DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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STRUCTURE OF GELFAND-MAZUR ALGEBRAS

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CONTENTS

LIST OF ORIGINAL PUBLICATIONS

1. Abel, Mart. *Topological section algebras.* Bachelor's Thesis. Tartu, 1998 (in Estonian).

2. Abel, Mart. *Description of closed maximal ideals in Gelfand-Mazur algebras.* In "General topological algebras (Tartu, 1999)" Math. Stud. (Tartu), **1**, Est. Math. Soc., Tartu, 2001. 7-13.

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4. Abel, Mart. *Description of closed maximal one-sided ideals in some Gelfand-Mazur algebras.* (To appear in the Proceedings of ICTAA 2).

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6. Abel, Mart. *Description of all closed maximal regular ideals in subalgebras of the algebra* $C(X, A; \sigma)$. (Submitted).

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Introduction

Mathematicians have been interested in the structure of algebras during many decades. To study the structure of an algebra it is helpful to know a description of its ideals.

Various representations of (not necessarily topological) algebras and modules have been studied by several mathematicians in [18], [48], [32], [39], [1], [2] and [3]. Sectional representations, section algebras and (vector) bundles have been studied in [26], [33], [34] and [1]. However, the space of maximal ideals of a section algebra still needs to be studied more deeply. The space of maximal ideals in topological algebras of vector-valued functions has been studied in case of Banach or locally convex algebras, e.g., in [27], [25], [28], [29], [9], [19], [20], [35], [24], [44], [21], and, for a more general case, in the present Thesis and in some papers, e.g., [4] and [6], by the author of the present Thesis.

For mathematicians, who deal with topological algebras, it is a well-known fact that every closed maximal regular ideal in a commutative Gelfand-Mazur algebra is the kernel of some nontrivial continuous linear multiplicative functional. In many cases it is relatively easy to describe all continuous linear multiplicative functionals of a given commutative topological algebra. Therefore nontrivial continuous linear multiplicative functionals are usually used to describe the structure of closed maximal ideals in several classes of commutative Gelfand-Mazur algebras. The same techniques cannot be used in a noncommutative case since even for noncommutative Banach algebras the set of nontrivial continuous linear multiplicative functionals may be empty.

In 1962 Lucien Waebroeck [49] showed that for studying the descriptions of maximal ideals in noncommutative unital locally convex Waelbroeck algebras it is possible to use the descriptions of maximal ideals of the center of the given topological algebra. In 1968 Graham Allan [17] developed a method for describing all maximal ideals in noncommutative unital Banach algebras by using extendible ideals of the center of this Banach algebra. It appeared (see [2]) that this method can be generalized for more general classes of (not necessarily commutative) Gelfand-Mazur algebras.

Allan considered only the unital case in his paper. By using and improving the ideas and methods presented by Stone and Naimark (see [38], VI, p. 182), it was possible (see [5]) to get a description of all closed maximal regular ideals, including the nonunital case.

Thus far, a description of closed maximal regular ideals of $C(X, A)$ (the algebra of all continuous functions from X to A, endowed with the compact-open topology) was given only in case, when A was a locally convex algebra, using for it the fact that $C(X,\mathbb{K}) \otimes A$ is dense in $C(X, A)$. Unfortunately, this fact is known to be true only if A is a locally convex algebra or if A has the approximation property (see [11], Theorem 1). By applying the method, developed by Allan, it was possible to avoid the density condition mentioned above. It allows to obtain a description of all closed maximal ideals of $C(X, A)$ for more general classes of Gelfand-Mazur algebras A.

Sometimes a description of ideals is known only for some subalgebra of the algebra we are interested in or for another algebra, which can be embedded into the algebra we are studying. The present Thesis gives a new method for describing all closed maximal regular ideals in a topological algebra A in case a description of all closed maximal regular ideals is known for some subalgebra B of the center of A. Moreover, it allows to describe all closed maximal regular ideals of a topological algebra A also in case when a description of all closed maximal regular ideals is known for some topological algebra B, which can be embedded into A as a closed subalgebra of the center of A.

In the First Chapter of the Thesis the definitions of the main terms are presented. The known results, used in Chapters 2, 3 and 4, are mentioned at the end of this Chapter.

The Second Chapter is dedicated to the problems and properties of the quotization and unitization of topological algebras. Here the results, which will be needed in Chapters 3 and 4, are proved. The main results of this Chapter are about the classes of topological algebras over $\mathbb{C}$ (the field of complex numbers) for which (a subalgebra of) the center is topologically isomorphic to $\mathbb{C}$.

At the beginning of the Third Chapter are found the sufficient conditions for a topological algebra A, for which the intersection $M \cap B$ of a closed maximal regular (not necessarily two-sided) ideal M of A and a subalgebra B of the center of A is a closed maximal regular ideal (not coinciding with B) in B. The ideas of Waelbroeck and Allan are generalized in the main part of the Third Chapter. A description of all closed maximal regular ideals in several classes of (not necessarily commutative) Gelfand-Mazur algebras is given. In the last sections of Chapter 3 are described two kinds of representations of topological algebras: *sectional representations* (that is, representations of topological algebras as subalgebras of a section algebra $\Gamma(\pi)$, defined in 1.4.) and *module representations* (that is, representations of topological algebras as subalgebras of an algebra $C(X, A)$ for some topological space X and topological algebra A). Therefore it is useful to have more information about the ideal structure of algebras $\Gamma(\pi)$ and $C(X, A)$.

In the last Chapter, the results of Chapters 2 and 3 are applied to a description of the ideal structure of algebra $C(X, A)$. A description of all closed maximal regular ideals of $C(X, A)$ is obtained for several classes of (not necessarily commutative and not necessarily unital) Gelfand-Mazur algebras A.

The results, presented in this Thesis, have been published in [2] and [3]. Papers [4], [5], [6], [7] and [8], dealing with the same kind of problems, will appear soon. The author has introduced these results at the following international conferences: "Congrès International de Mathématiques" (Rabat, 1999), "International Workshop on General Topological Algebras" (Tartu, 1999), "International Conference on Topological Algebras" (Rabat, 2000), "International Conference on Topological Algebras and its Applications" (Oulu, 2001), "International Conference on Topological Algebras and its Applications" (Oaxaca, 2002), "International Congress of Mathematicians" (Beijing, 2002).

Chapter I

Preliminaries

1.1. Terms, connected with topological linear spaces

Here we give some definitions of terms, which are connected with the linear and topological structure of topological linear spaces. Throughout of this Thesis $\mathbb{K}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers.

Topological linear space. A linear (or vector) space E over $\mathbb{K}$ is a *topological linear* (or *vector*) *space* over $\mathbb{K}$ (shortly, a *topological linear space*), if E is equipped with a topology so that

(a) for every neighbourhood O of zero of E there exists a neighbourhood U of zero of E such that $U + U \subset O$;

(b) for every neighbourhood O of zero of E there exist a neighbourhood U of zero of E and a neighbourhood V of zero of $\mathbb{K}$ such that $VU \subset O$.

Metrizable linear space. A topological linear space E is *metrizable*, if it has a countable base of neighbourhoods of zero.

Cauchy net. A net $(x_\lambda)_{\lambda \in \Lambda}$ of elements of E is a *Cauchy net*, if for every neighbourhood O of zero of E there is an index $\lambda_0 \in \Lambda$ such that $x_\lambda - x_\mu \in O$, whenever $\lambda > \mu > \lambda_0$.

Cauchy sequence. A sequence (x_n) of elements of E is a *Cauchy sequence*, if for every neighbourhood O of zero of E there is an index $n_0 \in \mathbb{N}$ such that $x_m - x_n \in O$, whenever $m > n > n_0$.

Complete topological linear space. A topological linear space E is *complete*, if every Cauchy net of elements of E converges in E.

Sequentially complete topological linear space. A topological linear space E is *sequentially complete*, if every Cauchy sequence in E converges in E.

Completion of a Hausdorff linear space. It is known (see, e.g., [30], Theorem 1, p. 131) that for every Hausdorff linear space E there exists a complete (sequentially complete) Hausdorff linear space $\tilde{E}$ such that E is topologically isomorphic to a dense subspace of $\tilde{E}$. The space $\tilde{E}$ is called the *completion* (*sequential completion, respectively*) of E.

Balanced set. A set U in a linear space E over $\mathbb{K}$ is *balanced*, if $\mu U \subset U$ for every $|\mu| \leqslant 1$.

Convex set. A set U in a linear space E over $\mathbb{K}$ is *convex*, if for each $x, y \in U$ and $0 \leqslant \lambda \leqslant 1$ holds $\lambda x + (1-\lambda)y \in U$.

Pseudoconvex set. A set U in a linear space E over $\mathbb{K}$ is *pseudoconvex*, if $U + U \subset \nu U$ for some $\nu > 0$.

Bounded set. A set U in a topological linear space E over $\mathbb{K}$ is *bounded* in E, if for every neighbourhood O of zero of E there exists an element $\mu_O \in \mathbb{C}\backslash\{0\}$ such that $U \subset \mu_O O$.

Homogeneous seminorm. Let E be a linear space over $\mathbb{K}$. The map $p : E \to \mathbb{R}^+$ (the set of nonnegative real numbers) is a *homogeneous seminorm* on E, if

a) $p(x) \geqslant 0$ for each $x \in E$;

b) $p(\lambda x) = |\lambda|\, p(x)$ for each $\lambda \in \mathbb{K}$ and $x \in E$;

c) $p(x+y) \leqslant p(x) + p(y)$ for each $x, y \in E$.

A homogeneous seminorm p is a *homogeneous norm* on E, in case $p(x) = 0$ if and only if $x = \theta_E$ (the zero element of E).

Nonhomogeneous seminorm. Let E be a linear space over $\mathbb{K}$ and let α be a positive number. The map $p : E \to \mathbb{R}^+$ is an *α-homogeneous seminorm* on E, if p satisfies the conditions a), c) and

d) $p(\lambda x) = |\lambda|^\alpha p(x)$ for each $\lambda \in \mathbb{K}$ and $x \in E$.

Again, p is an *α-homogeneous norm* on E, in case $p(x) = 0$ if and only if $x = \theta_E$.

Locally convex space. A topological linear space E is a *locally convex space*, if E has a base of neighbourhoods of zero consisting of convex sets. The topology of E is usually given by a system of homogeneous seminorms.

Locally bounded space. A topological linear space E is a *locally bounded space*, if E has a bounded neighbourhood of zero. The topology of E is usually given by a k-homogeneous norm for some $k \in (0, 1]$.

Locally pseudoconvex space. A topological linear space E is a *locally pseudoconvex space*, if E has a base of neighbourhoods of zero, consisting of balanced and pseudoconvex sets. The topology of E is usually given by a system $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous seminorms, where $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$.

Exponentially galbed space. A topological linear space E is an *exponentially galbed space*, if for every neighbourhood O of zero of E there exists a neighbourhood U of zero of E such that

$$\left\{\sum_{k=0}^{n} \frac{x_k}{2^k} : x_0, ..., x_n \in U\right\} \subset O$$

for each $n \in \mathbb{N}$.

1.2. Terms, connected with topological algebras

Next, we give some definitions of terms, which are connected with topological and algebraic structure of topological algebras.

Idempotent set. A subset U of a topological algebra A over $\mathbb{K}$ is *idempotent*, if $UU \subset U$.

Invertible elements. An element $a \in A$ is *invertible* in a unital algebra A, if there exists an element $b \in A$ such that $ab = ba = e_A$. Such element b is usually denoted by a^{-1}. The set of all invertible elements of A is denoted by $\mathrm{Inv}A$ and map $p_A : a \to a^{-1}$ is called the *inversion* in A.

Quasi-invertible elements. An element $a \in A$ is *quasi-invertible* in the algebra A, if there exists an element $b \in A$ such that

$$a + b - ab = a + b - ba = \theta_A.$$

Such element b is often denoted by a_q^{-1}. The set of all quasi-invertible elements of A is denoted by $\mathrm{Qinv}A$ and the map $p_A^q : a \to a_q^{-1}$ is called the *quasi-inversion* in A.

Division algebra. An algebra A over $\mathbb{K}$ is a *division algebra*, if every element $a \in A\backslash\{\theta_A\}$ is invertible in A.

Topological algebra. A topological linear space A over $\mathbb{K}$ is called a *topological algebra* over $\mathbb{K}$ (shortly, a *topological algebra*), if there has been defined an associative multiplication in A such that

(a) A is an algebra over $\mathbb{K}$;

(b) the multiplication in A is *separately continuous.*

The contition (b) means that for every $a \in A$ and every neighbourhood O of zero in A there exists a neighbourhood U of zero in A such that $Ua, aU \subset O$.

In case when the multiplication in A is *jointly continuous* (that is, for every neighbourhood O of zero in A there exists a neighbourhood U of zero in A such that $UU \subset O$), then A is said to be a *topological algebra with jointly continuous multiplication.*

Bounded element. An element x of a topological algebra A is *bounded*, if there exists a number $\lambda \in \mathbb{C}\backslash\{0\}$ such that the set

$$\left\{\left(\frac{x}{\lambda}\right)^n : n \in \mathbb{N}\right\}$$

is bounded in A.

Pair of topological algebras. Topological algebras A and B form a *pair* (A, B) *of topological algebras*, if B is dense subalgebra of A and the topology of B is not weaker than the topology on B, induced by the topology of A.

Unitization of a topological algebra. Let A be a topological algebra over $\mathbb{K}$ without a unit. The direct product $A \times \mathbb{K}$ in the product topology[1] is called the *unitization* of A, if the algebraic operations in $A \times \mathbb{K}$ are given by

$$(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu);$$

$$\nu(a, \lambda) = (\nu a, \nu\lambda);$$

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$

for each $a, b \in A$ and $\lambda, \mu, \nu \in \mathbb{K}$.

By Lemma 1.16 (see 1.6. Auxiliary results), we know that $A \times \mathbb{K}$ is a topological algebra in the product topology, if A is a topological algebra.

Sets $\hom(A, C)$ and $\hom_C(A, C)$. Let A and C be topological algebras. The set of all nontrivial continuous homomorphisms from A to C is denoted by $\hom(A, C)$. If $C = \mathbb{C}$, then we write just $\hom A$ instead of $\hom(A, \mathbb{C})$. The set $\hom_C(A, C)$ denotes the subset of $\hom(A, C)$ consisting of homomorphisms $\Psi \in \hom(A, C)$ for which $\Psi(ca) = c\Psi(a)$ and $\Psi(ac) = \Psi(a)c$ for all $c \in C$ and $a \in A$.

Gelfand space. The set $\hom A$ of all non-trivial continuous multiplicative linear functionals on A, endowed with the *Gelfand topology*, is called the *Gelfand space* of A. In the Gelfand topology on $\hom A$ a subbase of neighbourhoods of an element $\phi_0 \in \hom A$ consists of sets

$$\{\phi \in \hom A : |\phi(a) - \phi_0(a)| < \epsilon\}$$

with $a \in A$ and $\epsilon > 0$. In some cases (see, e.g., [50], Example 12.1, p. 40; [51], 14.2, p. 124 or [31], Example 21, p. 153) the set $\hom A$ might be empty. Therefore the assumption $\hom A \neq \emptyset$ is often necessary.

[1]If (X, τ_X) and (Y, τ_Y) are topological algebras, then the *product topology* on $X \times Y$ is the topology τ, the base of which is $\{O \times U : O \in \beta_X$ and $U \in \beta_Y\}$, where β_X and β_Y are the bases of topologies τ_X and τ_Y, respectively.

Equicontinuous set. A subset $H \subset \text{hom}A$ is equicontinuous at $a_0 \in A$, if for every $\epsilon > 0$ there exists a neighbourhood $O(a_0)$ of a_0 such that $|\phi(a) - \phi(a_0)| < \epsilon$ for each $\phi \in H$ and $a \in O(a_0)$. The set H is *equicontinuous*, if H is equicontinuous at every $a \in A$.

