DISSERTATIONES
MATHEMATICAE
UNIVERSITATIS
TARTUENSIS
140

ALVIN LEPIK

On Morita equivalence of semigroups

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

140

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
140

ALVIN LEPIK

On Morita equivalence
of semigroups

UNIVERSITY OF TARTU
Press

Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia

Dissertation has been accepted for the commencement of the degree of Doctor of Philosophy (PhD) in mathematics on April 28, 2023 by the Council of the Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu.

Supervisor:

Prof. Valdis Laan
Institute of Mathematics and Statistics
University of Tartu
Tartu, Estonia

Opponents:

Prof. Mark Verus Lawson
Department of Mathematics
Heriot-Watt University
Edinburgh, UK

Prof. Benjamin Steinberg
Mathematics Department
City College of New York
New York, USA

Commencement will take place on June 13, 2023 at 15:30 in Narva mnt 18-1020.

Publication of this dissertation has been granted by the Institute of Mathematics and Statistics, University of Tartu.

ISSN 1024-4212 (print)
ISBN 978-9949-27-225-1 (print)
ISSN 2806-240X (pdf)
ISBN 978-9949-27-226-8 (pdf)

University of Tartu Press
http://www.tyk.ee

Contents

Chapter 1

Introduction

The purpose of this thesis is to study the Morita equivalence of semigroups. Our approach consists in studying Morita equivalence classes of some 'nice' semigroups and obtaining various Morita invariants, i.e, properties that are shared by all members of the same equivalence class. Additionally, we study the lattices of relations of Morita equivalent semigroups. Finally, we dedicate a chapter to the study of perfection for semigroups in order to determine when this property is a Morita invariant.

1.1 Background

Two rings with identity are regarded to be equivalent if the categories of modules over those rings are equivalent. Such a relation was first investigated by Kiiti Morita in the 1950s and is nowadays called Morita equivalence in his honour.

Two semigroups are regarded to be the same if they are isomorphic but it is hopeless to try and characterise all semigroups up to isomorphism – even the characterisation problem of finite simple groups remained open for over a century and was solved only recently. The idea of Morita theory of semigroups is to find a (significantly) weaker equivalence relation than isomorphism on the class of all semigroups and attempt to study the equivalence classes. Morita equivalence of monoids was first independently investigated by Banaschewski [5] and Knauer [25] regarding two monoids to be equivalent if the categories of right acts, satisfying the identity $x1 = x$, are equivalent – similarly to how it was done by Morita in the case of rings with identity.

A useful tool in the study of Morita equivalence is a construction nowadays referred to as a 'Morita context', which is also implicit in Morita's work, and was formally introduced, as we use it now, by Bass [7, 8] for rings with identity under the name ‘set of pre-equivalence data’. The term ‘Morita context’ appeared for it around 1970 in works of Amitsur, Müller and Stenström. It was taken over to the semigroup case by Talwar [47, 48]. While equivalence of categories makes it clear that Morita equivalence is an equivalence relation, equivalence functors are sometimes difficult to work with. It is recently determined that Morita equivalence of factorisable semigroups can be expressed via certain kinds of Morita contexts. Morita contexts have proved useful, for instance, in determining various Morita invariants of semigroups.

Given a semigroup, one obtains a category called the Cauchy completion whose objects are the idempotents of that semigroup. A distinguishing aspect between rings and semigroups is that the Cauchy completions of semigroups are used to describe Morita equivalence of semigroups with local units [38, 30] and finite semigroups [36].

Enlargements of semigroups are certain factorisable semigroups, which again are used to describe Morita equivalence of semigroups with local units [38]. Idempotent rings are the ring theoretic equivalent of factorisable semigroups. For Morita theory of rings, enlargements provide a description of the Morita equivalence of all idempotent rings [51]. It is not known, whether an analogous description exists for all factorisable semigroups.

1.2 Overview of the thesis

This thesis consists of seven chapters. The first chapter is the introduction, where we outline our goals, make some historical remarks regarding Morita theory and give an overview of the thesis.

In Chapter 2 we shall go over some preliminaries which are necessary to understand the material presented throughout the thesis. We specify some subclasses of semigroups and recall various notions pertaining to acts over semigroups and category theoretic terminology. Additionally, we shall remind the reader of tensor products of acts over semigroups.

In Chapter 3 we shall define Morita equivalence of semigroups, Morita contexts connecting two semigroups as well as Cauchy completions and enlargements of semigroups.

Chapter 4 is based on the author’s article [40]. It was already known that two Morita equivalent monoids are enlargements of one another [5, 25]. As a generalisation, we shall show that the enlargements of a monoid are precisely the factorisable semigroups Morita equivalent to that monoid. It is also already known that the strong Morita equivalence class of groups contains completely simple semigroups [26, 38]. We shall show that the enlargements of a given group are precisely Rees matrix semigroups over that group. As a consequence, complete simplicity is a Morita invariant for factorisable semigroups.

Chapter 5 is based on the author's article [41]. We shall study strict local isomorphisms, which, for instance, are used to describe Morita equivalence of semigroups with local units [30]. We show that such morphisms emerge naturally from Morita contexts and give a recipe for constructing such morphisms for a smaller subclass of semigroups. Additionally, a Morita context gives rise to Morita semigroups [48], which we shall use to describe Morita equivalence of firm semigroups. We shall also show that the construction given in [30] can be used to describe Rees matrix semigroups over firm semigroups in terms of Morita semigroups. Finally, we show that some results for dual pairs obtained by Hotzel [17] in the case of monoids also hold for semigroups with weak local units. In particular, a Morita semigroup determined by a dual pair is identified with a semigroup of pairs of adjoint endomorphisms.

Chapter 6 is based on the author's article [42]. Galois connections emerging from Morita contexts connecting two rings have been studied by Kashu [20] and a semigroup analogue was later developed by Paseka [44]. The main results in both authors' articles are obtained for Morita contexts which satisfy an array of extra conditions while imposing no restrictions on the underlying rings or semigroups. We shall make use of some of the mappings defined in [44], however, we prefer to work with the more familiar unitary surjective Morita contexts. To compensate, we shall assume the presence of common weak local units in the semigroups. We shall see that such Morita equivalent semigroups must have isomorphic lattices of compatible relations, weak congruences, tolerances and congruences. A similar result for the lattices of congruences is also established in [31].

Chapter 7 is based on the joint work [29] with supervisor Valdis Laan. Perfect monoids were defined by Isbell [19] who showed that a monoid is left perfect if and only if it satisfies two conditions labelled as (A) and (D). Further descriptions for perfect monoids were provided by Fountain [13]. Since then perfectness of monoids and pomonoids have been described by many authors. We define right perfect semigroups and show that some descriptions known from the monoid case can be transferred to the case of factorisable semigroups. In addition, we show that a factorisable semigroup S is right perfect if and only if all right sequence acts over S are projective in the category of all unitary right S-acts. We conclude that the subclass of perfect semigroups contains all nilpotent semigroups, all completely (0-)simple semigroups and that perfectness is a Morita invariant for factorisable semigroups.

Chapter 2

Preliminaries

2.1 Semigroups, acts and categories

Let S be a semigroup and $U, V \subseteq S$. We denote

$$UV := \{uv \in S \mid u \in U,\ v \in V\}.$$

Recall that an element $e \in S$ is called an **idempotent** if $e^2 = e$. The subset of idempotents of S is denoted by $E(S)$. It is well known that every nonempty finite semigroup contains at least one idempotent. We shall often be considering the following subclasses of semigroups.

Definition 2.1. A semigroup S is said to

- be **factorisable** if $S^2 := SS = S$;
- have **weak local units** if for every $s \in S$, there exist $u, v \in S$ such that $us = s = sv$;
- have **local units** if for every $s \in S$, there exist idempotents $f, e \in S$ such that $fs = s = se$.

We recall the notion of an act over a semigroup, which is the semigroup theoretic analogue of module over a ring.

Definition 2.2. Let A be a set and S a semigroup. A map $\cdot : A \times S \longrightarrow A$ such that $(a, s) \mapsto a \cdot s$, is called a **right S-action** on A if

$$(\forall a \in A)(\forall s, s' \in S)((a \cdot s) \cdot s' = a \cdot (ss')).$$

A set A equipped with a right S-action is called a **right S-act** and denoted by A_S. **Left S-acts** ${}_SA$ are defined dually.

Let A_S be a right S-act equipped with a right S-action $\cdot$. Instead of writing $a \cdot s$, we usually omit the symbol $\cdot$ and write simply as.

Definition 2.3. Let A_S and B_S be right S-acts. A map $\varphi : A_S \longrightarrow B_S$ is called a **right S-homomorphism** if

$$(\forall a \in A)(\forall s \in S)(\varphi(as) = \varphi(a)s).$$

The category of all right S-acts with the right S-homomorphisms as morphisms is denoted by Act_S. **Left S-homomorphisms** for left S-acts are defined dually and the category of all left S-acts is denoted by ${}_S\mathsf{Act}$.

We shall omit the designation ‘left’ or ‘right’ whenever it is clear from context.

Definition 2.4. Let S and T be semigroups. A set A is called an (S,T)-**biact** if A is both a left S-act and a right T-act and also satisfies the condition

$$(\forall a \in A)(\forall s \in S)(\forall t \in T)((sa)t = s(at)).$$

An (S,T)-biact A is denoted by ${}_SA_T$.

In particular, there is a biact ${}_SS_S$ where the right and left S-action on the set S is defined using the multiplication of the semigroup S.

Definition 2.5. Let ${}_SA_T$ and ${}_SB_T$ be (S,T)-biacts, where S and T are semigroups. A map $\varphi : {}_SA_T \longrightarrow {}_SB_T$ is called an (S,T)-**homomorphism** if φ is both an S-homomorphism and a T-homomorphism. The category of all (S,T)-biacts with the (S,T)-homomorphisms as morphisms is denoted by ${}_S\mathsf{Act}_T$.

We shall call S-homomorphisms and (S,T)-homomorphisms respectively S-morphisms and (S,T)-morphisms for short. If the roles of the semigroups are clear then we sometimes also call them morphisms and biact morphisms respectively.

Definition 2.6. A right S-act A_S is called **unitary** if

$$A = AS := \{as \mid a \in A, s \in S\}.$$

We denote by UAct_S the full subcategory of Act_S, where the objects are all unitary right S-acts. Unitary left acts are defined dually. An (S,T)-biact ${}_SA_T$ is **unitary** if both ${}_SA$ and A_T are unitary.

In other words, unitarity for A_S means that for every $a \in A$, there exist $a' \in A$ and $s \in S$ such that $a = a's$.

Remark 2.7. Note that if S is a monoid with the identity element 1, then an act A_S is unitary if and only if $a1 = a$ for every $a \in A$.

We assume the reader is familiar with the notions category, subcategory and functor. Let $\mathcal{C}$ be a category. We denote the class of objects of $\mathcal{C}$ by $\mathcal{C}_0$ and denote the set of morphisms from A to B by $\mathcal{C}(A, B)$, where $A, B \in \mathcal{C}_0$. Compositions shall be denoted from right to left. So if $\varphi \in \mathcal{C}(A, B)$ and $\psi \in \mathcal{C}(B, C)$, then $\psi\varphi \in \mathcal{C}(A, C)$. We regard the following well known result as a definition.

Definition 2.8. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called an **equivalence functor** if F is fully faithful and essentially surjective. Two categories are called **equivalent** if there exists an equivalence functor between them. An equivalence functor is an **isomorphism** of categories if it also induces a one-to-one correspondence between the classes of objects.

We also recall the notion of skeleton [1].

Definition 2.9. Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a **skeleton** of $\mathcal{C}$ if

$$(\forall A \in \mathcal{C}_0)(\exists A' \in \mathcal{C}'_0)(A \cong A')$$

and any two objects of $\mathcal{C}'$ are isomorphic only if they are equal.

It is easy to see that the canonical embedding $\mathcal{C}' \longrightarrow \mathcal{C}$, where $\mathcal{C}'$ is a skeleton of a category $\mathcal{C}$, is an equivalence functor. The Axiom of Choice implies that every category has a skeleton. Any two skeleta of the same category must be isomorphic (cf. Proposition I.4.14 in [1]). We shall see that for factorisable semigroups, the equivalence of their Cauchy completions is necessary for the Morita equivalence of the semigroups. In some cases it is easy to see that the Cauchy completions cannot have isomorphic skeleta and therefore cannot be equivalent.

2.2 Tensor products

We recall the notion of tensor product of acts. This section is based on Chapter II Section 5 in [24] but the notions and results are adopted to the semigroup case. Let ${}_SP \in {}_S\mathsf{Act}$ and $Q_S \in \mathsf{Act}_S$, where S is a semigroup.

Definition 2.10. A mapping $\varphi : Q_S \times {}_SP \longrightarrow C$, where C is a set, is called S-**balanced** if $\varphi(qs, p) = \varphi(q, sp)$ for all $p \in P, q \in Q$ and $s \in S$.

Definition 2.11. Let A be a set and $\tau : Q_S \times {}_SP \longrightarrow T$ an S-balanced mapping. The pair (T, τ) is called a **tensor product** of Q_S and ${}_SP$ if for any set C and S-balanced mapping $\hat{\varphi} : Q_S \times {}_SP \longrightarrow C$ there exists a unique mapping $\varphi : T \longrightarrow C$ such that the diagram

$$\begin{tikzcd}
 & T \arrow[dr, "\varphi"] & \\
Q_S \times {}_SP \arrow[ur, "\tau"] \arrow[rr, "\hat{\varphi}"] & & C
\end{tikzcd}$$

commutes.

The map τ in the above definition is sometimes also called a **universal morphism** because it satisfies the specified **universal property**. Tensor products of acts are unique in the following sense.

Proposition 2.12. *Let (T, τ) and (T', τ') be tensor products of ${}_SP \in {}_S\mathsf{Act}$ and $Q_S \in \mathsf{Act}_S$. Then T and T' are isomorphic as sets.*

As in the case of modules over rings, there is a canonical way to construct the tensor product of two acts. Let ϑ be the equivalence relation on the direct product $Q_S \times {}_SP$ generated by the subset

$$\{((qs, p), (q, sp)) \mid p \in P, q \in Q, s \in S\}.$$

Denote the quotient set $(Q \times P)/\vartheta$ by $Q \otimes_S P$ and the equivalence class of (q, p) by $q \otimes p$, where $p \in P$ and $q \in Q$. So,

$$Q \otimes_S P = \{q \otimes p \mid p \in P, q \in Q\}.$$

In particular, $qs \otimes p = q \otimes sp$ holds for every $p \in P, q \in Q$ and $s \in S$. We denote the canonical projection onto the quotient by $\otimes_S$. It maps a pair (q, p) to $q \otimes p$. In case of acts over semigroups, there is a convenient way to determine, whether $q \otimes p = q' \otimes p'$ in $Q \otimes_S P$. We denote by S^1 the monoid obtained from S by adjoining the external identity 1. If Q_S is an S-act and $q \in Q$, then $q1$ will be interpreted as q.

Lemma 2.13. *Let ${}_SP \in {}_S\mathsf{Act}$, $Q_S \in \mathsf{Act}_S$ $p, p' \in P$ and $q, q' \in Q$. Then $q \otimes p = q' \otimes p'$ in $Q \otimes_S P$ if and only if there exist*

$$p_1, \ldots, p_n \in P, \quad q_1, \ldots, q_{n-1} \in Q, \quad \textit{and} \quad r_1, \ldots, r_n, s_1, \ldots, s_n \in S^1$$

such that

$$\begin{array}{lclcrcl} & & & \quad & r_1p_1 & = & p \\ qr_1 & = & q_1s_1 & & r_2p_2 & = & s_1p_1 \\ q_1r_2 & = & q_2s_2 & & r_3p_3 & = & s_2p_2 \\ & \dots & & & & \dots & \\ q_{n-2}r_{n-1} & = & q_{n-1}s_{n-1} & & r_np_n & = & s_{n-1}p_{n-1} \\ q_{n-1}r_n & = & q's_n & & p' & = & s_np_n. \end{array} \tag{2.1}$$

The scheme presented in (2.1) is called an S-**tossing** connecting the pairs (q,p) and (q',p') in $Q \otimes P$.

Theorem 2.14. *Let* ${}_SP \in {}_S\mathsf{Act}$ *and* $Q_S \in \mathsf{Act}_S$. *Then* $Q \otimes_S P$ *together with the canonical projection* $\otimes_S$ *is a tensor product of* ${}_SP$ *and* Q_S.

We shall refer to $Q \otimes_S P$ as the tensor product of Q_S and ${}_SP$. If the role of the semigroup S is clear, then we shall omit the subscript and write simply $Q \otimes P$ and denote the canonical projection by the symbol $\otimes$.

Maps defined on quotient sets must be verified to be well defined in the sense that images do not depend on the choice of representatives of the equivalence classes. In the case of tensor products of acts over semigroups, this verification can be achieved with the help of Lemma 2.13. In the case of modules over rings, however, no such analogous criterion is known. The universal property described before lends us the more common way of checking, whether a map defined on a tensor product is well defined.

Proposition 2.15. *Let* Q_S, ${}_SP$ *be* S*-acts and* $\hat{\varphi} : Q \times P \longrightarrow C$ *be an* S*-balanced map. Then there exists a unique map* $\varphi : Q \otimes_S P \longrightarrow C$ *such that* $\hat{\varphi} = \varphi \circ \otimes$, *i.e,* $\varphi(q \otimes p) = \hat{\varphi}(q,p)$ *for all* $q \in Q$ *and* $p \in P$.

In particular, we can form tensor products of morphisms.

Corollary 2.16. *Let* $\varphi : A_S \longrightarrow C_S$ *and* $\psi : {}_SB \longrightarrow {}_SD$ *be* S*-morphisms. Then the map*

$$\varphi \otimes \psi : A \otimes_S B \longrightarrow C \otimes_S D, \quad (\varphi \otimes \psi)(a \otimes b) := \varphi(a) \otimes \psi(b),$$

is well defined.

If ${}_SP \in {}_S\mathsf{Act}$ and ${}_TQ_S \in {}_T\mathsf{Act}_S$, then the tensor product $Q \otimes P$ can be viewed as a left T-act, defining $t(q \otimes p) := tq \otimes p$ whenever $p \in P, q \in Q$ and $t \in T$. The tensor product $Q \otimes P$ of biacts ${}_SP_W \in {}_S\mathsf{Act}_W$, where W is a semigroup, and ${}_TQ_S \in {}_T\mathsf{Act}_S$ can be regarded as a (T,W)-biact, where

$$t(q \otimes p) := tq \otimes p \quad \text{and} \quad (q \otimes p)w := q \otimes pw$$

for every $p \in P, q \in Q, t \in T$ and $w \in W$.

Chapter 3

Morita equivalence of semigroups

Two rings are Morita equivalent if the categories of modules over those rings are equivalent. For semigroups the idea of such category based equivalence was first independently explored by Banaschewski [5] and Knauer [25], who regarded two monoids to be Morita equivalent if the categories of acts over those monoids are equivalent. Banaschewski [5] also showed that if the categories Act_S and Act_T for semigroups S and T are equivalent, then the semigroups S and T are isomorphic. Thus, to have a meaningful notion of Morita equivalence for semigroups in terms of equivalence of categories of acts, these categories should be restricted. One possible approach is to consider the subcategories of 'fixed acts', which was done by Talwar [47]. Such acts, which are defined in a rather complicated manner, were later shown to coincide with 'firm' acts in the case of semigroups with local units [38] and factorisable semigroups [35].

Definition 3.1. A right S-act P_S is called **firm** if the map

$$\mu_P : P \otimes_S S \longrightarrow P, \quad p \otimes s \mapsto ps,$$

is bijective. A semigroup S is called **firm** if S_S is firm as an S-act, that is, the map

$$\mu_S : S \otimes_S S \longrightarrow S, \quad s \otimes s' \mapsto ss',$$

is bijective. We denote by FAct_S the full subcategory of Act_S where the objects are all firm acts.

Remark 3.2. An S-act P_S is unitary if and only if μ_P is surjective. Thus, all firm acts are unitary and FAct_S is a full subcategory of UAct_S.

The term ‘firm’ comes from ring theory [45]. Lawson [38] uses the term ‘closed’ for the same notion. For semigroups the following implications hold:

$$\text{monoid} \Rightarrow \text{local units} \Rightarrow \text{weak local units} \overset{(*)}{\Rightarrow} \text{firm} \Rightarrow \text{factorisable}.$$

The implication $(*)$ follows from Proposition 2.4 in [33]. The following definition of Morita equivalence was first used by Lawson [38].

Definition 3.3. Semigroups S and T are called **Morita equivalent** if the categories FAct_S and FAct_T are equivalent.

Remark 3.4. Let S be a monoid with identity 1 and P_S a unitary S-act. Then the equality $ps = p's'$ in P implies that

$$ps \otimes 1 = p's' \otimes 1 \Leftrightarrow p \otimes s1 = p' \otimes s'1 \Leftrightarrow p \otimes s = p' \otimes s'$$

in the tensor product $P \otimes_S S$. So μ_P is injective. By Remark 3.2, μ_P is surjective. Thus, unitary acts over monoids are firm as acts over semigroups. In categorical terms, $\mathsf{UAct}_S = \mathsf{FAct}_S$ for every monoid S.

Hence, two monoids are Morita equivalent as semigroups if and only if they are Morita equivalent in the sense of [5] and [25].

3.1 Cauchy completions and Morita contexts

In the case of semigroups, Morita equivalence has ties to the category equivalence of the Cauchy completions (also known as Karoubi envelopes) of the semigroups.

Definition 3.5. Let S be a semigroup. The **Cauchy completion** of S is the category $C(S)$ such that $C(S)_0 = E(S)$, the morphism sets are given by

$$C(S)(e,f) := \{(f,s,e) \in \{f\} \times S \times \{e\} \mid fse = s\}$$

for all idempotents $e, f \in E(S)$ and composition is defined with the equality

$$(g,s',f)(f,s,e) := (g,s's,e).$$

It is useful to know when two objects of a Cauchy completion are isomorphic. We recall some of the well known Green’s relations.

Definition 3.6. Elements $s, s' \in S$ are said to be $\mathcal{R}$-related ($\mathcal{L}$-related) if $sS^1 = s'S^1$ ($S^1s = S^1s'$). Elements $s, s' \in S$ are said to be $\mathcal{D}$-related if there exists $u \in S$ such that $s \,\mathcal{R}\, u \,\mathcal{L}\, s'$.

The following lemma is well known.

Lemma 3.7. *Let $e, f \in S$ be two objects of the category $C(S)$. Then e and f are isomorphic if and only if $e \mathcal{D} f$ in S.*

Remark 3.8. Two idempotents $e, f \in S$ are $\mathcal{D}$-related if and only if there exist $x, y \in S$ such that $xy = e$ and $yx = f$ (cf. Theorem 2.3.4 in [18]).

Example 3.9. The Cauchy completion of a nonempty semigroup with no idempotents is the empty category. Such a semigroup is necessarily infinite.

1. The Cauchy completion of the trivial group contains precisely one object and the identity morphism on it.
2. The Cauchy completion of $(\mathbb{Z}, \cdot)$ contains two nonisomorphic objects 0 and 1 with the morphisms as follows

$$\begin{tikzcd} 0 \arrow[loop left, "{(0,0,0)}"] \arrow[r, bend left, "{(1,0,0)}"] & 1 \arrow[loop right, "{(1,k,1),\ k \in \mathbb{Z}}"] \arrow[l, bend left, "{(0,0,1)}"] \end{tikzcd}$$

   The objects cannot be isomorphic because $\mathbb{Z}$ contains no zero divisors. Alternatively, one may note that

$$(1,0,0)(0,0,1) = (1,0,1) \neq (1,1,1),$$

   hence neither morphism between 0 and 1 can be an isomorphism.

3. Let S be a rectangular band, i.e, S satisfies the identity $xyx = x$. Given $e, f \in S$, we have the equalities

$$(f,x,e)(e,y,f) = (f,xy,f) = (f,fxyf,f) = (f,f,f)$$

   for all morphisms between e and f. Thus, all morphisms in $C(S)$ are isomorphisms. In particular, the endomorphisms are precisely the identities. Thus, every skeleton of $C(S)$ is isomorphic to the Cauchy completion of the trivial group.

Remark 3.10. It is known that factorisable semigroups Morita equivalent to the trivial group are precisely rectangular bands [26]. A finite n-element semigroup S is Morita equivalent to the trivial group if and only if the subsemigroup S^n of all products of length n is a rectangular band [36].

Now we recall the notion of Morita context, which is one of the main tools in this thesis.

Definition 3.11. A **Morita context** connecting semigroups S and T is a six-tuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, where ${}_SP_T$ is an (S,T)-biact, ${}_TQ_S$ is a (T,S)-biact and

$$\theta : {}_S(P \otimes_T Q)_S \longrightarrow {}_SS_S \quad \text{and} \quad \phi : {}_T(Q \otimes_S P)_T \longrightarrow {}_TT_T$$

are biact morphisms satisfying the identities

$$\theta(p \otimes q)p' = p\phi(q \otimes p') \quad \text{and} \quad q'\theta(p \otimes q) = \phi(q' \otimes p)q.$$

A Morita context is called

(1) **unitary** if the biacts are unitary;

(2) **surjective** if the biact morphisms are surjective;

(3) **bijective** if the biact morphisms are bijective.

Example 3.12. Let S be a rectangular band. The equality $S = SeS$ implies that S is an enlargement, in the sense of [39], of the singleton subsemigroup $eSe = \{e\}$. By the observation on p. 192 in [39], we have a unitary surjective Morita context connecting S and $\{e\}$, where ${}_SP_{\{e\}} := Se$, ${}_{\{e\}}Q_S := eS$ and the mappings $\theta : Se \otimes_{\{e\}} eS \longrightarrow S$ and $\phi : eS \otimes_S Se \longrightarrow \{e\}$ are defined by

$$\theta(se \otimes es') := ses' \quad \text{and} \quad \phi(es \otimes s'e) := e(ss')e = e.$$

Semigroups that are connected by a unitary surjective Morita context are called **strongly Morita equivalent** [48]. Strongly Morita equivalent semigroups are necessarily factorisable [30]. It is reasonable to expect there exist semigroups that are Morita equivalent but not strongly Morita equivalent.

Example 3.13. This is Example 3.7 in [32]. The trivial group is Morita equivalent to any zero semigroup by Proposition 3.5 in [32]. These are semigroups with zero in which all products are zero. On the other hand, such semigroups containing at least two elements are not factorisable and therefore they cannot be strongly Morita equivalent to any semigroup.

A natural question is whether Morita equivalence and strong Morita equivalence coincide on the class of factorisable semigroups. This question is answered in the affirmative in the case of semigroups with local units in [38] and firm semigroups in [33]. The following recent result permits us to work with unitary surjective Morita contexts instead of categories of acts.

Theorem 3.14 (Theorem 4.11 in [35]). *Two factorisable semigroups are Morita equivalent if and only if they are connected by a unitary surjective Morita context.*

By Theorem 1.1 in [38] it follows that if two semigroups with local units are strongly Morita equivalent then their Cauchy completions are equivalent categories. This is relaxed further in [30], requiring that the semigroups be strongly Morita equivalent. The next theorem shows that this implication is true under weaker assumptions. In particular, it is true when the Morita context is acceptable in the sense of [27].

Theorem 3.15. *Let S and T be semigroups that are connected by a Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ such that $E(S) \subseteq \operatorname{im}(\theta)$ and $E(T) \subseteq \operatorname{im}(\phi)$. Then the categories $C(S)$ and $C(T)$ are equivalent.*

Proof. The core of this proof is due to Laan [26]. We only require preimages for idempotents of the semigroups. Let us define an equivalence functor $F : C(S) \longrightarrow C(T)$. For every $e \in E(S)$ choose $p_e \in P, q_e \in Q$ such that $e = \theta(p_e \otimes q_e)$. Put

$$u_e := \phi(q_e \otimes ep_e).$$

Then

$$\begin{aligned} u_e^2 = \phi(q_e \otimes ep_e)\phi(q_e \otimes ep_e) &= \phi\left(q_e \otimes ep_e\phi(q_e \otimes ep_e)\right) \\ &= \phi\left(q_e \otimes e\theta(p_e \otimes q_e)ep_e\right) \\ &= \phi(q_e \otimes ep_e) \\ &= u_e. \end{aligned}$$

Define $F(e) := u_e$. If $(f, s, e) : e \longrightarrow f$ is a morphism in $C(S)$ then we define

$$F(f, s, e) := (u_f, \phi(q_f \otimes sp_e), u_e).$$

We have the equalities

$$\begin{aligned} u_f\phi(q_f \otimes sp_e) = \phi(q_f \otimes fp_f)\phi(q_f \otimes sp_e) &= \phi\left(q_f \otimes fp_f\phi(q_f \otimes sp_e)\right) \\ &= \phi\left(q_f \otimes f\theta(p_f \otimes q_f)sp_e\right) \\ &= \phi(q_f \otimes fsp_e) \\ &= \phi(q_f \otimes sp_e). \end{aligned}$$

Similarly, the equality $\phi(q_f \otimes sp_e)u_e = \phi(q_f \otimes sp_e)$ holds. Thus, F is well defined. We show F is functorial. Firstly, F preserves identity morphisms due to

$$F(\mathrm{id}_e) = F(e, e, e) = (u_e, \phi(q_e \otimes ep_e), u_e) = (u_e, u_e, u_e) = \mathrm{id}_{F(e)}$$

for every idempotent $e \in S$. Let $e, f, g \in E(S)$ and $s, s' \in S$ such that $fse = s$ and $gs'f = s'$. The equalities

$$\begin{aligned}\phi(q_g \otimes s'p_f)\phi(q_f \otimes sp_e) &= \phi(q_g \otimes s'p_f\phi(q_f \otimes sp_e)) \\ &= \phi(q_g \otimes s'fsp_e) \\ &= \phi(q_g \otimes s'sp_e)\end{aligned}$$

show that F respects morphism composition.

