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VITALI RETŠNOI

Vector fields and Lie group representations

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List of original publications

This Thesis is based on the following papers:

1. M. Rahula, V. Retšnoi, *Adjoint representations and movements*, Proc. AGMP, Springer-Verlag, Berlin, Heidelberg, 2009, 161 - 170.
2. M. Rahula, V. Retšnoi, *Total differentiation under jet composition*, Proc. AGMP, Journ. of Nonlin. Math. Ph., 2006, 102-109.
3. V. Retšnoi, *Integration of tensor fields*, BSG Proc., edited by Balkan Society of Geometers (to appear).

Other publications by the author:

4. M. Rahula, V. Retšnoi, *Dual structures: floors and jets*, Proc. Intern. Geom. Center, **1(1-2)** (2008), 131-154 (in Russian).
5. V. Retšnoi, *Existence theorems for commutative diagrams*, Lobachevskii Journal of Mathematics, **17** (2005), 211-228.

Introduction

General background

One of the most significant tools in differential geometry and global analysis, in continuous environment mechanics and dynamical systems is the notion of vector field. The following concepts are related to vector fields (some of them more studied others less):

1. trajectories and flows, interaction of flows;
2. phase portrait, gas-liquid flow, curls, turbulence, shock waves, separatrices and attractors;
3. dragging of tensor fields (including functions, vector fields and differential forms) along a flow;
4. coordinate-free differentiation of tensor fields, Lie derivatives, Lie-Cartan calculus;
5. linear vector fields, linear approximation of non-linear flows;
6. projective and projectable vector fields;
7. Lie groups and their representations, group operators;
8. nonholonomic object and nonholonomic basis, derivation formulas;
9. connections in bundles, curvature theory;
10. integration of tensor fields, differential equations with Lie derivatives;
11. symmetries and invariants, stability and conservation laws;
12. exponential law in jet space;
13. operators of total differentiation and map composition;
14. the study of motions: transformations of motions, motions of higher orders.

The topics 1–7 have been widely researched, cf. [1]–[14], [23]–[25], but the topics 8–14 are investigated a little. A systematical study of them is undertaken in [15] and [16], and in the monograph [2]. The author of this thesis took part in the preparation of the monograph. His contribution is contained in the topics 10, 12 and 13, see [17]–[20]. The paper [17] is attached to this thesis.

Let us introduce some well-known facts in the form convenient in what follows. Any vector can be interpreted as a stop-frame of a point moving along own its trajectory through space, and any vector field as a stop-frame of a flow generated by this field. Let M be a smooth manifold, and let X be a smooth vector field on M. Therefore a flow $a_t = \exp tX$ as a one-parameter group of transformations of M is associated with X. Choosing local coordinates u^i on a neighborhood $U \subset M$, the flow a_t is determined by the system of first-order ordinary differential equations (ODEs)

$$(u^i)' = x^i(u), \tag{1}$$

where the prime denotes differentiation with respect to a time-parameter t, and x^i are components of the vector field X at a point $u \in U$. More precisely, the flow a_t is a local pseudogroup of local transformations of M, because the theorem of uniqueness and existence of solutions of the system (1) has a local character. Such a relation between the local and the global should be kept in mind.

In the flow a_t points move along their own trajectories, and functions are dragged according to the composition law:

$$f \quad \rightsquigarrow \quad f_t = f \circ a_t.$$

Under some conditions of smoothness and convergence a dragged function f_t can be expanded in a Maclaurin series in terms of the powers of a parameter t:

$$f_t = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!},$$

where $f^{(k)} = X^k f$, $k = 0, 1, 2, \ldots$. Moreover, one can consider a dragging of a smooth tensor field S of a general type (p, q) along the flow a_t of X, described by a Lie-Maclaurin series

$$S_t = \sum_{k=0}^{\infty} S^{(k)} \frac{t^k}{k!},$$

with Lie derivatives $S^{(k)} = \mathcal{L}_X^{(k)} S$ as coefficients, where $k = 0, 1, 2, \ldots$.

The notion of Lie derivatives is coordinate-free, but it is possible to calculate them in any local coordinate system. Such calculations are carried out in a natural basis, i.e., in a frame and coframe consisting of partial derivative operators and differentials of coordinate functions, respectively. Therefore, the aim of this thesis is to develop Lie derivatives of tensor fields and their applications in a nonholonomic basis. The nonholonomy object (J. A. Schouten, [24]) appearing in the calculation formulas allows us to apply this technique to the theory of Lie groups. In particular, the structure constants are precisely the nonholonomy object of the left- or right-invariant basis in a Lie group.

The nonholonomy object is also important in the theory of connections in fiber bundles. The connection in fiber bundle $\pi : M_1 \to M$, with a smooth manifold M as its base, is determined by defining an n-dimensional horizontal distribution Δ_h supplemental to a r-dimensional vertical distribution $\Delta_v = \ker T\pi$ on a manifold M_1, where $\dim M = n$ and $\dim M_1 = n+r$. It means that in M_1 there is defined the structure $\Delta_h \oplus \Delta_v$, and the nonholonomy object in this structure are decomposed on subobjects. In the adopted basis are defined two subobjects – one of them determines transferring of fibers (an object of connection) and another one a curvature of the space (an object of curvature). In the case of a tangent bundle this structure underlies in the tensor analysis and covariant differentiation.

In this work the following situation is considered. Let X be a smooth vector field with canonical parameter s on a smooth manifold M, and suppose f is a smooth function on M. Let us form an infinite sequence $F = (f, f', f'', \ldots)$ consisting of f and its derivatives of all orders with respect to X. Then there is defined a triplet (X, s, F) on M. All possible triplets of such kind on manifolds form a category. In particular, two triples (X, s, F) and $(Y, \widetilde{s}, \widetilde{F})$ on manifolds M and $\widetilde{M}$, respectively, are linked together by a morphism $\varphi : M \to \widetilde{M}$ that is a smooth map for which $s = \widetilde{s} \circ \varphi$, $F = \widetilde{F} \circ \varphi$, and X and Y are φ-related vector fields, i.e., for any smooth function g on $\widetilde{M}$ the equality $X(g \circ \varphi) = (Yg) \circ \varphi$ is valid. In the category of triplets (X, s, F) the terminal object is precisely the triplet (D, t, U), where D is the total differentiation operator (TDO) in the space of infinite jets $\mathcal{J}_{1,1}$, t is a parameter (time) and U is a set of fiber coordinates $u, u', u'', \ldots$. Recall that the term *terminal object* refers that there exists exactly one morphism from each object (X, s, F) to (D, t, U).

In the jet space $\mathcal{J}_{1,1}$ there is defined an *exponential law* that appears in the following three implications (in matrix notations):

$$U' = CU \quad\Longrightarrow\quad U_t = e^{tC}U \quad\Longrightarrow\quad I = e^{-tC}U, \tag{I}$$

$$\omega' = C\omega \quad\Longrightarrow\quad \omega_t = e^{tC}\omega \quad\Longrightarrow\quad dI = e^{-tC}\omega, \tag{II}$$

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U}C \quad\Longrightarrow\quad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U}e^{-tC} \quad\Longrightarrow\quad \frac{\partial}{\partial I} = \frac{\partial}{\partial U}e^{tC}. \tag{III}$$

The implication (I) determines invariants I of the TDO D that are transformed by φ to the invariants $I \circ \varphi = e^{-sC}F$ of X on M. The implication (II) establishes the connection between the differentials dI and Cartan forms ω defined in $\mathcal{J}_{1,1}$. The forms $\omega = dU - U'dt$ are transformed by φ to the forms $\omega \circ T\varphi = dF - F'ds$ on M. The implication (III) determines infinitesimal symmetries of the operator D and thus gives the rule for construction of corresponding infinitesimal symmetries for the vector field X on M. All implications (I)–(III) can be naturally extended on a jet space $\mathcal{J}_{n,m}$ by using multi-indices technique.

Finally, the following situation is considered: analogously to the process when a smooth map induces an infinite jet, the composition of smooth maps induces a composition of jets. The problem is how to define a recurrence formula for jet composition in such a way that the definition does not depend on a choice of maps. The problem leads to another question – how the corresponding TDOs and Cartan forms are related under the jet composition. The answer is given in Chapter 4, see Propositions 4.1–4.12, in the form of convenient recurrence formulas.

Short review and technical notes

In Chapter 1 some basic notions from tensor algebra and global analysis are introduced for the purpose of having a complete picture of the subject . In particular, for a linear map $f : V_1 \to V_2$ from one vector space to another contravariant tensors of a type $(p, 0)$ (including vectors) are transformed from V_1 to V_2 (from left to right) and covariant tensors of a type $(0, q)$ (including covectors) from the dual space V_2^* to the dual space V_1^* (from right to left). It means that for a $(p, 0)$-tensor in V_1 or $(0, q)$-tensor in V_2^* the corresponding tensors f-related to them are determined uniquely. But for f-related tensors S and $\widetilde{S}$ of a mixed type (p, q) this claim is not true, because we can not express the coefficients of $\widetilde{S}$ in terms of coefficients of S, and vice versa. The notion of f-related tensors allows us further to define φ-related tensor fields for a smooth map φ between smooth manifolds, and then Lie derivatives. Some basic properties of vector fields and differential forms on smooth manifolds are also presented.

In Chapter 2, a special attention is paid to the notion of φ-related tensor fields on smooth manifolds, which allows us to define a dragging of tensor fields (including vector fields and differential forms) in a flow $a_t = \exp tX$ of a vector field X. Then there are can be defined Lie derivatives of tensor fields with respect to the vector field X. Also Lie differentiation in nonholonomic basis is developed, where an important role is played by derivation formulas together with the nonholonomy object. Finally, the integration of tensor fields as a reverse process to the Lie differentiation is introduced. A few geometrical examples included in the text clarify the topic being discussed.

Chapter 3 is devoted to the theory of Lie group representations. Considering vector fields as infinitesimal generators of flows on a manifold M, the Lie derivative is an infinitesimal version of a representation of a diffeomorphism group on tensor fields. In this Section the basic properties of a tangent group TG of a Lie group G are considered. The formulas for left- and right-invariant bases, and for operators of adjoint representations are derived. Some of these formulas are discussed for linear group $GL(2, \mathbb{R})$ in the original publication [17] attached to this thesis, see page 91.

Chapter 4 is devoted to the structure of jet space $\mathcal{J}_{n,m}$ of infinite jets of smooth maps $\mathbb{R}^n \to \mathbb{R}^m$. We begin with the case $\mathcal{J}_{1,1}$, where TDO D and Cartan forms ω are defined. The emergence of implications (I), (II) and (III) is explained. Then the notions of TDO and Cartan forms together with these implications are naturally generalized to the general case $\mathcal{J}_{n,m}$ by using multi-indices. A special attention is paid to the coupling of TDOs and Cartan forms under a jet composition. The above-mentioned Propositions containing recurrence formulas are proved.

We will use the following conventions for the rest of the thesis.

1. We use the *Einstein summation*, i.e., the convention that repeated indices are implicitly summed over. Every time when the summation in an expression must be made over an index which occurs twice, once as a superscript and once as a subscript, we use the Einstein summation convention. Unless we specify otherwise, any index that is to be summed over we write in the upper position. For example, using Einstein summation,

$$e_i x^i = \sum_i e_i x^i \quad \text{or} \quad a^i_k a^k_j = \sum_k a^i_k a^k_j .$$

2. The row index of a matrix is written as a superscript while the column index is written as a subscript, so that the general term a^i_j in matrix $A = (a^i_j)$ denotes the element in the i-th row and j-th column in A.

3. The smoothness of functions, vector fields and any tensor fields means that the relevant objects occurring will be assumed to be differentiable of sufficiently high class C^p or, if it is necessary, even C^∞ or C^ω. It is assumed that tensor fields are sufficiently smooth so that derivatives can be taken.

4. When it does not lead to misunderstandings, we use a prime in order to denote the Lie derivative with respect to the fixed vector field X: $S' = \mathcal{L}_X S$, where S is an arbitrary smooth tensor field on a manifold.

5. We use the following rule: *summation excludes differentiation*. This means that the expression $X_i x^j$ denotes the differentiation of a function x^j with respect to a vector field X_i, while the expression $X_i x^i$ denotes, according to the Einstein summation convention, the linear combination of vector fields X_i with the coefficients x^i.

6. The most essential formulas for the theory are framed in the text.

Chapter 1

Prerequisites

We begin with some basic concepts from multi-linear algebra (vectors, covectors, general tensors of a mixed type, their transformations under a linear map f between vector spaces, f-related tensors etc.), which allow us further to define φ-related tensors on manifolds for φ being a smooth map between smooth manifolds. The notion of φ-related tensors is important in the definition of the Lie derivative on manifolds, which is the key concept for this thesis. Then smooth manifolds are considered, and some basic properties of vector fields and differential forms are introduced in such notations that turn out to be very useful in what follows (cf. [10], [11]).

1.1 Preliminaries from linear algebra

Suppose V is an n-dimensional vector space over the field $\mathbb{R}$, and let V^* be its dual space. Let in V and V^* a dual basis (e, e^*) be given. The frame e is an n-dimensional row-matrix, i.e., $e = (e_i)$, $i = 1, \dots, n$, and coframe e^* is an n-dimensional column-matrix, i.e., $e^* = (e^j)$, $j = 1, \dots, n$. The duality of the basis (e, e^*) means that $e^*(e) = E$, where E is a unit matrix, or, what is equivalent, $e^j(e_i) = \delta_i^j$, where δ_i^j is the *Kronecker delta*, i.e.

$$\delta_i^j = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases} .$$

Any vector $X \in V$ and any covector $\Phi \in V^*$ can be presented in terms of the given basis as[1]

$$X = e_i x^i, \qquad \Phi = \varphi_j e^j.$$

In matrix notation, $X = ex$ and $\Phi = \varphi e^*$, where $x = (x^i)$ is an n-dimensional column matrix of components of the vector X in the frame e and $\varphi = (\varphi_i)$ is an n-dimensional row matrix of components of the covector Φ in the coframe e^*. The action of Φ on X is $\Phi(X) = \varphi_i x^i$ or $\Phi(X) = \varphi x$.

---

[1] According to the Einstein summation convention, see Introduction.

A *q-covariant and p-contravariant tensor* (briefly, a *(p,q)-tensor*) S is defined as an element of the tensor space

$$(\otimes^p V)\otimes(\otimes^q V^*) = \underbrace{V\otimes\cdots\otimes V}_{p\text{ times}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{q\text{ times}}.$$

Let in the spaces V and V^* a dual basis (e,e^*) be given. Then a frame in the space of all (p,q)-tensors $(\otimes^p V)\otimes(\otimes^q V^*)$ is given by

$$e_{i_1}\otimes\cdots\otimes e_{i_p}\otimes e^{j_1}\otimes\cdots\otimes e^{j_q},\quad 1\le i_0,\dots,i_p,j_0,\dots,j_q\le n,$$

and any tensor S is determined by its coefficients $s^{i_1\dots i_p}_{j_1\dots j_q}$ as

$$S = e_{i_1}\otimes\cdots\otimes e_{i_p}s^{i_1\dots i_p}_{j_1\dots j_q}e^{j_1}\otimes\cdots\otimes e^{j_q}.$$

This is a real-valued multi-linear form. Its value on p covectors $\Phi^1 = \varphi^1_{i_1}e^{i_1},\dots,$ $\Phi^p = \varphi^p_{i_p}e^{i_p}$ and q vectors $X_1 = e_{j_1}x^{j_1}_1,\dots,X_q = e_{j_q}x^{j_q}_q$ is defined by

$$S(\Phi^1,\dots,\Phi^p;X_1,\dots,X_q) = \varphi^1_{i_1}\dots\varphi^p_{i_p}s^{i_1\dots i_p}_{j_1\dots j_q}x^{j_1}_1\dots x^{j_q}_q.$$

Moreover, the coefficients of S are given by the action of S on the basis:

$$s^{i_1\dots i_p}_{j_1\dots j_q} = S(e^{i_1},\dots,e^{i_p};e_{j_1},\dots,e_{j_q}).$$

For instance, a vector X and covector Φ are tensors of the type $(1,0)$ and $(0,1)$, respectively. The expressions $\Phi(X)$ and $X(\Phi)$ are equivalent.

Let (e,e^*) and $(\widetilde{e},\widetilde{e^*})$ be dual bases in the space V, and let the transformation of one to the other is determined by a regular matrix $A = (a^i_j)$ of order n and its inverse $A^{-1} = (\overline{a}^i_j)$, such that

$$\widetilde{e} = eA^{-1},\qquad \widetilde{e^*} = Ae^*.$$

Then the coefficients of S are transformed according to the rule

$$\widetilde{s}^{i_1\dots i_p}_{j_1\dots j_q} = a^{i_1}_{k_1}\dots a^{i_p}_{k_p}s^{k_1\dots k_p}_{l_1\dots l_q}\overline{a}^{l_1}_{j_1}\dots\overline{a}^{l_q}_{j_q}. \tag{1.1}$$

For instance, for coefficients of a vector $X = ex$ and covector $\Phi = \varphi e^*$ we have the following transformation formulas:

$$\widetilde{x} = Ax,\qquad \widetilde{\varphi} = \varphi A^{-1}.$$

For an *affinor* (tensor of type $(1,1)$) $S = ese^*$ the rule (1.1) implies

$$\widetilde{s} = AsA^{-1}.$$

Note that, in general, it is impossible to write transformation formulas for coefficients of tensors of a mixed type (p,q) in matrix notations.

Let V_1 and V_2 be vector spaces of dimensions n and m, respectively, and let $f : V_1 \to V_2$ be a linear map. Then under f vectors from V_1 are being transformed to V_2 (from left to right), $X \mapsto Y = f(X)$, and covectors are being transformed from V_2^* to V_1^* (from right to left), $\Phi \mapsto \Phi \circ f$. So the comap f^* is induced:

$$f^* : V_2^* \to V_1^* : \Phi \mapsto \Psi = \Phi \circ f.$$

Now suppose (e, e^*) and $(\varepsilon, \varepsilon^*)$ are dual bases for the vector spaces V_1 and V_2, respectively. In these bases f can be presented by some $(m \times n)$-matrix $F = (f_i^\alpha)$, $i = 1, \dots, n$; $\alpha = 1, \dots, m$, such that

$$f(e) = \varepsilon F, \qquad \varepsilon^* \circ f = F e^*.$$

Given a linear map f, there exists a covector with vector values in the space V_2, briefly *vector-covector*, defined by

$$\mathcal{F} = \varepsilon F e^* \quad \text{or} \quad \mathcal{F} = \varepsilon_\alpha f_i^\alpha e^i. \tag{1.2}$$

Let $X = ex \in V_1$ and $\Psi = \psi e^* \in V_2^*$. Then $\mathcal{F}(X) = \varepsilon F x \in V_2$ and $\Psi(\mathcal{F}) = \psi F e^* \in V_1^*$. If $f : X = ex \mapsto Y = \varepsilon y$ and $f^* : \Psi = \psi\varepsilon^* \mapsto \Phi = \varphi e^*$, then $y = Fx$ and $\varphi = \psi F$. So the components are related by $y^\alpha = f_i^\alpha x^i$ and $\varphi_i = \psi_\alpha f_i^\alpha$.

The notion of the vector-covector $\mathcal{F}$ is independent of a choice of a basis, so $\mathcal{F}$ can be identified with the map f, although the matrix F, that depends on a choice of a basis, is presented in (1.2).

Let S and $\widetilde{S}$ be tensors of a type (p, q) in $(\otimes^p V_1) \otimes (\otimes^q V_1^*)$ and $(\otimes^p V_2) \otimes (\otimes^q V_2^*)$, respectively.

Definition 1.1. Two tensors S and $\widetilde{S}$ are said to be *f-related* if for any set of p covectors $\Psi^1, \dots, \Psi^p \in V_2^*$ and q vectors $X_1, \dots, X_q \in V_1$ the equality

$$S(\Psi^1 \circ f, \dots, \Psi^p \circ f; X_1, \dots, X_q) = \widetilde{S}(\Psi^1, \dots, \Psi^p; f(X_1), \dots, f(X_q)) \tag{1.3}$$

is valid.

Coefficients of f-related tensors S and $\widetilde{S}$ are related according to the formula

$$f_{i_1}^{\alpha_1} \dots f_{i_p}^{\alpha_p} s_{j_1 \dots j_q}^{i_1 \dots i_p} = \widetilde{s}_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} f_{j_1}^{\beta_1} \dots f_{j_q}^{\beta_q}. \tag{1.4}$$

Note that in general f transforms contravariant $(p, 0)$-tensors from V_1 to V_2 (from left to right) and covariant $(0, q)$-tensors from V_2^* to V_1^* (from right to left). It means that for a $(p, 0)$-tensor in V_1 or $(0, q)$-tensor in V_2^* the corresponding f-related to them tensors are determined uniquely. But for f-related tensors S and $\widetilde{S}$ of a mixed type (p, q) this claim is not true, because we can not express coefficients of $\widetilde{S}$ in terms of coefficients of S and vice versa. This follows from (1.4), where the matrix $F = (f_i^\alpha)$ is in general not invertible. The correspondence $S \leftrightarrow \widetilde{S}$ is one-to-one only under an isomorphism of vector spaces.

Nevertheless, the same tensor operations (tensor product, contraction, symmetrization, alternation and linear combinations with constant coefficients) over f-related tensors lead again to f-related tensors.

Let, for instance, f be an isomorphism between vector spaces V_1 and V_2. Then there exists a one-to-one correspondence between f-related tensors S and $\widetilde{S}$:

$$\widetilde{S}^{i_1 \dots i_p}_{j_1 \dots j_q} = f^{i_1}_{\alpha_1} \dots f^{i_p}_{\alpha_p} S^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \overline{f}^{\beta_1}_{j_1} \dots \overline{f}^{\beta_q}_{j_q},$$

where the matrix $F = (f^{\alpha}_{i})$ is regular and matrix $F^{-1} = (\overline{f}^{\alpha}_{i})$ is its inverse.

1.2 Vector fields, Lie brackets, flows and invariants

Let M be a smooth manifold and let f be a smooth function on M. Then any *vector field* X on M is defined as a linear differential operator which assigns to f the function Xf which is the *derivative* of f with respect to X. As it was mentioned in Introduction, we use prime in order to denote the derivative with respect to the fixed vector field X: $f' = Xf$. By definition X satisfies the *Leibniz rule*

$$(fg)' = f'g + fg',$$

where f and g are arbitrary differentiable functions on M. The *differential* of f as a 1-form on M is defined by

$$df(X) = Xf.$$

The vector field X is called *smooth* if for any smooth function f its derivative f' is also smooth.

For any two vector fields X and Y their compositions $X \circ Y$ and $Y \circ X$ or, briefly, XY and YX do not in general satisfy the Leibniz rule and, thus, cannot be vector fields. But the operator

$$[X, Y] = XY - YX \tag{1.5}$$

is a vector field called a *Lie bracket* of the vector fields X and Y.

Any one-parametric group of transformations (diffeomorphisms) $a_t : M \to M$ induces a *flow* in M – all points move along own trajectories and functions are dragged according to the composition law:

$$M \ni u \mapsto u_t = a_t(u), \qquad f \mapsto f_t = f \circ a_t.$$

For any smooth function f on M the flow a_t gives rise to a smooth vector field X as follows:

$$Xf = (f \circ a_t)'_{t=0}.$$

Conversely, with any smooth vector field X on M a local one-parametric group of transformations a_t of M is associated. It is said to be the flow of X. In both cases we denote

$$a_t = \exp tX.$$

Under some conditions of smoothness and convergence, the dragged function f_t can be expanded in a *Maclaurin series* in terms of the group parameter t powers:

$$f_t = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!}, \quad f^{(k)} = X^k f, \quad k = 0, 1, 2, \dots. \tag{1.6}$$

For instance, if $f_t = f$, i.e., $f' = 0$, then the function f remains constant on the trajectories of X and is called an *invariant* of X. In the case of n-dimensional manifold M the vector field X has $n - 1$ independent (*basic*) invariants. All other invariants of X can be presented as functions of the basic ones.

If $f_t = f + t$, i.e., $f' = 1$, then the function f is an antiderivative of 1 and is called a *canonical parameter* of X.

It can appear that in (1.6) the derivatives $f^{(k)}$ are somehow related and form an ordinary differential equation (ODE). Then the solution of an ODE gives the dragged function f_t immediately.

For example,

$$f'' + f = 0 \quad \Longrightarrow \quad f_t = f \cos t + f' \sin t.$$

So if the parameter t changes, then f_t describes the pulsation of f.

1.3 Differential forms

Let M be a smooth manifold of dimension n. Any skew-symmetric p-linear function on p vector fields on M is called an *exterior differential form of degree p* or simply *p-form*.

In-particular, a 1-form Φ on M is defined as a linear function $\Phi(X)$ with vector field X as its argument, or just as a covector field on M.

An *exterior derivative* of 1-form Φ is a 2-form defined by

$$\mathbf{d}\Phi(X, Y) = X(\Phi(Y)) - Y(\Phi(X)) - \Phi([X, Y]), \tag{1.7}$$

where X and Y are arbitrary vector fields on M. A *wedge product* of two 1-forms Φ and Ψ is a 2-form defined by

$$\Phi \wedge \Psi(X, Y) = \begin{vmatrix} \Phi(X) & \Phi(Y) \\ \Psi(X) & \Psi(Y) \end{vmatrix}. \tag{1.8}$$

A wedge product of three 1-forms Φ, Ψ and Θ is a 3-form defined by

$$\Phi \wedge \Psi \wedge \Theta(X, Y, Z) = \begin{vmatrix} \Phi(X) & \Phi(Y) & \Phi(Z) \\ \Psi(X) & \Psi(Y) & \Psi(Z) \\ \Theta(X) & \Theta(Y) & \Theta(Z) \end{vmatrix}, \tag{1.9}$$

where X, Y and Z are arbitrary vector fields on M. In such a way we can define the wedge product of any number of 1-forms.

Let $X_1, X_2, \dots, X_k$, $k \le n$, be a set of linearly independent vector fields on M. Then a set of 1-forms Φ_1, $\Phi_2, \dots, \Phi_k$ is said to be *linearly independent* if

$$\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_k(X_1, X_2, \dots, X_k) \neq 0.$$

Chapter 2

Lie derivatives in nonholonomic basis

The notion of the Lie derivative is coordinate-independent, but particular calculations are carried out in local coordinates. However, defining a Lie derivative of a tensor field, the following problem arises. As it was mentioned in the previous Chapter, any linear map $f : V_1 \to V_2$ transforms contravariant tensors from the vector space V_1 to the vector space V_2 (from left to right) and covariant tensors from V_2 to V_1 (from right to left). Unfortunately for tensors of a mixed type this claim is not true. Nevertheless, the key is f-related tensors in the spaces V_1 and V_2. Therefore we begin by defining φ-related tensor fields on smooth manifolds M_1 and M_2, where $\varphi : M_1 \to M_2$ is a smooth map. Then we consider the dragging of tensor fields along a flow of a vector field X and define the Lie derivatives.

We also develop Lie differentiation of tensor fields (including vector fields and differential forms) in a nonholonomic basis, where an important role is played by derivation formulas together with the nonholonomy object. The nonholonomy object appearing in computation formulas is a consequence of interaction of non-commuting basis operators, and allows us to apply this technique to the theory of Lie groups. In particular, in the case of a Lie group the structure constants are precisely nonholonomy objects of left-invariant and right-invariant bases, cf. Chapter 3.

In the end of the present Chapter is introduced the idea of integration of tensor field as a reverse process to the Lie differentiation. A few geometrical examples included in the text clarify the topic being discussed.

2.1 φ-related tensor fields

Let M_1 and M_2 be smooth manifolds, and let $\varphi : M_1 \to M_2$ be a smooth map.

Definition 2.1. Two functions f and $\widetilde{f}$ on M_1 and M_2, respectively, are said to be *φ-related*, if

$$f = \widetilde{f} \circ \varphi.$$

Definition 2.2. Two vector fields Y and $\widetilde{Y}$ on M_1 and M_2, respectively, are said to be *φ-related*, if for any φ-related differentiable functions f and $\widetilde{f}$ their derivatives Yf and $\widetilde{Y}\widetilde{f}$ are φ-related, i.e.,

$$Yf = (\widetilde{Y}\widetilde{f}) \circ \varphi,$$

or, equivalently, if for any smooth function g on M_2,

$$Y(g \circ \varphi) = (\widetilde{Y}g) \circ \varphi.$$

Definition 2.3. Two 1-forms Φ and $\widetilde{\Phi}$ on M_1 and M_2, respectively, are said to be *φ-related*, if for any φ-related vector fields Y and $\widetilde{Y}$ the values $\Phi(Y)$ and $\widetilde{\Phi}(\widetilde{Y})$ are φ-related, i.e.,

$$\Phi(Y) = \big(\widetilde{\Phi}(\widetilde{Y})\big) \circ \varphi.$$

Definition 2.4. Two tensor fields S and $\widetilde{S}$ of a mixed type (p, q) on M_1 and M_2, respectively, are said to be *φ-related*, if their values on any set of φ-related 1-forms $\Phi^1, \ldots, \Phi^p$ and $\widetilde{\Phi}^1, \ldots, \widetilde{\Phi}^p$, and φ-related vector fields $Y_1, \ldots, Y_q$ and $\widetilde{Y}_1, \ldots, \widetilde{Y}_q$ are φ-related, i.e.,

$$S(\Phi^1, \ldots, \Phi^p; Y_1, \ldots, Y_q) = (\widetilde{S}(\widetilde{\Phi}^1, \ldots, \widetilde{\Phi}^p; \widetilde{Y}_1, \ldots, \widetilde{Y}_q)) \circ \varphi.$$

Example 2.1. Let us consider a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : (u, v) \mapsto (x, y)$ defined by

$$\begin{cases} x \circ \varphi = \dfrac{1}{2}(u^2 + v^2), \\ y \circ \varphi = 2uv. \end{cases}$$

Let two functions $f = (u - v)^2$ and $\widetilde{f} = 2x - y$ be given on the uv and xy planes, respectively. Then

$$\widetilde{f} \circ \varphi = (2x - y) \circ \varphi = 2(x \circ \varphi) - y \circ \varphi = u^2 - 2uv + v^2 = (u - v)^2 = f.$$

Thus, from the Definition 2.1 it follows that f and $\widetilde{f}$ are φ-related.

Consider two vector fields

$$Y = v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} \quad \text{and} \quad \widetilde{Y} = y\frac{\partial}{\partial x} + 4x\frac{\partial}{\partial y}$$

on the uv and xy planes, respectively. Then from

$$Yf = 2(u - v)^2 = -2f \quad \text{and} \quad \widetilde{Y}\widetilde{f} = -2(2x - y) = -2\widetilde{f}$$

it follows that $Yf = (\widetilde{Y}\widetilde{f}) \circ \varphi$. But the last condition is not enough for Y and $\widetilde{Y}$ to be φ-related, because f and $\widetilde{f}$ are not arbitrary functions.

