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HANNA BRITT SOOTS

Collocation based approximations for fractional differential equations

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To my friends and family,
who asked polite questions and nodded supportively.

CONTENTS

1. INTRODUCTION

The study of integro-differential equations is a large branch of mathematics that connects various fields within and outside mathematics. Usually, when we talk about derivatives of a function $y = f(t)$, we think of the derivatives of the integer order, i.e. $y' = \frac{d}{dt}f(t)$, $y'' = \frac{d^2}{dt^2}f(t), \ldots$. However, the question arises - can the order of a derivative be a positive real number or even a complex number? It turns out it is possible (see e.g. [34]), and then we are talking about *fractional* derivatives. The concept of fractional derivatives has a rich historical background dating back to 1695 when l'Hospital posed the fundamental question about the meaning of a fractional order derivative to Leibniz. When Leibniz proposed the symbol $\frac{d^n}{dt^n}f(t)$ for the n-th order derivative of a function $f(t)$, l'Hospital asked, what does $\frac{d^n}{dt^n}f(t)$ mean if $n = \frac{1}{2}$. Leibniz's reply concluded with the remark "This is an apparent paradox from which, one day, useful consequences will be drawn." [61]. Because l'Hospital specifically asked for the case $n = 1/2$, which is a fraction, the term *fractional derivative* became common use.

L'Hospital's inquiry initiated the investigation into extending classical differential operators to non-integer (fractional) orders. For nearly three centuries, the development of fractional calculus remained predominantly theoretical, with limited application in real-world problems—thus, it was largely confined to pure mathematics. However, in recent decades, this perspective has shifted significantly, as fractional-order derivatives, and equations containing them (i.e. fractional differential equations), have demonstrated strong potential in accurately modelling memory-dependent and hereditary behaviours in complex materials and dynamic systems. For instance, fractional derivatives have shown great promise in biomedicine [30, 52, 55], fluid dynamics [3, 22] and modelling chaotic systems [6, 15, 32, 54]. For a broader overview of applications and usages to fractional derivatives, we refer the reader to [7, 23, 48, 53, 70, 71].

Since closed-form solutions for fractional differential equations are rarely available [51, 60] the analysis and development of numerical methods for fractional differential equations has become a very active area of research. For the fundamental theory of fractional derivatives and equations containing them, we refer to the monographs [16, 34, 60, 63] and comprehensive works [17, 18, 19, 45, 68, 76]. Some other recent studies on fractional differential equations can be found in [1, 7, 8, 9, 14, 31, 44, 77].

Fractional differential equations present significant analytical and numerical challenges due to their inherent nonlocality and possible singular behaviour of their solutions. The nonlocality leads to long-range dependencies, making standard discretisation techniques inefficient or inaccurate. Furthermore, the presence of singularities of the solution near domain boundaries results in a low regularity of solutions, demanding careful treatment in both theoretical analysis and numerical schemes [57, 62, 67, 68]. Therefore, meticulous analytical approaches for obtaining high-order numerical methods are needed to overcome those obstacles. One strategy involves reformulating fractional differential equations as integral equations. This strategy is often helpful in investigating the existence, uniqueness and regularity of exact solutions. Furthermore, integral equations are often more practical in the construction of numerical approximations [4, 10, 11, 74]. In particular, for fractional differential and integro-differential equations, several studies have used collocation-based techniques, see, for example, [25, 26, 27, 36, 43, 47, 59, 78].

Motivated by these observations, in the present dissertation we investigate various fractional differential and integro-differential equations and study the existence, uniqueness, and regularity of their solutions. Based on the information obtained, we construct and analyse numerical approximations of underlying problems. Throughout this thesis, we study different fractional problems applying the following steps.

Integral equation reformulation: Each fractional problem under consideration is converted into corresponding integral equation.

The study of existence, uniqueness and regularity: Rigorous theoretical analysis establishes conditions under which solutions to these integral equations exist uniquely and are of sufficient regularity. Such analysis is vital for ensuring the validity and reliability of subsequent numerical solutions.

Numerical approximation via collocation methods: Employing collocation-based numerical methods, we provide systematic and reliable numerical approaches to solve the integral formulations of fractional equations.

Convergence and error analysis: Comprehensive convergence analyses and precise error estimations are performed to assess the accuracy of the numerical methods.

Comparison of theoretical results with numerical experiments: Theoretical findings are validated through carefully constructed numerical examples. These examples illustrate the effectiveness of our methods and offer practical insights into their performance.

The present dissertation is organised into several chapters. **Chapters 1** and **2** have an introductory character. In **Chapter 2** we present the preliminary results that form the basis of the research by defining necessary functions, spaces, operators and presenting some of their properties. For next chapters we target specific classes of fractional differential equations which in **Chapters 3** and **4** involve Caputo fractional differential operators and in **Chapter 5** the inverse of Riemann-Liouville integral operators (for the exact definitions of the fractional differential operators, see Section 2.3).

In **Chapter 3**, we investigate a class of boundary value problems for Caputo fractional weakly singular integro-differential equations. We first reformulate the underlying problem into an equivalent integral equation. This reformulation facilitates a clearer understanding of the problem structure. It lays the foundation for high-order numerical methods tailored to handle the solution's potential singular behaviour near the origin.

Then, we establish theoretical aspects such as the existence, uniqueness, and regularity of the equation's solution under general conditions based on the equation's coefficients and data. Leveraging Fredholm theory we show that the associated operator is compact and that the integral equation has a unique solution under quite general assumptions. Furthermore, we characterise the solution's smoothness by suitable weighted spaces, which provide insight into the nature of possible singularities of the exact solution and how they affect numerical approximations.

Next, we develop and analyse a numerical scheme based on collocation with piecewise polynomial approximation. The method accommodates non-uniform mesh refinement to resolve singularities effectively. Then, we provide a detailed convergence analysis and derive error estimates, demonstrating how the method's accuracy depends on mesh grading, the smoothness of the data, and the choice of collocation points. The chapter concludes with numerical experiments validating the theoretical error bounds and showcasing the method's high-order accuracy. The main results of this chapter are given by Theorems 3.3, 3.5 and 3.6. Obtained results complement the corresponding results of the previous work of [56, 58, 78].

Chapter 4 introduces and analyses a numerical approach for solving the time-fractional sub-diffusion equation, a mathematical model that describes anomalous diffusion where particles spread more slowly than in classical diffusion. The governing equation includes the Caputo fractional derivative in time, making the problem nonlocal and capturing memory effects inherent in many real-world processes, such as transport in disordered media or diffusion in biological systems. The chapter begins by motivating the study through physical intuition and classical diffusion theory, then formulates the sub-diffusion model with appropriate initial and boundary conditions.

The method of lines is employed by discretising the spatial variable, leading to a system of fractional differential equations in time to solve the problem numerically. We reformulate this system as a system of weakly singular Volterra integral equations of the second kind. We study the existence, uniqueness and regularity of this system's solution. Then, we apply a collocation method, using piecewise polynomial basis functions, to approximate the solution of the underlying problem.

We analyse the convergence of the proposed numerical method. The results show that the error depends on both the spatial discretisation and the choice of collocation points in time. The method achieves optimal convergence rates despite possible singular behaviour of the solution near the initial time, a common feature of fractional models. Finally, we present a final error estimate combining spatial and temporal discretisation effects, confirming that the method is accurate and reliable for a broad class of sub-diffusion problems, which illustrate our findings with some numerical examples. The main results of this chapter are given by Theorems 4.4, 4.6, 4.7 and 4.8. The results enable us to construct a high-order numerical method, despite the fact that the temporal partial derivatives of solutions of time-fractional diffusion equations sometimes have weak singularities at the initial time. It is also worth mentioning that we can achieve a sufficiently high convergence order even when using polynomials of low degree.

Chapter 5 presents a rigorous study of a class of singular fractional integro-differential equations with non-constant coefficients, using cordial Volterra-type integral operators. The model equation incorporates the inverse of the Riemann–Liouville fractional integral operator and features integral operators that complicate the analysis due to their possible non-compactness. The reformulation of the equation into a cordial Volterra integral form allows for deeper analysis of the existence and uniqueness of the exact solution to our underlying problem. Due to the possible non-compactness of the integral operators, traditional approaches like the Fredholm alternative can not be applied.

We use these integral operators' spectral properties to study the existence and uniqueness of the problem's solution. We show that the spectrum of cordial Volterra-type integral operators with varying coefficients depends only on their values at the origin. We formulate the conditions for the existence and uniqueness of exact solutions of the underlying problem.

In the final part of the chapter, we develop a collocation-based numerical scheme. Rigorous convergence analysis is provided, including error estimates. And we include practical remarks to ease the verification of key assumptions in applied settings. Also, we present numerical examples to illustrate the key findings. The main results of this chapter are given by Theorems 5.13, 5.17 and 5.18. The results presented in this chapter extend the approach proposed in [38] to a wider class of equations with non-constant coefficients. Generalising to equations with non-constant coefficients presents several significant challenges since both the investigation of the unique solvability of the equation and the study of collocation based numerical methods require new ideas and auxiliary results compared with the constant case. The results of this chapter also complement and generalise [41] where only the unique solvability of singular fractional differential equations was considered.

In conclusion, **Chapters 3–5** illustrate a unified approach to studying fractional differential and integro-differential equations—bridging theoretical solvability with numerical methods.

The results obtained in this thesis are published in articles [39, 40, 65], and they have also been presented by the author at four international scientific conferences.

2. PRELIMINARY RESULTS

In this chapter we introduce basic notations and results which we use throughout this thesis. For a thorough overview we refer the reader to [5, 37, 66], more precisely for fractional analysis, we recommend the following sources [7, 16, 34, 60, 63].

2.1. Notations

Throughout this work $c, c_0, c_1, \dots$ denote positive constants that may have different values in different occurrences. We denote the set of natural numbers by $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, by $\mathbb{Z}$ the set of all integers $\{\dots, -1, 0, 1, \dots\}$, the set of real numbers is denoted by $\mathbb{R} = (-\infty, \infty)$. The set of all complex numbers is defined by $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, where $\lambda = \mathrm{Re}\lambda + i\mathrm{Im}\lambda$ for $\lambda \in \mathbb{C}$ and $i = \sqrt{-1}$ is the imaginary unit.
By $C[0, T]$, for $T \in (0, \infty)$, we denote the space of continuous functions $f : [0, T] \to \mathbb{C}$ and by $C^m[0, T]$, for $m \in \mathbb{N}$, the space of m-times continuously differentiable functions $f : [0, T] \to \mathbb{C}$ (for $m = 0$ we set $C^0[0, T] = C[0, T]$). The space $C[0, T]$ equipped with the norm

$$\|f\|_{C[0,T]} = \|f\|_\infty = \max_{t \in [0,T]} |f(t)|, \quad f \in C[0, T],$$

becomes a Banach space. And the space $C^m[0, T]$ with $m \in \mathbb{N}$ is a Banach space if it is equipped with the norm

$$\|f\|_{C^m[0,T]} = \max_{k=0,\dots,m} \|f^{(k)}\|_{C[0,T]}, \quad f \in C^m[0, T],$$

where $f^{(k)}$ denotes the k-th derivative of function $f \in C^m[0, T]$.
We will also use the space of m-times ($m \in \mathbb{N}$) continuously differentiable vector functions $C^m_n[0, T]$, $n \in \mathbb{N}$; for $m = 0$ we set $C^0_n[0, T] = C_n[0, T]$. Namely for

$$\vec{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} = (f_1(t), \dots, f_n(t))^T, \quad t \in [0, T],$$

the notation $\vec{f} \in C^m_n[0, T]$ means that $f_i \in C^m[0, T]$ for every $i = 1, \dots, n$. Note that $C^m_n[0, T]$ is a Banach space with respect to the norm

$$\|\vec{f}\|_{C^m_n[0,T]} = \max_{i=1,\dots,n} \max_{k=0,\dots,m} \max_{t \in [0,T]} |f_i^{(k)}(t)|, \quad \vec{f} \in C^m_n[0, T].$$

By $L^1(0, T)$, we denote the Banach space of measurable functions $f : [0, T] \to \mathbb{C}$ equipped with the norm

$$\|f\|_{L^1(0,T)} = \|f\|_1 = \int_0^T |f(t)| dt < \infty.$$

By $L^\infty(0,T)$, we denote the Banach space of measurable functions $f : [0,T] \to \mathbb{C}$ such that

$$\inf_{A\subset[0,T];\mu(A)=0} \sup_{t\in[0,T]\setminus A} |f(t)| < \infty, \tag{2.1}$$

where $\mu(A)$ is the Lebesgue measure of the set A. The space $L^\infty(0,T)$ is equipped with the norm

$$\|f\|_{L^\infty(0,T)} = \|f\|_\infty = \inf_{A\subset[0,T];\mu(A)=0} \sup_{t\in[0,T]\setminus A} |f(t)|.$$

For $\vec{f} = (f_1,\dots,f_n)^T$, the writing $\vec{f} \in L_n^\infty(0,T)$ means that $f_i \in L^\infty(0,T)$, $i = 1,\dots,n$. The space $L_n^\infty(0,T)$ is a Banach space with respect of the norm

$$\|\vec{f}\|_{L_n^\infty(0,T)} = \max_{i=1\dots,n} \|f_i\|_\infty, \quad \vec{f} \in L_n^\infty(0,T).$$

Let $T \in (0,\infty)$. A function $f : [0,T] \to \mathbb{C}$ is said to be absolutely continuous on $[0,T]$ (and we denote it by $f \in AC[0,T]$), if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\{(a_j,b_j)\}_{j=1}^n$, is a finite pairwise disjoint family of subintervals of $[0,T]$ satisfying

$$\sum_{j=1}^n (b_j - a_j) < \delta, \quad \text{then} \quad \sum_{j=1}^n |f(b_j) - f(a_j)| < \epsilon.$$

By $AC^n[0,T]$, where $n \in \mathbb{N}$, we denote the set of functions with an absolutely continuous $(n-1)$-st derivative. That means that $f \in AC^n[0,T]$ if $f^{(n-1)} \in AC[0,T]$. If $f \in AC^n[0,T]$, there exists a function $g \in L^1[0,T]$ such that

$$f^{(n-1)}(x) = f^{(n-1)}(0) + \int_0^x g(t)dt.$$

In this case we call g the generalised n-th derivative of f, and we simply write $g = f^{(n)}$.

In order to characterize the existence, uniqueness and regularity of a solution of a fractional differential equation under consideration, we introduce a weighted space $C^{q,\nu}(0,T]$ of functions on $(0,T]$.

More precisely, due to [13] (see also [74]), for given $q \in \mathbb{N}$, $\nu, T \in \mathbb{R}$ where $T > 0$ and $\nu < 1$, by $C^{q,\nu}(0,T]$ we denote the set of continuous functions $f : [0,T] \to \mathbb{R}$ which are q times continuously differentiable in $(0,T]$ such that for all $t \in (0,T]$ and $i = 1,\dots,q$, the following estimates hold:

$$|f^{(i)}(t)| \le c \begin{cases} 1 & \text{for} \quad i < 1-\nu, \\ 1 + |\log t| & \text{for} \quad i = 1-\nu, \\ t^{1-\nu-i} & \text{for} \quad i > 1-\nu. \end{cases}$$

Here $c = c(y)$ is a positive constant. The set $C^{q,\nu}(0,T]$ becomes a Banach space if it is equipped with the norm

$$\|f\|_{C^{q,\nu}(0,T]} = \|f\|_\infty + \sum_{i=1}^{q} \sup_{0<t\leq T} \omega_{i-1+\nu}(t)|f^{(i)}(t)|, \quad f \in C^{q,\nu}(0,T],$$

where, for $t > 0$, $\lambda \in \mathbb{R}$,

$$\omega_\lambda(t) = \begin{cases} 1 & \text{for} \quad \lambda < 0, \\ \dfrac{1}{1+|\log t|} & \text{for} \quad \lambda = 0, \\ t^\lambda & \text{for} \quad \lambda > 0. \end{cases}$$

Note that

$$C^n[0,T] \subset C^{n,\nu}(0,T] \subset C^{m,\mu}(0,T] \subset C[0,T], \quad n \geq m \geq 1, \quad \nu \leq \mu < 1.$$

For $\vec{f} = (f_1, \dots, f_n)^T$, the writing $\vec{f} \in C_n^{q,\nu}(0,T]$ means that $f_i \in C^{q,\nu}(0,T]$ for every $i = 1, \dots, n$. The set $C_n^{q,\nu}(0,T]$ is a Banach space with respect to the norm

$$\|\vec{f}\|_{C_n^{q,\nu}(0,T]} = \max_{i=1,\dots,n} \|f_i\|_{C^{q,\nu}(0,T]}, \quad \vec{f} \in C_n^{q,\nu}(0,T].$$

2.2. Gamma, beta and Mittag-Leffler functions

Having established the function spaces, we now define special functions essential to fractional calculus. More details can be found, for example, in [16, 60].

The gamma function $\Gamma(x)$, where $x \in (0,\infty)$, is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in (0,\infty). \tag{2.2}$$

The beta function $B(x,y)$, where $x, y \in (0,\infty)$, is defined by the formula

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds, \quad x, y \in (0,\infty).$$

Gamma and beta functions are related by the equality

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y \in (0,\infty).$$

The function E_α defined by

$$E_\alpha(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(1+j\alpha)}, \quad x \in \mathbb{R},$$

where Γ denotes the gamma function, is called the Mittag-Leffler function with parameter $\alpha > 0$. Note that the series converges for all values of $x \in \mathbb{R}$.

2.3. Fractional differential operators

For a thorough overview of fractional differential operators we refer the reader to [16, 34, 60, 63]. By I we denote the identity mapping and by $D^n = \left(\frac{d}{dt}\right)^n$ the classical differential operator.

Let $T \in (0, \infty)$. The Riemann-Liouville fractional integral operator J^α of order $\alpha > 0$ is defined by

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \ \ \alpha \in (0,\infty), \ \ x \in [0,T], \ \ f \in L^\infty(0,T).$$

We also set $J^0 = I$. Note that for any $f \in L^\infty(0,T)$, we have the semigroup property

$$J^\delta J^\eta f = J^\eta J^\delta f = J^{\delta+\eta} f, \quad \delta > 0, \quad \eta > 0, \tag{2.3}$$

and

$$D^k(J^\eta f) \in C[0,T], \quad (D^k J^\eta f)(0) = 0, \quad \eta > 0, \quad k = 0, \ldots, \lceil \eta \rceil - 1, \tag{2.4}$$

where $\lceil \eta \rceil$ denotes the smallest integer greater than or equal to real number η.

The Riemann-Liouville differential operator $D^\alpha_{\text{R-L}}$ of order $\alpha \in (n-1, n]$, for $n \in \mathbb{N}$, is determined by the formula

$$D^\alpha_{\text{R-L}} f = D^n J^{n-\alpha} f, \quad J^{n-\alpha} f \in C^n(0,T], \quad f \in C^{n-1}[0,T].$$

If $\alpha \in \mathbb{N}$ and $f \in C^\alpha[0,T]$, then $(D^\alpha_{\text{R-L}} f)(t) = f^{(\alpha)}(t)$, $t \in [0,T]$.

Let $n \in \mathbb{N}$. By $Q_{n-1}[f]$ we denote the Taylor polynomial of degree $n-1$ for the function $f \in C^{n-1}[0,T]$ at the point 0:

$$(Q_{n-1}[f])(s) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} s^j. \tag{2.5}$$

We can define the Caputo fractional differential operator D^α_{Cap} of order $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, via Riemann-Liouville fractional differential operator as follows:

$$(D^\alpha_{\text{Cap}} f)(t) = (D^\alpha_{\text{R-L}} (f - Q_{n-1}[f]))(t), \quad t \in (0,T], \quad f \in C^{n-1}[0,T]. \tag{2.6}$$

Observe that in (2.6) Caputo fractional derivative is defined for functions f for which the Riemann-Liouville fractional derivatives exists.

If $\alpha \in \mathbb{N}$ and $f \in C^\alpha[0,T]$, then $(D^\alpha_{\text{Cap}} f)(t) = f^{(\alpha)}(t)$, $t \in [0,T]$. Also note, that for any $f \in L^\infty(0,b)$ we have

$$D^n J^n f = f, \quad D^\eta_{\text{Cap}} J^\eta f = f, \quad n \in \mathbb{N}, \quad \eta \geq 0. \tag{2.7}$$

In **Chapter 3** we study differential equations with Caputo-type fractional differential operator defined by (2.6) of order $\alpha \in (1,2)$. Note that in [76] necessary and sufficient conditions for the existence of $D^\alpha_{\text{Cap}} f \in C[0,T]$ have been derived.

Lemma 2.1 (Theorem 5.2 in [76]). For $m < \alpha < m+1$, $m \in \mathbb{N}_0$, and $f \in C^m[0,T]$ the following conditions are equivalent:

(i) the fractional derivative $D^{\alpha}_{\mathrm{Cap}} f \in C[0,T]$ exists;

(ii) a finite limit $\lim_{t\to 0} t^{m-\alpha}(f^{(m)}(t) - f^{(m)}(0))$ exist, and

$$\sup_{0<t\le T} \left| \int_{\theta t}^{t} (t-s)^{m-\alpha-1}(f^{(m)}(t) - f^{(m)}(s))ds \right| \to 0, \quad \text{as } \theta \to 1. \tag{2.8}$$

The following lemma from [16] shows that the Caputo derivative is not in general the right inverse of the Riemann-Liouville integral and therefore in general $z = D^{\alpha}_{\mathrm{Cap}} f$ does *not* imply that $f = J^{\alpha} z$.

Lemma 2.2. (Theorem 3.8 in [16]) Assume that $n \in \mathbb{N}$, $\alpha \in (n-1, n]$ and $f \in AC^n[0,T]$. Then

$$\left(J^{\alpha} D^{\alpha}_{\mathrm{Cap}} f\right)(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k, \quad x \in [0,T]. \tag{2.9}$$

We have also the following result.

Lemma 2.3 (Theorem 2.1 in [34]). Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. If $f \in AC^n[0,T]$ then the Caputo fractional derivative $D^{\alpha}_{\mathrm{Cap}} f$ exists almost everywhere on $[0,T]$ and

$$D^{\alpha}_{\mathrm{Cap}} f = J^{n-\alpha} D^n f. \tag{2.10}$$

Sometimes (for example in [60]) this formula is used for the definition of the Caputo fractional derivative. In **Chapter 4** we use this definition and we will adopt the notation $D^{\alpha} f$ for the Caputo fractional derivative of $f \in AC^n[0,T]$. In particular, in the case $0 < \alpha < 1$, we have

$$(D^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad t \in (0,T], \quad \alpha \in (0,1), \tag{2.11}$$

where f is an absolutely continuous function on $[0,T]$.

In **Chapter 5** we study equations containing fractional differentiation operators D^{α}_0 of order $\alpha \in [0,\infty)$, which are defined as the inverse of the Riemann-Liouville integral operator J^{α} on $J^{\alpha}(C[0,T])$, i.e.

$$D^{\alpha}_0 v = (J^{\alpha})^{-1} v, \quad v \in J^{\alpha}(C[0,T]), \quad \alpha > 0; \quad D^0_0 := I. \tag{2.12}$$

Note that (2.12) is well defined. Indeed, we know, that for $\alpha = q \in \mathbb{N}$, the range of the operator J^q is the set

$$C^q_0[0,T] = \{u \in C^q[0,T] : u^{(k)}(0) = 0, \quad k = 0, \ldots, q-1\} \subset C[0,T],$$

and J^q is invertible on it: $(J^q)^{-1} v = D^q_0 v$, $v \in C^q_0[0,T]$, where the operator $D^q_0 : C^q_0[0,T] \to C[0,T]$ is the restriction of the classical differential operator

$D^q = \left(\frac{d}{dt}\right)^q : C^q[0,T] \to C[0,T]$ to the subspace $C_0^q[0,T]$ of $C[0,T]$. Due to the semigroup property (2.3), the operator J^α is invertible on its range $J^\alpha(C[0,T])$ also for fractional $\alpha > 0$. Indeed, if $J^\alpha u = 0$ for some $u \in C[0,T]$, then taking a $q \in \mathbb{N}$ $q > \alpha$, we have $J^q u = J^{q-\alpha} J^\alpha u = 0$, $u = 0$.
In theoretical considerations the use of D_0^α is often preferable to Riemann-Liouville and Caputo fractional differential operators that are more popular in literature. In particular, due to (2.3), operator D_0^α has the property

$$D_0^\alpha D_0^\beta = D_0^\beta D_0^\alpha = D_0^{\alpha+\beta}, \quad \alpha > 0, \beta > 0, \tag{2.13}$$

whereas for Riemann-Liouville and Caputo fractional differential operators this property is lost.
To describe the class of fractionally differentiable functions (i.e. the range of $J^\alpha(C[0,T])$), we present the following result.
Lemma 2.4. (Theorem 2.2. in [76]) For $m < \alpha < m+1$, $m \in \mathbb{N}_0$ and $v \in C[0,T]$, the following conditions are equivalent:

(i) $v \in J^\alpha(C[0,T])$, i.e. the fractional derivative $D_0^\alpha v \in C[0,T]$ exists,

(ii) $v \in C_0^m[0,T]$, a finite limit $\lim_{t\to 0} t^{m-\alpha} v^{(m)}(t)$ exists, and

$$\sup_{0<t\le T} \left| \int_{\theta t}^{t} (t-s)^{m-\alpha-1} (v^{(m)}(t) - v^{(m)}(s)) ds \right| \to 0 \text{ for } \theta \to 1.$$

2.4. Linear operators

We present the following results from the classical theory of operators, see, e.g., [5, 37, 64].

Let E and F be vector spaces. An operator $B : E \to F$ is called a linear operator if

$$B(\alpha x + \beta y) = \alpha B(x) + \beta B(y), \quad \forall x, y \in E, \quad \alpha, \beta \in \mathbb{C}.$$

Definition 2.5. Let E and F be normed spaces. A linear operator $B : E \to F$ is called bounded if there exists a constant $M \ge 0$ such that

$$\|Bx\|_F \le M \|x\|_E, \quad \forall x \in E.$$

For Banach spaces E, F, by $\mathcal{L}(E,F)$, we denote the Banach space of linear bounded operators $A : E \to F$ with the norm $\|A\|_{\mathcal{L}(E,F)} = \sup\{\|Ax\|_F : x \in E, \|x\|_E \le 1\}$. We also denote $\mathcal{L}(E) = \mathcal{L}(E,E)$.
Lemma 2.6. [37] Let E, F be Banach spaces. If $A, B \in \mathcal{L}(E,F)$, $A^{-1} \in \mathcal{L}(F,E)$ and $\|B\|_{\mathcal{L}(E,F)} \|A^{-1}\|_{\mathcal{L}(F,E)} < 1$, then operator $A+B$ has an inverse operator $(A+B)^{-1} \in \mathcal{L}(F,E)$ and

$$\|(A+B)^{-1}\|_{\mathcal{L}(F,E)} \le \frac{\|A^{-1}\|_{\mathcal{L}(F,E)}}{1 - \|B\|_{\mathcal{L}(E,F)} \|A^{-1}\|_{\mathcal{L}(F,E)}}. \tag{2.14}$$

Let E and F be normed spaces. A linear operator $B : A \to F$ is said to be compact if for every bounded sequence $(x_n)_{n\geq 1} \subset E$ the sequence $(Bx_n)_{n\geq 1} \subset F$ has a convergent subsequence in F.

Lemma 2.7. [5] Let E, F and G be normed spaces and let $A : E \to F$ and $B : F \to G$ be bounded linear operators. Then the product $BA : E \to G$ is compact if one of the two operators A or B is compact.