Next, fix idempotents $u_f, u_e \in E(T)$, where $e, f \in E(S)$. We show F is full. Let (u_f, t, u_e) be a morphism in $C(T)$. Put

$$s := \theta(p_f \otimes tq_e).$$

Then (f, fse, e) is a morphism in $C(S)$ and

$$\begin{aligned}F(f, fse, e) &= (u_f, \phi(q_f \otimes \theta(p_f \otimes tq_e)p_e), u_e) \\ &= (u_f, \phi(q_f \otimes p_f t\phi(q_e \otimes p_e)), u_e) \\ &= (u_f, u_f t u_e, u_e) \\ &= (u_f, t, u_e).\end{aligned}$$

We show F is faithful. Let $\phi(q_f \otimes sp_e) = \phi(q_f \otimes s'p_e)$ for some morphisms (f, s, e) and (f, s', e) in $C(S)$. It suffices to show $s = s'$. The following holds

$$\begin{aligned}& & \phi(q_f \otimes sp_e) &= \phi(q_f \otimes s'p_e) \\ &\Rightarrow & p_f\phi(q_f \otimes sp_e) &= p_f\phi(q_f \otimes s'p_e) \\ &\Rightarrow & fsp_e &= fs'p_e \\ &\Rightarrow & sp_e &= s'p_e \\ &\Rightarrow & sp_e \otimes q_e &= s'p_e \otimes q_e \\ &\Rightarrow & s\theta(p_e \otimes q_e) &= s'\theta(p_e \otimes q_e) \\ &\Rightarrow & se &= s'e \\ &\Rightarrow & s &= s'.\end{aligned}$$

Finally, we show that F is essentially surjective. Let $u \in E(T)$. We have $\phi(q \otimes p) = u$ for some $q \in Q$ and $p \in P$. We have the equalities

$$\begin{aligned}\theta(p \otimes uq)^2 = \theta\left(p \otimes uq\theta(p \otimes uq)\right) &= \theta(p \otimes u\phi(q \otimes p)uq) \\ &= \theta(p \otimes u^3 q) \\ &= \theta(p \otimes uq),\end{aligned}$$

which imply that $e := \theta(p \otimes uq) \in S$ is an idempotent. Note also that $\phi(q \otimes pu) = u^2 = u$ and

$$\phi(q \otimes ep) = \phi(q \otimes \theta(p \otimes uq)p) = \phi(q \otimes pu\phi(q \otimes p)) = \phi(q \otimes pu^2) = u.$$

Recall that we also fixed elements $p_e \in P$ and $q_e \in Q$ such that $e = \theta(p_e \otimes q_e)$. Consider the elements

$$x := \phi(q \otimes p_e) \quad \text{and} \quad y := \phi(q_e \otimes ep_e)\phi(q_e \otimes p)u.$$

We have the equalities

$$\begin{aligned} xy &= \phi(q \otimes p_e)\phi(q_e \otimes ep_e)\phi(q_e \otimes p)u \\ &= \phi(q \otimes p_e\phi(q_e \otimes ep_e)\phi(q_e \otimes p))u \\ &= \phi(q \otimes \theta(p_e \otimes q_e)ep_e\phi(q_e \otimes p))u \\ &= \phi(q \otimes e^2\theta(p_e \otimes q_e)p)u \\ &= \phi(q \otimes e^3p)u \\ &= \phi(q \otimes ep)u \\ &= u^2 \\ &= u \end{aligned}$$

as well as

$$\begin{aligned} yx &= \phi(q_e \otimes ep_e)\phi(q_e \otimes p)u\phi(q \otimes p_e) \\ &= \phi(q_e \otimes ep_e\phi(q_e \otimes p))u\phi(q \otimes p_e) \\ &= \phi(q_e \otimes e\theta(p_e \otimes q_e)pu\phi(q \otimes p_e)) \\ &= \phi(q_e \otimes e^2\theta(pu \otimes q)p_e) \\ &= \phi(q_e \otimes e^3p_e) \\ &= \phi(q_e \otimes ep_e) \\ &= F(e). \end{aligned}$$

By Lemma 3.7 and Remark 3.8 we have that $F(e)$ and u are isomorphic objects in $C(T)$. Alternatively, one can directly verify that the morphisms

$$(u, u\phi(q \otimes p_e), F(e)) \quad \text{and} \quad (F(e), \phi(q_e \otimes p)u, u)$$

are mutually inverse isomorphisms in $C(T)$. ■

Corollary 3.16. *Having at least one idempotent is a Morita invariant for factorisable semigroups.*

Sometimes the equivalence of Cauchy completions is equivalent to Morita equivalence.

Definition 3.17. A semigroup S is called a **sandwich semigroup** if $S = SES$, where $E = E(S)$.

In addition to semigroups with local units, all factorisable finite semigroups are sandwich semigroups (cf. Corollary 5.6 in [35]).

Theorem 3.18. *Let S and T be semigroups. Then Morita equivalence of S and T is equivalent to the equivalence of the categories $C(S)$ and $C(T)$ whenever any of the following conditions holds:*

(1) S and T have right local units (cf. Corollary 2.12 in [15]);

(2) S and T are sandwich semigroups (cf. Corollary 2.13 in [15]);

(3) S and T are finite (cf. Theorem 4.3 in [36]).

Since arbitrary semigroups need not contain idempotents, it is unlikely that their Morita equivalence can be determined by their Cauchy completions alone. Nevertheless, we are not aware of any counterexamples at this time.

Problem 3.19. Do there exist two (factorisable) semigroups with equivalent Cauchy completions such that the semigroups are not Morita equivalent? It can be seen from Theorem 3.18 that examples of such semigroups, assuming they exist, might not be easy to come up with.

3.2 Enlargements of semigroups

Definition 3.20. A semigroup T is called an **enlargement** of its subsemigroup S if $T = TST$ and $S = STS$. Additionally, T is called an **enlargement** of any semigroup isomorphic to such a subsemigroup $S \subseteq T$.

It is clear that enlargements are factorisable semigroups. Similarly to Proposition 4.2 in [51], we can deduce some basic properties of enlargements of semigroups.

Proposition 3.21. *Let T be an enlargement of its subsemigroup S. If either T is commutative or S is an ideal of T, then $S = T$.*

Proof. If T is commutative, then we have the equalities

$$T = TST = T(STS)T = S(TTT)S = STS = S.$$

If S is an ideal, we immediately obtain $T = TST \subseteq S$ by definition of ideal.■

Corollary 3.22. *A commutative semigroup T is an enlargement of a semigroup S only if $S \cong T$. In particular, S must be commutative.*

Remark 3.23. By Proposition 4.2 in [51], the one element ring has precisely one enlargement in the sense of rings – itself. In the case of semigroups, however, it is readily verified that any rectangular band is an enlargement of the trivial group.

Sandwich semigroups provide natural examples of enlargements.

Example 3.24. Let S be a sandwich semigroup. Then it is easy to see that S is an enlargement of its subsemigroup ESE. This subsemigroup has local units.

Enlargements provide a sufficient condition for Morita equivalence of factorisable semigroups.

Lemma 3.25. *Let a semigroup T be an enlargement of its factorisable subsemigroup S. Then T and S are strongly Morita equivalent.*

Proof. This proof is outlined on p. 192 in [39]. Put

$$ {}_SP_T := {}_SST_T \quad \text{and} \quad {}_TQ_S := {}_TTS_S. $$

Define the mappings $\theta : P \otimes Q \longrightarrow S$ and $\phi : Q \otimes P \longrightarrow T$ with the equalities

$$ \theta(st \otimes t's') := stt's' \in STS = S \quad \text{and} \quad \phi(ts \otimes s't') := tss't'. $$

The corresponding maps $\hat{\theta}$ and $\hat{\phi}$ are respectively T- and S-balanced due to associativity, therefore θ and ϕ are well defined by Proposition 2.15. Surjectivity of θ and ϕ follow from the equalities $S = STTS$ and $T = TSST$. The compatibility of θ and ϕ is clear. ■

Definition 3.26. A semigroup W is called a **joint enlargement** of semigroups S and T if W is an enlargement of both S and T.

By Lemma 3.25 and transitivity, two factorisable semigroups have a joint enlargement only if they are strongly Morita equivalent. It is not known, whether the converse holds for all factorisable semigroups. However, the converse is true by Theorem 1.1 in [38] under the assumption both semigroups contain local units.

Remark 3.27. A ring is called idempotent, if any element can be expressed as a finite sum of products of two elements. By Theorem 4.13 in [51], two idempotent rings are Morita equivalent if and only if they have a joint enlargement in the sense of rings. Furthermore, two factorisable Morita equivalent semigroups can be embedded as semigroups in an idempotent ring, which is a joint enlargement in the sense of rings for the subrings generated by the images of the semigroups (cf. Theorem 4.25 in [51]).

Chapter 4

Semigroups strongly Morita equivalent to monoids

This chapter is based on the author's article [40]. We show that the strong Morita equivalence class of a monoid consists of the enlargements of that monoid and also show that enlargements of a group are precisely all Rees matrix semigroups over that group.

Following ideas of the proof of Theorem 9 in [26], we can let S be a semigroup, but also leave the monoid M fixed.

Theorem 4.1. *Let M be a monoid and S a semigroup. The following are equivalent:*

(1) S and M are strongly Morita equivalent;

(2) there exists $e \in E(S)$ such that $S = SeS$ and $eSe \cong M$ as monoids;

(3) S is an enlargement of M.

PROOF. (1) ⇒ (2). Let $(S, M, {}_SP_M, {}_MQ_S, \theta, \phi)$ be a unitary surjective Morita context. Using surjectivity of ϕ, let $\phi(q_1 \otimes p_1) = 1_M =: 1$, where $q_1 \otimes p_1 \in Q \otimes P$. Put $e := \theta(p_1 \otimes q_1) \in S$. We show that $ee = e$. The M-acts P_M and ${}_MQ$ are unitary, thus for every $p \in P$ and $q \in Q$ we have $p1 = p$ and $1q = q$. Then

$$\begin{aligned} ee = \theta(p_1 \otimes q_1)\theta(p_1 \otimes q_1) &= \theta(p_1 \otimes q_1\theta(p_1 \otimes q_1)) \\ &= \theta(p_1 \otimes \phi(q_1 \otimes p_1)q_1) \\ &= \theta(p_1 \otimes 1q_1) \\ &= \theta(p_1 \otimes q_1) = e. \end{aligned}$$

To prove the equality $SeS = S$ we show that $S \subseteq SeS$. We have $SS = S$ since strong Morita equivalence implies factorisability. Fix $s \in S$ and let $s = s's''$ for some $s', s'' \in S$. Then

$$s' = \theta(p' \otimes q') \qquad \text{and} \qquad s'' = \theta(p'' \otimes q'')$$

for some $p' \otimes q', p'' \otimes q'' \in P \otimes Q$. Similarly to the proof of Theorem 9 in [26], we have

$$\begin{aligned}
s = s's'' &= \theta(p' \otimes q')\theta(p'' \otimes q'') \\
&= \theta\left(p' \otimes q'\theta(p'' \otimes q'')\right) \\
&= \theta\left(p' \otimes \phi(q' \otimes p'')q''\right) \\
&= \theta\left(p' \otimes \phi(q' \otimes p'')1^2q''\right) \\
&= \theta\left(p' \otimes \phi(q' \otimes p'')(\phi(q_1 \otimes p_1))^2q''\right) \\
&= \theta\left(p' \otimes q'\theta(p'' \otimes \phi(q_1 \otimes p_1)\phi(q_1 \otimes p_1)q'')\right) \\
&= \theta\left(p' \otimes q'\theta(p'' \otimes \phi(q_1 \otimes p_1)q_1\theta(p_1 \otimes q''))\right) \\
&= \theta\left(p' \otimes q'\theta(p'' \otimes q_1\theta(p_1 \otimes q_1)\theta(p_1 \otimes q''))\right) \\
&= \theta(p' \otimes q')\theta(p'' \otimes q_1)\theta(p_1 \otimes q_1)\theta(p_1 \otimes q'') \\
&= \theta(p' \otimes q')\theta(p'' \otimes q_1)\, e\, \theta(p_1 \otimes q'') \in SeS.
\end{aligned}$$

Note that for every $s \in S$ we have the equalities

$$\begin{aligned}
q_1 \otimes (ese)p_1 &= q_1 \otimes \theta(p_1 \otimes q_1)s\theta(p_1 \otimes q_1)p_1 \\
&= q_1\theta(p_1 \otimes q_1) \otimes s\theta(p_1 \otimes q_1)p_1 \\
&= \phi(q_1 \otimes p_1)q_1 \otimes sp_1\phi(q_1 \otimes p_1) \\
&= 1q_1 \otimes sp_1 1 \\
&= q_1 \otimes sp_1.
\end{aligned}$$

Thus, the mapping

$$\tau : eSe \longrightarrow M, \quad ese \mapsto \phi(q_1 \otimes sp_1),$$

is well defined. We show that τ is a semigroup morphism. Let $s, s' \in S$, then

$$\begin{aligned}
\tau((ese)(es'e)) = \tau(e(ses')e) &= \phi(q_1 \otimes ses'p_1) \\
&= \phi(q_1 \otimes s\theta(p_1 \otimes q_1)s'p_1) \\
&= \phi(q_1 \otimes sp_1\phi(q_1 \otimes s'p_1)) \\
&= \phi(q_1 \otimes sp_1)\phi(q_1 \otimes s'p_1) \\
&= \tau(ese)\tau(es'e).
\end{aligned}$$

For injectivity, we have, for every $s, s' \in S$,

$$\begin{aligned}
& & \phi(q_1 \otimes sp_1) &= \phi(q_1 \otimes s'p_1) \\
&\Rightarrow & p_1\phi(q_1 \otimes sp_1) &= p_1\phi(q_1 \otimes s'p_1) \\
&\Rightarrow & \theta(p_1 \otimes q_1)sp_1 &= \theta(p_1 \otimes q_1)s'p_1 \\
&\Rightarrow & esp_1 &= es'p_1 \\
&\Rightarrow & esp_1 \otimes q_1 &= es'p_1 \otimes q_1 \\
&\Rightarrow & \theta(esp_1 \otimes q_1) &= \theta(es'p_1 \otimes q_1) \\
&\Rightarrow & es\theta(p_1 \otimes q_1) &= es'\theta(p_1 \otimes q_1) \\
&\Rightarrow & ese &= es'e.
\end{aligned}$$

Finally, we show that τ is surjective. Let $m \in M$. Then $m = \phi(q_m \otimes p_m)$ for some $q_m \otimes p_m \in Q \otimes P$. Put

$$\theta(p_1 \otimes q_m)\theta(p_m \otimes q_1) =: s \in S.$$

Note that $ese = s$. Indeed, we have the equalities

$$\begin{aligned}
ese &= \theta(p_1 \otimes q_1)\theta(p_1 \otimes q_m)\theta(p_m \otimes q_1)\theta(p_1 \otimes q_1) \\
&= \theta(p_1 \otimes q_1\theta(p_1 \otimes q_m))\theta(\theta(p_m \otimes q_1)p_1 \otimes q_1) \\
&= \theta(p_1 \otimes \phi(q_1 \otimes p_1)q_m)\theta(p_m\phi(q_1 \otimes p_1) \otimes q_1) \\
&= \theta(p_1 \otimes 1q_m)\theta(p_m1 \otimes q_1) \\
&= \theta(p_1 \otimes q_m)\theta(p_m \otimes q_1) = s.
\end{aligned}$$

Applying τ, we obtain

$$\begin{aligned}
\tau(s) = \tau(ese) &= \phi(q_1 \otimes sp_1) = \phi(q_1 \otimes \theta(p_1 \otimes q_m)\theta(p_m \otimes q_1)p_1) \\
&= \phi(q_1\theta(p_1 \otimes q_m) \otimes \theta(p_m \otimes q_1)p_1) \\
&= \phi(\phi(q_1 \otimes p_1)q_m \otimes p_m\phi(q_1 \otimes p_1)) \\
&= \phi(1q_m \otimes p_m1) \\
&= \phi(q_m \otimes p_m) = m.
\end{aligned}$$

A semigroup isomorphism also preserves the identity. Thus, the monoids eSe and M are isomorphic.
(2) ⇒ (3). Due to the equality $S = SeS$, we have

$$S(eSe)S = SeS = S \qquad \text{and} \qquad (eSe)S(eSe) = eSe.$$

Therefore, S is an enlargement of $eSe \cong M$.
(3) ⇒ (1). Monoids are factorisable semigroups, hence S and M are strongly Morita equivalent by Lemma 3.25. ■

Corollary 4.2. *Let M be a monoid. Then M is strongly Morita equivalent to a semigroup S if and only if S is an enlargement of M.*

Thus, a monoid is a minimal semigroup in its strong Morita equivalence class with respect to cardinality. One might wonder when strong Morita equivalence degenerates to isomorphism. The following is also known in [25] when S and M are monoids.

Corollary 4.3. *Let a semigroup S and a monoid M be strongly Morita equivalent. If S is either a finite monoid or a commutative semigroup, then S and M are isomorphic.*

Proof. There exist $e \in E(S)$ such that $S = SeS$ and a monoid isomorphism $\psi : M \longrightarrow eSe$. Firstly, let S be a finite monoid. Then M must also be finite. There also exist $e' \in E(M)$ and a monoid isomorphism $\varphi : S \longrightarrow e'Me'$. It is readily verified that the map

$$M \longrightarrow e'Me', \quad m \mapsto \varphi(\psi(m)),$$

is injective. It follows that $e'Me' = M$, because M is finite. Secondly, let S be a commutative semigroup. We have the equalities $SeS = eSS = eS = eSe$. In both cases, S is a monoid isomorphic to M. ■

In particular, we obtain the following result. Recall that the notions of strong Morita equivalence and Morita equivalence are interchangeable in case of factorisable semigroups by Theorem 3.14.

Corollary 4.4. *Two finite monoids are (strongly) Morita equivalent if and only if they are isomorphic.*

Thus, up to isomorphism, every strong Morita equivalence class of a monoid can contain only one finite monoid and only one commutative monoid. If there are finite and commutative monoids in the same equivalence class, then they are necessarily isomorphic. However, the class may contain several infinite monoids. Examples of nonisomorphic infinite Morita equivalent monoids are given in [24] and [25].

We also see that the strong Morita equivalence class of a monoid contains precisely the enlargements of that monoid. Theorem 8 in [26] implies that the enlargements of monoids are sandwich semigroups.

We recall the notion of Rees matrix semigroup.

Definition 4.5. Let S be a semigroup and U, V nonempty sets. Let also $p : V \times U \longrightarrow S$ be a mapping. The set $U \times S \times V$ equipped with the multiplication

$$(u, s, v)(u', s', v') := (u, s\,p(v, u')\,s', v')$$

is called a **Rees matrix semigroup** over S with **sandwich matrix** p, which is often denoted as $\mathcal{M}(S, U, V, p)$.

A monoid M is strongly Morita equivalent to a Rees matrix semigroup $\mathcal{M}(M,U,V,p)$ over M if and only if $M = M\,\mathrm{im}(p)\,M$ (cf. Proposition 2 in [30]). Such Rees matrix semigroups are enlargements of that monoid. A Rees matrix semigroup over a monoid with this property need not have weak local units (cf. p. 443 in [31]).

Problem 4.6. Does there exist a nonfirm factorisable Rees matrix semigroup over a monoid?

Let us turn to groups now. It is well known that **completely simple semigroups** are up to isomorphism Rees matrix semigroups over groups. We make use of the following lemma, whose result is also stated in [26].

Lemma 4.7. *Let H be a group and $S := \mathcal{M}(H,U,V,p)$ a Rees matrix semigroup over H. Then every subsemigroup eSe of S, where e is an idempotent, is isomorphic to H.*

Proof. Denote $e = (u_e, h_e, v_e) \in U \times H \times V$, where $e \in E(S)$. By uniqueness of inverse it holds that $p(v_e, u_e) = h_e^{-1}$. We have

$$H_e := \{(u_e, h, v_e) \mid h \in H\} = eSe.$$

One readily verifies that H_e is a group with the identity e. It is clear that the map $H \longrightarrow H_e,\ h \mapsto (u_e, hh_e, v_e)$, is a group isomorphism. ∎

On the one hand, Talwar has shown that Rees matrix semigroups over G are strongly Morita equivalent to G (see p. 391 in [48]). For the converse, it is known that semigroups with local units Morita equivalent to groups are completely simple by Theorem 5.3 in [38]. By Theorem 12 in [26], the assumption regarding local units can be dropped.

Theorem 4.8. *Let G be a group and S a semigroup. The following are equivalent:*

(1) S and G are strongly Morita equivalent;

(2) S is isomorphic to a Rees matrix semigroup over G;

(3) S is simple and there exists $e \in E(S)$ such that $eSe \cong G$ as groups;

(4) S is an enlargement of G.

Proof. (1) $\Rightarrow$ (2). The semigroup S is completely simple by Theorem 12 in [26], thus S is isomorphic to a Rees matrix semigroup over a group H. By Theorem 4.1, there exists $e \in E(S)$ such that $G \cong eSe$. We have $eSe \cong H$ by Lemma 4.7. Thus, we have $G \cong H$ and S is isomorphic to a Rees matrix semigroup over G.

(2) $\Rightarrow$ (3). Since S is completely simple, it is simple and regular. By Lemma 4.7, $eSe \cong G$ for every $e \in E(S)$.

(3) $\Rightarrow$ (4). The equality $S = SeS$ holds due to simplicity. Thus, S is an enlargement of $eSe \cong G$.

(4) $\Rightarrow$ (1). This follows from Lemma 3.25 because G is factorisable. ■

Theorem 4.8 also shows that complete simplicity is a Morita invariant for factorisable semigroups. More precisely, we have the following.

Corollary 4.9. *Let S be a Rees matrix semigroup over a group G. Then a semigroup T is strongly Morita equivalent to S if and only if T is isomorphic to a Rees matrix semigroup over G.*

Proof. If T is strongly Morita equivalent to S, then by transitivity T is also Morita equivalent to G. Thus, T must be isomorphic to a Rees matrix semigroup over G. Conversely, T is Morita equivalent to G, which in turn is Morita equivalent to S. ■

Recall that a 0-**group** is any group with external zero adjoined. It is well known that a semigroup is **completely** 0-**simple** if and only if it is isomorphic to a Rees matrix semigroup with a regular sandwich matrix over a 0-group. A natural question is whether enlargements of 0-groups are necessarily completely 0-simple. This is not the case.

Example 4.10. Due to Reimaa, we know that the semigroup $S = \{0, 1, 2\}$, defined by

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array},$$

is strongly Morita equivalent to the 0-group $T = \{0, 1\}$ [28, p. 57–58]. The equalities $STS = S$ and $TST = T$ are readily verified. On the other hand, S is not 0-simple because S does contain a zero element. This example also shows that complete 0-simplicity is not a Morita invariant.

Chapter 5

On connections between Morita semigroups and Strong Morita equivalence

This chapter is based on the author's article [41]. We show that Morita contexts provide natural examples of strictly locally injective semigroup morphisms and give a recipe for constructing strict local isomorphisms for a subclass of semigroups, similarly to how it is done in [51]. Additionally, we use Morita semigroups to describe Morita equivalence of firm semigroups and also show that the construction given in [30] also works for identifying Rees matrix semigroups over S with Morita semigroups in the event S is firm. Finally, we show that certain semigroups of adjoint endomorphisms are identified with Morita semigroups determined by dual pairs and use it to deduce a sufficient condition for strong Morita equivalence for semigroups with weak local units, generalising some results in [17] obtained for monoids.

5.1 Strict local isomorphisms

One may assume that a unitary surjective Morita context between two Morita equivalent firm semigroups is bijective (cf. Theorem 5.9 in [33]). In general, the biact morphisms of a unitary surjective Morita context are strict local isomorphisms.

Definition 5.1. We say a semigroup morphism $\varphi : S \longrightarrow T$ is **strictly locally injective** if it is injective on all subsemigroups of the form aSb, where $a \in Sa$ and $b \in bS$. A strictly locally injective semigroup morphism is called a **strict local isomorphism** if it is also surjective. **Idempotents lift along** φ, if for every $f \in E(T)$, there exists $e \in E(S)$ such that $f = \varphi(e)$.

Remark 5.2. Given a strict local isomorphism along which idempotents lift, regular elements also lift by Lemma 3.1 in [39].

Strict local isomorphisms along which idempotents lift appear in a covering theorem by Rees matrix semigroups (cf. Theorem 3.2 in [39]). Such morphisms are also used in [30] to describe strong Morita equivalence of semigroups with local units.

Definition 5.3. A semigroup S is said to have **common weak left local units** if

$$(\forall s_1, s_2 \in S)(\exists u \in S)(us_1 = s_1 \quad \text{and} \quad us_2 = s_2).$$

Common weak right local units are defined dually. A semigroup S is said to have **common weak local units** if S has both common weak left local units and common weak right local units.

Clearly, semigroups with common weak local units have weak local units and are therefore firm.

Example 5.4. Every monoid is a semigroup with common weak local units.

1. Every lattice has common weak local units with respect to both joins and meets.

2. Let $e \in S$ be an idempotent. Then the principal right ideal eS has a left identity e. Hence, eS has common weak left local units.

Lemma 5.5. *Let $\varphi : S \longrightarrow T$ be a semigroup morphism. Assume that S has common weak local units. The following are equivalent:*

(1) φ is strictly locally injective;

(2) $\varphi|_{sS}$ is injective for every $s \in S$;

(3) $\varphi|_{Ss}$ is injective for every $s \in S$.

Proof. (1) $\Rightarrow$ (2). Let φ be strictly locally injective and take $s, s', s'' \in S$ such that $\varphi(ss') = \varphi(ss'')$. By assumption, let $u, v \in S$ be such that $us = s$, $s' = s'v$ and $s'' = s''v$. Then $ss', ss'' \in uSv$, where $u \in Su$ and $v \in vS$. Strict local injectivity implies that $ss' = ss''$. The implication (1) $\Rightarrow$ (3) is proved similarly.

(2) $\Rightarrow$ (1). This implication requires no additional assumptions on S. Let $u, v \in S$ and note that since φ is injective on uS, it must be injective on $uSv \subseteq uS$. The implication (3) $\Rightarrow$ (1) is proved similarly. ■

Remark 5.6. A ring R is called **s**-unital if, for every $s \in R$, there exist $u, v \in R$ such that $us = s = sv$. Tominaga [49] showed that this implies that every finite nonempty subset $F \subseteq R$ admits $u, v \in R$ such that $uf = f = fv$ for every $f \in F$ (cf. Theorem 1). In particular, in our present terminology, a ring has weak local units if and only if it has common weak local units. For semigroups, however, this is false. Rectangular bands containing at least two elements have local units but do not have common weak local units.

For an R-module M_R, where R is a ring (not necessarily with identity) and a morphism $\varphi : M_R \longrightarrow R_R$ of R-modules, the set M can be turned into a ring, where φ then becomes a locally injective morphism of rings and, conversely, every strict local isomorphism $S \longrightarrow R$ is, essentially, an R-valued linear functional (cf. Proposition 3.21 in [51]). A similar idea works in the semigroup case.

Proposition 5.7. *Let S be a semigroup, A_S an S-act and $\rho : A_S \longrightarrow S_S$ an S-morphism. The following statements hold.*

1. *The set A is a semigroup under the multiplication $a \cdot a' := a\rho(a')$. The S-morphism ρ is a strictly locally injective semigroup morphism. If ρ is also surjective, then idempotents lift along ρ.*

2. *If T is a semigroup with common weak local units, then all strict local isomorphisms $T \longrightarrow S$ arise in the manner specified in 1.*

Proof. For the first item, we have associativity due to the equalities

$$(a \cdot a') \cdot a'' = (a\rho(a')) \cdot a'' = a\rho(a')\rho(a'') = a\rho(a'\rho(a'')) = a \cdot (a' \cdot a''),$$

where $a, a', a'' \in A$. It is clear that ρ is a semigroup morphism. We show that ρ is injective on subsemigroups of the form $a \cdot A$, where $a \in A \cdot a$. Let $a = a' \cdot a$ for some $a' \in A$. Take $m, n \in A$ such that $\rho(a \cdot m) = \rho(a \cdot n)$. Then

$$a \cdot m = a' \cdot (a \cdot m) = a'\rho(a \cdot m) = a'\rho(a \cdot n) = a' \cdot (a \cdot n) = a \cdot n.$$

Assume ρ is surjective and let $e = \rho(a)$ be an idempotent. Then

$$a^4 = a \cdot a^3 = a\rho(a^3) = a(\rho(a))^3 = a\rho(a) = a^2 \in E(A)$$

and $\rho(a^2) = (\rho(a))^2 = e$.