The flow a_t of Y is defined by solutions of the corresponding system of differential equations:

$$\begin{cases} \dot{u} = v \\ \dot{v} = u \end{cases} \implies \begin{cases} u_t = u\cosh t + v\sinh t \\ v_t = u\sinh t + v\cosh t \end{cases},$$

and the flow $\widetilde{a}_t$ of $\widetilde{Y}$ is defined by

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = 4x \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{l} x_t = x\cosh 2t + \dfrac{1}{2} y \sinh 2t \\ y_t = 2x \sinh 2t + y \cosh 2t \end{array} \right. .$$

The functions $I = u^2 - v^2$ and $\widetilde{I} = 4x^2 - y^2$ are invariants of Y and $\widetilde{Y}$, respectively, and related by $I^2 = \widetilde{I} \circ \varphi$. Thus, the trajectories of Y (hyperboles) on the uv plane are mapped by φ onto the trajectories of $\widetilde{Y}$ (hyperboles too) on the xy plane.

Consider two 1-forms Φ and $\widetilde{\Phi}$ given on the uv and xy planes, respectively:

$$\Phi = (u+v)(du+dv), \qquad \widetilde{\Phi} = dx + \frac{1}{2} dy.$$

From

$$\Phi(Y) = (u+v)^2, \quad \widetilde{\Phi}(\widetilde{Y}) = y + 2x \quad \text{and} \quad (y+2x)\circ\varphi = (u+v)^2$$

it follows that $\Phi(Y) = \big(\widetilde{\Phi}(\widetilde{Y})\big) \circ \varphi$. But again the last condition is not enough for Φ and $\widetilde{\Phi}$ to be φ-related, because first we need to show that Y and $\widetilde{Y}$ are φ-related (see Example 2.2).

2.2 Basic properties of φ-related tensor fields

A number of non-obvious properties of φ-related tensor fields follows immediately from the Definitions 2.1–2.4. Let us list these properties with some proofs.

P. 2.1. The *differential* of the map φ at a point $u \in M_1$ is defined as a linear map $T_u\varphi$ between tangent spaces as follows:

$$T_u\varphi : T_uM_1 \to T_vM_2 : Y_u \mapsto \widetilde{Y}_v,$$

where $v = \varphi(u) \in M_2$. The vector $\widetilde{Y}_v \in T_vM_2$, as an image of $Y_u \in T_uM_1$ at the point $v \in M_2$, acts on arbitrary differentiable function f defined on a neighborhood of v according to the rule

$$\widetilde{Y}_v f = Y_u(f \circ \varphi).$$

This means that if vector fields Y and $\widetilde{Y}$ given on M_1 and M_2, respectively, are φ-related, then the tangent map[1]

$$T\varphi : TM_1 \to TM_2,$$

which acts at each point $u \in M_1$ as $T_u\varphi$, maps all vectors of Y to the vectors of $\widetilde{Y}$ on the image $\varphi(M_1) \subset M_2$. In this case Y is called a *φ-projectable* onto the image $\varphi(M_1)$, i.e., $\widetilde{Y} = T\varphi(Y)$ on $\varphi(M_1) \subset M_2$.

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[1] The detailed description of the tangent functor, tangent map and levels is introduced in [2].

Over neighborhoods $U \subset M_1$ and $V \subseteq \varphi(U) \subset M_2$ with local coordinates u^i and v^α, $i = 1, \dots, \dim M_1; \alpha = 1, \dots, \dim M_2$, respectively, the map φ is determined by functions φ^α φ-related to v^α, i.e. $v^\alpha \circ \varphi = \varphi^\alpha$. Then the tangent map $T\varphi$ is determined locally by the *Jacobi matrix* $(\varphi_i^\alpha) = \left(\dfrac{\partial \varphi^\alpha}{\partial u^i}\right)$.

Locally, the vector fields Y and $\widetilde{Y}$ can be written as $Y = y^i \dfrac{\partial}{\partial u^i}$ and $\widetilde{Y} = \widetilde{y}^\alpha \dfrac{\partial}{\partial v^\alpha}$, where y^i and $\widetilde{y}^\alpha$ are smooth functions of coordinates u^i and v^α, respectively. Then Y and $\widetilde{Y}$ are φ-related if and only if $\widetilde{y}^\alpha \circ \varphi = Y\varphi^\alpha$ or

$$\widetilde{y}^\alpha \circ \varphi = \varphi_i^\alpha y^i.$$

Indeed, suppose that Y and $\widetilde{Y}$ are φ-related, then for any differentiable function f on M_2 with partial derivatives $f_\alpha = \dfrac{\partial f}{\partial v^\alpha}$ the equality $Y(f \circ \varphi) = (\widetilde{Y} f) \circ \varphi$ is valid. The last equality is also valid for coordinate functions v^α, i.e. $Y(v^\alpha \circ \varphi) = (\widetilde{Y} v^\alpha) \circ \varphi$ or $Y\varphi^\alpha = \widetilde{y}^\alpha \circ \varphi$. On the other hand, suppose that $\widetilde{y}^\alpha \circ \varphi = Y\varphi^\alpha$ is valid. Then from

$$\begin{aligned}(\widetilde{Y} f) \circ \varphi &= (f_\alpha \widetilde{y}^\alpha) \circ \varphi = (f_\alpha \circ \varphi)(\widetilde{y}^\alpha \circ \varphi) = (f_\alpha \circ \varphi) Y\varphi^\alpha = \\ &= Y(f \circ \varphi)\end{aligned}$$

it follows that the vector fields Y and $\widetilde{Y}$ are φ-related.

P. 2.2. The map $T\varphi$ from the Property 2.1 transforms any 1-form $\widetilde{\Phi}$ on M_2 into a 1-form Φ on M_1, which is φ-related to $\widetilde{\Phi}$. Therefore the φ-related 1-forms Φ and $\widetilde{\Phi}$ as real-valued functions on TM_1 and TM_2 are $T\varphi$-related, i.e.

$$\Phi = \widetilde{\Phi} \circ T\varphi.$$

Locally, suppose $\Phi = \xi_i du^i$ and $\widetilde{\Phi} = \widetilde{\xi}_\alpha dv^\alpha$, where ξ_i and ξ_α are smooth functions on corresponding neighborhoods on M_1 and M_2, respectively, and $i = 1, \dots, \dim M_1$, $\alpha = 1, \dots, \dim M_2$. Then 1-forms Φ and $\widetilde{\Phi}$ are φ-related if and only if their components are related by

$$\xi_i = (\widetilde{\xi}_\alpha \circ \varphi)\varphi_i^\alpha,$$

where (φ_i^α) is the Jacob matrix of φ.

Indeed, suppose that 1-forms Φ and $\widetilde{\Phi}$ are φ-related. Then for any φ-related vector fields $Y = y^i \dfrac{\partial}{\partial u^i}$ and $\widetilde{Y} = \widetilde{y}^\alpha \dfrac{\partial}{\partial v^\alpha}$ we have $\Phi(Y) = \widetilde{\Phi}(\widetilde{Y}) \circ \varphi$, $\Phi(Y) = \xi_i y^i$, $\widetilde{\Phi}(\widetilde{Y}) = \widetilde{\xi}_\alpha \widetilde{y}^\alpha$ and

$$\xi_i y^i = (\widetilde{\xi}_\alpha \widetilde{y}^\alpha) \circ \varphi = (\widetilde{\xi}_\alpha \circ \varphi)(\widetilde{y}^\alpha \circ \varphi) = (\widetilde{\xi}_\alpha \circ \varphi)\varphi_i^\alpha y^i.$$

Example 2.2. Let us consider the situation from the Example 2.1. The Jacobi matrix of the map φ is

$$J = \begin{pmatrix} u & v \\ 2v & 2u \end{pmatrix}.$$

It is easy to see that the components of the vector fields Y and $\widetilde{Y}$ are related by

$$\begin{pmatrix} y \\ 4x \end{pmatrix}_{\circ\varphi} = \begin{pmatrix} u & v \\ 2v & 2u \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix}.$$

Thus, from the Property 2.1 it follows that Y and $\widetilde{Y}$ are φ-related.

Analogously, the components of the 1-forms Φ and $\widetilde{\Phi}$ are related by

$$\begin{pmatrix} u+v & u+v \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \end{pmatrix}_{\circ\varphi} \cdot \begin{pmatrix} u & v \\ 2v & 2u \end{pmatrix},$$

and from the Property 2.2 it follows that Φ and $\widetilde{\Phi}$ are φ-related.

P. 2.3. For any φ-related tensor fields S and $\widetilde{S}$ their tensors at the points $u \in M_1$ and $v = \varphi(u) \in M_2$, respectively, are $T_u\varphi$-related.

P. 2.4. For any functions f and g on M_2,

$$\begin{aligned} (f \pm g) \circ \varphi &= (f \circ \varphi) \pm (g \circ \varphi), \\ (f \cdot g) \circ \varphi &= (f \circ \varphi) \cdot (g \circ \varphi), \\ \left(\frac{f}{g}\right) \circ \varphi &= \frac{f \circ \varphi}{g \circ \varphi} \quad (g \neq 0). \end{aligned}$$

It means that the same arithmetic operations (addition, substraction, multiplications and division) over φ-related functions lead again to φ-related functions.

P. 2.5. The same linear combinations of φ-related tensor fields with φ-related coefficients are φ-related. For instance, let α and β be φ-related functions, i.e., $\alpha = \beta \circ \varphi$. Then for any φ-related vector fields Y and $\widetilde{Y}$ (1-forms Φ and $\widetilde{\Phi}$, *affinor fields* (tensor fields of type $(1,1)$) S and $\widetilde{S}$) vector fields αY and $\beta\widetilde{Y}$ (1-forms $\alpha\Phi$ and $\beta\widetilde{\Phi}$, affinor fields αS and $\beta\widetilde{S}$) are also φ-related, i.e.,

$$\begin{aligned} Y(f \circ \varphi) = (\widetilde{Y} f) \circ \varphi \quad &\Longrightarrow \quad (\alpha Y)(f \circ \varphi) = \big[(\beta\widetilde{Y})f\big] \circ \varphi, \\ \Phi(Y) = \big[\widetilde{\Phi}(\widetilde{Y})\big] \circ \varphi \quad &\Longrightarrow \quad (\alpha\Phi)(Y) = \big[(\beta\widetilde{\Phi})(\widetilde{Y})\big] \circ \varphi, \\ S(\Phi; Y) = \big[\widetilde{S}(\widetilde{\Phi}; \widetilde{Y})\big] \circ \varphi \quad &\Longrightarrow \quad (\alpha S)(\Phi; Y) = \big[(\beta\widetilde{S})(\widetilde{\Phi}; \widetilde{Y})\big] \circ \varphi, \end{aligned}$$

where f is an arbitrary function on M_2. Thus, it is possible to form the same linear combinations of any number of φ-related vector fields, 1-forms and, in general, tensor fields of a type (p, q) with corresponding φ-related coefficients, and in the result we have φ-related tensor fields.

For instance, for any φ-related functions α^i and $\widetilde{\alpha}^i$, functions β_i and $\widetilde{\beta}_i$, vector fields Y_i and $\widetilde{Y}_i$, and 1-forms Φ^i and $\widetilde{\Phi}^i$, where $i = 1, 2, ..., n$, $n \in \mathbb{N}$, we have[2]

$$\begin{aligned} \alpha^i = \widetilde{\alpha}^i \circ \varphi, \quad Y_i(f \circ \varphi) = (\widetilde{Y}_i f) \circ \varphi \quad &\Longrightarrow \quad (Y_i \alpha^i)(f \circ \varphi) = \big[(\widetilde{Y}_i \widetilde{\alpha}^i) f\big] \circ \varphi \\ \beta_i = \widetilde{\beta}_i \circ \varphi, \quad \Phi^i(Y) = (\widetilde{\Phi}^i(\widetilde{Y})) \circ \varphi \quad &\Longrightarrow \quad (\beta_i \Phi^i)(Y) = \big[(\widetilde{\beta}_i \widetilde{\Phi}^i)(\widetilde{Y})\big] \circ \varphi. \end{aligned}$$

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[2] Here we use the rule: *summation excludes differentiation*, see Introduction.

P. 2.6. The same tensor operations (tensor product, contraction, symmetrization and alternation) over φ-related tensor fields on M_1 and M_2 lead again to φ-related tensor fields.

For instance, the wedge products $\Phi \wedge \Psi$ and $\widetilde{\Phi} \wedge \widetilde{\Psi}$ of φ-related 1-forms on M_1 and M_2 are φ-related. Indeed, from (1.8) and Property 2.4 it follows

$$\begin{aligned}(\Phi \wedge \Psi)(Y, Z) &= \Phi(Y)\Psi(Z) - \Phi(Z)\Psi(Y) = \\ &= \big(\widetilde{\Phi}(\widetilde{Y}) \circ \varphi\big)\big(\widetilde{\Psi}(\widetilde{Z}) \circ \varphi\big) - \big(\widetilde{\Phi}(\widetilde{Z}) \circ \varphi\big)\big(\widetilde{\Psi}(\widetilde{Y}) \circ \varphi\big) = \\ &= \big(\widetilde{\Phi}(\widetilde{Y})\widetilde{\Psi}(\widetilde{Z})\big) \circ \varphi - \big(\widetilde{\Phi}(\widetilde{Z})\widetilde{\Psi}(\widetilde{Y})\big) \circ \varphi = \\ &= \big(\widetilde{\Phi}(\widetilde{Y})\widetilde{\Psi}(\widetilde{Z}) - \widetilde{\Phi}(\widetilde{Z})\widetilde{\Psi}(\widetilde{Y})\big) \circ \varphi = \\ &= \big[(\widetilde{\Phi} \wedge \widetilde{\Psi})(\widetilde{Y}, \widetilde{Z})\big] \circ \varphi,\end{aligned}$$

where Y, Z and $\widetilde{Y}$, $\widetilde{Z}$ are φ-related vector fields on M_1 and M_2, respectively.

P. 2.7. The differentials of φ-related functions f and $\widetilde{f}$, as 1-forms on M_1 and M_2, respectively, are φ-related and, as scalar functions on TM_1 and TM_2, respectively, are $T\varphi$-related,

$$f = \widetilde{f} \circ \varphi \quad \Longrightarrow \quad df = d\widetilde{f} \circ T\varphi.$$

The differential of a function f is defined by

$$df(X) = Xf,$$

where X is an arbitrary vector field on a manifold M. Thus, for any φ-related vector fields X and $\widetilde{X}$ we have

$$f = \widetilde{f} \circ \varphi, \quad Xf = (\widetilde{X}\widetilde{f}) \circ \varphi \quad \Longrightarrow \quad df(X) = \big[d\widetilde{f}(\widetilde{X})\big] \circ \varphi,$$

which implies that df and $d\widetilde{f}$ are indeed φ-related.

Note that the same result follows from the *chain rule* of the composition of tangent maps:

$$f = \widetilde{f} \circ \varphi \quad \Longrightarrow \quad Tf = T\widetilde{f} \circ T\varphi.$$

Example 2.3. Consider the φ-related functions $f = (u - v)^2$ and $\widetilde{f} = 2x - y$ from the Example 2.1, where the Jacobi matrix of the map φ is $J = \begin{pmatrix} u & v \\ 2v & 2u \end{pmatrix}$. The differentials of the functions f and $\widetilde{f}$ are

$$df = 2(u - v)(du - dv), \qquad d\widetilde{f} = 2dx - dy.$$

Thus, from the equality

$$\begin{pmatrix} 2(u - v) & 2(u - v) \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} u & v \\ 2v & 2u \end{pmatrix}$$

and from the Property 2.2 it follows that df and $d\widetilde{f}$ are φ-related.

P. 2.8. For any φ-related vector fields Y, $\widetilde{Y}$ and Z, $\widetilde{Z}$ on M_1 and M_2, respectively, the Lie brackets $[Y, Z]$ and $[\widetilde{Y}, \widetilde{Z}]$ are also φ-related.

Indeed, for φ-related functions f and $\widetilde{f}$ on M_1 and $M2$, respectively, and by assumption we have $f = \widetilde{f} \circ \varphi$, $Yf = (\widetilde{Y}\widetilde{f}) \circ \varphi$ and $Zf = (\widetilde{Z}\widetilde{f}) \circ \varphi$. From (1.5) and Property 2.4 it follows

$$\begin{aligned}[Y, Z]f &= (YZ - ZY)f = Y(Zf) - Z(Yf) = Y\big((\widetilde{Z}\widetilde{f}) \circ \varphi\big) - Z\big((\widetilde{Y}\widetilde{f}) \circ \varphi\big) = \\ &= \big(\widetilde{Y}(\widetilde{Z}\widetilde{f})\big) \circ \varphi - \big(\widetilde{Z}(\widetilde{Y}\widetilde{f})\big) \circ \varphi = \big((\widetilde{Y}\widetilde{Z} - \widetilde{Z}\widetilde{Y})\widetilde{f}\big) \circ \varphi = \\ &= \big([\widetilde{Y}, \widetilde{Z}]\widetilde{f}\big) \circ \varphi.\end{aligned}$$

P. 2.9. The exterior differentials of φ-related p-forms Θ and $\widetilde{\Theta}$ are φ-related.

For instance, for φ-related 1-forms Φ and $\widetilde{\Phi}$ the equality (1.7), Property 2.5 and Property 2.8 imply

$$\begin{aligned}\mathbf{d}\Phi(Y, Z) &= Y\big(\Phi(Z)\big) - Z\big(\Phi(Y)\big) - \Phi([Y, Z]) = \\ &= Y\big(\widetilde{\Phi}(\widetilde{Z}) \circ \varphi\big) - Z\big(\widetilde{\Phi}(\widetilde{Y}) \circ \varphi\big) - \Phi([\widetilde{Y}, \widetilde{Z}] \circ \varphi) = \\ &= \big[\widetilde{Y}\big(\widetilde{\Phi}(\widetilde{Z})\big)\big] \circ \varphi - \big[\widetilde{Z}\big(\widetilde{\Phi}(\widetilde{Y})\big)\big] \circ \varphi - \big[\widetilde{\Phi}([\widetilde{Y}, \widetilde{Z}])\big] \circ \varphi = \\ &= \big[\mathbf{d}\widetilde{\Phi}(\widetilde{Y}, \widetilde{Z})\big] \circ \varphi,\end{aligned}$$

where Y, $\widetilde{Y}$ and Z, $\widetilde{Z}$ are arbitrary pairs of φ-related vector fields on M_1 and M_2, respectively.

P. 2.10. For any φ-related differentiable functions f and $\widetilde{f}$,

$$d\widetilde{f} = 0 \quad \Longrightarrow \quad df = 0,$$

but the opposite is not true.

More precisely, if the function $\widetilde{f}$ is constant on a manifold M_2, then also the function f is constant on a manifold M_1. But if f is constant on M_1, then $\widetilde{f}$ is constant on the image $\varphi(M_1) \subset M_2$, not necessarily on the whole manifold M_2.

P. 2.11. For any φ-related p-forms Θ and $\widetilde{\Theta}$,

$$\mathbf{d}\widetilde{\Theta} = 0 \quad \Longrightarrow \quad \mathbf{d}\Theta = 0,$$

i.e., if $\widetilde{\Theta}$ is closed p-form, then Θ is also closed. But it is possible that not closed p-form $\widetilde{\Theta}$ on M_2 is transformed into a closed p-form Θ on M_1, as the next Example shows.

Example 2.4 (see [2], p. 60)**.** Suppose a map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$(u, v, w) \circ \varphi = \big(\frac{1}{2}(x^2 + y^2 - z^2), y - kz, 1 - k^2\big),$$

where $k \in \mathbb{R}$. Consider two 1-forms

$$\Theta = xdx + (y + 2)dy - (z + 2k)dz \quad \text{and} \quad \widetilde{\Theta} = du + 2dv - vdw$$

in the spaces xyz and uvw, respectively. From the equality

$$\begin{pmatrix} x & y+2 & -(z+2k) \end{pmatrix} = \begin{pmatrix} 1 & 2 & -v \end{pmatrix}_{\circ\varphi} \begin{pmatrix} x & y & -z \\ 0 & 1 & -k \\ 0 & 0 & 0 \end{pmatrix}$$

it follows that the local condition from Property 2.2 for 1-forms Θ and $\widetilde{\Theta}$ being φ-related is fulfilled, i.e. $\Theta = \widetilde{\Theta} \circ T\varphi$. The exterior derivatives are $\mathbf{d}\widetilde{\Theta} = dv \wedge dw \neq 0$ and $\mathbf{d}\Theta = 0$, which mean that $\widetilde{\Theta}$ is not closed 1-form and Θ is closed. Thus, it is possible, that the tangent map $T\varphi$ transforms not closed 1-form into the closed one.

P. 2.12. For any submersion φ and for any pair of φ-related vector fields Y, $\widetilde{Y}$ and Z, $\widetilde{Z}$ we have

$$[Y, Z] = 0 \quad \Longrightarrow \quad [\widetilde{Y}, \widetilde{Z}] = 0,$$

and for any immersion φ we have

$$[\widetilde{Y}, \widetilde{Z}] = 0 \quad \Longrightarrow \quad [Y, Z] = 0.$$

Example 2.5 (From the Connection theory, see [2], p. 82)**.** The horizontal vector fields

$$X_i = \partial_i + \Gamma_i^\alpha \partial_\alpha$$

are π-projectible onto the base into commuting partial differentiation operators ∂_i, i.e., $T\pi X_i = \partial_i$. We have $[\partial_i, \partial_j] = 0$, while the Lie brackets of X_i determine the curvature of connection, which is in general non-zero:

$$[X_i, X_j] = (X_i \Gamma_j^\alpha - X_j \Gamma_i^\alpha)\partial_\alpha.$$

P. 2.13. After we have defined the Lie derivative, the following important property will be added to this list: *The Lie derivatives $\mathcal{L}_X S$ and $\mathcal{L}_{\widetilde{X}} \widetilde{S}$ of φ-related tensor fields S and $\widetilde{S}$ with respect to φ-related vector fields X and $\widetilde{X}$ are φ-related.*

2.3 *a*-related tensor fields

Let M be a smooth manifold and $a : M \to M$ be a diffeomorphism. Then for any functions, vector fields, 1-forms and, in general, for any tensor fields defined on M there exist uniquely defined functions, vector fields, 1-forms and tensor fields on M that are a-related to the first ones, respectively. Let f be a smooth function on M. Let us denote a function which is a-related to f by the symbol $\hat{a}f$, i.e., $\hat{a}f = f \circ a$. Analogously, let a vector field Y, 1-form Φ and tensor field S be a-related to a vector field $\hat{a}Y$, 1-form $\hat{a}\Phi$ and tensor field $\hat{a}S$, respectively. From the Definitions 2.1–2.4

and Properties 2.1–2.13 we obtain the following ten formulas:

$$\begin{aligned}
\hat{a}f &= f \circ a, && (2.1)\\
\hat{a}(Yf) &= (\hat{a}Y)(\hat{a}f), && (2.2)\\
\hat{a}(\Phi(Y)) &= (\hat{a}\Phi)(\hat{a}Y), && (2.3)\\
\hat{a}(Y_i y^i) &= (\hat{a}Y_i)(\hat{a}y^i), && (2.4)\\
\hat{a}(\varphi_i \Phi^i) &= (\hat{a}\varphi_i)(\hat{a}\Phi^i), && (2.5)\\
\hat{a}(df) &= d(\hat{a}f), && (2.6)\\
\hat{a}(\mathbf{d}\Theta) &= \mathbf{d}(\hat{a}\Theta), && (2.7)\\
\hat{a}[Y,Z] &= [\hat{a}Y, \hat{a}Z], && (2.8)\\
\hat{a}(\mathcal{L}_Y S) &= \mathcal{L}_{\hat{a}Y}(\hat{a}S), && (2.9)\\
\hat{a}\big[S(\Phi^1, ..., \Phi^p; Y_1, ..., Y_q)\big] &= (\hat{a}S)(\hat{a}\Phi^1, ..., \hat{a}\Phi^p; \hat{a}Y_1, ..., \hat{a}Y_q). && (2.10)
\end{aligned}$$

2.4 Lie derivative

Now we are in a position to define Lie derivatives on a manifold.

Let X be a smooth vector field on M. With X there is associated a flow $a_t = \exp tX$ as a one-parameter group of transformations

$$a_t : M \to M.$$

In the case of the family of diffeomorphisms a_t, the expressions

$$\hat{a}_t f,\ \hat{a}_t Y,\ \hat{a}_t \Phi,\ \hat{a}_t(Y_i y^i),\ \hat{a}_t(\varphi_i \Phi^i),\ \hat{a}_t(df),\ \hat{a}_t(\mathbf{d}\Theta),\ \hat{a}_t[Y,Z],\ \hat{a}_t(\mathcal{L}_Y S),\ \hat{a}_t S$$

in (2.1)–(2.10) have sense and describe the dragging of the corresponding tensor fields along the flow of X. Therefore, each tensor field S of a type (p,q) on M may be dragged along by the flow a_t for each value of t to define a one-parameter family of tensor fields, indicated by the abbreviation

$$S \mapsto S_t = \hat{a}_t S.$$

The last relation describes the change of the tensor S_t at each point $u \in M$. Thus, it is possible to value the velocity of this change according to the formula

$$\boxed{S' = \lim_{t \to 0} \frac{S_t - S}{t}}, \qquad (2.11)$$

called a *Lie derivative* of S with respect to X. The tensor fields S and S' are of the same type.

The Lie derivative of S along the flow of X is usually denoted by the symbol $\mathcal{L}_X S$. But for the sake of convenience we use primes in order to denote the Lie derivative with respect to the *fixed* vector field X.

Now one can consider the operator

$$\hat{a}_t = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}.$$

Being applied to S, this operator determines the dragging of S along the flow a_t by[3]

$$S_t = \sum_{k=0}^{\infty} S^{(k)} \frac{t^k}{k!}. \tag{2.12}$$

The series expansion given by (2.12) is called a *Lie-Maclaurin series* with Lie derivatives $S^{(k)} = \mathcal{L}_X^{(k)} S$, $k = 0, 1, 2, \ldots$, as coefficients.

If the derivatives S, S', S'', … are somehow interconnected, then one can speak about ordinary differential equations (ODEs) in the flow a_t. For a such ODE, the solution is (2.12). On the other hand, the solution S_t of ODE describes the dragging of the field S along the flow of X. If in the series (2.12) $S^{(n)}$ is an invariant of X, i.e. $S^{(n+1)} = 0$, then S behaves along the flow a_t as a polynomial of the order n. For instance, for $n = 1$ the field S behaves as a linear function: $S_t = S + tS'$, for $n = 2$ – as a quadratic one: $S_t = S + tS' + \frac{t^2}{2} S''$, and s.o.

Example 2.6. Consider the ODE

$$S'' + S = 0.$$

Let us compute the solution of this ODE using Lie-Maclaurin series (2.12). Note that

$$S'' = -S, \quad S''' = -S', \quad S^{(IV)} = S, \quad S^{(V)} = S', \quad S^{(VI)} = S, \quad \ldots.$$

Thus, the series (2.12) is

$$\begin{aligned} S_t &= S + S't + S''\frac{t^2}{2} + S'''\frac{t^3}{3!} + S^{(IV)}\frac{t^4}{4!} + S^{(V)}\frac{t^5}{5!} + \ldots = \\ &= S + S't - S\frac{t^2}{2} - S'\frac{t^3}{3!} + S\frac{t^4}{4!} + S'\frac{t^5}{5!} - S\frac{t^6}{6!} - S'\frac{t^7}{7!} + \ldots = \\ &= S\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right) + S'\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right) = \\ &= S\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!} + S'\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!} = \\ &= S\cos t + S'\sin t. \end{aligned}$$

It means that the field S exhibits oscillation, i.e., at the fixed point $u \in M$ one can consider a periodic change of the field S.

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[3] The convergence is assumed.

From (2.1)–(2.10) one can obtain the main computation formulas for Lie derivatives. We summarize them in the following Proposition.

Proposition 2.1.

$$\mathcal{L}_X f = Xf, \tag{2.13}$$

$$\mathcal{L}_X Y = [X,Y], \tag{2.14}$$

$$X\big[\Phi(Y)\big] = (\mathcal{L}_X\Phi)(Y) + \Phi(\mathcal{L}_X Y), \tag{2.15}$$

$$\mathcal{L}_X\Phi = \boldsymbol{d}\Phi(X,.) + d\big[\Phi(X)\big],$$

$$\mathcal{L}_X(Y_i y^i) = (\mathcal{L}_X Y_i)y^i + Y_i(Xy^i), \tag{2.16}$$

$$\mathcal{L}_X(\varphi_i\Phi^i) = (X\varphi_i)\Phi^i + \varphi_i(\mathcal{L}_X\Phi^i), \tag{2.17}$$

$$\mathcal{L}_X(df) = d(Xf), \tag{2.18}$$

$$\mathcal{L}_X(\boldsymbol{d}\Theta) = \boldsymbol{d}(\mathcal{L}_X\Theta), \tag{2.19}$$

$$\mathcal{L}_X[Y,Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z], \tag{2.20}$$

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X, \tag{2.21}$$

$$\begin{aligned}\mathcal{L}_X S(\Phi^1,\dots,\Phi^p;Y_1,\dots,Y_q) &= X\big[S(\Phi^1,\dots,\Phi^p;Y_1,\dots,Y_q)\big]- \\ &- \sum_{i=1}^{p} S(\Phi^1,\dots,\mathcal{L}_X\Phi^i,\dots,\Phi^p;Y_1,\dots,Y_q)- \\ &- \sum_{j=1}^{q} S(\Phi^1,\dots,\Phi^p;Y_1,\dots,\mathcal{L}_X Y_j,\dots,Y_q).\end{aligned} \tag{2.22}$$

Proof. From (2.1) or $f_t = f\circ a_t$ and from(2.11) it follows that

$$\mathcal{L}_X f = \lim_{t\to 0}\frac{f_t - f}{t} = (f\circ a_t)'_{t=0} = Xf.$$

Thus, the Lie derivative of a function f is the ordinary derivative (2.13) along the vector field X.

(2.14) is obtained by differentiating (2.2) with respect to t and setting $t = 0$. Thus, from (2.11) and $(Yf)_t = Y_t f_t$ we have

$$\begin{aligned} X(Yf) &= \lim_{t\to 0}\frac{(Yf)_t - Yf}{t} = \lim_{t\to 0}\frac{Y_t f_t - Yf}{t} = \\ &= \lim_{t\to 0}\frac{Y_t f_t - Y_t f + Y_t f - Yf}{t} = \\ &= \lim_{t\to 0}\frac{Y_t(f_t - f) + (Y_t - Y)f}{t} = \\ &= \lim_{t\to 0}\frac{Y_t(f_t - f)}{t} + \lim_{t\to 0}\frac{(Y_t - Y)f}{t} = \\ &= (\mathcal{L}_X)Yf + Y(Xf).\end{aligned}$$

The last equality can be rewritten as

$$(Yf)' = Y'f + Yf',$$

which means that the Lie derivative of Yf satisfies the Leibniz rule. Since f is an arbitrary differentiable function, it follows immediately that

$$\mathcal{L}_X Y = XY - YX.$$

Thus, the Lie derivative of Y with respect to X is equal to the ordinary Lie bracket

$$\mathcal{L}_X Y = [X, Y].$$

The formulas (2.15), (2.16) and (2.17) can be proved in the analogous way. Namely, using the equality $(\Phi(Y))_t = \Phi_t(Y_t)$ we compute

$$\begin{aligned}\mathcal{L}_X\big(\Phi(Y)\big) &= \lim_{t\to 0}\frac{\big(\Phi(Y)\big)_t - \Phi(Y)}{t} = \lim_{t\to 0}\frac{\Phi_t(Y_t) - \Phi(Y)}{t} = \\ &= \lim_{t\to 0}\frac{\Phi_t(Y_t) - \Phi_t(Y) + \Phi_t(Y) - \Phi(Y)}{t} = \\ &= \lim_{t\to 0}\frac{\Phi_t(Y_t - Y) + (\Phi_t - \Phi)(Y)}{t} = \\ &= \lim_{t\to 0}\frac{\Phi_t(Y_t - Y)}{t} + \lim_{t\to 0}\frac{(\Phi_t - \Phi)(Y)}{t} = \\ &= (\mathcal{L}_X\Phi)(Y) + \Phi(\mathcal{L}_X Y),\end{aligned}$$

which means that the Lie derivative of $\Phi(Y)$ satisfies the Leibniz rule

$$\big(\Phi(Y)\big)' = \Phi'(Y) + \Phi(Y').$$

Comparing (2.15) with formula (1.7) for the exterior differential of 1-form Φ:

$$\begin{aligned}(\mathcal{L}_X\Phi)(Y) &= X(\Phi(Y)) - \Phi([X, Y]), \\ \mathbf{d}\Phi(X, Y) &= X(\Phi(Y)) - Y(\Phi(X)) - \Phi([X, Y]),\end{aligned}$$

and taking in account, that $Y(\Phi(X)) = d(\Phi(X))(Y)$, and omitting an arbitrary argument Y, we obtain the computation formula (2.15) for the Lie derivative of 1-form:

$$\mathcal{L}_X\Phi = \mathbf{d}\Phi(X, .) + d(\Phi(X)).$$

In the same way from (2.4) and (2.5) one can derive the formulas (2.16) and (2.17) for Lie derivatives of linear combinations of vector fields and 1-forms, respectively.