Locally equicontinuous set. A subset $H \subset \text{hom}A$ is *locally equicontinuous*, if every $\phi_0 \in \text{hom}A$ has an equicontinuous neighbourhood.

Topological module. Let A be an algebra over $\mathbb{K}$. The linear space E over $\mathbb{K}$ is a *left (right) A-module*, if the module multiplication $(a, x) \mapsto ax$ of $A \times E$ into E ($(x, a) \mapsto xa$ of $E \times A$ into E, respectively) is a bilinear map, which satisfies the condition $a_1(a_2x) = (a_1a_2)x$ (the condition $(xa_1)a_2 = x(a_1a_2)$, respectively) for each $a_1, a_2 \in A$ and $x \in E$. If A is a topological algebra over $\mathbb{K}$ and E is a topological linear space over $\mathbb{K}$ in which the module multiplication is separately continuous, then E is a *topological left (right) A-module.* A linear space E is a *topological A-bimodule*, if E is both a topological left A-module and a topological right A-module.

Topological module-algebra. Let C be a topological algebra. A topological algebra A is called *a topological C-algebra* or *a topological module-algebra with respect to C*, if

(i) A is a topological C-bimodule;

(ii) both left and right module multiplications are separately continuous;

(iii) $c(ab) = (ca)b$, $(ab)c = a(bc)$, $(ac)b = a(cb)$ for all $a, b \in A$ and $c \in C$;

(iv) $e_Ac = ce_A$ for all $c \in C$, in case A has a unit e_A;

(v) $e_Ca = a = ae_C$ for all $a \in A$, in case C has a unit e_C.

The B-extension property. Let A be a topological C-algebra and B a subalgebra of $Z(A)$ for which $\text{hom}B \neq \emptyset$. We say that A has a *B-extension property*, if[2]

(i) $C \otimes B$ is dense in A;

(ii) for each $\phi \in \text{hom}B$ there exists $\Psi_\phi \in \text{hom}_C(A, C)$ such that

$$\Psi_\phi\Big(\sum_{k=1}^{n} c_k b_k\Big) = \sum_{k=1}^{n} c_k \phi(b_k)$$

for all $n \in \mathbb{N}$, $c_1, ..., c_n \in C$ and $b_1, ..., b_n \in B$.

[2]For all topological algebras A and B we denote here and in the sequel by $A \otimes B$ the set

$$\Big\{\sum_{k=1}^{n} a_k b_k : n \in \mathbb{N},\ a_1, ..., a_n \in A,\ b_1, ..., b_n \in B\Big\}.$$

Algebra $C(X, A)$. Let X be a topological space and A a topological algebra. The set of all continuous functions $f : X \to A$ is denoted by $C(X, A)$. It is easy to verify that $C(X, A)$ is an algebra with respect to the *pointwise* algebraic operations (that is, $f + g, fg$ and λf are defined by

$$(f + g)(x) = f(x) + g(x),$$

$$(fg)(x) = f(x)g(x)$$

and

$$(\lambda f)(x) = \lambda(f(x))$$

for each $x \in X$). If for every neighbourhood O of zero of A there exists a compact subset $K_O \subset X$ such that $f(x) \in O$ for all $x \in X \backslash K_O$, then it is said that f *vanishes at infinity.* The set of all such functions $f \in C(X, A)$, which vanish at infinity, is denoted by $C_0(X, A)$. It can be easily shown that $C_0(X, A)$ is an algebra.

Algebra $C(X, A; \sigma)$. For every topological algebra A with jointly continuous multiplication[3], every topological space X and every cover σ of X let $C(X, A; \sigma)$ denote the set of all continuous functions $f : X \to A$ for which the closure of $f(S)$ in the topology of A is compact in A for each $S \in \sigma$. We define the algebraic operations on $C(X, A; \sigma)$ pointwise and endowe $C(X, A; \sigma)$ with the topology, whose subbase $\mathcal{B}$ of neighbourhoods of zero is

$$\{T(S, O) : S \in \sigma,\ O \text{ is a neighbourhood of zero in } A\},$$

where $T(S, O) = \{f \in C(X, A; \sigma) : f(S) \subset O\}$. Then $C(X, A; \sigma)$ is a topological algebra. It is easy to see that $C(X, A; \sigma) \subset C(X, A)$.

A-valued analytic function on $\mathbb{C} \cup \{\infty\}$. A map $\Psi : \mathbb{C} \cup \{\infty\} \to A$ is called an *A-valued analytic function on* $\mathbb{C} \cup \{\infty\}$, if the following condition holds. For every $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ there are numbers $\delta, R > 0$ and sequences (x_k) and (y_k) of elements of A such that

$$\Psi(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k \lambda^k,$$

if $|\lambda| < \delta$ and

$$\Psi(\lambda) = \sum_{k=0}^{\infty} \frac{y_k}{\lambda^k},$$

if $|\lambda| > R$.

[3]It is known (see [40], Proposition 3.1) that $C(X, A; \sigma)$ is an algebra if and only if A is a topological algebra with jointly continuous multiplication.

1.3. Subsets of topological algebras

Let A be an algebra over $\mathbb{K}$, θ_A denote the zero element of A and e_A be the unit element of A, if A is a unital algebra.

Center. The center of A is the set

$$Z(A) = \{z \in A : za = az \text{ for all } a \in A\}.$$

Ideals. A linear subspace I of A is

1) a *left ideal* of A if and only if $ai \in I$ for all $a \in A$ and $i \in I$;

2) a *right ideal* of A if and only if $ia \in I$ for all $a \in A$ and $i \in I$;

3) a *two-sided ideal* of A if and only if I is both left and right ideal of A.

Let I be a left (right or two-sided) ideal of A. If for every left (right or two-sided, respectively) ideal J of A from the conditions $I \subset J$ and $I \neq J$ follows that $J = A$, then I is a *maximal* left (right or two-sided, respectively) ideal.

Regular ideal. A left (right or two-sided) ideal I of A is *regular* (or *modular*) if and only if there exists an element $u \in A$ such that $a - au \in I$ ($a - ua \in I$ or $a - au, a - ua \in I$, respectively) for all $a \in A$. Such element u is called a *right* (*left* or *two-sided*, respectiely) *modular unit* for I.

Primitive ideal. Let M be a maximal regular left (right) ideal of A. Then the two-sided ideal $P = \{a \in A : aA \subset M\}$ ($P = \{a \in A : Aa \subset M\}$, respectively) is called a *primitive ideal* of A with respect to M.

Primitive algebra. An algebra A is *primitive* if and only if $\{\theta_A\}$ is a primitive ideal in A.

In the present Thesis we use the following denotations for every topological algebra A:

$m(A)$ is the set of all closed regular two-sided ideals I of A which are maximal as left or right ideals;

$m_l(A)$ ($m_r(A)$ or $m_t(A)$) is the set of all closed maximal regular left (right or two-sided, respectively) ideals of A;

$M_t(A)$ is the set of all maximal regular two-sided ideals of A;

$\mathrm{cl}_A(U)$ is the closure of a subset U of A in the topology of A.

Let S be a subset of $Z(A)$. Then

$$I(S) = \mathrm{cl}_A\Big\{\sum_{k=1}^{n} a_k s_k + \lambda_k s_k : n \in \mathbb{N}, a_k \in A, s_k \in S, \lambda_k \in \mathbb{K} \text{ for all } k\Big\}.$$

If A is a unital algebra, then

$$I(S) = \mathrm{cl}_A\Big\{\sum_{k=1}^{n} a_k s_k : n \in \mathbb{N}, a_1, \dots, a_n \in A, s_1, \dots, s_n \in S\Big\}.$$

Extendible ideal. Let A be a topological algebra and B a closed subalgebra of $Z(A)$. An ideal $M \in m(B)$ is called an *extendible ideal*, if $I(M) \neq A$. The set of all extendible ideals of B in A is denoted by $m_e(B)$.

Hull of an ideal. Let A be a topological algebra and I a closed regular two-sided ideal of A. The set

$$\mathrm{h}(I) = \{M \in m_t(A) : I \subset M\}$$

is the *hull* of I.

Kernel of the set of ideals. Let A be a topological algebra and S a subset of $m_t(A)$. The set

$$\mathrm{k}(S) = \bigcap_{M \in S} M$$

is the *kernel* of S.

Hull-kernel topology. The *hull-kernel topology* τ_{hk} on $m_t(A)$ is the topology in which $\mathrm{cl}_A(S) = h(k(S))$ for each $S \in m_t(A)$. It is known that $(m_t(A), \tau_{hk})$ is a T_1-space but not necessarily a Hausdorff space.

1.4. Main classes of topological algebras

In the sequel we will use the following classes of topological algebras.

Normed algebra. A topological algebra is called a *normed algebra*, if its topology is given by an *algebra norm*, i.e., by a norm p, which satisfies the condition $p(ab) \leqslant p(a)p(b)$ for each $a, b \in A$.

Banach algebra. A complete normed algebra is called a *Banach algebra*.

Spectral algebra. A topological algebra A is called a *spectral algebra*, if its topology is given by an *spectral seminorm*, i.e., by a *submultiplicative*[4] seminorm p, which satisfies the condition $r_A(a) \leqslant p(a)$ for each $a \in A$ (here $r_A(a)$ denotes the spectral radius (see, e.g., [22], p. 11) of $a \in A$). It is easy to see that every Banach algebra is a spectral algebra.

[4]A seminorm p is *submultiplicative*, if $p(ab) \leqslant p(a)p(b)$ for each $a, b \in A$.

Locally convex algebra. A topological algebra A is called a *locally convex algebra*, if the underlying topological linear space of A is locally convex.

Locally bounded algebra. A topological algebra A is called a *locally bounded algebra*, if the underlying topological linear space of A is locally bounded.

Locally pseudoconvex algebra. A topological algebra A is called a *locally pseudoconvex algebra*, if the underlying topological linear space of A is locally pseudoconvex.

It is easy to see that all locally convex algebras and all locally bounded algebras are locally pseudoconvex algebras.

Locally m-pseudoconvex algebra. A topological algebra A is called a *locally multiplicatively pseudoconvex* (shortly, *locally m-pseudoconvex*) *algebra*, if A has a base of neighbourhoods of zero consisting of balanced, pseudoconvex and idempotent sets. The topology of A is often given by a system $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous submultiplicative seminorms, where $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$.

k-normed algebra. A topological algebra is called a *k-normed algebra*, if its topology is given by a k-homogeneous submutiplicative norm with $k \in (0, 1]$.

k-Banach algebra. A complete k-normed algebra is called a *k-Banach algebra.*

Locally A-pseudoconvex algebra. A topological algebra A is called a *locally absorbingly pseudoconvex* (shortly, *locally A-pseudoconvex*) *algebra*, if A has a base $\mathcal{B}$ of neighbourhoods of zero, consisting of balanced and pseudoconvex sets such that for every $U \in \mathcal{B}$ and every $a \in A$ there exists a number $\nu = \nu(a, U) > 0$ such that $aU, Ua \subset \nu U$. The topology of A is often given by a system $\{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous *A-multiplicative*[5] seminorms, where $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$.

It is easy to see that every locally m-pseudoconvex algebra is locally A-pseudoconvex. Indeed, let A be a locally m-pseudoconvex algebra, $a \in A$ and U an element of a base of neighbourhoods of zero in A. Since every neighbourhood of zero absorbs every element of A (that is, for every $a \in A$ and every neighbourhood U of zero of A there exists a number

[5] A seminorm p is *A-multiplicative*, if for each $a \in A$ there exist positive numbers $M(a,p)$ and $N(a,p)$ such that $p(ab) \leqslant M(a,p)p(a)$ and $p(ab) \leqslant N(a,p)p(b)$ for each $b \in A$.

$\lambda > 0$ such that $a \in \lambda U$), then $aU \subset \lambda UU \subset \lambda U$ (similarily, $Ua \subset \lambda U$). Let $\nu(a, U) = \lambda$. Then A is an A-pseudoconvex algebra.

Gelfand-Mazur algebra. A topological algebra A (over $\mathbb{K}$) is called a *Gelfand-Mazur algebra*, if the quotient algebra A/M (in the quotient topology[6]) is topologically isomorphic to $\mathbb{K}$ for each $M \in m(A)$.

Fréchet algebra. A topological algebra A is called a *Fréchet algebra*, if A is complete and metrizable.

Q-algebra. A topological algebra A is called a *Q-algebra*, if the set $\text{Qinv}A$ of all quasi-invertible elements of A is open in A. In case when A is a unital algebra, then A is a Q-algebra if the set $\text{Inv}A$ of all invertible elements of A is open in A.

Waelbroeck algebra. A topological algebra A is called a *Waelbroeck algebra*, if A is a Q-algebra, in which the quasi-inversion (if A has the unit element, then inversion) is continuous.

Exponentially galbed algebra. A topological algebra A is called an *exponentially galbed algebra*, if the underlying topological linear space of A is exponentially galbed.

Simplicial (or **normal** in the sence of Michael) **algebra.** A topological algebra A is called a *simplicial* (or *normal* in the sence of Michael) algebra, if every closed regular left (right or two-sided) ideal of A is contained in some closed maximal regular left (right or two-sided, respectively) ideal of A. If every closed regular two-sided ideal of A is contained in some closed maximal regular two-sided ideal of A, then A is *simplicial* (*with respect to two-sided ideals*) algebra.

Topologically primitive algebra. A topological algebra A is *topologically primitive* if and only if there exists a closed maximal regular left (right) ideal M of A such that $\{a \in A : aA \subset M\} = \{\theta_A\}$ ($\{a \in A : Aa \subset M\} = \{\theta_A\}$, respectively).

Fiber bundle. The complex (B, π, X), where B and X are topological spaces and $\pi : B \to X$ is a continuous open surjection, is called *a fiber bundle.*

Section of a fiber bundle. A mapping $f : X \to B$ is said to be *a section* of the fiber bundle (B, π, X) (shortly, a *section of* π), if $\pi[f(x)] = x$ for every $x \in X$. The set of all continuous sections of π is denoted by $\Gamma(\pi)$.

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[6]If (A, τ) is a topological algebra and π denotes the canonical homomorphism from A onto A/M, then $\tau' = \{O \subset A/M : \pi^{-1}(O) \in \tau\}$ is the *quotient topology* on A/M.

Section algebra. Let (B, π, X) be a fiber bundle, for which all fibers $B_x = \{b \in B : \pi(b) = x\}$ are topological algebras. If we define the algebraic operations in $\Gamma(\pi)$ pointwise and the topology on $\Gamma(\pi)$ by giving a subbase of $f_0 \in \Gamma(\pi)$ by

$$B(f_0) = \{U_O(f_0) : O \in B(\theta_P)\},$$

where $B(\theta_P)$ is a base of neighbourhoods of zero of the algebra[7]

$$P = \prod_{x \in X} B_x$$

in the product topology and

$$U_O(f_0) = \{f \in \Gamma(\pi) : ((f - f_0)(x))_{x \in X} \in O\},$$

then $\Gamma(\pi)$ is a topological algebra which is called a *section algebra*.

1.5. Representations of topological algebras

Next, we give some definitions of terms, which are connected with representations of topological algebras.

Representation. Let A and B be topological algebras. Every continuous homomorphism f from A into B is called a *representation* of A in B. In case when B is a section algebra $\Gamma(\pi)$, then f is called a *sectional representation of* A. When B is $C(X, C; \sigma)$, for some topological space X and topological algebra C, then f is called a *functional representation* of A.