Now let $\tau : T \longrightarrow S$ be a strict local isomorphism, where T has common weak local units. We want to define a right S-action on T. Define

$$\star : T \times S \longrightarrow T, \quad t \star s' := tt',$$

where $t, t' \in T$ and $s' \in S$ are such that $\tau(t') = s'$. Suppose $\tau(t'') = s'$ for some $t'' \in T$. Then $\tau(tt') = \tau(t)s' = \tau(tt'')$. By Lemma 5.5, the map $\tau|_{tT}$ is injective, hence $tt' = tt''$. Thus, $\star$ is well defined. Now, take $s', s'' \in S$, $t \in T$ and assume $\tau(t') = s'$ and $\tau(t'') = s''$ for some $t', t'' \in T$. It follows that

$$(t \star s') \star s'' = tt' \star s'' = tt't'' = t \star (s's''),$$

where the last equality holds due to $s's'' = \tau(t't'')$. Thus, we have a right S-action on T. The equalities

$$\tau(t \star s') = \tau(tt') = \tau(t)\tau(t') = \tau(t)s'$$

show that τ is an S-morphism. ■

The following definition is due to Talwar [48]. Recall that given S-acts ${}_SP$ and Q_S, the direct product ${}_SP \times Q_S$ can be viewed as an (S,S)-biact with the actions $s(p,q) = (sp,q)$ and $(p,q)s = (p,qs)$, where $p \in P, q \in Q$ and $s \in S$.

Definition 5.8. Let ${}_SP$ and Q_S be S-acts and let

$$\langle\, , \rangle : {}_SP \times Q_S \longrightarrow {}_SS_S, \quad (p,q) \mapsto \langle\, p, q\, \rangle\,,$$

be an (S,S)-morphism, that is $s\,\langle\, p,q\,\rangle = \langle\, sp, q\,\rangle$ and $\langle\, p,q\,\rangle\, s = \langle\, p, qs\,\rangle$ for every $p \in P, q \in Q$ and $s \in S$. A **Morita semigroup** over S defined by $\langle\, , \rangle$ is the set $Q \otimes_S P$ with multiplication

$$(q \otimes p)(q' \otimes p') := q \otimes \langle\, p, q'\,\rangle\, p'.$$

The Morita semigroup is

(1) **unitary** if P and Q are unitary S-acts;

(2) **surjectively defined** if the map $\langle\, , \rangle$ is surjective.

Example 5.9. Every Morita context gives rise, in a natural way, to two Morita semigroups. Let arbitrary semigroups S and T be connected by a Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then using the biact morphism

$$\langle\, , \rangle : {}_SP \times Q_S \longrightarrow {}_SS_S, \ \ (p,q) \mapsto \theta(p \otimes q),$$

we can turn $Q \otimes_S P$ into a Morita semigroup with multiplication

$$(q \otimes p)(q' \otimes p') = q \otimes \theta(p \otimes q')p' = q \otimes p\phi(q' \otimes p').$$

The equalities

$$\phi((q \otimes p)(q' \otimes p')) = \phi(q \otimes p\phi(q' \otimes p')) = \phi(q \otimes p)\phi(q' \otimes p')$$

yield that ϕ is a semigroup morphism. In a similar way, $P \otimes_T Q$ is a Morita semigroup.

As we are naturally provided with two Morita semigroups, a Morita context also yields two strictly locally injective semigroup morphisms.

Theorem 5.10. *Let arbitrary nonempty semigroups* S *and* T *be connected by a Morita context* $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$*. Then* θ *and* ϕ *are strictly locally injective semigroup morphisms. If* θ *is surjective, then idempotents lift along* θ*. The same holds for* ϕ*.*

Proof. It suffices to prove these statements for θ. By definition, the map $\theta : (P \otimes_T Q)_S \longrightarrow S_S$ is an S-morphism. Additionally, for every $p \otimes q, p' \otimes q' \in P \otimes_T Q$, we have $(p \otimes q)(p' \otimes q') = p \otimes q\theta(p' \otimes q')$. Then θ is strictly locally injective by the first item of Proposition 5.7. Therefore, if θ is also surjective, idempotents lift along θ. ■

Remark 5.11. It is not required that θ is surjective for idempotents to lift along θ. If $\theta(p \otimes q) = e$ for some idempotent $e \in S$, then $\theta(p \otimes qe) = e^2 = e$ and

$$(p \otimes qe)^2 = p \otimes qe\theta(p \otimes qe) = p \otimes qe^3 = p \otimes qe \in E\left(P \otimes_T Q\right).$$

Strictly locally injective semigroup morphisms need not be injective.

Example 5.12. Example 2.3 in [33] yields a nonfirm factorisable semigroup. Let S be nonfirm factorisable. Then $S \otimes_S S$ is firm both as a biact and a semigroup by Theorem 2.6 in [35] and $\mu : S \otimes_{S \otimes S} S \longrightarrow S,\ s \otimes s' \mapsto ss'$, is a biact morphism in the Morita context

$$(S, S \otimes S, {}_SS \otimes S_{S \otimes S}, {}_{S \otimes S}S_S, \mu, \mathrm{id}_{S \otimes S})$$

connecting S and $S \otimes_S S$ (cf. Proposition 4.7 in [35]). The morphism μ is a strict local isomorphism by Theorem 5.10, but it cannot be injective because S is not firm.

Theorem 5.10 also generalises the necessity part of Theorem 13 in [30] from semigroups with local units to factorisable semigroups. It could also be considered as an analogue of Theorem 3 in [30], where Rees matrix semigroups are replaced by Morita semigroups.

Corollary 5.13. *If* S *and* T *are strongly Morita equivalent semigroups then there exists a surjectively defined unitary Morita semigroup* $Q \otimes_S P$ *and a strict local isomorphism from* $Q \otimes_S P$ *onto* T *along which idempotent and regular elements lift.*

The following strengthens considerably Theorem 13 in [30]. Firstly, only firmness is assumed of the semigroups instead of having local units and, secondly, strict local isomorphisms are replaced by isomorphisms. It can also be viewed as a semigroup theoretic analogue of Theorem 2.6 in [3].

Theorem 5.14. *Let S and T be firm semigroups. The following are equivalent:*

(1) S and T are strongly Morita equivalent;

(2) S is isomorphic to a surjectively defined Morita semigroup over T.

Proof. For (1) ⇒ (2), assume that the firm semigroups S and T are strongly Morita equivalent. By Theorem 5.9 in [33], they are connected by a unitary bijective Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then $P \otimes_T Q$ defined by $\phi \circ \otimes : Q \times P \longrightarrow T$ is a unitary surjectively defined Morita semigroup over T. Similarly to Example 5.9, $\theta : P \otimes_T Q \longrightarrow S$ also respects the semigroup structure and is therefore an isomorphism of semigroups.
For (2) ⇒ (1), assume that S is isomorphic to a surjectively defined Morita semigroup $P \otimes_T Q$ over T. By Theorem 5 in [48], the Morita semigroup $P \otimes_T Q$ is strongly Morita equivalent to T. Using transitivity, we conclude that the semigroups S and T are strongly Morita equivalent. Note that only factorisability of S and T is needed for this implication. ■

Hotzel [17] noted that a surjectively defined unitary Morita semigroup over a monoid with free acts is a coordinate-free copy of a Rees matrix semigroup over that monoid. Laan and Márki [30] showed that this is true of semigroups with weak local units (cf. Proposition 10). We make use of their construction to show the following.

Theorem 5.15. *Let S be a factorisable semigroup and $\mathcal{M} := \mathcal{M}(S, U, V, p)$ a Rees matrix semigroup over S. Then there exists a unitary Morita semigroup $Q \otimes_S P$ and a strict local isomorphism from $Q \otimes_S P$ onto $\mathcal{M}$ along which idempotents lift.*

Proof. Put $Q_S := (U \times S)_S$ and ${}_SP := {}_S(S \times V)$, where the S-action on Q is defined by $(u, s)s' := (u, ss')$ and similarly for P. Due to factorisability of S, they are unitary S-acts. Define

$$\langle\, ,\, \rangle : {}_SP \times Q_S \longrightarrow {}_SS_S, \quad \langle (s, v), (u, s') \rangle = s\, p(v, u)\, s'.$$

One readily verifies $\langle\, ,\, \rangle$ is an (S, S)-morphism. Define

$$\psi : Q \otimes_S P \longrightarrow \mathcal{M}, \quad (u, s) \otimes (t, v) \mapsto (u, st, v).$$

Consider the corresponding map $\hat{\psi} : Q \times P \longrightarrow \mathcal{M}$. The equalities

$$\hat{\psi}\,((u,s)s_0,(t,v)) = (u,(ss_0)t,v) = (u,s(s_0t),v) = \hat{\psi}\,((u,s),s_0(t,v))$$

show that $\hat{\psi}$ is S-balanced, therefore ψ is well defined by Proposition 2.15. Surjectivity of ψ follows from factorisability of S.

Since S is factorisable, every element in $\mathcal{M}$ can be written as (u_0, s_0t_0, v_0) for some $u_0 \in U, v_0 \in V$ and $s_0, t_0 \in S$. Define

$$\star : (Q \otimes_S P) \times \mathcal{M} \longrightarrow Q \otimes_S P$$

with the equality

$$\begin{aligned}((u,s)\otimes(t,v))\star(u_0,s_0t_0,v_0) &:= (u,s)\otimes\langle\,(t,v),(u_0,s_0)\,\rangle\,(t_0,v_0)\\ &= (u,s)\otimes tp(v,u_0)s_0(t_0,v_0).\end{aligned}$$

Take $(u_0, s_0t_0, v_0) \in \mathcal{M}$ and denote the multiplication of the Morita semigroup $Q \otimes_S P$ by the symbol $\cdot$. For every $(u,s)\otimes(t,v) \in Q\otimes_S P$ we have the equality

$$((u,s)\otimes(t,v))\star(u_0,s_0t_0,v_0) = ((u,s)\otimes(t,v))\,\cdot\,((u_0,s_0)\otimes(t_0,v_0))\,.$$

Thus, the maps $-\star(u_0,s_0t_0,v_0)$ and $-\cdot((u_0,s_0)\otimes(t_0,v_0))$ coincide on $Q\otimes_S P$. It follows that $\star$ is well defined on $Q\otimes_S P$. Take $(u,s)\otimes(t,v)\in Q\otimes_S P$. It is clear that in the event $s_0t_0 = s_0't_0'$, we have

$$((u,s)\otimes(t,v))\star(u_0,s_0t_0,v_0) = ((u,s)\otimes(t,v))\star(u_0,s_0't_0',v_0).$$

Take also $(u_1,s_1t_1,v_1)\in\mathcal{M}$. We have the equalities

$$\begin{aligned}&(((u,s)\otimes(t,v))\star(u_0,s_0t_0,v_0))\star(u_1,s_1t_1,v_1)\\ &= ((u,s)\otimes(t,v))\,\cdot\,((u_0,s_0)\otimes(t_0,v_0))\,\cdot\,((u_1,s_1)\otimes(t_1,v_1))\\ &= ((u,s)\otimes(t,v))\,\cdot\,((u_0,s_0)\otimes t_0\,p(v_0,u_1)\,(s_1t_1,v_1))\\ &= ((u,s)\otimes(t,v))\star(u_0,s_0t_0\,p(v_0,u_1)s_1t_1,v_1)\\ &= ((u,s)\otimes(t,v))\star((u_0,s_0t_0,v_0)(u_1,s_1t_1,v_1))\,.\end{aligned}$$

Therefore, $\star$ is a right $\mathcal{M}$-action on $Q\otimes_S P$. We also have the equalities

$$\begin{aligned}\psi\,(((u,s)\otimes(t,v))\star(u_0,s_0t_0,v_0)) &= \psi\,((u,s)\otimes t\,p(v,u_0)\,s_0(t_0,v_0))\\ &= (u,st\,p(v,u_0)\,s_0t_0,v_0)\\ &= (u,st,v)(u_0,s_0t_0,v_0)\\ &= \psi((u,s)\otimes(t,v))\,(u_0,s_0t_0,v_0),\end{aligned}$$

which imply that $\psi : (Q \otimes_S P)_{\mathcal{M}} \longrightarrow \mathcal{M}_{\mathcal{M}}$ is an $\mathcal{M}$-morphism. Due to the equality

$$((u,s)\otimes(t,v))\cdot((u_0,s_0)\otimes(t_0,v_0)) = ((u,s)\otimes(t,v))\star\psi((u_0,s_0)\otimes(t_0,v_0))$$

we have by Proposition 5.7 that ψ is a strict local isomorphism along which idempotents lift. ■

Corollary 5.16. *Let S be a factorisable semigroup and $\mathcal{M}$ a Rees matrix semigroup over S. Then $\mathcal{M}$ is a quotient of a unitary Morita semigroup over S.*

It also turns out that the construction given in Proposition 10 of [30] yields an isomorphism if S is firm.

Corollary 5.17. *Let S be a firm semigroup and $\mathcal{M} := \mathcal{M}(S,U,V,p)$ a Rees matrix semigroup over S. Then $\mathcal{M}$ is isomorphic to a unitary Morita semigroup over S. If $S = S\,\mathrm{im}(p)\,S$, then $\mathcal{M}$ is isomorphic to a surjectively defined unitary Morita semigroup over S.*

Proof. Assume the construction given in the proof of Theorem 5.15. It is clear that the map $\langle\, ,\,\rangle$ is surjective if and only if the equality $S = S\mathrm{im}(p)S$ holds. It remains to show that ψ is injective.

Let the equality $(u,st,v) = (u,s't',v)$ hold in $\mathcal{M}$. Then $st = s't'$ if and only if $s\otimes t = s'\otimes t'$ in $S\otimes_S S$ due to firmness of S. By Lemma 2.13, we have an S-tossing

$$\begin{array}{rclrcl}
 & & & r_1y_1 & = & t \\
sr_1 & = & x_1s_1 & r_2y_2 & = & s_1y_1 \\
x_1r_2 & = & x_2s_2 & r_3y_3 & = & s_2y_2 \\
 & \dots & & & \dots & \\
x_{n-2}r_{n-1} & = & x_{n-1}s_{n-1} & r_ny_n & = & s_{n-1}y_{n-1} \\
x_{n-1}r_n & = & s's_n & t' & = & s_ny_n
\end{array}$$

connecting $s\otimes t$ and $s'\otimes t'$, where $x_i, y_i \in S$ and $r_i, s_i \in S^1$. We extend the above to the following S-tossing

$$\begin{array}{rclrcl}
 & & & r_1(y_1,v) & = & (t,v) \\
(u,s)r_1 & = & (u,x_1)s_1 & r_2(y_2,v) & = & s_1(y_1,v) \\
(u,x_1)r_2 & = & (u,x_2)s_2 & r_3(y_3,v) & = & s_2(y_2,v) \\
 & \dots & & & \dots & \\
(u,x_{n-2})r_{n-1} & = & (u,x_{n-1})s_{n-1} & r_n(y_n,v) & = & s_{n-1}(y_{n-1},v) \\
(u,x_{n-1})r_n & = & (u,s')s_n & (t',v) & = & s_n(y_n,v),
\end{array}$$

which again by Lemma 2.13 implies that the equality

$$(u, s) \otimes (t, v) = (u, s') \otimes (t', v)$$

holds in $Q \otimes_S P$. Therefore, ψ is injective. ■

Remark 5.18. For a Rees matrix semigroup $\mathcal{M} := \mathcal{M}(S, U, V, p)$ over a factorisable semigroup S, the condition $S = S\,\mathrm{im}(p)\,S$ is equivalent to $\mathcal{M}$ being factorisable, which, in turn, is equivalent to S being strongly Morita equivalent to $\mathcal{M}$ by Proposition 2 in [30].

A semigroup S is factorisable if and only if $S \otimes S$ is a firm semigroup, where the multiplication is defined by

$$(s \otimes t)(s' \otimes t') := ss' \otimes tt'$$

(cf. [35]). In the same article it is shown that $A \otimes_S B = A \otimes_{S \otimes S} B$ holds for factorisable semigroups S. While we do not know whether Corollary 5.17 holds for factorisable semigroups, we can conclude the following.

Corollary 5.19. *Let S be a factorisable semigroup and $\mathcal{M}(S \otimes S, U, V, p)$ a Rees matrix semigroup over $S \otimes S$. Then $\mathcal{M}$ is isomorphic to a unitary Morita semigroup over S, which may be assumed to be surjectively defined if $\mathcal{M}$ is factorisable.*

5.2 Dual pairs and Morita semigroups

Hotzel [17] considers Morita semigroups over monoids with 0. The acts also must have a fixed zero element. We show that the mapping presented in Theorem 2.4 in [17] is a well-behaved morphism, in general. It also turns out that it is an isomorphism for dual pairs over a semigroup with weak local units.

Definition 5.20. A **pair** over a semigroup S consists of

- a left act ${}_SA$;
- a right act B_S;
- an (S, S)-morphism $\langle\, , \rangle : {}_SA \times B_S \longrightarrow {}_SS_S$.

Denote such a pair by β and write $\beta = ({}_SA, B_S)$.

Hotzel assumes the acts in a pair to be unitary, but we do not need to assume this. Every pair induces a Morita semigroup, which is denoted by $B \otimes_S^\beta A$ and defined by $\langle\, , \,\rangle$. Hotzel did not call these Morita semigroups. We use Talwar's terminology for the construction inspired by Hotzel's (cf. p. 386 in [48]). It is often clear $B \otimes_S^\beta A$ is taken with respect to β, so the superscript β is omitted. With β is associated the following subsemigroup of $\mathsf{End}({}_S A) \times \mathsf{End}(B_S)$:

$$\begin{aligned}\Omega^\beta := \{(\rho, \sigma) \mid \rho \in \mathsf{End}({}_S A),\ \sigma \in \mathsf{End}(B_S),& \\ \langle\, \rho(a), b \,\rangle = \langle\, a, \sigma(b) \,\rangle \text{ for all } a \in A, b \in B\}&\,.\end{aligned}$$

Multiplication on $\mathsf{End}({}_S A) \times \mathsf{End}(B_S)$ is given by the equality

$$(f, g)(f', g') := (f'f, gg').$$

Hotzel refers to such ρ and σ as linked endomorphisms. We call such ρ and σ **adjoint endomorphisms** as does Ánh [3] in the case of modules over rings.

Example 5.21. Every Morita context induces a number of pairs of adjoint endomorphisms. For a given Morita context $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$, we consider the biact morphism

$$\langle\, , \,\rangle : {}_S P \times Q_S \longrightarrow S, \quad (p, q) \mapsto \theta(p \otimes q).$$

For fixed elements $p_0 \in P$ and $q_0 \in Q$

$$\begin{aligned}\rho &:= \theta(- \otimes q_0)p_0 \in \mathsf{End}({}_S P) \text{ and} \\ \sigma &:= q_0\theta(p_0 \otimes -) \in \mathsf{End}(Q_S)\end{aligned}$$

are adjoint due to the equalities

$$\begin{aligned}\langle \rho(p), q \rangle = \theta(\theta(p \otimes q_0)p_0 \otimes q) &= \theta(p\phi(q_0 \otimes p_0) \otimes q) \\ &= \theta(p \otimes \phi(q_0 \otimes p_0)q) \\ &= \theta(p \otimes q_0\theta(p_0 \otimes q)) \\ &= \langle p, \sigma(q) \rangle,\end{aligned}$$

where $p \in P$ and $q \in Q$. A symmetric construction works for ϕ.

We have an ideal of Ω^β:

$$\Omega_1^\beta := \left\{(\rho, \sigma) \in \Omega^\beta \mid (\exists a \in A)(\exists b \in B)(\rho(A) \subseteq Sa \text{ and } \sigma(B) \subseteq bS)\right\}.$$

Elements of Ω_1^β are sometimes called **adjoint endomorphisms of rank one**. We also have another ideal of Ω^β:

$$\Sigma^\beta = \{(\rho, \sigma) \in \Omega^\beta \mid (\exists b \in B)(\exists a \in A)(\rho = \langle\, -, b \,\rangle\, a \text{ and } \sigma = b\,\langle\, a, - \,\rangle)\} \subseteq \Omega_1^\beta.$$

Such pairs are denoted with the symbol $[b,a]$. So,

$$[b,a] = (\langle -, b\rangle a, b\langle a, -\rangle).$$

Note that, for $[b,a],[b',a'] \in \Sigma^{\beta}$, we have

$$\begin{aligned}[b,a][b',a'] &= (\langle -, b\rangle a\,;\, b\langle a,-\rangle)\,(\langle -, b'\rangle a'\,;\, b'\langle a',-\rangle)\\ &= (\langle -, b'\rangle a' \circ \langle -, b\rangle a\,;\, b\langle a,-\rangle \circ b'\langle a',-\rangle)\\ &= (\langle\langle -, b\rangle a, b'\rangle a'\,;\, b\langle a, b'\langle a',-\rangle\rangle)\\ &= (\langle -, b\rangle\langle a, b'\rangle a'\,;\, b\langle a, b'\rangle\langle a,-\rangle)\\ &= (\langle -, b\rangle\langle a, b'\rangle a'\,;\, b\langle\langle a, b'\rangle a',-\rangle).\end{aligned}$$

Hence, it holds that

$$[b,a][b',a'] = [b, \langle a, b'\rangle a']. \tag{3.1}$$

Before we proceed, we will justify the above.

Proposition 5.22. *The following statements hold.*

1. *The subset Ω^{β} is a submonoid of $\mathsf{End}({}_SA) \times \mathsf{End}(B_S)$.*
2. *The subset Ω^{β}_1 is an ideal in Ω^{β}.*
3. *The subset $\Sigma^{\beta} \subseteq \Omega^{\beta}_1$ is an ideal in Ω^{β}.*

Proof. For the first item take $(\rho_1,\sigma_1),(\rho_2,\sigma_2) \in \Omega^{\beta}$ and let $a \in A, b \in B$. Then

$$\langle (\rho_2\rho_1)(a), b\rangle = \langle \rho_1(a), \sigma_2(b)\rangle = \langle a, (\sigma_1\sigma_2)(b)\rangle.$$

Thus, the morphisms $\rho_2\rho_1$ and $\sigma_1\sigma_2$ are adjoint. As id_A and id_B are clearly adjoint, Ω^{β} is a submonoid.

For the second item take $(\rho,\sigma) \in \Omega^{\beta}$ and $(\rho_1,\sigma_1) \in \Omega^{\beta}_1$. There exist $a_1 \in A$ and $b_1 \in B$ such that $\rho_1(A) \subseteq Sa_1$ and $\sigma_1(B) \subseteq b_1S$. Then $(\rho_1\rho)(A) \subseteq \rho_1(A) \subseteq Sa_1$ and $(\sigma\sigma_1)(B) \subseteq \sigma(b_1S) = \sigma(b_1)S$. So $(\rho,\sigma)(\rho_1,\sigma_1) = (\rho_1\rho, \sigma\sigma_1) \in \Omega^{\beta}_1$. Thus Ω^{β}_1 is a left ideal in Ω^{β}. The right ideal case is proved similarly.

For the third item let $(\rho_1,\sigma_1) \in \Sigma^{\beta}$, i.e, $\rho_1 = \langle -, b_1\rangle a_1$ and $\sigma_1 = b_1\langle a_1, -\rangle$ for some $a_1 \in A$ and $b_1 \in B$. The inclusion $(\rho_1,\sigma_1) \in \Omega^{\beta}_1$ is clear. Take $(\rho,\sigma) \in \Omega^{\beta}$ and note that, for every $x \in A$, we have

$$(\rho_1\rho)(x) = \langle \rho(x), b_1\rangle a_1 = \langle x, \sigma(b_1)\rangle a_1$$

and, on the other hand, for every $y \in B$, we have

$$(\sigma\sigma_1)(y) = \sigma\left(b_1 \langle a_1, y \rangle\right) = \sigma(b_1) \langle a_1, y \rangle .$$

Therefore, $\rho_1\rho = \langle -, \sigma(b_1) \rangle a_1$, $\sigma\sigma_1 = \sigma(b_1) \langle a_1, - \rangle$ and $(\rho, \sigma)(\rho_1, \sigma_1) = (\rho_1\rho, \sigma\sigma_1) = [\sigma(b_1), a_1] \in \Sigma^\beta$. Hence Σ^β is a left ideal in Ω^β. The right ideal case is proved similarly. ■

Definition 5.23. A pair $\beta = ({}_SA, B_S)$ is called **dual** if

(1) $(\forall a \in A)(\exists a' \in A)(a \in Sa'$ and $\langle a', B \rangle = S)$;

(2) $(\forall b \in B)(\exists b' \in B)(b \in b'S$ and $\langle A, b' \rangle = S)$.

Examples of dual pairs can be found on page 257 in [17]. Note that acts ${}_SA$ and B_S in a dual pair are necessarily unitary. By Theorem 2.5 in [17], the equality $\Sigma^\beta = \Omega_1^\beta$ holds for a dual pair over a monoid with zero. This is also true in case of a dual pair over a semigroup with weak local units.

Theorem 5.24 (cf. Theorem 2.5 in [17]). *Let $\beta = ({}_SA, B_S)$ be a dual pair over a semigroup S with weak local units. Then $\Sigma^\beta = \Omega_1^\beta$.*

Proof. The containment $\Sigma^\beta \subseteq \Omega_1^\beta$ is clear. We show $\Omega_1^\beta \subseteq \Sigma^\beta$. Take $(\rho, \sigma) \in \Omega_1^\beta$, that is, $\rho(A) \subseteq Sa_1$ and $\sigma(B) \subseteq b'S$ for some $a_1 \in A$ and $b' \in B$. We must show that there exist $a \in A$ and $b \in B$ such that $\rho = \langle -, b \rangle a$ and $\sigma = b \langle a, - \rangle$.

Since we have a dual pair, $a_1 = sa_2$ for some $s \in S$ and $a_2 \in A$. Due to the presence of weak local units in S take $s_r \in S$ such that $s = ss_r$ and let $\langle a_2, b_2 \rangle = s_r$ for some $b_2 \in B$. Similarly, $b' = b''t$, $t = t_\ell t$ for some $t_\ell \in S$ and $\langle a'', b'' \rangle = t_\ell$ for some $a'' \in A$. Then $\sigma(b_2) = b'v'$ and $\rho(a'') = u'a_1$ for some $u', v' \in S$. Putting $v := tv'$ and $u := u's$ we have

$$\sigma(b_2) = b'v' = b''tv' = b''v \quad \text{and} \quad \rho(a'') = u'a_1 = u'sa_2 = ua_2.$$

Hence

$$\begin{aligned} u = u's = u'ss_r = u's \langle a_2, b_2 \rangle = \langle u'sa_2, b_2 \rangle = \langle \rho(a''), b_2 \rangle \\ = \langle a'', \sigma(b_2) \rangle = \langle a'', b''tv' \rangle = \langle a'', b'' \rangle tv' = t_\ell tv' = tv' = v. \end{aligned}$$

Take $x \in A$, then $\rho(x) = za_1$ for some $z \in S$ and

$$\langle \rho(x), b_2 \rangle a_2 = \langle zsa_2, b_2 \rangle a_2 = zs \langle a_2, b_2 \rangle a_2 = zss_ra_2 = zsa_2 = za_1 = \rho(x)$$

and therefore, for every $x \in A$, we have

$$\rho(x) = \langle\, \rho(x), b_2 \,\rangle\, a_2 = \langle\, x, \sigma(b_2) \,\rangle\, a_2 = \langle\, x, b''v \,\rangle\, a_2 = \langle\, x, b'' \,\rangle\, va_2.$$

Similarly, for a fixed $y \in B$, we have $\sigma(y) = b'w$ for some $w \in S$ and

$$b'' \langle\, a'', b'w \,\rangle = b'' \langle\, a'', b''tw \,\rangle = b'' \langle\, a'', b'' \,\rangle\, tw = b''t_\ell tw = b''tw = b'w = \sigma(y).$$

Therefore, for every $y \in B$ we have

$$\sigma(y) = b'' \langle\, a'', \sigma(y) \,\rangle = b'' \langle\, \rho(a''), y \,\rangle = b'' \langle\, ua_2, y \,\rangle = b'' \langle\, va_2, y \,\rangle\,.$$

Thus, it suffices to take $b = b''$ and $a = ua_2 = va_2$. ∎

The following is a semigroup theoretic analogue of Proposition 2.2 in [3].

