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1. INTRODUCTION

1.1. Background and motivation

The geometry of infinite-dimensional Banach spaces exhibits many striking and sometimes counterintuitive phenomena. In contrast to finite-dimensional spaces, where the structure of the unit ball is relatively rigid, infinite-dimensional Banach spaces allow for a wide range of geometric behaviours, which often reflect the underlying structure of these spaces.

Among the phenomena that have attracted significant attention in recent years are the so-called diameter two properties and almost square properties. These properties capture that certain natural subsets of the unit ball have diameter two, the diameter of the unit ball itself.

The systematic study of diameter two properties began in the work of Abrahamsen, Lima, and Nygaard in [3]. Since then, these properties have been investigated in many classical Banach spaces and have been linked to other geometric properties such as octahedral norms and the Daugavet property. The diameter two properties are closely related to almost square properties, which impose stronger geometric constraints on Banach spaces. A Banach space admits an equivalent almost square norm if and only if it contains an isomorphic copy of c_0 (see [13, Corollary 2.4]).

This thesis investigates these geometric properties in spaces of Lipschitz functions and their canonical preduals, the Lipschitz-free spaces. The structure of these spaces reflects geometric properties of the underlying metric space, and thus forms a natural bridge between Banach space theory and the geometry of metric spaces.

Several fundamental results reveal a strong interplay between properties of Lipschitz-free spaces and the geometry of the underlying metric space. In particular, by [10], [20], and [30], the Daugavet property and several diameter two properties coincide in Lipschitz-free spaces and are characterised by the underlying metric space being a length space. Another result in [35] characterises octahedrality in Lipschitz-free spaces in terms of the long trapezoid property of the underlying metric space.

The aim of this thesis is to explore these connections further by studying diameter two properties and almost square properties in Lipschitz-free spaces and in spaces of Lipschitz functions. The results presented in the thesis contribute to distinguishing different geometric properties of Banach spaces and deepen the understanding of these properties in Lipschitz settings.

1.2. Preliminaries

This section presents the background material needed for the results of this thesis. It contains a collection of definitions and known results stated without proofs; appropriate references are provided throughout.

1.2.1. Notation

We consider only nontrivial real Banach spaces and use standard Banach space notation. For a Banach space X, we denote the closed unit ball by B_X, the unit sphere by S_X, and the dual space by X^*. A slice of B_X, determined by $x^* \in S_{X^*}$ and $\alpha > 0$, is

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\}.$$

For a metric space M, we set

$$\widetilde{M} = \big\{(x,y) \in M \times M : x \neq y\big\},$$

and define $\pi\colon \mathscr{P}(\widetilde{M}) \to \mathscr{P}(M)$ by

$$\pi(A) = \big\{x \in M\colon \text{there exists } y \in M \text{ such that } (x,y) \in A \text{ or } (y,x) \in A\big\}.$$

1.2.2. Diameter two properties

We begin by presenting one of the most well-studied diameter 2 properties. We say that a Banach space X has the *Daugavet property* if, for every $x \in S_X$, every slice S of B_X, and every $\varepsilon > 0$, there exists $y \in S$ such that

$$\|x - y\| \geq 2 - \varepsilon.$$

The investigation of this property dates back to the seminal paper [16] by Daugavet, where it was proved that the Banach space $C[0,1]$ satisfies this property. The Daugavet property was initially formulated in operator-theoretic terms, which is equivalent to the geometric characterisation presented above (see, e.g. [39]).

Since the Daugavet property is rather strong, it is natural to look for weaker variants. Following [3] and [4], we say that a Banach space X has the

- *local diameter* 2 *property* (briefly, *LD*2*P*) if every slice of B_X has diameter 2. This property is also called the slice-diameter 2 property (briefly, *slice-D*2*P*) (see [12]);
- *diameter* 2 *property* (briefly, *D*2*P*) if every nonempty relatively weakly open subset of B_X has diameter 2;
- *strong diameter* 2 *property* (briefly, *SD*2*P*) if every convex combination of slices of B_X has diameter 2, i.e. the diameter of $\sum_{i=1}^{n} \lambda_i S_i$ is 2 whenever $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, and $S_1, \dots, S_n$ are slices of B_X;

- *symmetric strong diameter* 2 *property* (briefly, *SSD2P*) if for every $n \in \mathbb{N}$, every family $\{S_1, \ldots, S_n\}$ of slices of B_X, and every $\varepsilon > 0$, there exist $f_1 \in S_1, \ldots, f_n \in S_n$, and $g \in B_X$ with $\|g\| > 1 - \varepsilon$ such that $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$.

In general,

$$\text{SSD2P} \Longrightarrow \text{SD2P} \Longrightarrow \text{D2P} \Longrightarrow \text{LD2P}.$$

The first implication was established in [3, Lemma 4.1], the second implication follows from Bourgain's lemma and the third implication holds trivially. However, all of these properties are different. The space $L_1[0,1]$ has the SD2P but lacks the SSD2P (see [4]). The space $c_0 \oplus_2 c_0$ has the D2P but not the SD2P (see [5]). Finally, a renorming of c_0 with the LD2P but without the D2P was constructed in [11].

If X is a dual space, then we also consider the weak* counterparts of these diameter two properties (w^*-LD2P, w^*-D2P, w^*-SD2P, and w^*-SSD2P), where slices and weakly open subsets in the above definitions are replaced by weak* slices and weak* open subsets, respectively. We have

$$w^*\text{-SSD2P} \Longrightarrow w^*\text{-SD2P} \Longrightarrow w^*\text{-D2P} \Longrightarrow w^*\text{-LD2P}.$$

The reverse implications do not hold in general. This follows from the fact that a Banach space X has a diameter two property if and only if its bidual X^{**} has the corresponding weak* property.