Theorem 2.8. *[5] (Fredholm alternative theorem). Let E be a Banach space and let $A \in \mathcal{L}(E, E)$ be a compact operator. Then the equation $z = Az + f$, $f \in E$ has a unique solution $z \in E$ if and only if the homogeneous equation $z = Az$ has only the trivial solution $z = 0$.*

Lemma 2.9. [13] Let $\eta \in (-\infty, 1)$, $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}$ and $U \in C(\Delta)$. Then operators S_1 and S_2 defined by

$$(S_1y)(t) = \int_0^t (t-s)^{-\eta} U(t,s)y(s)ds, \quad t \in [0,b], \tag{2.15}$$

$$(S_2y)(t) = \int_0^t [1 + \log(t-s)]U(t,s)y(s)ds, \quad t \in [0,b], \tag{2.16}$$

are both compact as operators from $L^\infty(0, b)$ into $C[0, b]$. If in addition $U \in C^q(\Delta)$, $q \in \mathbb{N}$, then S_1 is compact as an operator from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$, where $\eta \leq \nu < 1$, and S_2 is compact as an operator from from $C^{q,\nu}(0, b]$ into $C^{q,\nu}(0, b]$ for $0 \leq \nu < 1$.

Lemma 2.10. [13] If $y_1, y_2 \in C^{q,\nu}(0, b]$, $q \in \mathbb{N}$, $\nu < 1$, then $y_1, y_2 \in C^{q,\nu}(0, b]$ and

$$\|y_1 y_2\|_{C^{q,\nu}(0,b]} \leq c\|y_1\|_{C^{q,\nu}(0,b]}\|y_2\|_{C^{q,\nu}(0,b]},$$

with a constant c which is independent of y_1 and y_2.

2.5. Graded grids and interpolation operators

Throughout this dissertation we have chosen collocation based techniques as part of our numerical approach.

Let $N \in \mathbb{N}$, first we introduce a graded grid $\Pi_N = \{t_0, \ldots, t_N\}$ of the interval $[0, b]$ with the grid points

$$t_j = b\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \ldots, N, \quad N \in \mathbb{N}. \tag{2.17}$$

The so called grading exponent r belongs to $[1, \infty)$. If $r = 1$, then we have a uniform grid; for $r > 1$, the points (2.17) are more densely clustered near the left endpoint of the interval $[0, b]$.

Next, for a given integer $k \in \mathbb{N}_0$, by $S_k^{(-1)}(\Pi_N)$, we denote the space of piecewise polynomial functions

$$S_k^{(-1)}(\Pi_N) = \{v : \quad v|_{[t_{j-1},t_j]} \in \pi_k, j = 1, \ldots, N\}. \tag{2.18}$$

Here $v|_{[t_{j-1},t_j]}$ $(j = 1, \ldots, N)$ is the restriction of function $v : [0, b] \to \mathbb{R}$ onto the subinterval $[t_{j-1}, t_j] \subset [0, b]$ and π_k denotes the set of polynomials of degree not exceeding k. Note that the elements of the space $S_k^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \ldots, t_{N-1}$ of Π_N.
We will also fix $m \in \mathbb{N}$ collocation parameters $\eta_1, \ldots, \eta_m$, which satisfy inequality

$$0 \leq \eta_1 < \ldots < \eta_m \leq 1. \tag{2.19}$$

Using those collocation parameters we define m collocation points for every subinterval $[t_{j-1}, t_j]$:

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \ldots, m, \quad j = 1, \ldots, N. \tag{2.20}$$

For given $N, m \in \mathbb{N}$ and collocation points (2.20) we define for any function $v \in C[0, b]$ the interpolation operator $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0, b] \to S_{m-1}^{(-1)}(\Pi_N)$ by

$$\mathcal{P}_N v \in S_{m-1}^{(-1)}(\Pi_N),\ (\mathcal{P}_N v)(t_{jk}) = v(t_{jk}),\ j = 1, \ldots, N,\ k = 1, \ldots, m. \tag{2.21}$$

If $\eta_1 = 0$, then by $(\mathcal{P}_N v)(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} (\mathcal{P}_N v)(t)$. If $\eta_m = 1$, then by $(\mathcal{P}_N v)(t_{jm})$ we denote the left limit $\lim_{t \to t_j, t < t_j} (\mathcal{P}_N v)(t)$.

Let t_λ be defined by (2.17) and $t_{\lambda\mu}$ defined by (2.20). If $\eta_1 > 0$ or $\eta_m < 1$, we can use the Lagrange basis in the space $S_{m-1}^{(-1)}(\Pi_N)$. Lagrange fundamental polynomials are defined as follows:

$$l_{\lambda\mu}(\tau) = \begin{cases} 0 & \text{for} \quad \tau \notin [t_{\lambda-1}, t_\lambda], \\ \displaystyle\prod_{i=1, i\neq\mu}^{m} \frac{\tau - t_{\lambda i}}{t_{\lambda\mu} - t_{\lambda i}}, & \text{for} \quad \tau \in [t_{\lambda-1}, t_\lambda]. \end{cases} \tag{2.22}$$

for $\mu = 1, \ldots, m$ and $\lambda = 1, \ldots, N$. If $m = 1$, then we take $l_{\lambda,1}(\tau) = 1$ for $\tau \in [t_{\lambda-1}, t_\lambda]$ and $l_{\lambda,1}(\tau) = 0$ if $\tau \notin [t_{\lambda-1}, t_\lambda]$.

We now present few lemmas about the interpolation operator $\mathcal{P}_N$, $N \in \mathbb{N}$ given in (2.21).

Lemma 2.11. [13] Let $A : L^\infty(0, b) \to C[0, b]$ be a linear compact operator and $\mathcal{P}_N$ defined by (2.21). Then

$$\|A - \mathcal{P}_N A\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \to 0 \text{ as } N \to \infty. \tag{2.23}$$

Lemma 2.12. [13] The operators $\mathcal{P}_N$, $N \in \mathbb{N}$, defined by (2.21), belong to the space $\mathcal{L}(C[0, b], L^\infty(0, b))$ and $\|\mathcal{P}_N\|_{\mathcal{L}(C[0,b], L^\infty(0,b))} \leq c$, with a positive constant c which is independent of N. Moreover, for every $u \in C[0, b]$ we have

$$\|u - \mathcal{P}_N u\|_{L^\infty(0,b)} \to 0 \quad \text{as} \quad N \to \infty.$$

Lemma 2.13. [13, 35] Let $z \in C^{m,\mu}(0,b]$, where $m \in \mathbb{N}$ and $\mu \in (-\infty, 1)$. Let $\mathcal{P}_N$ be the interpolation operator defined in (2.21) using piecewise polynomials on the graded mesh Π_N with grading parameter $r \in [1, \infty)$.

Then there exists a constant $c > 0$, independent of N, such that

$$\|z - \mathcal{P}_N z\|_\infty \leq c\varepsilon_N^{(m,\mu,r)},$$

where

$$\varepsilon_N^{(m,\mu,r)} = \begin{cases} N^{-m} & \text{for} \quad m < 1-\mu,\ r \geq 1, \\ N^{-m}(1+\log N) & \text{for} \quad m = 1-\mu,\ r = 1, \\ N^{-m} & \text{for} \quad m = 1-\mu,\ r > 1, \\ N^{-r(1-\mu)} & \text{for} \quad m > 1-\mu,\ 1 \leq r < \frac{m}{1-\mu}, \\ N^{-m} & \text{for} \quad m > 1-\mu,\ r \geq \frac{m}{1-\mu}. \end{cases} \tag{2.24}$$

This estimate holds for arbitrary collocation parameters $\eta_1, \ldots, \eta_m \in (0,1)$ as defined in (2.19), and quantifies the interpolation error in $S_{m-1}^{(-1)}(\Pi_N)$.

Lemma 2.14. [35] Let $z \in C^{m+1,\nu}(0,b]$, $m \in \mathbb{N}$, $\nu \in (-\infty,1)$, $\alpha \in (0,1]$. Let $N \in \mathbb{N}$, $r \in [1,\infty)$, and assume that the mesh points are defined by relation (2.17) and the collocation points by relation (2.20). Assume that the collocation parameters $\eta_1, ..., \eta_m$ defined in (2.19) are chosen so that the quadrature formula

$$\int_0^1 F(x)dx = \sum_{k=1}^{m} w_k F(\eta_k) + R_m(F) \quad (0 \leq \eta_1 < ... < \eta_m \leq 1) \tag{2.25}$$

is exact for all polynomials F of degree m (this means that the remainder term $R_m(F)$ is zero if F is any polynomial of degree m).

Then for $\alpha \in (0,1)$ we have

$$\|J^\alpha(z - \mathcal{P}_N z)\|_\infty \leq c \begin{cases} N^{-m-\alpha} & \text{for } m < 1+\alpha-\nu, \quad r \geq 1, \\ N^{-m-\alpha}(1+\log N) & \text{for } m = 1+\alpha-\nu, \quad r = 1, \\ N^{-m-\alpha} & \text{for } m = 1+\alpha-\nu, \quad r > 1, \\ N^{-r(1+\alpha-\nu)} & \text{for } m > 1+\alpha-\nu, \quad 1 \leq r < \dfrac{m+1}{1+\alpha-\nu}, \\ N^{-m-\alpha} & \text{for } m > 1+\alpha-\nu, \quad r \geq \dfrac{m+1}{1+\alpha-\nu}, \end{cases}$$

and for $\alpha = 1$ we have

$$\|J^1(z-\mathcal{P}_N z)\|_\infty \le \tilde{c} \begin{cases} N^{-m-1} & \text{for} \quad m < 2-\nu, \qquad r \ge 1, \\ N^{-m-1}(1+\log N)^2 & \text{for} \quad m = 2-\nu, \qquad r = 1, \\ N^{-m-1}(1+\log N) & \text{for} \quad m = 2-\nu, \qquad r > 1, \\ N^{-r(2-\nu)} & \text{for} \quad m > 2-\nu, \qquad 1 \le r < \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for} \quad m > 2-\nu, \qquad r \ge \dfrac{m+1}{2-\nu}, \end{cases}$$

where $c, \tilde{c}$ are some positive constants independent of N.

3. A CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEMS

Fractional boundary value problems frequently arise in modelling physical and engineering processes. Solving these equations numerically is challenging due to the inherent singular behaviour of their solutions. In this chapter, we study a class of fractional integro-differential equations with weakly singular kernels. Here we provide a rigorous analysis of existence, uniqueness, and numerical approximation, using appropriate integral equation reformulation, for a case, where the highest order of the underlying fractional differential operator belongs to $(1, 2)$. This chapter is based on the article [65].

3.1. Problem formulation

We will study a class of boundary value problems for Caputo fractional integro-differential equations with weakly singular kernels. For the following problem we assume that the order α of Caputo differential operator D^{α}_{Cap} is between $(1, 2)$ and y is the unknown function. We are interested in solutions $y \in C^1[0, b]$ of problem (3.1)–(3.3) such that $D^{\alpha}_{\text{Cap}} y \in C[0, b]$. Note that in [76] necessary and sufficient conditions for the existence of $D^{\alpha}_{\text{Cap}} y \in C[0, b]$ for a function $y \in C^1[0, b]$ have been derived (see Lemma 2.1 in Chapter 2).

The equation under consideration is in the form

$$(D^{\alpha}_{\text{Cap}} y)(t) + h(t)y(t) + \int_0^t L_\kappa(t, s)y(s)ds = f(t), \quad t \in [0, b], \quad \alpha \in (1, 2), \tag{3.1}$$

subject to conditions

$$a_{11}y(0) + a_{12}y(b_1) = \gamma_1, \tag{3.2}$$

$$a_{21}y'(0) + a_{22}y(b_1) = \gamma_2. \tag{3.3}$$

Observe that for $a_{12} = a_{22} = 0$ the problem (3.1)–(3.3) takes the form of an initial value problem for equation (3.1) and for $b_1 = b$ a two-point boundary value problem for equation (3.1).
The function $L_\kappa(t, s)$, for $0 \le s < t \le b$, is defined by the formula

$$L_\kappa(t, s) = \begin{cases} [1 + \log(t - s)]K(t, s) & \text{for} \quad \kappa = 0, \\ (t - s)^{-\kappa}K(t, s) & \text{for} \quad 0 < \kappa < 1, \end{cases} \tag{3.4}$$

where $K \in C(\Delta)$ and

$$\Delta = \{(t, s) : 0 \le s \le t \le b\}.$$

Finally, we assume that functions h, f, K and constants $a_{11}, a_{21}, a_{12}, a_{22}, b, b_1$ satisfy the following conditions:

$$h, f \in C[0,b], \quad K \in C(\Delta), \quad a_{11}, a_{21}, a_{12}, a_{22} \in \mathbb{R}, \quad b \in (0,\infty), \quad b_1 \in (0,b]. \tag{3.5}$$

In some theorems we have a little bit stricter conditions for functions h, f and K where $q \in \mathbb{N}$ and $\mu < 1$

$$h, f \in C^{q,\mu}(0,b], \; K \in C^q(\Delta), \; a_{11}, a_{21}, a_{12}, a_{22} \in \mathbb{R}, \; b \in (0,\infty), \; b_1 \in (0,b]. \tag{3.6}$$

The Caputo fractional derivative $D^\alpha_{\text{Cap}} y$ of order $\alpha \in (1,2)$ for function y is defined by the formula (see (2.5) and (2.6))

$$(D^\alpha_{\text{Cap}} y)(t) = (D^\alpha_{\text{R-L}}(y - Q_1[y]))(t), \quad t \in (0,b], \quad y \in C^1[0,b], \tag{3.7}$$

where $Q_1[y]$ is the Taylor polynomial for the function y of degree 1 at the point 0.

3.2. Existence, uniqueness and smoothness of the solution

In general (see [16]), we cannot expect that a solution of a fractional differential equation with Caputo differential operators will be smooth on the closed interval of integration and this is a challenge for constructing high order methods for the numerical solution of such equations. Therefore we will reformulate the problem (3.1)–(3.3) as an integral equation to construct a high order method for solving the aforementioned problem which takes into account the possible singular behaviour of its exact solution.

Let $y \in C^1[0,b]$ be an arbitrary function such that its Caputo derivative $D^\alpha_{\text{Cap}} y \in C[0,b]$, where $\alpha \in (1,2)$. Denote

$$z = D^\alpha_{\text{Cap}} y. \tag{3.8}$$

Using Lemma 2.2, we see, that for $\alpha \in (1,2)$ and $z = D^\alpha_{\text{Cap}} y$ function y can be presented by the formula

$$y(t) = y(0) + y'(0)t + (J^\alpha z)(t), \quad t \in [0,b]. \tag{3.9}$$

Proposition 3.1. For $\alpha \in (1,2)$, the function y of the form (3.9) satisfies the conditions (3.2)–(3.3) if and only if

$$y(t) = (J^\alpha z)(b_1)k_1 + k_2 + [(J^\alpha z)(b_1)k_3 + k_4]t + (J^\alpha z)(t), \quad t \in [0,b]. \tag{3.10}$$

where $z = D^\alpha_{\text{Cap}} y$ and

$$\begin{aligned} k_1 &= \frac{-a_{21}a_{12}}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \quad & k_3 &= \frac{-a_{22}a_{11}}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \\ k_2 &= \frac{\gamma_1(b_1 a_{22} + a_{21}) - \gamma_2 a_{12} b_1}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \quad & k_4 &= \frac{-\gamma_1 a_{22} + \gamma_2(a_{11} + a_{12})}{b_1 a_{11} a_{22} + a_{11}a_{21} + a_{12}a_{21}}, \end{aligned} \tag{3.11}$$

for $b_1 a_{11} a_{22} + a_{11} a_{21} + a_{12} a_{21} \neq 0$.

Proof. We begin by observing that by (3.9), we have

$$y(b_1) = y(0) + y'(0) b_1 + (J^\alpha z)(b_1). \tag{3.12}$$

By rewriting conditions (3.2)–(3.3) we obtain:

$$a_{11} y(0) + a_{12}[y(0) + y'(0) b_1 + (J^\alpha z)(b_1)] = \gamma_1, \tag{3.13}$$
$$a_{21} y'(0) + a_{22}[y(0) + y'(0) b_1 + (J^\alpha z)(b_1)] = \gamma_2. \tag{3.14}$$

Thus we get the following system for unknown variables $y(0), y'(0)$:

$$\begin{pmatrix} a_{11} + a_{12} & a_{12} b_1 \\ a_{22} & a_{21} + a_{22} b_1 \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} \gamma_1 - a_{12}(J^\alpha z)(b_1) \\ \gamma_2 - a_{22}(J^\alpha z)(b_1) \end{pmatrix}. \tag{3.15}$$

We see that this system is uniquely solvable if the matrix $A = \begin{pmatrix} a_{11} + a_{12} & a_{12} b_1 \\ a_{22} & a_{21} + a_{22} b_1 \end{pmatrix}$ is invertible (determinant is not equal to 0):

$$|A| = \begin{vmatrix} a_{11} + a_{12} & a_{12} b_1 \\ a_{22} & a_{21} + a_{22} b_1 \end{vmatrix} = b_1 a_{11} a_{22} + a_{11} a_{21} + a_{12} a_{21} \neq 0. \tag{3.16}$$

Then the inverse A^{-1} of matrix A is given by the formula

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} a_{21} + a_{22} b_1 & -a_{12} b_1 \\ -a_{22} & a_{11} + a_{12} \end{pmatrix}. \tag{3.17}$$

Therefore,

$$\begin{aligned} y(0) &= \frac{(a_{21} + a_{22} b_1)(\gamma_1 - a_{12}(J^\alpha z)(b_1)) - a_{12} b_1 (\gamma_2 - a_{22}(J^\alpha z)(b_1))}{|A|} \\ &= \frac{\gamma_1 a_{21} + \gamma_1 a_{22} b_1 - a_{12} \gamma_2 b_1}{|A|} + \frac{-a_{21} a_{12}}{|A|} (J^\alpha z)(b_1) \\ &= k_2 + k_1 (J^\alpha z)(b_1), \\ y'(0) &= \frac{-a_{22}(\gamma_1 - a_{12}(J^\alpha z)(b_1)) + (a_{11} + a_{12})(\gamma_2 - a_{22}(J^\alpha z)(b_1)}{|A|} \\ &= \frac{-a_{22} \gamma_1 + a_{11} \gamma_2 + a_{12} \gamma_2}{|A|} + \frac{-a_{22} a_{11}}{|A|} (J^\alpha z)(b_1) \\ &= k_4 + k_3 (J^\alpha z)(b_1). \end{aligned}$$

where k_1, k_2, k_3, k_4 are given by (3.11). We see, that the function y of the form (3.9) satisfies the conditions (3.2)–(3.3) if and only if (3.10) holds. ∎

Let $y \in C^1[0,b]$ be the solution of problem (3.1)–(3.3) such that $D^{\alpha}_{\mathrm{Cap}} y \in C[0,b]$. Next we substitute (3.10) into (3.1) and obtain that $z = D^{\alpha}_{\mathrm{Cap}} y$ is a solution of the integral equation in the form

$$z = Tz + g, \tag{3.18}$$

where, for $t \in [0,b]$,

$$\begin{aligned}(Tz)(t) =& - h(t)[(J^{\alpha}z)(t) + (k_1 + k_3 t)(J^{\alpha}z)(b_1)] \\ & - k_1 (J^{\alpha}z)(b_1) \int_0^t L_{\kappa}(t,s)ds - k_3 (J^{\alpha}z)(b_1) \int_0^t s L_{\kappa}(t,s)ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^t z(s)(t-s)^{\alpha} \left(\int_0^1 L_{\kappa}(t,(t-s)\sigma + s)\sigma^{\alpha-1} d\sigma \right) ds,\end{aligned} \tag{3.19}$$

and

$$g(t) = f(t) - h(t)[k_2 + k_4 t] - k_2 \int_0^t L_{\kappa}(t,s)ds - k_4 \int_0^t s L_{\kappa}(t,s)ds. \tag{3.20}$$

If $z \in C[0,b]$ is a solution to (3.18) then y determined by formula (3.10) belongs to $C^1[0,b]$ and is a solution to (3.1)–(3.3). In this sense equation (3.18) is equivalent to problem (3.1)–(3.3).

We conclude this section by presenting two theorems that characterise existence, uniqueness and regularity properties of the solution to (3.1)–(3.3)

Theorem 3.2. *Assume that $\alpha \in (1,2)$, $\kappa \in [0,1)$, L_{κ} is defined by* (3.4) *and conditions* (3.5) *are satisfied with $b_1 a_{11} a_{22} + a_{11} a_{21} + a_{12} a_{21} \neq 0$. Moreover, assume that the problem* (3.1)–(3.3) *with $f = 0$ and $\gamma_1 = \gamma_2 = 0$ has in $C[0,b]$ only the trivial solution $y = 0$.*
Then the problem (3.1)–(3.3) *possesses a unique solution $y \in C^1[0,b]$ such that $D^{\alpha}_{Cap} y \in C[0,b]$.*

Proof. We begin by observing equation (3.18). For the proof we are going to use Theorem 2.8 (Fredholm alternative theorem), therefore we need to show that operator T defined by (3.19) is a compact operator from $C[0,b]$ into $C[0,b]$ and homogeneous equation $z = Tz$ has in $C[0,b]$ only the trivial solution $z = 0$. Operator T can be written in the form

$$T = -H(J^{\alpha} + G) - MG - B.$$

Here G, H, M and B are defined by the following formulas:

$$\begin{aligned}(Gz)(t) &= (k_1 + k_3 t)(J^{\alpha}z)(b_1), \\ (Hz)(t) &= h(t)z(t), \\ (Mz)(t) &= \int_0^t L_{\kappa}(t,s)z(s)ds,\end{aligned}$$

$$(Bz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^\alpha \left(\int_0^1 L_\kappa(t,(t-s)\sigma + s)\sigma^{\alpha-1} d\sigma \right) z(s)ds,$$

with $t \in [0,b]$ and $z \in C[0,b]$. All these operators are bounded linear operators from $C[0,b]$ into $C[0,b]$. We can show with the help of Lemma 2.9, that operators J^α and B are compact from $C[0,b]$ into $C[0,b]$ and clearly $G : C[0,T] \to C[0,T]$ is compact. Therefore with the help from Lemma 2.7 operator T is compact from $C[0,b]$ into $C[0,b]$.

Since $f, h \in C[0,b]$ and $K \in C(\Delta)$ we get $g \in C[0,b]$ and thus, we need to show that homogeneous equation $z = Tz$ has in $C[0,b]$ only trivial solution $z = 0$. If $f = 0$ and $\gamma_1 = \gamma_2 = 0$ then $k_2 = k_4 = 0$ and $g = 0$. Since problem (3.1)–(3.3) with $f = 0$, $\gamma_1 = \gamma_2 = 0$, has only trivial solution in $C[0,b]$, also the problem $z = Tz$ has only the trivial solution $z = 0$.

Now according to Theorem 2.8 we see that the equation $z = Tz + g$ possess a unique solution $z \in C[0,b]$. Therefore the problem (3.1)–(3.3) has a unique solution $y \in C^1[0,b]$ such that its Caputo derivative belongs to $C[0,b]$. ■

The next theorem characterises the unique solution, if we demand that functions h, f belong to the weighted space $C^{q,\mu}(0,b]$ and function K is q-times continuously differentiable on Δ. The proof of the theorem follows the same ideas presented in Theorem 3.2.

Theorem 3.3. *Assume that $\alpha \in (1,2)$, $\kappa \in [0,1)$, L_κ is defined by (3.4), conditions (3.6) are fulfilled for $q \in \mathbb{N}$, $\mu < 1$ and $b_1a_{11}a_{22} + a_{11}a_{21} + a_{12}a_{21} \neq 0$. Moreover, assume that the problem (3.1)–(3.3) with $f = 0$ and $\gamma_1 = \gamma_2 = 0$ has in $C[0,b]$ only the trivial solution $y = 0$.*

Then y, the solution of problem (3.1)–(3.3), and its derivative $D^\alpha_{Cap}y$ belong to $C^{q,\nu}(0,b]$, where

$$\nu = \begin{cases} \max\{\mu, 1-\alpha\}, & \text{if } K = 0 \text{ (K vanishes identically)}, \\ \max\{\mu, \kappa\}, & \text{if } K \neq 0. \end{cases} \tag{3.21}$$

Proof. We begin our proof by showing that operator T is a compact operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Since $1 - \alpha \leq \nu$, it follows from Lemma 2.9 that J^α is compact operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Also G is a compact operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Furthermore, H and M are bounded as operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$ (see Lemmas 2.9 and 2.10) and we obtain with the help from Lemma 2.7 that operators $-H(J^\alpha + G)$ and MG are compact operators from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$.

If K in L_κ vanishes identically then $B = 0$. Otherwise (if $K \neq 0$) based on our definition (3.21) we get that $\kappa - \alpha \leq \nu$ and therefore operator B is compact from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$. Thus T is compact as an operator from $C^{q,\nu}(0,b]$ into $C^{q,\nu}(0,b]$.

Function g also belongs to $C^{q,\nu}(0,b]$. Indeed, $h, f \in C^{q,\mu}(0,b]$ and $\mu \leq \nu$. If K vanishes identically, then it follows from (3.4) that L_κ (where $\kappa \in [0,1)$)

vanishes identically and therefore $\int_0^t (k_2 + k_4 s) L_\kappa(t,s) ds = 0$ for any $t \in [0,b]$ and thus $g \in C^{q,\nu}(0,b]$. If $K \neq 0$ then it follows from Lemma 2.9 that $\int_0^t (k_2 + k_4 s) L_\kappa(t,s) ds \in C^{q,\nu}(0,b]$ and therefore g belongs to $C^{q,\nu}(0,b]$.
Since the homogeneous equation $z = Tz$ has in $C^{q,\nu}(0,b] \subset C[0,b]$ only the trivial solution it follows from Fredholm alternative theorem (see Theorem 2.8) that equation $z = Tz + g$ has a unique solution $z \in C^{q,\nu}(0,b]$ and the problem (3.1)–(3.3) possesses a unique solution $y \in C^{q,\nu}(0,b]$ such that its Caputo derivative z belongs to $C^{q,\nu}(0,b]$. ■

3.3. Numerical approach for the problem (3.1)–(3.3)

For solving problem (3.1)–(3.3), we use notations and results described in Section 2.5. Let $m, N \in \mathbb{N}$ and $r \geq 1$ be the parameters for our graded grid Π_N.

We find approximations $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ to the exact solution z of equation $z = Tz + g$ by collocation conditions

$$z_N(t_{jk}) = (Tz_N)(t_{jk}) + g(t_{jk}), \quad k = 1, \ldots, m, \quad j = 1, \ldots, N, \tag{3.22}$$

where the collocation points $\{t_{jk}\}$ are defined by (2.20). Using operator $\mathcal{P}_N$ defined by (2.21) conditions (3.22) take the form

$$z_N = \mathcal{P}_N T z_N + \mathcal{P}_N g. \tag{3.23}$$

Conditions (3.22) lead to a system of linear equations to uniquely determine z_N. Using Lagrange fundamental polynomial representation (see (2.22)) we obtain

$$z_N(\tau) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} l_{\lambda\mu}(\tau), \quad \tau \in [0,b]. \tag{3.24}$$

Then $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and $z_N(t_{jk}) = c_{jk}$ for every $k = 1, \ldots, m$, $j = 1, \ldots, N$. To determine approximation z_N in the form (3.24) we have to solve a system of linear algebraic equations with respect to $\{c_{jk}\}$:

$$c_{jk} = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (T l_{\lambda\mu})(t_{jk}) c_{\lambda\mu} + g(t_{jk}), \quad k = 1, \ldots, m, \; j = 1, \ldots, N. \tag{3.25}$$

Note that in Section 2.5 we defined the basis of $S_{m-1}^{(-1)}(\Pi_N)$ using Lagrange fundamental polynomials in the case where $\eta_1 > 0$ or $\eta_m < 1$ in (2.19). However if $\eta_1 = 0$ and $\eta_m = 1$, we have due to (2.20) that $t_{jk} = t_{j+1,1} = t_j$, $c_{jk} = c_{j+1,1} = z_N(t_j)$, $(j = 1, \ldots, N-1)$, and hence in the system (3.25) there are $(m-1)N + 1$ equations and unknowns.

Having found $\{c_{jk}\}$ from (3.25) we get that for $\alpha \in (1,2)$

$$\begin{aligned} y_N(\tau) =& k_1 \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} (J^\alpha l_{\lambda\mu})(b_1) + k_2 + \left[\sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} (J^\alpha l_{\lambda\mu})(b_1) k_3 + k_4 \right] \tau + \\ & \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} (J^\alpha l_{\lambda\mu})(\tau), \quad \tau \in [0,b]. \end{aligned} \tag{3.26}$$

It follows from (2.4) and (2.7) that function y_N defined by (3.26) is continuous on $[0,b]$.