Lemma 5.25. *Let $\beta = ({}_SA, B_S)$ be a dual pair, where S is a semigroup with weak local units. Then the Morita semigroup $B \otimes_S^\beta A$ has weak local units. If S has local units, then $B \otimes_S^\beta A$ also has local units.*

Proof. Take $b \otimes a \in B \otimes_S A$. Since we have a dual pair, there exist $a_1 \in A$ and $s \in S$ such that $a = sa_1$. Since S has weak local units, we can write $s = su$ for some $u \in S$. By duality, we also have $u = \langle\, a_1, b_1 \,\rangle$ for some $b_1 \in B$. Then

$$a = sa_1 = sua_1 = s \langle\, a_1, b_1 \,\rangle\, a_1 = \langle\, sa_1, b_1 \,\rangle\, a_1 = \langle\, a, b_1 \,\rangle\, a_1,$$

hence

$$b \otimes a = b \otimes \langle\, a, b_1 \,\rangle\, a_1 = (b \otimes a)(b_1 \otimes a_1).$$

If S has local units, then we may assume u to be an idempotent and we have that

$$(b_1 \otimes a_1)^4 = b_1 \otimes \langle\, a_1, b_1 \,\rangle^3 a_1 = b_1 \otimes \langle\, a_1, b_1 \,\rangle\, a_1 = (b_1 \otimes a_1)^2 \in E\,(B \otimes_S A)\,.$$

Similarly, $b = b_2t$ and $t = vt$, where $v = \langle\, a_2, b_2 \,\rangle$ for some $a_2 \in A$. The equality $b \otimes a = (b_2 \otimes a_2)(b \otimes a)$ follows. ∎

It turns out that Hotzel's construction (cf. Theorem 2.4 in [17]) yields a semigroup morphism with good properties in case of an arbitrary pair.

Theorem 5.26. *For any pair $\beta = ({}_SA, B_S)$ over an arbitrary nonempty semigroup S, there exists a strict local isomorphism from $B \otimes_S^\beta A$ onto Σ^β along which idempotents lift.*

Proof. Define, in the same way as in [17],

$$\varphi : B \otimes_S A \longrightarrow \Sigma^\beta, \quad b \otimes a \mapsto [b, a].$$

The corresponding map

$$B \times A \longrightarrow \Sigma^\beta, \quad (b, a) \mapsto [b, a],$$

is S-balanced. Indeed, for any $s \in S$ we have the equalities

$$\langle -, bs \rangle\, a = \langle -, b \rangle\, sa \quad \text{and} \quad bs\, \langle a, - \rangle = b\, \langle sa, - \rangle,$$

so $[bs, a] = [b, sa]$. Thus, φ is well defined by Proposition 2.15. Surjectivity of φ is clear. Define

$$\star : (B \otimes_S A) \times \Sigma^\beta \longrightarrow B \otimes_S A$$

with the equality

$$(b \otimes a) \star [b', a'] := b \otimes \langle a, b' \rangle\, a'.$$

Let $b' \in B$ and $a' \in A$ be fixed, then, for every $b \otimes a \in B \otimes_S A$, we have the equality $(b \otimes a) \star [b', a'] = (b \otimes a)(b' \otimes a')$. Thus, the maps $- \star [b', a']$ and $- \cdot (b' \otimes a')$ coincide on $B \otimes_S A$. It follows that $\star$ is well defined. Take $b \otimes a \in B \otimes_S A$ and fix $b'' \in B$ and $a'' \in A$. The equalities

$$\begin{aligned}
((b \otimes a) \star [b', a']) \star [b'', a''] &= (b \otimes a)(b' \otimes a')(b'' \otimes a'') \\
&= (b \otimes a)\, (b' \otimes \langle a', b'' \rangle\, a'') \\
&= b \otimes \langle a, b' \rangle\, (\langle a', b'' \rangle\, a'') \\
&= (b \otimes a) \star [b', \langle a', b'' \rangle\, a''] \\
&= (b \otimes a) \star ([b', a']\, [b'', a'']) \qquad \text{(cf. 3.1)}
\end{aligned}$$

show that $\star$ is a right Σ^β-action on $B \otimes_S A$. We also have

$$\begin{aligned}
\varphi\, ((b \otimes a) \star [b', a']) &= \varphi\, (b \otimes \langle a, b' \rangle\, a') \\
&= [b, \langle a, b' \rangle\, a'] \\
&= [b, a]\, [b', a'] \\
&= \varphi(b \otimes a)\, [b', a'].
\end{aligned}$$

Thus, φ is a Σ^β-morphism. Due to the equality

$$(b \otimes a)(b' \otimes a') = (b \otimes a) \star \varphi(b' \otimes a')$$

we have by Proposition 5.7 that φ is a strict local isomorphism along which idempotents lift. ■

By Theorem 2.4 in [17], for a dual pair β over a monoid with zero, the Morita semigroup $B \otimes_S^\beta A$ is isomorphic to the semigroup Σ^β. This is also true for semigroups with weak local units.

Theorem 5.27. *Let S be a semigroup with weak local units and $\beta = ({}_SA, B_S)$ a dual pair. Then the Morita semigroup $B \otimes_S^\beta A$ is isomorphic to the semigroup Σ^β.*

Proof. It suffices to show that the map φ from the proof of Theorem 5.26 is injective. Suppose $[b, a] = [b', a']$ for some $b \otimes a, b' \otimes a'$ in $B \otimes_S A$. Said equality means that

$$\begin{aligned}
&(\forall a'' \in A)(\langle a'', b \rangle a = \langle a'', b' \rangle a'),\\
&(\forall b'' \in B)(b \langle a, b'' \rangle = b' \langle a', b'' \rangle).
\end{aligned}$$

By Lemma 5.25, take $b_1 \otimes a_1, b_2 \otimes a_2 \in B \otimes_S A$ such that

$$b \otimes a = (b \otimes a)(b_1 \otimes a_1) \quad \text{and} \quad b' \otimes a' = (b_2 \otimes a_2)(b' \otimes a').$$

We then have

$$\begin{aligned}
b \otimes a &= (b \otimes a)(b_1 \otimes a_1)\\
&= b \otimes \langle a, b_1 \rangle a_1\\
&= b \langle a, b_1 \rangle \otimes a_1\\
&= b' \langle a', b_1 \rangle \otimes a_1\\
&= (b' \otimes a')(b_1 \otimes a_1)
\end{aligned}$$

and therefore,

$$\begin{aligned}
b \otimes a &= (b_2 \otimes a_2)(b' \otimes a')(b_1 \otimes a_1)\\
&= (b_2 \otimes a_2)(b \otimes a)\\
&= b_2 \otimes \langle a_2, b \rangle a\\
&= b_2 \otimes \langle a_2, b' \rangle a'\\
&= (b_2 \otimes a_2)(b' \otimes a')\\
&= b' \otimes a'.
\end{aligned}$$ ■

A sufficient condition for strong Morita equivalence follows.

Corollary 5.28. *Let S and T be semigroups with weak local units. If $T \cong \Sigma^\beta$ for some dual pair $\beta = ({}_SA, B_S)$ then S and T are strongly Morita equivalent.*

Proof. By Theorem 5.27, $T \cong \Sigma^\beta \cong B \otimes_S^\beta A$, where $B \otimes_S^\beta A$ is surjectively defined. Hence T and S are strongly Morita equivalent by Theorem 5.14. ■

In particular, each dual pair β over S with weak local units gives rise to a semigroup Σ^β which is strongly Morita equivalent to S.

Chapter 6

Some Galois connections arising from Morita contexts of semigroups

This chapter is based on the author's article [42]. Galois connections arising from Morita contexts connecting two rings have been studied by Kashu [20] and later, for semigroups, by Paseka [44]. We make use of some of the mappings considered in [44]. The results in the mentioned articles are obtained for nondegenerate Morita contexts. Instead of requiring nondegeneracy, we work with unitary surjective Morita contexts connecting two semigroups. To compensate, we assume the presence of common weak local units. We obtain monotone Galois connections whenever either semigroup has common weak local units. If both semigroups have this property, then, among others, their lattices of compatible relations are isomorphic.

Definition 6.1. Let ${}_SA$ be a left S-act. We say a binary relation ρ on A is S-**compatible** if

$$(\forall a_1, a_2 \in A)(\forall s \in S)(a_1 \,\rho\, a_2 \Rightarrow sa_1 \,\rho\, sa_2).$$

We denote the lattice of S-compatible relations on A by $\mathsf{Comp}({}_SA)$ and dually, $\mathsf{Comp}(A_S)$ for a right act A_S.

Lattices of compatible relations of algebras have been studied by Chajda, Šešelja and Tepavčević [10]. In general, unions of compatible relations need not be compatible, but since acts over semigroups are algebras with only unary operations it is readily verified that both $\mathsf{Comp}({}_SA)$ and $\mathsf{Comp}(A_S)$ are sublattices of $(\mathcal{P}(A \times A), \subseteq)$.

Definition 6.2. An S-compatible relation ρ on an S-act A is called a

- **weak congruence** if ρ is symmetric and transitive;
- **tolerance** if ρ is reflexive and symmetric;
- **compatible preorder** if ρ is reflexive and transitive;
- **congruence** if ρ is an equivalence relation.

We have $\mathsf{Comp}({}_SA_T) = \mathsf{Comp}({}_SA) \cap \mathsf{Comp}(A_T)$, where ${}_SA_T$ is an (S,T)-biact. In case of an (S,S)-biact ${}_SA_S$, we call the elements of $\mathsf{Comp}({}_SA)$ **left** S**-compatible** and, dually, the elements of $\mathsf{Comp}(A_S)$ **right** S**-compatible**.

Definition 6.3. Let X and Y be partially ordered sets (posets) and $\varphi \in Y^X$ and $\psi \in X^Y$ a pair of mappings. It is said that the pair (φ, ψ) forms a **monotone Galois connection** if

$$(\forall x \in X)(\forall y \in Y)(\varphi(x) \leqslant y \Leftrightarrow x \leqslant \psi(y)).$$

In this case, φ is called the **left adjoint** of ψ and one writes $\varphi \dashv \psi$.

It holds that $\varphi \dashv \psi$ if and only if $\varphi\psi \leqslant \mathrm{id}_Y$ and $\mathrm{id}_X \leqslant \psi\varphi$ with respect to the pointwise order of mappings. It is clear that if $\varphi \dashv \psi \dashv \varphi$, then φ and ψ are mutually inverse order isomorphisms.

Recall that a left (right) S-act ${}_SA$ (A_S) is called **faithful** if for any $s, s' \in S$ it holds that $s = s'$ whenever $sa = s'a$ ($as = as'$) for all $a \in A$.

Problem 6.4. A Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ connecting semigroups S and T is called **nondegenerate** (cf. [44]) if ${}_SP, P_T, {}_TQ$ and Q_S are faithful and the following conditions are satisfied for any $p_1, p_2 \in P$ and $q_1, q_2 \in Q$:

(1) $p_1 = p_2$ whenever $\phi(q \otimes p_1) = \phi(q \otimes p_2)$ for all $q \in Q$;

(2) $p_1 = p_2$ whenever $\theta(p_1 \otimes q) = \theta(p_2 \otimes q)$ for all $q \in Q$;

(3) $q_1 = q_2$ whenever $\phi(q_1 \otimes p) = \phi(q_2 \otimes p)$ for all $p \in P$;

(4) $q_1 = q_2$ whenever $\theta(p \otimes q_1) = \theta(p \otimes q_2)$ for all $p \in P$.

When are two semigroups connected by a nondegenerate Morita context? What are examples of such Morita contexts, where at least one of the semigroups is not singleton?

Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. We consider the pairs of mappings (see p. 2251 in [44])

$$\begin{tikzcd} \mathcal{P}(P \times P) \arrow[r, shift left, "\alpha_P"] & \mathcal{P}(T \times T) \arrow[l, shift left, "\beta_P"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{P}(Q \times Q) \arrow[r, shift left, "\alpha_Q"] & \mathcal{P}(T \times T) \arrow[l, shift left, "\beta_Q"] \end{tikzcd}$$

that are defined by

$$\begin{aligned}
&\alpha_P(\sigma) := \{(t, t') \mid (\forall p \in P)(pt\,\sigma\,pt')\},\\
&\beta_P(\psi) := \{(p, p') \mid (\forall q \in Q)(\phi(q \otimes p)\,\psi\,\phi(q \otimes p'))\},\\
&\alpha_Q(\tau) := \{(t, t') \mid (\forall q \in Q)(tq\,\tau\,t'q)\},\\
&\beta_Q(\psi) := \{(q, q') \mid (\forall p \in P)(\phi(q \otimes p)\,\psi\,\phi(q' \otimes p))\}.
\end{aligned}$$

It is clear that the maps α_P, β_P, α_Q and β_Q are order preserving. These maps also have the following properties.

Proposition 6.5. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. Then all of the maps $\alpha_P, \beta_P, \alpha_Q$ and β_Q preserve reflexivity, symmetricity and transitivity. Additionally, the following assertions hold:*

(1) $\alpha_P(\mathcal{P}(P \times P)) \subseteq \mathsf{Comp}({}_TT)$ and $\alpha_P(\mathsf{Comp}(P_T)) \subseteq \mathsf{Comp}(T_T)$;

(2) $\beta_P(\mathcal{P}(T \times T)) \subseteq \mathsf{Comp}({}_SP)$ and $\beta_P(\mathsf{Comp}(T_T)) \subseteq \mathsf{Comp}(P_T)$;

(3) $\alpha_Q(\mathcal{P}(Q \times Q)) \subseteq \mathsf{Comp}(T_T)$ and $\alpha_Q(\mathsf{Comp}({}_TQ)) \subseteq \mathsf{Comp}({}_TT)$;

(4) $\beta_Q(\mathcal{P}(T \times T)) \subseteq \mathsf{Comp}(Q_S)$ and $\beta_Q(\mathsf{Comp}({}_TT)) \subseteq \mathsf{Comp}({}_TQ)$.

Proof. Preservation of symmetricity is clear in all cases. We verify preservation of reflexivity and transitivity for β_P. Preservation properties for the other maps are verified analogously. Let $\psi \subseteq T \times T$ be reflexive and take $p \in P$. Then $\phi(q \otimes p)\,\psi\,\phi(q \otimes p)$ for all $q \in Q$ due to reflexivity of ψ. Therefore, $(p, p) \in \beta_P(\psi)$. Now let ψ be transitive and take $(p_1, p_2), (p_2, p_3) \in \beta_P(\psi)$. It follows that

$$\phi(q \otimes p_1)\,\psi\,\phi(q \otimes p_2)\,\psi\,\phi(q \otimes p_3),$$

whence $\phi(q \otimes p_1)\,\psi\,\phi(q \otimes p_3)$ for all $q \in Q$ due to transitivity of ψ. Thus, $(p_1, p_3) \in \beta_P(\psi)$.

For item (1) let $\sigma \subseteq P \times P$. Take $(t_1, t_2) \in \alpha_P(\sigma)$ and $t \in T$. Note that $pt_1\,\sigma\,pt_2$ must hold for all $p \in P$. Firstly, we check $\alpha_P(\sigma)$ is left T-compatible.

Let $p \in P$. Then by the above we have $p(tt_1) = (pt)t_1 \, \sigma \, (pt)t_2 = p(tt_2)$. Thus, $(tt_1, tt_2) \in \alpha_P(\sigma)$.

Secondly, assume σ is T-compatible. From T-compatibility we conclude $(pt_1)t \, \sigma \, (pt_2)t$ for all $p \in P$. Thus, $(t_1t, t_2t) \in \alpha_P(\sigma)$.

For item (2) let $\psi \subseteq T \times T$. Take $(p_1, p_2) \in \beta_P(\psi)$ and $q \in Q$. Firstly, we check $\beta_P(\psi)$ is S-compatible. Take $s \in S$, then by definition of $\beta_P(\psi)$ we have

$$\phi(q \otimes sp_1) = \phi(qs \otimes p_1) \, \psi \, \phi(qs \otimes p_2) = \phi(q \otimes sp_2),$$

which implies that $(sp_1, sp_2) \in \beta_P(\psi)$. Secondly, assume ψ is right T-compatible. Then

$$\phi(q \otimes p_1t) = \phi(q \otimes p_1)t \, \psi \, \phi(q \otimes p_2)t = \phi(q \otimes p_2t)$$

for every $t \in T$, which implies that $(p_1t, p_2t) \in \beta_P(\psi)$. Items (3) and (4) are proved similarly. ■

The following result can be checked easily.

Lemma 6.6. *Let ${}_SA$ be a unitary S-act, where S is a semigroup with common weak left local units. Then for every $a_1, a_2 \in A$, there exists $u \in S$ such that $ua_1 = a_1$ and $ua_2 = a_2$.*

Proposition 6.7. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. The following assertions hold for the pair of maps*

$$\alpha_P : \mathsf{Comp}({}_SP) \longrightarrow \mathsf{Comp}({}_TT) \quad \textit{and} \quad \beta_P : \mathsf{Comp}({}_TT) \longrightarrow \mathsf{Comp}({}_SP).$$

1. *If T is a semigroup with common weak left local units, then $\alpha_P \dashv \beta_P$ and α_P is surjective.*

2. *If S is a semigroup with common weak left local units, then $\beta_P \dashv \alpha_P$ and β_P is surjective.*

Proof. We restrict the mapping α_P from Proposition 6.5 from $\mathcal{P}(P \times P)$ to its subset $\mathsf{Comp}({}_SP)$, but we still denote the restriction by α_P. We use a similar convention for β_P. The maps α_P and β_P are well defined by Proposition 6.5. Let $\sigma \in \mathsf{Comp}({}_SP)$ and $\psi \in \mathsf{Comp}({}_TT)$.

1. Assume $\alpha_P(\sigma) \subseteq \psi$ and let $(p_1, p_2) \in \sigma$. To show $(p_1, p_2) \in \beta_P(\psi)$, we must show that $\phi(q \otimes p_1) \, \psi \, \phi(q \otimes p_2)$ for every $q \in Q$. Let $q \in Q$ and take $p \in P$. Note that

$$p\phi(q \otimes p_1) = \theta(p \otimes q)p_1 \quad \text{and} \quad p\phi(q \otimes p_2) = \theta(p \otimes q)p_2.$$

Since $\theta(p \otimes q) \in S$ and σ is S-compatible, we have

$$p\phi(q \otimes p_1) \, \sigma \, p\phi(q \otimes p_2).$$

This implies that $(\phi(q \otimes p_1), \phi(q \otimes p_2)) \in \alpha_P(\sigma) \subseteq \psi$, as required.

Conversely, assume $\sigma \subseteq \beta_P(\psi)$ and let $(t_1, t_2) \in \alpha_P(\sigma)$. Then $pt_1 \, \sigma \, pt_2$ for every $p \in P$. By assumption, we have $(pt_1, pt_2) \in \beta_P(\psi)$, so

$$(\forall p \in P)(\forall q \in Q)(\phi(q \otimes pt_1) \, \psi \, \phi(q \otimes pt_2)).$$

Equivalently, $tt_1 \, \psi \, tt_2$ for every $t \in T$ due to surjectivity of ϕ. In particular, we have $t_1 = vt_1 \, \psi \, vt_2 = t_2$ for some $v \in T$ due to the presence of common weak local units in T. Hence, $\alpha_P(\sigma) \subseteq \psi$, as required.

We check surjectivity of α_P. Let $\psi \in \mathsf{Comp}({}_T T)$. The containment $\alpha_P(\beta_P(\psi)) \subseteq \psi$ holds due to $\alpha_P \dashv \beta_P$. Conversely, let $t_1 \, \psi \, t_2$. Left T-compatibility implies that

$$\phi(q \otimes pt_1) = \phi(q \otimes p)t_1 \, \psi \, \phi(q \otimes p)t_2 = \phi(q \otimes pt_2)$$

for every $p \in P$ and $q \in Q$. Equivalently, $(pt_1, pt_2) \in \beta_P(\psi)$ for every $p \in P$. So we have $\psi \subseteq \alpha_P(\beta_P(\psi))$. Therefore $\alpha_P(\beta_P(\psi)) = \psi$ and α_P is surjective.

2. Assume $\psi \subseteq \alpha_P(\sigma)$ and let $(p_1, p_2) \in \beta_P(\psi)$. It follows that

$$\begin{aligned} &(\forall q \in Q)(\phi(q \otimes p_1) \, \psi \, \phi(q \otimes p_2)) \\ \Rightarrow &(\forall q \in Q)(\phi(q \otimes p_1) \, \alpha_P(\sigma) \, \phi(q \otimes p_2)) \\ \Leftrightarrow &(\forall q \in Q)(\forall p \in P)(p\phi(q \otimes p_1) \, \sigma \, p\phi(q \otimes p_2)). \end{aligned}$$

By Lemma 6.6, take $u \in S$ such that $up_1 = p_1$ and $up_2 = p_2$. We have $u = \theta(p \otimes q)$ for some $p \in P$ and $q \in Q$ due to surjectivity of θ. It follows that

$$p_1 = up_1 = \theta(p \otimes q)p_1 = p\phi(q \otimes p_1) \, \sigma \, p\phi(q \otimes p_2) = p_2.$$

So $(p_1, p_2) \in \sigma$ and $\beta_P(\psi) \subseteq \sigma$.

Conversely, assume $\beta_P(\psi) \subseteq \sigma$ and let $(t_1, t_2) \in \psi$. To show $(t_1, t_2) \in \alpha_P(\sigma)$, we must show that $pt_1 \, \sigma \, pt_2$ for every $p \in P$. Let $p \in P$. Since ψ is left T-compatible, we have

$$(\forall q \in Q)(\phi(q \otimes pt_1) \, \psi \, \phi(q \otimes pt_2)).$$

This implies that $(pt_1, pt_2) \in \beta_P(\psi) \subseteq \sigma$, as required.

We check surjectivity of β_P. Let $\sigma \in \mathsf{Comp}({}_S P)$ and $p_1 \, \sigma \, p_2$. Take $p \in P$ and $q \in Q$. By S-compatibility we have $\theta(p \otimes q)p_1 \, \sigma \, \theta(p \otimes q)p_2$, which implies that

$$p\phi(q \otimes p_1) \, \sigma \, p\phi(q \otimes p_2).$$

It follows that $(\phi(q \otimes p_1), \phi(q \otimes p_2)) \in \alpha_P(\sigma)$ so $(p_1, p_2) \in \beta_P(\alpha_P(\sigma))$. Thus, $\sigma \subseteq \beta_P(\alpha_P(\sigma))$. The containment $\beta_P(\alpha_P(\sigma)) \subseteq \sigma$ follows from $\beta_P \dashv \alpha_P$. ■

Remark 6.8. It is well known that if $f \dashv g$, then f is surjective if and only if g is injective.

Similarly to the proof of Proposition 6.7, one can prove the following.

Proposition 6.9. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. The following assertions hold for the pair of maps*

$$\alpha_Q : \mathsf{Comp}(Q_S) \longrightarrow \mathsf{Comp}(T_T) \quad \textit{and} \quad \beta_Q : \mathsf{Comp}(T_T) \longrightarrow \mathsf{Comp}(Q_S).$$

1. *If T is a semigroup with common weak right local units, then $\alpha_Q \dashv \beta_Q$ and α_Q is surjective.*

2. *If S is a semigroup with common weak right local units, then $\beta_Q \dashv \alpha_Q$ and β_Q is surjective.*

The assertions made for one sided acts also hold in case of biacts.

Corollary 6.10. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. The following assertions hold for the pairs of maps*

$$\alpha_P : \mathsf{Comp}({}_SP_T) \longrightarrow \mathsf{Comp}({}_TT_T) \quad \textit{and} \quad \beta_P : \mathsf{Comp}({}_TT_T) \longrightarrow \mathsf{Comp}({}_SP_T),$$
$$\alpha_Q : \mathsf{Comp}({}_TQ_S) \longrightarrow \mathsf{Comp}({}_TT_T) \quad \textit{and} \quad \beta_Q : \mathsf{Comp}({}_TT_T) \longrightarrow \mathsf{Comp}({}_TQ_S).$$

1. *If T is a semigroup with common weak left local units, then $\alpha_P \dashv \beta_P$ and α_P is surjective.*

2. *If S is a semigroup with common weak left local units, then $\beta_P \dashv \alpha_P$ and β_P is surjective.*

3. *If T is a semigroup with common weak right local units, then $\alpha_Q \dashv \beta_Q$ and α_Q is surjective.*

4. *If S is a semigroup with common weak right local units, then $\beta_Q \dashv \alpha_Q$ and β_Q is surjective.*

PROOF. The maps are well defined by Proposition 6.5. The adjunctions pertaining to α_P and β_P follow from Proposition 6.7. The claims for α_Q and β_Q are proved analogously. ■

Results obtained with the Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ can be reproduced dually using the Morita context $(T, S, {}_TQ_S, {}_SP_T, \phi, \theta)$, for which we have the dual mappings

$$\begin{tikzcd}
\mathcal{P}(Q \times Q) \arrow[r, shift left, "\overline{\alpha}_Q"] & \mathcal{P}(S \times S) \arrow[l, shift left, "\overline{\beta}_Q"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathcal{P}(P \times P) \arrow[r, shift left, "\overline{\alpha}_P"] & \mathcal{P}(S \times S) \arrow[l, shift left, "\overline{\beta}_P"]
\end{tikzcd}$$

that are defined by

$$\begin{aligned}
\overline{\alpha}_Q(\tau) &:= \{(s, s') \mid (\forall q \in Q)(qs\, \tau\, qs')\}\,, \\
\overline{\beta}_Q(\rho) &:= \{(q, q') \mid (\forall p \in P)(\theta(p \otimes q)\, \rho\, \theta(p \otimes q'))\}\,, \\
\overline{\alpha}_P(\sigma) &:= \{(s, s') \mid (\forall p \in P)(sp\, \sigma\, s'p)\}\,, \\
\overline{\beta}_P(\rho) &:= \{(p, p') \mid (\forall q \in Q)(\theta(p \otimes q)\, \rho\, \theta(p' \otimes q))\}\,.
\end{aligned}$$

Theorem 6.11. *Let semigroups S and T with common weak local units be connected by a unitary surjective Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then we have the following pairs*

$$\begin{tikzcd}
\mathsf{Comp}({}_SP) \arrow[r, shift left, "\alpha_P"] & \mathsf{Comp}({}_TT) \arrow[l, shift left, "\beta_P"]
\end{tikzcd} \qquad \begin{tikzcd}
\mathsf{Comp}(P_T) \arrow[r, shift left, "\overline{\alpha}_P"] & \mathsf{Comp}(S_S) \arrow[l, shift left, "\overline{\beta}_P"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Comp}({}_TQ) \arrow[r, shift left, "\overline{\alpha}_Q"] & \mathsf{Comp}({}_SS) \arrow[l, shift left, "\overline{\beta}_Q"]
\end{tikzcd} \qquad \begin{tikzcd}
\mathsf{Comp}(Q_S) \arrow[r, shift left, "\alpha_Q"] & \mathsf{Comp}(T_T) \arrow[l, shift left, "\beta_Q"]
\end{tikzcd}$$

of mutually inverse lattice isomorphisms that preserve weak congruences, compatible preorders, tolerances and congruences.

Proof. Since both S and T have common weak local units, it follows by Proposition 6.7 that $\alpha_P \dashv \beta_P \dashv \alpha_P$. Therefore, α_P and β_P are mutually inverse order isomorphisms. The other pairs of order isomorphisms are similarly verified with the dual of Proposition 6.7, Proposition 6.9 and its dual. Preservation of weak congruences, compatible preorders, tolerances and congruences follows by Proposition 6.5 and its dual. It is well known that an order isomorphism between lattices is a lattice isomorphism. ■

Theorem 3 in [31] yields that the lattices of ideals of strongly Morita equivalent semigroups S and T are isomorphic whenever S and T have weak local units. In other words, the sublattices of Rees congruences on S and T are isomorphic. In view of Corollary 6.10 and its dual, we conclude the following.

Theorem 6.12. *Let semigroups S and T with common weak local units be connected by a unitary surjective Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then we have the following pairs*

$$\begin{tikzcd}
\mathsf{Comp}({}_SP_T) \arrow[r, shift left, "\alpha_P"] & \mathsf{Comp}({}_TT_T) \arrow[l, shift left, "\beta_P"] \arrow[r, shift left, "\beta_Q"] & \mathsf{Comp}({}_TQ_S) \arrow[l, shift left, "\alpha_Q"] \arrow[r, shift left, "\overline{\alpha}_Q"] & \mathsf{Comp}({}_SS_S) \arrow[l, shift left, "\overline{\beta}_Q"]
\end{tikzcd}$$

of mutually inverse lattice isomorphisms that preserve weak congruences, compatible preorders, tolerances and congruences.

A relation $\psi \subseteq T \times T$ is a congruence of the biact ${}_TT_T$ if and only if it is a congruence of the semigroup T. A similar assertion holds for compatible preorders. Hence we obtain the following result.