The formula (2.18) follows from $(df)_t = df_t$, i.e., Lie derivative and differential (of a function) commute. Briefly, $(df)' = df'$. This formula can be also obtained from (2.15) by setting $\Phi = df$, $\Phi(X) = f'$ and $\mathbf{d}\Phi = 0$.

Analogously, from (2.7) or $(\mathbf{d}\Theta)_t = \mathbf{d}\Theta_t$ it follows that $\mathcal{L}_X(\mathbf{d}\Theta) = \mathbf{d}(\mathcal{L}_X\Theta)$. Thus, the Lie derivative and exterior differential (of p-form) commute, i.e.,

$$\mathcal{L}_X\mathbf{d} = \mathbf{d}\,\mathcal{L}_X.$$

The formula (2.20) can be calculated using (2.11) and the equality $[Y,Z]_t = [Y_t, Z_t]$:

$$\begin{aligned}
\mathcal{L}_X[Y,Z] &= \lim_{t\to 0}\frac{[Y,Z]_t - [Y,Z]}{t} = \lim_{t\to 0}\frac{[Y_t,Z_t]-[Y,Z]}{t} = \\
&= \lim_{t\to 0}\frac{Y_tZ_t - Z_tY_t - YZ + ZY}{t} = \\
&= \lim_{t\to 0}\frac{Y_tZ_t - YZ}{t} - \lim_{t\to 0}\frac{Z_tY_t - ZY}{t} = \\
&= \lim_{t\to 0}\frac{Y_tZ_t - Y_tZ + Y_tZ - YZ}{t} - \lim_{t\to 0}\frac{Z_tY_t - Z_tY + Z_tY - ZY}{t} = \\
&= \lim_{t\to 0}\frac{Y_t(Z_t - Z) + (Y_t - Y)Z}{t} - \lim_{t\to 0}\frac{Z_t(Y_t - Y) + (Z_t - Z)Y}{t} = \\
&= \lim_{t\to 0}\frac{Y_t(Z_t - Z)}{t} + \lim_{t\to 0}\frac{(Y_t - Y)Z}{t} - \lim_{t\to 0}\frac{Z_t(Y_t - Y)}{t} - \lim_{t\to 0}\frac{(Z_t - Z)Y}{t} = \\
&= Y(\mathcal{L}_XZ) + (\mathcal{L}_XY)Z - Z(\mathcal{L}_XY) - (\mathcal{L}_XZ)Y.
\end{aligned}$$

Using (2.14), the last equality is

$$\big[X,[Y,Z]\big] = Y[X,Z] + [X,Y]Z - Z[X,Y] - [X,Z]Y = \big[Y,[X,Z]\big] + \big[[X,Y],Z\big],$$

which is the *Jacobi identity* for vector fields X, Y and Z:

$$\big[X,[Y,Z]\big] + \big[Z,[X,Y]\big] + \big[Y,[Z,X]\big] = 0.$$

The formula (2.21) is obtained by using $(\mathcal{L}_YS)_t = \mathcal{L}_{Y_t}(S_t)$, $(\mathcal{L}_{Y_t}S)'_{t=0} = \mathcal{L}_{Y'}S$ and the fact that the map $X \mapsto \mathcal{L}_X$ is additive and homogenous with respect to the scalar multiplication:

$$\begin{aligned}
\mathcal{L}_X(\mathcal{L}_YS) &= \lim_{t\to 0}\frac{(\mathcal{L}_YS)_t - \mathcal{L}_YS}{t} = \lim_{t\to 0}\frac{\mathcal{L}_{Y_t}S_t - \mathcal{L}_YS}{t} = \\
&= \lim_{t\to 0}\frac{\mathcal{L}_{Y_t}S_t - \mathcal{L}_{Y_t}S + \mathcal{L}_{Y_t}S - \mathcal{L}_YS}{t} = \\
&= \lim_{t\to 0}\frac{\mathcal{L}_{Y_t}(S_t - S) + (\mathcal{L}_{Y_t} - \mathcal{L}_Y)S}{t} = \\
&= \lim_{t\to 0}\frac{\mathcal{L}_{Y_t}(S_t - S)}{t} + \lim_{t\to 0}\frac{(\mathcal{L}_{Y_t} - \mathcal{L}_Y)S}{t} = \mathcal{L}_Y(\mathcal{L}_XS) + \mathcal{L}_{\mathcal{L}_XY}S \\
&= \mathcal{L}_Y(\mathcal{L}_XS) + \mathcal{L}_{[X,Y]}S.
\end{aligned}$$

Since S is an arbitrary differentiable tensor field, it follows immediately that

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X.$$

Omitting an arbitrary argument Z in (2.20), it is easy to see that the formulas (2.20) and (2.21) are equivalent.

The last formula (2.22) is obtained by applying the Leibniz rule to the Lie derivative of $S(\Phi^1,\dots,\Phi^p;Y_1,\dots,Y_q)$. As the result we obtain $p+q+1$ terms in (2.22). ■

Remark 2.1. Let Y be a vector field on a manifold M, and let b_τ be its flow. Then for any differentiable function f on M,

$$b_\tau = \exp \tau Y \quad \Longleftrightarrow \quad Yf = (f \circ b_\tau)'_{\tau=0}.$$

Let $a : M \to M$ be a diffeomorphism of M. Then the flow b_τ is transformed into the flow $ab_\tau a^{-1}$ and Y is transformed into TaY, i.e.,

$$ab_\tau a^{-1} = \exp \tau(TaY) \quad \Longleftrightarrow \quad TaYf = (f \circ ab_\tau a^{-1})'_{\tau=0}$$

Indeed, from

$$(f \circ ab_\tau a^{-1})'_{\tau=0} = (f \circ a \circ b_\tau)'_{\tau=0} \circ a^{-1} = (Y(f \circ a)) \circ a^{-1}$$

it follows that

$$TaYf = (Y(f \circ a)) \circ a^{-1}.$$

The last equality determines the vector field TaY.

Geometrically, two points on the trajectory of Y are moved by a into the points on the trajectory of TaY:

$$(u, b_\tau(u)) \mapsto (a(u), ab_\tau a^{-1}(a(u))).$$

It means that the flow b_τ is affected by an interior automorphism of the group of diffeomorphisms on M, i.e.,

$$b_\tau \mapsto ab_\tau a^{-1}.$$

2.5 Nonholonomic basis

Let M be an n-dimensional smooth manifold. Suppose that on M there given a *nonholonomic basis*[4] consisting of n vector fields R_j and n 1-forms θ^i, that form a frame and dual coframe in a tangent space T_uM, $\forall u \in M$, i.e., $\theta_i(R^j) = \delta_i^j$. Denote by

$$R = \begin{pmatrix} R_1 & R_2 & \dots & R_n \end{pmatrix}, \qquad \Theta = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \vdots \\ \theta^n \end{pmatrix}.$$

Then the nonholonomic basis in matrix notations is (R, Θ) with condition $\Theta(R) = E$, where E is the unit matrix of order n.

With the basis (R, Θ) a *nonholonomy object* c^i_{jk}, which appears in the structure equations

$$[R_j, R_k] = R_i c^i_{jk}, \qquad \mathbf{d}\theta^i = -\frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k, \tag{2.23}$$

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[4] The term *nonholonomic basis* means *non-coordinate basis*. Such a basis can be determined on a completely parallelizable manifold M. If M is not completely parallelizable, we assume that our formulas hold on a completely parallelizable domain of M.

is associated. From skew-symmetry of the Lie brackets follows the skew-symmetry of the nonholonomy object:

$$c^i_{jk} = -c^i_{kj}.$$

Using (1.7) and (1.8), we have

$$\begin{aligned} c^i_{pq}\theta^p \wedge \theta^q(R_j, R_k) &= c^i_{pq}\big(\theta^p(R_j)\theta^q(R_k) - \theta^p(R_k)\theta^q(R_j)\big) = \\ &= c^i_{pq}(\delta^p_j\delta^q_k - \delta^p_k\delta^q_j) = c^i_{jk} - c^i_{kj} = \\ &= 2c^i_{jk}, \end{aligned}$$

and duality of the equations (2.23) follows from

$$\begin{aligned} \mathbf{d}\theta^i(R_j, R_k) &= R_j\big(\theta^i(R_k)\big) - R_k\big(\theta^i(R_j)\big) - \theta^i\big([R_j, R_k]\big) = \\ &= -\theta^i([R_j, R_k]) = -c^i_{jk} = \\ &= -\frac{1}{2}c^i_{pq}\theta^p \wedge \theta^q(R_j, R_k). \end{aligned}$$

In the dual basis (R, Θ) a general tensor field S of a type (p, q) is defined by

$$S = R_{i_1} \otimes \cdots \otimes R_{i_p} s^{i_1 \dots i_p}_{j_1 \dots j_q} \theta^{j_1} \otimes \cdots \otimes \theta^{j_q},$$

where the coefficients $s^{i_1 \dots i_p}_{j_1 \dots j_q}$ are determined by

$$s^{i_1 \dots i_p}_{j_1 \dots j_q} = S(\theta^{i_1}, \dots, \theta^{i_p}; R_{j_1}, \dots, R_{j_q}).$$

For further needs we write two vector fields X and Y, and a 1-form Φ in the basis (R, Θ) as follows:

$$\begin{aligned} X &= R_i x^i, \quad x^i = \theta^i(X), \\ Y &= R_i y^i, \quad y^i = \theta^i(Y), \\ \Phi &= \varphi_i \theta^i, \quad \varphi_i = \Phi(R_i). \end{aligned}$$

It is also convenient to present these formulas in matrix notation:

$$X = Rx, \quad Y = Ry, \quad \Phi = \varphi\Theta,$$

where x and y are column-matrix of components x^i and y^i, respectively, and φ is a raw-matrix of components φ_i.

In the case of *holonomic basis* the nonholonomic object c^i_{jk} in (2.23) is equal to zero. Then 1-forms θ^i are closed and locally exact. Moreover, on some coordinate neighborhood $U \subset M$ 1-forms θ^i are equal to the differentials of some functions u^i, which can be declared as coordinate ones. Thus, we obtain a *natural basis* related to a coordinate system u^i. Then R_j coincide with operators of partial derivatives:

$$c^i_{jk} = 0 \quad \Longrightarrow \quad \theta^i = du^i, \quad R_j = \frac{\partial}{\partial u^j}.$$

2.6 Derivation formulas

Let X be a smooth vector field on M, and let a_t be its flow. The infinitesimal transformation of any differentiable tensor field S in the flow a_t is determined by its Lie derivative $S' = \mathcal{L}_X S$. If we want to calculate a Lie derivative of S along a vector field X in a nonholonomic basis (R, Θ), we should know how this basis itself changes in the flow of X.

Proposition 2.2. *Let a nonholonomic basis (R, Θ) be given on M, and suppose $X = R_i x^i$ in this basis. The change of (R, Θ) in the flow a_t of X is described by the derivation formulas*

$$\boxed{R' = -RC, \quad \Theta' = C\Theta}, \tag{2.24}$$

where the matrix C determines an infinitesimal turn in the direction of X of a frame and coframe at each point $u \in M$.

The matrix $C = (C^i_j)$ is determined by the nonholonomy object of the basis (R, Θ) and by the components of X in this basis,

$$\boxed{C^i_j = c^i_{jk} x^k + R_j x^i}. \tag{2.25}$$

Proof. Suppose that the matrices in (2.24) are different, i.e., $R' = R\widetilde{C}$ and $\Theta' = C\Theta$. Then differentiating the equality $\Theta(R) = E$ we have $\Theta'(R) + \Theta(R') = 0$, which implies $C + \widetilde{C} = 0$ or $\widetilde{C} = -C$. The formula (2.25) is obtained by straightforward calculating of the Lie brackets of the fields X and R_j, and using (2.16) and (2.23):

$$\begin{aligned} R'_j = [R_i x^i, R_j] = -[R_j, R_i x^i] = -[R_j, R_i] x^i - R_i(R_j x^i) = \\ = -R_k(c^k_{ji} x^i) - R_i(R_j x^i) = -R_i(c^i_{jk} x^k + R_j x^i). \end{aligned}$$ ■

Proposition 2.3. *Let a transformation of a nonholonomic basis $(R, \Theta) \to (\widetilde{R}, \widetilde{\Theta})$ is presented by a regular matrix $A = (a^i_j)$ and its inverse $A^{-1} = (\overline{a}^i_j)$:*

$$\widetilde{R} = RA^{-1}, \qquad \widetilde{\Theta} = A\Theta.$$

Then C and the nonholonomy object c^i_{jk} are related to $\widetilde{C}$ and $\widetilde{c}^i_{jk}$ by

$$\begin{gathered} \widetilde{C} = (XA + AC)A^{-1}, \\ a^k_s c^s_{ij} = \widetilde{c}^k_{pq} a^p_i a^q_j + R_i a^k_j - R_j a^k_i. \end{gathered}$$

In the particular case of holonomic basis $(\widetilde{R}, \widetilde{\Theta})$ the non-holonomy object of the basis (R, Θ) has a local expression in terms of nonholonomic frame

$$c^k_{ij} = \overline{a}^k_s (R_i a^s_j - R_j a^s_i).$$

Proof. From the equalities $\widetilde{\Theta} = A\Theta$, $\widetilde{\Theta}' = \widetilde{C}\widetilde{\Theta}$ and $\Theta' = C\Theta$ it follows that

$$\begin{aligned}
\widetilde{\Theta}' &= (XA)\Theta + A\Theta', \\
\widetilde{C}\widetilde{\Theta} &= (XA)\Theta + AC\Theta, \\
\widetilde{C}A\Theta &= (XA)\Theta + AC\Theta, \\
\widetilde{C}A &= XA + AC, \\
\widetilde{C} &= (XA + AC)A^{-1}.
\end{aligned}$$

The connecting formula for nonholonomy objects c^i_{jk} and $\widetilde{c}^i_{jk}$ follows from the equalities

$$[R_i, R_j] = R_k c^k_{ij}, \qquad [\widetilde{R}_i, \widetilde{R}_j] = \widetilde{R}_k \widetilde{c}^k_{ij}.$$

Namely, from $\widetilde{R} = RA^{-1}$ we have $R = \widetilde{R}A$ and straightforward calculation leads to

$$\begin{aligned}
[R_i, R_j] &= [R_i, \widetilde{R}_q a^q_j] = [R_i, \widetilde{R}_q]a^q_j + \widetilde{R}_q(R_i a^q_j) = \\
&= -[\widetilde{R}_q, R_i]a^q_j + \widetilde{R}_q(R_i a^q_j) = -[\widetilde{R}_q, \widetilde{R}_p a^p_i]a^q_j + \widetilde{R}_q(R_i a^q_j) = \\
&= -\big([\widetilde{R}_q, \widetilde{R}_p]a^p_i + \widetilde{R}_p(\widetilde{R}_q a^p_i)]\big)a^q_j + \widetilde{R}_q(R_i a^q_j) = \\
&= [\widetilde{R}_p, \widetilde{R}_q]a^p_i a^q_j + \widetilde{R}_q(R_i a^q_j) - \widetilde{R}_p(\widetilde{R}_q a^p_i)a^q_j.
\end{aligned}$$

Thus

$$R_k c^k_{ij} = \widetilde{R}_k \widetilde{c}^k_{pq} a^p_i a^q_j + \widetilde{R}_q(R_i a^q_j) - \widetilde{R}_p(\widetilde{R}_q a^p_i)a^q_j.$$

Taking in account that $R_k c^k_{ij} = \widetilde{R}_s a^s_k c^k_{ij}$ and

$$\begin{aligned}
\widetilde{R}_p(\widetilde{R}_q a^p_i)a^q_j &= \widetilde{R}_p\big((R_s \overline{a}^s_q)a^p_i\big)a^q_j = \widetilde{R}_p\big((R_s a^p_i)\overline{a}^s_q a^q_j\big) = \widetilde{R}_p\big((R_s a^p_i)\delta^s_j\big) = \\
&= \widetilde{R}_p(R_j a^p_i),
\end{aligned}$$

and after rearranging indices we have

$$\widetilde{R}_k a^k_s c^s_{ij} = \widetilde{R}_k \widetilde{c}^k_{pq} a^p_i a^q_j + \widetilde{R}_k(R_i a^k_j) - \widetilde{R}_k(R_j a^k_i).$$

Reducing by $\widetilde{R}_k$ from the left we obtain the desired formula

$$a^k_s c^s_{ij} = \widetilde{c}^k_{pq} a^p_i a^q_j + R_i a^k_j - R_j a^k_i.$$

Putting $\widetilde{c}^i_{pq} = 0$ in the last equality, we obtain the local expression for nonholonomy object

$$c^k_{ij} = \overline{a}^k_s(R_i a^s_j - R_j a^s_i). \qquad \blacksquare$$

2.7 Lie derivatives in nonholonomic and holonomic bases

Now, using the derivation formulas (2.24) for a nonholonomic basis (R, Θ), it is easy to calculate the Lie derivatives of a vector filed $Y = Ry$ and 1-form $\Phi = \varphi\Theta$ with respect to the vector field $X = Rx$. These formulas in matrix notation are

$$Y' = R'y + Ry' = -RCy + Ry' = R(y' - Cy) \tag{2.26}$$

$$\Phi' = \varphi'\Theta + \varphi\Theta' = \varphi'\Theta + \varphi C\Theta = (\varphi' + \varphi C)\Theta. \tag{2.27}$$

The Lie derivative of a tensor field S of an arbitrary type (p, q) is defined by

$$\mathcal{L}_X S(\theta^{i_1}, \ldots, \theta^{i_p}; R_{j_1}, \ldots, R_{j_q}) = X s^{i_1\ldots i_p}_{j_1\ldots j_q} - \sum_{k=1}^{p} C^{i_k}_{l} s^{i_1\ldots l\ldots i_p}_{j_1\ldots j_q} + \sum_{k=1}^{q} s^{i_1\ldots i_p}_{j_1\ldots l\ldots j_q} C^{l}_{j_k}. \tag{2.28}$$

The formulas (2.26) and (2.27) are particular cases of (2.28) for $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$, respectively.

In the holonomic case, the basis (R, Θ) becomes natural and consists of the frame $R_j = \dfrac{\partial}{\partial u^j}$ and coframe $\theta^i = du^i$. Then the derivation formulas (2.24) are

$$\mathcal{L}_X \frac{\partial}{\partial u^j} = -\frac{\partial}{\partial u^i}\frac{\partial x^i}{\partial u^j}, \qquad \mathcal{L}_X(du^i) = \frac{\partial x^i}{\partial u^j} du^j,$$

where the matrix C is just the Jacobi matrix of components of X:

$$C^i_j = \frac{\partial x^i}{\partial u^j}.$$

The computing formulas (2.26) and (2.27) in natural basis for Lie derivatives of a vector field $Y = \dfrac{\partial}{\partial u^i} y^i$ and 1-form $\Phi = \varphi_i du^i$ with respect to $X = \dfrac{\partial}{\partial u^i} x^i$ are

$$Y' = [X, Y] = \frac{\partial}{\partial u^i}(Xy^i - Yx^i), \tag{2.29}$$

$$\mathcal{L}_X \Phi = (X\varphi_i)du^i + \varphi_i dx^i.$$

What is the difference between nonholonomic basis (R, Θ) and natural basis?

1) All formulas written in terms of (R, Θ) are invariant in the sense that they are independent on any choice of local coordinates.

2) If (R, Θ) is invariant under the flow a_t of the vector field X, then in the derivation formulas (2.24) $C = 0$ and all other formulas become simpler. Taking the canonical parameter s, $s' = 1$, and independent invariants of X as coordinate functions, the natural basis becomes *invariant* on some neighborhood U. In this basis X coincides with $\dfrac{\partial}{\partial s}$. However, the invariant basis (R, Θ) for X can be constructed without any references to local coordinates.

Example 2.7. Consider two linear vector fields

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad Y = \frac{\partial}{\partial x}$$

on the xy plane. The field X is a rotation operator and Y is a translation one. Using (2.29), let us compute the Lie derivatives of Y with respect to X:

$$Y' = [X, Y] = -\frac{\partial}{\partial y}, \quad Y'' = [X, Y'] = -\frac{\partial}{\partial x} = -Y.$$

Thus we have the similar situation as in the Example 2.6 for $S = Y$:

$$Y'' + Y = 0 \quad \Longrightarrow \quad Y_t = Y\cos t + Y'\sin t.$$

Let us perform the same calculations in the polar coordinates. The Cartesian coordinates in terms of the polar ones are

$$x = r\cos\varphi, \quad y = r\sin\varphi,$$

where r is the radial distance from the origin, and φ is the counterclockwise angle from the x-axis. The coordinate transformation $(x, y) \mapsto (r, \varphi)$ naturally induces the transformation of the frame $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \to \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\right)$ defined by the inverse of the Jacobi matrix as follows:

$$\begin{aligned}\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix} &= \begin{pmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi\end{pmatrix}^{-1} = \\ &= \begin{pmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi\end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}, \\ \frac{\partial}{\partial y} &= \sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi},\end{aligned}$$

and the vector fields X and Y in polar coordinates are

$$X = \frac{\partial}{\partial \varphi}, \qquad Y = \cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}.$$

Calculating the Lie derivatives of Y with respect to X:

$$\begin{aligned}Y' &= [X, Y] = -\sin\varphi\frac{\partial}{\partial r} - \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}, \\ Y'' &= [X, Y'] = -\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi} = -Y\end{aligned}$$

we obtain the same result in polar coordinates:

$$Y'' + Y = 0 \quad \Longrightarrow \quad Y_t = Y\cos t + Y'\sin t.$$

It means that differential equations with Lie derivatives are invariant, i.e., do not depend on a choice of local coordinates.

2.8 The case of constant matrix C in derivation formulas

Suppose that in (2.24) the matrix C is constant[5]. Then X is a linear operator in terms of the basis, i.e., an operator of the affine group of transformation.

Consider this situation in the context of Lie groups. Suppose G is an n-dimensional Lie group. Then one can construct a nonholonomic left-invariant or right-invariant basis (R, Θ) in G. In such a basis left-invariant and right-invariant tensor fields have constant components. The matrix C remains constant under a flow a_t of a left-invariant or right-invariant vector field X, and in the tangent space TG the flow Ta_t is determined by the exponential of the matrix Ct:

$$R' = -RC \quad \Longrightarrow \quad R_t = Re^{-Ct}, \tag{2.30}$$

$$\Theta' = C\Theta \quad \Longrightarrow \quad \Theta_t = e^{Ct}\Theta. \tag{2.31}$$

The left equalities are ODEs and the corresponding solutions are presented by the right ones. The dragging of the basis (R, Θ) along the flow of X is determined by the exponential of the matrix tC. In this case C is an element of Lie algebra $gl(n, \mathbb{R})$ and the exponential e^{Ct} is understood as a one-parametric subgroup of the linear group $GL(n, \mathbb{R})$.

The draggings of a vector field Y and 1-form Φ in the flow of X are described by

$$Y = Ry \quad \Longrightarrow \quad Y_t = R_t y_t = Re^{-Ct}y_t,$$
$$\Phi = \varphi\Theta \quad \Longrightarrow \quad \Phi_t = \varphi_t\Phi_t = \varphi_t e^{Ct}\Theta.$$

Let us consider the following two cases.

Case 1. Let Y and Φ be invariant under the flow a_t of X. Then their components are dragged according to the rules

$$Y_t = Y \quad \Longrightarrow \quad y_t = e^{Ct}y,$$
$$\Phi_t = \Phi \quad \Longrightarrow \quad \varphi_t = \varphi e^{-Ct}.$$

Case 2. On the other hand, let the components y and φ be constant under the flow a_t of X. Then Y and Φ are dragged according to the formulas

$$y_t = y \quad \Longrightarrow \quad Y_t = Re^{-Ct}y,$$
$$\varphi_t = \varphi \quad \Longrightarrow \quad \Phi_t = \varphi e^{Ct}\Theta.$$

For instance, let C be of the order $n = 2$, i.e. X is a linear vector field on a plane. Then C satisfies the Hamilton-Cayley formula

$$C^2 - \mathrm{tr}C \cdot C + \det C \cdot E = 0$$

---

[5] There is enough, that the matrix C is constant on the trajectories of X, i.e., $C' = XC = 0$.

and for (R, Θ) we have

$$\begin{aligned} R' = -RC \quad &\Longrightarrow \quad R'' + \mathrm{tr}C \cdot R' + \det C \cdot R = 0, \\ \Theta' = C\Theta \quad &\Longrightarrow \quad \Theta'' - \mathrm{tr} \cdot \Theta' + \det C \cdot \Theta = 0. \end{aligned}$$

Thus, in the Case 1 we obtain the similar ODEs for the components of Y and Φ, and in the Case 2 we obtain the similar ODEs for Y and Φ themselves, i.e.,

$$\begin{aligned} Y' = 0 \quad &\Longrightarrow \quad y'' - \mathrm{tr}C \cdot y' + \det C \cdot y = 0, \\ \Phi' = 0 \quad &\Longrightarrow \quad \varphi'' + \mathrm{tr}C \cdot \varphi' + \det C \cdot \varphi = 0, \\ y' = 0 \quad &\Longrightarrow \quad Y'' + \mathrm{tr}C \cdot Y' + \det C \cdot Y = 0, \\ \varphi' = 0 \quad &\Longrightarrow \quad \Phi'' - \mathrm{tr}C \cdot \Phi' + \det C \cdot \Phi = 0. \end{aligned}$$

Solutions of these ODEs correspond to the classification of linear flows on a plane, see [2], p.197, [16], p.69. Namely, the exponential e^{Ct} depends on the solutions λ_1 and λ_2 of the characteristic equation

$$\lambda^2 - \mathrm{tr}C \cdot \lambda + \det C = 0.$$

Depending on the sign of the discriminant $\Delta = \mathrm{tr}^2 C - 4 \det C$, the eigenvalues may be either real $\lambda_{1,2} = \alpha \pm \beta$ or complex conjugate $\lambda_{1,2} = \alpha \pm i\beta$ or, either different $(\beta \neq 0)$, or equal $(\beta = 0)$, where α and β are given by

$$\mathrm{tr}C = \lambda_1 + \lambda_2 = 2\alpha, \qquad \det C = \lambda_1 \lambda_2 = \alpha^2 \pm \beta^2.$$

Here the plus sign corresponds to the case of complex eigenvalues and minus to the case of real ones. Thus

$$\Delta = \mathrm{tr}^2 C - 4 \det C = (\lambda_1 - \lambda_2)^2 = \pm 4\beta^2$$

and e^{Ct} can be written as

$$e^{Ct} = \begin{cases} e^{\alpha t}\left[E \cos\beta t + (C - \alpha E)\dfrac{\sin\beta t}{\beta}\right], & \text{if} \quad \Delta < 0, \\ e^{\alpha t}\left[E \cosh\beta t + (C - \alpha E)\dfrac{\sinh\beta t}{\beta}\right], & \text{if} \quad \Delta > 0, \\ e^{\alpha t}\big[E + (C - \alpha E)t\big], \quad \text{if} \quad \Delta = 0. \end{cases}$$

The various possible flow patterns on the plane can be summarized as follows:
if $\Delta < 0$ and $\det C < 0$, then one has *elliptic flow with focuses*;
if $\Delta < 0$ and $\det C > 0$, then one has *hyperbolic flow with saddles*;
if $\Delta > 0$, then one has *hyperbolic knots*;
if $\Delta = 0$, then one has *parabolic knots.*
Depending on the sign of α the knots and focuses may be *stable* $(\alpha < 0)$ or *unstable* $(\alpha > 0)$.

2.9 Integration of tensor fields

The main purpose of this Section is to introduce the idea of integration of tensor field as a reverse process to the Lie differentiation. In the previous Sections we have defined a Lie derivative of a tensor field in the flow $a_t = \exp tX$ of a vector field X. The dragging of any smooth function f along the flow of X is described by $f_t = f \circ a_t$. Thus one can speak about a derivative of f in the flow of X:

$$f' = Xf = (f \circ a_t)'_{t=0} = \lim_{t \to 0} \frac{1}{t}(f_t - f).$$

Analogously, one can speak about an *integration* of f. In particular, we need to recover a function from its known Lie derivative with respect to the vector field X.

An *indefinite integral* of a function f'_t with respect to the parameter t is defined as the set of all antiderivatives of f'_t along the flow a_t of X, symbolized by

$$\int f'_t dt = f_t + f_0, \tag{2.32}$$

where f_0 is an *invariant* of X, i.e., $Xf_0 = 0$.

A *definite integral* of f'_t on a closed interval $[a, b]$ is defined by the *Newton–Leibniz formula*

$$\int_a^b f'_t dt = f_t \Big|_a^b = f(b) - f(a). \tag{2.33}$$

If in (2.32) and (2.33) f is a tensor field, then along with the Lie differentiation one can speak about an *integration of tensor fields* along the flow of X.

Let S and Q be smooth tensor fields of the same type on M.

Definition 2.5. A tensor field Q is said to be an *antiderivative* of S along the flow a_t of X if

$$Q' = \mathcal{L}_X Q = S.$$

Let Q_1 and Q_2 be tensor fields of the same type and suppose one of them is an antiderivative of S. Then the second one is an antiderivative of S if and only if $Q_1 - Q_2 = Q_0$, where Q_0 is an invariant tensor field along the flow of X, i.e. $\mathcal{L}_X Q_0 = 0$.

Definition 2.6. An *indefinite integral* of the tensor field S with respect to t is defined as the set of all antiderivatives of S along the flow a_t of X, symbolized by

$$\int S_t dt = Q_t + Q_0, \tag{2.34}$$

where Q is an antiderivative of S and $\mathcal{L}_X Q_0 = 0$.