3.4. Convergence analysis

In this section we will analyse the convergence of the proposed numerical method by presenting three theorems describing the convergence and convergence order of our method.

Theorem 3.4. *Assume that $\alpha \in (1,2)$, $\kappa \in [0,1)$, L_κ is defined by* (3.4) *and conditions* (3.5) *are satisfied for $b_1 a_{11} a_{22} + a_{11} a_{21} + a_{12} a_{21} \neq 0$. Moreover, assume that the problem* (3.1)–(3.3) *with $f = 0$ and $\gamma_1 = \gamma_2 = 0$ has in $C[0,b]$ only the trivial solution $y = 0$. Let $m, N \in \mathbb{N}$ and assume that the collocation points* (2.20) *with arbitrary parameters $\eta_1, \dots, \eta_m$ satisfying $0 \le \eta_1 < \dots < \eta_m \le 1$ and grid points* (2.17) *are used.*
Then:

- *(i) problem* (3.1)–(3.3) *possesses a unique solution $y \in C^1[0,b]$ such that its Caputo derivative $D^\alpha_{Cap} y \in C[0,b]$;*
- *(ii) there exists an integer $N_0 > 0$ such that for $N \ge N_0$, equation $z_N = \mathcal{P}_N T z_N + \mathcal{P}_N g$ possesses a unique solution $z_N \in S^{(-1)}_{m-1}(\Pi_N)$, determining by* (3.26) *a unique approximation $y_N \in C[0,b]$ to y, the solution of* (3.1)–(3.3) *and*

$$\|y - y_N\|_\infty \to 0 \text{ as } N \to \infty. \tag{3.27}$$

Proof. The first part *(i)* is proven with Theorem 3.2. To prove part *(ii)* we first need to show that operator $(I - \mathcal{P}_N T)$ is invertible in $L^\infty(0,b)$. To do that we observe that $I - \mathcal{P}_N T = (I - T) + (T - \mathcal{P}_N T)$ and we study $(I - T)$ and $(T - \mathcal{P}_N T)$ separately . Firstly we show that operator $(I - T)$ is invertible in $L^\infty(0,b)$ and $(I-T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$. Similarly as in the proof of Theorem 3.2 we can show that operator T is a compact operator from $L^\infty(0,b)$ into $C[0,b]$ and because $C[0,b] \subset L^\infty(0,b)$, operator T is also compact operator from $L^\infty(0,b)$ into $L^\infty(0,b)$. Furthermore, $g \in C[0,b] \subset L^\infty(0,b)$ and the homogeneous equation $z = Tz$ has in $C[0,b]$ only a trivial solution $z = 0$. Because $T \in \mathcal{L}(L^\infty(0,b), C[0,b])$, equation $z = Tz$ possesses only a trivial solution in $L^\infty(0,b)$ and with the help from Fredholm alternative theorem (see

Theorem 2.8) we see that equation $z = Tz + g$, where $g \in L^\infty(0,b)$ has only one solution in $L^\infty(0,b)$. Therefore, operator $I-T$ is invertible in $L^\infty(0,b)$ and because $(I-T) \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ also $(I-T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$. Using Lemma 2.11 and the boundedness of operator $(I-T)^{-1}$ in $L^\infty(0,b)$ we obtain that there exists $N_0 \in \mathbb{N}$ so that for every $N \geq N_0$ the following inequality holds

$$\|(I-T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))}\|T - \mathcal{P}_N T\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} < 1. \tag{3.28}$$

From (3.28) and Lemma 2.6 we see that for $N \geq N_0$ operator $I - \mathcal{P}_N T$ is invertible in $L^\infty(0,b)$ and

$$\|(I-\mathcal{P}_N T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} = \|((I-T)+(T-\mathcal{P}_N T))^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c, \tag{3.29}$$

where $N \geq N_0$ and c is a constant independent of N. Therefore for every $N \geq N_0$ equation (3.23) has only one solution z_N in $S^{(-1)}_{m-1}(\Pi_N)$. We have shown that operator $(I - \mathcal{P}_N T)^{-1}$ exists. Next we need to show that

$$z - z_N = (I - \mathcal{P}_N T)^{-1}(z - \mathcal{P}_N z). \tag{3.30}$$

Indeed due to $z = Tz + g$ and $z_N = \mathcal{P}_N T z_N + \mathcal{P}_N g$, we have

$$\begin{aligned}(I - \mathcal{P}_N T)(z - z_N) &= z - z_N - \mathcal{P}_N T z + \mathcal{P}_N T z_N \\ &= z - (\mathcal{P}_N T z_N + \mathcal{P}_N g) - \mathcal{P}_N T z + \mathcal{P}_N T z_N \\ &= z - \mathcal{P}_N T z - \mathcal{P}_N g \\ &= z - \mathcal{P}_N(Tz + g) \\ &= z - \mathcal{P}_N z,\end{aligned}$$

and therefore 3.30 holds for every $N \geq N_0$. With the help from inequality (3.29) we see that

$$\|z - z_N\|_\infty \leq c\|z - \mathcal{P}_N z\|_\infty. \tag{3.31}$$

To evaluate $\|y - y_N\|_\infty$ we need to show that $\|y - y_N\|_\infty \leq c\|z - z_N\|_\infty$. For that we fix $t \in [0,b]$ and use (3.10). We then see that for $t \in [0,b]$

$$|y(t) - y_N(t)| = |(J^\alpha(z - z_N))(b_1)k_1 + t(J^\alpha(z - z_N))(b_1)k_3 + (J^\alpha(z - z_N)(t))| \tag{3.32}$$

Therefore

$$\|y - y_N\|_\infty \leq c_1\|z - z_N\|_\infty \leq c_2\|z - \mathcal{P}_N z\|_\infty. \tag{3.33}$$

Here c_1, c_2 are constants independent of N. Based on Lemma 2.12 and (3.33) the convergence (3.27) holds. ■

Under additional regularity conditions, we can evaluate the error of our numerical approximation.

Theorem 3.5. *Let the assumptions of Theorem 3.4 hold, and conditions* (3.6) *be fulfilled, where* $q = m$ *and* $\mu \in (-\infty, 1)$. *Then the problem* (3.1)–(3.3) *has a unique solution* $y \in C^1[0,b]$ *such that* $y, D^\alpha_{Cap} y \in C^{m,\nu}(0,b]$. *The norm* $\|y - y_N\|_\infty$ *is bounded from above for sufficiently large* $N \in \mathbb{N}$ *as follows:*

$$\|y - y_N\|_\infty \leq c E_N(m, \nu, r, \alpha), \tag{3.34}$$

where c *is a positive constant independent of* N, ν *is defined by* (3.21), r *is the grading exponent of the mesh* (2.17) *and*

$$E_N(m,\nu,r,\alpha) = \begin{cases} N^{-m} & \text{for} \quad m < 1-\nu, \quad r \geq 1, \\ N^{-m}(1+\log N) & \text{for} \quad m = 1-\nu, \quad r = 1, \\ N^{-m} & \text{for} \quad m = 1-\nu, \quad r > 1, \\ N^{-r(1-\nu)} & \text{for} \quad m > 1-\nu, \quad 1 \leq r < \dfrac{m}{1-\nu}, \\ N^{-m} & \text{for} \quad m > 1-\nu, \quad r \geq \dfrac{m}{1-\nu}. \end{cases}$$

Proof. Let $K \in C^m(\Delta)$, $h, f \in C^{m,\mu}(0,b]$, $m \in \mathbb{N}$, and $\mu \in (-\infty, 1)$. Then from Theorem 3.3 it follows that the problem (3.1)–(3.3) has a unique solution $y \in C^1[0,b]$ such that y and $z = D^\alpha_{\text{Cap}} y$ belong to $C^{m,\nu}(0,b]$, where ν is defined by (3.21). From the proof of Theorem 3.4, we know that for sufficiently large N,

$$\|y - y_N\|_\infty \leq c\|z - \mathcal{P}_N z\|_\infty.$$

Now it follows from Lemma 2.13, that the estimate (3.34) holds. ■

Note that in Theorems 3.4 and 3.5, the collocation parameters are freely chosen quantities that satisfy the condition $0 \leq \eta_1 < ... < \eta_m \leq 1$. The last theorem of present section (Theorem 3.6) shows that if we refine the choice of collocation parameters and require that the functions f, h, and K are slightly smoother than in Theorem 3.4, we can refine the error estimate (3.34).

Theorem 3.6. *Let the assumptions of Theorem 3.4 hold, and conditions* (3.6) *be fulfilled, where* $q = m+1$ *and* $\mu \in (-\infty, 1)$. *Additionally, let the collocation points be defined by* (2.20), *where the collocation parameters* $\eta_1, \ldots, \eta_m$ *are chosen such that the quadrature formula with appropriate weights* $\{w_k\}$

$$\int_0^1 F(x)\,dx = \sum_{k=1}^m w_k F(\eta_k) + R_m(F) \quad (0 \leq \eta_1 < \ldots < \eta_m \leq 1)$$

is exact for all polynomials F *of degree* m *(that means* $R_m(F) = 0$*).*

Then the problem (3.1)–(3.3) *has a unique solution* $y \in C^1[0,b]$ *such that* $y, D^\alpha_{Cap} y \in C^{m+1,\nu}(0,b]$. *For sufficiently large* $N \in \mathbb{N}$, *there exists a unique solution* $z_N \in S^{(-1)}_{m-1}(\Pi_N)$ *for the equation* (3.23), *which defines a unique approximation* y_N *of the solution* y *to the problem* (3.1)–(3.3) *based on the relation*

(3.26), *and the following error estimate holds:*

$$\|y - y_N\|_\infty \leq cE_N^*(m, \nu, r, \alpha), \tag{3.35}$$

where c *is a positive constant independent of* N, ν *is defined by* (3.21), r *is the grading exponent of the mesh* (2.17) *and*

$$E_N^*(m,\nu,r,\alpha) = \begin{cases} N^{-m-1} & \text{for } m < 2-\nu, \quad r \geq 1, \\ N^{-m-1}(1+\log N)^2 & \text{for } m = 2-\nu, \quad r = 1, \\ N^{-m-1}(1+\log N) & \text{for } m = 2-\nu, \quad r > 1, \\ N^{-r(2-\nu)} & \text{for } m > 2-\nu, \quad 1 \leq r < \dfrac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } m > 2-\nu, \quad r \geq \dfrac{m+1}{2-\nu}. \end{cases} \tag{3.36}$$

Proof. We only need to prove the error estimate, the rest follows directly from Theorem 3.5. If we can show that $\|y - y_N\|_\infty \leq c\|J^1(z - \mathcal{P}_N z)\|_\infty$ it follows directly from Lemma 2.14 that inequality (3.35) holds. It follows from the proof of Theorem 3.4 that there exists an $N_0 \in \mathbb{N}$ so that for every $N \geq N_0$ equation $z_N = \mathcal{P}_N T z_N + \mathcal{P}_N g$ possesses only one unique solution $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ and the estimate (3.29) holds. Denote

$$\hat{z}_N = T z_N + g, \quad N \geq N_0. \tag{3.37}$$

Until the end of the proof let $N \geq N_0$. From (3.23) and (3.37) we see that

$$z_N = \mathcal{P}_N(T z_N + g) = \mathcal{P}_N \hat{z}_N. \tag{3.38}$$

If we substitute z_N with $\mathcal{P}_N \hat{z}_N$ in (3.37) we get an equation with respect to $\hat{z}_N$

$$\hat{z}_N = T\mathcal{P}_N \hat{z}_N + g, \quad N \geq N_0. \tag{3.39}$$

If we apply operator $(I - T\mathcal{P}_N)$ to the $\hat{z}_N - z$ we get that $(I - T\mathcal{P}_N)(\hat{z}_N - z) = T(\mathcal{P}_N z - z)$. Indeed,

$$\begin{aligned} (I - T\mathcal{P}_N)(\hat{z}_N - z) &= \hat{z}_N - z - T\mathcal{P}_N \hat{z}_N + T\mathcal{P}_N z \\ &= \hat{z}_N - (Tz + g) - T\mathcal{P}_N \hat{z}_N + T\mathcal{P}_N z \\ &= \hat{z}_N - (Tz + g) - (\hat{z}_N - g) + T\mathcal{P}_N z \\ &= -Tz + T\mathcal{P}_N z \\ &= T(\mathcal{P}_N z - z). \end{aligned}$$

Therefore

$$\hat{z}_N - z = (I - T\mathcal{P}_N)^{-1} T(\mathcal{P}_N z - z).$$

Due to existence of the inverse $(I - \mathcal{P}_N T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ there exists also the inverse $(I - T\mathcal{P}_N)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ and we have

$$(I - T\mathcal{P}_N)^{-1} = I + T(I - \mathcal{P}_N T)^{-1}\mathcal{P}_N. \tag{3.40}$$

For $N \geq N_0$ and using Lemma 2.12 we can now evaluate

$$\|\hat{z}_N - z\|_\infty = \|(I - T\mathcal{P}_N)^{-1} T(\mathcal{P}_N z - z)\|_\infty \leq c\|T(\mathcal{P}_N z - z)\|_\infty, \tag{3.41}$$

where c is a positive constant independent of N. Further, on the basis of the definition of operator T (see (3.19)), we have

$$\|T(\mathcal{P}_N z - z)\|_\infty \leq c\|J^\alpha(\mathcal{P}_N z - z)\|_\infty, \quad N \geq N_0, \tag{3.42}$$

where c is a positive constant not depending on N. It follows from (3.42) and (3.41) that

$$\|\hat{z}_N - z\|_\infty \leq c_1\|J^\alpha(\mathcal{P}_N z - z)\|_\infty, \quad N \geq N_0, \tag{3.43}$$

where c_1 is a positive constant not depending on N. Since $z_N = \mathcal{P}_N \hat{z}_N$, we have $z_N - z = \mathcal{P}_N \hat{z}_N - z = \mathcal{P}_N(\hat{z}_N - z) + \mathcal{P}_N z - z$. This leads to the following estimate:

$$\begin{aligned}
|y_N(t) - y(t)| &= |(J^\alpha(z_N - z))(b_1)k_1 + t(J^\alpha(z_N - z))(b_1)k_3 + (J^\alpha(z_N - z))(t)| \\
&\leq |(J^\alpha\mathcal{P}_N(\hat{z}_N - z))(b_1)k_1 + t(J^\alpha\mathcal{P}_N(\hat{z}_N - z))(b_1)k_3 + (J^\alpha\mathcal{P}_N(\hat{z}_N - z))(t)| \\
&\quad + |(J^\alpha(\mathcal{P}_N z - z))(b_1))k_1 + t(J^\alpha(\mathcal{P}_N z - z))(b_1)k_3 + (J^\alpha(\mathcal{P}_N z - z))(t)|,
\end{aligned}$$

with $t \in [0,b]$. Thus, it follows from Lemma 2.12 and inequality (3.43) that

$$\|y_N - y\|_\infty \leq c_1\|\hat{z}_N - z\|_\infty + c_2\|J^\alpha(\mathcal{P}_N z - z)\|_\infty \leq c_3\|J^\alpha(\mathcal{P}_N z - z)\|_\infty, \tag{3.44}$$

where c_1, c_2 and c_3 are some positive constants not depending on $N \geq N_0$.

Since $\alpha \in (1,2)$ we get from (3.44), (2.3) and the boundedness of operator $J^{\alpha-1} : C[0,b] \to C[0,b]$ that

$$\|y_N - y\|_\infty \leq c_1\|J^{\alpha-1}J^1(\mathcal{P}_N z - z)\|_\infty \leq c_2\|J^1(\mathcal{P}_N z - z)\|_\infty, \quad N \geq N_0, \tag{3.45}$$

where c_1, c_2 are some positive constants independent of N. With the help of Lemma 2.14 we now see that the inequality (3.35) holds. ■

3.5. Numerical examples

In this section we present two numerical examples to illustrate our theoretical findings. In the examples below $N \in \mathbb{N}$, y is the exact solution to the problem (3.1)–(3.3) and y_N defined by (3.26) is the approximation found using our numerical approach. The error estimates ε_N are calculated as follows

$$\varepsilon_N = \max_{j=1,\dots,N} \max_{k=0,\dots,10} |y(\tau_{jk}) - y_N(\tau_{jk})|,$$

where

$$\tau_{jk} = t_{j-1} + k(t_j - t_{j-1})/10, \quad k = 0, \ldots, 10, \quad j = 1 \ldots, N, \tag{3.46}$$

with the gridpoints t_j defined by (2.17). In our examples we use collocation points (2.20), where

$$\eta_1 = \frac{3-\sqrt{3}}{6}, \quad \eta_2 = 1 - \eta_1 \quad (\text{if } m = 2) \tag{3.47}$$

and

$$\eta_1 = \frac{5-\sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = 1 - \eta_1 \quad (\text{if } m = 3) \tag{3.48}$$

are collocation parameters that satisfy the conditions of Theorem 3.6. The ratios

$$\Theta_N = \frac{\varepsilon_{N/2}}{\varepsilon_N}$$

characterising the observed convergence rate, are also presented.

3.5.1. Example 1

Consider the following problem:

$$(D^{\frac{21}{20}}_{\text{Cap}} y)(t) + h(t)y(t) + \int_0^t (t-s)^{-\frac{3}{4}} K(t,s)y(s)ds = f(t), \quad t \in [0,1], \tag{3.49}$$

$$y(0) + y\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^{\frac{11}{10}}, \tag{3.50}$$

$$y'(0) + y\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^{\frac{11}{10}}. \tag{3.51}$$

with

$$h(t) = t, \quad K(t,s) = ts, \quad f(t) = \frac{\Gamma\left(\frac{21}{10}\right)}{\Gamma\left(\frac{21}{20}\right)} t^{\frac{1}{20}} + t^{\frac{21}{10}} + \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{31}{10}\right)}{\Gamma\left(\frac{67}{20}\right)} t^{\frac{67}{20}},$$

where $t \in [0,1]$ and $s \in [0,t]$. We see that (3.49)–(3.51) is a problem of the form (3.1)–(3.3) with $\alpha = \frac{21}{20}$, $\kappa = \frac{3}{4}$, $b_1 = \frac{1}{10}$, $b = 1$, $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $\gamma_1 = \gamma_2 = \left(\frac{1}{10}\right)^{\frac{11}{10}}$ and that

$$y(t) = t^{\frac{11}{10}}, \quad t \in [0,1],$$

is its exact solution. Clearly, $f, h \in C^{q,\mu}(0,1]$ with $\mu = \frac{19}{20}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (3.21),

$$\nu = \max\{\kappa, \mu\} = \frac{19}{20}.$$

In the case $m = 2$ it follows from the error estimate (3.36) with $\nu = \frac{19}{20}$ that, for sufficiently large N,

$$\varepsilon_N \leq c \begin{cases} N^{-r\left(\frac{21}{20}\right)} & \text{for} \quad 1 \leq r < \dfrac{60}{21}, \\ N^{-3} & \text{for} \quad r \geq \dfrac{60}{21}, \end{cases} \tag{3.52}$$

where c is a positive constant not depending on N. Due to (3.52), the ratios Θ_N for $r = 1$, $r = 2$, $r = 60/21$ and $r = 4$ ought to be approximately $2^{\frac{21}{20}} \approx 2.07$, $2^{\frac{42}{20}} \approx 4.29$, $2^3 = 8$ and $2^3 = 8$, respectively. These values are given in the last row of Table 1. We see that the numerical results are in accordance with the theoretical estimates given by Theorem 3.6.

	$r = 1$		$r = 2$		$r = 60/21$		$r = 4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	8.35E-04		1.79E-04		2.47E-04		2.94E-04	
8	5.18E-04	1.61	4.90E-05	3.65	2.64E-05	9.37	4.07E-05	7.22
16	2.15E-04	2.41	1.03E-05	4.74	2.50E-06	10.55	5.57E-06	7.31
32	9.78E-05	2.20	2.13E-06	4.84	3.12E-07	8.02	6.63E-07	8.40
64	4.50E-05	2.17	4.43E-07	4.82	5.53E-08	5.64	8.04E-08	8.25
128	2.06E-05	2.18	9.35E-08	4.73	4.62E-09	11.96	9.82E-09	8.18
256	9.41E-06	2.19	2.16E-08	4.32	6.08E-10	7.60	1.21E-09	8.12
		2.07		4.29		8.00		8.00

Table 1. Numerical results for problem (3.49)–(3.51) with $m = 2$.

In the case $m = 3$ it follows from the error estimate (3.36) with $\nu = \frac{19}{20}$ that, for sufficiently large N,

$$\varepsilon_N \leq c \begin{cases} N^{-r\left(\frac{21}{20}\right)} & \text{for} \quad 1 \leq r < \dfrac{80}{21}, \\ N^{-4} & \text{for} \quad r \geq \dfrac{80}{21}, \end{cases} \tag{3.53}$$

where c is a positive constant not depending on N. Due to (3.53), the ratios Θ_N for $r = 1$, $r = 2$, $r = 80/21$, $r = 4$ ought to be approximately $2^{\frac{21}{20}} \approx 2.07$, $2^{\frac{42}{20}} \approx 4.29$, $2^4 = 16$ and $2^4 = 16$, respectively. These values are given in the last row of Table 2.

	$r=1$		$r=2$		$r=80/21$		$r=4$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	4.85E-04		8.77E-05		4.21E-05		4.73E-05	
8	2.01E-04	2.42	2.03E-05	4.31	3.06E-06	13.77	4.18E-06	11.33
16	9.55E-05	2.10	4.07E-06	5.00	1.94E-07	15.77	1.77E-07	23.65
32	4.37E-05	2.19	8.42E-07	4.83	1.47E-08	13.16	1.07E-08	16.49
64	2.00E-05	2.19	1.73E-07	4.87	5.93E-10	24.83	6.64E-10	16.12
128	9.11E-06	2.19	3.77E-08	4.58	3.78E-11	15.69	4.13E-11	16.10
256	4.15E-06	2.19	8.80E-09	4.29	2.33E-12	16.23	2.61E-12	15.84
		2.07		4.29		16.00		16.00

Table 2. Numerical results for problem (3.49)–(3.51) with $m=3$.

We see that the numerical results are in accordance with the theoretical estimates given by Theorem 3.6.

3.5.2. Example 2

Consider the following problem:

$$(D^{\frac{3}{2}}_{\text{Cap}}y)(t) + h(t)y(t) + \int_0^t [1+\log(t-s)]y(s)ds = f(t), \quad t \in [0,1], \tag{3.54}$$

$$y(0) = 1, \tag{3.55}$$

$$y'(0) = 1, \tag{3.56}$$

with

$$h(t) = 1, \quad f(t) = t^{\frac{1}{10}}, \quad t \in [0,1].$$

We see that (3.54)–(3.56) is a problem of the form (3.1)–(3.3) with $\alpha = \frac{3}{2}$, $\kappa = 0$, $K = 1$, $b = 1$, $a_{11} = a_{21} = \gamma_1 = \gamma_2 = 1$, $a_{12} = a_{22} = 0$. It is easy to see that $h, f \in C^{q,\mu}(0,1]$ with $\mu = \frac{9}{10}$ and arbitrary $q \in \mathbb{N}$. Therefore, by (3.21),

$$\nu = \max\{\kappa, \mu\} = \frac{9}{10}.$$

Here, the exact solution is not known. For the numerical tests we use approximation y_N obtained with $m=3$, $r=4$ and $N=2048$, i.e., $y(x) \approx y_{2048}(x)$ $(0 \le x \le 1)$.

In the case $m=2$ it follows from the error estimate (3.36) with $\nu = \frac{9}{10}$ that, for sufficiently large N,

$$\varepsilon_N \le c \begin{cases} N^{-r(\frac{11}{10})} & \text{for} \quad 1 \le r < \dfrac{30}{11}, \\ N^{-3} & \text{for} \quad r \ge \dfrac{30}{11}, \end{cases} \tag{3.57}$$

where c is a positive constant not depending on N. Due to (3.57), the ratios Θ_N for $r=1$, $r=2$ and $r=30/11$ ought to be approximately $2^{\frac{11}{10}} \approx 2.14$,

$2^{\frac{22}{10}} \approx 4.59$ and $2^3 = 8$, respectively. These values are given in the last row of Table 3.

	$r=1$		$r=2$		$r=30/11$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	1.69E-03		7.58E-04		1.15E-03	
8	6.96E-04	2.42	1.14E-04	6.65	1.31E-04	8.79
16	3.03E-04	2.30	1.96E-05	5.81	1.33E-05	10.03
32	1.36E-04	2.23	3.80E-06	5.16	1.33E-06	10.03
64	6.21E-05	2.19	7.85E-07	4.83	1.35E-07	9.84
128	2.86E-05	2.17	1.68E-07	4.69	1.42E-08	9.52
256	1.33E-05	2.16	3.62E-08	4.63	1.54E-09	9.20
		2.14		4.59		8.00

Table 3. Numerical results for problem (3.54)–(3.56) with $m = 2$.

In the case $m = 3$, it follows from (3.36) with $\nu = \frac{9}{10}$ that, for sufficiently large N,

$$\varepsilon_N \le c \begin{cases} N^{-r\left(\frac{11}{10}\right)} & \text{for} \quad 1 \le r < \dfrac{40}{11}, \\ N^{-4} & \text{for} \quad r \ge \dfrac{40}{11}, \end{cases} \tag{3.58}$$

where c is a positive constant not depending on N. Due to (3.58) the ratios Θ_N for $r = 1$, $r = 2$ and $r = 40/11$ ought to be approximately $2^{\frac{11}{10}} \approx 2.14$, $2^{\frac{22}{10}} \approx 4.59$ and $2^4 = 16$, respectively. These values are given in the last row of Table 4.

	$r=1$		$r=2$		$r=40/11$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
4	6.85E-04		1.43E-04		1.11E-04	
8	3.00E-04	2.28	2.95E-05	4.85	6.93E-06	16.05
16	1.35E-04	2.22	6.33E-06	4.66	4.01E-07	17.28
32	6.18E-05	2.19	1.37E-06	4.62	2.23E-08	18.01
64	2.85E-05	2.17	2.98E-07	4.60	1.21E-09	18.36
128	1.32E-05	2.16	6.48E-08	4.60	6.62E-11	18.33
256	6.15E-06	2.15	1.41E-08	4.60	3.66E-12	18.07
		2.14		4.59		16.00

Table 4. Numerical results for problem (3.54)–(3.56) with $m = 3$.

As we can see from Tables 3 and 4, the numerical results are in accordance with our theoretical estimates.

4. TIME-FRACTIONAL SUB-DIFFUSION EQUATIONS

Time-fractional sub-diffusion equations have been found useful in modelling many real-life processes where anomalous diffusion occurs [2, 24, 42, 49, 50, 80]. In this chapter, we study a time-fractional sub-diffusion problem, where the order of the Caputo time derivative belongs to $(0, 1)$. Our aim is to use space variable discretization technique called method of lines (see [7]) to create a system of fractional differential equations. To solve the obtained system of fractional differential equations, we reformulate it as a system of weakly singular Volterra integral equations of the second kind and employ a suitable collocation method for finding approximate solutions. This approach enables us to construct a high-order numerical method for solving the sub-diffusion problem, despite the fact (see [28, 67, 69]) that the temporal partial derivatives of solutions of time-fractional diffusion equations have, in general, weak singularities at the initial time $t = 0$. The chapter is based on the article [40].

4.1. Background

The study of diffusion processes originated from the field of statistical physics. They have been used to model many physical, biological, engineering, economic, and social phenomena because diffusion is one of the fundamental mechanisms for transport of materials in physical, chemical, and biological systems. In the fields where diffusion has been applied, it has been used to model phenomena evolving randomly and continuously in time under certain conditions, for example, the fluctuations in the prices of securities in a perfect market, variations of population growth in ideal conditions, and communication systems with noise [29].