Corollary 6.13. *If S and T are strongly Morita equivalent semigroups with common weak local units, then the lattices of weak congruences, compatible preorders, tolerances and congruences of the biacts ${}_SS_S$ and ${}_TT_T$ are isomorphic. Explicitly, if $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ is a unitary surjective Morita context, then the mutually inverse isomorphisms are the restrictions of the mappings*

$$\Theta : \mathsf{Comp}({}_SS_S) \longrightarrow \mathsf{Comp}({}_TT_T) \quad \textit{and} \quad \Phi : \mathsf{Comp}({}_TT_T) \longrightarrow \mathsf{Comp}({}_SS_S)$$

defined by

$$\begin{aligned}
\Theta(\rho) &:= \{(t, t') \mid (\forall q \in Q)(\forall p \in P)(\theta(pt \otimes q)\, \rho\, \theta(pt' \otimes q))\}\,,\\
\Phi(\psi) &:= \{(s, s') \mid (\forall q \in Q)(\forall p \in P)(\phi(q \otimes sp)\, \psi\, \phi(q \otimes s'p))\}\,.
\end{aligned}$$

In particular, the lattices of compatible preorders and congruences of the semigroups S and T are isomorphic.

These mappings are similar to the mappings Θ and Φ of Theorem 2 in [50].

Remark 6.14. Let S be a semigroup. Since tolerances are reflexive, it follows that every tolerance of the semigroup S is also a tolerance of ${}_SS_S$. Also, due to transitivity, every weak congruence of ${}_SS_S$ is a weak congruence of the semigroup S.

Remark 6.15. The above corollary generalises Theorem 6 in [31]. It turns out that the mappings in said paper coincide with the ones given in Corollary 6.13. Laan and Márki [31] consider the transitive closure of the relation

$$\Pi_\rho := \{(\phi(q \otimes sp), \phi(q \otimes s'p)) \mid s\, \rho\, s', p \in P, q \in Q\}\,,$$

where $\rho \in \mathsf{Con}(S)$. It holds that

$$\Pi_\rho = \Theta(\rho) = \{(t, t') \mid (\forall q \in Q)(\forall p \in P)(\theta(pt \otimes q)\, \rho\, \theta(pt' \otimes q))\}\,.$$

Proof. We show $\Theta(\rho) \subseteq \Pi_\rho$. Let $(t,t') \in \Theta(\rho)$. Since ϕ is surjective, there exist $p_t, p_{t'} \in P$ and $q_t, q_{t'} \in Q$ such that $t = \phi(q_t \otimes p_t)$ and $t' = \phi(q_{t'} \otimes p_{t'})$. Lemma 2.3 and its dual imply that there exist $u = \phi(q_u \otimes p_u) \in T$ and $v = \phi(q_v \otimes p_v)$ such that $uq_t = q_t$, $uq_{t'} = q_{t'}$, $p_t v = p_t$ ja $p_{t'}v = p_{t'}$. Denote

$$s := \theta(p_u t \otimes q_v) \;\rho\; \theta(p_u t' \otimes q_v) =: s'.$$

It follows that

$$\begin{aligned} t = \phi(q_t \otimes p_t) &= \phi(uq_t \otimes p_t v) \\ &= \phi(\phi(q_u \otimes p_u)q_t \otimes p_t\phi(q_v \otimes p_v)) \\ &= \phi(q_u \otimes p_u)\phi(q_t \otimes p_t)\phi(q_v \otimes p_v) \\ &= \phi(q_u \otimes p_u t\phi(q_v \otimes p_v)) \\ &= \phi(q_u \otimes \theta(p_u t \otimes q_v)p_v) \\ &= \phi(q_u \otimes sp_v) \end{aligned}$$

and similarly $t' = \phi(q_u \otimes s'p_v)$. Thus, we have $(t,t') \in \Pi_\rho$.

Conversely, let $\phi(q_0 \otimes sp_0) \,\Pi_\rho\, \phi(q_0 \otimes s'p_0)$, where $s \,\rho\, s'$. Take $q \in Q$ and $p \in P$. It follows that

$$\begin{aligned} \theta(p\phi(q_0 \otimes sp_0) \otimes q) &= \theta(\theta(p \otimes q_0)sp_0 \otimes q) \\ &= \theta(p \otimes q_0)\, s\, \theta(p_0 \otimes q) \\ &\;\rho\, \theta(p \otimes q_0)\, s'\, \theta(p_0 \otimes q) \\ &= \theta(p\phi(q_0 \otimes s'p_0) \otimes q). \end{aligned}$$

Thus, $\Pi_\rho \subseteq \Theta(\rho)$. ■

Corollary 6.13 implies that all properties defined in terms of the tolerance lattices of semigroups with common weak local units, regarded as biacts, are Morita invariant. One such property is **tolerance triviality** [9], i.e, every tolerance of the biact ${}_SS_S$ is a congruence.

Chapter 7

Perfection for semigroups

This chapter is based on the author's joint work [29] with supervisor Valdis Laan. Perfect monoids were defined in [19] by Isbell, who proved that a monoid is left perfect if and only if it satisfies two conditions referred to as (A) and (D). Fountain [13] gave more conditions that are equivalent to left perfectness. In particular, he proved that a monoid is left perfect if and only if all of its weakly flat (in the sense of being pullback flat) unitary left acts are projective. Subsequently, a number of other papers related to perfect monoids or pomonoids have appeared (e.g. [22], [23], [43], [16], [21], [4]). We define (right) perfect semigroups and show that many descriptions of right perfect monoids can be transferred to the case of factorisable semigroups. In our main theorem we will give six different conditions that are equivalent to right perfectness of a factorisable semigroup. Some of these are familiar from the monoid case, but some of them are new. For example, a factorisable semigroup is right perfect if and only if all right sequence acts over it are projective. It is well known that right perfectness and right semiperfectness are Morita invariants for rings with identity (Corollary 27.8 and Corollary 28.6 in [2]). We prove that right perfectness is also a Morita invariant for factorisable semigroups. Our results allow us to conclude that the class of perfect semigroups contains all completely (0-)simple semigroups and all nilpotent semigroups.

We will need the following result about firm acts (Corollary 2.4 in [35]).

Lemma 7.1. *If A_S is a unitary act over a factorisable semigroup, then $A \otimes_S S$ is a firm right S-act.*

Definition 7.2. Let A_S be a right S-act. **Finitely generated** subacts of A_S are those of the form $FS^1 := \{fs \mid f \in F, s \in S^1\}$, where $F \subseteq A$ is a finite subset. An act A_S is called **cyclic** if there exists $a \in A$ such that

$A = aS^1 = \{a\} \cup aS$ and **locally cyclic** if all of its finitely generated subacts are cyclic.

The following claim is readily verified.

Proposition 7.3. *An S-act A_S is locally cyclic if and only if*

$$(\forall a, a' \in A)(\exists a'' \in A)(\exists s, s' \in S^1)(a = a''s \quad \textit{and} \quad a' = a''s').$$

Essential epimorphisms are defined dually to essential monomorphisms.

Definition 7.4. An epimorphism $\varphi : P \longrightarrow A$ in a category $\mathcal{C}$ is called an **essential epimorphism** if, for every morphism $\psi : Q \longrightarrow P$ in $\mathcal{C}$,

$$\varphi\psi \text{ is an epimorphism} \Rightarrow \psi \text{ is an epimorphism.}$$

It is easy to see that the composite of two essential epimorphisms is an essential epimorphism.

Definition 7.5. An object P of a category $\mathcal{C}$ is called **projective** in $\mathcal{C}$ if for every epimorphism $\pi : A \longrightarrow B$ and for every morphism $\varphi : P \longrightarrow B$, there exists a morphism $\psi : P \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
 & P \arrow[d, "\varphi"] \arrow[dl, dashed, "\psi"'] \\
A \arrow[r, "\pi"'] & B
\end{tikzcd}$$

commutes, i.e, the equality $\varphi = \pi\psi$ holds.

Definition 7.6. Let $\varphi : P \longrightarrow A$ be an essential epimorphism in a category $\mathcal{C}$. The object P is called a **cover** of the object A. If the object P is projective in $\mathcal{C}$, then it is called a **projective cover** of A.

Similarly to Proposition 1.3 in [34], one can prove that epimorphisms in UAct_S are precisely the surjective morphisms.

Lemma 7.7. *Let S be a semigroup. An epimorphism $\varphi : P_S \longrightarrow A_S$ in UAct_S is essential if and only if $\varphi|_Q$ is not surjective for every unitary proper subact Q of P_S.*

Proof. **Necessity**. Assume that $\varphi : P_S \longrightarrow A_S$ is an essential epimorphism in UAct_S and let Q be a unitary proper subact of P_S. Then the inclusion map $\iota : Q \longrightarrow P$ is a morphism in UAct_S and $\varphi|_Q = f\iota$. If $\varphi\iota$ was surjective, then ι would be surjective due to essentiality of φ.

Sufficiency. Let $\varphi : P_S \longrightarrow A_S$ and $\psi : U_S \longrightarrow P_S$ be morphisms in UAct_S such that φ and $\varphi\psi$ are surjective. It is easy to see that $\psi(U)$ is a unitary subact of P_S and $\varphi|_{\psi(U)}$ is surjective. By assumption $\psi(U) = P$ so ψ is surjective and therefore φ is essential. ■

Definition 7.8. We say that a semigroup S is:

- **right perfect** if every object of UAct_S has a projective cover;
- **right semiperfect** if every unitary cyclic right S-act has a projective cover in UAct_S.

Dually, one can define left (semi)perfect semigroups. A semigroup is called **(semi)perfect** if it is both right and left (semi)perfect.

Remark 7.9. We have defined right semiperfect semigroups as nonadditive analogues of right semiperfect rings (see [6]).

Remark 7.10. Recall that if S is a monoid with identity 1, then an act A_S is unitary if and only if $a1 = a$ for every $a \in A$. Hence S is right perfect in the sense of Definition 7.8 if and only if it is right perfect in the sense of [19] and [13]. Thus, the results in those articles are special cases of our more general approach.

Definition 7.11 ([12]). A subsemigroup T of a semigroup S is called **left unitary** if, for every $s, t \in S$,

$$t, ts \in T \Rightarrow s \in T.$$

We shall make use of the following conditions on a semigroup S.

Definition 7.12. A semigroup S satisfies **Condition**:

- (A) if every right S-act satisfies the ascending chain condition (ACC) for cyclic subacts;
- (D) if every left unitary subsemigroup of S contains a minimal right ideal generated by an idempotent;
- (M_L) if S satisfies the descending chain condition (DCC) for principal left ideals of S.

7.1 Indecomposable and projective acts

Definition 7.13. An act is called **indecomposable** if it is not a disjoint union of two nonempty subacts.

The following is Lemma 4 in [11].

Proposition 7.14. *Every right S-act is a disjoint union of indecomposable subacts.*

In what follows, we will call the indecomposable subacts of A_S the **indecomposable components of** A_S. It is easy to see that each cyclic act is indecomposable. We will need the following result about projective acts, which is presented as Lemma 2 in [11].

Theorem 7.15. *Let $\mathcal{C}$ be a full subcategory of Act_S which is closed under coproducts. Let $P_S = \bigsqcup_{i\in I} P_i$, where $P_i \in \mathcal{C}$ for every $i \in I$. Then P_S is projective in $\mathcal{C}$ if and only if P_i is projective in $\mathcal{C}$ for every $i \in I$.*

Note that coproducts in Act_S are disjoint unions and both UAct_S and FAct_S are closed under disjoint unions.

Lemma 7.16. *Let e be an idempotent in a semigroup S. Then the right S-act eS is a projective object in Act_S, UAct_S and FAct_S.*

Proof. By the dual of Lemma 3.1(1) in [38], the act eS is firm, thus also unitary. In all these categories epimorphisms are precisely the surjective morphisms. Now, in any of these categories, we consider a morphism $\varphi : eS \twoheadrightarrow B_S$ and an epimorphism $\pi : A_S \twoheadrightarrow B_S$. Denote $b := \varphi(e)$ and choose $a \in A$ such that $\varphi(e) = \pi(a)$. Consider a homomorphism $\psi : eS \longrightarrow A_S$ defined by

$$\psi(es) := aes, \quad s \in S.$$

Then $\pi\psi = \varphi$, because $(\pi\psi)(es) = \pi(aes) = \pi(a)es = \varphi(e)es = \varphi(es)$. ■

The last two results allow us to prove the following result.

Corollary 7.17. *Every nilpotent semigroup is perfect. Every nilsemigroup is semiperfect.*

Proof. If S is nilpotent, then $S^n = 0$ for some $n \in \mathbb{N}$. If A_S is unitary, then, for any $a \in A$ we can find $a' \in A$ and $s_1, \dots, s_n \in S$ such that

$$a = a's_n \dots s_2 s_1 = a'0 = a'00 = a0.$$

So $aS^1 = \{a\}$ is a one-element subact, which is isomorphic to the act $0S = \{0\}$. The last is projective in UAct_S by Lemma 7.16. Thus,

$$A_S = \bigsqcup_{a\in A}\{a\} \cong \bigsqcup_{a\in A} 0S$$

is a projective act by Theorem 7.15, which is a projective cover of itself (with the identity mapping). Thus all objects of UAct_S (and similarly ${}_S\mathsf{UAct}$) have projective covers and S is perfect.

Assume now that S is a nilsemigroup. If aS^1 is a cyclic unitary act, then $a = as$ for some $s \in S$. For s, there exists $n \in \mathbb{N}$ such that $s^n = 0$. It follows that $a = as^n = a0$, and hence $aS^1 = \{a\} \cong 0S$, where $0S$ is projective. Thus S is right semiperfect and, similarly, it is also left semiperfect. ■

Over a factorisable semigroup, projective acts can be described as follows. The description relies on Theorem 2 in [11], which generalises a well-known characterisation of projective acts over monoids.

Theorem 7.18. *Let S be a factorisable semigroup. For $P_S \in \mathsf{Act}_S$, the following are equivalent:*

- *(1) P_S is a projective object in FAct_S;*
- *(2) P_S is a projective object in UAct_S;*
- *(3) $P \cong \bigsqcup_{i\in I} e_iS$ for some idempotents $e_i \in S$, $i \in I$.*

Proof. (1) ⇒ (2). Assume that P_S is a projective object in FAct_S. Clearly P_S is an object in UAct_S. We will prove that it is projective in UAct_S. Consider a morphism $\varphi : P_S \longrightarrow B_S$ and an epimorphism $\pi : A_S \longrightarrow B_S$ in UAct_S. By Lemma 7.1, the acts in the diagram

$$\begin{tikzcd}
P_S \arrow[r, "\mu_P^{-1}"] \arrow[d, dashed, "\psi"'] & P \otimes_S S \arrow[d, "\varphi \otimes \mathrm{id}_S"] \\
A \otimes_S S \arrow[r, "\pi \otimes \mathrm{id}_S"'] & B \otimes_S S
\end{tikzcd}$$

are firm, and $\pi \otimes \mathrm{id}_S$ is an epimorphism in FAct_S, because the functor $- \otimes S$, as a left adjoint, preserves epimorphisms. Using projectivity of P_S we can find a morphism $\psi : P_S \longrightarrow A \otimes_S S$ such that the square commutes. Now $\mu_A\psi : P_S \longrightarrow A_S$ is a morphism in UAct_S. Take any $p \in P$. Then there exist

$p_0 \in P, a \in A$ and $s_0, s \in S$ such that $p = p_0 s_0$ and $\psi(p) = a \otimes s$. Due to the commutativity of the square we have

$$\begin{aligned}\pi(a) \otimes s = (\pi \otimes \mathrm{id}_S)(\psi(p)) = (\varphi \otimes \mathrm{id}_S)(\mu_P^{-1}(p)) &= (\varphi \otimes \mathrm{id}_S)(p_0 \otimes s_0)\\ &= \varphi(p_0) \otimes s_0\end{aligned}$$

in $B \otimes_S S$. Applying the mapping μ_B to the equality $\pi(a) \otimes s = \varphi(p_0) \otimes s_0$ we obtain $\pi(a)s = \varphi(p_0)s_0$. Consequently,

$$(\pi\mu_A\psi)(p) = (\pi\mu_A)(a \otimes s) = \pi(as) = \pi(a)s = \varphi(p_0)s_0 = \varphi(p_0 s_0) = \varphi(p).$$

We have shown that $\pi(\mu_A\psi) = \varphi$, as needed.

(2) ⇒ (3). This is a consequence of Theorem 2 in [11], because UAct_S satisfies the assumptions of that theorem.

(3) ⇒ (1). By Lemma 7.16, the acts $e_i S$ are projective in FAct_S. A disjoint union of firm acts is firm, thus P_S is an object of FAct_S. It is projective in FAct_S due to Theorem 7.15. ■

Proposition 7.19. *The following are equivalent for an act A_S over a factorisable semigroup S:*

(1) A_S is an indecomposable projective object in UAct_S;

(2) $A_S \cong eS$ for some idempotent $e \in S$;

(3) there exists $a \in A$ and an idempotent $e \in S$ such that $A_S = aS^1$, $a = ae$ and

$$(\forall s, t \in S^1)(as = at \Rightarrow es = et).$$

PROOF. (1) $\iff$ (2). This follows from Theorem 7.18, because cyclic acts are indecomposable.

(2) $\iff$ (3). This is due to Lemma 1.1 and Corollary 1.2 in [14]. ■

The properties listed in item (3) may be assumed of any generator of a cyclic projective object in UAct_S.

Corollary 7.20. *An act $A_S = aS^1$ is a projective object in UAct_S if and only if there exists an idempotent $e \in S$ such that $a = ae$ and $as = at$ implies $es = et$ for all $s, t \in S^1$.*

PROOF. **Necessity.** (This proof is similar to the proof of implication (ii) ⇒ (iii) of Corollary 3.17.9 in [24].) By Proposition 7.19, $aS^1 \cong eS$ for some idempotent $e \in S$. Let $\varphi : eS \longrightarrow aS^1$ be an isomorphism. Then there exist $u \in S^1$ and $v \in S$ such that $\varphi(e) = au$ and $\varphi(ev) = a$. Now

$\varphi(e) = au = \varphi(ev)u = \varphi(evu)$ implies $e = evu$, because φ is injective. Also, $aue = \varphi(e)e = \varphi(e) = au$. Putting $z := uev \in S$, we have $z^2 = (uev)(uev) = u(evu)ev = ueev = z$, so z is an idempotent.

Suppose that $as = at$, where $s, t \in S^1$. Then $\varphi(evs) = \varphi(ev)s = \varphi(ev)t = \varphi(evt)$, which implies $evs = evt$. Therefore, $zs = uevs = uevt = zt$.

Sufficiency. This follows from Proposition 7.19. ■

7.2 Sequence acts

Let us recall the construction of a sequence act over a semigroup (cf. [37, Construction 3.9]). A similar construction in the case of a monoid appeared already in the proof of Lemma 1 in [13].

Let $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ be a sequence of elements of S. On the set

$$F = \mathbb{N} \times S^1 = \{(k, s) \mid k \in \mathbb{N}, s \in S^1\}$$

we define a right S-action by $(k, s)z := (k, sz)$, $k \in \mathbb{N}$, $s \in S^1$, $z \in S$. This gives us a right S-act F_S. On the set F we define a binary relation $\sim$ by

$$(k, s) \sim (k', s') \iff (\exists n \geqslant k, k')(s_n \dots s_k s = s_n \dots s_{k'} s').$$

Then $\sim$ is a congruence of F_S. Form the quotient act

$$M_S = F_S/{\sim} = \{[k, s] \mid k \in \mathbb{N}, s \in S^1\},$$

where $[k, s]$ denotes the $\sim$-class of (k, s).

Definition 7.21. Such quotient acts M_S are called **right sequence acts over** S.

Let M_S be a right sequence act. Note that

$$s_{k+1}s_k \cdot 1 = s_{k+1} \cdot s_k \Rightarrow [k, 1] = [k + 1, s_k] = [k + 1, 1]s_k \in [k + 1, 1]S^1$$

for each $k \in \mathbb{N}$, so we see that M_S is unitary and it is a union of a chain of its cyclic subacts

$$[1, 1]S^1 \subseteq [2, 1]S^1 \subseteq [3, 1]S^1 \subseteq \dots.$$

Lemma 7.22. *Right sequence acts are locally cyclic. A right sequence act M_S is cyclic if and only if $M_S = [k, 1]S^1$ for some $k \in \mathbb{N}$.*

Proof. Given any $[k, s], [\ell, t] \in M_S$, we have that $[k, s], [\ell, t] \in [\max\{k, \ell\}, 1]S^1$. Thus, right sequence acts are locally cyclic by Proposition 7.3.

If M_S is cyclic, then $M = [m, u]S^1$ for some $u \in S^1$. By the above,

$$M_S = [m, u]S^1 = [m + 1, s_m u]S^1 \subseteq [m + 1, 1]S^1,$$

so $M_S = [m + 1, 1]S^1$. The converse is obvious. ■

It is well known that locally cyclic acts are indecomposable.

Lemma 7.23. *Let S be a semigroup. Each act of the form eS, $e^2 = e \in S$, is isomorphic to a right sequence act.*

Proof. Let M_S be the right sequence act constructed using the constant sequence $(e, e, \ldots)$. For every $k \in \mathbb{N}$,

$$e \cdot 1 = \underbrace{e \ldots e}_{k} \cdot e \Rightarrow (k, 1) \sim (1, e) \Rightarrow [k, 1] = [1, 1]e \Rightarrow [k, 1]S^1 = [1, 1]S^1,$$

so $M_S = [1, 1]S^1$. Consider the S-act homomorphism

$$\varphi : eS \longrightarrow [1, 1]S^1, \quad es \mapsto [1, es].$$

It is injective, because

$$(1, es) \sim (1, et) \Rightarrow e \cdot es = e \cdot et \Rightarrow es = et$$

for every $s, t \in S$, and it is surjective, because

$$[1, s] = [2, es] = [2, e]es = [1, 1]es = [1, es] = \varphi(es)$$

for every $s \in S^1$. Thus φ is an isomorphism. ■

7.3 Condition (A)

Definition 7.24. A subset X of an act A_S is a **set of generators** if $A = XS^1$. A set X of generators is called **independent** ([13, p. 90]) if

$$(\forall x, x' \in X)(x \in x'S^1 \Rightarrow x = x').$$

Lemma 7.25 (cf. Lemma 2 in [13]). *Let A_S be an act over a semigroup S satisfying (ACC) for cyclic subacts. Then A_S has an independent set of generators.*

Proof. Denote

$$X := \{x \in A \mid (\forall x' \in A)(xS^1 \subseteq x'S^1 \Rightarrow xS^1 = x'S^1)\}\,.$$

We first show that X is a set of generators for A_S. It suffices to prove that $A \subseteq XS^1$. Take arbitrary $a \in A$ and consider the set

$$P_a = \{a'S^1 \mid a' \in A, aS^1 \subseteq a'S^1\}$$

as a poset with respect to inclusion. Since $aS^1 \in P_a$, by assumption, this poset must have a maximal element a_0S^1. In particular, $aS^1 \subseteq a_0S^1$. We show that $a_0 \in X$.

If $x' \in A$ and $a_0S^1 \subseteq x'S^1$, then also $aS^1 \subseteq x'S^1$, so $x'S^1 \in P_a$. Due to maximality of a_0S^1, we have $a_0S^1 = x'S^1$. Hence $a_0 \in X$ and $a \in a_0S^1 \subseteq XS^1$, as needed.

Define a relation $\approx$ on X by

$$x \approx x' \iff xS^1 \subseteq x'S^1\,.$$

It is clear by the choice of X that we have an equivalence relation.

From every $\approx$-class we choose a representative and form a set X' of those elements. We show that X' is an independent set of generators for A_S. By the definition of $\approx$, for every $x \in X$ there exist $x' \in X'$ and $s \in S^1$ such that $x = x's$. Since X is a set of generators, also X' is a set of generators.

To prove that X' is independent, we suppose that $x \in x'S^1$, where $x, x' \in X' \subseteq X$. Then $xS^1 \subseteq x'S^1$, which means that $x \approx x'$. Now $x = x'$, because each $\approx$-class contains precisely one element from X'. ■

Definition 7.26 (cf. Definition 4.1 in [4]). A semigroup S is called **right $\mathcal{IC}$-perfect** if every unitary act A_S has a cover $B_S \longrightarrow A_S$ where B_S is unitary and indecomposable components of B_S are cyclic. Such a cover will be called an **$\mathcal{IC}$-cover**.

Note that if $b = b's$, where $b, b' \in B$ and $s \in S$, then b and b' must be in the same indecomposable component. Therefore indecomposable components of B_S must be unitary. Hence,

$$B_S = \bigsqcup_{i \in I} b_iS^1,$$

where each b_iS^1 is a cyclic unitary act.

The following result is inspired by Result 1.2 in [19], by Lemma 1.3 in [21] and also by Lemma 2.2 and Theorem 5.2 in [4]. Among other things, it shows that Condition (A) can be given a description that does not refer to acts, but only uses the elements of S.

Theorem 7.27. *For a semigroup S the following are equivalent:*

(1) S satisfies Condition (A);

(2) every locally cyclic right S-act is cyclic;

(3) every right sequence act over S is cyclic;

(4) for every sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$,

$$(\exists k, m \in \mathbb{N})(\exists u \in S^1)(k \geqslant m+1 \text{ and } s_k \dots s_{m+1} = s_k \dots s_{m+1}s_m u);$$

(5) for every act A_S there exists a set $\{A_i \mid i \in I\}$ of cyclic subacts of A_S such that

$$A = \bigcup_{i\in I} A_i \quad \text{and} \quad (\forall j \in I)\left(A_j \not\subseteq \bigcup_{i\neq j} A_i\right); \tag{7.1}$$

(6) S is right $\mathcal{IC}$-perfect.

Proof. (1) ⇒ (2). Suppose that A_S is a locally cyclic act which is not cyclic. Choose $a_1 \in A$. Since A_S is not cyclic, there exists $b_1 \in A \setminus a_1S^1$. Using that A_S is locally cyclic, we can find $a_2 \in A$ and $u, v \in S^1$ such that $a_1 = a_2u$ and $b_1 = a_2v$. Note that $a_2 \notin a_1S^1$, because otherwise $b_1 \in a_1S^1$. Thus, $a_1S^1 \subset a_2S^1$.

Again, $a_2S^1 \neq A$, and we can find an element $b_2 \in A \setminus a_2S^1$. Continuing in this manner we can construct a strictly increasing sequence

$$a_1S^1 \subset a_2S^1 \subset a_3S^1 \subset \dots$$

of cyclic subacts of A_S, which contradicts Condition (A).

(2) ⇒ (3). Every right sequence act is locally cyclic, because it is the union of an ascending chain of its cyclic subacts.

(3) ⇒ (4). Assume that all right sequence acts over S are cyclic. Consider a sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ and the right sequence act

$$M_S = \left\{[k, s] \mid k \in \mathbb{N},\ s \in S^1\right\}$$

determined by it. By assumption, M_S must be cyclic. By Lemma 7.22, there exists $m \in \mathbb{N}$ such that $M = [m, 1]S^1$. Then $(m + 1, 1) \sim (m, u)$ for some $u \in S^1$, which means that there exists $k \geqslant m + 1$ such that $s_k \dots s_{m+1} = s_k \dots s_{m+1}s_m u$.

(4) ⇒ (1). Suppose that A_S does not satisfy (ACC) for cyclic subacts. Then there exists a sequence $(a_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that

$$a_1S^1 \subset a_2S^1 \subset a_3S^1 \subset \ldots .$$

Consequently, for every $i \in \mathbb{N}$, there exists $s_i \in S$ such that $a_i = a_{i+1}s_i$. For the sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ there exist $k, m \in \mathbb{N}$, $k \geqslant m+1$, and $u \in S^1$ such that $s_k \ldots s_{m+1} = s_k \ldots s_{m+1}s_m u$. Then

$$\begin{aligned} a_{m+1} = a_{m+2}s_{m+1} = a_{m+3}s_{m+2}s_{m+1} = a_{k+1}s_k \ldots s_{m+1} &= a_{k+1}s_k \ldots s_{m+1}s_m u \\ &= a_k s_{k-1} \ldots s_{m+1}s_m u = a_{m+1}s_m u = a_m u \in a_m S^1. \end{aligned}$$

It follows that $a_{m+1}S^1 = a_mS^1$, a contradiction.

(1) ⇒ (5). Assume that S satisfies Condition (A). Consider an act A_S. By Lemma 7.25, A_S has an independent set of generators X. Then, clearly, $A = \bigcup_{x\in X} xS^1$. Take $y \in X$ and suppose that $yS^1 \subseteq \bigcup_{x\neq y} xS^1$. Then there exists $z \in X \setminus \{y\}$ such that $y \in zS^1$. Independence of X implies $y = z$, a contradiction. Hence

$$yS^1 \nsubseteq \bigcup_{x\neq y} xS^1.$$

(5) ⇒ (6). Let A_S be a unitary act. By (5) there exist cyclic subacts $A_i = a_iS^1$, $i \in I$, such that (7.1) holds. It is easy to see that each a_iS^1 is unitary.

Now $B_S := \coprod_{i\in I} a_iS^1$ is an act whose indecomposable components are cyclic and unitary. Consider the S-act homomorphism

$$\varphi : B_S \longrightarrow A_S, \quad a_is \mapsto a_is.$$

Suppose that φ is not essential. Then there exists $j \in I$ and (possibly empty) proper subact C of a_jS^1 such that $\varphi : C \sqcup \coprod_{i\neq j} a_iS^1 \longrightarrow A$ is surjective. In particular, there exists $b \in C \sqcup \coprod_{i\neq j} a_iS^1$ such that $\varphi(b) = a_j$. We have two possibilities.