Algebra $\mathcal{L}$(X). Let X be a topological linear space over $\mathbb{K}$ and $\mathcal{L}(X)$ the set of all continuous endomorphisms on X (i.e., linear maps from X to X). If we define the addition of elements and the scalar multiplication in $\mathcal{L}(X)$ pointwise and the multiplication of elements in $\mathcal{L}(X)$ by the composition, then $\mathcal{L}(X)$ is an algebra over $\mathbb{K}$. We endow $\mathcal{L}(X)$ with the topology τ_s of simple convergence. A subbase of neighbourhoods of zero in τ_s consists of sets

$$T(S, O) = \{L \in \mathcal{L}(X) : L(S) \subset O\},$$

where S is a finite subset of X and O is a neighbourhood of zero in X. Then $\mathcal{L}_s(X)$ (i.e., $\mathcal{L}(X)$, endowed with τ_s) is a topological Hausdorff algebra, if X is a Hausdorff algebra.

Irreducible representation. Any homomorphism f of an algebra A into $\mathcal{L}(X)$ is called a *representation of* A *on* X. Every representation f

[7] See [1], p. 12.

of A on X defines on X a left (right) module multiplication $\cdot_f$ if we put $a\cdot_f x = f(a)(x)$ $(x\cdot_f a = f(a)(x)$, respectively) for each $a \in A$ and $x \in X$. In this case X becomes a left (right) A-module. We denote it by X_f. A left (right) A-module X is *nontrivial* if $AX \neq \{\theta_X\}$ $(XA \neq \{\theta_X\}$, respectively) and X is *irreducible* if it is a nontrivial left (right) A-module, where X and $\{\theta_X\}$ are the only A-submodules of X. A representation f of A on X is *irreducible* if X_f is an irreducible left (right) A-module.

Topological radical. Let A be a topological algebra. The intersection $\mathrm{rad}A$ of kernels of all continuous irreducible representations of A on topological linear spaces is the *topological radical* of a topological algebra A. In the present Thesis we will consider only such topological algebras for which $\mathrm{rad}A \neq A$. It is known (see [14], Theorem 1) that $\mathrm{rad}A$ is the intersection of all closed maximal regular left (right) ideals of A.

Topologically semi-simple algebra. A topological algebra A is *topologically semi-simple*, if $\mathrm{rad}A = \{\theta_A\}$.

Strongly topologically semi-simple algebra. A topological algebra A is *strongly topologically semi-simple*, if $k(m_t(A)) = \{\theta_A\}$.

1.6. Auxiliary results

For convenience of reference, we cite below the known results used in the present Thesis.

Lemma 1.1. *Any maximal regular two-sided ideal is primitive.*

Proof. See [42], Theorem 2.2.9 (ii), p. 54.

Lemma 1.2. *Let P be a primitive ideal of algebra A. Then A/P is a primitive algebra.*

Proof. See, e.g., [22], Proposition 9, p. 136.

Lemma 1.3. *The quotient space E/M of a topological linear space E modulo a linear subspace M is a Hausdorff space if and only if M is closed in E.*

Proof. See [30], Proposition 5, p. 105.

Lemma 1.4. *Let E be a complete metrizable topological linear space and M a closed subspace of E. Then the quotient space E/M is complete (and metrizable) in the quotient topology.*

Proof. See [30], Theorem 2, p. 138.

Lemma 1.5. *Every analytic map from* $\mathbb{C} \cup \{\infty\}$ *into an exponentially galbed Hausdorff linear space is a constant map.*

Proof. See [47], Corollary, p. 9.

Lemma 1.6. *Let* A *be a topological algebra and* I *a left (right or two-sided) ideal of* A. *If* $cl_A(I) \neq A$ *then* $cl_A(I)$ *is also a left (right or two-sided, respectively) ideal of* A.

Proof. See [38], Chapter II, § 8, VII, p. 169.

Lemma 1.7. *Let* A *be a topological algebra and* $f, g \in \mathrm{hom} A$. *Then* $f = g$ *if and only if* $\ker f = \ker g$.

Proof. See [36], Lemma 7.2, p. 69.

Lemma 1.8. *The inversion in a topological algebra* A *with unit* e_A *is continuous, if there exists a neighbourhood* $O(e_A)$ *of* e_A *such that every* $a \in O(e_A)$ *is invertible in* A *and the inversion in* A *is continuous at* e_A.

Proof. See [38], Chapter II, §8, p. 169–170.

Lemma 1.9. *The quasi-inversion in a topological algebra* A *is continuous, if there exists a neighbourhood* O *of* θ_A *such that every* $a \in O$ *is quasi-invertible in* A *and the quasi-inversion in* A *is continuous at* θ_A.

Proof. See [38], Chapter II, §8, p. 173–174.

Lemma 1.10. *Let* (A, B) *be a pair of topological algebras* A *and* B *with the same unit.*

(a) *If* A *is a Gelfand-Mazur algebra, then* $\mathrm{cl}_A(M \cap B) = M$ *for every* $M \in m(A)$.

(b) *If* A *and* B *are Gelfand-Mazur algebras and the topology of* B *is induced by the topology of* A, *then* $\mathrm{cl}_A(M) \cap B = M$ *for every* $M \in m(B)$.

Proof. See [7], Corollary 1.

Lemma 1.11. *Let* A *be one of the following topological algebras:*

(a) *a locally* A*-pseudoconvex algebra;*

(b) *a locally pseudoconvex Waelbroeck algebra;*

(c) *a locally pseudoconvex Fréchet algebra;*

(d) *an exponentially galbed algebra with bounded elements.*

Then A *is a Gelfand-Mazur algebra.*

Proof. See [13], Corollary 2, p. 125–126.

Lemma 1.12. *Let A be a topological algebra, for which the set $m(A)$ is non-empty. Then the following statements are equivalent:*

1) *A is a Gelfand-Mazur algebra;*

2) *there exists a topology τ on A such that (A, τ) is an exponentially galbed algebra with bounded elements and every $M \in m(A)$ is closed in the topology τ.*

Proof. See [16], Theorem 1 or (incomplete version) in [13], p. 123, parts (a) and (g) of Theorem 2.

Lemma 1.13. *Let A be a Q-algebra (a Waelbroeck algebra) and I a two-sided ideal of A. Then A/I is also a Q-algebra (a Waelbroeck algebra, respectively).*

Proof. See [21], Corollary 3.6.27, p. 155.

Lemma 1.14. *Let A be a unital primitive locally A-pseudoconvex Hausdorff algebra over $\mathbb{C}$ or a unital topologically primitive locally pseudoconvex Fréchet algebra over $\mathbb{C}$. Then the center $Z(A)$ of A is topologically isomorphic to $\mathbb{C}$.*

Proof. See [15], Theorem 2.

Lemma 1.15. *Let A be a locally convex Waelbroeck Hausdorff algebra. Then all elements of A are bounded.*

Proof. See [12], Remark 1, p. 20.

Lemma 1.16. *The unitization $A \times \mathbb{K}$ of a topological algebra A is a topological algebra in the product topology.*

Proof. See [21], Proposition 2.2.9, p. 87.

Lemma 1.17. *The unitization $A \times \mathbb{K}$ of A, in the product topology, is a Q-algebra (a Waelbroeck algebra) if and only if A is a Q-algebra (a Waelbroeck algebra, respectively).*

Proof. See [21], Lemma 3.6.26, p. 155.

Chapter II

Properties, connected with the quotization and unitization of topological algebras

2.1. Properties of quotient algebras and of the center of Gelfand-Mazur algebras

To describe all closed maximal regular ideals in a Gelfand-Mazur algebra A by closed maximal ideals of a subalgebra of the center of A, we need the following results.

Theorem 2.1. *Let A be a Gelfand-Mazur algebra and I a two-sided ideal of A. If $m(A) \neq \emptyset$, then there exists a topology τ on A such that A/I (in the quotient topology, defined by τ) and $Z(A/I)$ (in the subspace topology) are exponentially galbed algebras with bounded elements.*

Proof. Let A be a Gelfand-Mazur algebra for which $m(A) \neq \emptyset$. By Lemma 1.12, there exists a topology τ on A such that (A, τ) is an exponentially galbed algebra with bounded elements.

Let τ_I be the quotient topology on A/I, defined by τ. Let $\pi : A \to A/I$ be the canonical homomorphism and O' a neighbourhood of zero in $(A/I, \tau_I)$. Since π is a quotient map, then it is continuous and open. Therefore $O = \pi^{-1}(O')$ is a neighbourhood of zero in (A, τ). Now we can find a neighbourhood U of zero in (A, τ) such that

$$\Big\{\sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, ..., a_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Let $V' = \pi(U)$. Then we see that

$$\Big\{\sum_{k=0}^{n} \frac{x_k}{2^k} : x_0, ..., x_n \in V'\Big\} \subset O'$$

for each $n \in \mathbb{N}$, which implies that $(A/I, \tau_I)$ is exponentially galbed.

Since $\tau_Z = \{T' \cap Z(A/I) : T' \in \tau_I\}$ is the subspace topology on $Z(A/I)$, generated by τ_I, then every neighbourhood O'' of zero in $Z(A/I)$ is representable in the form $O'' = O' \cap Z(A/I)$, where O' is a neighbourhood of zero in A/I. As above, we find the neighbourhood V' of zero of A/I and take $V'' = V' \cap Z(A/I)$. Then

$$\left\{\sum_{k=0}^{n} \frac{y_k}{2^k} : y_0, y_1, ..., y_n \in V''\right\} \subset O''$$

for each $n \in \mathbb{N}$, which implies that $Z(A/I)$ is exponentially galbed in the topology τ_Z.

Let now $b \in A/I$ and V' be an arbitrary neighbourhood of zero in $(A/I, \tau_I)$. Then there exists an element $a \in A$ such that $b = \pi(a)$ and $V = \pi^{-1}(V')$ is a neighbourhood of zero in (A, τ). Since all elements of (A, τ) are bounded, then there are numbers $\lambda_a \in \mathbb{C}\backslash\{0\}$ and $\mu_V > 0$ such that

$$\left\{\left(\frac{a}{\lambda_a}\right)^n : n \in \mathbb{N}\right\} \subset \mu_V V.$$

For every fixed $n \in \mathbb{N}$ we have now

$$\left(\frac{b}{\lambda_a}\right)^n = \left(\frac{\pi(a)}{\lambda_a}\right)^n = \pi\left[\left(\frac{a}{\lambda_a}\right)^n\right] \in \pi(\mu_V V) = \mu_V V',$$

which implies

$$\left\{\left(\frac{b}{\lambda_a}\right)^n : n \in \mathbb{N}\right\} \subset \mu_V V'.$$

Hence b is bounded in $(A/I, \tau_I)$. Since b is an arbitrary element in A/I, then all elements of $(A/I, \tau_I)$ are bounded.

Let now c be an arbitrary element of $Z(A/I)$ and W'' an arbitrary neighbourhood of zero in $(Z(A/I), \tau_Z)$. Then $c \in A/I$ and there exists a neighbourhood W' of zero in A/I such that $W'' = W' \cap Z(A/I)$. As indicated above, there are numbers $\lambda_c \in \mathbb{C}\backslash\{0\}$ and $\mu'_W > 0$ such that

$$\left\{\left(\frac{c}{\lambda_c}\right)^n : n \in \mathbb{N}\right\} \subset \mu'_W W'.$$

Since $c \in Z(A/I)$, then

$$\left(\frac{c}{\lambda_c}\right)^n \in Z(A/I)$$

for each $n \in \mathbb{N}$. Therefore

$$\left\{\left(\frac{c}{\lambda_c}\right)^n : n \in \mathbb{N}\right\} \subset \mu'_W W''$$

which means that c is bounded in $(Z(A/I), \tau_Z)$. Consequently, every element in $(Z(A/I), \tau_Z)$ is bounded.

Lemma 2.2. *Let A be a locally pseudoconvex (locally A-pseudoconvex) algebra and I a two-sided ideal of A. Then A/I and $Z(A/I)$ are also locally pseudoconvex (locally A-pseudoconvex, respectively) algebras.*

Proof. Since A is a locally pseudoconvex algebra, then we can find a base $\mathcal{B}$ of neighbourhoods of zero of A consisting of balanced pseudoconvex neighbourhoods of zero. It is easy to see that $\mathcal{B}' = \pi(\mathcal{B})$ and

$$\mathcal{B}'' = \{U' \cap Z(A/I) : U' \in \mathcal{B}'\}$$

are suitable bases of neighbourhoods of zero for A/I and $Z(A/I)$, respectively.

Lemma 2.3. *Let A be a (locally pseudoconvex) Fréchet algebra and I a closed two-sided ideal of A. Then A/I and $Z(A/I)$ are also (locally pseudoconvex) Fréchet algebras.*

Proof. By Lemma 1.4, algebra A/I is a Fréchet algebra. Since $Z(A/I)$ is a closed linear subspace of A/I, then $Z(A/I)$ is complete and metrizable, hence a Fréchet algebra.

Using Lemma 2.2, we obtain the following lemma.

Lemma 2.4. *Let A be a (locally pseudoconvex) Waelbroeck algebra and I a two-sided ideal of A. Then A/I and $Z(A/I)$ are also (locally pseudoconvex) Waelbroeck algebras.*

Proof. By Lemma 1.13, we obtain that A/I is a Waelbroeck algebra. This implies that $\mathrm{Qinv}A/I$ is an open set in A/I. Since

$$\mathrm{Qinv}Z(A/I) = \mathrm{Qinv}A/I \cap Z(A/I),$$

then $\mathrm{Qinv}Z(A/I)$ is an open set in $Z(A/I)$. It is easy to see that the quasi-inversion is continuous in $Z(A/I)$ (because $\mathrm{Qinv}Z(A/I)$ is a subset of $\mathrm{Qinv}A/I$). Hence $Z(A/I)$ is a Waelbroeck algebra.

Lemma 2.5. *Let M be a maximal left (right) ideal in an algebra A for which $\{a \in A : aA \subset M\} = \{\theta_A\}$ ($\{a \in A : Aa \subset M\} = \{\theta_A\}$,* respectively). *Then $M \cap Z(A) = \{\theta_A\}$.*

Proof. Let $m \in M \cap Z(A)$. Then $mA = Am \subset M$. Hence $M \cap Z(A) = \{\theta_A\}$.

2.2. Properties of the unitization of topological algebras

To describe all closed maximal regular ideals of a topological algebra A in case when A has no unit, we need the following results.

Proposition 2.6. *The unitization $A\times\mathbb{K}$ of A in the product topology is an exponentially galbed algebra if and only if A is an exponentially galbed algebra.*

Proof. Let A be an exponentially galbed algebra and O any neighbourhood of zero in $A\times\mathbb{K}$. Then there are a neighbourhood U_1 of zero in A and a neighbourhood V_1 of zero in $\mathbb{K}$ such that $U_1\times V_1\subset O$. Since A is an exponentially galbed algebra, then there exist a neighbourhood U_2 of zero in A and a number $\epsilon>0$ such that

$$\Big\{\sum_{k=0}^{n}\frac{a_k}{2^k}:a_0,\dots,a_n\in U_2\Big\}\subset U_1$$

for each $n\in\mathbb{N}$ and

$$V_2=\{\lambda\in\mathbb{K}\ :|\lambda|<\epsilon\}\subset V_1.$$

Let $V_3=\frac{1}{2}V_2$. Then

$$\sum_{k=0}^{n}\frac{a_k}{2^k}\in U_1$$

and

$$\Big|\sum_{k=0}^{n}\frac{\lambda_k}{2^k}\Big|\leqslant\sum_{k=0}^{n}\frac{|\lambda_k|}{2^k}\leqslant\max_{k\in\{0,\dots,n\}}|\lambda_k|\sum_{k=0}^{n}\frac{1}{2^k}<\frac{\epsilon}{2}2=\epsilon$$

for each $n\in\mathbb{N}$ and $(a_0,\lambda_0),\dots,(a_n,\lambda_n)\in U_2\times V_3$. Hence

$$\sum_{k=0}^{n}\frac{(a_k,\lambda_k)}{2^k}=\Big(\sum_{k=0}^{n}\frac{a_k}{2^k},\sum_{k=0}^{n}\frac{\lambda_k}{2^k}\Big)\in U_1\times V_2\subset U_1\times V_1.$$

It means that

$$\Big\{\sum_{k=0}^{n}\frac{(a_k,\lambda_k)}{2^k}:(a_0,\lambda_0),\dots,(a_n,\lambda_n)\in U_2\times V_3\Big\}\subset O$$

for each $n\in\mathbb{N}$. Therefore $A\times\mathbb{K}$ is exponentially galbed.