Free path is the path travelled by a particle in the interval between two collisions. Since the path differs from collision to collision, the average value called mean free path, is used as as the basis of calculation. The displacement is the distance between the original position of a particle and its position after a certain period of time. Since, in the absence of a difference in concentration, positive and negative displacement are equally probable, the mean displacement is zero. For this reason, we introduce the mean square displacement $\langle x^2 \rangle$ which is a measure for the rate of diffusion, however it depends on time. Therefore, it is convenient to introduce another characteristic quantity, independent of time. This is the diffusion coefficient $\mathcal{D}$:

$$\mathcal{D} = \frac{\langle x^2 \rangle}{2t}. \tag{4.1}$$

The $\mathcal{D}$ describes the velocity of diffusion process - diffusion process is faster the greater $\mathcal{D}$ is.

Normal diffusion can be characterized by the following equation

$$\frac{\partial \rho(x,t)}{\partial t} = \mathcal{D}\frac{\partial^2 \rho(x,t)}{\partial x^2}. \tag{4.2}$$

For normal diffusion, the square of the particle position scales linearly with time-variable t and

$$\langle x^2 \rangle \sim t. \tag{4.3}$$

Sub-diffusion is a type of anomalous diffusion where the mean squared displacement of particles grows slower than linearly with time. For sub-diffusion process, the mean square displacement is proportional to t^α for $\alpha \in (0,1)$. That means that the particles diffuse in time in a nonlinear way and their velocity is slower than it would be in the normal diffusion process [23].

4.2. Problem formulation

Let us consider an equation of the form

$$(D_t^\alpha u)(x,t) - \mathcal{D}\frac{\partial^2 u(x,t)}{\partial x^2} + \theta(x)u(x,t) = f(x,t), \quad (x,t) \in Q = (0,L) \times (0,b] \tag{4.4}$$

with the boundary and initial conditions:

$$u(0,t) = 0, \quad t \in [0,b], \tag{4.5}$$
$$u(L,t) = 0, \quad t \in [0,b], \tag{4.6}$$
$$u(x,0) = \phi(x), \quad x \in [0,L]. \tag{4.7}$$

Here the unknown function is $u = u(x,t)$. The Caputo fractional derivative $(D_t^\alpha u)(x,t)$ of order $\alpha \in (0,1)$ of $u(x,t)$ with respect to variable t is given by (see (2.11))

$$(D_t^\alpha u)(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds, \quad (x,t) \in Q, \tag{4.8}$$

where Γ is the Euler-gamma function defined in (2.2). Additionally, $\mathcal{D} \geq 0$ is a general diffusion coefficient describing the velocity of the diffusion process and $\theta(x)$ is a non-negative continuous function on $[0,L]$ and ϕ is a continuous function on $[0,L]$. Lastly $f \in C(\overline{Q})$ where $\overline{Q} = [0,L] \times [0,b]$. To simplify the presentation, we formulate the conditions as follows:

$$L, b > 0, \quad \mathcal{D} \geq 0, \quad Q = (0,L) \times (0,b],$$
$$\theta \in C[0,L], \quad \theta \geq 0, \quad \phi \in C[0,L], \quad f \in C(\overline{Q}). \tag{4.9}$$

Note that without the loss of generality, we can consider equation (4.4) only with homogeneous boundary conditions (4.5) and (4.6). If we would have more general boundary conditions, for example

$$u(0,t) = \phi_0(t), \quad t \in [0,b], \tag{4.10}$$

$$u(L,t) = \phi_L(t), \quad t \in [0,b], \tag{4.11}$$

where ϕ_0 and ϕ_L are some sufficiently smooth functions on $[0,b]$, we could transform the problem into the form (4.4)–(4.7) using auxiliary function

$$v(x,t) = u(x,t) + \frac{x}{L}(\phi_0(t) - \phi_L(t)) - \phi_0(t). \tag{4.12}$$

The inhomogeneous boundary conditions (4.4), (4.7), (4.10), (4.11) for u are transformed to the homogeneous boundary conditions for $v : v(0,t) = v(L,t) = 0$ for every $t \in [0,b]$. Moreover, both equation (4.4) and the initial condition (4.7) maintain their original form with respect to the new unknown function v:

$$(D_t^\alpha v)(x,t) - \mathcal{D}\frac{\partial^2 v(x,t)}{\partial x^2} + \theta(x)v(x,t) = \hat{f}(x,t), \quad (x,t) \in Q, \tag{4.13}$$

where

$$\hat{f}(x,t) = f(x,t) + (D^\alpha \phi_L)(t) + \frac{x-L}{L}(D^\alpha \phi_0)(t) + \theta(x)\left(\frac{x}{L}(\phi_0(t) - \phi_L(t)) - \phi_0(t)\right)$$

and

$$v(x,0) = \hat{\phi}(x), \quad x \in [0,L],$$

where

$$\hat{\phi}(x) = \phi(x) + \frac{x}{L}(\phi_0(0) - \phi_L(0)) - \phi_0(0).$$

For the existence and uniqueness of a classical solution u to (4.4)–(4.7) (that is, $D_t^\alpha u$ and $\frac{\partial^2 u}{\partial x^2}$ both exist in Q and u satisfies (4.4)–(4.7) pointwise), we refer the reader to [46].

The regularity properties of solutions u to (4.4)–(4.7) are described in [69]. In particular, the smoothness of all the data of (4.4)–(4.7) does not imply the smoothness of the solution u in the closed domain $\overline{Q}$, and the essential feature of all typical solutions to (4.4)–(4.7) is that the first-order derivative, $\frac{\partial u(x,t)}{\partial t}$, in general, blows up as $t \to 0$ (see [69]). This is a significant obstacle for constructing high-order methods for the numerical solutions to (4.4)–(4.7).

On the other hand, in [69], it is shown that when the data of problem (4.4)–(4.7) has sufficient regularity, there exists a constant $C > 0$ such that for the spatial derivatives of the exact solution, $u(x,t)$, to (4.4)–(4.7), we have

$$\left|\frac{\partial^k u(x,t)}{\partial x^k}\right| \le C, \quad k = 0,1,2,3,4, \quad (x,t) \in [0,L] \times (0,b]. \tag{4.14}$$

In our approach below, we assume that the solution, u, to (4.4)–(4.7) satisfies the derivative bounds (4.14).

4.3. Space variable discretization

In this section we will develop a system of fractional differential equations from problem (4.4)–(4.7) by space variable discretization. Let $n \in \mathbb{N}$, $n \geq 2$. We introduce a uniform mesh on the space interval $[0, L]$ defined by $n+1$ gridpoints

$$x_i = ih, \quad i = 0, \dots, n, \quad h = \frac{L}{n}. \tag{4.15}$$

Using standard second-order difference formula and (4.15), we get

$$\frac{\partial^2 u(x_i,t)}{\partial x^2} = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h^2} + O(h^2),\ t \in (0,b],\ i=1,\dots,n-1, \tag{4.16}$$

and by denoting

$$\begin{aligned} y_i(t) &\approx u(x_i,t), && t \in (0,b], \quad i = 1,\dots,n-1,\\ y_0(t) &= u(0,t) = 0, && t \in [0,b],\\ y_n(t) &= u(L,t) = 0, && t \in [0,b], \end{aligned}$$

we approximate (4.4)–(4.7) using the system of equations

$$(D^\alpha y_i)(t) - \mathcal{D}\frac{y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)}{h^2} + \theta(x_i)y_i(t) = f(x_i,t), \quad i = 1,\dots,n-1. \tag{4.17}$$

Note that here $D^\alpha y$ is the Caputo fractional derivative of order α of function y given by (2.11). Thus, we have for finding $y_1(t),\dots,y_{n-1}(t)$, a system of fractional differential equations in the form

$$(D^\alpha y_i)(t) + \sum_{j=1}^{n-1} a_{ij}y_j(t) = v_i(t), \quad 0 < t \leq b \quad \alpha \in (0,1), \quad i = 1\dots,n-1, \tag{4.18}$$

subject to the initial conditions

$$y_i(0) = \phi(x_i), \quad i = 1,\dots,n-1, \tag{4.19}$$

where the functions $v_i \in C[0,b]$ are defined by

$$v_i(t) = f(x_i,t), \quad i = 1,\dots,n-1 \tag{4.20}$$

and the constants a_{ij} are determined by

$$a_{ij} = \begin{cases} \dfrac{2\mathcal{D}}{h^2} + \theta(x_i) & \text{for} \quad i = j,\\ -\dfrac{\mathcal{D}}{h^2} & \text{for} \quad |i-j| = 1,\\ 0 & \text{otherwise.} \end{cases}$$

For simplicity of presentation, we rewrite (4.18) and (4.19) in vector form

$$(\boldsymbol{D^{\alpha}}\vec{y})(t) + A\vec{y}(t) = \vec{v}(t), \quad t \in (0, b], \tag{4.21}$$

$$\vec{y}(0) = \vec{\beta}, \tag{4.22}$$

where $\vec{y}(t) = (y_1(t), \ldots, y_{n-1}(t))^T$ is unknown, its Caputo fractional derivative is defined componentwise by $(\boldsymbol{D^{\alpha}}\vec{y})(t) = ((D^{\alpha}y_1)(t), \ldots, (D^{\alpha}y_{n-1})(t))^T$, and

$$\vec{v}(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_{n-1}(t) \end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix} \phi(x_1) \\ \vdots \\ \phi(x_{n-1}) \end{pmatrix}, \quad A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

4.4. Integral equation reformulation

Let $\vec{y} = (y_1, \ldots y_{n-1})^T$ be a solution to (4.21)–(4.22) such that $\vec{y} \in C_{n-1}[0, b]$. and $\vec{z} = \boldsymbol{D^{\alpha}}\vec{y} \in C_{n-1}[0, b]$. Then by Lemma 2.2, we obtain

$$\boldsymbol{J^{\alpha}}\vec{z} = (J^{\alpha}z_1, \ldots, J^{\alpha}z_{n-1})^T = (y_1 - y_1(0), \ldots, y_{n-1} - y_{n-1}(0))^T = \vec{y} - \vec{y}(0). \tag{4.23}$$

As $\vec{y}(0) = \vec{\beta}$ (see (4.22)), we can rewrite $\vec{y}$ in the form

$$\vec{y} = \boldsymbol{J^{\alpha}}\vec{z} + \vec{\beta}. \tag{4.24}$$

Since $\vec{y}$ is the solution to (4.21)–(4.22), we obtain due to (4.21) and (4.24) that $\vec{z} = \boldsymbol{D^{\alpha}}\vec{y}$ satisfies the equation

$$\vec{z} = \boldsymbol{T}\vec{z} + \vec{g}, \tag{4.25}$$

where

$$(\boldsymbol{T}\vec{z})(t) = -A[(\boldsymbol{J}^{\alpha}z)(t)], \quad t \in [0, b], \tag{4.26}$$

$$\vec{g}(t) = \vec{v}(t) - A\vec{\beta}, \quad t \in [0, b]. \tag{4.27}$$

Conversely, if $\vec{z} \in C_{n-1}[0, b]$ is a solution to (4.25), then $\vec{y}$ defined by (4.24) is a solution to (4.21)–(4.22) and $\vec{y} \in C_{n-1}[0, b]$. In this sense, equation (4.25) is equivalent to problem (4.21)–(4.22).

4.5. Existence and uniqueness

In this section we present two theorems that characterise existence, uniqueness and regularity of the solution to (4.21)–(4.22). We first present a generalisation of Gronwall's result to integral inequalities with weakly singular kernels.

Lemma 4.1. (Lemma 1.3.13 in [12]) Let $b > 0$ and $\alpha \in (0, 1)$, assume that

(i) $f \in C[0,b]$, $f(t) \geq 0$ and f is non-decreasing on $[0,b]$;

(ii) the continuous, non-negative function z satisfies the inequality

$$z(t) \leq f(t) + M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s) ds, \quad t \in [0,b], \tag{4.28}$$

for some $M > 0$.

Then

$$z(t) \leq E_\alpha(Mt^\alpha) f(t), \quad t \in [0,b], \tag{4.29}$$

where E_α denotes the Mittag-Leffler function.

Now, we show that homogeneous equation $\boldsymbol{T}\vec{z} = \vec{z}$ (cf. (4.25)) possesses only a trivial solution in $C_{n-1}[0,b]$.

Proposition 4.2. Let $\boldsymbol{T}$ be defined by (4.26). Then equation $\boldsymbol{T}\vec{z} = \vec{z}$ has in $C_{n-1}[0,b]$ only a trivial solution $\vec{z} = (0, \dots, 0)^T$.

Proof. Let $\vec{u} = \boldsymbol{T}\vec{u}$, where $\vec{u} \in C_{n-1}[0,b]$. Then

$$\begin{aligned}
\vec{u} &= \boldsymbol{T}\vec{u} \\
&= -A\boldsymbol{J^\alpha}\vec{u} \\
&= -\begin{pmatrix} a_{1,1} & \dots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} J^\alpha u_1 \\ \vdots \\ J^\alpha u_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} -\sum_{k=1}^{n-1} a_{1,k} J^\alpha u_k \\ \vdots \\ -\sum_{k=1}^{n-1} a_{n-1,k} J^\alpha u_k \end{pmatrix}.
\end{aligned}$$

Therefore, we have that for every $t \in [0,b]$ and $i = 1, \dots, n-1$,

$$|u_i(t)| = \left| -\sum_{k=1}^{n-1} a_{ik} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u_k(s) ds \right| \leq \tilde{a} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\sum_{k=1}^{n-1} |u_k(s)| \right) ds,$$

where $\tilde{a} = \max_{i,k=1,\dots,n-1} |a_{ik}|$. Next,

$$\sum_{i=1}^{n-1} |u_i(t)| \leq (n-1) \frac{\tilde{a}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\sum_{k=1}^{n-1} |u_k(s)| ds \right), \quad \forall t \in [0,b].$$

Since $0 < \alpha < 1$, the generalised Gronwall inequality (see Lemma 4.1) yields that for every $t \in [0,b]$, we have

$$\sum_{i=1}^{n-1} |u_i(t)| \leq E_\alpha((n-1)\tilde{a}t^\alpha) \cdot 0 = 0$$

and therefore $\vec{u} = \vec{0}$. In conclusion, equation $\vec{z} = \boldsymbol{T}\vec{z}$ has in $C_{n-1}[0,b]$ only a trivial solution. ∎

The next theorem characterises the unique solvability of problem (4.21)–(4.22).

Theorem 4.3. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in (0,1)$ and conditions* (4.9) *hold. Then the problem* (4.21)–(4.22) *possesses a unique solution $\vec{y}$, such that it and its Caputo derivative $\vec{z} = \boldsymbol{D^\alpha}\vec{y}$ belong to $C_{n-1}[0,b]$.*

Proof. We observe that operator $\boldsymbol{T} = -A\boldsymbol{J^\alpha}$ is a compact operator from $C_{n-1}[0,b]$ to $C_{n-1}[0,b]$ since $\boldsymbol{J^\alpha} : C_{n-1}[0,b] \to C_{n-1}[0,b]$ is compact. Further, $\vec{g} = \vec{v} - A\vec{\beta} \in C_{n-1}[0,b]$ since $\vec{v} \in C_{n-1}[0,b]$ and $A\vec{\beta} \in C_{n-1}[0,b]$. By Proposition 4.2 equation $\vec{z} = \boldsymbol{T}\vec{z}$ has only a trivial solution in $C_{n-1}[0,b]$. Therefore, using the Fredholm alternative theorem (Theorem 2.8), we obtain that the equation $\vec{z} = \boldsymbol{T}\vec{z} + \vec{g}$ possesses in $C_{n-1}[0,b]$ a unique solution $\vec{z} \in C_{n-1}[0,b]$. Thus, problem (4.21)–(4.22) has a unique solution $\vec{y} = \boldsymbol{J^\alpha}\vec{z} + \vec{b} \in C_{n-1}[0,b]$. ■

The next theorem characterises the unique solution of problem (4.21)–(4.22) if we demand that $\vec{v} = (v_1, \ldots, v_{n-1})^T$ given by (4.20) belongs to weighted space $C^{q,\mu}_{n-1}(0,b]$.

Theorem 4.4. *Let $n \in \mathbb{N}$, $n \geq 2$, $\alpha \in (0,1)$, conditions* (4.9) *hold and $\vec{v} \in C^{q,\mu}_{n-1}(0,b]$, $q \in \mathbb{N}$, $\mu < 1$. Then the problem* (4.21)–(4.22) *possesses a unique solution $\vec{y}$, such that it and its Caputo derivative $\vec{z} = \boldsymbol{D^\alpha}\vec{y}$ belong to $C^{q,\nu}_{n-1}(0,b]$, where*

$$\nu = \max\{1-\alpha, \mu\}. \tag{4.30}$$

Proof. Let $\vec{v} \in C^{q,\mu}_{n-1}(0,b]$, $q \in \mathbb{N}$, $\mu < 1$. Then, clearly $\vec{g} = \vec{v} - A\vec{\beta} \in C^{q,\mu}_{n-1}(0,b] \subset C^{q,\nu}_{n-1}(0,b]$. Since $1-\alpha \leq \nu$, by Lemma 2.9 operator $\boldsymbol{J^\alpha}$ is a compact operator from $C^{q,\nu}_{n-1}(0,b]$ to $C^{q,\nu}_{n-1}(0,b]$. Therefore, $\boldsymbol{T} = -A\boldsymbol{J^\alpha}$ is also a compact operator from $C^{q,\nu}_{n-1}(0,b]$ to $C^{q,\nu}_{n-1}(0,b]$. Since the homogeneous equation $\vec{z} = \boldsymbol{T}\vec{z}$ has in $C^{q,\nu}_{n-1}(0,b] \subset C_{n-1}[0,b]$ only a trivial solution, it follows from the Fredholm alternative theorem (Theorem 2.8), that equation $\vec{z} = \boldsymbol{T}\vec{z} + \vec{g}$ has a unique solution $\vec{z} \in C^{q,\nu}_{n-1}(0,b]$. Thus, problem (4.21)–(4.22) possesses a unique solution $\vec{y} = \boldsymbol{J^\alpha}\vec{z} + \vec{\beta} \in C^{q,\nu}_{n-1}(0,b]$. ■

4.6. Numerical approach for the problem (4.21)–(4.22)

For solving problem (4.21)–(4.22), we use notations and results described in Section 2.5. Let $m, n, N \in \mathbb{N}$, where $n \geq 2$, and $r \geq 1$ is the parameter for our graded grid (2.17).

We find the approximation $\vec{z_N} = (z_{1,N}, \ldots, z_{n-1,N})^T$ for the exact solution $\vec{z}$ of equation $\vec{z} = \boldsymbol{T}\vec{z} + \vec{g}$ using collocation conditions

$$\vec{z_N}(t_{jk}) = \boldsymbol{T}\vec{z_N}(t_{jk}) + \vec{g}(t_{jk}), \quad k = 1, \ldots, m, \quad j = 1, \ldots, N, \tag{4.31}$$

where the collocation points $\{t_{jk}\}$ are defined by (2.20) and $z_{1,N}, \ldots, z_{n-1,N} \in S^{(-1)}_{m-1}(\Pi_N)$, $(m, N \in \mathbb{N})$.

The conditions (4.31) with respect to $\vec{z_N} = (z_{1,N}, \ldots, z_{n-1,N})^T$ lead to a system of linear algebraic equations to find $z_{i,N} \in S^{(-1)}_{m-1}(\Pi_N)$, $i = 1, \ldots, n-1$, the exact form of which is determined by the choice of a basis in the space $S^{(-1)}_{m-1}(\Pi_N)$.
Using Lagrange fundamental polynomial representation (see (2.22)) we obtain

$$z_{i,N}(t) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{i\lambda\mu} l_{\lambda\mu}(t), \quad t \in [0,b], \quad i = 1, \ldots, n-1, \tag{4.32}$$

where $c_{i\lambda\mu}$ are some constants.

Thus, we obtain a system of linear algebraic equations with respect to the coefficients $\{c_{ijk}\}$:

$$c_{ijk} = -\sum_{\gamma=1}^{n-1} a_{i\gamma} \left(\sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (J^\alpha l_{\lambda\mu})(t_{jk}) c_{\gamma\lambda\mu} \right) + v_i(t_{jk}) - \sum_{\gamma=1}^{n-1} a_{i\gamma}\beta_\gamma, \tag{4.33}$$

for $i = 1, \ldots, n-1$, $j = 1, \ldots, N$, $k = 1, \ldots, m$.

Having found $\{c_{ijk}\}$ by the system (4.33), we can determine $\vec{z_N}(t)$ with the help of (4.32). Thus, we obtain the approximation $\vec{y_N} = (y_{1,N}, \ldots, y_{n-1,N})^T$ to $\vec{y}$, the solution to problem (4.21)–(4.22), as follows:

$$\vec{y_N}(t) = (\boldsymbol{J^\alpha} \vec{z_N})(t) + \vec{\beta}, \quad t \in [0,b]. \tag{4.34}$$

4.7. Convergence analysis

In this section we will analyse the convergence of the proposed numerical method by presenting four theorems describing the convergence and convergence order of our numerical method.

Theorem 4.5. *Assume that $\alpha \in (0,1)$ and conditions (4.9) hold. Let $n, m, N \in \mathbb{N}$, $n \geq 2$, and assume that the collocation points (2.20) with parameters $\eta_1, \ldots, \eta_m$ satisfying $0 \leq \eta_1 < \ldots < \eta_m \leq 1$ and grid points (2.17) are used. Then:*

(i) the problem (4.21)–(4.22) possesses a unique solution $\vec{y} \in C_{n-1}[0,b]$ such that its Caputo derivative $\vec{z} = \boldsymbol{D^\alpha}\vec{y}$ belongs to $C_{n-1}[0,b]$;

(ii) there exists an integer $N_0 > 0$ such that for $N \geq N_0$, equation $\vec{z_N} = \boldsymbol{\mathcal{P}_N T}\vec{z_N} + \boldsymbol{\mathcal{P}_N}\vec{g}$ (see (4.31)) possesses a unique solution $\vec{z_N} = (z_{1,N}, \ldots, z_{n-1,N})^T$ where $z_{i,N} \in S^{(-1)}_{m-1}(\Pi_N)$, for $i = 1, \ldots, n-1$, determining by (4.34) a unique approximation $\vec{y_N} \in C_{n-1}[0,b]$ to $\vec{y}$ the solution to (4.21)–(4.22) and the said numerical approximation $\vec{y}_N$ converges to $\vec{y}$ i.e.

$$\|\vec{y} - \vec{y}_N\|_{C_{n-1}[0,b]} \to 0 \text{ as } N \to \infty. \tag{4.35}$$

Proof. Existence and uniqueness of the solution of problem (4.21)–(4.22) are already proven in Theorem 4.3; thus, we only need to prove the convergence (4.35). We note that $\boldsymbol{T}$ is a compact operator from $L^{\infty}_{n-1}(0,b)$ to $C_{n-1}[0,b]$, thus also from $L^{\infty}_{n-1}(0,b)$ to $L^{\infty}_{n-1}(0,b)$. Since $\vec{v} \in C_{n-1}[0,b]$, we have $\vec{g} = \vec{v} - A\vec{\beta} \in C_{n-1}[0,b] \subset L^{\infty}_{n-1}(0,b)$. Using the same proof idea as in Theorem 4.3, we can show that equation $\vec{z} = \boldsymbol{T}\vec{z} + \vec{g}$ possesses a unique solution $\vec{z} \in L^{\infty}_{n-1}(0,b)$. In other words, operator $\boldsymbol{I} - \boldsymbol{T}$ is invertible in $L^{\infty}_{n-1}(0,b)$ and its inverse is bounded: $(\boldsymbol{I} - \boldsymbol{T})^{-1} \in \mathcal{L}(L^{\infty}_{n-1}(0,b), L^{\infty}_{n-1}(0,b))$. From this and Lemma 2.11, we obtain that for all sufficiently large N, we can say that

$$\|(\boldsymbol{I} - \boldsymbol{T})^{-1}\|_{\mathcal{L}(L^{\infty}_{n-1}(0,b),L^{\infty}_{n-1}(0,b))}\|\boldsymbol{T} - \boldsymbol{\mathcal{P}_N}\boldsymbol{T}\|_{\mathcal{L}(L^{\infty}_{n-1}(0,b),L^{\infty}_{n-1}(0,b))} < 1. \quad (4.36)$$

Therefore, from (4.36) and Lemma 2.6, it follows that operator $\boldsymbol{I} - \boldsymbol{\mathcal{P}_N}\boldsymbol{T}$ is invertible in $L^{\infty}_{n-1}(0,b)$ for sufficiently large N and

$$\|(\boldsymbol{I} - \boldsymbol{\mathcal{P}_N}\boldsymbol{T})^{-1}\|_{\mathcal{L}(L^{\infty}_{n-1}(0,b),L^{\infty}_{n-1}(0,b))} \le c, \quad N \ge N_0, \quad (4.37)$$

where c is a constant independent of N. Thus, for $N \ge N_0$, equation $\vec{z_N} = \boldsymbol{\mathcal{P}_N}\boldsymbol{T}\vec{z_N} + \boldsymbol{\mathcal{P}_N}\vec{g}$ has a unique solution $\vec{z_N} = (z_{1,N}, \ldots, z_{n-1,N})^T$, where $z_{i,N} \in S^{(-1)}_{m-1}(\Pi_N)$ for every $i = 1, \ldots, n-1$. For $\vec{z}_N$ and $\vec{z}$, the solution to equation $\vec{z} = \boldsymbol{T}\vec{z} + \vec{g}$, we see that

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{\mathcal{P}_N}\boldsymbol{T})(\vec{z} - \vec{z_N}) &= \vec{z} - \vec{z_N} - \boldsymbol{\mathcal{P}_N}\boldsymbol{T}\vec{z} + \boldsymbol{\mathcal{P}_N}\boldsymbol{T}\vec{z_N} \\ &= \vec{z} - \boldsymbol{\mathcal{P}_N}(\boldsymbol{T}\vec{z} + \vec{g}) = \vec{z} - \boldsymbol{\mathcal{P}_N}\vec{z}.\end{aligned}$$

Therefore, by (4.37),

$$\|\vec{z} - \vec{z}_N\|_{L^{\infty}_{n-1}(0,b)} \le c\|\vec{z} - \boldsymbol{\mathcal{P}_N}\vec{z}\|_{L^{\infty}_{n-1}(0,b)}, \quad N \ge N_0, \quad (4.38)$$

where c is a positive constant independent of N. It follows from (4.38), (4.34) and Lemma 2.9 that

$$\|\vec{y} - \vec{y}_N\|_{C_{n-1}[0,b]} \le c_1\|\vec{z} - \boldsymbol{\mathcal{P}_N}\vec{z}\|_{C_{n-1}[0,b]},$$

where c_1 is a positive constant independent of N. Using Lemma 2.12, we see that convergence (4.35) holds. ■

We now evaluate the error of our numerical approximation.

Theorem 4.6. *Assume that $\alpha \in (0,1)$, conditions (4.9) hold and $v \in C^{q,\mu}_{n-1}(0,b]$, $q \in \mathbb{N}$, where $q = m$ and $\mu \in (-\infty, 1)$. Then the problem (4.21)–(4.22) has a unique solution $\vec{y}$, such that it and its Caputo derivative $\vec{z} = \boldsymbol{D^{\alpha}}\vec{y}$ belong to $C^{m,\nu}_{n-1}(0,b]$ where $\nu = \max\{1-\alpha, \mu\}$. Let the assumptions of Theorem 4.5 hold. The norm $\|\vec{y} - \vec{y_N}\|_{C_{n-1}[0,b]}$ is bounded from above for sufficiently large $N \in \mathbb{N}$ as follows:*

$$\|\vec{y} - \vec{y}_N\|_{C_{n-1}[0,b]} \le cE_N(m, \nu, r, \alpha), \quad (4.39)$$

where c is a positive constant independent of N, r is the grading of the mesh (2.17) *and*

$$E_N(m,\nu,r,\alpha) = \begin{cases} N^{-m} & \text{for} \quad m < 1-\nu,\ r \geq 1, \\ N^{-m}(1+\log N) & \text{for} \quad m = 1-\nu,\ r = 1, \\ N^{-m} & \text{for} \quad m = 1-\nu,\ r > 1, \\ N^{-r(1-\nu)} & \text{for} \quad m > 1-\nu,\ 1 \leq r < \frac{m}{1-\nu}, \\ N^{-m} & \text{for} \quad m > 1-\nu,\ r \geq \frac{m}{1-\nu}. \end{cases}$$

Proof. From Theorem 4.4, it follows that the problem (4.21)–(4.22) possesses a unique solution $\vec{y}$, such that it and its Caputo derivative $\vec{z}$ belong to $C^{m,\nu}_{n-1}(0,b]$. From the proof of Theorem 4.5 we know that for sufficiently large N, the estimate

$$\|\vec{y} - \vec{y_N}\|_{C_{n-1}[0,b]} \leq c\|\vec{z} - \boldsymbol{\mathcal{P}_N}\vec{z}\|_{C_{n-1}[0,b]}$$

holds. From Lemma 2.13, it follows that the estimate (4.39) holds. ■

Note that in Theorems 4.5 and 4.6, the collocation parameters are freely chosen quantities that satisfy the condition $0 \leq \eta_1 < ... < \eta_m \leq 1$. The next theorem of the present section (Theorem 4.7) shows that if we refine the choice of collocation parameters and require that $\vec{v}$ is slightly smoother than in Theorem 4.6, we can refine the error estimate (4.39).