1) $b \in C$. Then $b = a_js$ for some $s \in S^1$. Hence $a_j = \varphi(b) = \varphi(a_js) = a_js = b \in C$, which implies $a_jS^1 = C$, a contradiction.

2) $b \in \coprod_{i\neq j} a_iS^1$. Then $a_jS^1 = \varphi(b)S^1 = \varphi(bS^1) \subseteq \bigcup_{i\neq j} a_iS^1$, a contradiction.

(6) ⇒ (3). Assume that S is right $\mathcal{IC}$-perfect and consider a right sequence act M_S determined by a sequence $(s_i)_{i\in\mathbb{N}}$. Since M_S is unitary by the comment preceding Lemma 7.22, it has a cover $\varphi\colon B_S \longrightarrow M_S$, where

$$B_S = \coprod_{i\in I} b_iS^1.$$

Fix an index $j \in I$. Let $\varphi(b_j) = [k, s] \in M_S$. Then $\varphi(b_jS^1) \subseteq [k, 1]S^1$. Consider a number $\ell > k$. Suppose $[\ell, 1]$ has an φ-preimage in b_iS^1 for some $i \neq j$. Then $[\ell, 1]S^1 \subseteq \varphi(b_iS^1)$, which implies that φ restricted to $B \setminus b_jS^1$ is surjective. But that contradicts essentiality of φ. Thus, $[\ell, 1]$ must have a preimage in b_jS^1. Let $[\ell, 1] = \varphi(b_ju)$ for some $u \in S^1$. Now

$$[\ell, 1] = \varphi(b_ju) = \varphi(b_j)u = [k, s]u = [k, 1]su \in [k, 1]S^1,$$

which implies $[\ell, 1]S^1 = [k, 1]S^1$. Since this equality holds for every $\ell > k$, we have shown that $M_S = [k, 1]S^1$, so M_S is cyclic. ■

Remark 7.28. Condition (5) in Theorem 7.27 appears first in Lemma 2.2 of [4], but a similar condition was used in [21]. It is kind of interesting that this condition can be formulated in topological terms, as follows.

If A_S is an act, then the set of its subacts (here including the empty subact) is a topology on the set A, because any union or intersection of subacts is a subact. The set of all cyclic subacts is a basis for this topology. Now condition (5) says that this basis contains a subset which is a minimal cover (in the sense of topology) for A.

7.4 Condition (D)

In this section we will prove that a factorisable semigroup satisfies Condition (D) if and only if it is right semiperfect. The following result can be proved precisely as Lemma 4.4 in [16].

Lemma 7.29. *Let T be a left unitary subsemigroup of a semigroup S. Then*

$$(\forall u, v \in T)(uT^1 \subseteq vT^1 \iff uS^1 \subseteq vS^1).$$

Lemma 7.30. *If a subsemigroup T of a semigroup S is left unitary then it is a ρ-class of some right congruence ρ on S.*

Proof. Assume that T is a left unitary subsemigroup of S. Let ρ be the right congruence on S generated by the set $T \times T$. Take $t \in T$. Then $T \subseteq [t]_\rho$. We will prove that $[t]_\rho \subseteq T$. If $s \in [t]_\rho$, then $t \mathrel{\rho} s$ and

$$\begin{array}{rclclclclcl} t &=& t_1s_1 & & t'_2s_2 &=& t_3s_3 \ldots t'_{n-1}s_{n-1} &=& t_ns_n & & \\ & & t'_1s_1 &=& t_2s_2 & & & & t'_ns_n &=& s \end{array}$$

for some $t_i, t'_i \in T$ and $s_i \in S^1$. If $s_1 \neq 1$, then $s_1 \in T$, because T is left unitary. Hence $t'_1s_1 \in T$, both when $s_1 \neq 1$ and when $s_1 = 1$. If $s_2 \neq 1$, then $s_2 \in T$, and so on. Thus, for every $i \in \{1, \ldots, n\}$, either $s_i = 1$ or $s_i \in T$. It follows that $s = t'_ns_n \in T$. ■

Lemma 7.31. *If S is a semigroup and aS^1 is a unitary cyclic S-act, then*

$$T = \{s \in S \mid as = a\}$$

is a left unitary subsemigroup of S.

Proof. Since aS^1 is unitary, there exist $u \in S^1$ and $s \in S$ such that $a = (au)s = a(us)$. We see that $us \in T$, so T is nonempty, and it is clearly a subsemigroup of S. If $t, ts \in T$, then $as = (at)s = a(ts) = a$, so $s \in S$. Thus, T is left unitary. ∎

Theorem 7.32 (cf. Result 1.5 in [19] and Theorem 5.2 in [43]). *For a factorisable semigroup S the following assertions are equivalent:*

- *(1) S is right semiperfect;*
- *(2) S satisfies Condition (D).*

Proof. (1) ⇒ (2). (This is inspired by the proof of Proposition 4.6 in [16]. Differently from that proof, we cannot assume that R is the congruence class of the identity element, but with appropriate modifications the proof works.)

Let R be a left unitary subsemigroup of S. By Lemma 7.30, it is a ρ-class for some right congruence ρ on S, say $R = [t]$, $t \in R$. Then $[t]S^1$ is a cyclic subact of the right S-act S/ρ. It is a unitary S-act, because $t, t^2 \in R$ implies $[t] = [t]t$.

By assumption, $[t]S^1$ has a projective cover in UAct_S. This must be a unitary cyclic projective act (see the proof of Lemma 4.1 in [16]). Due to Theorem 7.18, it must be of the form eS, where $e \in S$ is an idempotent. Also, there exists an essential epimorphism $\varphi : eS \twoheadrightarrow [t]S^1$ in UAct_S. Because of surjectivity of φ there exists $eu \in eS$ such that $[t] = \varphi(eu)$. Since S is factorisable, euS is a unitary subact of eS. Now $eut \in euS$ and $[t] = [tt] = [t]t = \varphi(eu)t = \varphi(eut)$ imply that $\varphi|_{euS}$ is surjective. Due to essentiality we have the equality $euS = eS$. It follows that $e = eus'$ for some $s' \in S$. Putting $s := s'e \in S$ we have $e = eus$ and $s = se$. Now

$$(su)(su) = (seu)(su) = s(eus)u = seu = su,$$

so su is an idempotent. In addition,

$$[t] = \varphi(eu) = \varphi(eusu) = \varphi(eu)su = [t]su = [tsu].$$

Since R is left unitary, $t, tsu \in R$ implies $su \in R$. Let us prove that suR^1 is a minimal right ideal in R.

For this we show that, for every $r \in R$, $rR^1 \subseteq suR^1$ implies $rR^1 = suR^1$. Let $r \in R$. If $rR^1 \subseteq suR^1$, then $r = sur'$ for some $r' \in R^1$. Hence $r = (su)^2r' = (su)(sur') = sur$ and

$$\varphi(eurt) = \varphi(eu)rt = [t]rt = [trt] = [t],$$

because $trt \in R$. Since $eurS$ is a unitary subact of eS and $\varphi|_{eurS}$ is surjective, $eurS = eS = eS^1$. Now

$$sS = seS = seurS = surS = rS \subseteq rS^1 = surS^1 \subseteq suS^1 \subseteq sS,$$

which implies $rS^1 = suS^1$. Since $r, su \in R$, we conclude that $rR^1 = suR^1$ by Lemma 7.29.

(2) $\Rightarrow$ (1). Assume that S satisfies Condition (D) and consider a unitary cyclic S-act aS^1. The fact that aS^1 is unitary means that $a = at_0$ for some $t_0 \in S$. Now

$$T := \{t \in S \mid at = a\}$$

is a left unitary subsemigroup of S by Lemma 7.31. Since S satisfies Condition (D), there exists an idempotent $e \in T$ (so $ae = a$) such that eT is a minimal right ideal in T. We will show that the unitary S-act eS is a projective cover for aS^1 in UAct_S.

For this, we consider the surjective S-act homomorphism

$$\varphi : eS \longrightarrow aS^1, \quad es \mapsto aes = as.$$

Suppose that B is a unitary subact of eS such that $\varphi|_B$ is surjective. Then there exists $b \in B$ such that $\varphi(b) = a$. Since $B \subseteq eS$, there exists $s \in S$ such that $b = es$. Observe that $a = \varphi(es) = as$. Consequently, $s \in T$ and also $es \in T$. Now, using minimality of eT and Lemma 7.29,

$$esT^1 \subseteq eT \Rightarrow esT^1 = eT \Rightarrow esS^1 = eS \Rightarrow eS = bS^1 \subseteq B \subseteq eS \Rightarrow B = eS.$$

So $B = eS$, as required. ■

7.5 Condition (M_L)

Similarly to Condition (A), Condition (M_L) has a description in terms of sequences of elements of S.

Proposition 7.33. *A semigroup S satisfies Condition (M_L) if and only if*

$$(\forall (s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}})(\exists n \in \mathbb{N})(\forall m > n)(\exists u \in S^1)(us_m \dots s_1 = s_n \dots s_1).$$

Proof. **Necessity.** Assume that S satisfies Condition (M_L). Consider a sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ and denote $t_i := s_i s_{i-1}\dots s_1$ for each $i\in\mathbb{N}$. Since the descending chain

$$S^1t_1 \supseteq S^1t_2 \supseteq S^1t_3 \supseteq \dots$$

stabilises, there exists $n\in\mathbb{N}$ such that $S^1t_n = S^1t_m$ for every $m>n$. Hence $t_n \in S^1t_m$, which means that $us_m\dots s_1 = s_n\dots s_1$ for some $u\in S^1$.

Sufficiency. Suppose to the contrary that S does not satisfy Condition (M_L). Then we have a strictly decreasing chain of principal left ideals

$$S^1t_1 \supset S^1t_2 \supset S^1t_3 \supset \dots .$$

For every $i\in\mathbb{N}$, there exists $s_{i+1}\in S^1$ such that $t_{i+1} = s_{i+1}t_i$. Since we assumed that $t_i \neq t_{i+1}$, we have $s_{i+1}\in S$. Also, put $s_1 := t_1$. By our assumption, there exists $n\in\mathbb{N}$ such that for every $m>n$ there exists $u\in S^1$ such that $us_m\dots s_1 = s_n\dots s_1$. If $n=1$, then there exists $u\in S^1$ such that $us_2t_1 = t_1$, which implies $ut_2 = t_1$. Consequently, $S^1t_1 = S^1t_2$, a contradiction. Otherwise, let $m>n>1$. Then $s_n\dots s_1 = u's_m\dots s_1$ for some $u'\in S^1$ and

$$t_n = s_nt_{n-1} = s_ns_{n-1}t_{n-2} = \dots = s_ns_{n-1}\dots s_1 = u's_ms_{m-1}\dots s_1 = u't_m,$$

whence $S^1t_n \subseteq S^1t_m$, and therefore $S^1t_n = S^1t_m$, a contradiction. ■

We will also need the following lemma.

Lemma 7.34 (cf. Lemma 1 in [13]). *If all right sequence acts over a semigroup S are projective, then S satisfies Condition (M_L).*

Proof. We will employ Proposition 7.33. Consider a sequence $(s_i)_{i\in\mathbb{N}}$ and the right sequence act M_S determined by it. By assumption, M_S is projective. Hence, for the surjective morphism

$$\pi : \mathbb{N}\times S^1 = F_S \longrightarrow M_S,\quad (k,s)\mapsto [k,s],$$

there exists a morphism $\mu : M_S \longrightarrow F_S$ such that $\pi\mu = \mathrm{id}_M$.

$$\begin{tikzcd}
 & M_S \arrow[dl, dashed, "\mu"'] \arrow[d, "\mathrm{id}_M"] \\
F_S \arrow[r, "\pi"'] & M_S
\end{tikzcd}$$

Let $\mu([1,1]) = (k,s) \in \mathbb{N} \times S^1$. Then

$$[1,1] = (\pi\mu)([1,1]) = \pi(k,s) = [k,s].$$

Since $(1,1) \sim (k,s)$, there exists $n \geqslant k$ such that $s_n \dots s_1 \cdot 1 = s_n \dots s_k \cdot s$. Let $m > n$ and put $\mu([m+1,1]) = (r,c)$ for some $r \in \mathbb{N}$ and $c \in S^1$. Note that $(1,1) \sim (m+1, s_m \dots s_1)$, because

$$s_{m+1} \dots s_1 \cdot 1 = s_{m+1} \cdot s_m \dots s_1.$$

Now

$$\begin{aligned}(k,s) = \mu([1,1]) &= \mu([m+1, s_m \dots s_1]) = \mu([m+1,1])s_m \dots s_1 \\ &= (r,c)s_m \dots s_1 = (r, cs_m \dots s_1),\end{aligned}$$

whence $k = r$ and $s = cs_m \dots s_1$. Putting $u := s_n \dots s_k c \in S$ we have

$$s_n \dots s_1 = s_n \dots s_k s = s_n \dots s_k c s_m \dots s_1 = us_m \dots s_1,$$

as required. ■

7.6 Pullback flatness

Definition 7.35. A right S-act A_S is called **pullback flat**, if the functor $A \otimes_S - : {}_S\mathsf{Act} \longrightarrow \mathsf{Set}$ preserves pullbacks.

We will need the following conditions (introduced first for acts over monoids in [46]) and the following theorem.

Definition 7.36. An S-act A_S is said to satisfy **Condition**

(P) if whenever $as = a's'$ for some $a, a' \in A$ and $s, s' \in S$, there exist $a'' \in A$ and $u, v \in S$ such that

$$a = a''u, \quad a' = a''v, \quad us = vs';$$

(E) if whenever $as = as'$ for some $a \in A$ and $s, s' \in S$, there exist $a' \in A$ and $u \in S$ such that

$$a = a'u, \quad us = us'.$$

Theorem 7.37 (cf. Theorem 3.5 in [37]). *Let A_S be a unitary act over a semigroup S. Then A_S is pullback flat if and only if it satisfies both Condition (P) and Condition (E).*

Proposition 7.38. *A cyclic S-act aS^1 is unitary and pullback flat if and only if*

$$(\forall s,t \in S^1)(as = at \Rightarrow (\exists u \in S)(a = au \quad \textit{and} \quad us = ut)).$$

Proof. **Necessity**. Let $A = aS^1$ be unitary and pullback flat. Take $s, t \in S^1$ such that $as = at$. Since aS^1 is unitary, $a = av$ for some $v \in S$. Then $a(vs) = a(vt)$, where $vs, vt \in S$. By Theorem 7.37, aS^1 satisfies Condition (E), so there exists $w \in S$ such that $a = aw$ and $w(vs) = w(vt)$. Putting $u := wv$, we have $a = au$ and $us = ut$.

Sufficiency. By assumption, there exists $u \in S$ such that $a = au$, thus aS^1 is unitary. That aS^1 satisfies Condition (E) follows immediately from the assumption. To prove Condition (P), suppose that $(as_1)s = (as_2)s'$, where $s_1, s_2 \in S^1$ and $s, s' \in S$. By assumption, there exists $u \in S$ such that $a = au$ and $us_1s = us_2s'$. Now $as_1 = a(us_1)$ and $as_2 = a(us_2)$, where $us_1, us_2 \in S$.■

Lemma 7.39 (cf. Lemma 3 in [13]). *Let A_S be a unitary pullback flat act which satisfies (ACC) for cyclic subacts. If A_S is indecomposable, then it is cyclic.*

Proof. Take any $x \in A$ and consider the set

$$P_x := \{bS^1 \mid b \in A,\ xS^1 \subseteq bS^1\}$$

as a poset with respect to inclusion. Since $xS^1 \in P_x$, we have $P_x \neq \emptyset$, so by assumption, there exists a maximal element $aS^1 \in P_x$. In particular, $xS^1 \subseteq aS^1$. Consider the set $Y := A \setminus aS^1$. Assume for a contradiction that $Y \neq \emptyset$. Now, A_S is a union of its subacts aS^1 and YS^1. Since A_S is indecomposable,

$$aS^1 \cap YS^1 \neq \emptyset,$$

which means that there exist $y \in Y$ and $s,\ t \in S^1$ such that $as = yt$.

On the one hand, if $t = 1$, then $y = as \in aS^1$, a contradiction. On the other hand, suppose $s = 1$. The equality $a = yt$ implies that $xS^1 \subseteq aS^1 \subseteq yS^1$. It follows that $yS^1 \in P_x$ and $aS^1 = yS^1$ due to maximality. So again, we must have $y \in aS^1$, which is impossible. Thus, we must have $s, t \in S$.

Since A_S is a unitary pullback flat act, by Theorem 7.37 it satisfies Condition (P). Hence, there exist $a' \in A$ and $u, v \in S$ such that $a = a'u$, $y = a'v$ and $us = vt$. We conclude that $aS^1 \subseteq a'S^1$. It follows that $a'S^1 \in P_x$, so $aS^1 = a'S^1$ due to maximality. Now, for some $z \in S^1$, $a' = az$ and therefore $y = a'v = azv$, whence $y \in aS^1$, a contradiction. Thus, $Y = \emptyset$ and $A = aS^1$ is cyclic, as required. ■

Corollary 7.40. *If a semigroup S satisfies Condition (A), then every unitary pullback flat S-act is a disjoint union of cyclic unitary pullback flat subacts.*

Proof. Assume that S satisfies Condition (A) and let A_S be unitary and pullback flat. By Proposition 7.14,

$$A = \bigsqcup_{i \in I} A_i,$$

where A_i is an indecomposable subact of A_S for every $i \in I$. It is clear by Theorem 7.37 that the A_i are pullback flat. Since S satisfies Condition (A), each A_i satisfies (ACC) for cyclic subacts. We see that the A_i satisfy all assumptions of Lemma 7.39, so they must be cyclic. ■

Lemma 7.41 (cf. Lemmas 4 and 5 in [13]). *If a factorisable semigroup S satisfies either Condition (D) or Condition (M_L), then every unitary cyclic pullback flat right S-act is projective.*

Proof. Let $A_S = aS^1$ be a unitary cyclic pullback flat S-act. By Lemma 7.31, the set $T = \{s \in S \mid as = a\}$ is a left unitary subsemigroup of S. If S satisfies Condition (D), then T contains a minimal right ideal eT, where $e \in T$ is an idempotent. If S satisfies Condition (M_L), then the set

$$\mathcal{I} = \{S^1 t \mid t \in T\}$$

of principal left ideals of S contains a minimal element S^1e for some element $e \in T$. In both cases, it suffices to prove that condition (4) of Proposition 7.19 is satisfied.

Assume that $as = at$ for some $s, t \in S^1$. Since aS^1 is pullback flat, by Proposition 7.38 there exists $u \in S$ such that $a = au$ and $us = ut$. The equality $au = ae$ implies that $a = av$ and $vu = ve$ for some $v \in S$. Note that $u, v \in T$.

Firstly, let S satisfy Condition (D). By Lemma 8.12 in [12], Te is a minimal left ideal of T. It follows that

$$Tvu \subseteq Tu \cap Te \subseteq Te,$$

whence $Te = Tvu \subseteq Tu$ due to minimality. Therefore, $e = wu$ for some $w \in T$. We conclude that $es = wus = wut = et$, as required.

Secondly, assume S satisfies Condition (M_L). Since T is a subsemigroup, $vu \in T$ and $S^1vu \in \mathcal{I}$. The equality $vu = ve$ implies $S^1vu \subseteq S^1e$. We have $S^1vu = S^1e$ due to minimality and $e = wvu$ for some $w \in S^1$. This implies $es = wvus = wvut = et$. In particular, $a1 = ae$ implies that $e1 = ee$ by the previous argument, so e is an idempotent. ■

7.7 Condition (K)

The following is inspired by Definition 1.6 in [22].

Definition 7.42. We say that a subsemigroup T of S is **left collapsible** if T^1 is a left collapsible submonoid of S^1, that is,

$$(\forall t, t' \in T^1)(\exists u \in T^1)(ut = ut').$$

Condition (K) for monoids was introduced by Kilp in [22] and used later in [23]. We use it for semigroups in the following form.

Definition 7.43. A semigroup S satisfies **Condition** (K) if every left collapsible subsemigroup of S has a left zero.

Proposition 7.44 (cf. Theorem 2.3 in [22]). *A factorisable semigroup S satisfies Condition (K) if and only if every unitary cyclic pullback flat right S-act is projective.*

Proof. **Necessity.** Suppose S satisfies Condition (K). Let aS^1 be unitary pullback flat and let $T = \{s \in S \mid as = a\}$. If $t, t' \in T^1$, then $at = at'$. By Proposition 7.38, there exists $u \in S$ such that $a = au$ and $ut = ut'$. In particular, $u \in T$. Thus, T^1 is a left collapsible submonoid of S^1.

By assumption, T has a left zero e. We will check condition (4) of Proposition 7.19. We know that $a = ae$. Suppose that $as = at$, $s, t \in S^1$. Then there exists $u \in S$ such that $a = au$ and $us = ut$. Then $u \in T$, $eu = e$ and $es = eus = eut = et$.

Sufficiency. Assume that every unitary cyclic pullback flat right S-act is projective. Let $T \subseteq S$ be a left collapsible subsemigroup of S. Then by definition $P := T^1$ is a left collapsible submonoid of S^1. By Lemma 2.1 in [22], the relation ρ, defined by

$$s \,\rho\, t \iff (\exists p, q \in P)(ps = qt),$$

is a right congruence on the monoid S^1. In particular, $1 \,\rho\, p$ for all $p \in P$. The cyclic S^1-act $S^1/\rho = [1]_\rho S^1$ is pullback flat.

It turns out that $S^1/\rho = [1]_\rho S^1$ is also a cyclic unitary pullback flat S-act. Since T is nonempty, we can choose $t_0 \in T \subseteq S$. If $[1]_\rho s = [1]_\rho t$, $s, t \in S^1$, then Condition (E) implies that there exists $u \in S^1$ such that $[1]_\rho = [1]_\rho u$ and $us = ut$. If it happens that $u = 1$, then still $[1]_\rho = [1]_\rho t_0$ and $t_0 s = t_0 t$. By Proposition 7.38, S^1/ρ is a unitary pullback flat S-act.

By assumption, $[1]_\rho S^1$ is projective. Using Corollary 7.20 we know that there exists $e \in E(S)$ such that $[1]_\rho = [1]_\rho e$ and $[1]_\rho s = [1]_\rho t$ implies $es = et$

for all $s, t \in S^1$. In other words, $1 \, \rho \, e$, and $s \, \rho \, t$ implies $es = et$ for all $s, t \in S^1$. From $1 \, \rho \, e$ we obtain $p, q \in P$ such that $p = qe$. Let $r \in P$ be arbitrary. Then $1 \, \rho \, r$, which implies $e = er$, and therefore $pr = qer = qe = p$. Thus, p is a left zero for P. If $p = 1$, then $1 = qe = qee = e$, which contradicts the fact that $e \in S$. Thus, $p \in T$ and p is a left zero for T. ■

Corollary 7.45. *If S is factorisable, then Condition (M_L) implies Condition (K).*

Proof. This follows from Proposition 7.44 and Lemma 7.41. ■

7.8 The main theorem

Our main theorem is the following.

Theorem 7.46. *For a factorisable semigroup S, the following are equivalent:*

(1) S is right perfect;

(2) S is right $\mathcal{IC}$-perfect and right semiperfect;

(3) S satisfies both Condition (A) and Condition (D);

(4) S satisfies both Condition (A) and Condition (M_L);

(5) S satisfies both Condition (A) and Condition (K);

(6) every right sequence act over S is projective in UAct_S;

(7) every unitary pullback flat right S-act is projective in UAct_S.

Proof. (1) ⇒ (2). If a semigroup S is right perfect, then it is clearly right semiperfect. If $\bigsqcup_{i \in I} e_i S \twoheadrightarrow A_S$ is a projective cover for a unitary act A_S, then its indecomposable components $e_i S$, $i \in I$, are cyclic and unitary. Hence S is also right $\mathcal{IC}$-perfect.

(2) ⇒ (1) Let A_S be a unitary S-act. Since S is $\mathcal{IC}$-perfect, there exists a cover $\psi : B_S \twoheadrightarrow A_S$, where $B_S = \bigsqcup_{i \in I} b_i S^1$ is a disjoint union of unitary cyclic subacts. Semiperfectness of S gives a unitary projective cover $\varphi_i : P_i \twoheadrightarrow b_i S^1$ for each $i \in I$.

Then $P_S := \bigsqcup_{i \in I} P_i$ is a unitary projective act by Theorem 7.18 and

$$\varphi : P_S \twoheadrightarrow B_S, \quad x \mapsto \varphi_i(x) \text{ if } x \in P_i,$$

is a surjective morphism of right S-acts. We will show that φ is essential.

Suppose that Q is a proper subact of P_S. Then there exists $i \in I$ such that $Q_i := Q \cap P_i$ is a proper (possibly empty) subact of P_i. Hence $\varphi_i|_{Q_i} : Q_i \longrightarrow b_i S^1$ is not surjective, because φ_i is essential. But then also $\varphi|_Q$ is not surjective, because the preimages of elements of $b_i S^1$ can only come from P_i. Thus, φ is an essential epimorphism. Hence, $\psi\varphi : P_S \longrightarrow A_S$ is an essential epimorphism and P_S is a projective cover for A_S.

(2) $\Leftrightarrow$ (3). This follows from Theorem 7.32 and Theorem 7.27.

(6) $\Rightarrow$ (3). Assume that all right sequence acts are projective. Since they are indecomposable (see Lemma 7.22), unitary and projective, they must be cyclic. Hence S satisfies Condition (A) by Theorem 7.27.

We will prove that S satisfies Condition (D). The proof of this implication is inspired by the proof of Theorem 6.2 in [16].

Let T be a left unitary subsemigroup of S. Take any $t \in T$. We will prove that the principal right ideal tT^1 of T contains an idempotent. By the assumption, the right sequence act M_S, determined by the constant sequence $(t) \in S^{\mathbb{N}}$, is projective. Since M_S is indecomposable, it must be cyclic. From Lemma 7.22 we conclude that there exists $i \in \mathbb{N}$ such that $M = [i,1]S^1$. By Corollary 7.20, there exists $e^2 = e \in S$ such that $[i,1] = [i,e]$ and

$$(\forall x, y \in S^1)([i,x] = [i,y] \Rightarrow ex = ey). \tag{7.2}$$

From $[i,1]S^1 \subseteq [i+1,1]S^1 \subseteq M_S = [i,1]S^1$ we conclude that $[i,1]S^1 = [i+1,1]S^1$. Hence there exists $z \in S^1$ such that $[i+1,1] = [i,z]$. We have the diagram

$$\begin{tikzcd}
M = [i,1]S^1 \arrow[d, "\tau"'] & \\
(i,1)S^1 \arrow[r, "\alpha"'] & (i+1,1)S^1 \arrow[ul, "\beta"']
\end{tikzcd}$$

with 1-generated free S-acts $(i,1)S^1$ and $(i+1,1)S^1$, and with the right S-act morphisms defined by

$$\begin{aligned}
\alpha(i,s) &:= (i+1, ts),\\
\beta(i+1,s) &:= [i+1,s] = [i,zs],\\
\tau([i,s]) &:= (i, es),\\
\psi &:= \alpha\tau\beta : (i+1,1)S^1 \longrightarrow (i+1,1)S^1.
\end{aligned}$$

Note that τ is well defined due to the implication (7.2).

Denoting $h := tez$, since $[i+1,t]=[i,1]$, we have that

$$\begin{aligned}(\beta\alpha\tau)([i,z]) &= (\beta\alpha)(i,ez)=\beta(i+1,tez)=[i+1,tez]=[i+1,t]ez=[i,1]ez\\ &= [i,e]z=[i,1]z=[i,z]\end{aligned}$$

and

$$\begin{aligned}\psi^2(i+1,1) &= (\psi\alpha\tau\beta)(i+1,1)=(\psi\alpha\tau)([i+1,1])=(\alpha\tau\beta\alpha\tau)([i,z])\\ &= (\alpha\tau)([i,z])=(\alpha\tau\beta)(i+1,1)=\psi(i+1,1)=(\alpha\tau)([i,z])\\ &= \alpha(i,ez)=(i+1,tez)=(i+1,h).\end{aligned}$$

Hence,

$$\begin{aligned}(i+1,h)=\psi^2(i+1,1)=\psi(i+1,h)=\psi(i+1,1)h &= (i+1,h)h\\ &= (i+1,h^2),\end{aligned}$$

which yields $h^2=h$. The equalities $[i+1,1]=[i,z]$ and $[i,1]=[i,e]$ imply that $t^n=t^{n+1}z$ and $t^m=t^me$ for some $m,n\in\mathbb{N}$. Since T is a left unitary subsemigroup, $z,e\in T$ and we have an idempotent $h\in tT^1$.