Let now $A\times\mathbb{K}$ be an exponentially galbed algebra, U any neighbourhood of zero in A and V a neighbourhood of zero in $\mathbb{K}$. Then $O=U\times V$ is a neighbourhood of zero in $A\times\mathbb{K}$ in the product topology. Since $A\times\mathbb{K}$ is exponentially galbed, then there exists a neighbourhood O_1 of zero in $A\times\mathbb{K}$ such that

$$\Big\{\sum_{k=0}^{n}\frac{(a_k,\lambda_k)}{2^k}:(a_0,\lambda_0),\dots,(a_n,\lambda_n)\in O_1\Big\}\subset O$$

for each $n \in \mathbb{N}$. We can find a neighbourhood U_1 of zero in A and a neighbourhood V_1 of zero in $\mathbb{K}$ such that $U_1 \times V_1 \subset O_1$. Then

$$\sum_{k=0}^{n} \frac{(a_k, \lambda_k)}{2^k} = \Big(\sum_{k=0}^{n} \frac{a_k}{2^k}, \sum_{k=0}^{n} \frac{\lambda_k}{2^k}\Big) \in U \times V$$

for each $n \in \mathbb{N}$, if $a_0, ..., a_n \in U_1$ and $\lambda_0, ..., \lambda_n \in V_1$. Therefore

$$\Big\{\sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, ..., a_n \in U_1\Big\} \subset U$$

for each $n \in \mathbb{N}$. Consequently, A is also an exponentially galbed algebra.

Proposition 2.7. *The unitization $A \times \mathbb{K}$ of A in the product topology is an exponentially galbed algebra with bounded elements if and only if A is an exponentially galbed algebra with bounded elements.*

Proof. Let A be an exponentially galbed algebra with bounded elements. Then $A \times \mathbb{K}$ is also an exponentially galbed algebra, by Proposition 2.6. Let (a_0, λ_0) be an arbitrary element in $A \times \mathbb{K}$ and W a neighbourhood of zero in $A \times \mathbb{K}$. Then there exist a number $\mu_W > 0$, a balanced neighbourhood O of zero in $A \times \mathbb{K}$, a neighbourhood U of zero in A and a balanced neighbourhood V of zero in $\mathbb{K}$ such that $e_{A\times\mathbb{K}} \subset \mu_W W$ and $U \times V \subset O \subset W$. Since A is exponentially galbed, then there exists a balanced neigbourhood U_1 of zero in A such that

$$\Big\{\sum_{k=0}^{n-1} \frac{b_k}{2^k} : b_0, ..., b_{n-1} \in U_1\Big\} \subset U$$

for each $n \in \mathbb{N}$. By assumption, A is a topological algebra with bounded elements. Then there are numbers $\lambda_{a_0} \in \mathbb{C}\backslash\{0\}$ and $\mu_1 \geqslant 1$ such that

$$\Big\{\Big(\frac{a_0}{\lambda_{a_0}}\Big)^n : n \in \mathbb{N}\Big\} \subset \mu_1 U_1.$$

Now there exists an integer $r_V \in \mathbb{N}$ such that $r_V \geqslant 1$ and

$$\Big\{\lambda : \; |\lambda| \leqslant \frac{1}{2^{r_V}}\Big\} \subset V.$$

Let $\mu_2 = \max\{|\lambda_{a_0}|, 2^{r_V}\, |\lambda_0|\}$ and $\mu_O = 4\mu_2$. Since

$$2^k \binom{n}{k} \Big(\frac{a_0}{\mu_O}\Big)^k \Big(\frac{\lambda_0}{\mu_O}\Big)^{n-k} = \frac{2^k \binom{n}{k}}{2^{2n}} \Big(\frac{\lambda_{a_0}}{\mu_2}\Big)^k \Big(\frac{\lambda_0}{\mu_2}\Big)^{n-k} \Big(\frac{a_0}{\lambda_{a_0}}\Big)^k$$

and

$$\left|\frac{2^k\binom{n}{k}}{2^{2n}}\right|, \left|\left(\frac{\lambda_{a_0}}{\mu_2}\right)^k\right|, \left|\left(\frac{\lambda_0}{\mu_2}\right)^{n-k}\right| \leqslant 1,$$

then

$$2^k\binom{n}{k}\left(\frac{a_0}{\mu_O}\right)^k\left(\frac{\lambda_0}{\mu_O}\right)^{n-k} \in \mu_1 U_1.$$

Therefore

$$\sum_{k=0}^{n-1}\binom{n}{k}\left(\frac{a_0}{\mu_O}\right)^k\left(\frac{\lambda_0}{\mu_O}\right)^{n-k} \in \mu_1 U$$

and

$$\left(\frac{\lambda_0}{\mu_O}\right)^n \in V = \mu_1\left(\frac{1}{\mu_1}V\right) \subset \mu_1 V$$

for each $n \in \mathbb{N}\backslash\{0\}$, since V is balanced. Hence

$$\left(\frac{(a_0,\lambda_0)}{\mu_O}\right)^n = \left(\sum_{k=0}^{n-1}\binom{n}{k}\left(\frac{a_0}{\mu_O}\right)^k\left(\frac{\lambda_0}{\mu_O}\right)^{n-k}, \left(\frac{\lambda_0}{\mu_O}\right)^n\right) \in \mu_1 O \subset \mu_1 W$$

for each $n \in \mathbb{N}\backslash\{0\}$.

We know that

$$\left(\frac{(a_0,\lambda_0)}{\mu_O}\right)^0 = e_{A\times\mathbb{K}} \subset \mu_W W.$$

Let $\mu = \max\{\mu_1, \mu_W\}$. Then

$$\left(\frac{(a_0,\lambda_0)}{\mu_O}\right)^n \subset \mu W$$

for each $n \in \mathbb{N}$ which means that all elements in $A \times \mathbb{K}$ are bounded.

Let now $A \times \mathbb{K}$ be an exponentially galbed algebra with bounded elements. Then A is an exponentially galbed algebra, by Proposition 2.6. Let U be any neighbourhood of zero in A. Let V_0 be a fixed neighbourhood of zero in $\mathbb{K}$ and a_0 an arbitrary element of A. Then $(a_0, 0) \in A \times \mathbb{K}$ and $U \times V_0$ is a neighbourhood of zero in $A \times \mathbb{K}$. Since all elements in $A \times \mathbb{K}$ are bounded, then there exist numbers $\lambda_0 \in \mathbb{C}\backslash\{0\}$ and $\mu_{U,V_0} > 0$ such that

$$\left\{\left(\frac{(a_0,0)}{\lambda_0}\right)^n : n \in \mathbb{N}\right\} \subset \mu_{U,V_0}(U \times V_0).$$

Hence

$$\left\{\left(\frac{a_0}{\lambda_0}\right)^n : n \in \mathbb{N}\right\} \subset \mu_{U,V_0} U$$

which means that A is an exponentially galbed algebra with bounded elements.

Proposition 2.8. *The unitization $A \times \mathbb{K}$ of A in the product topology is a locally pseudoconvex algebra if and only if A is a locally pseudoconvex algebra.*

Proof. Let A be a locally pseudoconvex algebra and O a neighbourhood of zero in $A \times \mathbb{K}$. Then we can find a neighbourhood U_1 of zero in A and a balanced neighbourhood V_1 of zero in $\mathbb{K}$ such that $U_1 \times V_1 \subset O$. Since A is locally pseudoconvex, then there exists a balanced and pseudoconvex neighbourhood U_2 of zero in A such that $U_2 \subset U_1$. We can assume that

$$V_1 = \{\lambda \in \mathbb{K} : |\lambda| < \epsilon\}$$

for some $\epsilon > 0$. Let $O_1 = U_2 \times V_1$ and $(a, \alpha), (b, \beta) \in O_1$. Then

$$\mu(a, \alpha) \in \mu U_2 \times \mu V_1 \subset U_2 \times V_1 = O_1$$

for $| \mu | \leqslant 1$ which means that O_1 is balanced. Because U_2 is pseudoconvex, then there exists a constant $\nu > 0$ such that $U_2 + U_2 \subset \nu U_2$. Let $\nu_1 = \max\{\nu, 2\}$. Then

$$(a, \alpha) + (b, \beta) \in (\nu U_2) \times (2V_1) \subset (\nu_1 U_2) \times (\nu_1 V_1) = \nu_1 O_1.$$

Therefore O_1 is also pseudoconvex. Since $O_1 \subset O$, then $A \times \mathbb{K}$ is a locally pseudoconvex algebra.

Let now $A \times \mathbb{K}$ be a locally pseudoconvex algebra, U a neighbourhood of zero in A and V a neighbourhood of zero in $\mathbb{K}$. Then $U \times V$ is a neighbourhood of zero in $A \times \mathbb{K}$ in the product topology. Because $A \times \mathbb{K}$ is locally pseudoconvex, then we can find a balanced and pseudoconvex neighbourhood O of zero in $A \times \mathbb{K}$ such that $O \subset U \times V$. Now

$$O \cap (A \times \{0\}) = U_1 \times \{0\},$$

where U_1 is a subset of U. Since

$$(\lambda U_1) \times \{0\} = \lambda(O \cap (A \times \{0\})) \subset O \cap (A \times \{0\}) = U_1 \times \{0\}$$

for $|\lambda| \leqslant 1$ and

$$(U_1 + U_1) \times \{0\} = O \cap (A \times \{0\}) + O \cap (A \times \{0\}) \subset (O + O) \cap (A \times \{0\}) \subset$$

$$\subset (\nu O) \cap (A \times \{0\}) \subset \nu(O \cap (A \times \{0\})) = (\nu U_1) \times \{0\}$$

for some $\nu > 0$, then $U_1 + U_1 \subset \nu U_1$. Since $U_1 \subset U$, then A is locally pseudoconvex.

Proposition 2.9. *The unitization $A \times \mathbb{K}$ of A in the product topology is a locally A-pseudoconvex algebra if and only if A is a locally A-pseudoconvex algebra.*

Proof. Let A be a locally A-pseudoconvex algebra and O a neighbourhood of zero in $A \times \mathbb{K}$. Then (as in the previous proof) we can find a balanced and pseudoconvex neighbourhood U_2 of zero in A and $\epsilon > 0$ such that $U_2 \times V_1 \subset O$, where $V_1 = \{\lambda \in \mathbb{K} : |\lambda| < \epsilon\}$. Next, there exists a balanced and pseudoconvex neighbourhood U_3 of zero in A such that $U_3 \subset U_2$ and for every $a \in A$ there exists a number $\nu = \nu(a, U_3) \geqslant 1$ such that $aU_3, U_3a \subset \nu U_3$. It is easy to see that

$$O_1 = U_3 \times V_1 \subset U_2 \times V_1 \subset O$$

is a pseudoconvex and balanced set, since U_3 and V_1 are pseudoconvex and balanced. Therefore $O_1 + O_1 \subset pO_1$ for a $p > 0$ and $rO_1 \subset O_1$ for every $|r| \leqslant 1$. We have to show that for every fixed $(a, \mu) \in A \times \mathbb{K}$ there exists a number $\nu_1 = \nu_1(a, \mu, O_1) > 0$ such that $(a, \mu)O_1, O_1(a, \mu) \subset \nu_1 O_1$. For it, let (b, λ) be an arbitrary element in O_1. Then $b \in U_3$ and $|\lambda| < \epsilon$. If (a, μ) is fixed, then there exists a number $\gamma_a > 0$ such that $a \subset \gamma_a U_3$. Let $\rho = \max\ \{|\mu|, \epsilon\gamma_a\}$ and $\nu_1 = p \max\ \{\nu, p\rho\}$ (here $\nu = \nu(a, U_3)$). Then

$$(a, \mu)(b, \lambda) = (ab, 0) + \mu(b, \lambda) + \lambda(a, 0) \in (aU_3) \times V_1 + \mu O_1 + \epsilon\gamma_a O_1 \subset$$

$$\subset \nu(U_3 \times V_1) + \rho(O_1 + O_1) \subset \nu O_1 + p\rho O_1 \subset \max\{\nu, p\rho\}(O_1 + O_1) \subset \nu_1 O_1$$

for each $(b, \lambda) \in O_1$. Hence $(a, \mu)O_1 \subset \nu_1 O_1$. Similarily, we have $O_1(a, \mu) \subset \nu_1 O_1$. Therefore $A \times \mathbb{K}$ is locally A-pseudoconvex.

Let now $A \times \mathbb{K}$ be a locally A-pseudoconvex algebra, U a neighbourhood of zero in A and V a neighbourhood of zero in $\mathbb{K}$. Then $O = U \times V$ is a neighbourhood of zero in $A \times \mathbb{K}$ and there exists an A-pseudoconvex neighbourhood O_1 of zero in $A \times \mathbb{K}$ such that $O_1 \subset O$. Now, for every fixed $(a, \mu) \in A \times \mathbb{K}$ there exists a constant $\nu_1 = \nu_1(a, \mu, O_1) > 0$ such that $(a, \mu)O_1,\ O_1(a, \mu) \subset \nu_1 O_1$. Then

$$O_2 = U_1 \times \{0\} = O_1 \cap (A \times \{0\})$$

is (as in the proof of Proposition 2.8) a pseudoconvex neighbourhood of zero in $A \times \{0\}$, for which holds the claim: for every fixed $a \in A$ there exists a constant $\nu_2 = \nu_1(a, 0, O_1) > 0$ such that $(a, 0)O_2,\ O_2(a, 0) \subset \nu_2 O_2$. It is easy to see that $U_1 \subset U$ is an A-pseudoconvex neighbourhood of zero in A. Therefore A is a locally A-pseudoconvex algebra.

Proposition 2.10. *The unitization $A \times \mathbb{K}$ of A in the product topology is a Fréchet algebra if and only if A is a Fréchet algebra.*

Proof. Let A be a Fréchet algebra. Then there exists a countable base $\{B_n : n \in \mathbb{N}\}$ of neighbourhoods of zero in A. For each $m \in \mathbb{N}$, let

$$C_m = \{\lambda \in \mathbb{K} : \ |\lambda| \leqslant \frac{1}{m}\}.$$

Then $\{B_n \times C_m : n, m \in \mathbb{N}\}$ is a countable base of neighbourhoods of zero in $A \times \mathbb{K}$. Hence $A \times \mathbb{K}$ is metrizable in the product topology. Let $((a_n, \lambda_n))$ be a Cauchy sequence in $A \times \mathbb{K}$. Then (a_n) is a Cauchy sequence in A and (λ_n) is a Cauchy sequence in $\mathbb{K}$. Since both A and $\mathbb{K}$ are complete, then there are $a_0 \in A$ and $\lambda_0 \in \mathbb{K}$ such that (a_n) converges to a_0 and (λ_n) converges to λ_0. It is easy to see that then $((a_n, \lambda_n))$ converges to $(a_0, \lambda_0) \in A \times \mathbb{K}$. Thus $A \times \mathbb{K}$ is a Fréchet algebra.