Theorem 4.7. *Let the assumptions of Theorem 4.5 hold and $\vec{v} \in C^{q,\mu}_{n-1}(0,b]$, $q \in \mathbb{N}$, where $q = m+1$ and $\mu \in (-\infty, 1)$. Additionally, let the collocation points be defined by* (2.20)*, where the collocation parameters $\eta_1, \ldots, \eta_m$ are chosen such that the quadrature formula with appropriate weights $\{w_k\}$*

$$\int_0^1 F(x)\,dx = \sum_{k=1}^{m} w_k F(\eta_k) + R_m(F) \quad (0 \leq \eta_1 < \ldots < \eta_m \leq 1)$$

is exact for all polynomials F of degree m (i.e. $R_m(F) = 0$). Then the problem (4.21)–(4.22) *has a unique solution $\vec{y}$, such that it and its Caputo derivative $\vec{z} = \boldsymbol{D^{\alpha}}\vec{y}$ belong to $C^{m+1,\nu}_{n-1}(0,b]$ where $\nu = \max\{1-\alpha, \mu\}$. The norm $\|\vec{y} - \vec{y_N}\|_{C_{n-1}[0,b]}$, is bounded from above for sufficiently large $N \in \mathbb{N}$ as follows:*

$$\|\vec{y} - \vec{y}_N\|_{C_{n-1}[0,b]} \leq cE^*_N(m,\nu,r,\alpha), \tag{4.40}$$

where c is a positive constant independent of N, r is the grading of the mesh (2.17) *and*

$$E^*_N(m,\nu,r,\alpha) = \begin{cases} N^{-m-\alpha} & \text{for} \quad m < 1+\alpha-\nu, \quad r \geq 1, \\ N^{-m-\alpha}(1+\log N) & \text{for} \quad m = 1+\alpha-\nu, \quad r = 1, \\ N^{-m-\alpha} & \text{for} \quad m = 1+\alpha-\nu, \quad r > 1, \\ N^{-r(1+\alpha-\nu)} & \text{for} \quad m > 1+\alpha-\nu, \quad 1 \leq r < \dfrac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha} & \text{for} \quad m > 1+\alpha-\nu, \quad r \geq \dfrac{m+\alpha}{1+\alpha-\nu}. \end{cases} \tag{4.41}$$

Proof. It follows from Theorem 4.4 (with $q = m + 1$) that problem (4.21)–(4.22) has a unique solution $\vec{y}$, such that $\vec{y}, \vec{z} \in C^{q,\nu}_{n-1}(0, b]$. From the proof of Theorem 4.5 we know that there exists an integer N_0, such that for $N \geq N_0$, equation $\vec{z_N} = \boldsymbol{\mathcal{P}_N T} \vec{z_N} + \boldsymbol{\mathcal{P}_N} \vec{g}$ has a unique solution $\vec{z_N} = (z_{1,N}, \dots, z_{n-1,N})^T$, where for every $i = 1, \dots, n-1$, $z_{i,N} \in S^{(-1)}_{m-1}(\Pi_N)$. Denote

$$\widehat{\vec{z}}_N = \boldsymbol{T} \vec{z}_N + \vec{g}, \quad N \geq N_0. \tag{4.42}$$

With the help of $\vec{z_N} = \boldsymbol{\mathcal{P}_N T} \vec{z_N} + \boldsymbol{\mathcal{P}_N} \vec{g}$, we see that $\boldsymbol{\mathcal{P}_N} \hat{\vec{z}}_N = \vec{z}_N$, and therefore we obtain from (4.42) the following equation with respect to $\widehat{\vec{z}}_N$:

$$\widehat{\vec{z}}_N = \boldsymbol{T \mathcal{P}_N} \widehat{\vec{z}}_N + \vec{g}, \quad N \geq N_0. \tag{4.43}$$

Since $\vec{z} = \boldsymbol{T} \vec{z} + \vec{g}$, it follows from (4.43) for every $N \geq N_0$ that

$$(\boldsymbol{I} - \boldsymbol{T \mathcal{P}_N})(\widehat{\vec{z}}_N - \vec{z}) = \boldsymbol{T}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z}). \tag{4.44}$$

We know from the proof of Theorem 4.5 that $\boldsymbol{(I - \mathcal{P}_N T)}$ is invertible in $\mathcal{L}(L^\infty_{n-1}(0, b), L^\infty_{n-1}(0, b))$ for sufficiently large N and $\boldsymbol{(I - \mathcal{P}_N T)^{-1}}$ belongs to $\mathcal{L}(L^\infty_{n-1}(0, b), L^\infty_{n-1}(0, b))$ for all $N \geq N_0$. Thus, there exists also the inverse of $(\boldsymbol{I} - \boldsymbol{T \mathcal{P}_N})$ in $\mathcal{L}(L^\infty_{n-1}(0, b), L^\infty_{n-1}(0, b))$ for $N \geq N_0$ and

$$\boldsymbol{(I - T \mathcal{P}_N)^{-1}} = \boldsymbol{I} + \boldsymbol{T (I - \mathcal{P}_N T)^{-1} \mathcal{P}_N}, \quad N \geq N_0. \tag{4.45}$$

Using (4.44), (4.45), (4.37), and Lemma 2.12, we obtain

$$\begin{aligned}\|\widehat{\vec{z}}_N - \vec{z}\|_{L^\infty_{n-1}(0,b)} &= \|\boldsymbol{(I - T \mathcal{P}_N)^{-1} T}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z})\|_{L^\infty_{n-1}(0,b)} \\ &\leq c \|\boldsymbol{T}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z})\|_{L^\infty_{n-1}(0,b)}, \quad N \geq N_0,\end{aligned}$$

where c is a positive constant independent of N. From the definition of the operator $\boldsymbol{T}$ we see that

$$\|\boldsymbol{T}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z})\|_{L^\infty_{n-1}(0,b)} \leq c_1 \|\boldsymbol{J^\alpha}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z})\|_{L^\infty_{n-1}(0,b)}, \quad N \geq N_0,$$

and therefore,

$$\|\widehat{\vec{z}}_N - \vec{z}\|_{L^\infty_{n-1}(0,b)} \leq c_2 \|\boldsymbol{J^\alpha}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z})\|_{L^\infty_{n-1}(0,b)}, \quad N \geq N_0,$$

where c_1 and c_2 are some positive constants independent of N. Due to $\vec{z_N} = \boldsymbol{\mathcal{P}_N} \widehat{\vec{z}}_N$, we obtain

$$\vec{z_N} - \vec{z} = \boldsymbol{\mathcal{P}_N}(\widehat{\vec{z}}_N - \vec{z}) + \boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z}.$$

This leads to the estimate

$$\|\vec{y}_N - \vec{y}\|_{C_{n-1}[0,b]} \leq c_3 \|\boldsymbol{J^\alpha}(\boldsymbol{\mathcal{P}_N} \vec{z} - \vec{z}))\|_{C_{n-1}[0,b]},$$

where $c_3 > 0$ is a constant that is independent of N and $\vec{y_N}$ and $\vec{y}$ are defined with the help of (4.34) and (4.24), respectively. Using Lemma 2.14, we see that the error estimate (4.40) holds. ■

In Theorem 4.8 below, we present the error estimate of our numerical method for solving problem (4.4)–(4.7). We assume that the data of problem (4.4)–(4.7) satisfies the conditions laid out in Theorem 2.1 in [69]. Under these assumptions, it follows from [69] that problem (4.4)–(4.7) has a unique solution u that satisfies (4.4)–(4.7) pointwise, and there exists a constant C, such that

$$\left|\frac{\partial^k u(x,t)}{\partial x^k}\right| \le C, \quad (x,t) \in [0,L] \times (0,b], \quad k = 0,1,2,3,4, \tag{4.46}$$

$$\left|\frac{\partial^l u(x,t)}{\partial t^l}\right| \le C(1+t^{\alpha-l}), \quad (x,t) \in [0,L] \times (0,b], \quad l = 0,1,2. \tag{4.47}$$

Theorem 4.8. *Let the solution u to* (4.4)–(4.7) *satisfy the estimates* (4.46) *and* (4.47)*. Let the assumptions of Theorem 4.7 be fulfilled. Then, the following error estimate holds:*

$$\max_{(x_i,t)\in\bar{Q}} |u(x_i,t) - y_{i,N}(t)| \le c(h^2 + E_N^*(m,\nu,r,\alpha)), \quad i = 1,\dots,n-1.$$

Here, $h = \frac{L}{n}$ $(n \ge 2)$, $E_N^*(m,\nu,r,\alpha)$ is defined by (4.41)*, $N, m \in \mathbb{N}$, ν is given by formula* (4.30)*, r is a grading exponent given in* (2.20)*, $\vec{y_N} = (y_{1,N},\dots,y_{n-1,N})^T$ is given by* (4.34)*, and c is a positive constant that is independent of N.*

Proof. It follows from (4.16) and Theorem 4.7 that for $i = 1,\dots,n-1$, we have

$$\begin{aligned}\max_{(x_i,t)\in\bar{Q}} |u(x_i,t) - y_{i,N}(t)| &\le \max_{(x_i,t)\in\bar{Q}} |u(x_i,t) - y_i(t)| + \max_{t\in[0,b]} |y_i(t) - y_{i,N}(t)| \\ &\le c(h^2 + E_N^*(m,\nu,r,\alpha)).\end{aligned}$$

Note that x_i are fixed points defined by our numerical method, but t belongs to $[0,b]$. ■

4.8. Numerical examples

In this section we present two numerical examples to illustrate our theoretical findings. In the examples below let $N, n \in \mathbb{N}$ and $n \ge 2$, u is the exact solution to the problem (4.4)–(4.7) and $\vec{y}_N = (y_{1,N},\dots,y_{n-1,N})^T$ given by (4.34) is the approximation found using our numerical approach.

Let $n, m, N \in \mathbb{N}$ and $r \in [1,\infty)$. We present, in the tables below, some results of numerical experiments for different values of parameters n, m, N, and r. The errors, $\varepsilon_{n,N}$, are calculated as follows:

$$\varepsilon_{n,N} = \max_{i=1,\dots,n-1} \max_{j=1,\dots,N} \max_{k=0,\dots,10} |u(x_i,\tau_{jk}) - y_{i,N}(\tau_{jk})|, \tag{4.48}$$

where x_i is given in (4.15) and

$$\tau_{jk} = t_{j-1} + k(t_j - t_{j-1})/10, \quad k = 0, \dots, 10, \quad j = 1, \dots, N,$$

with the gridpoints, t_j, defined by (2.17). In our examples we use collocation points (2.20), where

$$\eta_1 = \frac{1}{2} \quad (\text{if } m = 1) \tag{4.49}$$

and

$$\eta_1 = \frac{3 - \sqrt{3}}{6}, \quad \eta_2 = 1 - \eta_1 \quad (\text{if } m = 2) \tag{4.50}$$

are collocation parameters. Note that (4.49) and (4.50) are actually the knots of the m-point Gaussian quadrature approximation for $m = 1$ and $m = 2$, respectively. In tables below, the ratios

$$\Theta_{n,N} = \frac{\varepsilon_{n/2,N/2}}{\varepsilon_{n,N}}, \tag{4.51}$$

characterizing the observed convergence rate, are presented.

4.8.1. Example 1

Consider the equation

$$(D_t^{0.2}u)(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\Gamma(1.5)}{\Gamma(1.3)} t^{0.3} \sin x + t^{0.5} \sin x \tag{4.52}$$

for $(x,t) \in Q := (0,\pi) \times (0,1]$ with

$$u(0,t) = 0, \quad t \in [0,1], \tag{4.53}$$
$$u(\pi,t) = 0, \quad t \in [0,1], \tag{4.54}$$
$$u(x,0) = 0, \quad x \in [0,\pi]. \tag{4.55}$$

We see that (4.52)–(4.55) is a problem of the form (4.4)–(4.7), where $\alpha = 0.2$, $L = \pi$, $b = 1$, $\theta = \phi = 0$, and $f(x,t) = \frac{\Gamma(1.5)}{\Gamma(1.3)} t^{0.3} \sin x + t^{0.5} \sin x$ for $(x,t) \in Q$. The exact solution to (4.52)–(4.55) is given by

$$u(x,t) = t^{0.5} \sin x, \quad (x,t) \in \overline{Q}.$$

Let $n \in \mathbb{N}$, $n \geq 2$. We introduce a uniform mesh on the interval $[0,\pi]$ with gridpoints $x_i = ih$, $i = 0, \dots, n$, where $h = \frac{\pi}{n}$. Using the space-variable discretization, we obtain a system of fractional differential equations in the form (4.18) with initial conditions (4.19) and functions

$$v_i(t) = \frac{\Gamma(1.5)}{\Gamma(1.3)} t^{0.3} \sin x_i + t^{0.5} \sin x_i, \quad t \in [0,1], \quad i = 1, \dots, n-1.$$

It is easy to see that functions v_i belong to $C^{q,\mu}(0,1]$, with $\mu = 0.7$ and arbitrary $q \in \mathbb{N}$. Therefore, by (4.30),

$$\nu = \max\{\mu, 1-\alpha\} = 0.8.$$

In Tables 5 and 6, the ratios $\Theta_{n,N}$ characterizing the observed convergence rate, are presented. Using (4.41), we see that for $m = 1$,

$$E_N^*(m,\nu,r,\alpha) = \begin{cases} N^{-0.4r} & \text{for} \quad r < 3, \\ N^{-1.2} & \text{for} \quad r \geq 3. \end{cases} \tag{4.56}$$

Based on Theorem 4.8 and (4.56), ratios $\Theta_{n,N}$ for $m = 1$ and for $r = 1$, $r = 2$, and $r = 3$ ought to be $2^{0.4} \approx 1.32$, $2^{0.8} \approx 1.74$, and $2^{1.2} \approx 2.30$, respectively. These values are given in the last row of Table 5. The numerical results in Table 5 indicate that the order of convergence of the method for $m = 1$ is based on $E_N^*(m,\nu,r,\alpha)$, which dominates the h^2 component of the error in Theorem 4.8.

		$r=1$		$r=2$		$r=3$	
n	N	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$
2	2	1.89E-01		1.57E-01		1.96E-01	
4	4	1.15E-01	1.64	5.78E-02	2.71	7.04E-02	2.78
8	8	7.87E-02	1.46	2.82E-02	2.05	2.71E-02	2.60
16	16	5.55E-02	1.42	1.41E-02	2.00	1.36E-02	1.99
32	32	3.94E-02	1.41	7.11E-03	1.99	6.40E-03	2.13
64	64	2.80E-02	1.41	3.58E-03	1.99	2.91E-03	2.20
128	128	1.99E-02	1.41	1.80E-03	1.99	1.29E-03	2.25
			1.32		1.74		2.30

Table 5. Numerical results for problem (4.52)–(4.55) using $m = 1$ and $n = N$.

Using (4.41), we see that for $m = 2$,

$$E_N^*(m,\nu,r,\alpha) = \begin{cases} N^{-0.4r} & \text{for} \quad r < 5.5, \\ N^{-2.2} & \text{for} \quad r \geq 5.5. \end{cases} \tag{4.57}$$

Based on Theorem 4.8 and (4.57), ratios $\Theta_{n,N}$ for $m = 2$ and for $r = 1$ and $r = 2$ ought to be $2^{0.4} \approx 1.32$ and $2^{0.8} \approx 1.74$, respectively. The numerical results in Table 6 indicate that the order of convergence of the method for $r = 1$ and $r = 2$ is based on $E_N^*(m,\nu,r,\alpha)$, which dominates the h^2 component of the error in Theorem 4.8. For $r = 5.5$, the h^2 component dominates the $E_N^*(m,\nu,r,\alpha)$ component of the error in Theorem 4.8 and therefore the ratio ought to be $2^2 = 4.00$. The obtained values for the ratios are given in the last row of Table 6. We see that the performed numerical experiments are in good accordance with the theoretical results.

		$r=1$		$r=2$		$r=5.5$	
n	N	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$
2	2	1.03E-01		1.04E-01		1.11E-01	
4	4	3.11E-02	3.30	2.61E-02	4.00	3.01E-02	3.68
8	8	2.04E-02	1.52	7.41E-03	3.52	7.79E-03	3.86
16	16	1.43E-02	1.43	3.71E-03	1.99	2.24E-03	3.48
32	32	1.02E-02	1.41	1.88E-03	1.98	5.83E-04	3.84
64	64	7.27E-03	1.40	9.51E-04	1.98	1.45E-04	4.02
128	128	5.19E-03	1.40	4.80E-04	1.98	3.53E-05	4.10
			1.32		1.74		4.00

Table 6. Numerical results for problem (4.52)–(4.55) using $m = 2$ and $n = N$.

4.8.2. Example 2

Consider the equation

$$(D_t^{0.25}u)(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + xu(x,t) = x(x-0.25)\frac{\Gamma(1.45)}{\Gamma(1.2)}t^{0.2} - 2t^{0.45} + x^2(x-0.25)t^{0.45} \tag{4.58}$$

for $(x,t) \in Q := (0, 0.25) \times (0, 3]$ with

$$u(0,t) = 0, \quad t \in [0,3], \tag{4.59}$$

$$u(0.25,t) = 0, \quad t \in [0,3], \tag{4.60}$$

$$u(x,0) = 0, \quad x \in [0, 0.25]. \tag{4.61}$$

We see that (4.58)–(4.61) is a problem of the form (4.4)–(4.7), where $\alpha = 0.25$, $L = 0.25$, $b = 3$, $\phi = 0$, $\theta(x) = x$ for $x \in (0, 0.25)$,

$$f(x,t) = x(x-0.25)\frac{\Gamma(1.45)}{\Gamma(1.2)}t^{0.2} - 2t^{0.45} + x^2(x-0.25)t^{0.45}, \quad (x,t) \in Q.$$

The exact solution to (4.58)–(4.61) is given by

$$u(x,t) = x(x-0.25)t^{0.45}, \quad (x,t) \in \overline{Q}. \tag{4.62}$$

To find the numerical solution to (4.58)–(4.61), we use the same approach as described in Example 1. For any uniform mesh on the interval $[0, 0.25]$ with gridpoints $x_i = ih$, $i = 0, \ldots, n$, where $h = \frac{0.25}{n}$, functions

$$v_i(t) = x_i(x_i - 0.25)\frac{\Gamma(1.45)}{\Gamma(1.2)}t^{0.2} - 2t^{0.45} + x_i^2(x_i - 0.25)t^{0.45}, \quad t \in [0,1], \quad i = 1, \ldots, n-1,$$

belong to $C^{q,\mu}(0,3]$, with $\mu = 0.8$ and arbitrary $q \in \mathbb{N}$. Therefore, by (4.30),

$$\nu = \max\{\mu, 1-\alpha\} = 0.8.$$

For Tables 7 and 8, the ratios, $\Theta_{n,N}$, are defined by (4.51). Using (4.41), we see that for $m = 1$,

$$E_N^*(m,\nu,r,\alpha) = \begin{cases} N^{-0.45r} & \text{for} \quad r < \frac{25}{9}, \\ N^{-1.25} & \text{for} \quad r \geq \frac{25}{9}. \end{cases} \tag{4.63}$$

Based on Theorem 4.8 and (4.63), ratios $\Theta_{n,N}$ for $m = 1$ and for $r = 1$, $r = 2$, and $r = \frac{25}{8}$ ought to be $2^{0.45} \approx 1.37$, $2^{0.9} \approx 1.87$, and $2^{1.25} \approx 2.38$, respectively. These values are given in the last row of Table 7. The numerical results in Table 7 indicate that the order of convergence of the method for $m = 1$ is based on $E_N^*(m,\nu,r,\alpha)$, which dominates the h^2 component of the error in Theorem 4.8.

		$r=1$		$r=2$		$r=25/9$	
n	N	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$
2	2	2.53E-03		1.93E-03		2.43E-03	
4	4	1.85E-03	1.37	1.03E-03	1.87	1.18E-03	2.06
8	8	1.36E-03	1.37	5.53E-04	1.87	5.19E-04	2.27
16	16	9.94E-04	1.37	2.96E-04	1.87	2.23E-04	2.33
32	32	7.28E-04	1.37	1.58E-04	1.87	9.45E-05	2.36
64	64	5.33E-04	1.37	8.46E-05	1.87	3.99E-05	2.37
128	128	3.90E-04	1.37	4.52E-05	1.87	1.68E-05	2.37
			1.37		1.87		2.38

Table 7. Numerical results for problem (4.58)–(4.61) using $m = 1$ and $n = N$.

Using (4.41), we see that for $m = 2$,

$$E_N^*(m,\nu,r,\alpha) = \begin{cases} N^{-0.45r} & \text{if } r < 5 \\ N^{-2.25} & \text{if } r \geq 5 \end{cases}. \tag{4.64}$$

Note that for equation (4.58), we have $\mathcal{O}(h^2) = 0$ in (4.16). Based on Theorem 4.8 and (4.64), ratios $\Theta_{n,N}$ for $m = 2$ and for $r = 1$, $r = 2$, and $r = 5$ ought to be $2^{0.45} \approx 1.37$, $2^{0.9} \approx 1.87$, and $2^{2.25} \approx 4.76$, respectively. These values are given in the last row of Table 8. The numerical results in Table 8 indicate that the order of convergence of the method is based on $E_N^*(m,\nu,r,\alpha)$, which dominates the h^2 component of the error in Theorem 4.8.

		$r=1$		$r=2$		$r=5$	
n	N	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$	$\varepsilon_{n,N}$	$\Theta_{n,N}$
2	2	6.42E-04		4.69E-04		7.39E-04	
4	4	4.70E-04	1.37	2.51E-04	1.87	2.94E-04	2.51
8	8	3.44E-04	1.37	1.34E-04	1.87	7.79E-05	3.78
16	16	2.51E-04	1.37	7.15E-05	1.88	1.79E-05	4.34
32	32	1.84E-04	1.37	3.85E-05	1.86	3.93E-06	4.57
64	64	1.34E-04	1.37	2.08E-05	1.85	8.40E-07	4.67
128	128	9.80E-05	1.37	1.13E-05	1.85	1.79E-07	4.69
			1.37		1.87		4.76

Table 8. Numerical results for problem (4.58)–(4.61) using $m = 2$ and $n = N$.

Finally, we consider what happens when $m = 2$, n is equal to 128, and $N \in \{2, 4, 8, 16, 32, 64, 128\}$. The ratios, $\widetilde{\Theta}_{128,N}$, in Table 9,

$$\widetilde{\Theta}_{128,N} = \frac{\varepsilon_{128,N/2}}{\varepsilon_{128,N}},$$

characterizing the observed convergence rate, are also presented. Based on Theorem 4.8 and (4.64), ratios $\widetilde{\Theta}_{128,N}$ for $m = 2$ and for $r = 1$, $r = 2$, and $r = 5$ ought to be $2^{0.45} \approx 1.37$, $2^{0.9} \approx 1.87$, and $2^{2.25} \approx 4.76$, respectively. These values are given in the last row of Table 9. We see that all of the performed numerical experiments are in good accordance with the theoretical results.

		$r=1$		$r=2$		$r=5$	
n	N	$\varepsilon_{n,N}$	$\widetilde{\Theta}_{128,N}$	$\varepsilon_{n,N}$	$\widetilde{\Theta}_{128,N}$	$\varepsilon_{n,N}$	$\widetilde{\Theta}_{128,N}$
128	2	6.43E-04		4.70E-04		7.40E-04	
128	4	4.70E-04	1.37	2.51E-04	1.87	2.94E-04	2.51
128	8	3.44E-04	1.37	1.34E-04	1.87	7.79E-05	3.78
128	16	2.51E-04	1.37	7.15E-05	1.88	1.79E-05	4.34
128	32	1.84E-04	1.37	3.85E-05	1.86	3.93E-06	4.57
128	64	1.34E-04	1.37	2.08E-05	1.85	8.40E-07	4.67
128	128	9.80E-05	1.37	1.13E-05	1.85	1.79E-07	4.69
			1.37		1.87		4.76

Table 9. Numerical results for problem (4.58)–(4.61) using $m = 2$ and $n = 128$.

5. SINGULAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-CONSTANT COEFFICIENTS

In this chapter, we consider a class of singular fractional integro-differential with non-constant coefficients. In previous chapters we studied fractional differential equations containing the Caputo fractional differential operators. In this chapter we study a singular fractional integro-differential equation with fractional differential operators $D_0^\alpha = (J^\alpha)^{-1}$ defined by (2.12) that are the inverses of Riemann-Liouville integral operators on $J^\alpha(C[0,T])$. However, due to the fact that the operators that we get after integral operator reformulation below, will be in general non-compact, we can not use the Fredholm alternative theorem for studying the existence and uniqueness of the solution of the underlying problem.

The main results of the present chapter extend the approach of [38] to a more wider class of equations with non-constant coefficients. Generalising to equations with non-constant coefficients presents several significant challenges since both the investigation of the unique solvability of the equation and the study of collocation based numerical methods require new ideas and auxiliary results compared with the constant case. The basis of our treatment will be the concept of cordial Volterra integral operators introduced by Vainikko in [72, 73], see also [12, 20, 79]. We will use also some ideas and results from [38], where the existence of a unique q times ($q \geq 0$) continuously differentiable solution to a singular fractional differential equation was considered. For similar equations in the spaces of analytic functions, see [33].

The structure of this chapter is as follows. In Sections 5.1 and 5.2, we introduce the model problem and reformulate the problem as a cordial Volterra integral equation. Section 5.3 provides theoretical results on the existence, uniqueness and regularity of solutions. Lastly we construct and analyse a collocation scheme on uniform meshes, proving convergence results and error estimates. This chapter is based on article [39].

5.1. Problem formulation

Let us consider the following equation

$$(D_0^\alpha M^\alpha u)(t) = \sum_{k=1}^{l} b_k(t)(D_0^{\alpha_k} M^{\alpha_k} u)(t) + b(t)(V_\psi u)(t) + f(t), \quad t \in (0,T]. \tag{5.1}$$

We assume that functions b_k, b, f and parameters α, α_k in (5.1) satisfy the following conditions

$$b_k, b, f \in C^q[0,T], \alpha \in (q, q+1], \quad \alpha > \alpha_k \geq 0, \quad k = 1, \dots, l, \quad q \in \mathbb{N}_0. \tag{5.2}$$

The notation M^α in (5.1) stands for the multiplication by the function t^α:

$$(M^\alpha v)(t) = t^\alpha v(t), \quad t \in (0,T], \quad v \in C[0,T]. \tag{5.3}$$

The operator V_ψ is an arbitrary cordial Volterra integral operator with a core $\psi \in L^1(0,1)$. The cordial integral operator is defined by

$$(V_\psi u)(t) = \int_0^t \frac{1}{t}\psi\left(\frac{s}{t}\right) u(s)ds, \quad t \in [0,T], \quad u \in C[0,T], \quad \psi \in L^1(0,1). \tag{5.4}$$

By a change of integration variables $x = \frac{s}{t}$ we obtain the second representation of V_ψ:

$$(V_\psi u)(t) = \int_0^t \frac{1}{t}\psi\left(\frac{s}{t}\right) u(s)ds = \int_0^1 \psi(x)u(tx)dx, \;\; t \in [0,T], \;\; u \in C[0,T], \;\; \psi \in L^1(0,1). \tag{5.5}$$

Remark 5.1. Notice that we do not permit any initial or boundary conditions for a unique solution $u \in C^q[0,T]$ of (5.1). Imposing them one usually determines a solution of lesser regularity.