It is also known that the dual space of $C[0,1]$ has the w^*-SD2P but lacks the LD2P (see [12]). It was unknown whether there exists a dual Banach space with the w^*-SSD2P but without the SSD2P.

1.2.3. Octahedral norms

We say that the norm of a Banach space X is (or simply that the Banach space X is)

- *locally octahedral* (briefly, *LOH*) if, for every $x \in S_X$ and every $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\|x \pm y\| \geq 2 - \varepsilon;$$
- *weakly octahedral* (briefly, *WOH*) if, for every finite-dimensional subspace E of X, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$, there exists $y \in S_X$ such that, for all $x \in E$,
$$\|x + y\| \geq (1 - \varepsilon)\big(|x^*(x)| + \|y\|\big).$$
- *octahedral* (briefly, *OH*) if, whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\|x_i + y\| > 2 - \varepsilon$$
for every $i \in \{1, \ldots, n\}$.

The study of octahedral norms is strongly motivated by the following characterisation by [18, Theorem III.2.5]:

> a Banach space admits an equivalent OH norm if and only if it contains an isomorphic copy of ℓ_1.

In general,

$$\text{OH} \Longrightarrow \text{WOH} \Longrightarrow \text{LOH},$$

and all of these implications are strict. By [17, 21, 27], diameter two properties admit elegant dual formulations in terms of octahedral norms. In particular, the following equivalences hold for any Banach space X:

$$\begin{array}{ll}
X \text{ has the SD2P} \Longleftrightarrow X^* \text{ is OH;} & X \text{ is OH} \Longleftrightarrow X^* \text{ has the } w^*\text{-SD2P;} \\
X \text{ has the D2P} \Longleftrightarrow X^* \text{ is WOH;} & X \text{ is WOH} \Longleftrightarrow X^* \text{ has the } w^*\text{-D2P;} \\
X \text{ has the LD2P} \Longleftrightarrow X^* \text{ is LOH;} & X \text{ is LOH} \Longleftrightarrow X^* \text{ has the } w^*\text{-LD2P.}
\end{array}$$

It is currently unknown whether an analogous dual characterisation exists for the SSD2P. A possible approach via decomposable octahedrality was proposed in [34], where such a characterisation was obtained for spaces of Lipschitz functions.

1.2.4. Almost square properties

Closely related to diameter two properties are almost square properties. A Banach space X is said to be

- *locally almost square* (briefly, LASQ) if, for every $x \in S_X$, there exists a sequence $(y_i) \subseteq B_X$ such that $\|y_i\| \to 1$ and
$$\|x \pm y_i\| \to 1;$$
- *weakly almost square* (briefly, WASQ) if, for every $x \in S_X$, there exists a sequence $(y_i) \subseteq B_X$ such that $\|y_i\| \to 1$, $y_i \to 0$ weakly, and
$$\|x \pm y_i\| \to 1;$$
- *almost square* (briefly, ASQ) if, whenever $n \in \mathbb{N}, x_1, \ldots, x_n \in S_X$, there exists a sequence $(y_i) \subseteq B_X$ such that $\|y_i\| \to 1$ and
$$\|x_j \pm y_i\| \to 1$$
for every $j \in \{1, \ldots, n\}$.

A key motivation for the study of almost square Banach spaces is the following characterisation due to [13]:

> a Banach space admits an equivalent ASQ norm if and only if it contains an isomorphic copy of c_0.

In general,

$$\mathrm{ASQ} \Longrightarrow \mathrm{WASQ} \Longrightarrow \mathrm{LASQ},$$

and it is known that ASQ is strictly stronger than WASQ (see [1]). It was unknown whether WASQ and LASQ are equivalent properties (see [1, Question 3.4]).

Almost square properties are tightly connected with diameter two behaviour. In particular, the following implications hold in any Banach space X, and they are strict:

$$\begin{aligned} \mathrm{ASQ} &\Longrightarrow \mathrm{SSD2P}; \\ \mathrm{WASQ} &\Longrightarrow \mathrm{D2P}; \\ \mathrm{LASQ} &\Longrightarrow \mathrm{LD2P}. \end{aligned}$$

To summarise the connections between diameter two properties and almost square properties in Banach spaces, we present the following diagram.

$$\begin{tikzcd}
\text{Daugavet} \arrow[dd, Rightarrow] & & \mathrm{ASQ} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
& \mathrm{SSD2P} \arrow[dl, Rightarrow] & & \mathrm{WASQ} \arrow[ddll, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathrm{SD2P} \arrow[dr, Rightarrow] & & & & \mathrm{LASQ} \arrow[ddll, Rightarrow] \\
& \mathrm{D2P} \arrow[dr, Rightarrow] & & & \\
& & \mathrm{LD2P} & &
\end{tikzcd}$$

1.2.5. Lipschitz-free spaces and spaces of Lipschitz functions

Let (M, d_M) and (N, d_N) be metric spaces. A mapping $f\colon M \to N$ is called a *Lipschitz function* if there exists a constant $L \geq 0$ such that for every pair of distinct points $p, q \in M$,

$$d_N\big(f(p), f(q)\big) \leq L d_M(p, q).$$

The minimal constant L for which this inequality holds is called the *Lipschitz constant* of f.

