaldavad ruumilisi gradiente ja on sümmetrilised singulaarsuse suhtes. Liikmed, mis on kasvavad Suure Paugu eelses Universumis, on kahanevad Suure Paugu järgses Universumis, ja vastupidi. Eksootilise mateeria korral, kui $\Gamma = 0$ või $\Gamma = 2$, kaovad kõik teist järku parandusliikmed Suure Paugu eelse epohhi lõpuks. Suure Paugu järgset Universumit kirjeldavad lahendid sisaldavad igal juhul liikmeid, mis on ajas kasvavad. Sellest võib järeldada, et lihtne mudel ilma dilatoni potentsiaalita või kosmoloogilise konstandita ei sobi Suure Paugu järgse Universumi kirjeldamiseks.

Eelkirjeldatust täiesti erinev, kuid samuti stringiteooriast inspireeritud mudel esitati 1999. aastal L. Randalli ja R. Sundrumi poolt. Lahendamaks standardmudeli hierarhia probleemi käsitlesid nad meie aegruumi kui 4-hüperpinda (nn. braan), mis asetseb 5-dimensionaalses mahtruumis ja kus kehtib Z_2 sümmeetria lisadimensiooni suhtes. Nad esitasid kaks erinevat kosmoloogilist mudelit, kusjuures esimeses mudelis liigub üks braan ja teises kaks braani 5-dimensionaalses mahtruumis.

Publikatsioonis [III] on kahe braani mudeli uurimiseks kasutatud gradientarendust 5. dimensiooni suhtes. Esimese lähenduse efektiivne teooria osutub spetsiifilise seosefunktsiooniga skalaar-tensor-tüüpi gravitatsiooniteooriaks, kusjuures skalaarväli (nn. radion) määrab braanidevahelise kauguse. Olgu mateeria kirjeldatud barotroopse ideaalse vedelikuga A-braanil ("meie 4-aegruum") ja fenomenoloogilise "kosmoloogilise konstandiga" B-braanil. On leitud lahendid A-braani mastaabikordaja ja skalaarvälja jaoks. Suvalise barotroopse indeksi väärtuse korral on hiline kosmoloogia A-braanil kirjeldatav Einsteini üldrelatiivsusteooriaga. Juhul kui A-braani aine on fenomenoloogiline kosmoloogiline konstant ($\Gamma = 0$, $p = -\rho$), siis on hiline Universum kirjeldatav de Sitteri tüüpi lahendiga. See lahend on iseloomulik ka teistele mudelitele, kus käsitletakse kosmoloogilisi lahendeid braanil, ja sobib hästi vaatlusliku tulemusega, et hiline Universum paisub kiirenevalt. Fenomenoloogilise teooriaga (nn. kvintessents) võrreldes pakub braanimaailma mudel seega paremini põhjendatud selgituse kiireneva paisumise kirjeldamiseks. On leitud lahendid A-braani mastaabikordaja ja skalaarvälja jaoks täiendaval eeldusel, et juhul kui $\Gamma \neq 0$, on A-braani mastaabikordaja esitatav astmefunktsioonina. Sellise eelduse korral kaob lahendist üks integreerimiskonstant, mis kirjeldab mahtruumi mõju braanile (nn. tume kiirgus), ja lahend ühtib Friedmanni kosmoloogia vastava lahendiga. Seega ei võimalda see lahend kuidagi eristada Friedmanni mudelit ja braanimaailma mudelit. On leitud mõned erilahendid erinevate barotroopsete indeksite väärtuste ($\Gamma = 0, 4/3, 2/3, 1$) korral ka siis, kui tume kiirgus on arvesse võetud. Uuritud on radioni dünaamikat ja järeldatud, et braanide põrge võib toimuda kindlal hetkel, mis on määratud aine olekuvõrrandiga braanidel. Põrkehetkel on mastaabikordaja ja radioni evolutsioon regulaarne.

Attached original publications

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Peamine töösuund:	teoreetiline kosmoloogia, Suure Paugu eelne Universum, braanimaailmad