5.2. Problem reformulation

We start by reformulating the problem (5.1) by a change of variables. Denote $v = D_0^\alpha M^\alpha u$. Then $u = M^{-\alpha}J^\alpha v$ and equation (5.1) takes the form

$$v = \sum_{k=1}^{l} b_k[D_0^{\alpha_k} M^{\alpha_k}][M^{-\alpha}J^\alpha]v + bV_\psi M^{-\alpha}J^\alpha v + f. \tag{5.6}$$

Note that for any $v \in C[0,T]$ and $k = 1,2,\ldots,l$, the function $M^{-\alpha}J^\alpha v$ belongs to the domain of the operator $D_0^{\alpha_k}M^{\alpha_k}$ or to the range of $(D_0^{\alpha_k}M^{\alpha_k})^{-1} = M^{-\alpha_k}J^{\alpha_k}$. Therefore, there exists a $w \in C[0,T]$ such that $M^{-\alpha}J^\alpha v = M^{-\alpha_k}J^{\alpha_k}w$. In [41] it was shown that this equality holds for $w = V_{\varphi_{\alpha,\alpha_k}}v$, where $V_{\varphi_{\alpha,\alpha_k}}$ is a cordial Volterra integral operator with the core $\varphi_{\alpha,\alpha_k} \in L^1(0,1)$ defined by

$$\varphi_{\alpha,\alpha_k}(x) = \frac{1}{\Gamma(\alpha-\alpha_k)}(1-x)^{\alpha-\alpha_k-1}x^{\alpha_k}, \quad x \in (0,1). \tag{5.7}$$

Therefore, for $k = 1,\ldots,l$, $v \in C[0,T]$,

$$b_k[D_0^{\alpha_k}M^{\alpha_k}][M^{-\alpha}J^\alpha]v = b_k[D_0^{\alpha_k}M^{\alpha_k}][M^{-\alpha_k}J^{\alpha_k}V_{\varphi_{\alpha,\alpha_k}}]v = b_kV_{\varphi_{\alpha,\alpha_k}}v. \tag{5.8}$$

Also notice that

$$(V_\psi M^{-\alpha}J^\alpha u)(t) = \int_0^t \frac{1}{t}\psi\left(\frac{s}{t}\right)\left(\frac{s^{-\alpha}}{\Gamma(\alpha)}\int_0^s (s-x)^{\alpha-1}u(x)dx\right)ds$$

$$= \int_0^t \frac{1}{t} \left(\int_x^t \frac{s^{-\alpha}}{\Gamma(\alpha)} \psi\left(\frac{s}{t}\right) (s-x)^{\alpha-1} ds \right) u(x) dx$$
$$= \int_0^t \frac{1}{t} \left(\int_{x/t}^1 \frac{y^{-\alpha}}{\Gamma(\alpha)} \psi(y) \left(y - \frac{x}{t}\right)^{\alpha-1} dy \right) u(x) dx$$
$$= (V_{\psi_\alpha} u)(t),$$

where ψ_α is defined as follows

$$\psi_\alpha(x) = \int_x^1 \frac{1}{\Gamma(\alpha)} y^{-\alpha} \psi(y)(y-x)^{\alpha-1} dy, \quad \psi_\alpha \in L^1(0,1), \quad x \in (0,1). \quad (5.9)$$

Thus we can rewrite equation (5.6) as the cordial Volterra integral equation

$$v = \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} v + b V_{\psi_\alpha} v + f, \quad (5.10)$$

where $\varphi_{\alpha,\alpha_k}, \psi_\alpha \in L^1(0,1)$ are defined by (5.7), (5.9) respectively.

Note also that for $\alpha > 0$ operator $M^{-\alpha} J^\alpha$ is a cordial Volterra integral operator:

$$(M^{-\alpha} J^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{t} \left(1 - \frac{s}{t}\right)^{\alpha-1} u(s) ds$$
$$= (V_{\varphi_\alpha} u)(t), \ 0 \le t \le T, \ u \in C[0,T],$$

where

$$\varphi_\alpha(x) = \frac{1}{\Gamma(\alpha)} (1-x)^{\alpha-1}, \ 0 < x < 1, \quad \varphi_\alpha \in L^1(0,1). \quad (5.11)$$

5.3. Existence and uniqueness of the solution

In this section, we study the unique solvability of equation (5.1). We first introduce some definitions and results regarding spectrums and resolvent sets.

Definition 5.2. We denote the resolvent set $\rho_{\mathcal{L}(X)}(A)$ of an operator $A \in \mathcal{L}(X)$ as the set of those $\lambda \in \mathbb{C}$ for which $\lambda I - A$ has the inverse $(\lambda I - T)^{-1} \in \mathcal{L}(X)$ and the spectrum $\sigma_{\mathcal{L}(X)}(A)$ of A is the complement set to $\rho_{\mathcal{L}(X)}(A)$, i.e.,

$$\rho_{\mathcal{L}(X)}(A) = \mathbb{C} \setminus \sigma_{\mathcal{L}(X)}(A).$$

In particular, $\lambda_0 \in \sigma_{\mathcal{L}(X)}(A)$ is an eigenvalue of A if there is an element $x_0 \in X$ called an eigenfunction of X to λ_0 such that

$$x_0 \neq 0, \quad (\lambda_0 I - A) x_0 = 0.$$

If $X = C^q[0,T]$ for $q \in \mathbb{N}_0$, we use the following notations $\rho_{\mathcal{L}(C^q[0,T])}(V) = \rho_q(V)$ and $\sigma_{\mathcal{L}(C^q[0,T])}(V) = \sigma_q(V)$.

We present the following result from the theory of cordial Volterra integral operators.

Lemma 5.3. [72, 73] For given $\varphi \in L^1(0,1)$ let V_φ be defined by (5.4) and set

$$\hat{\varphi}(\lambda) = \int_0^1 x^\lambda \varphi(x)dx \tag{5.12}$$

for every $\lambda \in \mathbb{C}$ where the integral converges. For $\varphi \in L^1(0,1)$, $m \geq 0$, it holds that $V_\varphi \in \mathcal{L}(C^m[0,T])$ and

$$\sigma_0(V_\varphi) = \{\hat{\varphi}(\lambda) | \lambda \in \mathbb{C}, \quad Re\lambda \geq 0\} \cup \{0\}; \tag{5.13}$$
$$\sigma_m(V_\varphi) = \{\hat{\varphi}(\lambda) | Re\lambda \geq m\} \cup \{\hat{\varphi}(j) | j = 0, 1 \dots, m-1\}, \quad m \geq 1. \tag{5.14}$$

Lemma 5.4. [73] Assume that $\varphi \in L^1(0,1)$ and $b \in C[0,T]$. If $b(0) = 0$, then $bV_\varphi : L^\infty(0,T) \to C[0,T]$ is compact and $\sigma_0(bV_\varphi) = \{0\}$.

An operator $A \in \mathcal{L}(X,Y)$ between Banach spaces X and Y is called Fredholm operator if its null space $\mathcal{N}(A) = \{x \in X : Ax = 0\}$ is finite dimensional and its range $\mathcal{R}(A) = \{Ax : \quad x \in X\}$ is closed and of finite codimension in X. The integer $\text{ind}(A) = \dim\mathcal{N}(A) - \text{codim}\mathcal{R}(A)$ is called the index of a Fredholm operator A.

Lemma 5.5. [64] Let $\Phi_0(X)$ be the class of Fredholm operators of index 0 and $A \in \mathcal{L}(X)$. The following conditions are equivalent:

1. $A \in \Phi_0(X)$;
2. A admits a representation $A = B + K$ where $B \in \mathcal{L}(X)$ possesses the inverse $B^{-1} \in \mathcal{L}(X)$ and $K \in \mathcal{L}(X)$ is compact.

Lemma 5.6. [41] Let $\mu I - A \in \Phi_0(X)$ for a $\mu \in \mathbb{C}$. If $\mathcal{N}(\mu I - A) = \{0\}$ then $\mu \in \rho_{\mathcal{L}(X)}(A)$.

Lemma 5.7. [72] For $\varphi \in L^1(0,1)$, $\mu \in \sigma_0(V_\varphi)$, $\mu \neq \widehat{\varphi}(0)$, the set $(\mu I - V_\varphi)(C[0,T])$ is dense in $C[0,T]$. For $\mu = \widehat{\varphi}(0)$, the functions $f \in (\mu I - V_\varphi)(C[0,T])$ satisfy $f(0) = 0$, hence the set $(\widehat{\varphi}(0)I - V_\varphi)(C[0,T])$ is not dense in $C[0,T]$.

Lemma 5.8. [72] For $\varphi \in L^1(0,1)$, $\mu \neq 0$, the operator $\mu I - V_\varphi : C[0,T] \to C[0,T]$ has the right hand inverse if and only if $\mu - \widehat{\varphi}(\mathrm{i}\xi) \neq 0$ for any $\xi \in \mathbb{R}$; further, $\mu I - V_\varphi : C[0,T] \to C[0,T]$ has the (two side) inverse if and only if, in addition, $\arg[\mu - \widehat{\varphi}(\mathrm{i}\xi)]_{\xi=-\infty}^{\infty} = 0$.

Lemma 5.9. [72] For $\varphi \in L^1(0,1)$, $\mu \notin \sigma_0(V_\varphi)$ it holds

$$(\mu I - V_\varphi)^{-1} = \mu^{-1} I + V_\psi,$$

where $\psi \in L^1(0,1)$ is uniquely determined by μ and φ.

Lemma 5.10. [41] Suppose that $\mu I - A \notin \Phi_0(X)$ for a $\mu \in \mathbb{C}$. Then $\mu \in \sigma_{\mathcal{L}(X)}(A)$.

In order to study the unique solvability of equation (5.1) we first present Lemma 5.11.

Lemma 5.11. If $a_k \in C[0,T]$, $k = 1, \dots, n$, it holds that

$$\sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right) = \sigma_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right), \tag{5.15}$$

where V_{ψ_k}, $k = 1, \dots, n$, is a cordial Volterra operator with core $\psi_k \in L^1(0,1)$.

Proof. We divide the proof into three parts:

1. First we prove the inclusion $\sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right) \subset \sigma_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$ for $a_k \in C^1[0,T]$ and $\psi_k \in C^1[0,1]$ with $k = 1, 2, \dots, n$.
2. Secondly, we show that $\sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right) \subset \sigma_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$ for $a_k \in C[0,T]$ and $\psi_k \in L^1(0,1)$ with $k = 1, 2, \dots, n$.
3. Lastly, we prove the inclusion $\sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right) \supset \sigma_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$ for $a_k \in C[0,T]$ and $\psi_k \in L^1(0,1)$ with $k = 1, 2, \dots, n$.

1. First part

Let $\psi_k \in C^1[0,1]$ and $a_k \in C^1[0,T]$ for $k = 1, 2, \dots, n$. Our goal is to show that if $\mu \in \sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right)$ then $\mu \in \sigma_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$. To achieve this, we show that $\mu \in \rho_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$ and $\mu \in \sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right)$ leads to a contradiction. We first observe that $0 \in \sigma_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$. Indeed this holds, because of (5.13) in Lemma 5.3, therefore we can assume that $\mu \neq 0$. Now

$$\mu I - \sum_{k=1}^{n} a_k V_{\psi_k} = \mu I - \sum_{k=1}^{n} a_k(0) V_{\psi_k} + \sum_{k=1}^{n} \left[a_k - a_k(0)\right] V_{\psi_k}.$$

By Lemma 5.4 operator $\sum_{k=1}^{n} \left[a_k - a_k(0)\right] V_{\psi_k}$ is compact in $C[0,T]$. Since $\mu \in \rho_0\left(\sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)$, we get that $\mu I - \sum_{k=1}^{n} a_k(0) V_{\psi_k}$ is invertible in $C[0,T]$. Hence by Lemma 5.5 operator $\mu I - \sum_{k=1}^{n} a_k V_{\psi_k}$ belongs to $\Phi_0(C[0,T])$. Based on Lemma 5.6 and $\mu \in \sigma_0(\sum_{k=1}^{n} a_k(0) V_{\psi_k})$ we observe that μ is the eigenvalue of operator $\sum_{k=1}^{n} a_k V_{\psi_k}$. Let $u_0 \in C[0,T]$, with $\|u_0\|_\infty = 1$, be the corresponding eigenfunction:

$$\left(\mu I - \sum_{k=1}^{n} a_k V_{\psi_k}\right) u_0 = 0 \tag{5.16}$$

or

$$u_0 = \left(\mu I - \sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)^{-1} \left(\sum_{k=1}^{n} [a_k - a_k(0)] V_{\psi_k}\right) u_0. \tag{5.17}$$

By Lemma 5.9 there exists $\theta \in L^1(0,1)$ for which

$$\left(\mu I - \sum_{k=1}^{n} a_k(0) V_{\psi_k}\right)^{-1} = \mu^{-1} I + V_\theta. \tag{5.18}$$

Therefore, we can rewrite (5.17) as

$$u_0 = (\mu^{-1} I + V_\theta) \left(\sum_{k=1}^{n} [a_k - a_k(0)] V_{\psi_k} \right) u_0. \tag{5.19}$$

We assumed that $a_k \in C^1[0, T]$ for $k = 1, 2, \ldots, n$, therefore we can evaluate $|a_k(t) - a(0)| \le c_k t$ for $t \in [0, T]$ with some constants $c_k > 0$. We now evaluate $|u_0(t)|$ step-by-step. The first step is

$$\begin{aligned}\left| \left(\left(\sum_{k=1}^{n} [a_k - a_k(0)] V_{\psi_k} \right) u_0 \right)(t) \right| &\le \sum_{k=1}^{n} |a_k(t) - a_k(0)| \int_0^t t^{-1} \left| \psi_k \left(\frac{s}{t} \right) \right| |u_0(s)| ds \\ &\le t \sum_{k=1}^{n} c_k \|\psi_k\|_1 \le ct \sum_{k=1}^{n} \|\psi_k\|_1,\end{aligned}$$

$$\begin{aligned}\left| \left(V_\theta \left(\sum_{k=1}^{n} [a_k - a_k(0)] V_{\psi_k} \right) u_0 \right)(t) \right| &\le ct \|\theta\|_1 \sum_{k=1}^{n} \|\psi_k\|_1, \\ |u_0(t)| &\le ct \left(|\mu^{-1}| + \|\theta\|_1 \right) \sum_{k=1}^{n} \|\psi_k\|_1,\end{aligned}$$

with a constant $c > 0$ independent of $t \in [0, T]$. Let us assume that after the m-th step we have the estimate $|u_0(t)| \le \hat{c}_m t^m$, $0 \le t \le T$ with a constant $\hat{c}_m$. Then for $t \in [0, T]$

$$\begin{aligned}\left| \left(\left(\sum_{k=1}^{n} [a_k - a_k(0)] V_{\psi_k} \right) u_0 \right)(t) \right| &\le \sum_{k=1}^{n} |a_k(t) - a_k(0)| \int_0^t \left| \psi_k \left(\frac{s}{t} \right) \right| \frac{s^m \hat{c}_m}{t} ds \\ &\le c \hat{c}_m t^{m+1} \sum_{k=1}^{n} \|\psi_k^{[m]}\|_1,\end{aligned}$$

$$\begin{aligned}\left| \left(V_\theta \left(\sum_{k=1}^{n} [a_k - a_k(0)] V_{\psi_k} \right) u_0 \right)(t) \right| &\le c \hat{c}_m t^{m+1} \|\theta\|_1 \sum_{k=1}^{n} \|\psi_k^{[m]}\|_1, \\ |u_0(t)| &\le c \hat{c}_m t^{m+1} \left(|\mu^{-1}| + \|\theta\|_1 \right) \sum_{k=1}^{n} \|\psi_k^{[m]}\|_1,\end{aligned}$$

where

$$\psi_k^{[m]}(x) = \psi_k(x) x^m, \quad x \in (0, 1). \tag{5.20}$$

Therefore, $|u_0(t)|$ upper bound for $t \in [0, T]$ is $\hat{c}_{m+1} t^{m+1}$ with a constant

$$\hat{c}_{m+1} = c \hat{c}_m \left(|\mu^{-1}| + \|\theta\|_1 \right) \sum_{k=1}^{n} \|\psi_k^{[m]}\|_1. \tag{5.21}$$

Since

$$\|\psi_k^{[m]}\|_1 = \int_0^1 x^m |\psi_k(x)| dx \leq \|\psi_k^{[1]}\|_\infty \int_0^1 x^{m-1} dx = m^{-1} \|\psi_k^{[1]}\|_\infty, \tag{5.22}$$

we have

$$\begin{aligned}
\hat{c}_{m+1} \leq & \hat{c}_m \frac{c(|\mu^{-1}| + \|\theta\|_1) \sum_{k=1}^n \|\psi_k^{[1]}\|_\infty}{m} \\
\leq & \hat{c}_{m-1} \frac{c^2(|\mu^{-1}| + \|\theta\|_1)^2 (\sum_{k=1}^n \|\psi_k^{[1]}\|_\infty)^2}{m(m-1)} \\
& \vdots \\
\leq & \ \hat{c}_1 \frac{c^m \left(|\mu^{-1}| + \|\theta\|_1\right)^m \left(\sum_{k=1}^n \|\psi_k^{[1]}\|_\infty\right)^m}{m!}.
\end{aligned}$$

Hence for $m \in \mathbb{N}_0$ and $t \in [0, T]$ we get

$$|u_0(t)| \leq \hat{c}_1 T^{m+1} \frac{c^m \left(|\mu^{-1}| + \|\theta\|_1\right)^m \left(\sum_{k=1}^n \|\psi_k^{[1]}\|_\infty\right)^m}{m!}.$$

Lastly, we replace $m!$ with the help of Stirling formula $m! \approx \sqrt{2\pi m}(\frac{m}{e})^m$ and consider the limit process $m \to \infty$. We get that $u_0(t) = 0$ for $t \in [0, T]$. This contradicts the fact that $u_0 \in C[0, T]$ is an eigenfunction with $\|u\|_\infty = 1$ and therefore for $a_k \in C^1[0, T]$, $\psi_k \in C^1[0, 1]$ for $k = 1, 2, \ldots, n$ inclusion $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k}) \subset \sigma_0(\sum_{k=1}^n a_k(0) V_{\psi_k})$ holds.

2. Second part

Let now $\psi_k \in L^1(0, 1)$ and $a_k \in C[0, T]$ for $k = 1, 2, \ldots, n$. We begin the discussion as in the first part and observe relation (5.19) as follows: the eigenvalue problem

$$\lambda u = (\mu^{-1} I + V_\theta) \left(\sum_{k=1}^n [a_k - a_k(0)] V_{\psi_k} \right) u \tag{5.23}$$

with compact operator $(\mu^{-1} I + V_\theta)\left(\sum_{k=1}^n [a_k - a_k(0)] V_{\psi_k}\right)$ has an eigensolution (λ_0, u_0), $\lambda_0 = 1$, $u_0 \in C[0, T]$, $\|u_0\|_\infty = 1$. We approximate functions ψ_k and a_k by $\psi_k^\epsilon \in C^1[0, 1]$ and $a_k^\epsilon \in C^1[0, T]$ so that

$$\begin{aligned}
a_k^\epsilon(0) &= a_k(0), \\
\|a_k - a_k^\epsilon\|_\infty &\leq \epsilon, \\
\|\psi_k - \psi_k^\epsilon\|_1 &\leq \epsilon,
\end{aligned}$$

for $k = 1, 2, \ldots, n$, where $\epsilon > 0$ is a given small number. The operator $\mu I - \sum_{k=1}^n a_k(0) V_{\psi_k^\epsilon}$ is still invertible in $C[0, T]$ and by Lemma 5.9 its inverse can be expressed as

$$\left(\mu I - \sum_{k=1}^{n} a_k(0) V_{\psi_k^\epsilon}\right)^{-1} = \mu^{-1} I + V_{\theta^\epsilon} \tag{5.24}$$

with a $\theta^\epsilon \in L^1(0,1)$ and where $\|\theta - \theta_\epsilon\|_1 \le c'\epsilon$, $c' > 0$. We get

$$\left\|(\mu^{-1}I+V_\theta)(\sum_{k=1}^{n}[a_k-a_k(0)]V_{\psi_k})-(\mu^{-1}I+V_{\psi_\epsilon})(\sum_{k=1}^{l}[a_k^\epsilon-a_k^\epsilon(0)]V_{\psi_k^\epsilon})\right\|_{\mathcal{L}(C[0,T])} \le c''\epsilon. \tag{5.25}$$

For a sufficiently small $\epsilon > 0$, the eigenvalue problem

$$\lambda u = (\mu^{-1} I + V_\theta)\left(\sum_{k=1}^{n}[a_k^\epsilon - a_k^\epsilon(0)] V_{\psi_k^\epsilon}\right) u \tag{5.26}$$

has a solution $(\lambda_\epsilon, u_\epsilon)$, $\|u_\epsilon\|_\infty = 1$ such that $\lambda_\epsilon \to 1$ as $\epsilon \to 0$. Using a similar discussion as in the first part we get that $u_\epsilon = 0$ which is a contradictions because we assumed that $\|u_\epsilon\|_\infty = 1$. Therefore, for $a_k \in C[0,T]$, $\psi_k \in L^1(0,1)$, $k = 1, 2, \ldots, n$, inclusion $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k}) \subset \sigma_0(\sum_{k=1}^n a_k(0) V_{\psi_k})$ holds.
3. Third part
We now prove the inclusion $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k}) \supset \sigma_0(\sum_{k=1}^n a_k(0) V_{\psi_k})$. According to Lemma 5.3, the inclusion is equivalent to the following inclusion

$$\left\{\sum_{k=1}^{n} a_k(0)\hat{\psi}_k(\lambda) | \mathrm{Re}\lambda \ge 0\right\} \cup \{0\} \subset \sigma_0\left(\sum_{k=1}^{n} a_k V_{\psi_k}\right). \tag{5.27}$$

Based on proven inclusion $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k}) \subset \sigma_0(\sum_{k=1}^n a_k(0) V_{\psi_k})$ the spectrum $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k})$ contains points $\sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda)$ for arbitrary large $\lambda \in \mathbb{R}$. Therefore, (since $\hat{\psi}_k(\lambda) \to 0$ as $\lambda \to \infty$) $\sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda) \to 0$ as $\lambda \to \infty$. From the closedness of the spectrum $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k})$ it now follows that $0 \in \sigma_0\left(\sum_{k=1}^n a_k V_{\psi_k}\right)$.
Let now $\mu \in \{\sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda) | \mathrm{Re}\lambda \ge 0\}$.
If we can show that $\mu I - \sum_{k=1}^n a_k V_{\psi_k} \notin \Phi_0(C^q[0,T])$ then by Lemma 5.10 we have $\mu \in \sigma_0(\sum_{k=1}^n a_k V_{\psi_k})$. For $\mu I - \sum_{k=1}^n a_k V_{\psi_k} \notin \Phi_0(C^q[0,T])$, it is sufficient to show that

$$\mu I - \sum_{k=1}^{n} a_k(0) V_{\psi_k} \notin \Phi_0(C^q[0,T]), \quad \forall \mu = \sum_{k=1}^{n} a_k(0)\hat{\psi}_k(\lambda) \text{ with } \mathrm{Re}\lambda \ge 0. \tag{5.28}$$

For $\mu = \sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda)$, $\mathrm{Re}\lambda > 0$, according to Lemmas 5.7, 5.8, $(\mu I - \sum_{k=1}^n a_k(0) V_{\psi_k}) C[0,T] = C[0,T]$ holds and μ is an eigenvalue of operator $\sum_{k=1}^n a_k(0) V_{\psi_k}$ with eigenfunction t^λ in $C[0,T]$, so $\mu I - \sum_{k=1}^n a_k(0) V_{\psi_k} \notin$

$\Phi_0(C^q[0,T])$.
For $\mu_0 = \sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda_0)$, $\mathrm{Re}\lambda_0 = 0$ relation $\mu_0 I - \sum_{k=1}^n a_k(0)V_{\psi_k} \in \Phi(C[0,T])$ cannot hold since otherwise it would also be true for a λ with $\mathrm{Re}\lambda > 0$ that is close to λ_0 but this is not the case. Thus inclusion $\sigma_0(\sum_{k=1}^n a_k V_{\psi_k}) \supset \sigma_0(\sum_{k=1}^n a_k(0)V_{\psi_k})$ holds. ■

Actually we can extend Lemma 5.11 to the Lemma 5.12 where we assume, that functions $a_k \in C^q[0,T]$ with $q \geq 1$ and $k = 1,\dots,n$.

Lemma 5.12. If $a_k \in C^q[0,T]$, $q \geq 0$, $k = 1,\dots,n$, it holds that

$$\sigma_q\left(\sum_{k=1}^n a_k V_{\psi_k}\right) = \sigma_q\left(\sum_{k=1}^n a_k(0)V_{\psi_k}\right), \tag{5.29}$$

where for $k = 1,\dots,n$, $\psi_k \in L^1(0,1)$ is the core of the cordial Volterra integral operator V_{ψ_k}.

The proof of Lemma 5.12 (see Lemma 4.2 in [39]) follows the similar steps presented in [41] where it is shown that with $\varphi_{\alpha,\alpha_k}$ defined by (5.7) and $q \geq 0$ $\sigma_q(\sum_{k=1}^n a_k V_{\varphi_{\alpha,\alpha_k}}) = \sigma_q(\sum_{k=1}^n a_k(0)V_{\varphi_{\alpha,\alpha_k}})$. We have already shown that for $q = 0$ equality(5.29) holds. For $q \geq 1$ we first can show that $\sigma_q(\sum_{k=1}^n a_k V_{\psi_k}) \subset \sigma_q(\sum_{k=1}^n a_k(0)V_{\psi_k})$ and then show, that inclusions

$$\left\{\sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda)|\mathrm{Re}\lambda \geq q\right\} \subset \sigma_q\left(\sum_{k=1}^n a_k V_{\psi_k}\right), \quad q \geq 1,$$

and

$$\left\{\sum_{k=1}^n a_k(0)\hat{\psi}_k(\lambda)|\lambda = 0,1,\dots,q-1\right\} \cup \{0\} \subset \sigma_q\left(\sum_{k=1}^n a_k V_{\psi_k}\right), \quad q \geq 1,$$

hold.

The following theorem characterises the existence and uniqueness of solution to the problem (5.1).