Let now $A \times \mathbb{K}$ be a Fréchet algebra. Then there exists a countable base $\{B_n : n \in \mathbb{N}\}$ of neighbourhoods of zero in $A \times \mathbb{K}$. It is clear that $\{B_n \cap (A \times \{0\}) : n \in \mathbb{N}\}$ is a countable base of neighbourhoods of zero in $A \times \{0\}$ in the subspace topology. Since A and $A \times \{0\}$ are homomorphic, then A has also a countable base of neighbourhoods of zero. Hence A is metrizable. If (a_n) is a Cauchy sequence in A, then $((a_n, 0))$ is a Cauchy sequence in $A \times \mathbb{K}$. As $A \times \mathbb{K}$ is a Fréchet algebra, then there exists an element $(a_0, \lambda_0) \in A \times \mathbb{K}$ such that $((a_n, 0))$ converges to (a_0, λ_0). Since $a_0 \in A$ and (a_n) converges to a_0, then A is a Fréchet algebra.

Lemma 2.11. *Let A be a topological algebra without a unit. Then the set $A \times \{0\}$ is a closed two-sided ideal of $A \times \mathbb{K}$, which is maximal as left (right or two-sided) ideal.*

Proof. It is easy to see that $A \times \{0\}$ is a closed two-sided ideal of $A \times \mathbb{K}$. Let M be a left (right or two-sided, respectively) ideal of $A \times \mathbb{K}$ such that $A \times \{0\} \subset M$. If $A \times \{0\} \neq M$, then $(a, \lambda) \in M$ for some $\lambda \neq 0$. Let (b, μ) be an arbitrary element of $A \times \mathbb{K}$. Since

$$(\theta_A, \lambda) = (a, \lambda) + (-a, 0) \in M,$$

then also

$$(b, \mu) = \frac{\mu}{\lambda}(\theta_A, \lambda) + (b, 0) \in M.$$

Therefore $M = A \times \mathbb{K}$, which is not possible. Hence $A \times \{0\}$ is a maximal left (right or two-sided, respectively) ideal of $A \times \mathbb{K}$.

The following Theorem has been proved in [38] (p. 158–160) and [21] (p. 17–18) for algebras which are not neccessarily topological. We present it here in a form more convenient for us. For the sake of completeness, we give our detailed version of the proof.

Theorem 2.12. *Let A be a topological algebra without a unit, $J(A)$ the set of all closed regular left (right) ideals of $A \times \{0\}$, $J(A \times \mathbb{K})$ the set of all closed left (right) ideals J of $A \times \mathbb{K}$, for which $J \not\subset (A \times \{0\})$. Let f be a map defined by $f(J) = J \cap (A \times \{0\})$ for each $J \in J(A \times \mathbb{K})$. Then f maps $J(A \times \mathbb{K})$ onto $J(A)$. Moreover, f is an one-to-one map between:*

a) *the set of all closed two-sided ideals I' of $A \times \mathbb{K}$ such that $I' \not\subset (A \times \{0\})$ and the set of all closed regular two-sided ideals of $A \times \{0\}$;*

b) *the set of all closed maximal left (right) ideals M' of $A \times \mathbb{K}$ such that $M' \not\subset (A \times \{0\})$ and the set of all closed maximal regular left (right) ideals of $A \times \{0\}$.*

Remark. The proof will be given only for left ideals; for right ideals the proof is similar.

Proof. Let $I' \in J(A \times \mathbb{K})$ and $I = I' \cap (A \times \{0\})$. Then there is an element $(a, k) \in I'$ such that $a \in A$ and $k \in \mathbb{K} \backslash \{0\}$. Since $A \times \{0\} \not\subset I'$, by Lemma 2.11, then $I \neq A \times \{0\}$. Hence I is a closed left ideal of $A \times \{0\}$. Let now $u = -\frac{1}{k}a$, $u_1 = (u, -1)$, $e = (\theta_A, 1)$ and $a' = (a, 0)$ for each $a \in A$. Then from

$$u_1 = u' - e = (\theta_A, -\frac{1}{k})(a, k) \in I'$$

we get

$$x'u' - x' = (xu - x, 0) = x'u_1 \in I$$

for each $x' \in A \times \{0\}$. Therefore $I = f(I')$ is a closed regular left ideal of $A \times \{0\}$ and u' is a right unit in $A \times \{0\}$ for I.

Let now I be a closed regular left ideal of $A \times \{0\}$. Then there exists an element $u' = (u, 0) \in (A \times \{0\}) \backslash I$ such that $x'u' - x' \in I$ for each $x' \in A \times \{0\}$. Let $y'_k = (y, k)$ for each $y \in A$ and $k \in \mathbb{K}$ and let

$$I' = \{y'_k \in A \times \mathbb{K} : \ y'_k u' \in I\}.$$

Then $I' \neq A \times \mathbb{K}$, because $(\theta_A, 1) \notin I'$. Moreover, I' is a closed subset of $A \times \mathbb{K}$. Indeed, if (y_0, k_0) is an element of the closure of I' in the product topology of $A \times \mathbb{K}$, then there exists a net $((a_\lambda, k_\lambda))_{\lambda \in \Lambda}$ in I' such that $((a_\lambda, k_\lambda))_{\lambda \in \Lambda}$ converges to (y_0, k_0) in the product topology. Since $(a_\lambda, k_\lambda)u' \in I$ for every $\lambda \in \Lambda$, then $(y_0, k_0)u' \in I$, because I is closed in $A \times \{0\}$. Hence $(y_0, k_0) \in I'$ and I' is closed. We will show now that $I' \in I(A \times \mathbb{K})$. For it, let $y'_k, z'_l \in I'$ and $\lambda \in \mathbb{K}$. Since

$$(y'_k + z'_l)u' = y'_k u' + z'_l u' \in I,$$

we have $y'_k + z'_l \in I'$. Similarily, $\lambda y'_k \in I'$ for each $\lambda \in \mathbb{K}$.

Let now w'_m be an arbitrary element of $A \times \mathbb{K}$ and $y'_k \in I'$. As

$$(w'_m y'_k)u' = (wyu + kwu, 0) + (myu + mku, 0) =$$

$$= (w,0)(yu + ku, 0) + m(yu + ku, 0) = w'(y'_k u') + m(y'_k u') \in I,$$

because I is a left ideal of $A \times \{0\}$, then $w'_m y'_k \in I'$. Thus I' is a closed left ideal of $A \times \mathbb{K}$ and $I' \not\subset (A \times \{0\})$, because $(u' - e)u' = u'u' - u' \in I$ implies $u' - e = (u, -1) \in I' \backslash (A \times \{0\})$.

To show that $I = I' \cap (A \times \{0\})$, let $i' \in I$. Then $i'u' = (i'u' - i') + i' \in I$. Hence $I \subset I'$ and thus $I \subset I' \cap (A \times \{0\})$. If $z \in I' \cap (A \times \{0\})$, then $z = a'$ for some $a \in A$. Therefore $z = a'u' - (a'u' - a') \in I$. Hence $I = I' \cap (A \times \{0\})$. Consequently, f is a surjection.

a) Let $I', J' \in A \times \mathbb{K}$ be closed two-sided ideals of $A \times \mathbb{K}$ such that $I', J' \not\subset (A \times \{0\})$ and

$$f(I') = I' \cap (A \times \{0\}) = J' \cap (A \times \{0\}) = f(J').$$

Then (as shown above) there are elements $u' = (u, 0), v' = (v, 0) \in A \times \{0\}$ such that $u' - e \in I', v' - e \in J'$. First, we will show that $I' \subset J'$. It is clear that

$$v'u' - v' = (vu - v, 0) = v'(u' - e) \in f(I') = f(J') \subset J'$$

and $v'u' - u' = (v' - e)u' \in J'$. Therefore

$$v' - u' = (v'u' - u') - (v'u' - v') \in J'.$$

Now, let (y, λ) be an arbitrary element of I'. Since

$$(uy + \lambda u, 0) = u'(y, \lambda) \in f(I')$$

and

$$(uy - y, 0) = (u' - e)y' \in f(I'),$$

then

$$(y + \lambda u, 0) = (uy + \lambda u, 0) - (uy - y, 0) \in f(I') = f(J') \subset J'.$$

Hence

$$(y, \lambda) = (y + \lambda u, 0) + \lambda(v' - u') - \lambda(v' - e) \in J'.$$

It means that $I' \subset J'$. Analogously, we can show that $J' \subset I'$. Therefore $I' = J'$ and f is one-to-one.

b) First, we will show that any closed maximal regular left ideal M of $A \times \{0\}$ defines a closed maximal left ideal J' in $A \times \mathbb{K}$ such that $J' \not\subset A \times \{0\}$ and $f(J') = M$. Then we show that any closed maximal left ideal J'' such that $J'' \not\subset A \times \{0\}$ and $f(J'') = M$, coincides with J'. For it, let M be a closed maximal regular left ideal in $A \times \{0\}$ and u', v' be elements of $A \times \{0\}$ such that $x'u' - x', x'v' - x' \in M$ for every $x' \in A \times \{0\}$. Then $v' - u' \in M$. To show it, let

$$W = \{w' \in A \times \{0\} : \ x'w' \in M \text{ for each } x' \in A \times \{0\}\}.$$

Now $M \subset W$. Suppose that $u' \in W$. Then $x'u' \in M$ for each $x' \in A \times \{0\}$. Therefore

$$x' = x'u' - (x'u' - x') \in M$$

for each $x' \in A \times \{0\}$, which contradicts our assumption that M is an ideal in $A \times \{0\}$. Hence $u' \notin W$. Consequently, $W \neq A \times \{0\}$.

If $w_1', w_2' \in W$ and $\lambda \in \mathbb{K}$, then $w_1' + w_2', \lambda w_1' \in W$. Moreover, if $w' \in W$ and r' is an arbitrary element of $A \times \{0\}$, then from

$$x'(r'w') = (x'r')w' \in M$$

follows $r'w' \in W$. Herewith, $x'u' - x' \in M \subset W$ for each $x' \in A \times \{0\}$. Hence W is a regular left ideal of $A \times \{0\}$. Since M is maximal as a left ideal in $A \times \{0\}$, then $W = M$. Now

$$x'(v' - u') = (x'v' - x') - (x'u' - x') \in M$$

for each $x \in A$. Therefore from $v' - u' \in W$ follows $v' - u' \in M$.

Let now

$$J' = \{y_k' \in A \times \mathbb{K} : y_k'u' \in M\}.$$

Then J' is a left ideal of $A \times \mathbb{K}$ (as shown above). Since $v' - u' \in M$, then $x'(v' - u') \in M$ for each $x' \in A \times \{0\}$. Hence $x'u' \in M$ if and only if $x'v' \in M$. It means that J' is independent of the selection of u' such that $x'u' - x' \in M$ for each $x' \in A \times \{0\}$.

Next, we show that J' is a closed maximal left ideal of $A \times \mathbb{K}$, for which $J' \cap (A \times \{0\}) = M$ and $J' \not\subset (A \times \{0\})$. For it, let m' be an arbitrary element of M. Then $m'u' - m' \in M$ and from

$$m'u' = (m'u' - m') + m' \in M$$

follows $m' \in J'$. Therefore $M \subset J' \cap (A \times \{0\})$. If now $z' \in J' \cap (A \times \{0\})$, then $z'u' \in M$ and $z'u' - z' \in M$. Thus

$$z' = z'u' - (z'u' - z') \in M.$$

So we have showed that $M = J' \cap (A \times \{0\})$. Since

$$(u, -1)u' = (uu - u, 0) = u'u' - u' \in M,$$

then $(u, -1) \in J' \backslash (A \times \{0\})$. Hence $J' \not\subset (A \times \{0\})$.

To show that J' is maximal, let M' be a left ideal of $A \times \mathbb{K}$ such that $J' \subset M'$. Then

$$M = J' \cap (A \times \{0\}) \subset M' \cap (A \times \{0\}).$$

Since M is a maximal left ideal of $A \times \{0\}$, then

$$M' \cap (A \times \{0\}) = M \tag{1}$$

or

$$M' \cap (A \times \{0\}) = A \times \{0\}. \tag{2}$$

In case (1), let (y, λ) be any element of M'. If $\lambda = 0$, then

$$(y, \lambda) \in M' \cap (A \times \{0\}) = M \subset J'.$$

Let $\lambda \neq 0$. Then

$$(y, \lambda) = -\lambda(g_1, -1) = -\lambda(g_1' - e),$$

where $g_1 = -\frac{1}{\lambda}\, y \in A$ and $g_1' - e = -\frac{1}{\lambda}(y, \lambda) \in M'$. Thus

$$z' g_1' - z' = z'(g_1' - e) \in M' \cap (A \times \{0\}) = M$$

for each $z' \in A \times \{0\}$. Therefore $w_1' = g_1' - u' \in M$, as shown above. Hence

$$(y, \lambda)u' = -\lambda(g_1 u - u, 0) = -\lambda[(w_1' u' - w_1') + w_1' + (u'u' - u')] \in M.$$

Therefore $(y, \lambda) \in J'$ and thus $M' \subset J'$, which gives us $M' = J'$.

The case (2) implies $M' = A \times \{0\}$, by Lemma 2.11, which is not possible, because $(u, -1) \in J' \backslash (A \times \{0\})$. Consequently, J' is a maximal left ideal of $A \times \mathbb{K}$ and $J' \not\subset A \times \{0\}$. Using exactly the same techniques as we used in the beginning of this proof for showing that I' is a closed ideal, it is easy to show that J' is closed.

Let now J'' be another closed maximal left ideal of $A \times \mathbb{K}$ such that $J'' \not\subset (A \times \{0\})$ and $f(J'') = M$. From $J'' \cap (A \times \{0\}) = M$ we get (as in case (1)) that $J'' \subset J'$. Since J'' is a maximal left ideal, then we have $J'' = J'$. Therefore f is one-to-one.

Finally, we show that $M' \cap (A \times \{0\})$ is a closed maximal regular left ideal in $A \times \{0\}$ for any such closed maximal left ideal M' of $A \times \mathbb{K}$, for which $M' \not\subset A \times \{0\}$. For it, let M' be an arbitrary closed maximal left ideal of $A \times \mathbb{K}$ such that $M' \not\subset (A \times \{0\})$ and $M' \cap (A \times \{0\}) = I$. Then I is a closed regular left ideal in $A \times \{0\}$, by the first part of the proof. Let M be a maximal regular left ideal of $A \times \{0\}$ such that $I \subset M$. According to the last part of the proof, M defines a maximal left ideal J' of $A \times \mathbb{K}$ such that $J' \not\subset A \times \{0\}$ and $J' \cap (A \times \{0\}) = M$. Since $I \subset M$, then we can show similarily, as in case (1), that $M' \subset J'$. As M' is a maximal ideal in $A \times \mathbb{K}$, then it implies $M' = J'$. Therefore $M = I$, which completes the proof.

Remark. It is easy to see that this theorem holds, if we drop the word "closed" everywhere in Theorem 2.12.

Corollary 2.13. *Let A be a topological algebra without a unit, S_1 the set of all closed maximal two-sided ideals M' of $A \times \mathbb{K}$ such that $M' \not\subset (A \times \{0\})$ and S_2 the set of all closed maximal regular two-sided ideals M of $A \times \{0\}$. Then the map $f : S_1 \to S_2$, defined by $f(J) = J \cap (A \times \{0\})$ for each $J \in S_1$, is a bijection.*

Proof. Let M' be a closed maximal two-sided ideal of $A \times \mathbb{K}$ such that $M' \not\subset (A \times \{0\})$ and $M = M' \cap (A \times \{0\})$. Then M is a closed regular two-sided ideal of $A \times \{0\}$, by Theorem 2.12. Let L be a maximal two-sided ideal of $A \times \{0\}$ such that $L \supset M$. Similarily, as in the proof of Theorem 2.12, there exists a two-sided ideal L' in $A \times \mathbb{K}$ such that $L' \not\subset (A \times \{0\})$ and $L' \cap (A \times \{0\}) = L$. But now, as in case (1) in the proof of Theorem 2.12, we can show that $L \supset M$ implies $L' \supset M'$. Since M' is a maximal two-sided ideal of $A \times \mathbb{K}$, then $L' = A \times \mathbb{K}$ or $L' = M'$. The first case implies $L = A \times \{0\}$, which is a contradiction. The second case implies $L = M$. Thus M is a closed maximal regular two-sided ideal of $A \times \{0\}$.