Proposition 2.4. *Let Q be an antiderivative of S along the flow a_t of X and suppose S is continuous on a closed interval $[a, b]$. Then a definite integral of S is defined by*

$$\int_a^b S_t dt = Q_b - Q_a. \tag{2.35}$$

Proof. Let the closed interval $[a, b]$ be partitioned by points

$$a = t_0 < t_1 < ... < t_{i-1} < t_i < t_{i+1} < ... < t_{n-1} < t_n = b.$$

Then the definite integral of S is defined by taking a limit of a sum

$$\int_a^b S_t dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} S_{\xi_i} \Delta t_i,$$

where S_{ξ_i} is a value of S at an arbitrary point $\xi_i \in (t_{i-1}, t_i)$ and $\Delta t_i = t_i - t_{i-1}$ is the length of the subinterval, $i = 1, 2, \ldots, n$. According to the Mean Value Theorem there is a one point ξ_i in each open interval (t_{i-1}, t_i) such that

$$S_{\xi_i} \Delta t_i = Q_{t_i} - Q_{t_{i-1}}.$$

We have

$$Q_b - Q_a = \sum_{i=1}^{n} (Q_{t_i} - Q_{t_{i-1}}),$$

which can be rewritten as

$$Q_b - Q_a = \sum_{i=1}^{n} S_{\xi_i} \Delta t_i.$$

Then taking a limit of the sum in the right-hand side of the last equality as $n \to \infty$ we obtain (2.35). ■

Let Y be a differentiable vector field on M. Then (2.35) yields

$$\int_a^b [X, Y]_t dt = Y_b - Y_a. \tag{2.36}$$

Example 2.8. Let us consider the linear vector field

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

on the xy plane. The flow a_t of X is a uniform circular motion around the origin:

$$a_t : (x, y) \mapsto (x \cos t - y \sin t, y \cos t + x \sin t).$$

The indefinite integral of a function along the flow a_t is defined by (2.32), where $f_0 = f_0(I)$ is a function of invariant $I = x^2 + y^2$ of X.

From (2.34) it follows that an indefinite integral of a vector field $[X, Y]_t$ is of the form

$$\int [X, Y]_t dt = Y_t + Y_0,$$

where Y is a differentiable vector field on the xy plane and Y_0 is of the form

$$Y_0 = \zeta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}.$$

According to the condition

$$[X, Y_0] = (X\zeta + \eta)\frac{\partial}{\partial x} + (X\eta - \zeta)\frac{\partial}{\partial y} = 0,$$

the functions ζ and η must satisfy the system of linear ODEs

$$\begin{cases} \zeta'' + \zeta = 0 \\ \eta'' + \eta = 0 \end{cases},$$

where prime denotes the derivative with respect to X.

Suppose two functions $f = x$ and $g = y$ be given on the xy plane. The draggings of these functions and the function $f + g = x + y$ along the flow of X are described by $f_t = x_t$, $g_t = y_t$ and $(f + g)_t = (x + y)_t$, respectively. Let us calculate the corresponding definite integrals on the closed interval $[a, b] = \left[0, \frac{\pi}{2}\right]$. By (2.33) we have

$$\int_0^{\frac{\pi}{2}} f'_t dt = \int_0^{\frac{\pi}{2}} x'_t dt = x_t\Big|_0^{\frac{\pi}{2}} = -x - y,$$
$$\int_0^{\frac{\pi}{2}} g'_t dt = \int_0^{\frac{\pi}{2}} y'_t dt = y_t\Big|_0^{\frac{\pi}{2}} = x - y,$$
$$\int_0^{\frac{\pi}{2}} (f + g)'_t dt = \int_0^{\frac{\pi}{2}} (x_t + y_t)' dt = (x_t + y_t)\Big|_0^{\frac{\pi}{2}} = -2y.$$

Consider the vector field $Y = \dfrac{\partial}{\partial y}$. The Lie derivatives of Y with respect to X are

$$Y' = [X, Y] = \frac{\partial}{\partial x}, \qquad Y'' = [X, Y'] = -\frac{\partial}{\partial y} = -Y.$$

Thus, we have $Y'' + Y = 0$ and the dragging of Y along the flow of X is described by the vector-function

$$Y_t = Ta_t Y = Y \cos t + Y' \sin t = \sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y}.$$

Then using (2.36) we obtain the definite integral of the field $Y' = \dfrac{\partial}{\partial x}$ on the closed interval $\left[0, \dfrac{\pi}{2}\right]$:

$$\int_0^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x}\right)_t dt = Y_t\Big|_0^{\frac{\pi}{2}} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$

where

$$\left(\frac{\partial}{\partial x}\right)_t = \cos t\frac{\partial}{\partial x} - \sin t\frac{\partial}{\partial y}.$$

The Figures 2.1–2.3 illustrate the meaning of the definite integral of a vector field on the 1st quarter of the xy plane.

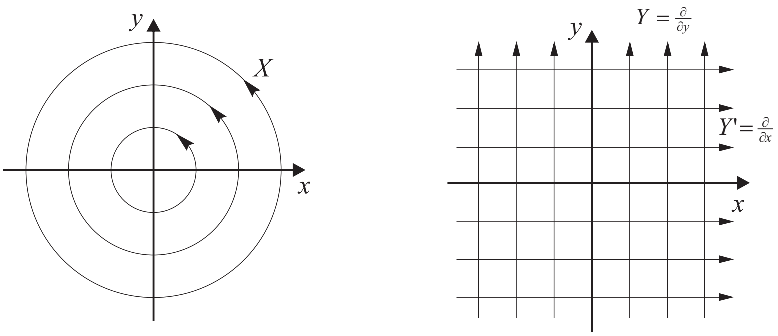

Figure 2.1: The flow of X is the uniform circular motion around the origin in the counterclockwise direction. The Lie derivative of $Y=\dfrac{\partial}{\partial y}$ (*south wind*) with respect to X is the field $Y'=\dfrac{\partial}{\partial x}$ (*west wind*).

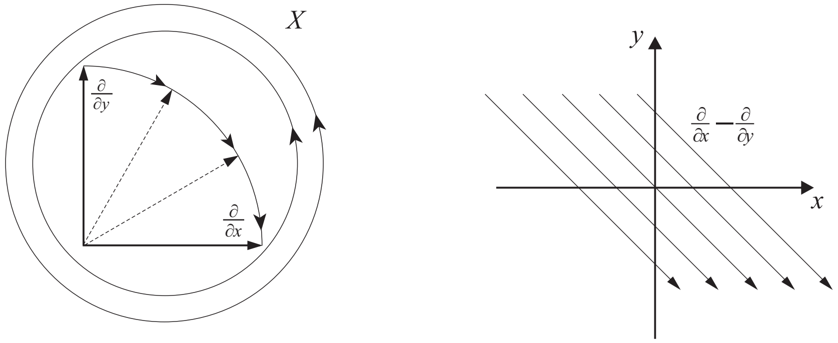

Figure 2.2: The field Y is rotated in moving frame according to the law $Y_t = Y\cos t + Y'\sin t$ (the wind changes own direction rotating clockwise). The calculating of definite integral $\displaystyle\int_0^{\frac{\pi}{2}}[X,Y]_t dt$ yields the field $\dfrac{\partial}{\partial x}-\dfrac{\partial}{\partial y}$ (*north-west wind*).

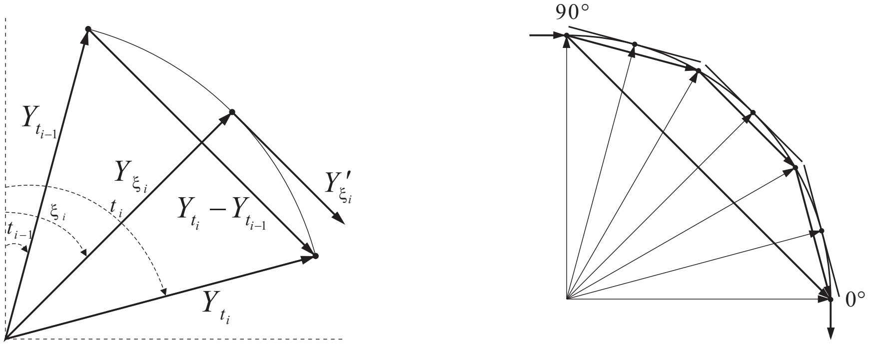

Figure 2.3: The summands for the integral sum are defined by the Mean Value Theorem. Taking a limit of the integral sum we obtain the closing line to the hodograph of Y_t.

The *hodograph* is a plot of the velocity as a function of time. The hodograph of the vector-function Y_t has the same trajectory as X but with opposite direction. The integral sum $\sum Y'_{\xi_i}\Delta t_i$ is a broken line to the hodograph and the integral $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ is a straight line closing this broken line, see Figure 2.3.

Remark 2.2. Note that the formulas in the Example 2.8 can be obtained as follows. Consider the dragging of the natural frame $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and coframe (dx, dy) in the flow of X. The derivation formulas (2.24) are

$$\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}' = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix} \cdot \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \begin{pmatrix}dx\\ dy\end{pmatrix}' = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \cdot \begin{pmatrix}dx\\ dy\end{pmatrix},$$

where $C = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ is the Jacobi matrix of the components of X. From this it follows, using (2.30) and (2.31), that

$$\begin{aligned}\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}_t &= \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix} \cdot \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix},\\ \begin{pmatrix}dx\\ dy\end{pmatrix}_t &= \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} \cdot \begin{pmatrix}dx\\ dy\end{pmatrix}.\end{aligned}$$

Thus, the definite integrals on the closed interval $\left[0, \frac{\pi}{2}\right]$ of the corresponding vector fields and 1-forms can be calculated as follows:

$$\begin{aligned}\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}_0^{\frac{\pi}{2}} &= \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}_{\frac{\pi}{2}} - \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}_0 = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix} \cdot \begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix},\\ \begin{pmatrix}dx\\ dy\end{pmatrix}_0^{\frac{\pi}{2}} &= \begin{pmatrix}dx\\ dy\end{pmatrix}_{\frac{\pi}{2}} - \begin{pmatrix}dx\\ dy\end{pmatrix}_0 = \begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix} \cdot \begin{pmatrix}dx\\ dy\end{pmatrix}.\end{aligned}$$

Example 2.9. Let three vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \qquad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

be given in the space xyz. The flows of X, Y and Z are rotations about three axes x, y and z, respectively.

Let us consider the dragging of Y in the flow of X, the dragging of Z in the flow of Y and the dragging of X in the flow of Z, respectively:

$$\begin{aligned} Y' &= [X, Y] = Z, \quad Y'' + Y = 0 \quad \Longrightarrow \quad Y_t = Y\cos t + Z\sin t,\\ Z' &= [Y, Z] = X, \quad Z'' + Z = 0 \quad \Longrightarrow \quad Z_t = Z\cos t + X\sin t,\\ X' &= [Z, Y] = Y, \quad X'' + X = 0 \quad \Longrightarrow \quad X_t = X\cos t + Y\sin t.\end{aligned}$$

Let us calculate the integrals of X, Y and Z on a closed interval $[a,b]$:

$$\int_a^b Z_t dt = \int_a^b [X,Y]_t dt = Y_b - Y_a = 2\sin\frac{a-b}{2}\left(Y\sin\frac{a+b}{2} - Z\cos\frac{a+b}{2}\right),$$
$$\int_a^b X_t dt = \int_a^b [Y,Z]_t dt = Z_b - Z_a = 2\sin\frac{a-b}{2}\left(Z\sin\frac{a+b}{2} - X\cos\frac{a+b}{2}\right),$$
$$\int_a^b Y_t dt = \int_a^b [Z,X]_t dt = X_b - X_a = 2\sin\frac{a-b}{2}\left(X\sin\frac{a+b}{2} - Z\cos\frac{a+b}{2}\right).$$

Taking $a=0$ and $b=\frac{\pi}{2}$ we have three vector fields

$$\int_0^{\frac{\pi}{2}} Z_t dt = (y+z)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \tag{2.37}$$
$$\int_0^{\frac{\pi}{2}} X_t dt = -y\frac{\partial}{\partial x} + (x+z)\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \tag{2.38}$$
$$\int_0^{\frac{\pi}{2}} Y_t dt = y\frac{\partial}{\partial x} - (x+z)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}. \tag{2.39}$$

The flow of (2.37) is

$$(x,y,z) \mapsto \begin{cases} x_t = x\cos\sqrt{2}t + (y+z)\dfrac{\sin\sqrt{2}t}{\sqrt{2}} \\ y_t = y - x\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (y+z)\dfrac{1-\cos\sqrt{2}t}{2} \\ z_t = z - x\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (y+z)\dfrac{1-\cos\sqrt{2}t}{2} \end{cases}.$$

From the equalities $y_t - z_t = y - z$ and $2x_t^2 + (y_t+z_t)^2 = 2x^2 + (y+z)^2$ we obtain two invariants

$$I_1 = 2x^2 + (y+z)^2, \qquad I_2 = y - z.$$

It means that the level surfaces of trajectories of the field (2.37) are elliptic cylinders with axis of rotation $\begin{cases} y+z=0 \\ x=0 \end{cases}$. The trajectories are ellipses on the intersections of the cylinders $I_1 = c > 0$ with planes $I_2 = c \geq 0$ perpendicular to the axis.

The flow of the field (2.38) is

$$(x,y,z) \mapsto \begin{cases} x_t = x - y\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (x+z)\dfrac{1-\cos\sqrt{2}t}{2} \\ y_t = y\cos\sqrt{2}t + (x+z)\dfrac{\sin\sqrt{2}t}{\sqrt{2}} \\ z_t = z - y\dfrac{\sin\sqrt{2}t}{\sqrt{2}} - (x+z)\dfrac{1-\cos\sqrt{2}t}{2} \end{cases},$$

and the invariants are

$$I_1 = 2y^2 + (x+z)^2, \qquad I_2 = x - z.$$

The level surfaces of trajectories of the field (2.38) are elliptic cylinders with axis of rotation $\left\{ \begin{array}{l} x+z=0 \\ y=0 \end{array} \right.$. The trajectories are ellipses on the intersections of the cylinders $I_1 = c > 0$ with planes $I_2 = c \geq 0$ perpendicular to the axis .

From $\displaystyle\int_0^{\frac{\pi}{2}} Y_t dt = -\int_0^{\frac{\pi}{2}} X_t dt$ it follows that the flows and invariants of the fields (2.38) and (2.39) are the same, but the trajectories of these field are opposite directed.

In the Example 2.8 the definite integral on the xy plane is a straight line that closing the integral sum $\sum Y'_{\xi_i} \Delta t_i$, see Figure 2.3. In the xyz space we have analogous situation, but the closing line is a part of an ellipse.

2.10 Conclusion

In the present Chapter the dragging of tensor fields in a flow of vector field X is studied, and the basic formulas of the Lie-Cartan calculus are derived. The formulas of Lie derivatives are presented in nonholonomic basis (R, Θ), taking in account that the basis itself may be dragged along the flow of X. In this case the derivation formulas

$$R' = -RC, \qquad \Theta' = C\Theta$$

for the frame R and coframe Θ are used. In the case of nonholonomic basis the matrix C is determined by components of the vector field X and by the nonholonomy object of the basis. In the case of a holonomic basis the matrix C is just the Jacobi matrix of components of X. If X is a linear vector field, then C is a constant matrix.

Note that in this Chapter appear differential equations with Lie derivatives, that represent a quite new topic in mathematics. In order to find solutions of such differential equations we introduce the idea of integration of tensor fields as a reverse process to the Lie differentiation. As we have seen, the definitions of indefinite and definite integrals for tensor fields are similar to the analogous definitions for integrable functions in undergraduate differential calculus. In our definition the definite integral of a function is also a function, not a number, and the definite integral of a general tensor field is also a tensor field of the same type. Our main purpose has been to explain the geometrical meaning of integrals by simple examples.

Chapter 3

Lie group representations

The present Chapter is devoted to the theory of Lie group representations. In particular, considering vector fields as infinitesimal generators of flows on a smooth manifold, the Lie derivatives are infinitesimal versions of representation of a diffeomorphism group on tensor fields.

In this Chapter the basic properties of a tangent group TG of a Lie group G is considered, cf. [11], [23]. The formulas for left- and right-invariant bases and for operators of adjoint representations are derived. Some of the formulas considered in this Chapter are discussed in the case of the linear group $GL(n,\mathbb{R})$ in the original publication [17] attached to this thesis, see page 91.

3.1 Generalized Leibniz rule

Let M_1, M_2 and M be smooth manifolds, and let

$$\lambda : M_1 \times M_2 \to M, \quad (u,v) \mapsto w = u \cdot v$$

be a smooth map. For any fixed elements $u \in M_1$ and $v \in M_2$ there are two smooth maps

$$\begin{aligned} \lambda_v &: M_1 \to M, \quad u \mapsto w = u \cdot v, \quad \forall v \in M_2, \\ \lambda_u &: M_2 \to M, \quad v \mapsto w = u \cdot v, \quad \forall u \in M_1 \end{aligned}$$

associated with λ. Their tangent maps

$$\begin{aligned} T\lambda_v &: TM_1 \to TM, \quad u_1 \mapsto u_1 \cdot v, \\ T\lambda_u &: TM_2 \to TM, \quad v_1 \mapsto u \cdot v_1 \end{aligned}$$

determine the tangent map $T\lambda$ as follows:

$$\begin{aligned} T\lambda : T(M_1 \times M_2) = (TM_1 \times M_2) \otimes (M_1 \times TM_2) \to TM, \\ (u_1, v_1) \mapsto w_1 = u_1 \cdot v + u \cdot v_1. \end{aligned}$$

Namely, two vectors $u_1 \in T_u M_1$ and $v_1 \in T_v M_2$ at $u \in M_1$ and $v \in M_2$, respectively, are mapped to the vectors $T\lambda_v(u_1) = u_1 \cdot v$ and $T\lambda_u(v_1) = u \cdot v_1$ at $\omega \in M$, see Figure 3.1. The sum of these images is equal to the image of (u_1, v_1) under $T\lambda$:

$$\boxed{w = u \cdot v \quad \Longrightarrow \quad w_1 = u_1 \cdot v + u \cdot v_1}. \tag{3.1}$$

The formula (3.1) generalizes the ordinary *Leibniz rule*.

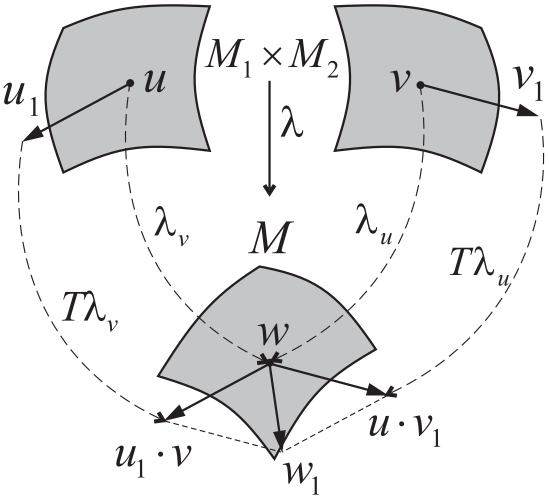

Figure 3.1: Leibniz rule:
$w = \lambda_v(u) = \lambda_u(v) = u \cdot v \quad \Longrightarrow \quad w_1 = T\lambda_v(u_1) + T\lambda_u(v_1) = u_1 \cdot v + u \cdot v_1.$

Locally, $w = u \cdot v$ can be presented as the system

$$w^\rho \circ \lambda = \lambda^\rho(u^i, v^\alpha),$$

where u^i, v^α and w^ρ are coordinates of u, v and w, respectively, on corresponding neighborhoods $U_1 \subset M_1$, $U_2 \subset M_2$ and $U \subset M$, $i = 1, ..., \dim M_1$, $\alpha = 1, ..., \dim M_2$, $\rho = 1, ..., \dim M$. Then $T\lambda$ is determined by

$$dw^\rho \circ T\lambda = \frac{\partial \lambda^\rho}{\partial u^i} du^i + \frac{\partial \lambda^\rho}{\partial v^\alpha} dv^\alpha$$

or in matrix notation,

$$dw^\rho \circ T\lambda = \begin{pmatrix} \dfrac{\partial \lambda^\rho}{\partial u^i} & \dfrac{\partial \lambda^\rho}{\partial v^\alpha} \end{pmatrix} \cdot \begin{pmatrix} du^i \\ dv^\alpha \end{pmatrix}.$$

Here the Jacobi matrix of $T\lambda$ is the block matrix in which the left block corresponds to $T\lambda_v$ and the right one to $T\lambda_u$. Let u_1^i, v_1^α and w_1^ρ be components of u_1, v_1 and w_1, respectively. Then

$$w_1^\rho = \frac{\partial \lambda^\rho}{\partial u^i} u_1^i + \frac{\partial \lambda^\rho}{\partial v^\alpha} v_1^\alpha,$$

where the partial derivatives are fixed at the point $(u, v) \in M_1 \times M_2$.

Remark 3.1. The notation of tangent vector u_1 at a point $u \in M$ follows from the notion of higher order tangent bundles, cf. [2], [6], [21], [22], [26]. Namely, kth order tangent bundle is induced by k iterations of *covariant tangent functor* T, which assigns to a smooth manifold M its tangent bundle $TM = \bigcup\limits_{u\in M} T_uM$ and to a smooth map φ its tangent map $T\varphi$. Thus we have k *levels* of M:

$$TM, \quad T^2M, \quad T^3M, \quad \dots, \quad T^kM.$$

The element of the *first level* TM is a pair (u, u_1), where u_1 is a tangent vector at a point $u \in M$. The element of the *second level* T^2M is a quadruple (u, u_1, u_2, u_{12}), where (u_2, u_{12}) is a tangent vector on TM at a point (u, u_1). The element of the *third level* T^3M is a tuple $(u, u_1, u_2, u_{12}, u_3, u_{13}, u_{23}, u_{123})$, where $(u_3, u_{13}, u_{23}, u_{123})$ is a tangent vector on T^2M at a point (u, u_1, u_2, u_{12}) and s.o. Thus, the dimension of kth order tangent bundle T^kM is

$$\dim T^kM = 2^k \dim M, \quad k = 1, 2, \dots.$$

Due to the formula (3.1) we have

$$T\lambda : \big((u, u_1), (v, v_1)\big) \mapsto \big(w = u \cdot v, w_1 = u_1 \cdot v + u \cdot v_1\big),$$

and the second tangent map $T^2\lambda$ is determined by the Leibniz rule as follows:

$$T^2\lambda : \big((u, u_1, u_2, u_{12}), (v, v_1, v_2, v_{12})\big) \mapsto (w, w_1, w_2, w_{12}),$$

where

$$\begin{aligned} w &= u \cdot v, \\ w_1 &= u_1 \cdot v + u \cdot v_1, \\ w_2 &= u_2 \cdot v + u \cdot v_2, \\ w_{12} &= u_{12} \cdot v + u_1 \cdot v_2 + u_2 \cdot v_1 + u \cdot v_{12}. \end{aligned}$$

In the same way one can express all higher order tangent maps $T^k\lambda$, $k = 3, 4, \dots$.

3.2 Tangent group TG

Let G be a Lie group endowed with a composition law[1]

$$\gamma : G \times G \to G, \quad (a, b) \mapsto c = ab,$$

where γ is a smooth map. The group G acts on itself by *right* and *left translations*

$$\begin{aligned} r_b &: G \to G, \quad a \mapsto ab, \quad \forall b \in G, \\ l_a &: G \to G, \quad b \mapsto ab, \quad \forall a \in G. \end{aligned}$$

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[1] The group operation in an arbitrary Lie group will be denoted by juxtaposition ab. In other cases will be used the notation $u \cdot v$ or $a \cdot u$.

The tangent bundle (*first level*) TG becomes automatically a Lie group with the composition law[2]

$$T\gamma : TG \times TG \to TG, \quad (a_1, b_1) \mapsto c_1 = a_1 b + a b_1,$$

where $a_1 b = Tr_b(a_1)$ and $ab_1 = Tl_a(b_1)$. Two vectors $a_1 \in T_aG$ and $b_1 \in T_bG$ are mapped by $T\gamma$ to the vector $c_1 \in T_cG$:

$$\boxed{c = ab \quad \rightsquigarrow \quad c_1 = a_1 b + a b_1}. \tag{3.2}$$

The group TG is called a *tangent group* of G.

Let us list the main properties of the group TG.

P. 3.1. The *unity* of TG is precisely the zero vector at the unity $e \in G$.

P. 3.2. The inversion map $\rho : G \to G$ given by $a \mapsto a^{-1}$ yields the *inversion map* in TG:

$$T\rho : TG \to TG, \quad a_1 \mapsto a_1^{-1} = -a^{-1} a_1 a^{-1}.$$

Indeed, from (3.2) we have

$$e = aa^{-1} \quad \Rightarrow \quad 0 = a_1 a^{-1} + a a_1^{-1} \quad \Rightarrow \quad a a_1^{-1} = -a_1 a^{-1} \quad \Rightarrow \quad a_1^{-1} = -a^{-1} a_1 a^{-1}.$$

The same result can be obtained by applying (3.2) to the equality $e = a^{-1}a$. Thus, two vectors a_1 at $a \in G$ and $-a^{-1}a_1a^{-1}$ at $a^{-1} \in G$ are mutually inverse elements in TG.

P. 3.3. Left and right translations l_a and r_b, and inner automorphism $A_a : G \to G$ defined by $b \mapsto aba^{-1}$ are transformations (diffeomorphisms) of G. Their tangent maps Tl_a, Tr_b and TA_a are transformations of TG. More precisely, they are morphisms of the tangent bundle TG. Thus, the following maps are homomorphisms of G to the groups of diffeomorphisms of G and TG, respectively:

$$\begin{array}{lll} a \mapsto l_a, & b \mapsto r_b, & a \mapsto A_a, \\ a \mapsto Tl_a, & b \mapsto Tr_b, & a \mapsto TA_a. \end{array}$$

Let us show, that the last two maps $a \mapsto A_a$ and $a \mapsto TA_a$ are homomorphisms. Indeed, from

$$\begin{aligned} A_{ab}(c) &= (ab)c(ab)^{-1} = (ab)c(b^{-1}a^{-1}) = a(bcb^{-1})a^{-1} = aA_b(c)a^{-1} = A_a(A_b(c)) = \\ &= (A_a \circ A_b)(c) \end{aligned}$$

it follows that $ab \mapsto A_{ab} = A_a \circ A_b$ and $ab \mapsto TA_{ab} = TA_a \circ TA_b$.

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[2] Although the notation a_1 for a tangent vector at a can lead to confusion with the notation of one-parametric group of transformations a_t, the context typically eliminates any such ambiguity.

P. 3.4. The following relations hold in G:

$$\begin{aligned} l_{a^{-1}} &= l_a^{-1}, & r_{b^{-1}} &= r_b^{-1}, & \rho^{-1} &= \rho, \\ l_a \circ r_b &= r_b \circ l_a, & \rho \circ l_a &= r_{a^{-1}} \circ \rho, & \rho \circ r_a &= l_{a^{-1}} \circ \rho, \\ A_{a^{-1}} &= A_a^{-1}, & A_a &= l_a \circ r_a^{-1}, & \rho \circ A_a &= A_a \circ \rho. \end{aligned}$$

These relations yield the corresponding relations in TG.

Let us show, that $A_{a^{-1}} = A_a^{-1}$ and $\rho \circ A_a = A_a \circ \rho$. The first relation holds due to the result from the Property 3.3:

$$\begin{aligned} A_a \circ A_{a^{-1}} &= A_{aa^{-1}} = A_e = id, \\ A_{a^{-1}} \circ A_a &= A_{a^{-1}a} = A_e = id. \end{aligned}$$

The second relation follows from

$$\begin{aligned} (\rho \circ A_a)(b) = \rho(A_a(b)) = (aba^{-1})^{-1} = a(ab)^{-1} = ab^{-1}a^{-1} = A_a(b^{-1}) = A_a(\rho(b)) = \\ = (A_a \circ \rho)(b). \end{aligned}$$

P. 3.5. Any vector $a_1 \in TG$ has two representatives at the unity $e \in G$ – the *left representative* $a^{-1}a_1$ and *right representative* a_1a^{-1} in T_eG. Then

$$\begin{aligned} T\rho(a^{-1}a_1) &= -a^{-1}a_1, & TA_a(a^{-1}a_1) &= a_1a^{-1}, \\ T\rho(a_1a^{-1}) &= -a_1a^{-1}, & TA_{a^{-1}}(a_1a^{-1}) &= a^{-1}a_1. \end{aligned}$$

The relations with A_a are valid due to the straightforward calculations

$$\begin{aligned} TA_a(a^{-1}a_1) &= (Tl_a \circ Tr_a^{-1})(a^{-1}a_1) = Tl_a\big(Tr_{a^{-1}}(a^{-1}a_1)\big) = Tl_a(a^{-1}a_1a^{-1}) = \\ &= a_1a^{-1}, \\ TA_{a^{-1}}(a_1a^{-1}) &= (Tl_{a^{-1}} \circ Tr_a)(a_1a^{-1}) = Tl_{a^{-1}}\big(Tr_a(a_1a^{-1})\big) = Tl_{a^{-1}}(a_1a^{-1}a) = \\ &= a^{-1}a_1. \end{aligned}$$

P. 3.6. Systematically applying left and right translations l_a and r_a to any vector $e_1 \in T_eG$ one can obtain *left-invariant* and *right-invariant* vector fields ae_1 and e_1a, respectively. Left-invariance and right-invariance mean that $b(ae_1) = (ba)e_1$ and $(e_1a)b = e_1(ab)$, $\forall b \in G$, respectively. The following relations take place:

$$\begin{aligned} T\rho(ae_1) &= -e_1a^{-1}, & TA_a(e_1a) &= ae_1, \\ T\rho(e_1a) &= -a^{-1}e_1, & TA_{a^{-1}}(ae_1) &= e_1a, \end{aligned}$$

since left-invariance and right-invariance of the fields ae_1 and e_1a, respectively, and $TA_a = Tl_a \circ Tr_{a^{-1}}$, $TA_{-1} = Tr_a \circ Tl_{a^{-1}}$ imply

$$\begin{aligned} T\rho(ae_1) &= -a^{-1}(ae_1)a^{-1} = -(a^{-1}a)e_1a^{-1} = -e_1a^{-1}, \\ T\rho(e_1a) &= -a^{-1}(e_1a)a^{-1} = -a^{-1}e_1(aa^{-1}) = -a^{-1}e_1, \\ TA_a(e_1a) &= (Tl_a \circ Tr_{a^{-1}})(e_1a) = Tl_a(Tr_{a^{-1}}(e_1a)) = Tl_a((e_1a)a^{-1}) = ae_1(aa^{-1}) = \\ &= ae_1, \\ TA_{a^{-1}}(ae_1) &= (Tr_a \circ Tl_{a^{-1}})(ae_1) = Tr_a(Tl_{a^{-1}}(ae_1)) = Tr_a(a^{-1}(ae_1)) = (a^{-1}a)e_1a = \\ &= e_1a. \end{aligned}$$

Let $Z \subset G$ be a *center* of G, i.e., $Z = \{a \in G : ab = ba \text{ for all } b \in G\}$. Then for $a \in Z$, $ae_1 = e_1 a$.