Theorem 5.13. *Let $\psi \in L^1(0,1)$ in equation (5.1) and conditions (5.2) hold. Equation (5.10) has a unique solution $v \in C^q[0,T]$ for any $f \in C^q[0,T]$ (hence equation (5.1) has a unique solution $u = M^{-\alpha}J^\alpha v \in C^q[0,T]$ for any $f \in C^q[0,T]$) iff for $q = 0$ we have*

$$\sum_{k=1}^l b_k(0)\frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)} + b(0)\hat{\psi}(\lambda)\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)} \neq 1, \forall\lambda \in \mathbb{C} \text{ with } Re\lambda \geq 0, \tag{5.30}$$

or, for $q > 0$, we have

$$\sum_{k=1}^l b_k(0)\frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)} + b(0)\hat{\psi}(\lambda)\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)} \neq 1, \forall\lambda \in \mathbb{C} \text{ with } Re\lambda \geq q, \tag{5.31}$$

and

$$\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k+j+1)}{\Gamma(\alpha+j+1)} + b(0)\hat{\psi}(j)\frac{\Gamma(j+1)}{\Gamma(\alpha+j+1)} \neq 1, \quad j=0,1,\ldots,q-1. \tag{5.32}$$

Proof. Equation (5.10) has a unique solution $v \in C^q[0,T]$ if and only if $1 \notin \sigma_q\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right)$. Therefore, due to Lemma 5.12, we need to establish conditions when $1 \notin \sigma_q\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}} + b(0)V_{\psi_\alpha}\right)$. By Lemma 5.3, this means that in the case $q=0$ we must have

$$\sum_{k=1}^{l} b_k(0)\widehat{\varphi}_{\alpha,\alpha_k}(\lambda) + b(0)\widehat{\psi}_\alpha(\lambda) \neq 1, \quad \forall\lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq 0,$$

whereas in the case $q>0$ we must have

$$\sum_{k=1}^{l} b_k(0)\widehat{\varphi}_{\alpha,\alpha_k}(\lambda) + b(0)\widehat{\psi}_\alpha(\lambda) \neq 1, \quad \forall\lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq q,$$

and

$$\sum_{k=1}^{l} b_k(0)\widehat{\varphi}_{\alpha,\alpha_k}(j) + b(0)\widehat{\psi}_\alpha(j) \neq 1, \quad j=0,1,\ldots,q-1,$$

where $\widehat{\varphi}_{\alpha,\alpha_k}$ $(k=1,\ldots,l)$ and $\widehat{\psi}_\alpha$ can be found as follows:

$$\begin{aligned}
\widehat{\varphi}_{\alpha,\alpha_k}(\lambda) &= \int_0^1 x^\lambda \varphi_{\alpha,\alpha_k}(x)dx \\
&= \int_0^1 x^\lambda \frac{1}{\Gamma(\alpha-\alpha_k)}(1-x)^{\alpha-\alpha_k-1}x^{\alpha_k}dx \\
&= \frac{1}{\Gamma(\alpha-\alpha_k)}\int_0^1 x^{\lambda+\alpha_k}(1-x)^{\alpha-\alpha_k-1}dx \\
&= \frac{1}{\Gamma(\alpha-\alpha_k)}B(\lambda+\alpha_k+1, \alpha-\alpha_k) \\
&= \frac{\Gamma(\lambda+\alpha_k+1)\Gamma(\alpha-\alpha_k)}{\Gamma(\alpha-\alpha_k)\Gamma(\lambda+\alpha+1)} \\
&= \frac{\Gamma(\lambda+\alpha_k+1)}{\Gamma(\lambda+\alpha+1)}, \quad \mathrm{Re}\lambda \geq 0,
\end{aligned}$$

$$\begin{aligned}
\widehat{\psi}_\alpha(\lambda) &= \int_0^1 x^\lambda \psi_\alpha(x)dx \\
&= \int_0^1 x^\lambda \int_x^1 \frac{1}{\Gamma(\alpha)}y^{-\alpha}\psi(y)(y-x)^{\alpha-1}dydx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_0^1 \psi(y) y^\lambda \int_0^1 s^\lambda (1-s)^{\alpha-1} ds dy \\
&= \frac{1}{\Gamma(\alpha)} B(\lambda+1, \alpha) \widehat{\psi}(\lambda) \\
&= \frac{\Gamma(\lambda+1)\Gamma(\alpha)}{\Gamma(\lambda+1+\alpha)\Gamma(\alpha)} \widehat{\psi}(\lambda) \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+1+\lambda)} \widehat{\psi}(\lambda), \quad \mathrm{Re}\lambda \geq 0.
\end{aligned}
$$

Therefore, we arrive at conditions (5.30) and (5.31)-(5.32). Furthermore, operator $M^{-\alpha} J^\alpha$ is a cordial Volterra integral operator and thus $v \in C^q[0,T]$ implies $u = M^{-\alpha} J^\alpha v \in C^q[0,T]$; recall that $M^{-\alpha} J^\alpha v$ belongs to the domain of $D_0^{\alpha_k} M^{\alpha_k}$ and u really satisfies (5.1).

■

5.4. Numerical approach for the problem (5.1)

For solving equation (5.1), we use the notations and results described in Section 2.5. Let $m, N \in \mathbb{N}$ and $r = 1$ in (2.17), i.e. $t_j = T\left(\frac{j}{N}\right)$. We are looking for approximations $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ for the solution of (5.10) that satisfy the collocation conditions

$$
v_N(t_{ij}) = \sum_{k=1}^{l} b_k(t_{ij}) \left(V_{\varphi_{\alpha,\alpha_k}} v_N \right)(t_{ij}) + b(t_{ij}) \left(V_{\psi_\alpha} v_N \right)(t_{ij}) + f(t_{ij}), \quad (5.33)
$$

where $j = 1, \ldots, m$ and $i = 1, \ldots, N$. Here $\{t_{ij}\}$ are defined by (2.20). Observe that conditions (5.33) have an operator equation representation

$$
v_N = \sum_{k=1}^{l} \mathcal{P}_N b_k V_{\varphi_{\alpha,\alpha_k}} v_N + \mathcal{P}_N b V_{\psi_\alpha} v_N + \mathcal{P}_N f, \quad (5.34)
$$

where $\mathcal{P}_n$ is defined by (2.21). Conditions (5.33) lead to a system of linear equations to uniquely determine v_N. Using Lagrange fundamental polynomial representation (see (2.22)), we get

$$
v_N(t) = \sum_{\kappa=1}^{N} \sum_{\mu=1}^{m} c_{\kappa\mu} l_{\kappa\mu}(t), \quad t \in [0,T], \quad (5.35)
$$

with some coefficients $\{c_{\kappa\mu}\}$. Note that $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ and $v_N(t_{ij}) = c_{ij}$ for $j = 1 \ldots, m$, $i = 1, \ldots, N$. To find coefficients $\{c_{\kappa\mu}\}$ we have to solve a system of linear algebraic equations

$$
c_{ij} = \sum_{\kappa=1}^{N} \sum_{\mu=1}^{m} \left[\sum_{k=1}^{l} b_k(t_{ij}) (V_{\varphi_{\alpha,\alpha_k}} l_{\kappa\mu})(t_{ij}) + b(t_{ij}) (V_{\psi_\alpha} l_{\kappa\mu})(t_{ij}) \right] c_{\kappa\mu} + f(t_{ij}), \quad (5.36)
$$

with respect to c_{ij}, $j = 1, \ldots, m$, $i = 1. \ldots, N$.
To find the approximate solution u_N of (5.1) we use the equality $u = M^{-\alpha} J^{\alpha} v$:

$$u_N(t) = (M^{-\alpha} J^{\alpha} v_N)(t) = (V_{\varphi_\alpha} v_N)(t) = \sum_{j=1}^{N} \sum_{\mu=1}^{m} c_{j\mu} \left(V_{\varphi_\alpha} l_{j\mu}\right)(t), \quad 0 \le t \le T, \tag{5.37}$$

where φ_α is defined by (5.11).

5.5. Convergence analysis

In this section, we analyse the convergence and convergence order of the proposed numerical method. We first present some lemmas that we need in our discussion below.

Lemma 5.14 ([73])**.** For $\varphi \in L^1(0,1)$ and $u \in L^\infty(0,T)$ having a finite limit $u(0) := \lim_{t\to 0} u(t)$, it holds that $V_\varphi u \in C[0,T]$, $(V_\varphi u)(0) = u(0) \int_0^1 \varphi(x)dx$.

Lemma 5.15 ([75])**.** Let $\varphi \in L^1(0,1)$, $f \in C[0,T]$, $1 \in \rho_0(V_\varphi)$ for equation

$$u = V_\varphi u + f. \tag{5.38}$$

Also let $m \in \mathbb{N}$ and assume that

$$\det(I_m - D_i) \neq 0, \quad i = 1, \ldots, N-1, \tag{5.39}$$

where I_m is the $m \times m$ identity matrix and $D_i = (d_{j,k}^{i,i})_{j,k=1}^{m}$ is an $m \times m$ matrix with the elements defined by the formula

$$d_{j,k}^{i,i} = \int_{i/(i+\tau_j)}^{1} \varphi(x) l_k((i+\tau_j)x - i)dx, \quad j,k = 1, \ldots, m, \quad i = 0, \ldots, N-1,$$

where $l_k \in \pi_{m-1}$ are the Lagrange fundamental polynomials associated with the knots (collocation parameters) $0 \le \tau_1 < \ldots < \tau_m \le 1$.
Then collocation equation $u_N = \mathcal{P}_N V_\varphi u_N + \mathcal{P}_N f$ has for $N \ge 1$ a unique solution $u_N \in S_{m-1}^{(-1)}(\Pi_N)$, and

$$\|u - u_N\|_\infty \le c_0 \|u - \mathcal{P}_N u\|_\infty \to 0 \text{ as } N \to \infty,$$

where $u \in C[0,T]$ is the unique solution of equation (5.38).

If $f \in C^m[0,T]$, then also $u \in C^m[0,T]$ and

$$\|u - u_N\|_\infty \le c_m h^m \|u^{(m)}\|_\infty.$$

The constants c_0 and c_m are independent of N and f.

Lemma 5.16 ([75])**.** Let the assumptions of Lemma 5.15 hold. Then operator $I - \mathcal{P}_N V_\varphi : S_{m-1}^{(-1)}(\Pi_N) \to S_{m-1}^{(-1)}(\Pi_N)$ is invertible; moreover equipping $S_{m-1}^{(-1)}(\Pi_N) \subset L^\infty(0,T)$ with the norm from $L^\infty(0,T)$, we get

$$\|(I - \mathcal{P}_N V_\varphi)^{-1}\|_{S_{m-1}^{(-1)}(\Pi_N) \to S_{m-1}^{(-1)}(\Pi_N)} \le c, \quad N \ge 1.$$

We now present the theorem describing the convergence of our numerical method.

Theorem 5.17. *Assume that in equation* (5.1) *function* $\psi \in L^1(0,1)$ *and conditions* (5.2) *hold. Further, assume that conditions* (5.30) *are fulfilled and either* (*a*) *or* (*b*) *hold, where*

(a) $b_k(0) = 0$ *for* $k = 1, \ldots, l$ *and* $b(0) = 0$*,*

(b) for $i = 1, \ldots, N-1$

$$\det(I_m - D_i) \neq 0,$$

where $D_i = (d_{j,k}^{i,i})_{j,k=1}^m$ *is the* $m \times m$ *matrix with elements*

$$d_{j,k}^{i,i} = \int_{i/(i+\tau_j)}^1 \left[\sum_{r=1}^l b_r(0)\varphi_{\alpha,\alpha_r}(x) + b(0)\psi_\alpha(x) \right] l_k((i+\tau_j)x - i)dx,$$

with $j, k = 1, \ldots, m$ *and* $i = 1, \ldots, N-1$ *and where* $l_k \in \pi_{m-1}$ *are the Lagrange fundamental polynomials associated with the knots (collocation parameters)* $0 \leq \tau_1 < \ldots < \tau_m \leq 1$.

Then there exists an $N_0 \in \mathbb{N}$ *such that equation* (5.34) *has for* $N \geq N_0$ *a unique solution* $v_N \in S_{m-1}^{(-1)}(\Pi_N)$*, determining by* (5.37) *a unique approximation* u_N *to* u*, the solution of* (5.1) *and*

$$\|u - u_N\|_\infty \to 0 \text{ as } N \to \infty. \tag{5.40}$$

Proof. With conditions (5.2) we know that $b_k, b \in C[0,T]$ for $k = 1, \ldots, l$. According to Lemma 5.14, $V_{\varphi_{\alpha,\alpha_k}} v_N, V_{\psi_\alpha} v_N \in C[0,T]$ for $\varphi_{\alpha,\alpha_k}, \psi_\alpha \in L^1(0,1)$ and $v_N \in S_{m-1}^{(-1)}(\Pi_N)$, thus $\mathcal{P}_N b_k V_{\varphi_{\alpha,\alpha_k}} v_N$, $\mathcal{P}_N b V_{\psi_\alpha} v_N$ are well defined in (5.34). We begin our discussions with the simpler case, where $b_k(0) = b(0) = 0$ for $k = 1, \ldots, l$. According to Lemma 5.4 operators $b_k V_{\varphi_{\alpha,\alpha_k}}$, $k = 1, \ldots, l$, and bV_{ψ_α} are compact from $L^\infty(0,T) \to C[0,T]$ and $\sigma_0(\sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}) = \{0\}$. Therefore, the sum of these operators is also compact and with Lemma 2.11 gives us that

$$\left\| (I - \mathcal{P}_N) \left(\sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha} \right) \right\|_{L^\infty(0,T) \to L^\infty(0,T)} \to 0 \text{ as } N \to \infty. \tag{5.41}$$

Thus collocation equation (5.34) has for sufficiently large N a unique solution $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ and

$$\|v - v_N\|_\infty \leq c_0 \|v - \mathcal{P}_N v\|_\infty \to 0 \text{ as } N \to \infty. \tag{5.42}$$

Since $u = M^{-\alpha} J^\alpha v$, we have

$$u = V_{\varphi_\alpha} v, \quad u_N = V_{\varphi_\alpha} v_N, \quad N \geq 1, \tag{5.43}$$

where φ_α is defined by (5.11) and $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ is the solution of equation (5.34), u_N is defined by (5.37), v is the solution of (5.10) and u is the solution of (5.1). We now see that for $N \geq 1$,

$$||u - u_N||_\infty = \|V_{\varphi_\alpha}(v - v_N)\| \leq c\|v - v_N\|_\infty. \tag{5.44}$$

In conclusion estimates (5.44) and (5.42) imply that for the case $b_k(0) = b(0) = 0$ $(k = 1, \ldots, l)$ the proposed numerical method converges.
We now look at the more general case, where $b_k(0)$ and $b(0)$ are not equal to zero. In such case, the operators in equation $v = \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} v + bV_{\psi_\alpha} v + f$ are generally non-compact and thus the analysis of collocation methods becomes more complicated. In order to examine convergence of our numerical approach, we first rewrite the operators in (5.10) as follows:

$$\begin{aligned}\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} v + bV_{\psi_\alpha} v =& \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} v + b(0)V_{\psi_\alpha} v \\ &+ \sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}} + [b - b(0)]V_{\psi_\alpha} v.\end{aligned}$$

Then we consider the auxiliary equation

$$v = \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} + b(0)V_{\psi_\alpha} v + f. \tag{5.45}$$

Assume that for $i = 1, \ldots, N-1$

$$\det(I_m - D_i) \neq 0, \tag{5.46}$$

where $D_i = (d_{j,k}^{i,i})_{j,k=1}^{m}$ is the $m \times m$ matrix with elements

$$d_{j,k}^{i,i} = \int_{i/(i+\tau_j)}^{1} \left[\sum_{r=1}^{l} b_r(0)\varphi_{\alpha,\alpha_r}(x) + b(0)\psi_\alpha(x) \right] l_k((i+\tau_j)x - i)dx, \tag{5.47}$$

with $j, k = 1, \ldots, m$ and $i = 1, \ldots, N-1$ and $l_k \in \pi_{m-1}$ are the Lagrange fundamental polynomials associated with the knots (collocation parameters) $0 \leq \tau_1 < \ldots < \tau_m \leq 1$.
The assumptions of Lemma 5.15 and 5.16 are now fulfilled for equation (5.45) with $\varphi(x) = \sum_{r=1}^{l} b_r(0)\varphi_{\alpha,\alpha_r}(x) + b(0)\psi_\alpha(x)$ $(0 < x < 1)$ in Lemma 5.15. Thus, based on Lemma 5.16 we have

$$\left\| \left(I - \mathcal{P}_N \left(\sum_{k=1}^{l} b_k(0) V_{\varphi,\varphi_k} + b(0)V_{\psi_\alpha} \right) \right)^{-1} \right\|_{S_{m-1}^{(-1)}(\Pi_N) \to S_{m-1}^{(-1)}(\Pi_N)} \leq c, \quad N \geq 1. \tag{5.48}$$

We also know that, by Lemma 5.4, operator $[b_k - b_k(0)]V_{\varphi_{\alpha,\alpha_k}} : L^\infty(0,T) \to C[0,T]$ is compact for $k = 1,\dots,l$ and $[b - b(0)]V_{\psi_\alpha} : L^\infty(0,T) \to C[0,T]$ is compact. Therefore, the sum of those operators is also compact, and it holds that

$$\left\|(I-\mathcal{P}_N)\left[\sum_{k=1}^{l}[b_k-b_k(0)]V_{\varphi_{\alpha,\alpha_k}}+[b-b(0)]V_{\psi_\alpha}\right]\right\|_{L^\infty(0,T)\to L^\infty(0,T)} \to 0 \text{ as } N\to\infty.$$

From last two relations we obtain that for sufficiently large N, the inverses to

$$I - \mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right] : S_{m-1}^{(-1)}(\Pi_N) \to S_{m-1}^{(-1)}(\Pi_N)$$

exist and are uniformly bounded:

$$\left\|\left(I - \mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right]\right)^{-1}\right\|_{S_{m-1}^{(-1)}(\Pi_N)\to S_{m-1}^{(-1)}(\Pi_N)} \le 2c, \quad N \ge N_0. \tag{5.49}$$

For solutions v of equation (5.10) and v_N the solution of equation (5.34) we have

$$\left(I - \mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right]\right)\mathcal{P}_N v = \mathcal{P}_N\left(I - \left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right]\right)\mathcal{P}_N v$$

$$= \left(I - \mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right]\right)v_N + \mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right](v - \mathcal{P}_N v)$$

implying

$$\mathcal{P}_N v - v_N = \left(I - \mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right]\right)^{-1}\mathcal{P}_N\left[\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + bV_{\psi_\alpha}\right](v - \mathcal{P}_N v).$$

Using the last equality, (5.49) and Lemma 2.12, we get

$$\|\mathcal{P}_N v - v_N\|_\infty \le c\|v - \mathcal{P}_N v\|_\infty.$$

Thus

$$\|v - v_N\|_\infty \le \|v - \mathcal{P}_N v\|_\infty + \|\mathcal{P}_N v - v_N\|_\infty \le c\|v - \mathcal{P}_N v\|_\infty.$$

Based on (5.43), we now see that

$$\|u - u_N\|_\infty = \|V_{\varphi_\alpha}(v - v_N)\|_\infty \le c_0\|v - v_N\|_\infty \le c\|v - \mathcal{P}_N v\|_\infty \to 0 \text{ as } N \to \infty.$$

■

We now present the theorem describing the convergence order of the numerical method.

Theorem 5.18. *Assume that in equation* (5.1) *function* $\psi \in L^1(0,1)$, *conditions* (5.2) *hold and let* $b_k, b, f \in C^m[0,T]$, $m \in \mathbb{N}$. *Further, assume that conditions* (5.30) *are fulfilled and either (a) or (b) hold, where*

(a) $b_k(0) = 0$ *for* $k = 1, \ldots, l$ *and* $b(0) = 0$,

(b) for $i = 1, \ldots, N-1$

$$\det(I_m - D_i) \neq 0,$$

where $D_i = (d_{j,k}^{i,i})_{j,k=1}^m$ *is the* $m \times m$ *matrix with elements*

$$d_{j,k}^{i,i} = \int_{i/(i+\tau_j)}^1 \left[\sum_{r=1}^l b_r(0)\varphi_{\alpha,\alpha_r}(x) + b(0)\psi_\alpha(x)\right] l_k((i+\tau_j)x - i)dx,$$

with $j,k = 1, \ldots, m$ *and* $i = 1, \ldots, N-1$ *and where* $l_k \in \pi_{m-1}$ *are the Lagrange fundamental polynomials associated with the knots (collocation parameters)* $0 \leq \tau_1 < \ldots < \tau_m \leq 1$.

Then there exists an $N_0 \in \mathbb{N}$ *such that equation* (5.34) *has for* $N \geq N_0$ *a unique solution* $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ *for* $q = m$, *determining by* (5.37) *a unique approximation* u_N *to* u, *the solution of* (5.1) *and*

$$\|u - u_N\|_\infty \leq cN^{-m}, \tag{5.50}$$

where $c > 0$ *is an independent constant of* N.

Proof. The existence of the unique solution of (5.34) is proven in Theorem 5.17 and thus we only have to show that (5.50) holds. First let $b_k(0) = b(0) = 0$ for $k = 1, \ldots, l$. If $b_k, b, f \in C^m[0,T]$ then $v \in C^m[0,T]$ and

$$\|v - v_N\|_\infty \leq c_m h^m \|v^{(m)}\|_\infty.$$

where c_m are some positive constants independent of N. Due to estimate (5.44),

$$\|u - u_N\|_\infty \leq c\|v - v_N\|_\infty \leq \tilde{c}N^{-m}, \tag{5.51}$$

where $c, \tilde{c}$ are some positive constants independent of N. If $b_k(0), b(0)$ are not equal to zero, then based on (5.43), we can see that

$$\|u - u_N\|_\infty = \|V_{\varphi_\alpha}(v - v_N)\|_\infty \leq c_0\|v - v_N\|_\infty \leq c\|v - \mathcal{P}_N v\|_\infty.$$

If $b_k, b, f \in C^m[0,T]$ then $v \in C^m[0,T]$ and by estimating $\|v - \mathcal{P}_N v\|_\infty$ by standard estimates for the spline interpolation we arrive at the following error estimate:

$$\|u - u_N\|_\infty \leq cN^{-m},$$

where constant $c > 0$ is independent of N. ■

In practice it is sometimes quite difficult to check if conditions (5.46) hold for a specific singular fractional integro-differential equation. Fortunately, for piecewise constant and piecewise linear approximation the situation is somewhat easier.

Remark 5.19. For $m = 1$, we can show (cf. [38, 75]) that conditions (5.46) follow directly from the other assumptions of Theorems 5.17 and 5.18 if

$$\sum_{k=1}^{l} b_k(0)\varphi_{\alpha,\alpha_k}(x) + b(0)\psi_\alpha(x) \geq 0, \quad 0 < x < 1. \tag{5.52}$$

Similarly, for $m = 2$, we can show (cf. [21, 38]) that if conditions (5.52) and

$$\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k + 1)}{\Gamma(\alpha + 1)} + b(0)\frac{\widehat{\psi}(0)}{\Gamma(\alpha + 1)} < 1, \tag{5.53}$$

are fulfilled, then the conditions (5.46) follow from the other assumptions of Theorem 5.17 and 5.18.

5.6. Numerical examples

In this section we will present three numerical examples to illustrate our theoretical results. We use the numerical method described in Section 5.4 to find the approximate solution of equation (5.1).
Let $N, m \in \mathbb{N}$. We introduce for the interval $[0, T]$ a uniform grid

$$t_j = j\frac{T}{N}, \quad j = 0, \ldots, N, \tag{5.54}$$

and use collocation points $t_{i-1} + \tau_\kappa \frac{T}{N}$, $\kappa = 1, \ldots, m$, $i = 1, \ldots, N$, where the collocation parameters satisfy $0 \leq \tau_1 < \ldots < \tau_m \leq 1$. To find the approximate solution of equation (5.1) we use (5.37), where $\{c_{i\kappa}\}$ is the solution of (5.36).

We present, in the tables below, some results of numerical experiments for different values of parameters N, m. The errors, ε_N, are calculated as follows:

$$\varepsilon_N = \max_{j=1,\ldots,N} \max_{k=0,\ldots,10} |u(\tau_{jk}) - u_N(\tau_{jk})|, \tag{5.55}$$

where u is the exact solution to our problem and u_N is the approximate solution found by (5.37) and

$$\tau_{jk} = t_{j-1} + k(t_j - t_{j-1})/10, \quad k = 0, \ldots, 10, \quad j = 1, \ldots, N,$$

with the gridpoints, t_j, defined by (5.54). In tables below, the ratios

$$\Theta_N = \frac{\varepsilon_{N/2}}{\varepsilon_N}, \tag{5.56}$$

characterizing the observed convergence rate, are presented.

5.6.1. Example 1

Consider the singular fractional integro-differential equation

$$(D_0^{\frac{7}{3}} M^{\frac{7}{3}} u)(t) = \cos(t)(D_0^{\frac{4}{3}} M^{\frac{4}{3}} u)(t) + (1-t)^{\frac{5}{2}} (V_\psi u)(t) + \cos(t) + t^{\frac{28}{10}}, \quad 0 < t \le 1, \tag{5.57}$$

where $\psi(x) = x^{\frac{7}{3}}$, $0 < x < 1$. We see that (5.57) is an equation of the form (5.1) with conditions (5.2), where

$$\alpha = \frac{7}{3},\ q = 2,\ l = 1,\ \alpha_1 = \frac{4}{3},\ T = 1,$$

and

$$b_1(t) = \cos(t), \quad b(t) = (1-t)^{\frac{5}{2}}, \quad f(t) = \cos(t) + t^{\frac{28}{10}}, \quad 0 \le t \le 1. \tag{5.58}$$

By a change of variables $v = D_0^{\frac{7}{3}} M^{\frac{7}{3}} u$, equation (5.57) takes the form

$$v = b_1 V_{\varphi_{\alpha,\alpha_1}} v + b V_{\psi_\alpha} v + f, \tag{5.59}$$

where

$$\varphi_{\alpha,\alpha_1}(x) = x^{\frac{4}{3}},\ \psi_\alpha(x) = \int_x^1 \frac{1}{\Gamma\left(\frac{7}{3}\right)} (y-x)^{\frac{4}{3}} dy, \quad 0 < x < 1. \tag{5.60}$$

Here, the exact solution is not known. For the numerical tests we use approximation u_N obtained with $m = 2$, $N = 1024$, i.e., $u(t) \approx u_{1024}(t)$ $(0 \le t \le 1)$.

According to Theorem 5.13 equation (5.59) (thus also equation (5.57)) is uniquely solvable in $C^2[0,1]$. Since $f \in C^2[0,1]$, we take $m = 2$.

Using (5.58) and (5.60), we have

$$b_1(0)\varphi_{\alpha,\alpha_1}(x) + b(0)\psi_\alpha(x) = x^{\frac{4}{3}} + \int_x^1 \frac{1}{\Gamma\left(\frac{7}{3}\right)} (y-x)^{\frac{4}{3}} dy \ge 0, \quad 0 < x < 1,$$

and

$$b_1(0) \frac{\Gamma(\alpha_1 + 1)}{\Gamma(\alpha + 1)} + b(0) \frac{\widehat{\psi}(0)}{\Gamma(\alpha + 1)} = \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{10}{3}\right)} + \frac{\frac{3}{10}}{\Gamma\left(\frac{10}{3}\right)} \approx 0.54 < 1.$$

Thus conditions (5.52) and (5.53) are fulfilled. The errors ε_N are calculated by (5.55). Based on Theorem 5.18 and Remark 5.19 that, for $m = 2$ and for sufficiently large N

$$\varepsilon_N \le cN^{-2}, \tag{5.61}$$

where $c > 0$ is a positive constant independent of N. The ratio Θ_N ought to be $2^2 = 4$. In Table 10 some results of numerical experiments for different values of collocation parameters τ_1, τ_2 and parameter N are presented. Table 10 shows that numerical results agree with the theoretical estimates.

	$\tau_1 = 1/4,\ \tau_2 = 3/4$		$\tau_1 = 1/3,\ \tau_2 = 2/3$		$\tau_1 = 1/4,\ \tau_2 = 4/9$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
8	5.41E-04		6.51E-04		3.75E-04	
16	1.80E-04	3.01	2.17E-04	3.00	1.13E-04	3.30
32	5.14E-05	3.50	6.20E-05	3.50	3.11E-05	3.64
64	1.38E-05	3.74	1.66E-05	3.74	8.17E-06	3.81
128	3.55E-06	3.88	4.28E-06	3.88	2.09E-06	3.91
256	8.72E-07	4.07	1.05E-06	4.08	5.12E-07	4.08
		4.00		4.00		4.00

Table 10. Numerical results for problem (5.57) for $m = 2$.

5.6.2. Example 2

Consider the singular fractional integro-differential equation

$$(D_0^{\frac{12}{10}} M^{\frac{12}{10}} u)(t) = \sin(t)(D_0^{\frac{1}{2}} M^{\frac{1}{2}} u)(t) + (V_\psi u)(t) + \frac{\Gamma\left(\frac{18}{5}\right)}{\Gamma\left(\frac{12}{5}\right)} t^{\frac{7}{5}} - \sin(t) \frac{\Gamma\left(\frac{29}{10}\right)}{\Gamma\left(\frac{12}{5}\right)} t^{\frac{7}{5}} - \frac{5}{18} t^{\frac{7}{5}}, \tag{5.62}$$

where $0 < t \le 1$ and $\psi(x) = x^{\frac{12}{10}}$, $0 < x < 1$. We see that (5.62) is an equation of the form (5.1) with conditions (5.2), where

$$\alpha = \frac{12}{10},\ q = 1,\ l = 1,\ \alpha_1 = \frac{1}{2},\ T = 1,$$

and

$$b_1(t) = \sin(t), \quad b(t) = 1, \quad f(t) = \frac{\Gamma\left(\frac{18}{5}\right)}{\Gamma\left(\frac{12}{5}\right)} t^{\frac{7}{5}} - \sin(t) \frac{\Gamma\left(\frac{29}{10}\right)}{\Gamma\left(\frac{12}{5}\right)} t^{\frac{7}{5}} - \frac{5}{18} t^{\frac{7}{5}},\ 0 \le t \le 1.$$

The exact solution of equation (5.62) is $u(t) = t^{\frac{7}{5}}$ $(0 \le t \le 1)$. Since $f \in C^1[0, 1]$, we take $m = 1$.