Let (M, d) be a pointed metric space with a distinguished point 0. We denote by $\mathrm{Lip}_0(M)$ the Banach space of all Lipschitz functions $f\colon M \to \mathbb{R}$ satisfying $f(0) = 0$, equipped with the Lipschitz constant norm, i.e.

$$\|f\| = \sup\left\{ \frac{|f(p) - f(q)|}{d(p, q)} \colon p, q \in M,\ p \neq q \right\}.$$

The space $\mathrm{Lip}_0(M)$ does not depend on the choice of the base point in M. If the base point 0 is replaced by another point $0' \in M$, then $T\colon \mathrm{Lip}_0(M) \to \mathrm{Lip}_{0'}(M)$, defined by

$$(Tf)(p) = f(p) - f(0'), \qquad f \in \mathrm{Lip}_0(M),\ p \in M,$$

is an isometric isomorphism between $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_{0'}(M)$.

Let $\delta\colon M \to \mathrm{Lip}_0(M)^*$ be the canonical isometric embedding defined by

$$\langle f, \delta(p) \rangle = f(p), \qquad p \in M,\ f \in \mathrm{Lip}_0(M).$$

The norm-closed linear span of $\delta(M)$ in $\mathrm{Lip}_0(M)^*$ is called the *Lipschitz-free space* over M and is denoted by $\mathscr{F}(M)$. It is well known that

$$\mathscr{F}(M)^* = \mathrm{Lip}_0(M).$$

Since the Lipschitz-free space over a metric space M is isometric to the Lipschitz-free space over the completion of M, we may assume without loss of generality that M is complete. For $p, q \in M$, one has $\|\delta(p) - \delta(q)\| = d(p,q)$. For $p, q \in M$ with $p \neq q$, we denote by $m_{p,q}$ the norm-one element $\frac{\delta(p)-\delta(q)}{d(p,q)}$.

Let us briefly recall some background on Lipschitz-free spaces. Lipschitz-free spaces are universal in the following sense. For every Banach space X and every Lipschitz map $f\colon M \to X$ satisfying $f(0) = 0$, there exists a unique bounded linear operator $F\colon \mathscr{F}(M) \to X$ such that

$$F \circ \delta = f \quad \text{and} \quad \|F\| = \|f\|,$$

implying that the following diagram commutes.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\delta"'] & X \\
\mathscr{F}(M) \arrow[ur, "F"'] &
\end{tikzcd}$$

This universal property shows that Lipschitz-free spaces provide a canonical linearisation of Lipschitz maps between metric spaces. More precisely, if M and N are pointed metric spaces and $f\colon M \to N$ is a Lipschitz map with $f(0) = 0$, then there exists a unique bounded linear operator $F\colon \mathscr{F}(M) \to \mathscr{F}(N)$ such that for the canonical embeddings $\delta_M\colon M \to \mathrm{Lip}_0(M)^*$ and $\delta_N\colon N \to \mathrm{Lip}_0(N)^*$,

$$F \circ \delta_M = \delta_N \circ f \quad \text{and} \quad \|F\| = \|f\|,$$

implying that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\delta_M"'] & N \arrow[d, "\delta_N"] \\ \mathscr{F}(M) \arrow[r, "F"] & \mathscr{F}(N) \end{tikzcd}$$

For more background on Lipschitz-free spaces and spaces of Lipschitz functions, we refer to [22] and [38].

1.2.6. De Leeuw transform

When studying diameter two properties in spaces of Lipschitz functions, it is often necessary to work in the dual space $\mathrm{Lip}_0(M)^*$. To facilitate this, we make use of the de Leeuw transform.

The de Leeuw transform is the map $\Phi\colon \mathrm{Lip}_0(M) \to \ell_\infty(\widetilde{M})$ (see [38, Definition 2.31 and Theorem 2.35]) defined by $\Phi f = \tilde{f}$, where

$$\tilde{f}(x,y) = \frac{f(x) - f(y)}{d(x,y)}.$$

This mapping is a linear isometry. It follows from [19, Theorem IV.8.16] that $\ell_\infty(\widetilde{M})^* = ba(\widetilde{M})$, the Banach space of bounded finitely additive signed measures on the power set of $\widetilde{M}$, endowed with the norm $\|\mu\| = |\mu|(\widetilde{M})$.

We call a measure $\mu \in ba(\widetilde{M})$ *optimal* if it is positive and satisfies $\|\Phi^*\mu\| = \|\mu\|$. By [25, Proposition 2.4], for every $F \in \mathrm{Lip}_0(M)^*$ there exists a positive measure $\mu \in ba(\widetilde{M})$ with $\|\mu\| = \|F\|$ such that $\Phi^*\mu = F$, that is,

$$F(f) = \int_{\widetilde{M}} \tilde{f}\, d\mu \qquad \text{for all } f \in \mathrm{Lip}_0(M).$$

Note that the de Leeuw transform may also be viewed as an embedding of $\mathrm{Lip}_0(M)$ into the Banach space $C(\beta\widetilde{M})$ of continuous functions on the Stone–Čech compactification of $\widetilde{M}$ (see [38, p. 102]). This approach allows one to represent elements of $\mathrm{Lip}_0(M)^*$ as countably additive Radon measures on $\beta\widetilde{M}$. See, e.g. [6, 8, 7, 36] for recent developments using this method.