Now consider a chain

$$e_1T^1\supseteq e_2T^1\supseteq e_3T^1\supseteq\dots$$

of principal right ideals of T generated by idempotents $e_i\in T$. By Lemma 7.29, we have a chain

$$e_1S^1\supseteq e_2S^1\supseteq e_3S^1\supseteq\dots.$$

The following is standard semigroup trickery. We put $g_1:=e_1$, then $g_1S^1=e_1S^1$. Define inductively $g_{i+1}:=e_{i+1}g_i$ for every $i>1$. Since $e_{i+1}S^1\subseteq e_iS^1$, we have $e_ie_{i+1}=e_{i+1}$. Using induction, we assume that g_i is an idempotent such that $g_iS^1=e_iS^1$. Then $e_i=g_ie_i$. Also, $g_ie_{i+1}=e_{i+1}$, because $e_{i+1}\in g_iS^1$. It follows that $g_{i+1}^2=e_{i+1}g_ie_{i+1}g_i=e_{i+1}e_{i+1}g_i=g_{i+1}$. Clearly, $g_{i+1}S^1\subseteq e_{i+1}S^1$. Conversely, from

$$e_{i+1}=e_{i+1}e_{i+1}=e_{i+1}e_ie_{i+1}=e_{i+1}(g_ie_i)e_{i+1}=(e_{i+1}g_i)(e_ie_{i+1})=g_{i+1}e_{i+1}$$

we conclude that $e_{i+1}S^1\subseteq g_{i+1}S^1$. Thus, $e_{i+1}S^1=g_{i+1}S^1$. We also have $g_ig_{i+1}=g_{i+1}$ (because $g_{i+1}\in g_iS^1$) and

$$g_{i+1}g_i=e_{i+1}g_ig_i=e_{i+1}g_i=g_{i+1}.$$

Hence, we have a chain

$$S^1g_1\supseteq S^1g_2\supseteq S^1g_3\supseteq\dots,$$

where $g_{i+1}g_i = g_{i+1} = g_ig_{i+1}$ for every $i \in \mathbb{N}$.

By Lemma 7.34, S satisfies Condition (M_L). Thus, there exists $n \in \mathbb{N}$ such that $S^1g_i = S^1g_{i+1}$ for every $i \geqslant n$. Now $g_i = g_ig_{i+1} = g_{i+1}$, therefore $g_i = g_{i+1}$ and $e_iS^1 = e_{i+1}S^1$ for every $i \geqslant n$. By Lemma 7.29, it follows that $e_iT^1 = e_{i+1}T^1$ for every $i \geqslant n$. We have shown that T satisfies the descending chain condition for principal right ideals generated by idempotents.

Suppose that T does not have a minimal principal right ideal. Take any $b_1 \in T$. Then b_1T^1 contains and idempotent e_1 and $b_1T^1 \supseteq e_1T^1$. Since e_1T^1 is not a minimal ideal, there exists $b_2 \in T$ such that $e_1T^1 \supset b_2T^1$. In this way we get an infinite descending chain

$$b_1T^1 \supseteq e_1T^1 \supset b_2T^1 \supseteq e_2T^1 \supset b_3T^1 \supseteq e_3T^1 \supset \dots$$

of principal right ideals of T, where $e_i \in E(T)$. Thus, we also have a chain

$$e_1T^1 \supset e_2T^1 \supset e_3T^1 \supset \dots,$$

a contradiction. It follows that T has a minimal principal right ideal bT^1. As it contains an idempotent e, we have $bT^1 = eT^1$ due to minimality.

(4) ⇒ (5). By Corollary 7.45.

(5) ⇒ (6). Assume that S satisfies both Condition (A) and Condition (K). Let M_S be a right sequence act over S. By Theorem 7.27, M_S is cyclic. Due to Corollary 3.11 in [37], M_S is pullback flat. By Proposition 7.44, M_S is projective.

(6) ⇒ (4). By Lemma 7.34, S satisfies Condition (M_L). We show that S satisfies Condition (A) using Theorem 7.27. Take any sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ and consider the right sequence act M_S determined by it. By assumption, it is projective. It is also indecomposable, hence, by Proposition 7.19, there exists an S-isomorphism $\varphi : M_S \longrightarrow eS$, where $e \in E(S)$ is some idempotent. Then there exists $[m, s] \in M$ such that $\varphi([m, s]) = e$. Also, there exists $s' \in S$ such that $\varphi([m + 1, 1]) = es'$. Now

$$\varphi([m + 1, 1]) = \varphi([m, s])s' = \varphi([m, ss']) \Rightarrow [m + 1, 1] = [m, ss'].$$

Thus, there exists $k \geqslant m + 1$ such that

$$s_k \dots s_{m+1} = s_k \dots s_{m+1}s_mss'.$$

(3) ∨ (4) ⇒ (7). Let A_S be a unitary pullback flat act. Using Condition (A), by Corollary 7.40 we conclude that $A_S = \bigsqcup_{i\in I} A_i$, where A_i is a cyclic unitary pullback flat act for every $i \in I$. We have by Lemma 7.41 that each A_i is projective, when S satisfies either Condition (D) or Condition (M_L). Thus, A_S is projective due to Theorem 7.15.

(7) $\Rightarrow$ (6). Every right sequence act M_S satisfies Condition (E) by Lemma 3.10 in [37]. It is easy to see that M_S satisfies Condition (LC): if $a, a' \in M$, then there exist $a'' \in M$ and $u, v \in S$ such that $a = a''u$ and $a' = a''v$. By Theorem 4.2 in [37], right sequence acts are pullback flat. We also know that they are unitary. By assumption, they are projective. ■

Remark 7.47. 1. Condition (4) in Theorem 7.46 is important, because together with Theorem 7.27 and Proposition 7.33 it shows that perfectness of a factorisable semigroup S can be verified by checking two conditions that are formulated in terms of sequences of elements of S. These conditions are internal to S and do not refer to any categories.

2. If S is factorisable and right perfect, then any right sequence act over S is projective. In particular, S must contain at least one idempotent.

Corollary 7.48 (cf. Corollary 3.13 in [4]). *Every completely* (0-)*simple semigroup is perfect.*

Proof. We show that a completely 0-simple semigroup $S := \mathcal{M}^0(G, U, V, p)$, where G is a group, is right perfect. A dual argument shows that it is left perfect. Perfectness for completely simple semigroups follows similarly.

It suffices to prove that S satisfies both Condition (A) and Condition (M_L). To verify Condition (A), we use item (4) in Theorem 7.27. It is readily verified that

$$(\forall x, y \in S \setminus \{0\})(\exists u, v \in S \setminus \{0\})(x = xyu \text{ and } vyx = x).$$

Take a sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$. We consider three cases:

(1) $s_i \neq 0$ for every $i \in \mathbb{N}$. Then $s_2 = s_2 s_1 u$ for some $u \in S$;

(2) $s_1 = 0$, but $s_i \neq 0$ for every $i > 1$. Then $s_3 = s_3 s_2 u$ for some $u \in S$;

(3) $s_i = 0$ for some $i > 1$. Then $s_i = s_i s_{i-1}$ and we take $u = 1$.

To verify Condition (M_L), we will check the condition of Proposition 7.33 for the same sequence. If the sequence contains no zero, then we can take $n = 1$. If $s_n = 0$, then $s_m \dots s_n \dots s_1 = s_n \dots s_1$ clearly holds. ■

7.9 Morita invariance

In this section we will prove that right semiperfectness, right $\mathcal{IC}$-perfectness and right perfectness are Morita invariants on the class of factorisable semigroups. For this, we need to examine essential epimorphisms in the categories UAct_S and FAct_S.

Proposition 7.49. *Let S be a semigroup and $A_S \in \mathsf{UAct}_S$. Then μ_A is an essential epimorphism in UAct_S.*

Proof. Since A_S is unitary, μ_A is surjective, and it is easy to see that $A \otimes_S S$ is unitary. Let $U \subseteq A \otimes S$ be a unitary subact and consider the diagram

$$U_S \xrightarrow{\iota} A \otimes_S S \xrightarrow{\mu_A} A_S$$

in UAct_S, where ι is the embedding. Suppose $\mu_A \iota$ is surjective. By Lemma 7.7, it suffices to show that ι is surjective, i.e, $U = A \otimes S$. Take $a \otimes s \in A \otimes S$. By surjectivity of $\mu_A \iota$, there exist $a' \in A$ and $s' \in S$ such that $a' \otimes s' \in U$ and $a = (\mu_A \iota)(a' \otimes s') = a's'$. Now

$$a \otimes s = a's' \otimes s = a' \otimes s's = (a' \otimes s')s \in US \subseteq U.$$

Thus, ι is surjective, as required. ■

Proposition 7.50. *Let S be a factorisable semigroup. For a morphism $\varphi : P_S \longrightarrow A_S$ in FAct_S, the following are equivalent:*

(1) φ is an essential epimorphism in UAct_S;

(2) φ is an essential epimorphism in FAct_S.

Proof. (1) ⇒ (2). This is clear.

(2) ⇒ (1). Let $U \subseteq P_S$ be a unitary subact and let $\iota : U \longrightarrow P$ be the embedding. Suppose $\varphi\iota$ is surjective. By Lemma 7.7, it suffices to show that ι is surjective, i.e., $U = P$. According to Lemma 7.1, $U \otimes_S S$ is a firm right S-act. Consider the diagram

$$U \otimes_S S \xrightarrow{\mu_U} U_S \xrightarrow{\iota} P_S \xrightarrow{\varphi} A_S.$$

Since μ_U is surjective, $\varphi\iota\mu_U$ is a surjective morphism in FAct_S. Also, $\iota\mu_U$ is a morphism in FAct_S. Then $\iota\mu_U$ is surjective due to essentiality of φ. It follows that ι is surjective. ■

Next we consider cyclic acts. Note that a cyclic unitary act need not be firm. An example of such an act is given in Example 2.12 of [34].

Proposition 7.51. *The following are equivalent for a factorisable semigroup S:*

(1) every cyclic act in UAct_S has a projective cover;

(2) every cyclic act in FAct_S has a projective cover.

Proof. (1) ⇒ (2). Take a cyclic act $aS^1 \in \mathsf{FAct}_S$. Since aS^1 is unitary, it has a cyclic projective cover $\varphi : eS \twoheadrightarrow aS^1$ in UAct_S (note that a cover of a cyclic act must be cyclic). Recall that eS is firm (see Lemma 7.16). Consider a diagram

$$B_S \xrightarrow{\psi} eS \xrightarrow{\varphi} aS^1$$

in FAct_S. If $\varphi\psi$ is surjective, then ψ must be surjective, because this diagram is also in UAct_S and φ is an essential epimorphism in UAct_S. Thus, φ is an essential epimorphism in FAct_S.

(2) ⇒ (1). Take a cyclic act $aS^1 \in \mathsf{UAct}_S$. By Lemma 7.1, $aS^1 \otimes_S S$ is firm. Since aS^1 is unitary, there exist $s \in S$ and $u \in S^1$ such that $a = (au)s$. Then

$$aS^1 \otimes_S S = (au \otimes s)S^1.$$

The inclusion $\supseteq$ is clear. On the other hand, if $v \in S^1$ and $t \in S$, then

$$av \otimes t = (au)sv \otimes t = au \otimes svt = (au \otimes s)vt \in (au \otimes s)S^1.$$

Therefore, $aS^1 \otimes_S S$ is a cyclic right S-act.

By assumption, there exists a projective cover $\varphi : eS \twoheadrightarrow aS^1 \otimes S$ in FAct_S. By Proposition 7.50, φ is also an essential epimorphism in UAct_S. Proposition 7.49 implies that $\mu_{aS^1} : aS^1 \otimes S \twoheadrightarrow aS^1$ is an essential epimorphism in UAct_S. Therefore,

$$eS \xrightarrow{\mu_{aS^1}\varphi} aS^1$$

is a projective cover in UAct_S. ■

Having a projective cover is a purely categorical property, thus it is preserved by equivalence functors. Also, indecomposability is a categorical property: an act is indecomposable if it is not a coproduct of two noninitial objects (the initial object is the empty act). However, for cyclicity there is no obvious categorical description. Still, cyclic acts are preserved under tensor multiplication functors coming from certain Morita contexts.

Proposition 7.52. *Let S and T be firm semigroups connected by a unitary surjective Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then the functors*

$$- \otimes {}_SP_T : \mathsf{FAct}_S \twoheadrightarrow \mathsf{FAct}_T \quad \text{and} \quad - \otimes_T Q_S : \mathsf{FAct}_T \twoheadrightarrow \mathsf{FAct}_S$$

take cyclic acts to cyclic acts.

Proof. As in Proposition 3.16 of [38], one can prove that the functors $- \otimes {}_SP_T : \mathsf{FAct}_S \twoheadrightarrow \mathsf{FAct}_T$ and $- \otimes {}_TQ_S : \mathsf{FAct}_T \twoheadrightarrow \mathsf{FAct}_S$ are mutually inverse equivalence functors (see also Theorem 5.9 in [33]). We will prove that $- \otimes$

${}_SP_T$ takes cyclic acts to cyclic acts. The same is true for $-\otimes {}_TQ_S$, so cyclic acts correspond to each other under these functors.

Consider a cyclic act $aS^1 \in \mathsf{FAct}_S$. We will prove that $aS^1 \otimes_S P_T$ is cyclic. Since aS^1 is unitary, there exists $s \in S$ such that $as = a$. Using surjectivity of θ we can find $p_s \in P$ and $q_s \in Q$ such that $s = \theta(p_s \otimes q_s)$. We will prove that

$$aS^1 \otimes_S P_T = (a \otimes p_s)T^1.$$

The inclusion $(a \otimes p_s)T^1 \subseteq aS^1 \otimes_S P_T$ is clear. To prove the converse, we note that

$$\begin{aligned} au \otimes p = asu \otimes p = a \otimes sup = a \otimes \theta(p_s \otimes q_s)up = a \otimes p_s\phi(q_s \otimes up) \\ = (a \otimes p_s)\phi(q_s \otimes up) \in (a \otimes p_s)T^1 \end{aligned}$$

for every $u \in S^1$ and $p \in P$. ■

Proposition 7.53. *Let S be a factorisable semigroup. Then the functors*

$$-\otimes_S S_{S\otimes S} : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_{S\otimes S} \quad \textit{and} \quad -\otimes_{S\otimes S} S_S : \mathsf{FAct}_{S\otimes S} \longrightarrow \mathsf{FAct}_S$$

are mutually inverse equivalence functors which take cyclic acts to cyclic acts.

Proof. By Proposition 4.9 in [35], these functors are mutually inverse equivalence functors. We also recall that $S \otimes S$ is considered as a semigroup with the multiplication $(s \otimes t)(u \otimes v) = st \otimes uv$ and the action of $S_{S\otimes S}$ is $s(u \otimes v) = suv$.

Let $aS^1 \in \mathsf{FAct}_S$ be cyclic. We want to show that $aS^1 \otimes_S S_{S\otimes S}$ is a cyclic $S \otimes S$-act. Suppose $a = (au)s$ for some $u \in S^1$ and $s \in S$. We will prove that

$$aS^1 \otimes_S S_{S\otimes S} = (au \otimes s)(S \otimes S)^1.$$

On the one hand

$$(au \otimes s)(S \otimes S)^1 \subseteq (aS^1 \otimes_S S)(S \otimes S)^1 = aS^1 \otimes_S S.$$

Conversely, suppose that $av \otimes t \in aS^1 \otimes_S S_{S\otimes S}$. Since S is factorisable, $t = s_1s_2$ for some $s_1, s_2 \in S$. Hence,

$$\begin{aligned} av \otimes t = ausv \otimes t = au \otimes svt = au \otimes svs_1s_2 = au \otimes (s(vs_1 \otimes s_2)) \\ = (au \otimes s)(vs_1 \otimes s_2) \in (au \otimes s)(S \otimes S)^1. \end{aligned}$$

Let now $a(S \otimes S)^1 \in \mathsf{FAct}_{S\otimes S}$ be cyclic. We want to show that $a(S \otimes S)^1 \otimes_{S\otimes S} S_S$ is a cyclic S-act. Since $a(S \otimes S)^1$ is unitary, $a = a(s_1 \otimes s_2)$ for some $s_1, s_2 \in S$. We will prove that

$$a(S \otimes S)^1 \otimes_{S\otimes S} S_S = (a \otimes s_1s_2)S^1.$$

The inclusion $\supseteq$ is clear. Conversely, let $a(u_1 \otimes u_2) \otimes s \in a(S \otimes S)^1 \otimes_{S \otimes S} S_S$, where $u_1, u_2, s \in S$. Then

$$\begin{aligned} a(u_1 \otimes u_2) \otimes s = a(s_1 \otimes s_2)(u_1 \otimes u_2) \otimes s &= a(s_1 s_2 \otimes u_1 u_2) \otimes s \\ &= a \otimes (s_1 s_2 \otimes u_1 u_2)s \\ &= a \otimes s_1 s_2 u_1 u_2 s \\ &= (a \otimes s_1 s_2) u_1 u_2 s. \end{aligned}$$

So $a(u_1 \otimes u_2) \otimes s = (a \otimes s_1 s_2) u_1 u_2 s \in (a \otimes s_1 s_2) S^1$. ■

Theorem 7.54. *Right semiperfectness is a Morita invariant for factorisable semigroups.*

Proof. Suppose S and T are Morita equivalent factorisable semigroups. Equivalently, by Proposition 4.9 in [35], $S \otimes S$ and $T \otimes T$ are Morita equivalent firm semigroups. According to Theorem 5.9 in [33], there exists a unitary Morita context with bijective mappings connecting $S \otimes S$ and $T \otimes T$. Let $\mathcal{P}$-cover stand for 'projective cover', then we have that

$$\begin{aligned} S \text{ is right semiperfect} &\iff \text{all cyclic objects in } \mathsf{UAct}_S \text{ have a } \mathcal{P}\text{-cover} \\ \text{(by Proposition 7.51)} &\iff \text{all cyclic objects in } \mathsf{FAct}_S \text{ have a } \mathcal{P}\text{-cover} \\ \text{(by Proposition 7.53)} &\iff \text{all cyclic objects in } \mathsf{FAct}_{S \otimes S} \text{ have a } \mathcal{P}\text{-cover} \\ \text{(by Proposition 7.52)} &\iff \text{all cyclic objects in } \mathsf{FAct}_{T \otimes T} \text{ have a } \mathcal{P}\text{-cover} \\ \text{(by Proposition 7.53)} &\iff \text{all cyclic objects in } \mathsf{FAct}_T \text{ have a } \mathcal{P}\text{-cover} \\ \text{(by Proposition 7.51)} &\iff \text{all cyclic objects in } \mathsf{UAct}_T \text{ have a } \mathcal{P}\text{-cover} \\ &\iff T \text{ is right semiperfect.} \end{aligned}$$

■

Proposition 7.55. *The following are equivalent for a factorisable semigroup S:*

(1) every act in UAct_S has an $\mathcal{IC}$-cover;

(2) every act in FAct_S has an $\mathcal{IC}$-cover.

Proof. (1) ⇒ (2) Take $A_S \in \mathsf{FAct}_S$ and let

$$B_S := \bigsqcup_{i \in I} b_i S^1 \xrightarrow{\varphi} A_S$$

be an $\mathcal{IC}$-cover in UAct_S. By Lemma 7.1, $B \otimes_S S$ is firm. The indecomposable components of B are unitary. Hence, for every $i \in I$, there exist $u_i \in S^1$ and $s_i \in S$ such that $b_i = (b_i u_i) s_i$. Then

$$B \otimes_S S = \left(\bigsqcup_{i \in I} b_i S^1 \right) \otimes_S S = \bigsqcup_{i \in I} (b_i S^1 \otimes_S S) = \bigsqcup_{i \in I} (b_i u_i \otimes s_i) S^1.$$

The acts $(b_i u_i \otimes s_i) S^1$ are indecomposable, because they are cyclic.

Consider the diagram

$$B \otimes S \xrightarrow{\mu_B} B_S \xrightarrow{\varphi} A_S.$$

By Proposition 7.49, μ_B is essential in UAct_S. Thus, the composition $\varphi\mu_B$ is an essential epimorphism in UAct_S. Since $\varphi\mu_B$ is also a morphism in FAct_S, it is essential in FAct_S by Proposition 7.50.

(2) ⇒ (1) Take $A_S \in \mathsf{UAct}_S$. Then $A \otimes_S S$ is firm by Lemma 7.1. By assumption, there exists an $\mathcal{IC}$-cover

$$C_S := \bigsqcup_{i \in I} c_i S^1 \xrightarrow{\varphi} A \otimes_S S$$

in FAct_S. Then φ is an essential epimorphism in UAct_S by Proposition 7.50. By Proposition 7.49, μ_A is essential in UAct_S. Therefore, $\mu_A \varphi : C_S \longrightarrow A_S$ is an $\mathcal{IC}$-cover in UAct_S. ∎

Theorem 7.56. *Right $\mathcal{IC}$-perfectness is a Morita invariant for factorisable semigroups.*

Proof. Let S and T be factorisable semigroups and $F : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_T$ an equivalence functor that takes cyclic acts to cyclic acts. Suppose that $A_S \in \mathsf{FAct}_S$ has an $\mathcal{IC}$-cover

$$\varphi : \bigsqcup_{i \in I} b_i S^1 \longrightarrow A_S.$$

Since F preserves coproducts and essential epimorphisms,

$$F(\varphi) : \bigsqcup_{i \in I} F(b_i S^1) \cong F \left(\bigsqcup_{i \in I} b_i S^1 \right) \longrightarrow F(A_S)$$

is an $\mathcal{IC}$-cover of $F(A_S)$, where $F(b_i S^1) \in \mathsf{FAct}_S$ are cyclic acts (and hence indecomposable). So $F(A_S)$ also has an $\mathcal{IC}$-cover. If F has an inverse equivalence functor G, which also takes cyclic acts to cyclic acts, then it holds that A_S has an $\mathcal{IC}$-cover if and only if $F(A_S)$ has an $\mathcal{IC}$-cover.

Let now S and T be as in the proof of Theorem 7.54. Then

$$\begin{array}{rcl} S \text{ is right } \mathcal{IC}\text{-perfect} & \iff & \text{all objects in } \mathsf{UAct}_S \text{ have an } \mathcal{IC}\text{-cover} \\ \text{(by Proposition 7.55)} & \iff & \text{all objects in } \mathsf{FAct}_S \text{ have an } \mathcal{IC}\text{-cover} \\ \text{(by Proposition 7.53)} & \iff & \text{all objects in } \mathsf{FAct}_{S\otimes S} \text{ have an } \mathcal{IC}\text{-cover} \\ \text{(by Proposition 7.52)} & \iff & \text{all objects in } \mathsf{FAct}_{T\otimes T} \text{ have an } \mathcal{IC}\text{-cover} \\ \text{(by Proposition 7.53)} & \iff & \text{all objects in } \mathsf{FAct}_T \text{ have an } \mathcal{IC}\text{-cover} \\ \text{(by Proposition 7.55)} & \iff & \text{all objects in } \mathsf{UAct}_T \text{ have an } \mathcal{IC}\text{-cover} \\ & \iff & T \text{ is right } \mathcal{IC}\text{-perfect.} \end{array}$$

■

Corollary 7.57. *Right perfectness and perfectness are Morita invariants for factorisable semigroups.*

Proof. Let S and T be Morita equivalent factorisable semigroups. Then FAct_S and FAct_T are equivalent categories. By Theorem 7.54, Theorem 7.56 and Theorem 7.46 it follows that right perfectness is a Morita invariant. From Remark 4.12 in [35] we know that also ${}_S\mathsf{FAct}$ and ${}_T\mathsf{FAct}$ are equivalent categories. If now S is left perfect, then the duals of the proofs of Theorem 7.54 and Theorem 7.56 yield that T is also left perfect. It follows that perfectness is a Morita invariant. ■

We saw that completely simple semigroups are perfect. This fact can also be concluded from Morita invariance of perfectness in case of factorisable semigroups.

Corollary 7.58. *Completely simple semigroups are perfect.*

Proof. Clearly, groups are perfect semigroups. Factorisable semigroups Morita equivalent to a given group are precisely Rees matrix semigroups over that group by Theorem 4.8. ■

Summary

This thesis is a study of Morita equivalence of semigroups. Describing all semigroups up to isomorphism is an insurmountable task - there are too many of them. Morita equivalence is a significantly weaker equivalence relation than isomorphism. Of particular interest is the Morita equivalence of factorisable semigroups - these are semigroups in which every element can be expressed as a product of two elements.

Two semigroups are said to be Morita equivalent if the categories of firm right acts over them are equivalent. This makes it clear that this relation is an equivalence relation on the class of all semigroups. However, equivalence functors are sometimes difficult to work with. Some objectives of Morita theory are to describe Morita equivalence in terms of (computationally) more feasible conditions, attempt to study its equivalence classes and find Morita invariants, i.e, properties, which are shared by all semigroups in the same Morita equivalence class.

The objective of this thesis is to study Morita equivalence of semigroups in terms of various algebraic constructions that have already been used by other authors to describe Morita equivalence for certain subclasses of semigroups. Additionally, a study on perfection for semigroups is included with one of the goals as determining, whether perfection is a Morita invariant.

This thesis consists of seven chapters. The first chapter is the introduction, which includes a few historical remarks regarding Morita theory and an overview of the structure of the thesis.

The second chapter includes preliminaries about semigroups, acts over semigroups, categories and tensor products.

The third chapter includes preliminaries on Morita equivalence, specifically. Concepts such as Cauchy completions, Morita contexts and enlargements of semigroups are recalled. Brief commentary on some differences between Morita theory of rings and semigroups is included.

The fourth chapter is based on the author's article [40]. It is shown that the strong Morita equivalence class of a monoid consists of the enlargements of that monoid. As a consequence, the enlargements of a group are pre-

cisely Rees matrix semigroups over that group. It also follows that complete simplicity is a Morita invariant for factorisable semigroups.

The fifth chapter is based on the author's article [41]. Strict local isomorphisms have already been used, among other methods, to describe Morita equivalence of semigroups with local units [30]. It is shown that such morphisms emerge naturally from surjective Morita contexts and for a subclass of semigroups, a recipe for producing strict local isomorphisms is given. A Morita context also induces two Morita semigroups, which are used to describe Morita equivalence of firm semigroups. Finally, Morita semigroups induced by dual pairs of acts are identified with certain subsemigroups of adjoint endomorphisms, which provides a sufficient condition for Morita equivalence of semigroups with weak local units.

The sixth chapter is based on the author's article [42]. It is shown that a unitary surjective Morita context connecting two semigroups induces a number of monotone Galois connections in the event either semigroup contains common weak local units, e.g, is a monoid. It is shown among other things that if both semigroups have this property, then the lattices of compatible relations on those semigroups are isomorphic.

The seventh and final chapter is based on the author's joint work [29] with Valdis Laan. Perfect semigroups are defined. It is also shown that many descriptions of perfect monoids established by other authors can be transferred to the case of factorisable semigroups. A condition pertaining to projectivity of sequence acts is added to the list of descriptions. Perfectness is also shown to be a Morita invariant for factorisable semigroups. It is concluded that all nilpotent semigroups and all completely (0-)simple semigroups are perfect.

Kokkuvõte

Poolrühmade Morita ekvivalentsusest

Käesolevas töös uuritakse poolrühmade Morita ekvivalentsust. Kõikide poolrühmade kirjeldamine isomorfismi täpsuseni on praktiliselt võimatu - neid on liiga palju. Morita ekvivalentsus on isomorfismiseosest oluliselt nõrgem ekvivalentsiseos. Suur rõhk on faktoriseeruvate poolrühmade Morita ekvivalentsuse uurimisel - need on poolrühmad, mille iga element esitub kahe elemendi korrutisena.

Öeldakse, et kaks poolrühma on Morita ekvivalentsed, kui kõikide püsivate parempoolsete polügoonide kategooriad üle nende poolrühmade on ekvivalentsed. Definitsioonist on selge, et tegemist on ekvivalentsiseosega kõikide poolrühmade klassil. Teisalt, ekvivalentsifunktoritega on mõnikord keeruline ringi käia. Morita teooria mõned eesmärgid on kirjeldada Morita ekvivalentsust kasutades selleks (algoritmiliselt) jõukohasemaid tingimusi, uurida Morita ekvivalentsiklasse ning leida Morita invariante - need on parajasti kõikide ühest ja samast ekvivalentsiklassist pärit poolrühmade ühised omadused.

Käesoleva töö eesmärk on uurida poolrühmade Morita ekvivalentsust kasutades selleks mitmeid algebralisi konstruktsioone, mida teised autorid on kasutanud Morita ekvivalentsuse kirjeldamiseks poolrühmade teatud alamklassidel. Lisaks on uuritud poolrühmade perfektsust, et muu hulgas välja selgitada, kas tegemist on Morita invariandiga.

Töö koosneb seitsmest peatükist. Esimeses peatükis räägitakse lühidalt Morita teooria taustast ning esitatakse töö ülesehitus.

Teises peatükis tuletatakse meelde mõisted poolrühmade, polügoonide, kategooriate ja tensorkorrutiste kohta.

Kolmandas peatükis antakse eelteadmised Morita ekvivalentsusega seotud algebraliste konstruktsioonide kohta. Tuletatakse meelde mõisted Cauchy täield, Morita kontekst ja poolrühma laiend. Lühidalt kommenteeritakse Morita teooria mõningaid erinevusi ringide ja poolrühmade korral.