P. 3.7. With any one-parameter subgroup a_t of G is associated one-parameter groups of translations r_{a_t}, l_{a_t} and A_{a_t} of G that in their turn are generated by left-invariant vector field X, right-invariant vector field $\widetilde{X}$ and the field $\widetilde{X} - X$, respectively, i.e.,

$$r_{a_t} = \exp tX, \quad l_{a_t} = \exp t\widetilde{X}, \quad A_{a_t} = \exp t(\widetilde{X} - X).$$

This follows from the formulas

$$Xf = (f \circ r_{a_t})'_{t=0}, \quad \widetilde{X}f = (f \circ l_{a_t})'_{t=0}, \quad (\widetilde{X} - X)f = (f \circ A_{a_t})'_{t=0},$$

where f is an arbitrary smooth function on G. Namely, from Remark 2.1 and equalities $l_a \circ r_b = r_b \circ l_a$ it follows that

$$\begin{aligned} (Tl_b X)f &= (f \circ l_b r_{a_t} l_b^{-1})'_{t=0} = (f \circ r_{a_t})'_{t=0} = Xf \quad &\Longrightarrow \quad & Tl_b X = X, \forall b \in G, \\ (Tr_b \widetilde{X})f &= (f \circ r_b l_{a_t} r_b^{-1})'_{t=0} = (f \circ l_{a_t})'_{t=0} = \widetilde{X}f \quad &\Longrightarrow \quad & Tr_b \widetilde{X} = \widetilde{X}, \forall b \in G. \end{aligned}$$

Since $A_a = l_a \circ r_a^{-1}$ and $a_t^{-1} = a_{-t}$ the straightforward computation leads to

$$\begin{aligned} (f \circ A_{a_t})'_{t=0} &= (f \circ l_a r_{a_{-t}})'_{t=0} = \\ &= \lim_{t \to 0} \frac{1}{t}(f \circ l_{a_t} r_{a_{-t}} - f) = \\ &= \lim_{t \to 0} \frac{1}{t}(f \circ l_{a_t} r_{a_{-t}} - f \circ r_{a_{-t}} + f \circ r_{a_{-t}} - f) = \\ &= \lim_{t \to 0} \frac{1}{t}\big((f \circ l_{a_t} - f) r_{a_{-t}}\big) - \lim_{t \to 0} \frac{1}{-t}(f \circ r_{a_{-t}} - f) = \\ &= (f \circ l_{a_t})'_{t=0} - (f \circ r_{a_t})'_{t=0} = \\ &= (\widetilde{X} - X)f. \end{aligned}$$

Note that from $\rho \circ r_a \circ \rho^{-1} = l_a^{-1}$ and $\rho \circ l_a \circ \rho^{-1} = r_a^{-1}$ it follows that

$$T\rho X = -\widetilde{X}, \quad T\rho \widetilde{X} = -X, \quad T\rho(\widetilde{X} - X) = \widetilde{X} - X.$$

If $a_t \in Z \subset G$, $\forall t \in \mathbb{R}$, then $X = \widetilde{X}$.

P. 3.8. Let (R_e, Θ_e) be a dual basis (frame and coframe) in $T_e G$. Then applying left translations Tl_a (right translations Tr_a) to this frame and coframe one can obtain a left-invariant basis (R, Θ) (right-invariant basis $(\widetilde{R}, \widetilde{\Theta})$). In the case of a non-commutative group G these two bases are nonholonomic. On the center $Z \subset G$ they agree.

There exists a non-degenerate matrix $A = A(a)$, such that

$$\widetilde{R} = RA^{-1}, \quad \widetilde{\Theta} = A\Theta.$$

The map $a \mapsto A(a)$ is a homomorphism of G into the linear group $GL(p, \mathbb{R})$, where $p = \dim G$. The kernel of this homomorphism is precisely the center $Z \subset G$, and according to the First Isomorphism Theorem its image in $GL(p, \mathbb{R})$ is isomorphic to a quotient group G/Z that is a group G_1 of inner automorphisms of G.

P. 3.9. Any tensor field S is a left-invariant (right-invariant) if and only if its components are constant with respect to the left-invariant basis (R, Θ) (right-invariant basis $(\widetilde{R}, \widetilde{\Theta})$).

For example, let $Y = Ry$ be a vector field with components y in the left-invariant frame R. Then applying a left translation l_a to Y, we obtain the field $Tl_aY = R(y \circ l_a)$. Assume that Y is left-invariant. Then $Tl_aY = Y$ and $y \circ l_a = y$. The last equality implies $y(a) = y(e)$ at $e \in G$. Since $a \in G$ is an arbitrary element, y are constant in G. The opposite is obvious.

For instance, the nonholonomy objects associated with (R, Θ) and $(\widetilde{R}, \widetilde{\Theta})$ are constant in G. From the structure equations

$$[R_j, R_k] = R_i c^i_{jk}, \qquad d\theta^i = -\frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k,$$

$$[\widetilde{R}_j, \widetilde{R}_k] = -\widetilde{R}_i c^i_{jk}, \qquad d\widetilde{\theta}^i = \frac{1}{2} c^i_{jk} \widetilde{\theta}^j \wedge \widetilde{\theta}^k,$$

it follows that the nonholonomy objects for (R, Θ) and $(\widetilde{R}, \widetilde{\Theta})$ are equal up to a sign. The formulas in the second raw are obtained by applying $T\rho$ (see Property 3.7) to the formulas in the first one and vice versa.

3.3 Basic formula of Lie group representation

Suppose G is a Lie group and M is a smooth manifold.

Definition 3.1. Any smooth map

$$\lambda : G \times M \to M, \quad (a, u) \mapsto v = a \cdot u,$$

is said to be a *left action* of G on M if each map

$$\lambda_a : M \to M, \quad u \mapsto a \cdot u, \quad \forall a \in G,$$

is a diffeomorphism, and each map $a \mapsto \lambda_a$ is a homomorphism of G into the group of transformations of M.

Any left action satisfies $(ab) \cdot u = a \cdot (b \cdot u)$, $\forall a, b \in G$.

A *right action* of G on M is defined analogously as a map $(a, u) \mapsto v = u \cdot a$ satisfying $u \cdot (ab) = (u \cdot a) \cdot b$.

The homomorphism $a \mapsto \lambda_a$ for a left action read $\lambda_{ab} = \lambda_a \circ \lambda_b$, while for a right action $\lambda_{ab} = \lambda_b \circ \lambda_a$.

Definition 3.2. An action of G on M is said to be an *effective* if $a \mapsto \lambda_a$ is a monomorphism, i.e., if $e \in G$ is the unique element of its kernel.

Definition 3.3. For any fixed point $u \in M$, the image of the map

$$\lambda_u : G \to M, \quad a \mapsto a \cdot u$$

is called an *orbit* of u under the left action.

Analogously, one can define an orbit of $u \in M$ under the right action, i.e., the image of the map $\lambda_u : G \to M, a \mapsto u \cdot a$.

Thus, M is divided into orbits $\lambda_u(G)$. An action is said to be *transitive* if the orbit of any point $u \in M$ is all of M, i.e., $\lambda_u(G) = M$. A transitive action is said to be *simply transitive* if $\dim G = \dim M$. Simply transitive action determines an *exact representation* of G.

For instance, G acts simply transitively on itself by left and right translations. An adjoint representation $a \mapsto A_a$ is not transitive because its kernel $Z \subset G$ is not trivial and G itself is divided into orbits.

Definition 3.4. Given a point $u \in M$, an *isotropy group* of u under the action of G is a subgroup of G defined by

$$H = \{h \in G \,|\, h \cdot u = u\}.$$

H is also called a *stabilizer* of u.

Any left action λ yields a left action of the tangent space TG on the first level TM,

$$T\lambda : TG \times TM \to TM, \quad (a_1, u_1) \mapsto v_1 = a_1 \cdot u + a \cdot u_1.$$

Thus, we have

$$\boxed{v = a \cdot u \quad \rightsquigarrow \quad v_1 = a_1 \cdot u + a \cdot u_1}. \tag{3.3}$$

For instance, for $a_1 = 0$ (3.3) determines an action of G on TM:

$$a_1 = 0 \quad \Longrightarrow \quad u_1 \mapsto v_1 = a \cdot u_1,$$

while for $u_1 = 0$ an action of TG on M:

$$u_1 = 0 \quad \Longrightarrow \quad e_1 = a_1 a^{-1} \mapsto v_1 = a_1 \cdot u = a_1 a^{-1} \cdot v = e_1 \cdot v.$$

The formula

$$v_1 = a_1 a^{-1} \cdot v, \tag{3.4}$$

called the *basic formula of a Lie group representation*, assigns to any vector $e_1 = a_1 a^{-1} \in T_e G$, as an element of Lie algebra $\mathfrak{g}$, the vector $v_1 \in T_v M$ at an arbitrary point $v \in M$. This assignment determines a vector field called an operator of G or simply *group operator* on M. All these operators are tangent to the orbits.

Suppose $v_1 = 0$ and let $v \in M$ be fixed. Then (3.4) determines a Pfaffian system $a_1 a^{-1} \cdot v = 0$ or $e_1 \cdot v = 0$. One of the solutions of this system is precisely the stabilizer $H \subset G$ of v.

For a right action $v = u \cdot a$ we have

$$v_1 = u_1 \cdot a + u \cdot a_1 = u_1 \cdot a + v \cdot a^{-1} a_1,$$

and (3.4) is replaced by

$$v_1 = v \cdot a^{-1} a_1.$$

Given local or global coordinates v^α on M, $\alpha = 1, 2, ..., \dim M$, there are two formulas

$$dv^\alpha = \xi^\alpha_i \omega^i, \quad X_i = \xi^\alpha_i \frac{\partial}{\partial v^\alpha}, \quad i = 1, 2, ..., \dim G,$$

associated with (3.4). Here ω^i are 1-forms of a left-invariant coframe in G, and X_i are basic group operators on M tangent to orbits. Suppose $dv^\alpha = 0$ and let $v \in M$ be fixed. Then the Pfaffian system $\xi^\alpha_i \omega^i = 0$ is completely integrable and determines the stabilizer $H \subset G$ of v.

We can do no more then touch on the theory of group representations here. But some of these formulas are discussed in the case of linear group $GL(2, \mathbb{R})$ in the original publication [17] attached to this Thesis.

3.4 Action of Lie group on itself

Let us consider actions of Lie groups G and TG on itself by left translations:

$$b \mapsto c = ab, \quad b_1 \mapsto c_1 = a_1 b + a b_1 = (a_1 a^{-1})c + a b_1.$$

For $b_1 = 0$ the last formula gives the *basic formula of a left action*

$$c_1 = (a_1 a^{-1})c. \tag{3.5}$$

If we let G and TG act on itself by right translations, then

$$b \mapsto c = ba, \quad b_1 \mapsto c_1 = b_1 a + b a_1 = b_1 a + c(a^{-1} a_1),$$

and the *basic formula of a right action* is

$$c_1 = c(a^{-1} a_1). \tag{3.6}$$

Finally, let G and TG act on itself by an inner automorphism A_a:

$$\begin{aligned} &b \mapsto c = aba^{-1}, \\ &b_1 \mapsto c_1 = a_1 b a^{-1} + a b a^{-1}_1 + a b_1 a^{-1} = (a_1 a^{-1})c - c(a_1 a^{-1}) + a b_1 a^{-1}. \end{aligned}$$

Taking $b_1 = 0$ we obtain the *basic formula of an adjoint representation* of G:

$$c_1 = (a_1 a^{-1})c - c(a_1 a^{-1}). \tag{3.7}$$

Example 3.1 (see [2], p.203, [17])**.** Consider the linear group $GL(2, \mathbb{R})$. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL(2, \mathbb{R}).$$

The group $GL(2, \mathbb{R})$ is a 4-dimensional manifold with coordinates a_i, $i = 1, 2, 3, 4$. Then the group operators $X_i = \xi^\alpha_i \dfrac{\partial}{\partial v^\alpha}$ associated with (3.5) and (3.6) can be determined by the linearly independent left-invariant and right-invariant vector fields

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}, \quad \begin{pmatrix} \widetilde{X}_1 & \widetilde{X}_2 \\ \widetilde{X}_3 & \widetilde{X}_4 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix},$$

respectively, where $\partial_i = \dfrac{\partial}{\partial a_i}$, $i = 1, 2, 3, 4$.

Let $GL(2, \mathbb{R})$ acts on itself by inner automorphisms. Then there exist four operators $Y_i = \widetilde{X}_i - X_i$, $i = 1, 2, 3, 4$, of the adjoint representation of $GL(2, \mathbb{R})$,

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} - \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix},$$

associated with (3.7). These operators are not linearly independent and their common invariants are precisely the trace and determinant of A, i.e.,

$$\mathrm{tr} A = a_1 + a_4, \qquad \det A = a_1 a_4 - a_2 a_3.$$

Indeed,

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \mathrm{tr} A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} - \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \det A = \begin{pmatrix} a_4 & -a_3 \\ -a_2 & a_1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} - \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \cdot \begin{pmatrix} a_4 & -a_3 \\ -a_2 & a_1 \end{pmatrix} = 0.$$

Chapter 4

The structure of Jet space

The present Chapter is devoted to the jet space first introduced by Ch. Ehresmann in [7]. In this thesis we consider only infinite jets, introduced by A. Vinogradov, cf. [12]. The reason of it is that in the case of finite jets the corresponding Cartan distributions are not involutive and are of growing dimension when passing to higher order jet spaces, but when passing to the space of infinite order jets this distribution is involutive and finite dimensional.

The set $\mathcal{J}_{n,m}$ of infinite jets of smooth maps $\mathbb{R}^n \to \mathbb{R}^m$ is a trivial bundle space $\mathbb{R}^n \times \mathbb{R}^\infty$ with n-dimensional base $\mathbb{R}^n$ and infinite dimensional fibers $\mathbb{R}^\infty$. In $\mathcal{J}_{n,m}$ total differentiation operators (TDOs) D and Cartan forms ω are defined. At the same time the additive group $\mathbb{R}^n$ acts in $\mathcal{J}_{n,m}$ along orbits that are integral manifolds of distribution spanned by D. In $\mathcal{J}_{n,m}$ the operators D are linear vector fields and their flows and invariants are determined by the common exponential law, see [2], [16], [18] and [19]. The present Chapter is based on results published in [19].

4.1 Jet space $\mathcal{J}_{1,1}$

Let us begin with the case $n = m = 1$, which corresponds to the jet space $\mathcal{J}_{1,1}$. A *pure jet* as an element of $\mathcal{J}_{1,1}$ is an infinite sequence of symbols

$$t, u, u', u'', \dots . \tag{4.1}$$

In general, these symbols are not connected with each other. $\mathcal{J}_{1,1}$ is a bundle space $\mathbb{R} \times \mathbb{R}^\infty$ with time axis $\mathbb{R}$ as its base and infinite dimensional fibers $\mathbb{R}^\infty$. The symbols (4.1) are considered as coordinates in $\mathcal{J}_{1,1}$, where t is a base coordinate and $u, u', u'', \dots$ are fiber ones.

If the symbols $u^{(k)}$, $k = 0, 1, 2, \dots$, in (4.1) are replaced by a smooth function $u = u(t)$ of one independent variable t and its derivatives, then (4.1) becomes a jet of the given function, a section of the bundle $\mathbb{R} \times \mathbb{R}^\infty$. Any relation between the quantities in (4.1) can be interpreted as an ordinary differential equation (ODE). After prolongation this ODE can be considered as a surface in $\mathcal{J}_{1,1}$. For instance, being prolonged, an ODE of order n

$$u^{(n)} = \mathcal{F}(t, u, u', u'', \dots, u^{(n-1)})$$

allows to express $u^{(n)}$, $u^{(n+1)}, \dots$ by means of $n+1$ quantities $t, u, u', u'', \dots, u^{(n-1)}$. One can speak about parametric equations of an $(n+1)$-dimensional surface in $\mathcal{J}_{1,1}$. If the jet of the function $u(t)$, being considered as the section of the bundle $\mathbb{R} \times \mathbb{R}^\infty$, lies entirely on this surface, then $u(t)$ is called a solution of the ODE.

Note that in the present Section we study pure jets (4.1) which are in general not connected with a certain map.

Let us define in $\mathcal{J}_{1,1}$ the following infinite matrices:

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad e^{Ct} = \begin{pmatrix} 1 & t & \dfrac{t^2}{2} & \dots \\ 0 & 1 & t & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$U = \begin{pmatrix} u \\ u' \\ u'' \\ \vdots \end{pmatrix}, \quad U' = \begin{pmatrix} u' \\ u'' \\ u''' \\ \vdots \end{pmatrix}, \quad U_t = \begin{pmatrix} u_t \\ u_t' \\ u_t'' \\ \vdots \end{pmatrix}, \quad I = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ \vdots \end{pmatrix},$$

where E is the unit matrix, C is the *shift matrix*, e^{Ct} is the exponential of Ct, i.e.,

$$e^{Ct} = E + Ct + C^2\frac{t^2}{2} + C^3\frac{t^3}{6} + \dots = \sum_{k=0}^{\infty} \frac{(Ct)^k}{k!},$$

and U, U', U_t, I are infinite column-matrices. The matrices U, U' and U_t are points in fibers $\mathbb{R}^\infty$. Define also the infinite matrices

$$\frac{\partial}{\partial U} = \begin{pmatrix} \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial u'} & \dfrac{\partial}{\partial u''} & \dots \end{pmatrix} \quad \text{and} \quad dU = \begin{pmatrix} du \\ du' \\ du'' \\ \vdots \end{pmatrix}.$$

Then the infinite sequence of symbols (4.1) in matrix notation is $(t, U) \in \mathcal{J}_{1,1}$. Note that $U' = CU$ can be considered as a system of ODEs and the solution of this system in matrix notation is $U_t = e^{Ct}U$. Thus, we have the following implication

$$U' = CU \quad \Longrightarrow \quad U_t = e^{Ct}U \tag{4.2}$$

called the *exponential law* in jet space $\mathcal{J}_{1,1}$. In (4.2) we have

$$u_t = \sum_{k=0}^{\infty} u^{(k)}\frac{t^k}{k!}, \qquad u_t^{(l)} = \sum_{k=0}^{\infty} u^{(k+l)}\frac{t^k}{k!}, \quad l = 0, 1, 2, \dots. \tag{4.3}$$

For instance, if $u = u(t)$ is a smooth function of t, then the series expansions u_t and $u_t^{(l)}$ in (4.3) are the Maclaurin series of u and its derivatives $u^{(l)}$, where $l = 0, 1, 2, \ldots$, respectively. A one-parametric group of transformations $a_t : (0, U) \mapsto (t, U_t)$ in $\mathcal{J}_{1,1}$ induces a flow of a linear vector field

$$\boxed{D = \frac{\partial}{\partial t} + u' \frac{\partial}{\partial u} + u'' \frac{\partial}{\partial u'} + u''' \frac{\partial}{\partial u''} + \ldots}, \tag{4.4}$$

called the *total differentiation operator* (TDO). (4.4) in matrix notation is

$$D = \frac{\partial}{\partial t} + \frac{\partial}{\partial U} U' = \begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial U} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ U' \end{pmatrix}. \tag{4.5}$$

The occurrence of TDO (4.4) can be explained as follows. Let $u = u(t)$ and $f = f(t, u(t), \dot{u}(t), \ddot{u}(t), \ldots)$ be smooth functions. From

$$\frac{df(t, U(t))}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} \dot{u} + \frac{\partial f}{\partial \dot{u}} \ddot{u} + \ldots = \left(\frac{\partial}{\partial t} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}} + \ldots \right) f$$

it follows that $f' = Df$. Here we use primes in order to denote derivatives[1] with respect to D.

Remark 4.1. Note that $u' = Du$, $u'' = Du', \ldots$, or in matrix notation: $U' = DU$, i.e., D acts on fiber coordinates as an endomorphism, that allows us to replace differentiations by algebraic operations.

The TDO (4.4) is a linear vector field, i.e., its components are linear and homogeneous functions of (4.1). The correspondence $U \mapsto U_t$ determines a flow on the initial fiber $\mathbb{R}^\infty$. The point U_t for $t = 0$ is situated in the initial position $U = U_0$. As t changes the point moves along its trajectory. The vector field D is not tangential to the fibers and its trajectories are curves (t, U_t). A point which moves on the fiber according to the law $U_t = e^{Ct} U$ is transported simultaneously with the fiber and forms the trajectory of D, see Figure 4.1.

The time t is a *canonical parameter* of D, i.e., $Dt = 1$. The next Proposition determines *invariants* of D.

Proposition 4.1. *The infinite system of functions*

$$\boxed{I = e^{-Ct} U} \tag{4.6}$$

---

[1] In order to denote differentiation of any function or, in general, of any tensor field with respect to D (Lie differentiation), we use ordinary primes.

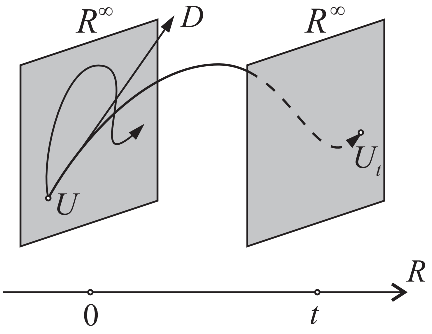

Figure 4.1: Jet bundle $\mathcal{J}_{1,1}$; U_t is a curve on a fiber and (t, U_t) is a trajectory of the operator D.

or in the detailed form: $i_k = \sum_{l=0}^{\infty} u^{(k+l)} \frac{(-t)^l}{l!}$, $k = 0, 1, 2, \ldots$,

$$\begin{cases} i_0 = u - u't + u''\dfrac{t^2}{2} - u'''\dfrac{t^3}{6} + \ldots \\ i_1 = u' - u''t + u'''\dfrac{t^2}{2} - u^{(IV)}\dfrac{t^3}{6} + \ldots \\ i_2 = u'' - u'''t + u^{(IV)}\dfrac{t^2}{2} - u^{(V)}\dfrac{t^3}{6} + \ldots \\ \quad \vdots \end{cases}$$

consists of invariants of the operator D.

Proof. The equality (4.6) is obtained from (4.2) by the substitution $t \rightsquigarrow -t$, i.e., $I = U_{-t}$. From

$$I' = e^{-Ct}U' - Ce^{-Ct}U = e^{-Ct}(U' - CU) = 0$$

it follows that I remains constant on the trajectories of the operator D. ■

The formula $U_t = e^{Ct}U$ in (4.2) determines an action of the additive group $\mathbb{R}$ on a fiber. The exponential e^{Ct} is understood as 1-parameter subgroup of the infinite linear group $GL(\mathbb{R})$ and C as the element of the corresponding Lie algebra $gl(\mathbb{R})$.

The *natural basis* in $\mathcal{J}_{1,1}$ consists of the frame $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial U}\right)$ and coframe (dt, dU). The next Proposition determines an *adapted basis*[2] in $\mathcal{J}_{1,1}$.

Proposition 4.2. *The adapted basis in* $\mathcal{J}_{1,1}$ *is defined by the frame* $\left(D, \frac{\partial}{\partial U}\right)$ *and dual coframe* (dt, ω), *where*

$$\boxed{\omega = dU - U'dt} \tag{4.7}$$

---

[2] The concept of the adapted basis is coming from the *theory of connections*, see [2], [15] and [16].

is the family of Cartan forms[3]

$$\begin{aligned}\omega^0 &= du - u'dt,\\ \omega^1 &= du' - u''dt,\\ \omega^2 &= du'' - u'''dt,\\ &\ \ \vdots\end{aligned}$$

Proof. Let us replace the operator $\dfrac{\partial}{\partial t}$ in natural frame by the operator D. Then using (4.5) we have

$$\left(D, \frac{\partial}{\partial U}\right) = \left(\left(\frac{\partial}{\partial t} \ \ \frac{\partial}{\partial U}\right) \cdot \begin{pmatrix}1\\ U'\end{pmatrix}, \frac{\partial}{\partial U}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial U}\right) \cdot \begin{pmatrix}1 & 0\\ U' & E\end{pmatrix}, \tag{4.8}$$

where the regular infinite matrix $\begin{pmatrix}1 & 0\\ U' & E\end{pmatrix}$ determines the transformation of natural frame to the adapted one. Thus, the transformation of the dual coframe is determined by the inverse matrix

$$\begin{pmatrix}1 & 0\\ U' & E\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ -U' & E\end{pmatrix}$$

as follows

$$\begin{pmatrix}1 & 0\\ U' & E\end{pmatrix}^{-1} \cdot \begin{pmatrix}dt\\ dU\end{pmatrix} = \begin{pmatrix}1 & 0\\ -U' & E\end{pmatrix} \cdot \begin{pmatrix}dt\\ dU\end{pmatrix} = \begin{pmatrix}dt\\ dU - U'dt\end{pmatrix} = \begin{pmatrix}dt\\ \omega\end{pmatrix}. \tag{4.9}$$

The duality follows from

$$\begin{pmatrix}dt\\ \omega\end{pmatrix} \left(D \ \ \frac{\partial}{\partial U}\right) = \begin{pmatrix}dt(D) & dt\left(\dfrac{\partial}{\partial U}\right)\\ \omega(D) & \omega\left(\dfrac{\partial}{\partial U}\right)\end{pmatrix} = E. \qquad \blacksquare$$

Proposition 4.3. *The dragging of the Cartan forms* ω *along the flow of* D *in* $\mathcal{J}_{1,1}$ *is described by the law*

$$\omega' = C\omega \quad \Longrightarrow \quad \omega_t = e^{Ct}\omega. \tag{4.10}$$

The forms ω *considered as a whole become exact differentials as follows:*

$$\boxed{dI = e^{-Ct}\omega}. \tag{4.11}$$

Proof. Note that in the infinite sequences $\omega' = C\omega$ each form is the Lie derivative of the previous one with respect to D, i.e.,

$$\omega^1 = \mathcal{L}_D\omega^0, \quad \omega^2 = \mathcal{L}_D\omega^1 = \mathcal{L}_D(\mathcal{L}_D\omega^0), \quad \omega^3 = \mathcal{L}_D\omega^2 = \mathcal{L}_D(\mathcal{L}_D(\mathcal{L}_D\omega^0)), \quad \ldots.$$

---

[3] In literature they are often called *contact forms*.

Using $U' = CU$ and $U'' = CU'$ in matrix notations we have

$$\omega' = \mathcal{L}_D\omega = \mathcal{L}_D(dU - U'dt) = dU' - U''dt = C(dU - U'dt) = C\omega.$$

Consider $\omega' = C\omega$ as an ODE. Then the solution of this equation is $\omega_t = e^{Ct}\omega$ and (4.10) holds.

(4.11) is obtained by calculating of differential of (4.6):

$$dI = d(e^{-Ct}U) = e^{-Ct}dU - Ce^{-Ct}Udt = e^{-Ct}(dU - U'dt) = e^{-Ct}\omega.$$

The forms ω are not exact: $\mathbf{d}\omega = -dU' \wedge dt \neq 0$. From (4.11) it follows that the exponential e^{-Ct} is an *integrating matrix*, so the Cartan forms considered as a whole become the exact differentials[4]. ■

Proposition 4.4. *The dragging of the vertical frame* $\dfrac{\partial}{\partial U}$ *along the flow of* D *is determined by the law*

$$\boxed{\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U}C \quad \Longrightarrow \quad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U}e^{-Ct}.} \tag{4.12}$$

Proof. The derivation formula for the vertical frame $\dfrac{\partial}{\partial U}$ is obtained by the straightforward calculation of the Lie derivative with respect to D:

$$\begin{aligned}\left(\frac{\partial}{\partial U}\right)' &= \left[D, \frac{\partial}{\partial U}\right] = \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial U}U', \frac{\partial}{\partial U}\right] = \\ &= \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial U}\right] + \left[\frac{\partial}{\partial U}U', \frac{\partial}{\partial U}\right] = \\ &= -\left[\frac{\partial}{\partial U}(U')\right]\frac{\partial}{\partial U} = -C\left[\frac{\partial}{\partial U}(U)\right]\frac{\partial}{\partial U} = \\ &= -\frac{\partial}{\partial U}C.\end{aligned}$$

Considering the left equality in (4.12) as ODE, the solution is presented by the right one and, thus, (4.12) is valid. ■

Proposition 4.5. *In the jet space* $\mathcal{J}_{1,1}$ *there is defined an invariant basis by*

$$\begin{pmatrix} D & \dfrac{\partial}{\partial I}\end{pmatrix} = \begin{pmatrix} D & \dfrac{\partial}{\partial U}\end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{Ct}\end{pmatrix}, \quad \begin{pmatrix} dt \\ dI \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-Ct}\end{pmatrix} \cdot \begin{pmatrix} dt \\ \omega \end{pmatrix}. \tag{4.13}$$

Proof. In the formula (4.6) the matrix e^{-Ct} is invertible. Thus, the fiber coordinates U can be expressed by means of the invariants I. Note that the replacement of U with I is accompanied by the corresponding transformation of the frame and

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[4] This is a good reason to study infinite jets instead of finite ones, because in the finite case there is not such an integrating factor that allows inexact Cartan forms to be made into exact ones.

coframe. From (4.11) it follows that the transformation of the adapted coframe (dt, ω) is defined by (4.13). Thus, the transformation of the adapted frame $\left(D, \dfrac{\partial}{\partial U}\right)$ is defined by the inverse matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{-Ct} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & e^{Ct} \end{pmatrix}. \qquad \blacksquare$$

The first four operators in the infinite sequence

$$\boxed{\frac{\partial}{\partial I} = \frac{\partial}{\partial U} e^{Ct}} \tag{4.14}$$

are

$$\begin{aligned}
\frac{\partial}{\partial i_0} &= \frac{\partial}{\partial u}, \\
\frac{\partial}{\partial i_1} &= t\frac{\partial}{\partial u} + \frac{\partial}{\partial u'}, \\
\frac{\partial}{\partial i_2} &= \frac{t^2}{2}\frac{\partial}{\partial u} + t\frac{\partial}{\partial u'} + \frac{\partial}{\partial u''}, \\
\frac{\partial}{\partial i_3} &= \frac{t^3}{6}\frac{\partial}{\partial u} + \frac{t^2}{2}\frac{\partial}{\partial u'} + t\frac{\partial}{\partial u''} + \frac{\partial}{\partial u'''}, \\
&\vdots
\end{aligned}$$

These operators commute pairwise and each of them commutes with D, i.e.

$$\left[D, \frac{\partial}{\partial I}\right] = 0, \quad \left[\frac{\partial}{\partial i_p}, \frac{\partial}{\partial i_q}\right] = 0, \quad p, q = 0, 1, 2, \ldots.$$

Consider a vertical vector field P in $\mathcal{J}_{1,1}$. Denote by μ and ν its components (infinite columns) in the natural and invariant frames, respectively. Calculate the Lie derivative with respect to D:

$$P = \frac{\partial}{\partial U}\mu = \frac{\partial}{\partial I}\nu \quad \Longrightarrow \quad P' = \frac{\partial}{\partial U}(\mu' - C\mu) = \frac{\partial}{\partial I}\nu'.$$

The vector fields P and D commute, i.e., $P' = 0$, iff
– either the components of ν are invariant with respect to D, i.e., $\nu' = 0$,
– or, equivalently, the components of μ satisfy the condition $\mu' = C\mu$.
The last condition $\mu' = C\mu$ means that in the sequence of functions μ_0, μ_1, $\mu_2, \ldots$ (the elements of the column μ) each function is the derivative of the previous one, i.e.

$$\mu_k = \mu'_{k-1}, \quad \text{or} \quad \mu_k = \mu_0^{(k)}, \quad k = 1, 2, \ldots.$$

The function μ_0 is said to be a *generating function* for P, and the vector field P with the condition $P' = 0$ is said to be a vertical *Lie vector field*.