Let now M be a closed maximal regular two-sided ideal of $A \times \{0\}$. Similarily, as in Theorem 2.12, there exists a closed two-sided ideal M' of $A \times \mathbb{K}$ such that $M' \not\subset (A \times \{0\})$ and $M = M' \cap (A \times \{0\})$. Suppose that there exists a two-sided ideal L' of $A \times \mathbb{K}$ such that $L' \not\subset (A \times \{0\})$ and $L' \supset M'$. Then

$$L = L' \cap (A \times \{0\}) \supset M' \cap (A \times \{0\}) = M$$

implies $L = A \times \{0\}$ or $L = M$. Since $L' \not\subset (A \times \{0\})$, then the first case implies $L' = A \times \mathbb{K}$, which is a contradiction. The second case implies $L' = M'$. Thus M' is a closed maximal two-sided ideal of $A \times \mathbb{K}$.

So we have showed that f maps S_1 onto S_2. By the part a) of Theorem 2.12, the map f is one-to-one. Therefore f is a bijection between the sets S_1 and S_2.

2.3. Properties of unital exponentially galbed algebras with bounded elements

Next, we study certain analytical properties of unital exponentially galbed algebras with bounded elements.

Proposition 2.14. *Let A be a unital exponentially galbed Hausdorff algebra over $\mathbb{C}$ with bounded elements. Let $\lambda_0 \in \mathbb{C}$ and $a_0 \in A$. Then there exists a neighbourhood $O(\lambda_0)$ of λ_0 such that*

$$\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k a_0^k$$

converges in A and

$$(e_A+(\lambda-\lambda_0)a_0)^{-1}=\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k a_0^k$$

for each $\lambda \in O(\lambda_0)$.

Proof. Let $a_0 \in A$ and $\lambda_0 \in \mathbb{C}$ be fixed. Moreover, let A be an exponentially galbed Hausdorff algebra over $\mathbb{C}$ with bounded elements and O an arbitrary neighbourhood of zero in A. There exist a closed and balanced neighbourhood O' of zero in A and a closed neighbourhood O'' of zero in $\mathbb{C}$ such that $O''O' \subset O$. Now we can find a balanced neighbourhood V of zero in A such that

$$\left\{\sum_{k=0}^{n}\frac{v_k}{2^k} \;:\; v_1,\dots,v_n \in V\right\} \subset O'$$

for all $n \in \mathbb{N}$. Since every element in A is bounded, then there is a number $\mu_0 \in \mathbb{C}\backslash\{0\}$ such that

$$\left\{\left(\frac{a_0}{\mu_0}\right)^n : n \in \mathbb{N}\right\}$$

is bounded in A. Therefore there exists a number $\rho_0 > 1$ such that

$$\left(\frac{a_0}{\mu_0}\right)^n \in \rho_0 V$$

for all $n \in \mathbb{N}$.

Let

$$U_{\mathbb{C}}=\left\{\lambda \in \mathbb{C}:\ |\lambda|<\frac{1}{2|\mu_0|}\right\},$$

$U(\lambda_0) = \lambda_0 + U_{\mathbb{C}}$ and

$$S_n(\lambda) = \sum_{k=1}^{n} (\lambda - \lambda_0)^k a_0^k$$

for each $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then

$$S_m(\lambda) - S_n(\lambda) = \sum_{k=n+1}^{m} (\lambda - \lambda_0)^k a_0^k = \sum_{k=0}^{m-n-1} (\lambda - \lambda_0)^{n+k+1} a_0^{n+k+1}$$

for each $\lambda \in \mathbb{C}$ and $m, n \in \mathbb{N}$ with $m > n$. Now $|2\mu_0(\lambda - \lambda_0)| < 1$ for each $\lambda \in U(\lambda_0)$. If we take

$$v_{n,k}(\lambda) = \frac{2^k (\lambda - \lambda_0)^k {a_0}^{n+k+1}}{\rho_0 {\mu_0}^{n+1}},$$

then

$$S_m(\lambda) - S_n(\lambda) = (\lambda - \lambda_0)^{n+1} {\mu_0}^{n+1} \rho_0 \sum_{k=0}^{m-n-1} \frac{v_{n,k}(\lambda)}{2^k}$$

for each $n, m \in \mathbb{N}$ with $m > n$ and $\lambda \in U(\lambda_0)$. Moreover,

$$v_{n,k}(\lambda) = \frac{1}{\rho_0}(2(\lambda - \lambda_0)\mu_0)^k \left(\frac{a_0}{\mu_0}\right)^{n+k+1} \in \frac{1}{\rho_0}(2\mu_0(\lambda - \lambda_0))^k \rho_0 V \subset V$$

for each $n, k \in \mathbb{N}$ and $\lambda \in U(\lambda_0)$, because $|2\mu_0(\lambda - \lambda_0)| < 1$. Hence

$$S_m(\lambda) - S_n(\lambda) \in \frac{(2\mu_0(\lambda - \lambda_0))^{n+1}}{2^{n+1}} \rho_0 O',$$

if $m > n$ and $\lambda \in U(\lambda_0)$. Since again $|2\mu_0(\lambda - \lambda_0)| < 1$, then there exists a number $n_0 \in \mathbb{N}$ such that

$$(2\mu_0(\lambda - \lambda_0))^{n+1} \in \frac{1}{\rho_0} O''$$

if $n > n_0$. Therefore

$$S_m(\lambda) - S_n(\lambda) \in \frac{1}{2^{n+1}} \frac{1}{\rho_0} O'' \rho_0 O' \subset O'' O' \subset O$$

for each $m > n > n_0$ and $\lambda \in U(\lambda_0)$, since O' is balanced. Hence $(S_n(\lambda))$ is a Cauchy sequence in A.

In case A is sequentially complete, the sequence $(S_n(\lambda))$ converges in A. If A is not sequentially complete, then let $\tilde{A}$ be the sequential completion of A and $\nu : A \to \tilde{A}$ the topological space isomorphism from A into $\tilde{A}$ such that $\nu(A)$ is dense in the Hausdorff linear space $\tilde{A}$. Then $\nu(A)$ is an exponentially galbed space, for each $a \in A$ there is a number $\lambda_a \in \mathbb{C}\backslash\{0\}$ such that

$$\Big\{\nu\Big[\Big(\frac{a}{\lambda_a}\Big)^n\Big] : n \in \mathbb{N}\Big\}$$

is bounded in $\nu(A)$, and $\nu(S_n(\lambda))$ is a Cauchy sequence in $\tilde{A}$. Since $\tilde{A}$ is complete, then there exists an element $b \in \tilde{A}$ such that $\nu(S_n(\lambda))$ converges to b for every $\lambda \in U(\lambda_0)$.

Next, we show that there exists a neighbourhood $O'(\lambda_0)$ of λ_0 such that

$$b = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k \nu(a_0^k) \in \nu(A)$$

for each $\lambda \in O'(\lambda_0)$. For it, let O_1 be an arbitrary closed neighbourhood of zero in $\nu(A)$. Then $e_{\tilde{A}} + O_1 = \nu(e_A) + O_1$ is closed neighbourhood of $e_{\tilde{A}}$ in $\nu(A)$. Since $\nu(A)$ is exponentially galbed, then there exists a balanced neighbourhood V_1 of zero in $\nu(A)$ such that

$$\Big\{\sum_{k=0}^{n}\frac{v_k}{2^k} \ : \ v_1, \dots, v_n \in V_1\Big\} \subset O_1$$

for all $n \in \mathbb{N}$ and a number $\mu_1 \in \mathbb{C}\backslash\{0\}$ such that

$$\Big\{\nu\Big[\Big(\frac{a_0}{\mu_1}\Big)^n\Big] : n \in \mathbb{N}\Big\}$$

is bounded in $\nu(A)$. Therefore there is a number $\rho_1 > 0$, for which

$$\nu\Big[\Big(\frac{a_0}{\mu_1}\Big)^n\Big] \in \rho_1 V_1$$

for all $n \in \mathbb{N}$. Let

$$O_{\mathbb{C}} = \Big\{\lambda \in \mathbb{C} : \ |\lambda| < \min\Big\{\frac{1}{2|\mu_1|}, \frac{1}{2|\mu_1|\rho_1}\Big\}\Big\}$$

and $O'(\lambda_0) = \lambda_0 + O_{\mathbb{C}}$. Then

$$v_k(\lambda) = 2^k(\lambda - \lambda_0)^k \nu({a_0}^k) = (2(\lambda - \lambda_0)\mu_1)^k \nu\Big[\Big(\frac{a_0}{\mu_1}\Big)^k\Big] \in$$

$$\in (2(\lambda - \lambda_0)\mu_1)^k \rho_1 V_1 = (2(\lambda - \lambda_0)\mu_1)^{k-1}(2(\lambda - \lambda_0)\mu_1\rho_1)V_1 \subset V_1,$$

if $k \geqslant 1$ and $\lambda \in O'(\lambda_0)$, since V is balanced. Let now $v_0 = \nu(\theta_A) = \theta_{\nu(A)}$. Then

$$R_n(\lambda) = \nu(S_n(\lambda)) = \sum_{k=0}^{n}(\lambda - \lambda_0)^k \nu({a_0}^k) = \nu(e_A) + \sum_{k=1}^{n} \frac{v_k(\lambda)}{2^k} =$$

$$= \nu(e_A) + \sum_{k=0}^{n} \frac{v_k(\lambda)}{2^k} \in \nu(e_A) + O_1$$

for each $n \in \mathbb{N}$ and $\lambda \in O'(\lambda_0)$. Since $\nu(e_A) + O_1$ is a closed set, then

$$b = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k \nu({a_0}^k) = \lim_{n\to\infty} R_n(\lambda) \in \nu(e_A) + O_1 \subset \nu(A)$$

for each $\lambda \in O'(\lambda_0)$. If now $\lambda \in O(\lambda_0) = O'(\lambda_0) \cap U(\lambda_0)$, then $(S_n(\lambda))$ converges in A. It is easy to see that

$$(e_A + (\lambda - \lambda_0)a_0)\Big(\sum_{k=0}^{\infty}(\lambda - \lambda_0)^k {a_0}^k\Big) =$$

$$= \Big(\sum_{k=0}^{\infty}(\lambda - \lambda_0)^k {a_0}^k\Big)(e_A + (\lambda - \lambda_0)a_0) = e_A$$

for each $\lambda \in O(\lambda_0)$. Hence

$$(e_A + (\lambda - \lambda_0)a_0)^{-1} = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k a_0^k \in A$$

for each $\lambda \in O(\lambda_0)$.

Corollary 2.15. *Let A be a unital exponentially galbed algebra over $\mathbb{C}$ with bounded elements. For every $a_0 \in A$ there exists a constant $R > 0$ such that*

$$\sum_{k=0}^{\infty} \frac{a_0^k}{\mu^{k+1}}$$

converges in A, whenever $|\mu| > R$.

Proof. If we take $\lambda_0 = 0$ in Proposition 2.14, then we get that

$$\sum_{k=0}^{\infty} \lambda^k a_0^k$$

converges in A for every λ with $|\lambda| < \delta$ for some number $\delta > 0$. If $\mu > R = \frac{1}{\delta}$, then $|\mu^{-1}| < \delta$ which means that

$$\sum_{k=0}^{\infty} \frac{a_0^k}{\mu^k}$$

converges in A. Hence

$$\sum_{k=0}^{\infty} \frac{a_0^k}{\mu^{k+1}} = \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{a_0^k}{\mu^k}$$

converges in A for every $\mu > R$.

2.4. Topological algebras, whose center is topologically isomorphic to $\mathbb{C}$

Next, we show that the centers of several topological algebras are topologically isomorphic to $\mathbb{C}$.

Theorem 2.16. *Let*[8] *A be a unital topologically primitive exponentially galbed Hausdorff algebra over $\mathbb{C}$ with bounded elements. Then $Z(A)$ is topologically isomorphic to $\mathbb{C}$.*

Proof. Let A be a unital topologically primitive exponentially galbed Hausdorff algebra over $\mathbb{C}$ with bounded elements. Let M be a closed maximal left ideal in algebra A (if M is a closed maximal right ideal, then the proof is similar), for which $\{a \in A : aA \subset M\} = \{\theta_A\}$, let π_M be a canonical homomorphism from A onto the quotient space A–M with respect to M and $z \in Z(A)\backslash\{\theta_A\} = Z(A)\backslash M$ (by Lemma 2.5). It is easy to see that $K_z = \{a \in A : az \in M\}$ is a left ideal of A for all $z \in Z(A)\backslash\{\theta_A\}$. Because $mz = zm \in M$ for each $m \in M$ and $e_A z = z \notin M$, then $M \subset K_z \neq A$. Since ideal M is maximal, then $M = K_z$ for all $z \in Z(A)\backslash\{\theta_A\}$.

We will show that every $z \in Z(A)$ defines a number $\lambda_z \in \mathbb{C}$ such that $z = \lambda_z e_A$. If $z = \theta_A$, then we take $\lambda_z = 0$. Suppose that there exists a $z \in Z(A)$ such that $z(\lambda) = \lambda e_A - z \neq \theta_A$ for all $\lambda \in \mathbb{C}$. Then $z(\lambda) \in Z(A)\backslash\{\theta_A\}$ means that $z(\lambda) \notin M$ for all $\lambda \in \mathbb{C}$. It is easy to see that $M + Az(\lambda)$ is a left ideal of A, $M \subset M + Az(\lambda)$ and

$$z(\lambda) = \theta_A + e_A z(\lambda) \in (M + Az(\lambda))\backslash M$$

[8]Similar results have been proved for Banach algebras in [42], Corollary 2.45 (see also [22], p. 127; [41], Theorem 1.2.11 and [23], Theorem 2.6.26(ii)); for k-Banach algebras in [21], Corollary 9.3.7; for locally m-convex Q-algebras in [45], Corollary 2; for locally A-convex algebras, in which all maximal ideals are closed, in [46], Corollary 2; for locally A-pseudoconvex algebras and for locally pseudoconvex Fréchet' algebras in [15], Theorem 3.1.

for all $\lambda \in \mathbb{C}$. Since M is a maximal left ideal in A, then $M + Az(\lambda) = A$ for all $\lambda \in \mathbb{C}$. Therefore, for every $\lambda \in \mathbb{C}$, there are elements $m(\lambda) \in M$ and $a(\lambda) \in A$ such that $e_A = m(\lambda) + a(\lambda)z(\lambda)$. Thus $a(\lambda)z(\lambda) - e_A \in M$. Let $a'(\lambda)$ be another element of A such that $a'(\lambda)z(\lambda) - e_A \in M$. Then from

$$[a(\lambda) - a'(\lambda)]z(\lambda) = a(\lambda)z(\lambda) - a'(\lambda)z(\lambda) \in M$$

follows $[a(\lambda) - a'(\lambda)] \in K_z(\lambda) = M$. Therefore $\pi_M(a(\lambda)) = \pi_M(a'(\lambda))$ for each $\lambda \in \mathbb{C}$.