Neljas peatükk põhineb autori artiklil [40]. Näidatakse ära, et monoidi tugev Morita ekvivalentsiklass koosneb parajasti selle monoidi laienditest.

Järeldusena näidatakse ka ära, et rühma laiendid on parajasti Rees'i maatrikspoolrühmad üle selle rühma ning et täiesti lihtsus on faktoriseeruvate poolrühmade Morita invariant.

Viies peatükk põhineb autori artiklil [41]. Varasemalt on kirjeldatud lokaalsete ühikelementidega poolrühmade Morita ekvivalentsus muu hulgas rangelt lokaalsete isomorfismide kaudu [30]. Näidatakse, et sellised morfismid tekivad loomulikul viisil sürjektiivsetest Morita kontekstidest ning teatud poolrühmade alamklassil antakse eeskiri, kuidas selliseid morfisme konstrueerida. Morita kontekst tekitab lisaks kaks Morita poolrühma, mille abil kirjeldatakse püsivate poolrühmade Morita ekvivalentsust. Viimaks, Morita poolrühmad, mis on tekitatud duaalsete paaridega, samastatakse teatud kaasendomorfismide alampoolrühmadega, mis annab piisava tingimuse nõrkade lokaalsete ühikelementidega poolrühmade Morita ekvivalentsiks.

Kuues peatükk põhineb autori artiklil [42]. Näidatakse, et kahe poolrühma vaheline unitaarne sürjektiivne Morita kontekst tekitab mitmeid monotoonseid Galois' seoseid, kui vähemalt ühel poolrühmadest on ühised nõrgad lokaalsed ühikud - on, näiteks, monoid. Kui mõlemal poolrühmal on selline omadus, siis nende poolrühmade kooskõlaliste seoste võred on isomorfsed.

Seitsmes ehk viimane peatükk põhineb autori artiklil [29], mis valmis koostöös Valdis Laanega. Defineeritakse paremperfektsed poolrühmad ning näidatakse, et paljud perfektsete monoidide kirjeldused saab üle kanda faktoriseeruvate poolrühmade juhule. Lisatakse ka uus kirjeldus jadapolügoonide projektiivsuse kaudu. Veel näidatakse ära, et perfektsus on faktoriseeruvate poolrühmade Morita invariant ning et kõik nilpotentsed ja täiesti (0-)lihtsad poolrühmad on perfektsed.

Bibliography

[1] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories. The Joy of Cats.* John Wiley and Sons, Inc, New York, 1990. Reprint available at `http://www.tac.mta.ca/tac/reprints/articles/17/tr17.pdf`.

[2] F. W. Anderson and K. R. Fuller. *Rings and categories of modules.* Springer-Verlag, New York-Heidelberg, 1974.

[3] P. N. Ánh. Morita equivalence and tensor product rings. *Commun. Algebra*, 17:2717–2737, 1989.

[4] A. Bailey, V. Gould, M. Hartmann, J. Renshaw, and L. Shaheen. Covers for S-acts and Condition (A) for a monoid S. *Glasg. Math. J.*, 57:323–341, 2015.

[5] B. Banaschewski. Functors into categories of M-sets. *Abh. Math. Semin. Univ. Hamburg.*, 8:49–64, 1972.

[6] H. Bass. Finitistic dimension and a homological generalization of semiprimary rings. *Trans. Amer. Math. Soc.*, 95:466–488, 1960.

[7] H. Bass. *The Morita Theorems.* Univ. Oregon Lecture Notes, 1962.

[8] H. Bass. *Algebraic K-Theory.* W.A. Benjamin, New-York – Amsterdam, 1968.

[9] I. Chajda. *Algebraic theory of tolerance relations.* Univerzita Palackého Olomouc, Olomouc, 1991.

[10] I. Chajda, B. Šešelja, and A. Tepavčević. Lattices of compatible relations satisfying a set of formulas. *Algebra Univer.*, 40:51–58, 1998.

[11] Y. Chen and K. P. Shum. Projective and indecomposable S-acts. *Sci. China Ser. A*, 42:593–599, 1999.

[12] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups Vol. II.* Providence, Rhode Island, 1967.

[13] J. Fountain. Perfect semigroups. *Proc. Edinb. Math. Soc. (2)*, 20(2):87–93, 1976.

[14] J. Fountain. Abundant semigroups. *Proc. London Math. Soc. (3)*, 44:103–129, 1982.

[15] J. Funk, M. V. Lawson, and B. Steinberg. Characterizations of Morita equivalent inverse semigroups. *J. Pure Appl. Algebra*, 215(9):2262–2279, 2011.

[16] V. Gould and L. Shaheen. Perfection for pomonoids. *Semigroup Forum*, 81:102–127, 2010.

[17] E. Hotzel. Dual D-operands and the Rees theorem. In *Algebraic Theory of Semigroups (Szeged 1976)*, pages 247–275. North-Holland, Amsterdam, 1979.

[18] J. Howie. *Fundamentals of Semigroup Theory.* Clarendon Press, Oxford, 1995.

[19] J. R. Isbell. Perfect monoids. *Semigroup Forum*, 2:95–118, 1971.

[20] A. I. Kashu. The composition of dualities in a nondegenerate Morita context. *J. Pure Appl. Algebra*, 133(1–2):143–149, 1998.

[21] R. Khosravi, M. Ershand, and M. Sedaghatjoo. Strongly flat and condition (P) covers of acts over monoids. *Comm. Algebra*, 38:4520–4530, 2010.

[22] M. Kilp. On monoids over which all strongly flat cyclic right acts are projective. *Semigroup Forum*, 52:241–245, 1996.

[23] M. Kilp. Perfect monoids revisited. *Semigroup Forum*, 53:225–229, 1996.

[24] M. Kilp, U. Knauer, and A. V. Mikhalev. *Monoids, Acts and Categories.* Walter de Gruytor, Berlin–New-York, 2000.

[25] U. Knauer. Projectivity of acts and Morita equivalence of monoids. *Semigroup Forum*, 3:359–370, 1972.

[26] V. Laan. Context equivalence of semigroups. *Period. Math. Hungar.*, 60:81–94, 2010.

[27] V. Laan. Acceptable Morita Contexts for Semigroups. *ISRN Algebra*, 2012. Article ID 725627.

[28] V. Laan. Poolrühmade Morita ekvivalentsusest. In A. Kivinukk and K. Lätt, editors, *EMS Aastaraamat 2019*, pages 51–60. Eesti Matemaatika Selts, Tartu, 2022.

[29] V. Laan and A. Lepik. Perfection for semigroups. *Proc. Edinburgh Math. Soc.*, publish online on April 24, 2023. doi:10.1017/S0013091523000159.

[30] V. Laan and L. Márki. Strong Morita equivalence of semigroups with local units. *J. Pure Appl. Algebra*, 215:2538–2546, 2011.

[31] V. Laan and L. Márki. Morita invariants for semigroups with local units. *Monatsh. Math.*, 166:441–451, 2012.

[32] V. Laan and L. Márki. Fair semigroups and Morita equivalence. *Semigroup Forum*, 92(3):633–644, 2016.

[33] V. Laan, L. Márki, and Ü. Reimaa. Morita equivalence of semigroups revisited: Firm semigroups. *J. Algebra*, 505:247–270, 2018.

[34] V. Laan and Ü. Reimaa. Monomorphisms in categories of firm acts. *Studia Sci. Math. Hungar.*, 56:267–279, 2019.

[35] V. Laan and Ü. Reimaa. Morita equivalence of factorizable semigroups. *Internat. J. Algebra Comput.*, 29:723–741, 2019.

[36] V. Laan, Ü. Reimaa, and L. Tart. Morita equivalence of finite semigroups. *Semigroup Forum*, 102(3):842–860, 2021.

[37] V. Laan, Ü. Reimaa, L. Tart, and E. Teor. Flatness properties of acts over semigroups. *Categ. Gen. Algebr. Struct. Appl.*, 15:59–92, 2021.

[38] M. V. Lawson. Morita equivalence of semigroups with local units. *J. Pure Appl. Algebra*, 215:455–470, 2011.

[39] M. V. Lawson and L. Márki. Enlargements and coverings by Rees matrix semigroups. *Monatsh. Math.*, 129:191–195, 2000.

[40] A. Lepik. Semigroups strongly Morita equivalent to monoids. *Period. Math. Hungar.*, 85:171–176, 2022.

[41] A. Lepik. On connections between Morita semigroups and strong Morita equivalence. *Comm. Algebra*, to appear.

[42] A. Lepik. Some Galois connections arising from Morita contexts of semigroups. Under review.

[43] M. Mahmoudi and J. Renshaw. On covers of cyclic acts over monoids. *Semigroup Forum*, 77:325–338, 2008.

[44] J. Paseka. Morita contexts and their lattices of relations. *Internat. J. Theoret. Phys.*, 44(12):2249–2258, 2005.

[45] D. Quillen. Module theory over nonunital rings, notes. 1996. Available at `http://www.claymath.org/library/Quillen/Working_papers/quillen%201996/1996-2.pdf`.

[46] B. Stenström. Flatness and localization over monoids. *Math. Nachr.*, 48:315–334, 1971.

[47] S. Talwar. Morita equivalence for semigroups. *J. Aust. Math. Soc. A*, 59:81–111, 1995.

[48] S. Talwar. Strong Morita Equivalence and a Generalisation of the Rees Theorem. *J. Algebra*, 181:371–394, 1996.

[49] H. Tominaga. On s-unital rings. *Math. J. Okayama Univ.*, 18(2):117–134, 1975/76.

[50] L. Tooming. Morita contexts, ideals, and congruences for semirings with local units. *Proc. Est. Acad. Sci.*, 67(3):252–259, 1998.

[51] K. Väljako. *On the Morita equivalence of idempotent rings and monomorphisms of firm bimodules.* Univ. Tartu Press, 2022.

List of author’s publications

1. A. Lepik, Semigroups strongly Morita equivalent to monoids. *Period. Math. Hung.* 85:171–176, 2022.
2. A. Lepik, On connections between Morita semigroups and strong Morita equivalence. *Comm. Algebra*, to appear.
3. V. Laan, A. Lepik, Perfection for semigroups. *Proc. Edinburgh Math. Soc.*, published online on April 24, 2023. doi:10.1017/S0013091523000159
4. A. Lepik, Some Galois connections arising from Morita contexts of semigroups. *Proc. Est. Acad. Sci.*, under review.

Curriculum vitae

NAME: ALVIN LEPIK
DATE AND PLACE OF BIRTH: December 1, 1990, Tartu, Estonia
NATIONALITY: Estonian
ADDRESS: Institute of Mathematics and Statistics, Narva mnt 18, 51009 Tartu, Estonia
E-MAIL: alvin.lepik@gmail.com

EDUCATION:

2007 - 2010	Hugo Treffner Gymnasium
2014 - 2017	University of Tartu, bachelor studies in mathematics
2017 - 2019	University of Tartu, master studies in mathematics
2019 -	University of Tartu, doctoral studies in mathematics

EMPLOYMENT:

2017 -	teaching assistant in University of Tartu

LANGUAGES: Estonian, English

SCIENTIFIC INTERESTS: semigroup theory, category theory

Curriculum vitae

NIMI: ALVIN LEPIK
SÜNNIAEG JA -KOHT: 1. detsember 1990, Tartu, Eesti
KODAKONDSUS: Eesti
AADRESS: Tartu Ülikooli matemaatika ja statistika instituut, Narva mnt. 18, 51009 Tartu, Eesti
E-POST: alvin.lepik@gmail.com

HARIDUS:

2007 - 2010	Hugo Treffneri Gümnaasium
2014 - 2017	Tartu Ülikool, matemaatika bakalaureuseõpe
2017 - 2019	Tartu Ülikool, matemaatika magistriõpe
2019 -	Tartu Ülikool, matemaatika doktoriõpe

TEENISTUSKÄIK:

2017 -	õppeassistent Tartu Ülikoolis

KEELTEOSKUS: eesti, inglise

TEADUSLIKUD HUVID: poolrühmateooria, kategooriateooria

Acknowledgements

I would like to thank my supervisor Valdis Laan whom I have been working with since my master's studies. This thesis would not exist without his continued efforts and criticisms.

The production of this thesis was partially supported by the Estonian Research Council grant PRG1204 and the Estonian Doctoral School in Mathematics and Statistics.

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

1. **Mati Heinloo.** The design of nonhomogeneous spherical vessels, cylindrical tubes and circular discs. Tartu, 1991, 23 p.
2. **Boris Komrakov.** Primitive actions and the Sophus Lie problem. Tartu, 1991, 14 p.
3. **Jaak Heinloo.** Phenomenological (continuum) theory of turbulence. Tartu, 1992, 47 p.
4. **Ants Tauts.** Infinite formulae in intuitionistic logic of higher order. Tartu, 1992, 15 p.
5. **Tarmo Soomere.** Kinetic theory of Rossby waves. Tartu, 1992, 32 p.
6. **Jüri Majak.** Optimization of plastic axisymmetric plates and shells in the case of Von Mises yield condition. Tartu, 1992, 32 p.
7. **Ants Aasma.** Matrix transformations of summability and absolute summability fields of matrix methods. Tartu, 1993, 32 p.
8. **Helle Hein.** Optimization of plastic axisymmetric plates and shells with piece-wise constant thickness. Tartu, 1993, 28 p.
9. **Toomas Kiho.** Study of optimality of iterated Lavrentiev method and its generalizations. Tartu, 1994, 23 p.
10. **Arne Kokk.** Joint spectral theory and extension of non-trivial multiplicative linear functionals. Tartu, 1995, 165 p.
11. **Toomas Lepikult.** Automated calculation of dynamically loaded rigid-plastic structures. Tartu, 1995, 93 p, (in Russian).
12. **Sander Hannus.** Parametrical optimization of the plastic cylindrical shells by taking into account geometrical and physical nonlinearities. Tartu, 1995, 74 p, (in Russian).
13. **Sergei Tupailo.** Hilbert’s epsilon-symbol in predicative subsystems of analysis. Tartu, 1996, 134 p.
14. **Enno Saks.** Analysis and optimization of elastic-plastic shafts in torsion. Tartu, 1996, 96 p.
15. **Valdis Laan.** Pullbacks and flatness properties of acts. Tartu, 1999, 90 p.
16. **Märt Põldvere.** Subspaces of Banach spaces having Phelps’ uniqueness property. Tartu, 1999, 74 p.
17. **Jelena Ausekle.** Compactness of operators in Lorentz and Orlicz sequence spaces. Tartu, 1999, 72 p.
18. **Krista Fischer.** Structural mean models for analyzing the effect of compliance in clinical trials. Tartu, 1999, 124 p.
19. **Helger Lipmaa.** Secure and efficient time-stamping systems. Tartu, 1999, 56 p.
20. **Jüri Lember.** Consistency of empirical k-centres. Tartu, 1999, 148 p.
21. **Ella Puman.** Optimization of plastic conical shells. Tartu, 2000, 102 p.
22. **Kaili Müürisep.** Eesti keele arvutigrammatika: süntaks. Tartu, 2000, 107 lk.

23. **Varmo Vene.** Categorical programming with inductive and coinductive types. Tartu, 2000, 116 p.
24. **Olga Sokratova.** Ω-rings, their flat and projective acts with some applications. Tartu, 2000, 120 p.
25. **Maria Zeltser.** Investigation of double sequence spaces by soft and hard analitical methods. Tartu, 2001, 154 p.
26. **Ernst Tungel.** Optimization of plastic spherical shells. Tartu, 2001, 90 p.
27. **Tiina Puolakainen.** Eesti keele arvutigrammatika: morfoloogiline ühestamine. Tartu, 2001, 138 p.
28. **Rainis Haller.** *M(r,s)*-inequalities. Tartu, 2002, 78 p.
29. **Jan Villemson.** Size-efficient interval time stamps. Tartu, 2002, 82 p.
30. Töö kaitsmata.
31. **Mart Abel.** Structure of Gelfand-Mazur algebras. Tartu, 2003. 94 p.
32. **Vladimir Kuchmei.** Affine completeness of some ockham algebras. Tartu, 2003. 100 p.
33. **Olga Dunajeva.** Asymptotic matrix methods in statistical inference problems. Tartu 2003. 78 p.
34. **Mare Tarang.** Stability of the spline collocation method for volterra integro-differential equations. Tartu 2004. 90 p.
35. **Tatjana Nahtman.** Permutation invariance and reparameterizations in linear models. Tartu 2004. 91 p.
36. **Märt Möls.** Linear mixed models with equivalent predictors. Tartu 2004. 70 p.
37. **Kristiina Hakk.** Approximation methods for weakly singular integral equations with discontinuous coefficients. Tartu 2004, 137 p.
38. **Meelis Käärik.** Fitting sets to probability distributions. Tartu 2005, 90 p.
39. **Inga Parts.** Piecewise polynomial collocation methods for solving weakly singular integro-differential equations. Tartu 2005, 140 p.
40. **Natalia Saealle.** Convergence and summability with speed of functional series. Tartu 2005, 91 p.
41. **Tanel Kaart.** The reliability of linear mixed models in genetic studies. Tartu 2006, 124 p.
42. **Kadre Torn.** Shear and bending response of inelastic structures to dynamic load. Tartu 2006, 142 p.
43. **Kristel Mikkor.** Uniform factorisation for compact subsets of Banach spaces of operators. Tartu 2006, 72 p.
44. **Darja Saveljeva.** Quadratic and cubic spline collocation for Volterra integral equations. Tartu 2006, 117 p.
45. **Kristo Heero.** Path planning and learning strategies for mobile robots in dynamic partially unknown environments. Tartu 2006, 123 p.
46. **Annely Mürk.** Optimization of inelastic plates with cracks. Tartu 2006. 137 p.
47. **Annemai Raidjõe.** Sequence spaces defined by modulus functions and superposition operators. Tartu 2006, 97 p.
48. **Olga Panova.** Real Gelfand-Mazur algebras. Tartu 2006, 82 p.

49. **Härmel Nestra.** Iteratively defined transfinite trace semantics and program slicing with respect to them. Tartu 2006, 116 p.
50. **Margus Pihlak.** Approximation of multivariate distribution functions. Tartu 2007, 82 p.
51. **Ene Käärik.** Handling dropouts in repeated measurements using copulas. Tartu 2007, 99 p.
52. **Artur Sepp.** Affine models in mathematical finance: an analytical approach. Tartu 2007, 147 p.
53. **Marina Issakova.** Solving of linear equations, linear inequalities and systems of linear equations in interactive learning environment. Tartu 2007, 170 p.
54. **Kaja Sõstra.** Restriction estimator for domains. Tartu 2007, 104 p.
55. **Kaarel Kaljurand.** Attempto controlled English as a Semantic Web language. Tartu 2007, 162 p.
56. **Mart Anton.** Mechanical modeling of IPMC actuators at large deformations. Tartu 2008, 123 p.
57. **Evely Leetma.** Solution of smoothing problems with obstacles. Tartu 2009, 81 p.
58. **Ants Kaasik.** Estimating ruin probabilities in the Cramér-Lundberg model with heavy-tailed claims. Tartu 2009, 139 p.
59. **Reimo Palm.** Numerical Comparison of Regularization Algorithms for Solving Ill-Posed Problems. Tartu 2010, 105 p.
60. **Indrek Zolk.** The commuting bounded approximation property of Banach spaces. Tartu 2010, 107 p.
61. **Jüri Reimand.** Functional analysis of gene lists, networks and regulatory systems. Tartu 2010, 153 p.
62. **Ahti Peder.** Superpositional Graphs and Finding the Description of Structure by Counting Method. Tartu 2010, 87 p.
63. **Marek Kolk.** Piecewise Polynomial Collocation for Volterra Integral Equations with Singularities. Tartu 2010, 134 p.
64. **Vesal Vojdani.** Static Data Race Analysis of Heap-Manipulating C Programs. Tartu 2010, 137 p.
65. **Larissa Roots.** Free vibrations of stepped cylindrical shells containing cracks. Tartu 2010, 94 p.
66. **Mark Fišel.** Optimizing Statistical Machine Translation via Input Modification. Tartu 2011, 104 p.
67. **Margus Niitsoo**. Black-box Oracle Separation Techniques with Applications in Time-stamping. Tartu 2011, 174 p.
68. **Olga Liivapuu.** Graded q-differential algebras and algebraic models in noncommutative geometry. Tartu 2011, 112 p.
69. **Aleksei Lissitsin.** Convex approximation properties of Banach spaces. Tartu 2011, 107 p.
70. **Lauri Tart.** Morita equivalence of partially ordered semigroups. Tartu 2011, 101 p.
71. **Siim Karus.** Maintainability of XML Transformations. Tartu 2011, 142 p.

72. **Margus Treumuth.** A Framework for Asynchronous Dialogue Systems: Concepts, Issues and Design Aspects. Tartu 2011, 95 p.
73. **Dmitri Lepp.** Solving simplification problems in the domain of exponents, monomials and polynomials in interactive learning environment T-algebra. Tartu 2011, 202 p.
74. **Meelis Kull.** Statistical enrichment analysis in algorithms for studying gene regulation. Tartu 2011, 151 p.
75. **Nadežda Bazunova.** Differential calculus $d^3 = 0$ on binary and ternary associative algebras. Tartu 2011, 99 p.
76. **Natalja Lepik.** Estimation of domains under restrictions built upon generalized regression and synthetic estimators. Tartu 2011, 133 p.
77. **Bingsheng Zhang.** Efficient cryptographic protocols for secure and private remote databases. Tartu 2011, 206 p.
78. **Reina Uba.** Merging business process models. Tartu 2011, 166 p.
79. **Uuno Puus.** Structural performance as a success factor in software development projects – Estonian experience. Tartu 2012, 106 p.
80. **Marje Johanson.** $M(r, s)$-ideals of compact operators. Tartu 2012, 103 p.
81. **Georg Singer.** Web search engines and complex information needs. Tartu 2012, 218 p.
82. **Vitali Retšnoi.** Vector fields and Lie group representations. Tartu 2012, 108 p.
83. **Dan Bogdanov.** Sharemind: programmable secure computations with practical applications. Tartu 2013, 191 p.
84. **Jevgeni Kabanov.** Towards a more productive Java EE ecosystem. Tartu 2013, 151 p.
85. **Erge Ideon**. Rational spline collocation for boundary value problems. Tartu, 2013, 111 p.
86. **Esta Kägo.** Natural vibrations of elastic stepped plates with cracks. Tartu, 2013, 114 p.
87. **Margus Freudenthal.** Simpl: A toolkit for Domain-Specific Language development in enterprise information systems. Tartu, 2013, 151 p.
88. **Boriss Vlassov.** Optimization of stepped plates in the case of smooth yield surfaces. Tartu, 2013, 104 p.
89. **Elina Safiulina.** Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space. Tartu, 2013, 85 p.
90. **Raivo Kolde.** Methods for re-using public gene expression data. Tartu, 2014, 121 p.
91. **Vladimir Šor.** Statistical Approach for Memory Leak Detection in Java Applications. Tartu, 2014, 155 p.
92. **Naved Ahmed.** Deriving Security Requirements from Business Process Models. Tartu, 2014, 171 p.
93. **Kerli Orav-Puurand.** Central Part Interpolation Schemes for Weakly Singular Integral Equations. Tartu, 2014, 109 p.

94. **Liina Kamm.** Privacy-preserving statistical analysis using secure multiparty computation. Tartu, 2015, 201 p.
95. **Kaido Lätt.** Singular fractional differential equations and cordial Volterra integral operators. Tartu, 2015, 93 p.
96. **Oleg Košik.** Categorical equivalence in algebra. Tartu, 2015, 84 p.
97. **Kati Ain.** Compactness and null sequences defined by ℓ_p spaces. Tartu, 2015, 90 p.
98. **Helle Hallik.** Rational spline histopolation. Tartu, 2015, 100 p.
99. **Johann Langemets.** Geometrical structure in diameter 2 Banach spaces. Tartu, 2015, 132 p.
100. **Abel Armas Cervantes.** Diagnosing Behavioral Differences between Business Process Models. Tartu, 2015, 193 p.
101. **Fredrik Milani.** On Sub-Processes, Process Variation and their Interplay: An Integrated Divide-and-Conquer Method for Modeling Business Processes with Variation. Tartu, 2015, 164 p.
102. **Huber Raul Flores Macario.** Service-Oriented and Evidence-aware Mobile Cloud Computing. Tartu, 2015, 163 p.
103. **Tauno Metsalu.** Statistical analysis of multivariate data in bioinformatics. Tartu, 2016, 197 p.
104. **Riivo Talviste.** Applying Secure Multi-party Computation in Practice. Tartu, 2016, 144 p.
105. **Md Raknuzzaman.** Noncommutative Galois Extension Approach to Ternary Grassmann Algebra and Graded q-Differential Algebra. Tartu, 2016, 110 p.
106. **Alexander Liyvapuu.** Natural vibrations of elastic stepped arches with cracks. Tartu, 2016, 110 p.
107. **Julia Polikarpus.** Elastic plastic analysis and optimization of axisymmetric plates. Tartu, 2016, 114 p.
108. **Siim Orasmaa.** Explorations of the Problem of Broad-coverage and General Domain Event Analysis: The Estonian Experience. Tartu, 2016, 186 p.
109. **Prastudy Mungkas Fauzi.** Efficient Non-interactive Zero-knowledge Protocols in the CRS Model. Tartu, 2017, 193 p.
110. **Pelle Jakovits.** Adapting Scientific Computing Algorithms to Distributed Computing Frameworks. Tartu, 2017, 168 p.
111. **Anna Leontjeva.** Using Generative Models to Combine Static and Sequential Features for Classification. Tartu, 2017, 167 p.
112. **Mozhgan Pourmoradnasseri.** Some Problems Related to Extensions of Polytopes. Tartu, 2017, 168 p.
113. **Jaak Randmets.** Programming Languages for Secure Multi-party Computation Application Development. Tartu, 2017, 172 p.
114. **Alisa Pankova.** Efficient Multiparty Computation Secure against Covert and Active Adversaries. Tartu, 2017, 316 p.
115. **Tiina Kraav.** Stability of elastic stepped beams with cracks. Tartu, 2017, 126 p.

116. **Toomas Saarsen.** On the Structure and Use of Process Models and Their Interplay. Tartu, 2017, 123 p.
117. **Silja Veidenberg.** Lifting bounded approximation properties from Banach spaces to their dual spaces. Tartu, 2017, 112 p.
118. **Liivika Tee.** Stochastic Chain-Ladder Methods in Non-Life Insurance. Tartu, 2017, 110 p.
119. **Ülo Reimaa.** Non-unital Morita equivalence in a bicategorical setting. Tartu, 2017, 86 p.
120. **Rauni Lillemets.** Generating Systems of Sets and Sequences. Tartu, 2017, 181 p.
121. **Kristjan Korjus.** Analyzing EEG Data and Improving Data Partitioning for Machine Learning Algorithms. Tartu, 2017, 106 p.
122. **Eno Tõnisson.** Differences between Expected Answers and the Answers Offered by Computer Algebra Systems to School Mathematics Equations. Tartu, 2017, 195 p.
123. **Kaur Lumiste.** Improving accuracy of survey estimators by using auxiliary information in data collection and estimation stages. Tartu, 2018, 112 p.
124. **Paul Tammo.** Closed maximal regular one-sided ideals in topological algebras. Tartu, 2018, 112 p.
125. **Mart Kals.** Computational and statistical methods for DNA sequencing data analysis and applications in the Estonian Biobank cohort. Tartu, 2018, 174 p.
126. **Annika Krutto.** Empirical Cumulant Function Based Parameter Estimation in Stable Distributions. Tartu, 2019, 140 p.
127. **Kristi Läll.** Risk scores and their predictive ability for common complex diseases. Tartu, 2019, 118 p.
128. **Gul Wali Shah.** Spline approximations. Tartu, 2019, 85 p.
129. **Mikk Vikerpuur.** Numerical solution of fractional differential equations. Tartu, 2019, 125 p.
130. **Priit Lätt.** Induced 3-Lie superalgebras and their applications in superspace. Tartu, 2020, 114 p.
131. **Sumaira Rehman.** Fast and quasi-fast solvers for weakly singular Fredholm integral equation of the second kind. Tartu, 2020, 105 p.
132. **Rihhard Nadel.** Big slices of the unit ball in Banach spaces. Tartu, 2020, 109 p.
133. **Katriin Pirk.** Diametral diameter two properties, Daugavet-, and Δ-points in Banach spaces. Tartu, 2020, 106 p.
134. **Zahra Alijani**. Fuzzy integral equations of the second kind. Tartu, 2020, 103 1p.
135. **Hina Arif.** Stability analysis of stepped nanobeams with defects. Tartu, 2021, 165 p.
136. **Joonas Sova.** Pairwise Markov Models. Tartu, 2021, 166 p.
137. **Kristo Väljako.** On the Morita equivalence of idempotent rings and monomorphisms of firm bimodules. Tartu, 2022, 139 p.

138. **Andre Ostrak.** Diameter two properties in spaces of Lipschitz functions. Tartu, 2022, 77 p.
139. **Mohammed Mainul Hossain.** Numerical analysis of vibrations of nano-beams. Tartu, 2022, 158 p.