2. RELEVANT PREVIOUS WORK AND NEW CONTRIBUTIONS

To place the results of this thesis into context, we recall several important results from the literature that are closely related to the topic. These results provide both the conceptual background and the main motivation for the problems studied in this thesis.

2.1. Almost square properties and diameter two properties in Lipschitz-free spaces

A key feature of a Lipschitz-free space is that many of its geometric properties are closely connected to the geometry of the underlying metric space. Regarding diameter two properties of Lipschitz-free spaces, length spaces play an important role. Recall that a complete metric space (M,d) is called a *length space* if, for every $p,q \in M$, the distance $d(p,q)$ equals the infimum of the lengths of rectifiable paths connecting them. Length spaces appear in the following characterisation of certain geometric properties of $\mathscr{F}(M)$ and $\mathrm{Lip}_0(M)$, obtained in [10] and based on results from [30, 20].

Theorem 2.1.1 ([10, Theorem 1.5])**.** *Let M be a complete metric space. Then the following assertions are equivalent:*

(1) M is a length space;

(2) $\mathrm{Lip}_0(M)$*, i.e.,* $\mathscr{F}(M)^*$*, has the Daugavet property;*

(3) $\mathscr{F}(M)$ *has the Daugavet property;*

(4) $\mathscr{F}(M)$ *has the SD2P;*

(5) $\mathscr{F}(M)$ *has D2P;*

(6) $\mathscr{F}(M)$ *has the LD2P;*

(7) the unit ball of $\mathscr{F}(M)$ *does not have strongly exposed points;*

(8) M has property (Z).

The main contribution of [10] is the implication $(8) \Rightarrow (1)$. The implications $(3) \Rightarrow (1)$ and $(7) \Leftrightarrow (8)$ were previously obtained in [20], while $(1) \Rightarrow (2)$ and $(1) \Rightarrow (8)$ were established in [30]. The chain of implications

$$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$$

holds in general, when replacing $\mathscr{F}(M)$ by an arbitrary Banach space.

In this thesis, we show that locally almost square Lipschitz-free spaces also fall under the same characterisation, thereby extending the previous theorem.

Theorem 2.1.2 ([23, Theorem 3.1])**.** *Let M be a complete metric space. The Lipschitz-free space* $\mathscr{F}(M)$ *is LASQ if and only if M is a length space.*

On the other hand, we show that almost squareness behaves differently in Lipschitz-free spaces.

Theorem 2.1.3 ([23, Theorem 4.1])**.** *No Lipschitz-free space is ASQ. In fact, no Lipschitz-free space is s-ASQ for any* $0 < s \leq 1$.

This naturally raises the question whether Lipschitz-free spaces over length spaces are also WASQ. For several years it remained open whether WASQ and LASQ are equivalent properties (see [1, Question 3.4]). In [31], we resolve this problem by studying it in the setting of Lipschitz-free spaces.

Theorem 2.1.4 ([31, Theorem 2.3])**.** *There exists a complete metric space* M *such that* M *is a length space and* $\mathscr{F}(M)$ *is not WASQ.*

Recall that a complete metric space (M,d) is called a *geodesic space* if, for every $p,q \in M$, there exists a path connecting them whose length equals $d(p,q)$. In light of the previous result, it is natural to ask whether geodesic metric spaces provide a characterisation for WASQ in Lipschitz-free spaces. We provide a partial positive answer in this direction by showing that finitely supported elements in Lipschitz-free spaces over geodesic metric spaces are so-called WASQ-points.

Proposition 2.1.5. *Let* M *be a geodesic metric space. Then for every* $x \in S_{\mathscr{F}(M)}$ *with finite support, there exists a sequence* $(y_i) \subseteq B_{\mathscr{F}(M)}$ *such that* $\|x \pm y_i\| \to 1$, $\|y_i\| \to 1$*, and* $y_i \to 0$ *weakly.*

A full characterisation of WASQ Lipschitz-free spaces remains an open problem.

As a final result in [31], we strengthen Theorem 2.1.3.

Theorem 2.1.6 ([31, Theorem 4.1])**.** *No Lipschitz-free space has the SSD2P. In fact, no Lipschitz-free space has the SSD(2d)P for any* $0 < d \leq 1$.

2.2. Diameter two properties in spaces of Lipschitz functions

To the best of our knowledge, the work in [29] is the first published result on diameter two properties in spaces of Lipschitz functions. Since then, subsequent studies [35, 14, 26, 32, 33, 34, 28] on $\mathrm{Lip}_0(M)$ have made remarkable progress. In this section, we present the main results motivating our new results, but refer to the introduction of [28] and references therein for a more detailed background on the topic.

2.2.1. Characterising weak* diameter two properties in spaces of Lipschitz functions

We begin by investigating metric characterisations of weak* diameter two properties. In particular, the w^*-SD2P in $\mathrm{Lip}_0(M)$ admits a metric characterisation in terms of the long trapezoid property of the underlying metric space M, introduced in [35].

Definition 2.2.1. A metric space M is said to have the *long trapezoid property* (briefly, *LTP*) if, for every $\varepsilon > 0$ and every finite subset $N \subset M$, there exist elements $u,v \in M$ with $u \neq v$ such that, for all $x,y \in N$,

$$(1-\varepsilon)\big(d(x,y)+d(u,v)\big) \leq d(x,u)+d(y,v).$$

The following theorem provides the corresponding characterisation.