Let now $\lambda_0 \in \mathbb{C}$ and $d(\lambda) = e_A + (\lambda - \lambda_0)a(\lambda_0)$ for each $\lambda \in \mathbb{C}$. By Proposition 2.14, there exists a neighbourhood $O(\lambda_0)$ of λ_0 such that

$$\sum_{k=0}^{\infty}(\lambda - \lambda_0)^k a(\lambda_0)^k$$

converges in A and

$$d(\lambda)\sum_{k=0}^{\infty}(\lambda - \lambda_0)^k a(\lambda_0)^k = \Big(\sum_{k=0}^{\infty}(\lambda - \lambda_0)^k a(\lambda_0)^k\Big)d(\lambda) = e_A$$

for each $\lambda \in O(\lambda_0)$. Thus, if $\lambda \in O(\lambda_0)$, then

$$d(\lambda)^{-1} = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k a(\lambda_0)^k.$$

Now

$$a(\lambda_0)d(\lambda)^{-1}z(\lambda) - e_A = a(\lambda_0)d(\lambda)^{-1}z(\lambda) - [a(\lambda_0)z(\lambda_0) + m(\lambda_0)] =$$

$$= a(\lambda_0)d(\lambda)^{-1}z(\lambda) - a(\lambda_0)d(\lambda)^{-1}d(\lambda)z(\lambda_0) - m(\lambda_0) =$$

$$= -a(\lambda_0)d(\lambda)^{-1}[-z(\lambda) + d(\lambda)z(\lambda_0)] - m(\lambda_0) =$$

$$= -a(\lambda_0)d(\lambda)^{-1}[(z - \lambda e_A) + (e_A + (\lambda - \lambda_0)a(\lambda_0))(\lambda_0 e_A - z)] - m(\lambda_0) =$$

$$= -a(\lambda_0)d(\lambda)^{-1}[(\lambda_0 - \lambda)(e_A - a(\lambda_0)z(\lambda_0))] - m(\lambda_0) =$$

$$= -a(\lambda_0)d(\lambda)^{-1}(\lambda_0 - \lambda)m(\lambda_0) - m(\lambda_0) \in M.$$

Therefore $\pi_M(a(\lambda)) = \pi_M(a(\lambda_0)d(\lambda)^{-1})$ for all $\lambda \in O(\lambda_0)$.

Let $\Psi(\lambda) = \pi_M(a(\lambda))$ for all $\lambda \in \mathbb{C}$. Next, we show that Ψ is an $(A\text{–}M)$-valued analytic function on $\mathbb{C} \cup \{\infty\}$. For it, let $\lambda_0 \in \mathbb{C}$. Then

$$\Psi(\lambda) = \pi_M(a(\lambda_0)d(\lambda)^{-1})$$

for each $\lambda \in O(\lambda_0)$. Now there exists a neighbourhood O_δ of zero in $\mathbb{C}$ such that $\lambda_0 + O_\delta \subset O(\lambda_0)$. Moreover, $h = \lambda - \lambda_0 \in O_\delta$ implies that $\lambda_0 + h \in O(\lambda_0)$ and

$$\Psi(\lambda_0 + h) = \pi_M(a(\lambda_0)d(\lambda_0 + h)^{-1}) = \pi_M\Big(a(\lambda_0)\sum_{k=0}^{\infty} h^k a(\lambda_0)^k\Big) =$$

$$= \pi_M\Big(\sum_{k=0}^{\infty} h^k a(\lambda_0)^{k+1}\Big) = \sum_{k=0}^{\infty} h^k \pi_M(a(\lambda_0)^{k+1}),$$

where $\pi_M(a(\lambda_0)^{k+1}) \in A–M$ for all $k \in \mathbb{N}$.

By Corollary 2.15, there exists a number $R > 0$ such that

$$\sum_{k=0}^{\infty} \frac{z^k}{\lambda^{k+1}}$$

converges in A for every λ with $|\lambda| > R$. Easy calculation shows that

$$z(\lambda)\sum_{k=0}^{\infty} \frac{z^k}{\lambda^{k+1}} = \Big(\sum_{k=0}^{\infty} \frac{z^k}{\lambda^{k+1}}\Big)z(\lambda) = e_A.$$

Therefore

$$z(\lambda)^{-1} = \sum_{k=0}^{\infty} \frac{z^k}{\lambda^{k+1}}$$

for every λ with $|\lambda| > R$. Since $z(\lambda)^{-1}z(\lambda) - e_A \in M$ for every λ with $|\lambda| > R$, then

$$\Psi(\lambda) = \pi_M(a(\lambda)) = \pi_M(z(\lambda)^{-1}) = \pi_M\Big(\sum_{k=0}^{\infty} \frac{z^k}{\lambda^{k+1}}\Big) = \sum_{k=0}^{\infty} \frac{\pi_M(z^k)}{\lambda^{k+1}}$$

for every λ with $|\lambda| > R$. Because $\pi_M(z^k) \in A-M$ for all $n \in \mathbb{N}$, then Ψ is an analytic $(A–M)$-valued function on $\mathbb{C} \cup \{\infty\}$. Hence $\Psi(\lambda)$ is a constant map, by Lemma 1.5. To show that $\Psi(\lambda) = \theta_{A-M}$ for each $\lambda \in \mathbb{C}$, let O be any neighbourhood of zero in A. Then there exist a closed neighbourhood O' and a neighbourhood V of zero in A such that $O' \subset O$ and

$$\Big\{\sum_{k=0}^{n} \frac{v_k}{2^k} : v_0, \dots, v_n \in V\Big\} \subset O'$$

for all $n \in \mathbb{N}$. Moreover, there are $\mu_z \in \mathbb{C}\backslash\{0\}$ and $\rho_V > 1$ such that

$$\Big(\frac{z}{\mu_z}\Big)^k \in \rho_V V$$

for all $k \in \mathbb{N}$. If now $|\lambda| > \max\{2|\mu_z|, \rho_V\}$ and

$$v_k(\lambda) = \frac{2^k z^k}{\lambda^{k+1}},$$

then

$$v_k(\lambda) = \frac{1}{\rho_V}\frac{\rho_V}{\lambda}\left(\frac{2\mu_z}{\lambda}\right)^k\left(\frac{z}{\mu_z}\right)^k \in \frac{1}{\rho_V}\left[\frac{\rho_V}{\lambda}\left(\frac{2\mu_z}{\lambda}\right)^k\right]\rho_V V \subset V$$

for each $k \in \mathbb{N}$. Therefore

$$\sum_{k=0}^{n}\frac{z^k}{\lambda^{k+1}} = \sum_{k=0}^{n}\frac{v_k(\lambda)}{2^k} \subset O'$$

for every $n \in \mathbb{N}$. Since O' is closed, then

$$z(\lambda)^{-1} = \sum_{k=0}^{\infty}\frac{z^k}{\lambda^{k+1}} = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{v_k(\lambda)}{2^k} \in O' \subset O,$$

if $|\lambda| > \max\{2|\mu_z|, \rho_V, R\}$. Hence

$$\lim_{|\lambda|\to\infty} z(\lambda)^{-1} = \theta_A$$

and

$$\lim_{|\lambda|\to\infty}\Psi(\lambda) = \lim_{|\lambda|\to\infty}\pi_M(z(\lambda)^{-1}) = \pi_M(\lim_{|\lambda|\to\infty} z(\lambda)^{-1}) = \pi_M(\theta_A) = \theta_{A-M}.$$

Thus $\Psi(\lambda) = \theta_{A-M}$ or $a(\lambda) \in M$ for all $\lambda \in \mathbb{C}$. Therefore

$$e_A = -(a(\lambda)z(\lambda) - e_A) + a(\lambda)z(\lambda) \in M,$$

which is not possible. Consequently, every $z \in Z(A)$ defines a $\lambda_z \in \mathbb{C}$ such that $z = \lambda_z e_A$. Hence $Z(A)$ is isomorphic to $\mathbb{C}$.

To show that the isomorphism ρ, defined by $\rho(z) = \lambda_z$ for every $z \in Z(A)$, is continuous, let O be a neighbourhood of zero in $\mathbb{C}$. Then there exists an $\epsilon > 0$ such that $O_\epsilon = \{\lambda \in \mathbb{C} : |\lambda| < \epsilon\} \subset O$. Let $\lambda_0 \in O_\epsilon\backslash\{0\}$. Since A is a Hausdorff space, then there exists a balanced neighbourhood V of zero of A such that $\lambda_0 e_A \notin V$. Hence $\lambda_0 e_A \notin V' = V \cap Z(A)$. If $|\lambda_z| \geqslant |\lambda_0|$, then $|\lambda_0\lambda_z^{-1}| \leqslant 1$ and therefore $\lambda_0 e_A = (\lambda_0\lambda_z^{-1})z \in V'$ for all $z \in V'$, which is not possible. Hence $\lambda_z \in O$ for each $z \in V'$. Thus ρ is continuous (ρ^{-1} is continuous, because $Z(A)$ is a topological linear space in the subspace topology). Consequently, $Z(A)$ is topologically isomorphic to $\mathbb{C}$.

Now we are ready to prove the following theorem.

Theorem 2.17. *Let A be a unital topological algebra over $\mathbb{C}$, P a primitive ideal of A, defined by a closed maximal regular left (right) ideal of A and let one of the following statements be true.*

a) *A is a locally A-pseudoconvex algebra;*

b) *A is a locally pseudoconvex Fréchet algebra;*

c) *A is an exponentially galbed algebra with bounded elements.*

Then $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.

Proof. Since P is a primitive ideal of A, defined by a closed maximal regular left (right) ideal of A, then P is closed in A. Therefore A/P is a topologically primitive Hausdorff algebra, by Lemma 1.2 and Lemma 1.3. In case a), the quotient algebra A/P is locally A-pseudoconvex, by Lemma 2.2; in case b), the quotient algebra A/P is a locally pseudoconvex Frèchet algebra, by Lemma 2.3. Hence (in cases a) and b)) $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$, by Lemma 1.14.

In case c) we get similarily, as in Theorem 2.1 that A/P, in the quotient topology, is an exponentially galbed algebra with bounded elements. Consequently, $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$, by the Theorem 2.16.

Corollary 2.18. *Let A be a unital locally m-pseudoconvex algebra over $\mathbb{C}$ and P a primitive ideal of A, defined by a closed maximal regular left (right) ideal of A. Then $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.*

Proof. According to the definitions, every locally m-pseudoconvex algebra is locally A-pseudoconvex. Therefore the condition a) of Theorem 2.17 is fulfilled. Thus $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.

Corollary 2.19. *Let A be a unital Waelbroeck algebra over $\mathbb{C}$ and P a primitive ideal of A, defined by a closed maximal regular left (right) ideal of A. If A is a locally convex Hausdorff algebra, A is a locally pseudoconvex algebra and all elements of A are bounded or A is a locally pseudoconvex algebra and $m(A) \neq \emptyset$, then $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.*

Proof. Every locally pseudoconvex algebra is exponentially galbed. If A is locally convex Hausdorff algebra, then all elements of A are bounded, by Lemma 1.15. If A is locally pseudoconvex and $m(A) \neq \emptyset$, then A is a Gelfand-Mazur algebra for which $m(A) \neq \emptyset$, by Lemma 1.11. Hence there exists a topology τ on A such that (A, τ) is an exponentially galbed algebra with bounded elements, by Lemma 1.12, and P is a primitive ideal in the topology τ, defined by a closed maximal regular left (right) ideal of A.

Therefore we have in all cases the situation c) of the Theorem 2.17. Thus $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.

Corollary 2.20. *Let A be a unital Gelfand-Mazur algebra over $\mathbb{C}$, for which $m(A) \neq \emptyset$, and P a closed primitive ideal of A, defined by a closed maximal regular left (right) ideal of A. Then $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.*

Proof. Let A be a unital Gelfand-Mazur algebra over $\mathbb{C}$, for which $m(A) \neq \emptyset$. Then there exists a topology τ on A such that (A, τ) is an exponentially galbed algebra with bounded elements and P is a primitive ideal, defined by a closed (in the topology τ) maximal regular left (right) ideal of A, by Lemma 1.12. Therefore we have the situation c) of Theorem 2.17 and thus $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$.

Chapter III

Description of closed maximal regular ideals in Gelfand-Mazur algebras

3.1. Extendible ideals

Here we show that, under certain conditions, the intersection of any closed maximal regular ideal of a topological algebra A with a closed subalgebra B of $Z(A)$ is an extendible ideal in B. For it we use some results obtained in Chapter II.

Proposition 3.1. *Let*[9] *A be a topological algebra over $\mathbb{C}$ with unit e_A, M a closed maximal left (right or two-sided) ideal of A and B a closed subalgebra of $Z(A)$, containing e_A. If at least one of the statements* a) - c) *of Theorem* 2.17 *holds, then*

1) *every $b \in B$ defines a number $\lambda \in \mathbb{C}$ such that $b - \lambda e_A \in M$;*
2) $M \cap B \in m_e(B)$.

Proof. 1) Let $b \in B$, P be the closed primitive ideal of A defined by M (if M is a two-sided ideal, then M is primitive by Lemma 1.1 and thus we can take $P = M$) and $\pi : A \to A/P$ the canonical homomorphism. Then $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$ by Theorem 2.17. We will denote this isomorphism by μ. Since $\pi(b) \in Z(A/P)$, then we can find a number $\lambda \in \mathbb{C}$ such that $\mu(\pi(b)) = \lambda = \mu(\pi(\lambda e_A))$. Therefore $\pi(b) = \pi(\lambda e_A)$ implies $b - \lambda e_A \in P \subset M$.

2) Let $M_B = M \cap B$. Then M_B is a closed subset of B, because M is closed in A. Since M is a left ideal of A (the proof for a right or a two-sided ideal is similar), then $e_A \notin M$. Hence $M_B \neq B$. It is easy to see that M_B is a linear subset of B. If $z \in M_B$, then $bz \in M_B$ for every $b \in B$. This implies that M_B is a closed left ideal of B. Let I be a left ideal of B such that $M_B \subset I$. If $I \neq M_B$ then there exists an element $b \in I \backslash M_B$ and (by the statement 1)) a number $\lambda \in \mathbb{C}$ such that $b - \lambda e_A \in M_B$. Since $b \notin M_B$, then $\lambda \neq 0$. Now from $b - \lambda e_A \in I$ follows that $e_A = \lambda^{-1}[b - (b - \lambda e_A)] \in I$. Therefore $I = B$, which is not possible. Consequently, $M_B \in m(B)$. As $M_B \subset M$ and M is a closed left ideal of A, then $I(M_B) \subset M \neq A$. Hence $M_B \in m_e(B)$.

[9] This result has been proved for Banach algebras in [17], Theorem 2.3 and Corollary 2.4, p. 195–196, and for locally convex Waelbroeck algebras with jointly continuous multiplication in [49], p. 82–83.

Corollary 3.2. *Let A be a topological algebra over $\mathbb{C}$ with unit e_A and M a closed maximal left (right or two-sided) ideal of A. If (at least) one of the statements* a) - c) *of Theorem* 2.17 *is true, then:*

1) *every $b \in Z(A)$ defines a number $\lambda \in \mathbb{C}$ such that $b - \lambda e_A \in M$;*

2) $M \cap Z(A) \in m_e(Z(A))$.

Proof. If we take $B = Z(A)$, then we have exactly the situation of Proposition 3.1.

Corollary 3.3. *Let A be a topological algebra over $\mathbb{C}$, M a closed maximal regular left (right or two-sided) ideal of A and B a closed subalgebra of $Z(A)$ such that $I(B) = A$. If (at least) one of the statements* a) - c) *of Theorem* 2.17 *holds, then $M \cap B \in m_e(B)$.*

Proof. By Theorem 2.12 (or Corollary 2.13 in case of two-sided ideal), there exists a uniquely defined closed maximal left (right or two-sided) ideal M' of $A \times \mathbb{C}$ such that $M' \not\subset (A \times \{0\})$ and

$$M' \cap (A \times \{0\}) = M \times \{0\} \neq A \times \{0\}.$$

Moreover, by Propositions 2.7, 2.8, 2.9 and 2.10, the algebra $A \times \mathbb{C}$ satisfies one of the conditions a) - c) of Theorem 2.17 and $B \times \mathbb{C}$ is a closed subalgebra of $Z(A \times \mathbb{C})$. Hence $A \times \mathbb{C}$ satisfies all conditions of Proposition 3.1. Therefore $M' \cap (B \times \mathbb{C}) \in m_e(B \times \mathbb{C})$, by statement 2) of Proposition 3.1, and $M' \not\subset (B \times \{0\})$.