Similarly as in the first example we can show that (5.52) is satisfied for problem (5.62). The errors ε_N are calculated by (5.55). According to Theorem 5.18 and Remark 5.19, for $m = 1$ and for sufficiently large N

$$\varepsilon_N \le cN^{-1}, \tag{5.63}$$

where $c > 0$ is a positive constant independent of N. The ratio Θ_N ought to be $2^1 = 2$. In Table 11 some results of numerical experiments for different values of the collocation parameter τ_1 and parameter N are given. As we can see, the numerical results are in agreement with the theoretical estimates.

	$\tau_1 = 2/5$		$\tau_1 = 3/5$		$\tau_1 = 4/5$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
8	6.39E-02		1.17E-01		2.61E-01	
16	2.37E-02	2.70	4.45E-02	2.63	1.15E-01	2.27
32	1.34E-02	1.77	1.97E-02	2.26	5.37E-02	2.15
64	7.30E-03	1.83	9.11E-02	2.16	2.58E-02	2.08
128	3.83E-03	1.91	4.35E-03	2.09	1.26E-02	2.05
256	1.97E-03	1.95	2.12E-03	2.05	6.23E-03	2.03
		2.00		2.00		2.00

Table 11. Numerical results for problem (5.62) for $m = 1$.

5.6.3. Example 3

Consider the singular fractional integro-differential equation

$$(D_0^{\frac{5}{2}} M^{\frac{5}{2}} u)(t) = t^{\frac{5}{2}} (D_0^{\frac{5}{3}} M^{\frac{5}{3}} u)(t) + t^{\frac{7}{3}} \sin(t)(V_\psi u)(t) + \frac{\Gamma\left(\frac{37}{6}\right)}{\Gamma\left(\frac{11}{3}\right)} t^{\frac{8}{3}} - \frac{\Gamma\left(\frac{16}{3}\right)}{\Gamma\left(\frac{11}{3}\right)} t^{\frac{31}{6}} - \frac{6}{37} t^5 \sin(t), \tag{5.64}$$

where $0 < t \le \frac{1}{2}$ and $\psi(x) = x^{\frac{5}{2}}$, $0 < x < 1$. We see that (5.64) is an equation of the form (5.1) with conditions (5.2), where

$$\alpha = \frac{5}{2},\ q = 2,\ l = 1,\ \alpha_1 = \frac{5}{3},\ T = \frac{1}{2},$$

and

$$b_1(t) = t^{\frac{5}{2}}, \quad b(t) = t^{\frac{7}{3}} \sin(t), \quad f(t) = \frac{\Gamma\left(\frac{37}{6}\right)}{\Gamma\left(\frac{11}{3}\right)} t^{\frac{8}{3}} - \frac{\Gamma\left(\frac{16}{3}\right)}{\Gamma\left(\frac{11}{3}\right)} t^{\frac{31}{6}} - \frac{6}{37} t^5 \sin(t),$$

where $0 \le t \le \frac{1}{2}$.

The exact solution of equation (5.64) is $u(t) = t^{\frac{8}{3}}$ $(0 \le t \le \frac{1}{2})$. Since $b(0) = b_1(0) = 0$, Theorem 5.13 implies that equation (5.64) has a unique solution in $C^2\left[0, \frac{1}{2}\right]$. As $f \in C^2\left[0, \frac{1}{2}\right]$, we take $m = 2$. The errors ε_N are calculated by (5.55). Since $b(0) = b_1(0) = 0$, it follows from Theorem 5.18 that for $m = 2$ and for sufficiently large N

$$\varepsilon_N \le c N^{-2}, \tag{5.65}$$

where $c > 0$ is a positive constant independent of N. The ratio Θ_N ought to be $2^2 = 4$. Table 12 presents results of numerical experiments for different values of collocation parameters τ_1, τ_2 and parameter N. We can see that the numerical results are in good accordance with the theoretical estimates.

	$\tau_1 = 1/4,\ \tau_2 = 3/4$		$\tau_1 = 1/10,\ \tau_2 = 9/10$		$\tau_1 = 6/11,\ \tau_2 = 9/11$	
N	ε_N	Θ_N	ε_N	Θ_N	ε_N	Θ_N
8	1.25E-03		2.07E-03		3.94E-03	
16	1.97E-04	6.34	5.15E-04	4.03	7.00E-04	5.63
32	3.49E-05	5.66	1.28E-04	4.01	1.69E-04	4.14
64	8.72E-06	4.00	3.21E-05	4.00	4.16E-05	4.07
128	2.18E-06	4.00	8.02E-06	4.00	1.03E-05	4.03
256	5.45E-07	4.00	2.00E-06	4.00	2.57E-06	4.02
		4.00		4.00		4.00

Table 12. Numerical results for problem (5.64) for $m = 2$.

BIBLIOGRAPHY

[1] P. Amodio, L. Brugnano, and F. Iavernaro. "A note on a stable algorithm for computing the fractional integrals of orthogonal polynomials". In: *Applied Mathematics Letters* 134 (Dec. 2022), p. 108338.

[2] J. M. Angulo, M. D. Ruiz-Medina, V. V. Anh, and W. Grecksch. "Fractional diffusion and fractional heat equation". In: *Advances in Applied Probability* 32.4 (Dec. 2000), pp. 1077–1099.

[3] M. Arif, P. Kumam, W. Kumam, A. Akgul, and T. Sutthibutpong. "Analysis of newly developed fractal-fractional derivative with power law kernel for MHD couple stress fluid in channel embedded in a porous medium". In: *Scientific Reports* 11.1 (Oct. 2021).

[4] K. E. Atkinson. *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge University Press, June 1997.

[5] K. E. Atkinson and W. Han. *Theoretical Numerical Analysis: A Functional Analysis Framework*. Springer-Verlag, New York, 2001.

[6] L. F. Ávalos-Ruiz, J. F. Gómez-Aguilar, A. Atangana, and K. M. Owolabi. "On the dynamics of fractional maps with power-law, exponential decay and Mittag–Leffler memory". In: *Chaos, Solitons & Fractals* 127 (Oct. 2019), pp. 364–388.

[7] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo. *Fractional Calculus: Models and Numerical Methods*. World Scientific, Mar. 2016.

[8] O. Boichuk and V. Feruk. "Terminal value problem for the system of fractional differential equations with additional restrictions". In: *Mathematical Modelling and Analysis* 30.1 (Jan. 2025), pp. 120–141.

[9] L. Brugnano, K. Burrage, P. Burrage, and F. Iavernaro. "A Spectrally Accurate Step-by-Step Method for the Numerical Solution of Fractional Differential Equations". In: *Journal of Scientific Computing* 99.2 (Apr. 2024).

[10] L. Brugnano and F. Iavernaro. *Line Integral Methods for Conservative Problems*. Chapman and Hall/CRC, Mar. 2016.

[11] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Differential Equations*. Cambridge University Press, Nov. 2004.

[12] H. Brunner. *Volterra Integral Equations: An Introduction to Theory and Applications*. Cambridge University Press, Jan. 2017.

[13] H. Brunner, A. Pedas, and G. Vainikko. "Piecewise Polynomial Collocation Methods for Linear Volterra Integro-Differential Equations with Weakly Singular Kernels". In: *SIAM Journal on Numerical Analysis* 39.3 (2001), pp. 957–982.

[14] A. Cardone, D. Conte, and B. Paternoster. "Stability of two-step spline collocation methods for initial value problems for fractional differen-

tial equations". In: *Communications in Nonlinear Science and Numerical Simulation* 115 (Dec. 2022), p. 106726.

[15] S. Deepika and P. Veeresha. "Dynamics of chaotic waterwheel model with the asymmetric flow within the frame of Caputo fractional operator". In: *Chaos, Solitons & Fractals* 169 (Apr. 2023), p. 113298.

[16] K. Diethelm. *The Analysis of Fractional Differential Equations.* Springer Berlin Heidelberg, 2010.

[17] K. Diethelm and N. J. Ford. "Analysis of Fractional Differential Equations". In: *Journal of Mathematical Analysis and Applications* 265.2 (Jan. 2002), pp. 229–248.

[18] K. Diethelm, R. Garrappa, A. Giusti, and M. Stynes. "Why Fractional Derivatives with Nonsingular Kernels Should Not Be Used". In: *Fractional Calculus and Applied Analysis* 23.3 (June 2020), pp. 610–634.

[19] K. Diethelm, V. Kiryakova, Y. Luchko, J. A. T. Machado, and V. E. Tarasov. "Trends, directions for further research, and some open problems of fractional calculus". In: *Nonlinear Dynamics* 107.4 (Jan. 2022), pp. 3245–3270.

[20] T. Diogo and P. Lima. "Superconvergence of collocation methods for a class of weakly singular Volterra integral equations". In: *Journal of Computational and Applied Mathematics* 218.2 (Sept. 2008), pp. 307–316.

[21] T. Diogo and G. Vainikko. "Applicability of Spline Collocation to Cordial Volterra Equations". In: *Math. Model. Anal.* 18.1 (2013), pp. 1–21.

[22] E. F. Doungmo Goufo and J. J. Nieto. "Attractors for fractional differential problems of transition to turbulent flows". In: *Journal of Computational and Applied Mathematics* 339 (Sept. 2018), pp. 329–342.

[23] L. R. Evangelista and E. K. Lenzi. *An Introduction to Anomalous Diffusion and Relaxation.* Springer International Publishing, 2023.

[24] L. R. Evangelista and E. K. Lenzi. *Fractional Diffusion Equations and Anomalous Diffusion.* Cambridge University Press, Jan. 2018.

[25] L. L. Ferrás, N. J. Ford, M. L. Morgado, and M. Rebelo. "High-Order Methods for Systems of Fractional Ordinary Differential Equations and Their Application to Time-Fractional Diffusion Equations". In: *Mathematics in Computer Science* 15.4 (Jan. 2020), pp. 535–551.

[26] N. J. Ford, M. L. Morgado, and M. Rebelo. "A nonpolynomial collocation method for fractional terminal value problems". In: *Journal of Computational and Applied Mathematics* 275 (Feb. 2015), pp. 392–402.

[27] N. J. Ford, A. Pedas, and M. Vikerpuur. "High order approximations of solutions to initial value problems for linear fractional integro-differential equations". In: *Fractional Calculus and Applied Analysis* 26.5 (July 2023), pp. 2069–2100.

[28] N. J. Ford and Y. Yan. “An approach to construct higher order time discretisation schemes for time fractional partial differential equations with nonsmooth data”. In: *Fractional Calculus and Applied Analysis* 20.5 (Oct. 2017), pp. 1076–1105.

[29] O. Ibe. *Markov Processes for Stochastic Modeling*. Elsevier, 2013.

[30] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado, and J. H. T. Bates. “The role of fractional calculus in modeling biological phenomena: A review”. In: *Communications in Nonlinear Science and Numerical Simulation* 51 (Oct. 2017), pp. 141–159.

[31] J. Janno and Y. Kian. “Inverse source problem with a posteriori boundary measurement for fractional diffusion equations”. In: *Mathematical Methods in the Applied Sciences* 46.14 (June 2023), pp. 15868–15882.

[32] K. B. Kachhia. “Chaos in fractional order financial model with fractal–fractional derivatives”. In: *Partial Differential Equations in Applied Mathematics* 7 (June 2023), p. 100502.

[33] U. Kangro. “Cordial Volterra Integral Equations and Singular Fractional Integro-Differential Equations in Spaces of Analytic Functions”. In: *Mathematical Modelling and Analysis* 22.4 (July 2017), pp. 548–567.

[34] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*. London, England: Elsevier Science, Jan. 2006.

[35] M. Kolk, A. Pedas, and E. Tamme. “Smoothing transformation and spline collocation for linear fractional boundary value problems”. In: *Applied Mathematics and Computation* 283 (2016), pp. 234–250.

[36] N. Kopteva and M. Stynes. “An efficient collocation method for a Caputo two-point boundary value problem”. In: *BIT Numerical Mathematics* 55.4 (2015), pp. 1105–1123.

[37] R. Kress. *Linear Integral Equations*. Springer New York, 1999.

[38] K. Lätt and A. Pedas. “Numerical schemes for a class of singular fractional integro-differential equations”. In: *Appl. Numer. Math.* 200 (2024), pp. 331–343.

[39] K. Lätt, A. Pedas, and H. B. Soots. “Singular fractional integro-differential equations with non-constant coefficients”. In: *Applied Numerical Mathematics* 211 (May 2025), pp. 179–192.

[40] K. Lätt, A. Pedas, H. B. Soots, and M. Vikerpuur. “Collocation-Based Approximation for a Time-Fractional Sub-Diffusion Model”. In: *Fractal and Fractional* 7.9 (Aug. 2023), p. 657.

[41] K. Lätt, A. Pedas, and G. Vainikko. “A Smooth Solution of a Singular Fractional Differential Equation”. In: *Zeitschrift für Analysis und ihre Anwendungen* 34.2 (Apr. 2015), pp. 127–146.

[42] X. Li and Y. Tang. “Interpolated Coefficient Mixed Finite Elements for Semilinear Time Fractional Diffusion Equations”. In: *Fractal and Fractional* 7.6 (June 2023), p. 482.

[43] H. Liang and M. Stynes. “Collocation methods for general Caputo two-point boundary value problems”. In: *Journal of Scientific Computing* 76 (2018), pp. 390–425.

[44] M. Lillemäe, A. Pedas, and M. Vikerpuur. “Central part interpolation schemes for fractional differential equations”. In: *Applied Numerical Mathematics* 200 (June 2024), pp. 318–330.

[45] Y. Luchko. “Fractional Derivatives and the Fundamental Theorem of Fractional Calculus”. In: *Fractional Calculus and Applied Analysis* 23.4 (Aug. 2020), pp. 939–966.

[46] Y. Luchko. “Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation”. In: *Fractional Calculus and Applied Analysis* 15.1 (Dec. 2011), pp. 141–160.

[47] E.-J. Määrits, A. Pedas, and M. Vikerpuur. “On the construction of iterated collocation-type approximations for linear fractional differential equations”. In: *Acta et Commentationes Universitatis Tartuensis de Mathematica* 28.2 (Nov. 2024), pp. 271–288.

[48] F. Mainardi. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Imperial College Press, May 2010.

[49] R. Metzler and J. Klafter. “The random walk’s guide to anomalous diffusion: a fractional dynamics approach”. In: *Physics Reports* 339.1 (Dec. 2000), pp. 1–77.

[50] M. Mozafarifard and D. Toghraie. “Time-fractional subdiffusion model for thin metal films under femtosecond laser pulses based on Caputo fractional derivative to examine anomalous diffusion process”. In: *International Journal of Heat and Mass Transfer* 153 (June 2020), p. 119592.

[51] Z. Navickas, T. Telksnys, I. Timofejeva, R. Marcinkevičius, and M. Ragulskis. “An operator-based approach for the construction of closed-form solutions to fractional differential equations”. In: *Mathematical Modelling and Analysis* 23.4 (2018), pp. 665–685.

[52] H. Niu, S. Zhao, C. I. Muresan, and C. M. Ionescu. “Enhanced Fractional-Order Nonsingular Terminal Sliding Mode Control for Fully Submerged Hydrofoil Craft with Actuator Saturation”. In: *Journal of Marine Science and Application* (Jan. 2025).

[53] M. D. Ortigueira. *Fractional Calculus for Scientists and Engineers*. Springer Netherlands, 2011.

[54] K. M. Owolabi. “Computational techniques for highly oscillatory and chaotic wave problems with fractional-order operator”. In: *The European Physical Journal Plus* 135.10 (Oct. 2020).

[55] A. Padder, L. Almutairi, S. Qureshi, A. Soomro, A. Afroz, E. Hincal, and A. Tassaddiq. "Dynamical Analysis of Generalized Tumor Model with Caputo Fractional-Order Derivative". In: *Fractal and Fractional* 7.3 (Mar. 2023), p. 258.

[56] A. Pedas, E. Tamme, and M. Vikerpuur. "Spline collocation for fractional weakly singular integro-differential equations". In: *Applied Numerical Mathematics* 110 (Dec. 2016), pp. 204–214.

[57] A. Pedas and E. Tamme. "Piecewise polynomial collocation for linear boundary value problems of fractional differential equations". In: *Journal of Computational and Applied Mathematics* 236.13 (July 2012), pp. 3349–3359.

[58] A. Pedas, E. Tamme, and M. Vikerpuur. "Numerical solution of linear fractional weakly singular integro-differential equations with integral boundary conditions". In: *Applied Numerical Mathematics* 149 (Mar. 2020), pp. 124–140.

[59] A. Pedas, E. Tamme, and M. Vikerpuur. "Smoothing transformation and spline collocation for nonlinear fractional initial and boundary value problems". In: *Journal of Computational and Applied Mathematics* 317 (2017), pp. 1–16.

[60] I. Podlubny. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. Mathematics in science and engineering. London: Academic Press, 1999.

[61] B. Ross. "The development of fractional calculus 1695–1900". In: *Historia Mathematica* 4.1 (Feb. 1977), pp. 75–89.

[62] K. Sakamoto and M. Yamamoto. "Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems". In: *Journal of Mathematical Analysis and Applications* 382.1 (Oct. 2011), pp. 426–447.

[63] S. Samko, A. A. Kilbas, and O. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. London, England: Taylor & Francis, Dec. 1993.

[64] J. Saranen and G. Vainikko. *Periodic Integral and Pseudodifferential Equations with Numerical Approximation*. Springer Berlin Heidelberg, 2002.

[65] H. B. Soots, K. Lätt, and A. Pedas. "Collocation based approximations for a class of fractional boundary value problems". In: *Mathematical Modelling and Analysis* 28.2 (Mar. 2023), pp. 218–236.

[66] K. R. Stromberg. *An introduction to classical real analysis*. AMS Chelsea Publishing. Providence, RI: American Mathematical Society, Dec. 2015.

[67] M. Stynes. "Singularities". In: *Numerical Methods*. De Gruyter, Apr. 2019, pp. 287–306.

[68] M. Stynes. "Too Much Regularity May Force Too Much Uniqueness". In: *Fractional Calculus and Applied Analysis* 19.6 (Dec. 2016), pp. 1554–1562.

[69] M. Stynes, E. O'Riordan, and J. L. Gracia. "Error Analysis of a Finite Difference Method on Graded Meshes for a Time-Fractional Diffusion Equation". In: *SIAM Journal on Numerical Analysis* 55.2 (Jan. 2017), pp. 1057–1079.

[70] H. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Chen. "A new collection of real world applications of fractional calculus in science and engineering". In: *Communications in Nonlinear Science and Numerical Simulation* 64 (Nov. 2018), pp. 213–231.

[71] V. E. Tarasov. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer Berlin Heidelberg, 2010.

[72] G. Vainikko. "Cordial Volterra Integral Equations 1". In: *Numerical Functional Analysis and Optimization* 30.9–10 (Dec. 2009), pp. 1145–1172.

[73] G. Vainikko. "Cordial Volterra Integral Equations 2". In: *Numerical Functional Analysis and Optimization* 31.2 (Mar. 2010), pp. 191–219.

[74] G. Vainikko. *Multidimensional Weakly Singular Integral Equations*. Springer-Verlag, Berlin, Heidelberg, 1993.

[75] G. Vainikko. "Spline Collocation for Cordial Volterra Integral Equations". In: *Numerical Functional Analysis and Optimization* 31.3 (Apr. 2010), pp. 313–338.

[76] G. Vainikko. "Which Functions are Fractionally Differentiable?" In: *Zeitschrift für Analysis und ihre Anwendungen* 35.4 (Oct. 2016), pp. 465–487.

[77] M. P. Velasco, D. Usero, S. Jiménez, L. Vázquez, J. L. Vázquez-Poletti, and M. Mortazavi. "About Some Possible Implementations of the Fractional Calculus". In: *Mathematics* 8.6 (June 2020), p. 893.

[78] M. Vikerpuur. "Two Collocation Type Methods for Fractional Differential Equations with Non-Local Boundary Conditions". In: *Mathematical Modelling and Analysis* 22.5 (2017), pp. 654–670.

[79] Z. W. Yang. "Second-Kind Linear Volterra Integral Equations with Noncompact Operators". In: *Numerical Functional Analysis and Optimization* 36.1 (Oct. 2014), pp. 104–131.

[80] M. Zayernouri and G. E. Karniadakis. "Fractional spectral collocation methods for linear and nonlinear variable order FPDEs". In: *Journal of Computational Physics* 293 (July 2015), pp. 312–338.

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SISUKOKKUVÕTE

Kollokatsioonitüüpi meetodid murruliste tuletistega diferentsiaalvõrrandite ligikaudseks lahendamiseks

Kui me räägime funktsiooni $y = f(t)$ tuletistest, siis peame tavaliselt silmas selle funktsiooni täisarvulist järku tuletist: $y' = \frac{d}{dt}f(t)$, $y'' = \frac{d^2}{dt^2}f(t)$ jne. Tekib loomulik küsimus, kas tuletise mõistet on võimalik laiendada nii, et tuletise järk on reaalarv või isegi kompleksarv. Selle küsimuse esitas l'Hospital Leibnizele juba 1695. aastal tundes huvi tähistuse $\frac{d^n}{dt^n}f(t)$ kohta juhul, kui $n = \frac{1}{2}$. Osutub, et tuletise mõiste laiendamine on võimalik ning tuletisi, mille järk ei ole täisarv, on hakatud nimetama murrulist järku tuletisteks või ka lihtsalt murrulisteks tuletisteks.

Murrulist järku tuletisi sisaldavad diferentsiaalvõrrandid on osutunud väga tõhusaks vahendiks just selliste materjalide ja nähtuste kirjeldamisel, mille käitumine sõltub nende varasemast olekust. Näiteks on murrulised tuletised ja neid sisaldavad diferentsiaalvõrrandid leidnud rakendust majanduses, meditsiinis, mehaanikas ja kaootiliste süsteemide uurimisel. Murrulisi tuletisi sisaldavate diferentsiaalvõrrandite täpse lahendi leidmine ei ole reeglina võimalik ning seetõttu on nende ligikaudne lahendamine väga aktuaalne uurimissuund. Osutub, et murruliste tuletistega diferentsiaalvõrrandite lahendamine on märgatavalt keerulisem ülesanne kui täisarvuliste tuletistega diferentsiaalvõrrandite lahendamine. Põhiline raskus siin on seotud murruliste diferentsiaalvõrrandite lahendite võimaliku singulaarse käitumisega.

Käesolevas väitekirjas uuritakse erinevat tüüpi murrulisi tuletisi sisaldavate diferentsiaalvõrrandite lahendite olemasolu, ühesust ja siledust ning konstrueeritakse kõrget järku täpsusega ligikaudsed meetodid, mis arvestavad lahendi võimalikku singulaarset käitumist. Käsitletakse kolme erinevat tüüpi murruliste tuletistega diferentsiaalvõrrandeid järgmiste sammude abil.

- Ülesande ümberformuleerimine integraalvõrrandina.
- Lahendi olemasolu, ühesuse ja sileduse uurimine.
- Lähismeetodi konstrueerimine tükiti polünomiaalse kollokatsioonimeetodi abil.
- Vaadeldava meetodi koondumise ja koondumiskiiruse uurimine.
- Tulemuste illustreerimine numbriliste näidetega.

Doktritöö koosneb viiest peatükist. Peatükis 1 antakse ülevaade doktoritööst. Peatükis 2 esitatakse rida mõisteid ja abitulemusi, mida läheb vaja järgmises kolmes peatükis.

Peatükis 3 käsitletakse Caputo murrulise tuletisega nõrgalt singulaarse integrodiferentsiaalvõrrandi rajaülesande (3.1)–(3.3) ligikaudsete lahendite leidmist tuginedes kollokatsioonimeetodile tükiti polünomiaalsete splainide korral. Peatüki põhitulemused on esitatud teoreemidega 3.3, 3.5 ja 3.6.

Peatükis 4 vaadeldakse difusioonivõrrandit sisaldava ülesande (4.4)–(4.7) ligikaudset lahendamist järgmisel viisil. Tuginedes sirgete meetodile, saadakse teatav murruliste tuletistega diferentsiaalvõrrandite süsteem. See süsteem formuleeritakse ümber nõrgalt singulaarsete Volterra integraalvõrrandite süsteemina ning uuritakse vaadeldava süsteemi lahendi olemasolu, ühesust ja siledust. Saadud informatsiooni põhjal konstrueeritakse lähendid esialgse ülesande täpsele lahendile ning uuritakse nende võimalikku viga. Peatüki põhitulemused on esitatud teoreemidega 4.4, 4.6, 4.7 ja 4.8.

Peatükk 5 on pühendatud mittekonstantsete kordajatega singulaarsete murruliste tuletistega integro-diferentsiaalvõrrandite (5.1) uurimisele. Erinevalt peatükkidest 3 ja 4 vaadeldakse selles peatükis murrulise diferentsiaaloperaatori rollis Riemann-Liouville'i integraaloperatori pöördoperaatorit. Uuritakse vaadeldava ülesande lahendi olemasolu, ühesust ja siledust. Ülesande ligikaudse lahendi leidmiseks konstrueeritakse kollokatsioonitüüpi meetod ning uuritakse selle meetodi koondumist ja koondumiskiirust. Peatüki põhitulemused on esitatud teoreemidega 5.13, 5.17 ja 5.18.

Peatükkides 3–5 vaadeldud ligikaudsete meetodite abil saadud lähislahendite veahinnanguid on kontrollitud ulatuslike numbriliste eksperimentidega. Leitud numbrilised tulemused on kooskõlas teoreetiliste hinnangutega.

Doktoritöö tulemused on publitseeritud teadusartiklitena [39, 40, 65] ja ettekantud neljal rahvusvahelisel teaduskonverentsil.

CURRICULUM VITAE

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Conferences attended with a presentation

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Administrative work

2025 –	University of Tartu Institute of Mathematics and Statistics, development advisor
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Teadustegevus

Peamised uurimisvaldkonnad on murrulised diferentsiaalvõrrandid ning nõrgalt singulaarsed integraalvõrrandid.

Ettekannetega konverentsid

2024	*The 27th International Conference Mathematical Modelling and Analysis*, Pärnu, Eesti
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LIST OF ORIGINAL PUBLICATIONS

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I **H.B. Soots**, K. Lätt, and A. Pedas. “Collocation based approximations for a class of fractional boundary value problems”. In: *Mathe- matical Modelling and Analysis* 28.2 (Mar. 2023), pp. 218–236.

II K. Lätt, A. Pedas, **H.B. Soots**, and M. Vikerpuur. “Collocation- Based Approximation for a Time-Fractional Sub-Diffusion Model”. In: *Fractal and Fractional 7.9* (Aug. 2023), p. 657.

III K. Lätt, A. Pedas, and **H.B. Soots**. “Singular fractional integro- differential equations with non-constant coefficients”. *In: Applied Numerical Mathematics* 211 (May 2025), pp. 179–192.

Other published works by the author

I G.W. Scott, M. Costa, M. Ewen, C. Lillie, H. Peterbauer, O. Vet- tori, C. Weiß, J. Warm, A. Valk, M. Karm, **H.B. Soots**, G. Geirs- dóttir, F. Magni, L.S. Agrati, N. Basbas, D. Căprioară, M. Girtu, G.L. Torstensdotter, P. Dalenius, J. Kilborn, B. Yanıkkaya, M. Ko rman, D. Şimşek, and T.C. Savaş. “A pan-European perspective of third space professionals located in learning and teaching centres”. In: *Journal of Learning Development in Higher Education 33* (Jan. 2025).

Popular science articles and opinion pieces by the author

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