Theorem 2.2.2 ([35, Theorem 3.1]). *Let M be a metric space. The following statements are equivalent:*

1. $\mathrm{Lip}_0(M)$ *has the w^*-SD2P;*
2. $\mathscr{F}(M)$ *is OH;*
3. *M has the LTP.*

In a subsequent study [33], an analogous characterisation of the w^*-SSD2P for $\mathrm{Lip}_0(M)$ was obtained in terms of a related metric condition called the *strong long trapezoid property*.

Definition 2.2.3. A metric space M is said to have the *strong long trapezoid property* (briefly, *SLTP*) if, for every finite subset $N \subset M$ and every $\varepsilon > 0$, there exist $u,v \in M$ with $u \neq v$ such that, for all $x,y \in N$,

$$(1-\varepsilon)\big(d(x,y)+d(u,v)\big) \leq d(x,u)+d(y,v),$$

and, for all $x,y,z,w \in N$,

$$(1-\varepsilon)\big(2d(u,v)+d(x,y)+d(z,w)\big) \leq d(x,u)+d(y,u)+d(z,v)+d(w,v).$$

Theorem 2.2.4 ([33, Theorem 2.1]). *Let M be a metric space. The space* $\mathrm{Lip}_0(M)$ *has the w^*-SSD2P if and only if M has the SLTP.*

For some time it remained unknown whether analogous metric characterisations exist for the w^*-D2P and w^*-LD2P. Motivated by the results above, in [25] we obtained such characterisations.

Definition 2.2.5. We say that M has the *Lip-LTP* if, given a finite subset $N \subset M$, $\varepsilon > 0$, and $f \in B_{\mathrm{Lip}_0(M)}$, there exist $u,v \in M$ with $u \neq v$ such that for all $x,y \in N$,

$$(1-\varepsilon)\big(|f(x)-f(y)|+d(u,v)\big) \leq d(x,u)+d(y,v). \tag{2.1}$$

The following result provides a characterisation of the weak-star diameter two property.

Theorem 2.2.6 ([25, Proposition 4.2]). *Let M be a metric space. The space* $\mathrm{Lip}_0(M)$ *has the w^*-D2P if and only if M has the Lip-LTP.*

For the weak-star local diameter two property we obtain a similar characterisation.

Definition 2.2.7. We say that M has the *2-Lip-LTP* if, given a finite cyclically monotonic subset A of $\widetilde{M}$ and $\varepsilon > 0$, there exist $f,g \in B_{\mathrm{Lip}_0(M)}$ and $u,v \in M$ with $u \neq v$ such that for all $(x,y) \in A$,

$$f(m_{x,y}) \geq 1-\varepsilon, \qquad g(m_{x,y}) \geq 1-\varepsilon,$$

and for all $x, y \in \pi(A)$,

$$\max\{f(x)-f(y), g(y)-g(x)\}+(1-\varepsilon)d(u,v) \leq d(x,u)+d(y,v).$$

Theorem 2.2.8 ([25, Proposition 4.5]). *Let M be a pointed metric space. The space $\operatorname{Lip}_0(M)$ has the w^*-LD2P if and only if M has the 2-Lip-LTP.*

2.2.2. Separating all diameter two properties in spaces of Lipschitz functions

Another natural question about diameter two properties in spaces of Lipschitz functions is whether these properties are all distinct. By results from [33], it is known that the w^*-SSD2P and the w^*-SD2P are different. Furthermore, in [28] it was shown that the w^*-SSD2P differs from the SD2P and that the w^*-D2P differs from the SD2P.

It remained open whether the LD2P and the D2P or the w^*-LD2P and the w^*-D2P are equivalent in spaces of Lipschitz functions. It was unknown whether the SSD2P, the SD2P, the D2P, and the LD2P are different from their weak* counterparts in these spaces. The question of whether the SSD2P and the w^*-SSD2P are equivalent was open for dual Banach spaces in general (see [26, Question 6.2]).

In [24] and [25] we resolve these questions by showing that all of these properties are distinct in spaces of Lipschitz functions. We begin with the following result.

Theorem 2.2.9 ([24, Theorem 1.1 and Example 3.2]). *There exists a metric space M such that $\operatorname{Lip}_0(M)$ has the w^*-SSD2P but fails the LD2P.*

This example also produces an octahedral Lipschitz-free space $\mathscr{F}(M)$ whose bidual $\mathscr{F}(M)^{**} = \operatorname{Lip}_0(M)^*$ is not even locally octahedral. Consequently, it solves [32, Problem 1.1] (see also [14, p. 1681]), which asks whether there exists an octahedral Lipschitz-free space whose bidual is not octahedral.

Our second result is the following.

Theorem 2.2.10 ([25, Example 5.2]). *There exists a metric space M such that $\operatorname{Lip}_0(M)$ has the LD2P but fails the w^*-D2P.*

Examples of Banach spaces with the LD2P but without the D2P are rare. To the best of our knowledge, the only known examples appear in [2, 11, 15], and it is not known whether any of them is a dual Banach space.

2.2.3. Sufficient and equivalent conditions for diameter two properties in spaces of Lipschitz functions

Several sufficient conditions have been established ensuring that the space $\operatorname{Lip}_0(M)$ possesses various diameter two properties.

In [29] Ivakhno proved that if a metric space M is unbounded or not uniformly discrete, then the space $\operatorname{Lip}_0(M)$ has the LD2P.

In [14] Cascales et al. showed that if a metric space M has infinitely many cluster points or if M is discrete but not uniformly discrete, then $\operatorname{Lip}_0(M)$ even has the

SSD2P. Later, Langemets and Rueda Zoca [32] generalised this result by proving that the same conclusion holds whenever M is unbounded or not uniformly discrete.