Note that the operators $\dfrac{\partial}{\partial I}$, see (4.14), are vertical Lie vector fields with generating functions $1, t, \dfrac{t^2}{2}, \ldots$, respectively.

In general, a vector field P is said to be a *Lie vector field* iff P is an infinitesimal symmetry of the operator D, i.e., $P' \| D$, where $\|$ means equality up to a coefficient of proportionality that is some matrix. Such a vector field has both vertical and horizontal components.

Example 4.1 (see [2], p.245)**.** Consider the linear group $GL(3,\mathbb{R})$. Taking elements of an arbitrary matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in GL(3,\mathbb{R})$$

as coordinates the group $GL(3,\mathbb{R})$ can be identified with $\mathbb{R}^9$. In $GL(3,\mathbb{R})$ there are defined the left-invariant and right-invariant frames by

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} = \begin{pmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_4 & \partial_5 & \partial_6 \\ \partial_7 & \partial_8 & \partial_9 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{X}_1 & \widetilde{X}_2 & \widetilde{X}_3 \\ \widetilde{X}_4 & \widetilde{X}_5 & \widetilde{X}_6 \\ \widetilde{X}_7 & \widetilde{X}_8 & \widetilde{X}_9 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_4 & \partial_5 & \partial_6 \\ \partial_7 & \partial_8 & \partial_9 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{pmatrix},$$

respectively, where $\partial_i = \dfrac{\partial}{\partial a_i}$, $i = 1, \ldots, 9$.

Let $GL(3,\mathbb{R})$ acts on itself by inner automorphisms. Then the corresponding adjoint representation of $GL(3,\mathbb{R})$ is determined by the operators $Y_i = \widetilde{X}_i - X_i$, $i = 1, \ldots, 9$ (we do not write them out). The common invariants of the adjoint representation[5] are

$$u = \frac{1}{6} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix},$$

$$u' = \frac{1}{6} \left(\begin{vmatrix} a_1 & a_2 \\ a_4 & a_5 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ a_7 & a_9 \end{vmatrix} + \begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix} \right),$$

$$u'' = \frac{1}{3}(a_1 + a_5 + a_9).$$

The center of $GL(3,\mathbb{R})$ consists of diagonal matrices, i.e., $Z = \{\tau E : \tau \in \mathbb{R}\}$, and the operator of Z at the unity $E \in GL(3,\mathbb{R})$ is

$$P = \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_5} + \frac{\partial}{\partial a_9}$$

---

[5] The general linear group $GL(n,\mathbb{R})$ acts on itself by inner automorphisms $A \mapsto BAB^{-1}$. The invariants of this action are precisely the sums of diagonal minors of the matrix A.

with the flow $a_t : A \mapsto A_t = A + tE$. Note that $Pu = u'$, $Pu' = u''$, $Pu'' = 1$, and P can be expressed in the coordinates (u, u', u'') as follows:

$$P = u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \frac{\partial}{\partial u''}.$$

Taking $t = u''$ the operator P can be connected with the total differentiation operator D in $\mathcal{J}_{1,1}$, because $Pu'' = 1$ and $Dt = 1$. Then one can obtain the flow and invariants of P by (4.2) and (4.6), respectively,

$$\begin{cases} u_t = u + u't + u''\dfrac{t^2}{2} + \dfrac{t^3}{6} \\ u_t' = u' + u''t + \dfrac{t^2}{2} \\ u_t'' = u'' + t \end{cases}, \qquad \begin{cases} \sigma_0 = u - u'u'' + \dfrac{(u'')^3}{3} \\ \sigma_1 = u' - \dfrac{(u'')^2}{2} \end{cases}.$$

The invariants σ_0 and σ_1 are not stable, they can be perturbed by infinitesimal symmetries of P such as the operators

$$Q_1 = \frac{\partial}{\partial u}, \qquad Q_2 = u''\frac{\partial}{\partial u} + \frac{\partial}{\partial u'},$$

cf. (4.14), that correspond to $\dfrac{\partial}{\partial \sigma_0}$ and $\dfrac{\partial}{\partial \sigma_1}$, respectively. From $[Q_1, P] = [Q_2, P] = 0$ it follows that Q_1 and Q_2 are indeed infinitesimal symmetries of P. Being projected by the map $\mathbb{R}^3 \to \mathbb{R}^2 : (u, u', u'') \mapsto (\sigma_0, \sigma_1)$ these operators yield the natural frame in $\sigma_0\sigma_1$ plane, i.e.,

$$\begin{pmatrix} Q_1\sigma_0 & Q_1\sigma_1 \\ Q_2\sigma_0 & Q_2\sigma_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The invariants σ_0 and σ_1 are not stable, but the cubic discriminant of the polynomial u_t gives the stable invariant

$$I = (3\sigma_0)^2 + (2\sigma_1)^3$$

with respect to infinitesimal symmetry of P, which is the operator

$$Q = 4\sigma_1^2 Q_1 - 3\sigma_0 Q_2,$$

that corresponds to $4\sigma_1^2\dfrac{\partial}{\partial \sigma_0} - 3\sigma_0\dfrac{\partial}{\partial \sigma_1}$. Indeed, $dI = 6(3\sigma_0 d\sigma_0 - 4\sigma_1^2 d\sigma_1)$ and $QI = 0$.

4.2 Jet spaces $\mathcal{J}_{2,1}$ and $\mathcal{J}_{n,1}$

Let us consider the case $n = 2$, $m = 1$, that corresponds to the two-dimensional time $t = (t^1, t^2)$. Then the quantities in the sequence (4.1) can be rewritten as

$$u' = (u_1, u_2), \quad u'' = (u_{11}, u_{12}, u_{22}), \quad u''' = (u_{111}, u_{112}, u_{122}, u_{222}), \quad \ldots,$$

and the coordinates in $\mathcal{J}_{2,1}$ are

$$t^1, t^2, u, u_1, u_2, u_{11}, u_{12}, u_{22}, u_{111}, u_{112}, u_{122}, u_{222}, \dots . \tag{4.15}$$

For instance, if u is a smooth function of two independent variables $u = u(t^1, t^2)$, then

$$u_i = \frac{\partial u}{\partial t^i}, \quad u_{ij} = \frac{\partial^2 u}{\partial t^i \partial t^j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial t^i \partial t^j \partial t^k}, \quad \dots,$$

where $i, j, k, \dots = 1, 2$, and (4.15) becomes a jet of the given function, which is the section of the trivial bundle $\mathbb{R}^2 \times \mathbb{R}^\infty$. The sequence (4.15) can be presented briefly as (t, U), where the infinite matrix

$$U = \begin{pmatrix} u & u_2 & u_{22} & u_{222} & \dots \\ u_1 & u_{12} & u_{122} & u_{1222} & \dots \\ u_{11} & u_{112} & u_{1122} & u_{11222} & \dots \\ u_{111} & u_{1112} & u_{11122} & u_{111222} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

consists of fiber coordinates.

In $\mathcal{J}_{2,1}$ there are defined two TDOs

$$\begin{aligned} D_1 &= \frac{\partial}{\partial t^1} + u_1 \frac{\partial}{\partial u} + u_{11} \frac{\partial}{\partial u_1} + u_{12} \frac{\partial}{\partial u_2} + u_{111} \frac{\partial}{\partial u_{11}} + u_{112} \frac{\partial}{\partial u_{12}} + u_{122} \frac{\partial}{\partial u_{22}} + \dots, \\ D_2 &= \frac{\partial}{\partial t^2} + u_2 \frac{\partial}{\partial u} + u_{12} \frac{\partial}{\partial u_1} + u_{22} \frac{\partial}{\partial u_2} + u_{112} \frac{\partial}{\partial u_{11}} + u_{122} \frac{\partial}{\partial u_{12}} + u_{222} \frac{\partial}{\partial u_{22}} + \dots, \end{aligned}$$

or

$$\boxed{D_i = \frac{\partial}{\partial t^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_i} + u_{ijk} \frac{\partial}{\partial u_{ij}} + u_{ijkl} \frac{\partial}{\partial u_{ijk}} + \dots}, \tag{4.16}$$

where $i, j, k, l, \dots = 1, 2$. The action of D_1 and D_2 on the fiber coordinates is determined by

$$U_1 = D_1 U = \begin{pmatrix} u_1 & u_{12} & u_{122} & \dots \\ u_{11} & u_{112} & u_{1122} & \dots \\ u_{111} & u_{1112} & u_{11122} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \; U_2 = D_2 U = \begin{pmatrix} u_2 & u_{22} & u_{222} & \dots \\ u_{12} & u_{122} & u_{1222} & \dots \\ u_{112} & u_{1122} & u_{11222} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$U_{11} = D_1 U_1 = \begin{pmatrix} u_{11} & u_{112} & u_{1122} & \dots \\ u_{111} & u_{1112} & u_{11122} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \; U_{22} = D_2 U_2 = \begin{pmatrix} u_{22} & u_{222} & \dots \\ u_{122} & u_{1222} & \dots \\ u_{1122} & u_{11222} & \dots \\ \dots & \dots & \dots \end{pmatrix},$$

$$U_{12} = D_1 U_2 = D_2 U_1 = \begin{pmatrix} u_{12} & u_{122} & u_{1222} & \dots \\ u_{112} & u_{1122} & u_{11222} & \dots \\ u_{1112} & u_{11122} & u_{111222} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad \dots,$$

or $U_i = D_i U$, $U_{ij} = D_i D_j U$, …, where $i, j, \ldots = 1, 2$.

TDOs D_1 and D_2 are linearly independent and commute: $[D_1, D_2] = 0$. Thus, D_1 and D_2 span a two-dimensional completely integrable distribution $D = (D_1, D_2)$. A flow of D is determined by the correspondence $U \mapsto U_t$, where

$$\begin{aligned} U_t = U + U_1 t^1 + U_2 t^2 + \frac{1}{2}\big(U_{11}(t^1)^2 + 2U_{12}t^1t^2 + U_{22}(t^2)^2\big) + \\ + \frac{1}{6}\big(U_{111}(t^1)^3 + 3U_{112}(t^1)^2t^2 + 3U_{122}t^1(t^2)^2 + U_{222}(t^2)^3\big) + \ldots . \end{aligned} \tag{4.17}$$

The first element u_t in (4.17) can be obtained as a Maclaurin series of a smooth function $u = u(t^1, t^2)$ in two variables, and the others elements are the corresponding partial derivatives of all orders of u_t with respect to t^1 and t^2, respectively.

The Cartan forms in $\mathcal{J}_{2,1}$ are elements of the following infinite-dimensional matrix:

$$\boxed{\omega = dU - U_1 dt^1 - U_2 dt^2}, \tag{4.18}$$

or in the detailed form

$$\omega = \begin{pmatrix} \omega_0 & \omega_2 & \omega_{22} & \cdots \\ \omega_1 & \omega_{12} & \omega_{122} & \cdots \\ \omega_{11} & \omega_{112} & \omega_{1122} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \text{where} \quad \begin{aligned} \omega_0 &= du - u_1 dt^1 - u_2 dt^2, \\ \omega_1 &= du_1 - u_{11} dt^1 - u_{12} dt^2, \\ \omega_2 &= du_2 - u_{12} dt^1 - u_{22} dt^2, \\ \omega_{11} &= du_{11} - u_{111} dt^1 - u_{112} dt^2, \\ \omega_{12} &= du_{12} - u_{112} dt^1 - u_{122} dt^2, \\ \omega_{22} &= du_{22} - u_{122} dt^1 - u_{222} dt^2, \\ &\vdots \end{aligned}$$

These forms can be rewritten as follows:

$$\omega_0 = du - u_i dt^i, \quad \omega_i = du_i - u_{ij} dt^j, \quad \omega_{ij} = du_{ij} - u_{ijk} dt^k, \quad \ldots,$$

or, using Lie derivatives,

$$\omega_i = \mathcal{L}_{D_i}\omega_0, \quad \omega_{ij} = \mathcal{L}_{D_i}\mathcal{L}_{D_j}\omega_0, \quad \omega_{ijk} = \mathcal{L}_{D_i}\mathcal{L}_{D_j}\mathcal{L}_{D_k}\omega_0, \quad \ldots, \tag{4.19}$$

where $i, j, k, \ldots = 1, 2$.

Let us generalize this situation to the case of n-dimensional time

$$t = (t^1, t^2, \ldots, t^n).$$

Then the corresponding coordinates in jet space $\mathcal{J}_{n,1}$ are

$$t^i, u, u_i, u_{ij}, u_{ijk}, \ldots, \quad i, j, k, \ldots = 1, 2, \ldots, n. \tag{4.20}$$

In $\mathcal{J}_{n,1}$ there are defined n linearly independent TDOs $D_1, D_2, \ldots, D_n$. It is easy to see that these TDOs are of the form (4.16) for all indices $i, j, k, \ldots$ run over $1, 2, \ldots, n$. Analogously, we have the same patterns (4.18) and (4.19) for the sequence of Cartan forms in $\mathcal{J}_{n,1}$ for $i, j, k, \ldots = 1, 2, \ldots, n$.

4.3 Jet spaces $\mathcal{J}_{1,2}$ and $\mathcal{J}_{1,m}$

Suppose in $\mathcal{J}_{n,m}$ we have $n=1$ and $m=2$. Then the time t is one-dimensional and in addition to the function u we have one more function v. The coordinate functions in $\mathcal{J}_{1,2}$ form the infinite sequence

$$t, u, v, u', v', u'', v'', u''', v''', \ldots. \tag{4.21}$$

The sequence (4.21) can be considered as a jet of a smooth vector-valued function $t \mapsto (u, v)$ or $\begin{cases} u = u(t), \\ v = v(t), \end{cases}$ $t \in \mathbb{R}$. In $\mathcal{J}_{1,2}$ there are defined one TDO

$$D = \frac{\partial}{\partial t} + u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v} + u''\frac{\partial}{\partial u'} + v''\frac{\partial}{\partial v'} + \ldots \tag{4.22}$$

and two infinite sequences of Cartan forms

$$\begin{array}{ll} \omega_1 = du - u'dt & \omega_2 = dv - v'dt \\ \omega_1' = du' - u''dt & \omega_2' = dv' - v''dt \\ \omega_1'' = du'' - u'''dt & \omega_2'' = dv'' - v'''dt \\ \vdots & \vdots \end{array} \tag{4.23}$$

Being in the space $\mathcal{J}_{1,m}$ it is easy to rewrite (4.21), (4.22) and (4.23) in the case of vector-valued function $t \mapsto (u^1, u^2, \ldots, u^m)$ with $(m+1)$-dimensional range. In $\mathcal{J}_{1,m}$ we have one TDO D and m infinite sequences (4.23) of Cartan forms $\omega_1, \omega_2, \ldots, \omega_m$ and their Lie derivatives of all orders with respect to D.

4.4 The general case $\mathcal{J}_{n,m}$

Let us consider the general case $\mathcal{J}_{n,m}$ – the space of infinite jets of smooth maps $\mathbb{R}^n \to \mathbb{R}^m$. In $\mathcal{J}_{n,m}$ the time t is n-dimensional: $t = (t^1, \ldots, t^n)$, and u in the sequence (4.1) is a system on m quantities $u = (u^1, \ldots, u^m)$. Thus, the coordinate functions in $\mathcal{J}_{n,m}$ form the infinite sequence

$$t^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, u^\alpha_{ijk}, \ldots, \tag{4.24}$$

where $i, j, k, \ldots = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$.

Combining (4.16), (4.18), (4.22) and (4.23), we obtain the formulas for TDOs and Cartan forms in $\mathcal{J}_{n,m}$:

$$\boxed{D_i = \frac{\partial}{\partial t^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij}\frac{\partial}{\partial u^\alpha_i} + u^\alpha_{ijk}\frac{\partial}{\partial u^\alpha_{ij}} + u^\alpha_{ijkl}\frac{\partial}{\partial u^\alpha_{ijk}} + \ldots}, \tag{4.25}$$

$$\boxed{\omega^\alpha_0 = du^\alpha - u^\alpha_i dt^i,\ \omega^\alpha_i = du^\alpha_i - u^\alpha_{ij}dt^j,\ \omega^\alpha_{ij} = du^\alpha_{ij} - u^\alpha_{ijk}dt^k,\ \ldots}, \tag{4.26}$$

where $i, j, k, l, \ldots = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$.

Proposition 4.6. *The TDOs* (4.25) *commute pairwise, i.e.,*

$$[D_i, D_j] = 0, \quad \forall i, j = 1, 2, \dots, n, \tag{4.27}$$

and form in $\mathcal{J}_{n,m}$ an n-dimensional completely integrable distribution

$$D = (D_1, D_2, \dots, D_n).$$

Proof. Using (2.29) we have

$$[D_i, D_j] = 0 \cdot \frac{\partial}{\partial t^k} + (D_i u_j^\alpha - D_j u_i^\alpha)\frac{\partial}{\partial u^\alpha} + (D_i u_{jk}^\alpha - D_j u_{ik}^\alpha)\frac{\partial}{\partial u_k^\alpha} + \dots .$$

Due to

$$D_i u_j^\alpha - D_j u_i^\alpha = u_{ij}^\alpha - u_{ij}^\alpha = 0, \quad D_i u_{jk}^\alpha - D_j u_{ik}^\alpha = u_{ijk}^\alpha - u_{ijk}^\alpha = 0, \quad \dots,$$

where $i, j, k, \dots = 1, 2, \dots, n$, we have $[D_i, D_j] = 0$. The TDOs (4.25) are linearly independent and indeed form an n-dimensional completely integrable distribution $D = (D_1, D_2, \dots, D_n)$. ■

From Proposition 4.6 it follows that in $\mathcal{J}_{n,m}$ there is defined an action of the additive group $\mathbb{R}^n$ with orbits that are n-dimensional integral surfaces of the distribution D.

From (4.27) it follows that the derivatives

$$D_i u^\alpha = u_i^\alpha, \quad D_i u_j^\alpha = u_{ij}^\alpha, \quad D_i u_{jk}^\alpha = u_{ijk}^\alpha, \quad \dots,$$

where $i, j, k, \dots = 1, 2, \dots, n; \alpha = 1, 2, \dots, m$, are symmetric in all subscripts. Since in the sequence of Cartan forms (4.26) each subsequent form is the Lie derivative of the previous one with respect to the corresponding TDO and the TDOs commute pairwise, the symmetry in all subscripts in (4.26) holds.

4.5 Total differentiation under jet composition

4.5.1 Statement of the problem

Let four smooth manifolds $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be given with dimensions $\dim \mathcal{A} = n_1$, $\dim \mathcal{B} = n_2$, $\dim \mathcal{C} = n_3$ and $\dim \mathcal{D} = n_4$. Suppose that local coordinates (t^i), (u^α), (v^λ) and (w^ρ) are given on corresponding local neighborhoods in the spaces $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, respectively. For the sake of convenience assume that

i, j, k run over $1, 2, \dots, n_1$;
α, β, γ run over $n_1 + 1, n_1 + 2, \dots, n_1 + n_2$;
λ, μ, ν run over $n_1 + n_2 + 1, n_1 + n_2 + 2, \dots, n_1 + n_2 + n_3$;
ρ, σ run over $n_1 + n_2 + n_3 + 1, n_1 + n_2 + n_3 + 2, \dots, n_1 + n_2 + n_3 + n_4$.

Consider three smooth maps

$$f : \mathcal{A} \to \mathcal{B}, \quad g : \mathcal{B} \to \mathcal{C}, \quad h : \mathcal{A} \to \mathcal{C},$$

where $h = g \circ f$. Then f, g and h are presented locally by the functions

$$u^\alpha \circ f = f^\alpha, \quad v^\lambda \circ g = g^\lambda, \quad v^\lambda \circ h = h^\lambda.$$

Hence, the function g^λ and h^λ are related by $h^\lambda = g^\lambda \circ f$. According to the chain rule the infinite sequence of partial derivatives of all orders of h^λ with respect to the coordinates t^i begins as follows:

$$\begin{aligned} h^\lambda_i &= (g^\lambda_\alpha \circ f) f^\alpha_i, \\ h^\lambda_{ij} &= (g^\lambda_{\alpha\beta} \circ f) f^\alpha_i f^\beta_j + (g^\lambda_\alpha \circ f) f^\alpha_{ij}, \\ h^\lambda_{ijk} &= (g^\lambda_{\alpha\beta\gamma} \circ f) f^\alpha_i f^\beta_j f^\gamma_k + (g^\lambda_{\alpha\beta} \circ f)(f^\alpha_i f^\beta_{jk} + f^\alpha_j f^\beta_{ik} + f^\alpha_k f^\beta_{ij}) + (g^\lambda_\alpha \circ f) f^\alpha_{ijk}, \\ &\vdots \end{aligned} \tag{4.28}$$

In the present Section the following problem is considered. Analogously to the process, when a smooth map induces an infinite jet, the map composition $h = g \circ f$ induces a *composition of jets* denoted symbolically by $J_h = J_g \circ J_f$. The problem is how to define a jet composition without any mention of concrete maps in such a way that the definition does not depend on the choice of maps. The problem leads to other questions – *how the corresponding TDOs and Cartan forms are related under the jet composition and how to derive the convenient recurrence formulas.* The following scheme illustrates this situation:

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & & C & & A \arrow[rr, "J_h"] \arrow[dr, "J_f"'] & & C \\
& B \arrow[ur, "g"'] & & & & B \arrow[ur, "J_g"'] & \\
& h = g \circ f & & \rightsquigarrow & & J_h = J_g \circ J_f &
\end{tikzcd}$$

$$(D_h, \omega_h) = (D_g, \omega_g) \overset{?}{\star} (D_g, \omega_g).$$

For the composition of smooth maps

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D}$$

there are defined jet spaces $\mathcal{J}(\mathcal{A}, \mathcal{B})$, $\mathcal{J}(\mathcal{B}, \mathcal{C})$, $\mathcal{J}(\mathcal{C}, \mathcal{D})$, $\mathcal{J}(\mathcal{A}, \mathcal{C})$ and $\mathcal{J}(\mathcal{A}, \mathcal{D})$. Provide them with coordinates:

$$\begin{aligned} \text{in } \mathcal{J}(\mathcal{A}, \mathcal{B}): &\quad t^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, u^\alpha_{ijk}, \dots; \\ \text{in } \mathcal{J}(\mathcal{B}, \mathcal{C}): &\quad u^\alpha, v^\lambda, v^\lambda_\alpha, v^\lambda_{\alpha\beta}, v^\lambda_{\alpha\beta\gamma}, \dots; \\ \text{in } \mathcal{J}(\mathcal{A}, \mathcal{C}): &\quad t^i, v^\lambda, v^\lambda_i, v^\lambda_{ij}, v^\lambda_{ijk}, \dots; \\ \text{in } \mathcal{J}(\mathcal{C}, \mathcal{D}): &\quad v^\lambda, w^\rho, w^\rho_\lambda, w^\rho_{\lambda\mu}, w^\rho_{\lambda\mu\nu}, \dots; \\ \text{in } \mathcal{J}(\mathcal{A}, \mathcal{D}): &\quad t^i, w^\rho, w^\rho_i, w^\rho_{ij}, w^\rho_{ijk}, \dots, \end{aligned}$$

define the TDOs:

$$
\begin{array}{ll}
\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{B}): & X_i = \dfrac{\partial}{\partial t^i} + u^\alpha_i \dfrac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \dfrac{\partial}{\partial u^\alpha_j} + u^\alpha_{ijk} \dfrac{\partial}{\partial u^\alpha_{jk}} + \dots; \\
\text{in}\,\mathcal{J}(\mathcal{B},\mathcal{C}): & Y_\alpha = \dfrac{\partial}{\partial u^\alpha} + v^\lambda_\alpha \dfrac{\partial}{\partial v^\lambda} + v^\lambda_{\alpha\beta} \dfrac{\partial}{\partial v^\lambda_\beta} + v^\lambda_{\alpha\beta\gamma} \dfrac{\partial}{\partial v^\lambda_{\beta\gamma}} + \dots; \\
\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{C}): & Y_i = \dfrac{\partial}{\partial t^i} + v^\lambda_i \dfrac{\partial}{\partial v^\lambda} + v^\lambda_{ij} \dfrac{\partial}{\partial v^\lambda_j} + v^\lambda_{ijk} \dfrac{\partial}{\partial v^\lambda_{jk}} + \dots; \\
\text{in}\,\mathcal{J}(\mathcal{C},\mathcal{D}): & Z_\lambda = \dfrac{\partial}{\partial v^\lambda} + w^\rho_\lambda \dfrac{\partial}{\partial w^\rho} + w^\rho_{\lambda\mu} \dfrac{\partial}{\partial w^\rho_\mu} + w^\rho_{\lambda\mu\nu} \dfrac{\partial}{\partial w^\rho_{\mu\nu}} + \dots; \\
\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{D}): & Z_i = \dfrac{\partial}{\partial t^i} + w^\rho_i \dfrac{\partial}{\partial w^\rho} + w^\rho_{ij} \dfrac{\partial}{\partial w^\rho_j} + w^\rho_{ijk} \dfrac{\partial}{\partial w^\rho_{jk}} + \dots;
\end{array}
$$

and Cartan forms:

$$
\begin{array}{ccc}
\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{B}): & \text{in}\,\mathcal{J}(\mathcal{B},\mathcal{C}): & \text{in}\,\mathcal{J}(\mathcal{A},\mathcal{C}): \\
\omega^\alpha = du^\alpha - u^\alpha_i dt^i, & \theta^\lambda = dv^\lambda - v^\lambda_\alpha du^\alpha, & \theta^\lambda = dv^\lambda - v^\lambda_i dt^i, \\
\omega^\alpha_i = du^\alpha_i - u^\alpha_{ij} dt^j, & \theta^\lambda_\alpha = dv^\lambda_\alpha - v^\lambda_{\alpha\beta} du^\beta, & \theta^\lambda_i = dv^\lambda_i - v^\lambda_{ij} dt^j, \\
\omega^\alpha_{ij} = du^\alpha_{ij} - u^\alpha_{ijk} dt^k, & \theta^\lambda_{\alpha\beta} = dv^\lambda_{\alpha\beta} - v^\lambda_{\alpha\beta\gamma} du^\gamma, & \theta^\lambda_{ij} = dv^\lambda_{ij} - v^\lambda_{ijk} dt^k, \\
\vdots & \vdots & \vdots
\end{array}
$$

$$
\begin{array}{cc}
\text{in}\,\mathcal{J}(\mathcal{C},\mathcal{D}): & \text{in}\,\mathcal{J}(\mathcal{A},\mathcal{D}): \\
\vartheta^\rho = dw^\rho - w^\rho_\lambda dv^\lambda, & \vartheta^\rho = dw^\rho - w^\rho_i dt^i, \\
\vartheta^\rho_\lambda = dw^\rho_\lambda - w^\rho_{\lambda\mu} dv^\mu, & \vartheta^\rho_i = dw^\rho_i - w^\rho_{ij} dt^j, \\
\vartheta^\rho_{\lambda\mu} = dw^\rho_{\lambda\mu} - w^\rho_{\lambda\mu\nu} dv^\nu, & \vartheta^\rho_{ij} = dw^\rho_{ij} - w^\rho_{ijk} dt^k, \\
\vdots & \vdots
\end{array}
$$

Note that in each sequence of Cartan forms each next form is the Lie derivative of the previous one with respect to the corresponding TDO:

$$
\begin{array}{ccccc}
\omega^\alpha_i = \mathcal{L}_{X_i}\omega^\alpha, & \theta^\lambda_\alpha = \mathcal{L}_{Y_\alpha}\theta^\lambda, & \theta^\lambda_i = \mathcal{L}_{Y_i}\theta^\lambda, & \vartheta^\rho_\lambda = \mathcal{L}_{Z_\lambda}\vartheta^\rho, & \vartheta^\rho_i = \mathcal{L}_{Z_i}\vartheta^\rho, \\
\omega^\alpha_{ij} = \mathcal{L}_{X_j}\omega^\alpha_i, & \theta^\lambda_{\alpha\beta} = \mathcal{L}_{Y_\beta}\theta^\lambda_\alpha, & \theta^\lambda_{ij} = \mathcal{L}_{Y_j}\theta^\lambda_i, & \vartheta^\rho_{\lambda\mu} = \mathcal{L}_{Z_\mu}\vartheta^\rho_\lambda, & \vartheta^\rho_{ij} = \mathcal{L}_{Z_j}\vartheta^\rho_i, \\
\omega^\alpha_{ijk} = \mathcal{L}_{X_k}\omega^\alpha_{ij}, & \theta^\lambda_{\alpha\beta\gamma} = \mathcal{L}_{Y_\gamma}\theta^\lambda_{\alpha\beta}, & \theta^\lambda_{ijk} = \mathcal{L}_{Y_k}\theta^\lambda_{ij}, & \vartheta^\rho_{\lambda\mu\nu} = \mathcal{L}_{Z_\nu}\vartheta^\rho_{\lambda\mu}, & \vartheta^\rho_{ijk} = \mathcal{L}_{Z_k}\vartheta^\rho_{ij}, \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

In the case of multi-variable formulas it is convenient to use the multi-index notation. Recall that a *multi-index* of range m is an ordered m-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ of non-negative integers. Thus, for multi-indices

$$
(j) = j_1 j_2 \dots j_p, \quad (\beta) = \beta_1 \beta_2 \dots \beta_p, \quad (\mu) = \mu_1 \mu_2 \dots \mu_p, \quad p = 0, 1, 2, \dots,
$$

the coordinates, TDOs and Cartan forms in the corresponding jet spaces can be rewritten in the compact form as follows:

$$\begin{aligned}
&\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{B}): && (t^i, u^\alpha_{(j)}), && X_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial u^\alpha_{(j)}} u^\alpha_{i(j)}, && \omega^\alpha_{(j)} = du^\alpha_{(j)} - u^\alpha_{i(j)} dt^i;\\
&\text{in}\,\mathcal{J}(\mathcal{B},\mathcal{C}): && (u^\alpha, v^\lambda_{(\beta)}), && Y_\alpha = \frac{\partial}{\partial u^\alpha} + \frac{\partial}{\partial v^\lambda_{(\beta)}} v^\lambda_{\alpha(\beta)}, && \theta^\lambda_{(\beta)} = dv^\lambda_{(\beta)} - v^\lambda_{\alpha(\beta)} du^\alpha;\\
&\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{C}): && (t^i, v^\lambda_{(j)}), && Y_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial v^\lambda_{(j)}} v^\lambda_{i(j)}, && \theta^\lambda_{(j)} = dv^\lambda_{(j)} - v^\lambda_{i(j)} dt^i;\\
&\text{in}\,\mathcal{J}(\mathcal{C},\mathcal{D}): && (v^\lambda, w^\rho_{(\mu)}), && Z_\lambda = \frac{\partial}{\partial v^\lambda} + \frac{\partial}{\partial w^\rho_{(\mu)}} w^\rho_{\lambda(\mu)}, && \vartheta^\rho_{(\mu)} = dw^\rho_{(\mu)} - w^\rho_{\lambda(\mu)} dv^\lambda;\\
&\text{in}\,\mathcal{J}(\mathcal{A},\mathcal{D}): && (t^i, w^\rho_{(j)}), && Z_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial w^\rho_{(j)}} w^\rho_{i(j)}, && \vartheta^\rho_{(j)} = dw^\rho_{(j)} - w^\rho_{i(j)} dt^i.
\end{aligned}$$

4.5.2 The intermediate space

In order to define a composition of jets first we need to determine the *intermediate* space $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ with coordinates

$$t^i, u^\alpha_i, u^\alpha_{ij}, u^\alpha_{ijk}, \ldots, v^\lambda_\alpha, v^\lambda_{\alpha\beta}, v^\lambda_{\alpha\beta\gamma}, \ldots$$

or in the compact form $(t^i, u^\alpha_{(j)}, v^\lambda_{(\beta)})$. The spaces $\mathcal{J}(\mathcal{A},\mathcal{B})$ and $\mathcal{J}(\mathcal{B},\mathcal{C})$ have the common coordinates u^α, and are, as it were, *glued*[6] along the common axes u^α to get the space $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$. As distinguished from the Cartesian product $\mathcal{J}(\mathcal{A},\mathcal{B}) \times \mathcal{J}(\mathcal{B},\mathcal{C})$, where u^α appears twice, in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ it appears only once like in a sum of two intersecting vector spaces.