A more general approach was developed in [28]. Two metric conditions were introduced, namely the sequential strong long trapezoid property and the sequential long trapezoid property, which imply that the space of Lipschitz functions has the SSD2P and the SD2P, respectively.

In [24] and [25] we further improve upon several of these results. We begin with results from [24].

Definition 2.2.11. We say that M has the *function LTP* (*FLTP*) if, given $\mu \in ba(\widetilde{M})$ with non-negative values, $\varepsilon > 0$, $n \in \mathbb{N}$, and $f_1, \ldots, f_n \in B_{\mathrm{Lip}_0(M)}$, there exist a subset $A \subset M$ with $\mu(\pi(A)) < \varepsilon$ and elements $u, v \in A$ with $u \neq v$ such that

$$(1-\varepsilon)\big(f_i(x) - f_i(y) + d(u,v)\big) \leq d(x,u) + d(y,v) \tag{2.2}$$

for all $x, y \in M \setminus A$ and $i \in \{1, \ldots, n\}$.

Theorem 2.2.12. *If M has the FLTP, then* $\mathrm{Lip}_0(M)$ *has the SD2P.*

Next, we provide a sufficient condition for the space $\mathrm{Lip}_0(M)$ to have the SSD2P.

Definition 2.2.13. We say that M has the *function SLTP* (*FSLTP*) if, given $\mu \in ba(\widetilde{M})$ with non-negative values, $\varepsilon > 0$, $n \in \mathbb{N}$, and $f_1, \ldots, f_n \in B_{\mathrm{Lip}_0(M)}$, there exist a subset $A \subset M$ with $\mu(\pi(A)) < \varepsilon$ and elements $u, v \in A$ with $u \neq v$ such that the inequalities (2.2) and

$$\begin{aligned}(1-\varepsilon)\big(f_i(x) - f_i(y) + f_j(z) - f_j(w) + 2d(u,v)\big)& \\ \leq d(x,u) + d(y,u) + d(z,v) + d(w,v)&\end{aligned}$$

hold for all $x, y, z, w \in M \setminus A$ and $i, j \in \{1, \ldots, n\}$.

Theorem 2.2.14. *If M has the FSLTP, then* $\mathrm{Lip}_0(M)$ *has the SSD2P.*

Clearly, the FLTP follows from the FSLTP. Moreover, both of these properties differ from the corresponding sequential variants introduced in [28].

In a subsequent study [25], we obtained the following characterisations for the SD2P and the LD2P for spaces of Lipschitz functions.

Theorem 2.2.15. *The space* $\mathrm{Lip}_0(M)$ *has the LD2P if and only if for every optimal $\mu \in S_{ba(\widetilde{M})}$ and $\gamma \in (0,1)$ there exist a subset $A \subset \widetilde{M}$ with $\mu(A) \geq \gamma$, functionals $f, g \in B_{\mathrm{Lip}_0(M)}$, and elements $u, v \in M$ with $u \neq v$ satisfying the following: for all $(x,y) \in A$ we have $f(m_{x,y}) \geq \gamma$ and $g(m_{x,y}) \geq \gamma$, and for all $x, y \in \pi(A)$,*

$$\max\{f(x) - f(y), g(y) - g(x)\} + \gamma d(u,v) \leq d(x,u) + d(y,v).$$

Theorem 2.2.16. *The space* $\mathrm{Lip}_0(M)$ *has the SD2P if and only if for all optimal $\mu_1, \ldots, \mu_n \in S_{ba(\widetilde{M})}$ and $\gamma \in (0,1)$ there exist subsets $A_1, \ldots, A_n \subset \widetilde{M}$ with $\mu_i(A_i) \geq \gamma$, functionals $f_1, g_1, \ldots, f_n, g_n \in B_{\mathrm{Lip}_0(M)}$, and elements $u, v \in M$ with*

$u \neq v$ satisfying the following: for every $i \in \{1,\ldots,n\}$ and $(x,y) \in A_i$ we have $f_i(m_{x,y}) \geq \gamma$ and $g_i(m_{x,y}) \geq \gamma$, and for all $x,y \in \pi(A_i)$,

$$\max\{f_i(x)-f_i(y), g_i(y)-g_i(x)\}+\gamma d(u,v) \leq d(x,u)+d(y,v).$$

It remains unknown whether FLTP is equivalent to the previous condition.

To prove these results, we introduce a generalised notion of cyclical monotonicity. Recall that a subset A of $\widetilde{M}$ is called *cyclically monotonic* (see, e.g. [9, 37]) if for any finite sequence of pairs $(x_1,y_1),\ldots,(x_n,y_n) \in A$ we have

$$\sum_{i=1}^{n} d(x_i,y_{i+1}) \geq \sum_{i=1}^{n} d(x_i,y_i),$$

where $y_{n+1}=y_1$.

We introduce the following generalisation.

Definition 2.2.17. Let $\gamma \in (0,1]$ and let $A \subset \widetilde{M}$. We say that A is *γ-cyclically monotonic* if for any finite sequence $(x_1,y_1),\ldots,(x_n,y_n) \in A$ we have

$$\sum_{i=1}^{n} \min\{d(x_i,y_{i+1})-\gamma d(x_i,y_i), d(y_i,y_{i+1})\} \geq 0,$$

where $y_{n+1}=y_1$.

Note that a subset A of $\widetilde{M}$ is cyclically monotonic if and only if it is 1-cyclically monotonic. The following result provides a useful generalisation of [9, Theorem 2.4 (i)⇔(iv)].

Theorem 2.2.18 ([25, Proposition 2.2])**.** *Let $A \subset \widetilde{M}$ and let $\gamma \in (0,1]$. Then A is γ-cyclically monotonic if and only if there exists $f \in B_{\mathrm{Lip}_0(M)}$ such that $f(m_{x,y}) \geq \gamma$ for all $(x,y) \in A$.*

SUMMARY

The main aim of this thesis is to investigate diameter two properties and almost square properties in Lipschitz-free spaces and in spaces of Lipschitz functions. These geometric properties play an important role in the geometry of Banach spaces and provide insight into their structure.

The thesis is organised as follows. The introductory chapter provides the background and motivation for the topic and introduces the necessary preliminary material. The second chapter reviews relevant previous work and outlines new contributions.

The remainder of the thesis consists of published journal papers containing the main results of this work.

The first paper included in the thesis is [23], where it is proven that the Lipschitz-free space over a length metric space is locally almost square. Furthermore, it is shown that no Lipschitz-free space can be almost square.

The second paper of the thesis is [31]. There a Lipschitz-free space is constructed that is locally almost square but not weakly almost square, thereby solving [1, Question 3.4]. A previous result from [23] is strengthened by showing that no Lipschitz-free space can have the SSD2P.

The third paper is the online-first version of [24]. In this work a dual Banach space is constructed that has the weak* symmetric strong diameter two property but does not have the local diameter two property, thereby solving [26, Question 6.2]. This result separates diameter two properties from their weak* counterparts in spaces of Lipschitz functions. Moreover, it solves [32, Problem 1.1] (see also [14, p. 1681]) by providing an example of an octahedral Lipschitz-free space whose bidual is not octahedral. Additionally, improved sufficient conditions for spaces of Lipschitz functions to have the strong diameter two property or the symmetric strong diameter two property are obtained. Finally, it is shown that the space of Lipschitz functions over any infinite metric subspace of ℓ_1 has the symmetric strong diameter two property.

The fourth paper is [25]. In this article equivalent conditions for a space of Lipschitz functions to have the weak* local diameter two property and the weak* diameter two property are provided. Characterisations of the strong diameter two property and the local diameter two property are obtained for spaces of Lipschitz functions. To achieve these results, a generalisation of cyclical monotonicity is introduced.

SISUKOKKUVÕTE

Diameeter-2 omadused ja peaaegu ruudu omadused Lipschitzi-vabades ruumides ja Lipschitzi funktsiooniruumides

Käesoleva väitekirja eesmärk on uurida Banachi ruumide geomeetrilisi omadusi, täpsemalt diameeter-2 omadusi ja peaaegu ruudu omadusi, ning analüüsida nende omaduste esinemist Lipschitzi funktsiooniruumides ja Lipschitzi-vabades ruumides. Need omadused kirjeldavad olukordi, kus Banachi ruumi kinnise ühikkera teatud osahulgad (näiteks viilud) on maksimaalse võimaliku diameetriga. Need omadused moodustavad Banachi ruumide geomeetrias olulise uurimissuuna, kuna annavad lisateadmisi Banachi ruumide struktuuri kohta. Vaadeldavate omadustega Banachi ruumid on alati lõpmatumõõtmelised, sest igal lõplikumõõtmelisel Banachi ruumil on Radon–Nikodými omadus, mistõttu leidub ühikkera kuitahes väikese diameetriga viile.

Selles teadustöös uuritakse Lipschitzi funktsiooniruume ja nendega seotud Lipschitzi-vabasid ruume. Lipschitzi funktsioonid meetriliste ruumide vahel on loomulik mittelineaarne analoog pidevatele lineaarsetele operaatoritele normeeritud ruumide vahel. Iga meetrilise ruumi M korral moodustavad kõik nullpunkti fikseerivad Lipschitzi funktsioonid Banachi ruumi $\mathrm{Lip}_0(M)$, kus funktsiooni normiks on tema Lipschitzi konstant. Osutub, et Banachi ruum $\mathrm{Lip}_0(M)$ on kaasruum, mille eelruumiks on Lipschitzi-vaba ruum $\mathscr{F}(M)$. See ruum sisaldab loomulikul viisil algset meetrilist ruumi M ning võimaldab käsitleda meetriliste ruumide vahel defineeritud Lipschitzi kujutusi pidevate lineaarsete operaatoritena vastavate Lipschitzi-vabade ruumide vahel. Lipschitzi-vabad ruumid moodustavad olulise silla Banachi ruumide teooria ja meetriliste ruumide geomeetria vahel. Mitmed Banachi ruumide omadused Lipschitzi ruumides peegeldavad aluseks oleva meetrilise ruumi struktuuri ning vastupidi.

Väitekirjas uuritakse diameeter-2 omadusi ja peaaegu ruudu omadusi Lipschitzi ruumides. Töö lähtepunktiks on järgnevad teadustulemused.

Esmalt on teada (vt [10, Teoreem 1.5], mis toetub ka artiklitele [30, 20]), et Lipschitzi-vabades ruumides $\mathscr{F}(M)$ langevad tuntud Daugaveti omadus ja teised diameeter-2 omadused omavahel kokku ning esinevad parajasti siis, kui aluseks olev meetriline ruum M on liinkaugusega ruum. Tekib loomulik küsimus, kas ka mõned teised Banachi ruumi omadused langevad Lipschitzi-vabades ruumides selle samaväärsuse alla.