One has a generalization of the following situation here. The usual composition of two binary relations can be obtained by taking their join, leading to a ternary relation, followed by a projection that removes the middle component[7]. Thus, the composition of jets is defined by taking the join of two sequences of coordinates $(t^i, u^\alpha_{(j)})$ and $(u^\alpha, v^\lambda_{(\beta)})$, leading to the sequence $(t^i, u^\alpha_{(j)}, v^\lambda_{(\beta)})$, followed by the map (the projection)

$$\varphi : \mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C}) \to \mathcal{J}(\mathcal{A},\mathcal{C}),$$

that removes the components u^α, cf. Proposition 4.7 below.

Define in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ the following operator:

$$\begin{aligned}
{}^{(1)}X_i = \frac{\partial}{\partial t^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + u^\alpha_{ijk} \frac{\partial}{\partial u^\alpha_{jk}} + \ldots +\\
+ \left(v^\lambda_\alpha \frac{\partial}{\partial v^\lambda} + v^\lambda_{\alpha\beta} \frac{\partial}{\partial v^\lambda_\beta} + v^\lambda_{\alpha\beta\gamma} \frac{\partial}{\partial v^\lambda_{\beta\gamma}} + \ldots \right) u^\alpha_i,
\end{aligned}$$

---

[6] It is similar to the situation when in the Euclidian xyz-space the xz and yz coordinate planes are *glued* along axis z.

[7] For the *rule of the third projection* in the theory of binary relations, see [2], p.3, or [20].

or in the compact form

$$\boxed{{}^{(1)}X_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial u^\alpha_{(j)}} u^\alpha_{i(j)} + \frac{\partial}{\partial v^\lambda_{(\beta)}} v^\lambda_{\alpha(\beta)} u^\alpha_i}. \tag{4.29}$$

One can see that

$${}^{(1)}X_i = X_i + \left(Y_\alpha - \frac{\partial}{\partial u^\alpha}\right) u^\alpha_i = \left(X_i - u^\alpha_i \frac{\partial}{\partial u^\alpha}\right) + Y_\alpha u^\alpha_i,$$

i.e., ${}^{(1)}X_i$ is similar to the sum $X_i + Y_\alpha u^\alpha_i$, but one term is omitted[8] – the first term of Y_α or, what is the same, the second term of X_i. The projection of ${}^{(1)}X_i$ onto $\mathcal{J}(\mathcal{A}, \mathcal{B})$ is X_i and onto $\mathcal{J}(\mathcal{B}, \mathcal{C})$ is $Y_\alpha u^\alpha_i$.

4.5.3 Composition of pure jets

The space $\mathcal{J}(\mathcal{A}, \mathcal{B})$ is extended to $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and the operators X_i are prolonged to the operators ${}^{(1)}X_i$. In the beginning of this Chapter it was mentioned that if one applies X_i to a function of the form $F(t^i, u^\alpha_{(j)})$ in the space $\mathcal{J}(\mathcal{A}, \mathcal{B})$, then one extends the differentiation rule for composite function on pure jets. Indeed, assuming that the quantities $u^\alpha_{(j)}$ depend on t^i, one has $\dfrac{\partial F}{\partial t^i} = X_i F$. Analogously, if one applies the operator ${}^{(1)}X_i$ to a function of the form $F(t^i, u^\alpha_{(j)}, v^\lambda_{(\beta)})$ in the space $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$, then the chain rule for composite function is naturally generalized, i.e., assuming that $u^\alpha_{(j)}$ depend on t^i and $v^\lambda_{(\beta)}$ depend on u^α, one has $\dfrac{\partial F}{\partial t^i} = {}^{(1)}X_i F$.

The operators (4.29) generalize the differentiation rule for composite functions and are said to be the *total differentiation operators of map compositions.*

Proposition 4.7. *A composition of pure jets from the spaces $\mathcal{J}(\mathcal{A}, \mathcal{B})$ and $\mathcal{J}(\mathcal{B}, \mathcal{C})$ is determined as the map*

$$\varphi : \mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to \mathcal{J}(\mathcal{A}, \mathcal{C}) \tag{4.30}$$

by relations $t^i \circ \varphi = t^i$ and recurrence relations

$$(Y_{(i)} v^\lambda) \circ \varphi = {}^{(1)}X_{(i)} v^\lambda, \tag{4.31}$$

where $Y_{(i)} = Y_{i_1} \dots Y_{i_p}$, ${}^{(1)}X_{(i)} = {}^{(1)}X_{i_1} \dots {}^{(1)}X_{i_p}$, $p = 0, 1, 2, \dots$, denote the consequent derivatives with respect to Y_i and ${}^{(1)}X_i$, respectively.

---

[8] The analogous situation appears in the xyz-space, where the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ has the projection $\mathbf{i}+\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$ onto the xz and yz coordinate planes, respectively, but is not equal to the sum of these projections.

Proof. Two first relations $t^i \circ \varphi = t^i$ and $v^\lambda \circ \varphi = v^\lambda$ are valid because jets in the spaces $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ and $\mathcal{J}(\mathcal{A},\mathcal{C})$ have the common source space $\mathcal{A}$ and common target space $\mathcal{C}$. For $p = 1, 2, \ldots$ (4.31) gives the system

$$\begin{aligned}
v_i^\lambda \circ \varphi &= v_\alpha^\lambda u_i^\alpha,\\
v_{ij}^\lambda \circ \varphi &= v_{\alpha\beta}^\lambda u_i^\alpha u_j^\beta + v_\alpha^\lambda u_{ij}^\alpha,\\
v_{ijk}^\lambda \circ \varphi &= v_{\alpha\beta\gamma}^\lambda u_i^\alpha u_j^\beta u_k^\gamma + v_{\alpha\beta}^\lambda (u_i^\alpha u_{jk}^\beta + u_j^\alpha u_{ik}^\beta + u_k^\alpha u_{ij}^\beta) + v_\alpha^\lambda u_{ijk}^\alpha,\\
&\vdots
\end{aligned}$$

which corresponds to the differentiation of a map composition (4.28). ■

Proposition 4.8. *Under the map* (4.30) *the operators* ${}^{(1)}X_i$ *in* $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ *and* Y_i *in* $\mathcal{J}(\mathcal{A},\mathcal{C})$ *are* φ*-related for each* i.

Proof. The proof is based on the Property 2.1 from the Section 2.2. In general, if the map $\varphi : M_1 \to M_2$ is locally determined by the functions φ^α on the manifold M_1, that are φ-related to coordinates functions v^α on the manifold M_2, i.e., $v^\alpha \circ \varphi = \varphi^\alpha$, then two vector fields Y and $\widetilde{Y}$ defined on M_1 and M_2, respectively, are said to be φ-related if and only if the derivatives $Y\varphi^\alpha$ are φ-related to components $\widetilde{y}^\alpha$ of $\widetilde{Y}$, i.e., $\widetilde{y}^\alpha \circ \varphi = Y\varphi^\alpha$.

For the operators ${}^{(1)}X_i$ and Y_i one has exactly the same situation. The map (4.30) is determined by the functions $t^i, v^\lambda, {}^{(1)}X_i v^\lambda, {}^{(1)}X_i {}^{(1)}X_j v^\lambda, \ldots$ on $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ that are φ-related to coordinates $(t^i, v_{(\beta)}^\lambda)$ in $\mathcal{J}(\mathcal{A},\mathcal{C})$, i.e.,

$$t^i \circ \varphi = t^i, \quad v^\lambda \circ \varphi = v^\lambda, \quad v_i^\lambda \circ \varphi = {}^{(1)}X_i v^\lambda, \quad v_{ij}^\lambda \circ \varphi = {}^{(1)}X_i {}^{(1)}X_j v^\lambda, \quad \ldots .$$

The derivatives

$${}^{(1)}X_i t^j = \delta_i^j, \quad {}^{(1)}X_i v^\lambda = v_\alpha^\lambda u_i^\alpha, \quad {}^{(1)}X_i v_\alpha^\lambda = v_{\alpha\beta}^\lambda u_i^\beta, \quad {}^{(1)}X_i v_{\alpha\beta}^\lambda = v_{\alpha\beta\gamma}^\lambda u_i^\gamma, \quad \ldots$$

are φ-related to components of Y_i. Thus, the operators ${}^{(1)}X_i$ in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ are φ-related to Y_i in $\mathcal{J}(\mathcal{A},\mathcal{C})$ for each index i. ■

Proposition 4.9. *The map* (4.30) *transforms the Cartan forms* $\theta_{(j)}^\lambda$ *from the space* $\mathcal{J}(\mathcal{A},\mathcal{C})$ *to the space* $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ *according to the recurrence formula*

$$(\mathcal{L}_{Y_{(i)}} \theta^\lambda) \circ T\varphi = \mathcal{L}_{{}^{(1)}X_{(i)}} (\theta^\lambda + v_\alpha^\lambda \omega^\alpha), \tag{4.32}$$

where $\mathcal{L}_{Y_{(i)}} = \mathcal{L}_{Y_{i_1}} \ldots \mathcal{L}_{Y_{i_p}}$, $\mathcal{L}_{{}^{(1)}X_{(i)}} = \mathcal{L}_{{}^{(1)}X_{i_1}} \ldots \mathcal{L}_{{}^{(1)}X_{i_p}}$, $p = 0, 1, 2, \ldots$, *denote the consequent Lie derivatives with respect to the* φ*-related operators* Y_i *and* ${}^{(1)}X_i$, *respectively.* (4.32) *in the explicit form is*

$$\begin{aligned}
\theta^\lambda \circ T\varphi &= \theta^\lambda + v_\alpha^\lambda \omega^\alpha,\\
\theta_i^\lambda \circ T\varphi &= \theta_\alpha^\lambda u_i^\alpha + v_\alpha^\lambda \omega_i^\alpha + v_{\alpha\beta}^\lambda \omega^\alpha u_i^\beta,\\
\theta_{ij}^\lambda \circ T\varphi &= \theta_\alpha^\lambda u_{ij}^\alpha + \theta_{\alpha\beta}^\lambda u_i^\alpha u_j^\beta + v_\alpha^\lambda \omega_{ij}^\alpha +\\
&\quad + v_{\alpha\beta}^\lambda (\omega_i^\alpha u_j^\beta + \omega_j^\alpha u_i^\beta + \omega^\alpha u_{ij}^\beta) + v_{\alpha\beta\gamma}^\lambda \omega^\alpha u_i^\beta u_j^\gamma,\\
&\vdots
\end{aligned}$$

Proof. Consider the Cartan forms $\theta^\lambda = dv^\lambda - v^\lambda_i dt^i$ in $\mathcal{J}(\mathcal{A},\mathcal{C})$. Taking in account that $t^i \circ \varphi = t^i$, $v^\lambda \circ \varphi = v^\lambda$, $v^\lambda_i \circ \varphi = v^\lambda_\alpha u^\alpha_i$, the forms θ^λ are transformed by (4.30) to the forms in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$:

$$\begin{aligned}\theta^\lambda \circ T\varphi = (dv^\lambda - v^\lambda_i dt^i) \circ T\varphi = dv^\lambda - v^\lambda_\alpha u^\alpha_i dt^i = \\ = dv^\lambda - v^\lambda_\alpha du^\alpha + v^\lambda_\alpha du^\alpha - v^\lambda_\alpha u^\alpha_i dt^i = \\ = \theta^\lambda + v^\lambda_\alpha \omega^\alpha,\end{aligned} \tag{4.33}$$

where $\theta^\lambda = dv^\lambda - v^\lambda_\alpha du^\alpha$ and $\omega^\alpha = du^\alpha - u^\alpha_i dt^i$ are forms in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$. (4.33) corresponds to (4.32) for $p = 0$. From (4.33) it follows that $\theta^\lambda + v^\lambda_\alpha \omega^\alpha$ in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ are φ-related to θ^λ in $\mathcal{J}(\mathcal{A},\mathcal{C})$, cf. Section 2.2, Property 2.2. Note that the Lie derivatives in the left-hand side in (4.32) form the sequence of Cartan forms $\theta^\lambda_{(j)}$. The forms in the right-hand side in (4.32) are φ-related to these forms, since the Lie derivatives of φ-related tensor fields (including differential forms) with respect to φ-related vector fields are φ-related, cf. Section 2.2, Property 2.13. Thus, the recurrence formula (4.32) is valid. ■

4.5.4 Chain rule for double jet composition

Given a double composition of smooth maps $\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D}$, the intermediate space $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ is extended to the space $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ with coordinates $(t^i, u^\alpha_{(j)}, v^\lambda_{(\beta)}, w^\rho_{(\mu)})$ and the operators ${}^{(1)}X_i$ are prolonged to the operators

$$\boxed{{}^{(2)}X_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial u^\alpha_{(j)}} u^\alpha_{i(j)} + \frac{\partial}{\partial v^\lambda_{(\beta)}} v^\lambda_{\alpha(\beta)} u^\alpha_i + \frac{\partial}{\partial w^\rho_{(\mu)}} w^\rho_{\lambda(\mu)} v^\lambda_\alpha u^\alpha_i}\,. \tag{4.34}$$

In the assumption that the quantities u^α depend on t^i, v^λ depend on u^α and w^ρ depend on v^λ the usual chain rule for coordinate functions gives

$$v^\lambda_i = v^\lambda_\alpha u^\alpha_i, \quad w^\rho_\alpha = w^\rho_\lambda v^\lambda_\alpha, \quad w^\rho_i = w^\rho_\lambda v^\lambda_\alpha u^\alpha_i.$$

The operators (4.34) determine the *chain rule for double jet composition.*

The next three Propositions are consequences of the Propositions 4.7, 4.8 and 4.9, respectively.

Proposition 4.10. *The double composition of pure jets from the spaces $\mathcal{J}(\mathcal{A},\mathcal{B})$, $\mathcal{J}(\mathcal{B},\mathcal{C})$ and $\mathcal{J}(\mathcal{C},\mathcal{D})$ is determined as the map*

$$\varphi : \mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \to \mathcal{J}(\mathcal{A},\mathcal{D}) \tag{4.35}$$

by relations $t^i \circ \varphi = t^i$ and recurrence relations

$$(Z_{(i)} w^\rho) \circ \varphi = {}^{(2)}X_{(i)} w^\rho, \tag{4.36}$$

where $Z_{(i)} = Z_{i_1} \ldots Z_{i_p}$, ${}^{(2)}X_{(i)} = {}^{(2)}X_{i_1} \ldots {}^{(2)}X_{i_p}$, $p = 0, 1, 2, \ldots$, denote the consequent derivatives with respect to Z_i and ${}^{(2)}X_i$, respectively.

Proof. The first two relations $t^i \circ \varphi = t^i$ and $w^\rho \circ \varphi = w^\rho$ are valid because jets in the spaces $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ and $\mathcal{J}(\mathcal{A},\mathcal{D})$ have the common source space $\mathcal{A}$ and the common target space $\mathcal{D}$. For $p = 1, 2, \dots$ (4.36) gives the infinite sequence of relations

$$\begin{aligned}
w^\rho_i &= w^\rho_\lambda v^\lambda_\alpha u^\alpha_i,\\
w^\rho_{ij} &= w^\rho_{\lambda\mu} v^\lambda_\alpha v^\mu_\beta u^\alpha_i u^\beta_j + w^\rho_\lambda (v^\lambda_{\alpha\beta} u^\alpha_i u^\beta_j + v^\lambda_\alpha u^\alpha_{ij}),\\
w^\rho_{ijk} &= w^\rho_{\lambda\mu\nu} v^\lambda_\alpha v^\mu_\beta v^\nu_\gamma u^\alpha_i u^\beta_j u^\gamma_k + w^\rho_{\lambda\mu}\big[v^\lambda_\alpha v^\mu_{\beta\gamma} u^\alpha_i u^\beta_j u^\gamma_k + v^\lambda_\beta v^\mu_{\alpha\gamma} u^\beta_i u^\alpha_j u^\gamma_k +\\
&\quad + v^\lambda_\gamma v^\mu_{\alpha\beta} u^\gamma_i u^\alpha_j u^\beta_k + (v^\lambda_\alpha v^\mu_\beta + v^\lambda_{\alpha\beta})(u^\alpha_i u^\beta_{jk} + u^\alpha_j u^\beta_{ik} + u^\alpha_k u^\beta_{ij})\big] +\\
&\quad + w^\rho_\lambda (v^\lambda_{\alpha\beta\gamma} u^\alpha_i u^\beta_j u^\gamma_k + v^\lambda_\alpha u^\alpha_{ijk}),\\
&\ \ \vdots
\end{aligned}$$

that correspond to the differentiation of a double composition of smooth maps $f_1 : \mathcal{A} \to \mathcal{B}$, $f_2 : \mathcal{B} \to \mathcal{C}$, $f_3 : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{A} \to \mathcal{D}$ defined locally by the functions

$$u^\alpha \circ f_1 = f^\alpha_1, \quad v^\lambda \circ f_2 = f^\lambda_2, \quad w^\rho \circ f_3 = f^\rho_3, \quad w^\rho \circ g = g^\rho$$

that are related by $g^\rho = f^\rho_3 \circ f_2 \circ f_1$ (we do not write out the corresponding sequence of partial derivatives). ■

Note that the associativity of map composition

$$(\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to \mathcal{D} = \mathcal{A} \to (\mathcal{B} \to \mathcal{C} \to \mathcal{D})$$

leads to the *associativity of jet composition* as follows:

$$\begin{aligned}
&w^\rho_\lambda (v^\lambda_\alpha u^\alpha_i) = (w^\rho_\lambda v^\lambda_\alpha) u^\alpha_i,\\
&w^\rho_{\lambda\mu} v^\lambda_\alpha v^\mu_\beta u^\alpha_i u^\beta_j + w^\rho_\lambda (v^\lambda_{\alpha\beta} u^\alpha_i u^\beta_j + v^\lambda_\alpha u^\alpha_{ij}) = (w^\rho_{\lambda\mu} v^\lambda_\alpha v^\mu_\beta u^\alpha_i u^\beta_j + w^\rho_\lambda v^\lambda_{\alpha\beta}) u^\alpha_i u^\beta_j + w^\rho_\lambda v^\lambda_\alpha u^\alpha_{ij},\\
&w^\rho_{\lambda\mu\nu} v^\lambda_\alpha v^\mu_\beta v^\nu_\gamma u^\alpha_i u^\beta_j u^\gamma_k + w^\rho_{\lambda\mu}\big[v^\lambda_\alpha v^\mu_{\beta\gamma} u^\alpha_i u^\beta_j u^\gamma_k + v^\lambda_\beta v^\mu_{\alpha\gamma} u^\beta_i u^\alpha_j u^\gamma_k + v^\lambda_\gamma v^\mu_{\alpha\beta} u^\gamma_i u^\alpha_j u^\beta_k +\\
&\qquad\qquad + (v^\lambda_\alpha v^\mu_\beta + v^\lambda_{\alpha\beta})(u^\alpha_i u^\beta_{jk} + u^\alpha_j u^\beta_{ik} + u^\alpha_k u^\beta_{ij})\big] + w^\rho_\lambda (v^\lambda_{\alpha\beta\gamma} u^\alpha_i u^\beta_j u^\gamma_k + v^\lambda_\alpha u^\alpha_{ijk}) =\\
&= (w^\rho_{\lambda\mu\nu} v^\lambda_\alpha v^\mu_\beta v^\nu_\gamma + w^\rho_{\lambda\mu} v^\lambda_\alpha v^\mu_{\beta\gamma} + w^\rho_\lambda v^\lambda_{\alpha\beta\gamma}) u^\alpha_i u^\beta_j u^\gamma_k + w^\rho_{\lambda\mu} v^\lambda_\beta v^\mu_{\alpha\gamma} u^\beta_i u^\alpha_j u^\gamma_k +\\
&\quad + w^\rho_{\lambda\mu} v^\lambda_\gamma v^\mu_{\alpha\beta} u^\gamma_i u^\alpha_j u^\beta_k + (w^\rho_{\lambda\mu} v^\lambda_\alpha v^\mu_\beta + w^\rho_{\lambda\mu} v^\lambda_{\alpha\beta})(u^\alpha_i u^\beta_{jk} + u^\alpha_j u^\beta_{ik} + u^\alpha_k u^\beta_{ij}) + w^\rho_\lambda v^\lambda_\alpha u^\alpha_{ijk},
\end{aligned}$$

etc.

Proposition 4.11. *Under the map* (4.35) *the operators* ${}^{(2)}X_i$ *in* $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ *and* Z_i *in* $\mathcal{J}(\mathcal{A},\mathcal{D})$ *are* φ*-related for each* i*.*

Proof. The proof is analogous to the proof of the Proposition 4.8. The map (4.35) is determined by the functions

$$t^i, \quad w^\rho, \quad {}^{(2)}X_i w^\rho, \quad {}^{(2)}X_i {}^{(2)}X_j w^\rho, \quad \dots$$

in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$, which are φ-related to coordinate functions $(t^i, w^\rho_{(j)})$ in $\mathcal{J}(\mathcal{A},\mathcal{D})$, i.e.,

$$t^i \circ \varphi = t^i, \quad w^\rho \circ \varphi = w^\rho, \quad w^\rho_i \circ \varphi = {}^{(2)}X_i w^\rho, \quad w^\rho_{ij} \circ \varphi = {}^{(2)}X_i {}^{(2)}X_j w^\rho, \quad \dots.$$

The derivatives

$${}^{(2)}X_i t^j = \delta^j_i, \quad {}^{(2)}X_i w^\rho = w^\rho_\lambda v^\lambda_\alpha u^\alpha_i, \quad {}^{(2)}X_i w^\rho_\lambda = w^\rho_{\lambda\mu} v^\mu_\alpha u^\alpha_i, \quad {}^{(2)}X_i w^\rho_{\lambda\mu} = w^\rho_{\lambda\mu\nu} v^\nu_\alpha u^\alpha_i, \dots$$

are φ-related to components of Z_i. Thus, the operators ${}^{(2)}X_i$ and Z_i are φ-related for each index i. ■

Proposition 4.12. *The map* (4.35) *transforms the Cartan forms ϑ^ρ from the space $\mathcal{J}(\mathcal{A},\mathcal{D})$ to the space $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ according to the recurrence formula*

$$(\mathcal{L}_{Z_{(i)}} \vartheta^\rho) \circ T\varphi = \mathcal{L}_{{}^{(2)}X_{(i)}} (\vartheta^\rho + w^\rho_\lambda \theta^\lambda + w^\rho_\lambda v^\lambda_\alpha \omega^\alpha), \tag{4.37}$$

where $\mathcal{L}_{Z_{(i)}} = \mathcal{L}_{Z_{i_1}} \dots \mathcal{L}_{Z_{i_p}}$, $\mathcal{L}_{{}^{(2)}X_{(i)}} = \mathcal{L}_{{}^{(2)}X_{i_1}} \dots \mathcal{L}_{{}^{(2)}X_{i_p}}$, $p = 0, 1, 2, \dots$, denote the consequent Lie derivatives with respect to the φ-related operators Z_i and ${}^{(2)}X_i$, respectively. (4.37) *in the explicit form is*

$$\vartheta^\rho \circ T\varphi = \vartheta^\rho + w^\rho_\lambda \theta^\lambda + w^\rho_\lambda v^\lambda_\alpha \omega^\alpha,$$
$$\vartheta^\rho_i \circ T\varphi = \vartheta^\rho_\lambda v^\lambda_\alpha u^\alpha_i + w^\rho_{\lambda\mu} \theta^\lambda v^\mu_\alpha u^\alpha_i + w^\rho_\lambda \theta^\lambda_\alpha u^\alpha_i + w^\rho_{\lambda\mu} v^\lambda_\alpha v^\mu_\beta \omega^\alpha u^\beta_i + w^\rho_\lambda v^\lambda_{\alpha\beta} \omega^\alpha u^\beta_i + w^\rho_\lambda v^\lambda_\alpha \omega^\alpha_i,$$

etc.

Proof. Consider the Cartan forms $\vartheta^\rho = dw^\rho - w^\rho_i dt^i$ in $\mathcal{J}(\mathcal{A},\mathcal{D})$. Taking in account that $t^i \circ \varphi = t^i$, $w^\rho \circ \varphi = w^\rho$, $w^\rho_i \circ \varphi = w^\rho_\lambda v^\lambda_\alpha u^\alpha_i$, the forms ϑ^ρ are transformed by (4.35) to the forms in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$:

$$\begin{aligned}\vartheta^\rho \circ T\varphi &= (dw^\rho - w^\rho_i dt^i) \circ T\varphi = dw^\rho - w^\rho_\lambda v^\lambda_\alpha u^\alpha_i dt^i = \\ &= dw^\rho - w^\rho_\lambda dv^\lambda + w^\rho_\lambda dv^\lambda - w^\rho_\lambda v^\lambda_\alpha du^\alpha + w^\rho_\lambda v^\lambda_\alpha du^\alpha - w^\rho_\lambda v^\lambda_\alpha u^\alpha_i dt^i = \\ &= \vartheta^\rho + w^\rho_\lambda \theta^\lambda + w^\rho_\lambda v^\lambda_\alpha \omega^\alpha,\end{aligned}$$

which corresponds to (4.37) for $p = 0$. Thus, $\vartheta^\rho + w^\rho_\lambda \theta^\lambda + w^\rho_\lambda v^\lambda_\alpha \omega^\alpha$ in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ are φ-related to ϑ^ρ in $\mathcal{J}(\mathcal{A},\mathcal{D})$, and the Lie derivatives in the left-hand side in (4.37), that is the sequence of Cartan forms $\vartheta^\rho_{(j)}$, are φ-related to the forms in the right-hand side, cf. Section 2.2, Property 2.13. ■

Remark 4.2. Note that the sequence of smooth maps

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D} \to \dots$$

is accompanied by natural extensions of the corresponding intermediate spaces

$$\mathcal{J}(\mathcal{A},\mathcal{B}) \rightsquigarrow \mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C}) \rightsquigarrow \mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \rightsquigarrow \dots$$

and prolongations of operators

$$X_i \rightsquigarrow {}^{(1)}X_i \rightsquigarrow {}^{(2)}X_i \rightsquigarrow \dots.$$

But the answers to the questions put in the beginning of the present Section are received on the first two prolongations.

4.5.5 Invertible jets

Let us consider a transformation (diffeomorphism) $a : M \to M$ of a smooth manifold M and its inverse $a^{-1} : M \to M$, $a \circ a^{-1} = a^{-1} \circ a = 1_M$. Two infinite jets

$$(t^i, a^i_j, a^i_{jk}, a^i_{jkl}, \ldots) \text{ and } (t^i, \bar{a}^i_j, \bar{a}^i_{jk}, \bar{a}^i_{jkl}, \ldots), \;\; i, j, k, l, \ldots = 1, 2, \ldots, \dim M,$$

induced by a and a^{-1}, respectively, are mutually inverse. It means that the corresponding Jacobi matrices (a^i_j) and $(\bar{a}^i_j)$ are mutually inverse, i.e., $a^i_k \bar{a}^k_j = \delta^i_j$, and the composition of these jets gives the jet $(t^i, \delta^i_j, 0, 0, \ldots)$ of the identity map 1_M.

In the jet space $\mathcal{J}(M, M)$ there are defined two sets of coordinates $(t^i, a^i_{(j)})$ and $(t^i, \bar{a}^i_{(j)})$, and two TDOs

$$X_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial \bar{a}^k_{(j)}} \bar{a}^k_{i(j)}, \qquad Y_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial a^k_{(j)}} a^k_{i(j)}.$$

For the multi-index $(j) = j_1 \ldots j_p$ assume that $p = 1, 2, \ldots$. Consider the intermediate space $\mathcal{J}(M, M, M)$ with coordinates $(t^i, a^i_{(j)}, \bar{a}^i_{(j)})$ and operator

$$\boxed{Z_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial \bar{a}^k_{(j)}} \bar{a}^k_{i(j)} + \frac{\partial}{\partial a^k_{(j)}} a^k_{l(j)} \bar{a}^l_i}. \tag{4.38}$$

The repeated application of Z_i to the equalities $a^i_k \bar{a}^k_j = \delta^j_i$ gives the relations between the jet and its inverse:

$$\begin{aligned} a^i_k \bar{a}^k_j &= \delta^j_i, \\ a^i_{pq} \bar{a}^p_j \bar{a}^q_k + a^i_s \bar{a}^s_{jk} &= 0, \\ a^i_{pqs} \bar{a}^p_j \bar{a}^q_k \bar{a}^s_l + a^i_{pq} (\bar{a}^p_j \bar{a}^q_{kl} + \bar{a}^p_k \bar{a}^q_{lj} + \bar{a}^p_l \bar{a}^q_{jk}) + a^i_s \bar{a}^s_{jkl} &= 0, \\ &\vdots \end{aligned}$$

Thus, the components of the inverse jet are expressed recursively in terms of the components of the initial one as follows:

$$\begin{aligned} \bar{a}^i_{jk} &= -\bar{a}^i_l a^l_{pq} \bar{a}^p_j \bar{a}^q_k, \\ \bar{a}^i_{jkl} &= \bar{a}^i_h (a^h_{ps} \bar{a}^s_t a^t_{qr} + a^h_{qs} \bar{a}^s_t a^t_{rp} + a^h_{rs} \bar{a}^s_t a^t_{pq} - a^h_{pqr}) \bar{a}^p_j \bar{a}^q_k \bar{a}^r_l, \\ &\vdots \end{aligned}$$

4.6 Conclusion

Summarizing, let us repeat the text fragment from Introduction on page 13. In the beginning of the present Chapter we introduced an algebraic scheme for calculation of invariants and symmetries of TDO D in $\mathcal{J}_{1,1}$. This scheme is universal in the following sense.

Let X be a smooth vector field with canonical parameter s, i.e. $Xs = 1$, on a smooth manifold M, and suppose f is a smooth function on M. Let us form an infinite sequence $F = (f, f', f'', f''', \ldots)$ consisting of f and its derivatives of all orders with respect to X. Then there is defined a triplet (X, s, F) on M. All possible triplets of such kind on smooth manifolds form a category. In particular, two triplets (X, s, F) and $(Y, \widetilde{s}, \widetilde{F})$ on manifolds M and $\widetilde{M}$, respectively, are linked together by a morphism $\varphi : M \to \widetilde{M}$ that is a smooth map for which $s = \widetilde{s} \circ \varphi$, $F = \widetilde{F} \circ \varphi$, and X and Y are φ-related. In the category of triplets (X, s, F) the terminal object is precisely the triplet (D, t, U) in $\mathcal{J}_{1,1}$, where D is the total differentiation operator, t is a parameter (time) and U is a set of fiber coordinates $(u, u', u'', u''', \ldots)$.