Teisalt on Lipschitzi funktsiooniruumides $\mathrm{Lip}_0(M)$ leitud $*$-nõrgale sümmeetrilisele tugevale diameeter-2 omadusele ja $*$-nõrgale tugevale diameeter-2 omadusele samaväärsed meetrilised kirjeldused (vt [35, 33]). Kuid $*$-nõrgale diameeter-2 omadusele ja $*$-nõrgale lokaalsele diameeter-2 omadusele vastavad meetrilised kirjeldused olid seni teadmata.

Kolmandaks on avastatud mitmeid piisavaid tingimusi, mille korral Lipschitzi funktsiooniruum $\mathrm{Lip}_0(M)$ omab erinevaid diameeter-2 omadusi (vt lähemalt [28] sissejuhatust ja seal esitatud viiteid). Näiteks on teada, et teatud tingimused aluseks oleva meetrilise ruumi M struktuurile (näiteks kui M on tõkestamata või pole ühtlaselt diskreetne) tagavad ruumile $\mathrm{Lip}_0(M)$ sümmeetrilise tugeva diameeter-2 omaduse. Hilisemas teadustöös [28] on neid tulemusi üldistatud uute meetriliste omaduste abil. Seal on ka näidatud, et sümmeetriline tugev diameeter-2 omadus, tugev diameeter-2 omadus ja diameeter-2 omadus ei lange Lipschitzi funktsiooniruumides kokku. Taas tekib loomulik küsimus, millised seosed kehtivad ülejäänud diameeter-2 omaduste vahel Lipschitzi funktsiooniruumides.

Käesolevas teadustöös esitatakse uued teadustulemused kõigis kolmes kirjeldatud uurimissuunas.

Väitekirja ülesehitus on järgmine. Sissejuhatavas peatükis tutvustatakse töö teemat ning vajalikke eelteadmisi. Teises peatükis esitatakse selles uurimisteemas varasemad teadustulemused, millele väitekiri tugineb. Selle kõrval pannakse konteksti väitekirjas esitatavad uued teadustulemused. Ülejäänud osa väitekirjast koosneb avaldatud teadusartiklitest, milles on esitatud töö peamised tulemused.

Väitekirja esimene artikkel on [23]. Selles tõestatakse, et Lipschitzi-vaba ruum $\mathscr{F}(M)$ on lokaalse peaaegu ruudu omadusega parajasti siis, kui meetriline ruum M on liinkaugusega ruum. Lisaks näidatakse, et ükski Lipschitzi-vaba ruum ei saa olla peaaegu ruudu omadusega.

Väitekirja teine artikkel on [31]. Selles konstrueeritakse Lipschitzi-vaba ruum, mis on lokaalse peaaegu ruudu omadusega, kuid ei ole nõrga peaaegu ruudu omadusega, millega lahendatakse teadustöös [1, Question 3.4] püstitatud probleem. Samuti tugevdatakse artiklis [23] saadud tulemust, näidates, et ükski Lipschitzi-vaba ruum ei saa omada sümmeetrilist tugevat diameeter-2 omadust.

Väitekirja kolmas artikkel on teadustöö [24, online–first version]. Selles esitatakse näide Lipschitzi funktsiooniruumist $\mathrm{Lip}_0(M)$, millel on $*$-nõrk sümmeetriline tugev diameeter-2 omadus, kuid millel puudub lokaalne diameeter-2 omadus, lahendades sellega küsimuse [26, Küsimus 6.2]. See tulemus näitab, et Lipschitzi funktsiooniruumides ei lange diameeter-2 omadused kokku oma $*$-nõrk vastetega. Lisaks annab see vastuse probleemile [32, Küsimus 1.1] (vt ka [14, p. 1681]), esitades näite oktaeedrilisest Lipschitzi-vabast ruumist, mille teine kaasruum ei ole oktaeedriline. Veel esitatakse parendatud piisavad tingimused selleks, et Lipschitzi funktsiooniruumil oleks tugev diameeter-2 omadus või sümmeetriline tugev diameeter-2 omadus. Lõpuks näidatakse, et Lipschitzi funktsiooniruumil $\mathrm{Lip}_0(M)$ on sümmeetriline tugev diameeter-2 omadus iga ruumi ℓ_1 lõpmatu meetrilise alamruumi M korral.

Väitekirja neljas artikkel on [25]. Selles esitatakse meetrilised kirjeldused selleks, et Lipschitzi funktsiooniruumil oleks $*$-nõrk lokaalne diameeter-2 omadus või $*$-nõrk diameeter-2 omadus, ning esitatakse näide, mis neid omadusi eristab. Samuti antakse Lipschitzi funktsiooniruumide tugeva diameeter-2 omaduse ja lokaalse diameeter-2 omaduse jaoks meetrilised kirjeldused. Nende tulemuste tõestamiseks üldistatakse tsüklilise monotoonsuse mõistet.

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2017–2020	Tartu Ülikool, matemaatika bakalaureuseõpe, Loodusteaduse bakalaureus (matemaatika)
2020–2022	Tartu Ülikool, matemaatika magistriõpe, Loodusteaduse magister (matemaatika ja statistika), cum laude
2022–	Tartu Ülikool, matemaatika doktoriõpe

Teenistuskäik

2021–2022	Cybernetica AS, teaduri abi
2022–	Tartu Ülikool, matemaatika nooremteadur

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