Since TDO D is a linear vector field in $\mathcal{J}_{1,1}$, the flow generated by D is determined by the common *exponential law*, that appears in formulas (4.2), (4.6), (4.10), (4.11), (4.12) and (4.14). Let us summarize these formulas as follows:

$$U' = CU \quad \Longrightarrow \quad U_t = e^{tC}U \quad \Longrightarrow \quad I = e^{-tC}U, \tag{4.39}$$

$$\omega' = C\omega \quad \Longrightarrow \quad \omega_t = e^{tC}\omega \quad \Longrightarrow \quad dI = e^{-tC}\omega, \tag{4.40}$$

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U}C \quad \Longrightarrow \quad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U}e^{-tC} \quad \Longrightarrow \quad \frac{\partial}{\partial I} = \frac{\partial}{\partial U}e^{tC}. \tag{4.41}$$

The implication (4.39) determines invariants I of D that are transformed by φ to the invariants $I \circ \varphi = e^{-sC}F$ of X on M. The implication (4.40) establishes the connection between differentials dI and Cartan forms ω defined in $\mathcal{J}_{1,1}$. The forms $\omega = dU - U'dt$ are transformed by φ to the forms $\omega \circ T\varphi = dF - F'ds$ on M. The implication (4.41) determines infinitesimal symmetries of the operator D and, thus, gives the rule for construction of corresponding infinitesimal symmetries for a vector field X on M. All implications (4.39)–(4.41) can be naturally extended on a jet space $\mathcal{J}_{n,m}$ by using multi-indices technique.

The described universal scheme is applicable to prolongations of differential equations and group operators, and to classification of singularities of smooth maps, see [4], [9], [12], [13], [14].

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Vektorväljad ja Lie rühma esitused

Kokkuvõte

Käesoleva väitekirja uurimisobjektiks on vektorväli, mis on üks olulisematest mõistetest diferentsiaalgeomeetrias ja globaalanalüüsis ning mis leiab rakendust pideva keskkonna mehaanikas ja dünaamilistes süsteemides. Töös on uurimisel vektorvälja kolm omapära. Esiteks, vektorväli on diferentsiaaloperaator ja me tõlgendame vektorvälja liikumise stopp-kaadrina. Teiseks, vektorväli projetseerib muutkonna invariantide ruumi ja sellega kaasneb singulaarsuste klassifitseerimine. Kolmandaks, vektorväli võib ise olla projetseeritav ja see võimaldab teistest struktuuridest (näit. lõpmatute džettide ruumist) saada vajalikke invariante ja sümmeetriaid.

Väitekirja esimene peatükk sisaldab lühiülevaadet f-seotud tensoritest kahe vektorrumi vahelise lineaarkujutuse f korral ning nende tensorite põhiomadustest. Hiljem kasutatakse seda Lie tuletiste defineerimisel. Tutvustatakse vektorväljade ja diferentsiaalvormide põhiomadusi.

Väitekirja teises peatükis defineeritakse φ-seotud ja seejärel a-seotud tensorväljad sileda kujutuse φ ja difeomorfismi a korral. Tugenedes a-seotud tensorväljade mõistele defineeritakse tensorväljade (vektorväljad ja diferentsiaalvormid kaasarvatud) Lie tuletised ning tuletatakse Lie diferentsiaalarvutuse põhivalemid mitteholonoomses (koordinaatidevabas) baasis (reeper + koreeper). Lie tuletiste abil kirjeldatakse, kuidas vektorvälja voos muutub mitteholonoomne baas (derivatsioonivalemid).

Teise peatüki viimases paragrahvis defineeritakse tensorväljade integreerimine. Siin vaadeldakse integreerimist kui Lie diferentseerimise pöördoperatsiooni. Nimelt, tavalise integreeritava funktsiooni määramata ja määratud integraali mõisted on laiendatud üldisema tensorvälja juhule, kus valemites esinevad Lie tuletised. Lihtsamatel juhtudel on antud tensorvälja integraali geomeetriline tõlgendus. Nimelt, on arvutatud nihete ja pöörete operaatorite integraalid pöörete voos tasandil ja ruumis.

Kolmas peatükk on pühendatud Lie rühma esituste teooriale. Selles peatükis on uurimisel Lie rühm G ja selle puutujarühm TG. Rühma TG toimet esitusruumis M kirjeldatakse üldistatud Leibnizi reegli abil. Esitatud materjal on sissejuhatavaks teoreetiliseks osaks artiklile [17], kus, kasutades infinitesimaalmeetodeid (S. Lie mõttes), on näidatud, kuidas lineaarrühm $GL(n, \mathbb{R})$ toimib iseendal vasak- ja paremnihetega ning siseautomorfismidega (adjungeeritud esitus). Adjungeeritud esituse invariantideks on vastava maatriksi diagonaalmiinorite summad ning orbiitideks vastavad kvadrikud. Selle artikli täistekst on lisatud väitekirja lõpus.

Väitekirja neljas peatükk põhineb artiklil [19]. Peatüki alguses uuritakse siledate kujutuste lõpmatute džettide ruumi $\mathcal{J}_{n,m}$ struktuuri. Ruumis $\mathcal{J}_{n,m}$ defineeritakse täisdiferentsiaaloperaatorid (TDO) D ja Cartan'i välisdiferentsiaalvormid ω. TDO-id D on lineaarsed vektorväljad ning nende voog ja invariandid määratakse eksponentsiaalseadusega. Lähemalt uuritakse ruumide $\mathcal{J}_{1,1}$, $\mathcal{J}_{1,2}$, $\mathcal{J}_{2,1}$, $\mathcal{J}_{1,m}$ ja $\mathcal{J}_{n,1}$ struktuuri.

Neljanda peatüki põhiteemaks on järmine probleem. On teada, et analoogiliselt protsessile, kuidas sile kujutus tekitab lõpmatu džeti, tekitab siledate kujutuste kompositsioon lõpmatute džettide kompositsiooni. Seoses sellega tekivad järgmised küsimused: kuidas defineerida lõpmatute džettide kompositsioon rekurrentse valemi abil nii, et definitsioon ei sõltuks konkreetsete kujutuste valikust, ja kuidas vastavad täisdiferentsiaaloperaatorid ja Cartan'i diferentsiaalvormid on seotud džettide kompositsioonil? Vastused püstitatud küsimustele on sõnastatud lausetena koos tõestustega. Tuletatud rekurrentsed valemid määravad lõpmatute džettide kompositsiooni suvaliste kujutuste puhul. Kahekordne kujutuste kompositsioon tekitab kahekordse džettide kompositsiooni ning ahelreegelit siledate kujutuste puhul on võimalik üle kanda lõpmatute džettide kompositsioonile. Kujutuste kompositsiooni assotsiatiivsus indutseerib džettide kompositsiooni assotsiatiivsuse. Difeomorfismide korral on võimalik defineerida vastava lõpmatu džeti pöörddžett.

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Chapter 15
Adjoint Representations and Movements

Maido Rahula and Vitali Retšnoi

Abstract The aim of this paper is to develop the theory of higher order movements by using the structure of multiple tangent bundles. In this context the meaning of invariants of polynomials and of matrices is explained. The connection with central and raw moments appearing in probability theory is established

Introduction

Let M be a smooth manifold and $b : M \to M$ be a smooth map. When a transformation (diffeomorphism) $a : M \to M$ is given, b is transformed to $\widetilde{b} = aba^{-1}$ so that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "b"] \arrow[d, "a"'] & M \arrow[d, "a"] \\
M \arrow[r, "\widetilde{b}"] & M
\end{tikzcd}$$

Suppose G is a group of transformations of M and $b \in G$. Then one can speak about the interior automorphism and adjoint representation of G. In particular, given a vector field X and its flow a_t, b is continuously transformed as follows: $b \rightsquigarrow b_t = a_t b a_t^{-1}$. Any other vector field Y is transformed to $\widetilde{Y} = Ta_t Y$, and the flow of Y is transformed according to the rule $b_s \rightsquigarrow a_t b_s a_t^{-1}$. Thus one can speak about a *movement of movement* and about *movements of higher orders*, or as we say movements in higher floors, see [4], [5], [8].

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15.1 Generalized Leibnitz Rule

Let M_1, M_2 and M be smooth manifolds. The tangent functor T assigns to a Cartesian product $M_1 \times M_2$ a vector space

$$T(M_1 \times M_2) = (TM_1 \times M_2) \otimes (M_1 \times TM_2),$$

and to a smooth map $\lambda : M_1 \times M_2 \to M, (u,v) \mapsto w = u \cdot v$, its tangent map

$$T\lambda : T(M_1 \times M_2) \to TM, \quad (u_1, v_1) \mapsto w_1 = u_1 \cdot v + u \cdot v_1.$$

There are two smooth maps

$$\begin{aligned} \lambda_v : M_1 \to M, \quad u \mapsto w = u \cdot v, \quad \forall v \in M_2, \\ \lambda_u : M_2 \to M, \quad v \mapsto w = u \cdot v, \quad \forall u \in M_1, \end{aligned}$$

associated with λ. Namely, two vectors $u_1 \in T_u M_1$ and $v_1 \in T_v M_2$ at $u \in M_1$ and $v \in M_2$, respectively, are mapped into vectors $T\lambda_v(u_1) = u_1 \cdot v$ and $T\lambda_u(v_1) = u \cdot v_1$ at $w \in M$. The sum of these images is equal to the image of (u_1, v_1) under $T\lambda$:

$$w = u \cdot v \quad \Longrightarrow \quad w_1 = u_1 \cdot v + u \cdot v_1. \tag{15.1}$$

This formula generalizes the ordinary *Leibnitz rule*.

Locally, (15.1) can be presented as the system

$$w^\rho \circ \lambda = \lambda^\rho(u^i, v^\alpha) \quad \Longrightarrow \quad w_1^\rho = \frac{\partial \lambda^\rho}{\partial u^i} u_1^i + \frac{\partial \lambda^\rho}{\partial v^\alpha} v_1^\alpha,$$

where u^i, v^α, w^ρ are coordinates of u, v, w, and $u_1^i, v_1^\alpha, w_1^\rho$ are components of u_1, v_1, w_1, respectively, on corresponding coordinate neighborhoods $U_1 \subset M_1, U_2 \subset M_2$ and $U \subset M$, $i = 1, \ldots, \dim M_1$, $\alpha = 1, \ldots, \dim M_2$, $\rho = 1, \ldots, \dim M$.

15.2 Tangent Group

Let G be a Lie group with the composition law $\gamma : (a,b) \mapsto c = ab$. This group acts on itself by right and left translations:

$$\begin{aligned} r_b : G \to G, \quad a \mapsto ab, \quad \forall b \in G, \\ l_a : G \to G, \quad b \mapsto ab, \quad \forall a \in G. \end{aligned}$$

The tangent bundle (*first floor*) TG becomes automatically a Lie group with the composition law

$$T\gamma : TG \times TG \to TG, \quad (a_1, b_1) \mapsto c_1 = a_1 b + a b_1.$$

Two vectors $a_1 \in T_aG$ and $b_1 \in T_bG$ are mapped by $T\gamma$ to the vector $a_1b + ab_1 = Tr_b(a_1) + Tl_a(b_1) \in T_cG$:

$$c = ab \quad \Longrightarrow \quad c_1 = a_1b + ab_1. \tag{15.2}$$

The group TG is called a *tangent group* of G. The unit of TG is precisely the zero vector at the unit $e \in G$. The inverse element for $a_1 \in T_aG$ is defined by

$$a_1^{-1} = -a^{-1}a_1a^{-1} \in T_{a^{-1}}G.$$

Systematically applying left and right translations l_a and r_a to any vector $e_1 \in T_eG$ one can obtain left-invariant and right-invariant vector fields ae_1 and e_1a, respectively. The interior automorphism $A_a = l_a \circ r_a^{-1}$ yields the transformation $T_eA_a : e_1 \mapsto ae_1a^{-1}$ in T_eG. Let a basis (frame) be given in T_eG. Then there exists a homomorphism $a \mapsto A(a)$ from G to the general linear group $GL(\dim G, \mathbb{R})$ with center of G $Z \subset G$ as a kernel. An adjoint representation of TG is defined as follows:

$$c = aba^{-1} \quad \Longrightarrow \quad c_1 = (a_1a^{-1})c - c(a_1a^{-1}) + ab_1a^{-1}. \tag{15.3}$$

For $b_1 = 0$ this formula determines an action of TG on G, and for $a_1 = 0$ – an action of G on the group TG.

With any one-parameter subgroup a_t of G we associate one-parameter groups of transformations r_{a_t}, l_{a_t} and A_{a_t} of G that in their turn are generated by a left-invariant vector field X, a right-invariant vector field $\widetilde{X}$ and the field $\widetilde{X} - X$, respectively, i.e.

$$r_{a_t} = \exp tX, \quad l_{a_t} = \exp t\widetilde{X}, \quad A_{a_t} = \exp t(\widetilde{X} - X).$$

This follows from the formulas

$$Xf = (f \circ r_{a_t})'_{t=0}, \quad \widetilde{X}f = (f \circ l_{a_t})'_{t=0}, \quad (\widetilde{X} - X)f = (f \circ A_{a_t})'_{t=0},$$

where f is an arbitrary function on G, and from the fact that left and right translations commute.

15.3 Linear Group $GL(2, \mathbb{R})$

Consider the linear group $GL = GL(2, \mathbb{R})$ and its Lie algebra $gl = gl(2, \mathbb{R})$. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL$ and $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in gl$. The exponential map $\kappa : gl \to GL$ takes each additive subgroup $Ct, t \in \mathbb{R}$, to the one-parameter subgroup e^{Ct}, called the exponential of Ct. The left-invariant basis (X_i, ω^i) and right-invariant basis $(\widetilde{X}_i, \omega^i)$, i.e. the corresponding fields of frames and coframes in GL, can be defined as follows:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}, \quad \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1} \begin{pmatrix} da_1 & da_2 \\ da_3 & da_4 \end{pmatrix},$$
$$\begin{pmatrix} \widetilde{X}_1 & \widetilde{X}_2 \\ \widetilde{X}_3 & \widetilde{X}_4 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix} \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}, \quad \begin{pmatrix} \widetilde{\omega}^1 & \widetilde{\omega}^2 \\ \widetilde{\omega}^3 & \widetilde{\omega}^4 \end{pmatrix} = \begin{pmatrix} da_1 & da_2 \\ da_3 & da_4 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1},$$

where $\partial_i = \frac{\partial}{\partial a_i}$, $i = 1,2,3,4$.

Let GL acts on itself by interior automorphisms. Then there exist four basic operators $Y_i = \widetilde{X}_i - X_i$, $i = 1,2,3,4$, that are corresponded to the adjoint representation of GL. Actually, they are linearly dependent: $Y_1 + Y_4 = 0$, $(a_1 - a_4)Y_1 + a_2Y_2 + a_3Y_3 = 0$. Note that the functions

$$\det A = a_1a_4 - a_2a_3, \quad \operatorname{tr} A = a_1 + a_2$$

are common invariants of Y_i, $i = 1,2,3,4$. Thus the integral distribution spanned by Y_i is two-dimensional and its integral surfaces are precisely two-dimensional quadrics (the family of hyperboloids). The operator in general form

$$Y = c_1Y_1 + c_2Y_2 + c_3Y_3 + c_4Y_4,$$

as a linear vector field in GL, determines the system of ODEs and the flow (see [5], p.49, [6]):

$$A' = CA - AC \quad \Longrightarrow \quad A_t = e^{Ct}Ae^{-Ct}.$$

The corresponding table of commutators looks as follows:

↱	Y_1	Y_2	Y_3	Y_4
Y_1	0	Y_2	$-Y_3$	0
Y_2	$-Y_2$	0	$Y_1 - Y_4$	Y_2
Y_3	Y_3	$Y_4 - Y_1$	0	Y_3
Y_4	0	$-Y_2$	Y_3	0

It allows to describe the dragging of Y_i in the flow of Y (primes here denote the Lie derivative with respect to Y):

$$\begin{aligned}
\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}' &= \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}, \\
\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}'' &= \operatorname{tr} C \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}', \\
\operatorname{tr} C \neq 0 \Longrightarrow \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}_t &= \frac{e^{t \operatorname{tr} C} - 1}{\operatorname{tr} C} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}' + \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}, \\
\operatorname{tr} C = 0 \Longrightarrow \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}_t &= t \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}' + \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}.
\end{aligned}$$

Proposition 15.1. *For each operator* $\frac{\partial}{\partial c_i}$, $i = 1,2,3,4$, *in gl there is defined a one-parameter subgroup of GL such that there are operators* X_i, $\widetilde{X}_i$ *and* Y_i, $i = 1,2,3,4$, *associated with it and induced by the corresponding left and right translations, and interior automorphisms, respectively:*

$$\frac{\partial}{\partial c_1} \rightsquigarrow X_1 = a_1\partial_1 + a_3\partial_3, \quad \widetilde{X}_1 = a_1\partial_1 + a_2\partial_2, \quad Y_1 = a_2\partial_2 - a_3\partial_3;$$
$$\frac{\partial}{\partial c_2} \rightsquigarrow X_2 = a_1\partial_2 + a_3\partial_4, \quad \widetilde{X}_2 = a_3\partial_1 + a_4\partial_2, \quad Y_2 = a_3(\partial_1 - \partial_4) - (a_1 - a_4)\partial_2;$$
$$\frac{\partial}{\partial c_3} \rightsquigarrow X_3 = a_2\partial_1 + a_4\partial_3, \quad \widetilde{X}_3 = a_1\partial_3 + a_2\partial_4, \quad Y_3 = -a_2(\partial_1 - \partial_4) + (a_1 - a_4)\partial_3;$$
$$\frac{\partial}{\partial c_4} \rightsquigarrow X_4 = a_2\partial_2 + a_4\partial_4, \quad \widetilde{X}_4 = a_3\partial_3 + a_4\partial_4, \quad Y_4 = -a_2\partial_2 + a_3\partial_3.$$

The proof of this proposition is based on the exponential map κ.

15.4 The Operator of Center

Consider the vector field

$$P = \frac{\partial}{\partial c_1} + \frac{\partial}{\partial c_4},$$

which corresponds to the center of the Lie group gl. The image of P under $T\kappa$ is the homothety operator

$$T\kappa P = a_1\partial_1 + a_2\partial_2 + a_3\partial_3 + a_4\partial_4$$

in GL. Let $s = \frac{1}{2}\,\mathrm{tr}\,C$ be the canonical parameter of P. Then using the implication $U' = CU \;\Rightarrow\; U_t = e^{Ct}U \;\Rightarrow\; I = e^{-Cs}U$, see [5], p.49, [6], we obtain its flow and three basic invariants of P:

$$C' = E \quad\Longrightarrow\quad C_t = C + tE \quad\Longrightarrow\quad I = C - sE,$$

where E is the unit matrix. Define the projection $\pi : gl \to \mathbb{R}^3$ by

$$\begin{cases} x \circ \pi = \frac{1}{2}(c_1 - c_4), \\ y \circ \pi = c_2, \\ z \circ \pi = c_3. \end{cases}$$

Thus gl is projected by π onto the xyz space of invariants of P. Using the fact that $\mathrm{tr}\,I = 0$ together with $\det e^C = e^{\mathrm{tr}\,C}$ we obtain $\det e^I = 1$. It means that the xyz space is the tangent space to the subgroup of matrices with determinant 1. From

$$P(\det C) = \mathrm{tr}\,C, \quad P^2(\det C) = P(\mathrm{tr}\,C) = 2$$

it follows that $\det C$ behaves in the flow of P as a quadratic polynomial on t and $\mathrm{tr}\,C$ as a linear one, i.e.

$$(\det C)_t = \det C + \operatorname{tr} C \cdot t + t^2, \quad (\operatorname{tr} C)_t = \operatorname{tr} C + 2t.$$

The substitution of $t = -s$ into $(\det C)_t$ gives us the important invariant

$$\Delta = \frac{1}{4}(\operatorname{tr}^2 C - 4\det C).$$

After regrouping of terms we get $\Delta = (x^2 + yz) \circ \pi$. Note that 4Δ is the discriminant of the quadratic polynomial $(\det C)_t$. The eigenvalues of C are equal up to a sign to the roots of the polynomial $(\det C)_t$. Consider the dragging of a hyper-quadric $\det C = k$, $k \in \mathbb{R}$, along the flow of P in $gl \equiv \mathbb{R}^4$. Then we obtain a one-parameter family of hyper-quadrics defined by $(\det C)_t = k$. The intersection of $\det C = k$ with the plane $\operatorname{tr} C = 0$ corresponds to the characteristic which consists of points at that the trajectories of P are tangent to $\det C = k$. The image of this intersection in xyz space is a hyperboloid $x^2 + yz = -k$, which is either hyperboloid of one sheet $(k < 0)$ or two sheets $(k > 0)$, or the light cone $(k = 0)$. The envelope of the family of surfaces is defined by $\Delta = -k$, which is the projecting cylinder with the trajectories of P as generating lines tangent to the characteristic. One can imagine an illumination of the surface $\det C = k$ along the trajectories of P and a shadow on the xyz–screen, the border of which is precisely the hyperboloid $x^2 + yz = -k$. In this case we deal with the cusp singularity of the first type called *fold*, see [1], p.157.

The hyperboloids $x^2 + yz = \text{const}$ in xyz space are precisely the orbits of a generalized group of rotations (see [5], p.77) with operators

$$\widetilde{Y}_1 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}, \quad \widetilde{Y}_2 = z\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}, \quad \widetilde{Y}_3 = -y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}.$$

Here $\widetilde{Y}_2$ and $\widetilde{Y}_3$ have parabolic flows and $\widetilde{Y}_1$ has a hyperbolic one. The field $\widetilde{Y}_1$ is an *infinitesimal symmetry* for $\widetilde{Y}_2$ and $\widetilde{Y}_3$. This way we have described the structure of the quotient group $G_1 = G/Z$.

15.5 Discriminant Parabola

Define the map $\zeta : gl \to \mathbb{R}^2$ by $\begin{cases} u \circ \zeta &= \frac{1}{2}\det C, \\ u' \circ \zeta &= \frac{1}{2}\operatorname{tr} C, \end{cases}$ where (u, u') denote the coordinates in $\mathbb{R}^2$. The operator P is projected by $T\zeta$ to the vector field $T\zeta P = u'\frac{\partial}{\partial u} + \frac{\partial}{\partial u'}$. Thus the flow $C_t = C + tE$ of P is mapped by ζ to the flow of $T\zeta P$ determined by the system

$$\begin{cases} u_t &= u + u't + \frac{t^2}{2}, \\ u'_t &= u' + t. \end{cases}$$

From $((u')^2 - 2u) \circ \zeta = \Delta$ it follows that the discriminant of the quadratic function u_t is ζ–related to the invariant Δ. Being mapped by ζ, the generating lines of the cylinder $\Delta = 0$ lay down onto the *discriminant parabola* $(u')^2 - 2u = 0$ in $\mathbb{R}^2$.

The classification of linear flows in dimension 2 takes place in the uu' plane with respect to the discriminant parabola, see [2], p.86, [5], p.73. From $U' = CU \Rightarrow U_t = e^{Ct}U$ it follows that a linear flow is determined by the exponential e^{Ct}. The eigenvalues of C satisfy the quadratic equation

$$\lambda^2 - \operatorname{tr} C \cdot \lambda + \det C = 0.$$

Depending on the sign of Δ, the eigenvalues may be real: $\lambda_{1,2} = \alpha \pm \beta$, or complex conjugate: $\lambda_{1,2} = \alpha \pm i\beta$, or equal: $\lambda_1 = \lambda_2 = \alpha$, where $\alpha, \beta \in \mathbb{R}$ are given by

$$\operatorname{tr} C = \lambda_1 + \lambda_2 = 2\alpha, \quad \det C = \lambda_1 \lambda_2 = \alpha^2 \pm \beta^2.$$

Here the plus sign corresponds to the case of complex roots and minus to the case of real ones. Thus

$$\Delta = \operatorname{tr}^2 C - 4 \det C = (\lambda_1 - \lambda_2)^2 = \mp 4\beta^2$$

and e^{Ct} depends on the sign of Δ as follows:

$$e^{Ct} = \begin{cases} e^{\alpha t}\left[E \cos \beta t + (C - \alpha E)\frac{\sin \beta t}{\beta}\right], & \text{if } \Delta < 0, \\ e^{\alpha t}\left[E \cosh \beta t + (C - \alpha E)\frac{\sinh \beta t}{\beta}\right], & \text{if } \Delta > 0, \\ e^{\alpha t}\left[E + (C - \alpha E)t\right], & \text{if } \Delta = 0. \end{cases}$$

The various possible flow patterns can be summarized as follows:
$\Delta < 0, \det C < 0$ -Ű-elliptic flow with focuses;
$\Delta < 0, \det C > 0$ –Űhyperbolic flow with saddles;
$\Delta > 0$ Ű–hyperbolic knots;
$\Delta = 0$ –Űparabolic knots.
Depending on the sign of α the knots and focuses may be stable $(\alpha < 0)$ or unstable $(\alpha > 0)$.

15.6 Relations to Moments in Probability Theory

Consider the raw and central moments

$$\nu_k = EX^k, \quad \mu_k = E(X - EX)^k,$$

where X is a random variable and E denotes the expectation value. The central moments μ_k, $k = 1, 2, \ldots$, can be expressed in terms of ν_k. The first few cases are given by

$$\begin{aligned}
\mu_1 &= 0,\\
\mu_2 &= \nu_2 - \nu_1^2,\\
\mu_3 &= \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3\\
\mu_4 &= \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4,\\
&\ \vdots
\end{aligned}$$

Let us show for $k = 2,3,4$ that if we identify (up to a constant multiplier) the coefficients in u_t with raw moments, then the invariants of u_t agree with the central ones.

The case $k = 2$. Denote

$$\begin{aligned}
u_t &= \frac{1}{2}E(X+t)^2, & u &= \frac{1}{2}EX^2 = \frac{1}{2}\nu_2,\\
u_t' &= E(X+t), & u' &= EX = \nu_1.
\end{aligned}$$

Then

$$\begin{aligned}
u_t &= u + u't + \frac{t^2}{2},\\
u_t' &= u' + t.
\end{aligned}$$

The substitution $t = -u'$ gives us the fiber invariant that is equal (up to the coefficient) to the central moment μ_2:

$$\Delta = u - \frac{1}{2}(u')^2, \quad \Delta = \frac{1}{2}\mu_2.$$

The case $k = 3$. Denote

$$\begin{aligned}
u_t &= \frac{1}{3!}E(X+t)^3, & u &= \frac{1}{3!}EX^3 = \frac{1}{3!}\nu_3,\\
u_t' &= \frac{1}{2}E(X+t)^2, & u' &= \frac{1}{2}EX^2 = \frac{1}{2}\nu_2,\\
u_t'' &= E(X+t), & u'' &= EX = \nu_1.
\end{aligned}$$

Then

$$\begin{aligned}
u_t &= u + u't + u''\frac{t^2}{2} + \frac{t^3}{3!},\\
u_t' &= u' + u''t + \frac{t^2}{2},\\
u_t'' &= u'' + t.
\end{aligned}$$

The substitution $t = -u''$ gives us two fiber invariants equal (up to the coefficients) to the central moments μ_3 and μ_2, respectively:

$$
\begin{aligned}
i_0 &= u - u'u'' + \frac{1}{3}(u'')^3, &\qquad i_0 &= \frac{1}{3!}\mu_3,\\
i_1 &= u' - \frac{1}{2}(u'')^2, &\qquad i_1 &= \frac{1}{2}\mu_2.
\end{aligned}
$$

Note that the cubic discriminant of u_t expressed in terms of i_0 and i_1 is

$$I = (3i_0)^2 + (2i_1)^3.$$

The case $k = 4$. Denote

$$
\begin{aligned}
u_t &= \frac{1}{4!}E(X+t)^4, &\qquad u &= \frac{1}{4!}EX^4 = \frac{1}{4!}\nu_4,\\
u_t' &= \frac{1}{3!}E(X+t)^3, &\qquad u' &= \frac{1}{3!}EX^3 = \frac{1}{3!}\nu_3,\\
u_t'' &= \frac{1}{2}E(X+t)^2, &\qquad u'' &= \frac{1}{2}EX^2 = \frac{1}{2}\nu_2,\\
u_t''' &= E(X+t), &\qquad u''' &= EX = \nu_1.
\end{aligned}
$$

Then

$$
\begin{aligned}
u_t &= u + u't + u''\frac{t^2}{2} + u'''\frac{t^3}{3!} + \frac{t^4}{4!},\\
u_t' &= u' + u''t + u'''\frac{t^2}{2} + \frac{t^3}{3!},\\
u_t'' &= u'' + u'''t + \frac{t^2}{2},\\
u_t''' &= u''' + t.
\end{aligned}
$$

The substitution $t = -u'''$ gives us three fiber invariants I_0, I_1, I_2 equal (up to the coefficients) to the central moments μ_4, μ_3 and μ_2, respectively:

$$
\begin{aligned}
I_0 &= u - u'u''' + \frac{1}{2}u''(u''')^2 - \frac{1}{8}(u''')^4, &\qquad I_0 &= \frac{1}{4!}\mu_4,\\
I_1 &= u' - u''u''' + \frac{1}{3}(u''')^3, &\qquad I_1 &= \frac{1}{3!}\mu_3,\\
I_2 &= u'' - \frac{1}{2}(u''')^2, &\qquad I_2 &= \frac{1}{2}\mu_2.
\end{aligned}
$$

These three cases are united by the common scheme (*exponential law*)

$$U' = CU \quad\Longrightarrow\quad U_t = e^{Ct}U \quad\Longrightarrow\quad I = e^{-Ct}U$$

for calculating invariants of total differentiation operator

$$D = \frac{\partial}{\partial t} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + u'''\frac{\partial}{\partial u''} + \cdots,$$

taking into account $u' = 1$, $u'' = 1$, $u''' = 1, \ldots$, respectively, see [5], p.49, [6].

15.7 Conclusion

In the previous section we have shown that the central moments from probability theory are invariants of polynomials given by the raw moments. The analogous situation can be observed with moments appearing in mechanics. For instance, the statistic moment in mechanics corresponds to the raw moment $\nu_1 = EX$, the moment of inertia to the central moment $\mu_2 = E(X - EX)^2$ (dispersion) and so on. Rewrite the discriminant of the quadratic function u_t in the form $\Delta = uu'' - \frac{1}{2}(u')^2$, and suppose that u, u' and u'' denote initial path, velocity and acceleration, respectively. Then the first summand in Δ is understood as a potential energy and the second one as a kinetic energy. Thus the equality $\Delta' = 0$ presents the conservation law of energy.

Any matrix C is understood as a linear element of movement. Any perturbation of a movement leads to an interior automorphism of C. The coefficients appearing in Hamilton-Cayley formula for C gives us the matrix invariants under interior automorphisms, including $\det C$ and $\operatorname{tr} C$. The invariants of the next perturbation caused by the vector field P (or D after the map ζ) are precisely the discriminant Δ in dimension 2 $(k = 2)$ and the discriminant I in dimension 3 $(k = 3)$, and so on.

In general, a tangent vector as an element of a tangent bundle TM of a manifold M is understood as a stop-frame of a movement of order 1, and an element of a k–th order tangent bundle T^kM as a stop-frame of a movement of order k.

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