Suppose that $M' \cap (B \times \mathbb{C}) \subset (B \times \{0\})$. Since $B \times \{0\} \in m(B \times \mathbb{C})$, by Lemma 2.11, and $M' \cap (B \times \mathbb{C}) \in m(B \times \mathbb{C})$, then $M' \cap (B \times \mathbb{C}) = B \times \{0\}$. Hence $B \times \{0\} \subset M'$. Now

$$A \times \{0\} = I(B) \times \{0\} = I(B \times \{0\}) \subset M' \not\subset (A \times \{0\}).$$

As $A \times \{0\} \in m(A \times \mathbb{C})$, by Lemma 2.11, then $M' = A \times \mathbb{C}$, but this is not possible. Hence $M' \cap (B \times \mathbb{C}) \in m_e(B \times \mathbb{C})$, but $M' \cap (B \times \mathbb{C}) \not\subset (B \times \{0\})$. Thus

$$M' \cap (B \times \{0\}) = \left[M' \cap (B \times \mathbb{C})\right] \cap (B \times \{0\}) \in m(B \times \{0\}),$$

by Theorem 2.12, and

$$I(M' \cap (B \times \{0\})) \subset I(B \times \{0\}) \subset A \times \{0\}.$$

Therefore from $I(M' \cap (B \times \{0\})) \neq B \times \mathbb{C}$ follows

$$(M \cap B) \times \{0\} = (M \times \{0\}) \cap (B \times \{0\}) =$$

$$= [M' \cap (A \times \{0\})] \cap (B \times \{0\}) = M' \cap (B \times \{0\}) \in m_e(B \times \{0\}).$$

Hence $M \cap B \in m(B)$ (because the map μ, defined by $\mu(b) = (b, 0)$, is a topological isomorphism from B onto $B \times \{0\}$). Since $I(M \cap B) \subset M \neq A$, then $M \cap B \in m_e(B)$.

Corollary 3.4. *Let A be a topological algebra over $\mathbb{C}$, M a closed maximal regular left (right or two-sided) ideal of A and $I(Z(A)) = A$. If (at least) one of the statements* a) - c) *of Theorem* 2.17 *holds, then*

$$M \cap Z(A) \in m_e(Z(A)).$$

Proof. If we take $B = Z(A)$, then we have exactly the situation of Corollary 3.3.

Remark. Note that Proposition 3.1 and Corollaries 3.2, 3.3, 3.4 will remain true also in case of locally m-pseudoconvex algebras, of locally convex Waelbroeck Hausdorff algebras, of locally pseudoconvex Waelbroeck algebras, for which all elements are bounded or $m(A) \neq \emptyset$, and of Gelfand-Mazur algebras, for which $m(A) \neq \emptyset$.

3.2. Description of all closed maximal regular ideals and of the topological radical

Let A be a topological algebra over $\mathbb{C}$ and B a subalgebra of $Z(A)$. For each $M \in m_e(B)$ and $a \in A$ let $A_M = A/I(M)$, $\kappa_M : A \to A_M$ be the canonical homomorphism and $a\hat{}(M) = \kappa_M(a)$.

Lemma 3.5. *Let A be a topological algebra over $\mathbb{C}$, I a closed maximal regular left (right or two-sided) ideal of A, B a subalgebra of $Z(A)$ and $M \in m_e(B)$. Then*

$$\kappa_M(I) = \{a\hat{}(M) : a \in I\}$$

is a closed maximal regular left (right or two-sided, respectively) ideal of A_M. If J is a closed maximal regular left (right or two-sided, respectively) ideal of A_M, then $\kappa_M^{-1}(J)$ is a closed maximal regular left (right or two-sided, respectively) ideal of A.

Proof. Let $b_1, b_2 \in \kappa_M(I)$, $\lambda \in \mathbb{C}$ and $d \in A_M$. Then there are elements $a_1, a_2 \in I$ and $c \in A$ such that $\kappa_M(a_i) = b_i$ for $i \in \{1, 2\}$ and $\kappa_M(c) = d$. Since I is a left (right or two-sided, respectively) ideal of A, then $a_1 + a_2$, λa_1, $ca_1 \in I$ (similarily, $a_1 c \in I$). Therefore

$$b_1 + b_2 = \kappa_M(a_1) + \kappa_M(a_2) = \kappa_M(a_1 + a_2) \in \kappa_M(I),$$

$$\lambda b_1 = \lambda\kappa_M(a_1) = \kappa_M(\lambda a_1) \in \kappa_M(I)$$

and

$$db_1 = \kappa_M(c)\kappa_M(a_1) = \kappa_M(ca_1) \in \kappa_M(I)$$

(similarily, $b_1 d \in \kappa_M(I)$). If u denotes the right (left or two-sided, respectively) regular unit for I, then $\kappa_M(u)$ is the right (left or two-sided, respectively) regular unit for $\kappa_M(I)$ and therefore $\kappa_M(I)$ is a regular left (right or two-sided, respectively) ideal of A_M.

Let now $c_1, c_2 \in \kappa_M^{-1}(J)$ and $\lambda \in \mathbb{C}$. Then we can find $b_1, b_2 \in J$ such that $\kappa_M(c_i) = b_i$ for $i \in \{1,2\}$. Since J is a left (right or two-sided, respectively) ideal in A_M, then $\kappa_M(c_1 + c_2) = b_1 + b_2 \in J$ and $\kappa_M(\lambda c_1) = \lambda b_1 \in J$. Therefore $c_1 + c_2, \lambda c_1 \in \kappa_M^{-1}(J)$. If $a \in A$ and $d = \kappa_M(a) \in A_M$, then $\kappa_M(ac_1) = db_1 \in J$ or $ac_1 \in \kappa_M^{-1}(J)$ (similarily, $c_1 a \in \kappa_M^{-1}(J)$). It is easy to see that $\kappa_M^{-1}(J)$ is closed in A (because J is closed). If v is the right (left or two-sided, respectively) regular unit for J, then every $v' \in \kappa_M^{-1}(v)$ is a right (left or two-sided, respectively) regular unit for $\kappa_M^{-1}(J)$. Thus $\kappa_M^{-1}(J)$ is a closed regular left (right or two-sided, respectively) ideal of A.

If $\kappa_M(I) \subset W$ for a left (right or two-sided, respectively) ideal W of A_M, then

$$I \subset \kappa_M^{-1}(\kappa_M(I)) \subset \kappa_M^{-1}(W).$$

Since W is a left (right or two-sided, respectively) ideal of A_M, then (as above) $\kappa_M^{-1}(W)$ is also a left (right or two-sided, respectively) ideal of A. We have two possibilities: $\kappa_M^{-1}(W) = A$, which gives us a contradiction $W = A_M$, or $\kappa_M^{-1}(W) = I$, which gives us $\kappa_M(I) = W$. Thus $\kappa_M(I)$ is a maximal regular left (right or two-sided, respectively) ideal of A_M.

Next, we show that $\kappa_M(I)$ is closed. If $\mathrm{cl}_{A_M}(\kappa_M(I)) \neq A_M$, then $\mathrm{cl}_{A_M}(\kappa_M(I))$ is also a left (right or two-sided, respectively) ideal in A_M, by Lemma 1.6. Hence $\kappa_M(I) = \mathrm{cl}_{A_M}(\kappa_M(I))$. If $\mathrm{cl}_{A_M}(\kappa_M(I)) = A_M$, then $\kappa_M(u) \in \mathrm{cl}_{A_M}(\kappa_M(I))$. It means that there is a family $(m_\lambda)_{\lambda\in\Lambda} \in I$ such that $(\kappa_M(m_\lambda))_{\lambda\in\Lambda}$ converges to $\kappa_M(u)$. Let O' be an arbitrary neighbourhood of zero in A. Then $O = \kappa_M(O')$ is a neighbourhood of zero in A_M. We can find an index $\mu \in \Lambda$ such that $\kappa_M(m_\lambda - u) \in O$ for each $\lambda > \mu$. Now $\kappa_M(I(M)) = \theta_{A_M} \subset \kappa_M(I)$ implies $I(M) \subset \kappa_M^{-1}(\kappa_M(I)) = I$. If $\lambda_0 > \mu$, then

$$m_{\lambda_0} - u \in \kappa_M^{-1}(\kappa_M(O')) = I(M) + O' \subset I + O'$$

and therefore

$$u = (u - m_{\lambda_0}) + m_{\lambda_0} \in I + O' + I \subset I + O'.$$

Hence

$$u \in \bigcap\{I + O' : O' \text{ is a neighbourhood of zero in } A\} = \mathrm{cl}_{A_M}(I) = I$$

(see [43], p. 13). In this case $I = A$, which is not possible. Consequently, $\kappa_M(I)$ is a closed maximal regular left (right or two-sided, respectively) ideal of A_M.

If now $\kappa_M^{-1}(J) \subset H$ for a left (right or two-sided, respectively) ideal H of A, then

$$J \subset \kappa_M(\kappa_M^{-1}(J)) \subset \kappa_M(H).$$

Again, as above, we see that $\kappa_M(H)$ is a left (right or two-sided, respectively) ideal of A_M. Therefore we have two possibilities: $\kappa_M(H) = A_M$ or $\kappa_M(H) = J$. In the first case there is an element $h \in H$ such that

$$\kappa_M(h) = v = \kappa_M(v')$$

(here v and v' are the same as above), because of what

$$h - v' \in \kappa_M^{-1}(\theta_{A_M}) \subset \kappa_M^{-1}(J).$$

Since v' is a right (left or two-sided, respectively) regular unit for $\kappa_M^{-1}(J)$, then $a - av' \in \kappa_M^{-1}(J) \subset H$ ($a - v'a \in \kappa_M^{-1}(J) \subset H$, respectively) for each $a \in A$. As $v' = h - (h - v') \in H$, then $av' \in H$ ($v'a \in H$, respectively) for each $a \in A$. Thus $a = av' - (a - av') \in H$ ($a = v'a - (a - v'a) \in H$, respectively) for every $a \in A$ implies $H = A$, which is a contradiction. Therefore $\kappa_M(H) = J$ and from $\kappa_M^{-1}(J) = \kappa_M^{-1}(\kappa_M(H)) \supset H$ follows $\kappa_M^{-1}(J) = H$. Hence $\kappa_M^{-1}(J)$ is a closed maximal regular left (right or two-sided, respectively) ideal of A, which proves the Lemma.

For each $M_{B\times\mathbb{C}} \in m_e(B \times \mathbb{C})$, let

$$\rho = \rho_{M_{B\times\mathbb{C}}} : A \times \mathbb{C} \to (A \times \mathbb{C})/I(M_{B\times\mathbb{C}})$$

be a canonical homomorphism and $\mu : A \to A \times \{0\}$ again a topological isomorphism, defined by $\mu(a) = (a, 0)$ for every $a \in A$.

Theorem 3.6. *Let A be a topological algebra over $\mathbb{C}$, which satisfies (at least) one of the conditions* a) - c) *of Theorem* 2.17, *B a closed subalgebra of $Z(A)$ and I a closed maximal regular left (right or two-sided) ideal of A. If*

a) *A is a unital algebra and B has the same unit*

or

b) $I(B) = A$,

then

1) $I = \kappa_M^{-1}(J_M) = \{a \in A : a\hat{\ }(M) \in J_M\}$ *for an ideal $M \in m_e(B)$ and a closed maximal regular left (right or two-sided, respectively) ideal J_M of A_M;*

2) *there exists a bijection*

$$\Lambda_k : \bigcup_{M \in m_e(B)} \{M\} \times m_k(A_M) \longrightarrow m_k(A),$$

where $k = l$, $k = r$ *or* $k = t$.

Proof. 1) Let I be a closed maximal regular left (right or two-sided) ideal in A. By Proposition 3.1 for unital case and by Corollary 3.3 for nonunital case, $M = I \cap B \in m_e(B)$. Let $J_M = \kappa_M(I)$. By Lemma 3.5, J_M is a closed maximal regular left (right or two-sided, respectively) ideal of A_M and $I \subset \kappa_M^{-1}(J_M)$, where $\kappa_M^{-1}(J_M)$ is a closed maximal regular left (right or two-sided, respectively) ideal of A. Since I is also a maximal regular left (right or two-sided) ideal in A, then $I = \kappa_M^{-1}(J_M)$, which proves the part 1).

2) Let Λ_k be a map, defined by

$$\Lambda_k((M, J_M)) = \kappa_M^{-1}(J_M)$$

for each $M \in m_e(B)$ and $J_M \in m_k(A_M)$. Then Λ_k maps every pair (M, J_M) into $m_k(A)$, by Lemma 3.5, and for each $I \in m_k(A)$ there exists a pair (M, J_M) such that $\Lambda_k((M, J_M)) = I$, by the statement 1) of Theorem 3.6. Hence Λ_k is an onto map.

Now we will show that Λ_k is one-to-one. For it, let

$$I = \kappa_{M_1}^{-1}(J_{M_1}) = \kappa_{M_2}^{-1}(J_{M_2})$$

for some $M_1, M_2 \in m_e(B)$. Then $I \in m_k(A)$, by Lemma 3.5, and

$$I_B = I \cap B \in m_e(B) \subset m(B),$$

by Proposition 3.1 (and Corollary 3.3). Since

$$\kappa_M(I(M)) = \theta_{A_M} \in J_M$$

for each $M \in m_e(B)$, then

$$M_1 \subset I(M_1) \subset \kappa_{M_1}^{-1}(J_{M_1}) = I$$

and

$$M_2 \subset I(M_2) \subset \kappa_{M_2}^{-1}(J_{M_2}) = I.$$

Hence $M_1, M_2 \subset I_B$. As M_1 and M_2 are maximal left (right or two-sided, respectively) ideals of B, then $M_1 = M_2 = I_B$ and

$$J_{M_1} = \kappa_{M_1}(\kappa_{M_1}^{-1}(J_{M_1})) = \kappa_{M_2}(\kappa_{M_2}^{-1}(J_{M_2})) = J_{M_2}.$$

Therefore from $\Lambda_k((M_1, J_{M_1})) = \Lambda_k((M_2, J_{M_2}))$ follows that

$$(M_1, J_{M_1}) = (M_2, J_{M_2}).$$

Hence Λ_k is an one-to-one map. Consequently, Λ_k is a bijection.

Remark A. The condition $I(B) = A$ is automatically fulfilled in case $m(B) \neq m_e(B)$. The case $I(B) \neq A$ remains open and should be studied separately.

Remark B. Note that Corollary 3.3 and Theorem 3.6 will remain true also in case of locally m-pseudoconvex algebras, of locally convex Waelbroeck Hausdorff algebras, of locally pseudoconvex Waelbroeck algebras, for which all elements are bounded or $m(A) \neq \emptyset$, and of Gelfand-Mazur algebras, for which $m(A) \neq \emptyset$.

Proposition 3.7. *Let A be a topological algebra over $\mathbb{C}$, for which (at least) one of the conditions* a) - c) *of Theorem* 2.17 *holds. Let B be a closed subalgebra of $Z(A)$. Suppose that there exists a closed maximal regular left (right) ideal in A and that either condition* a) *or condition* b) *of Theorem* 3.6 *holds. Then*

$$\mathrm{rad}A = \bigcap\{\kappa_M^{-1}(\mathrm{rad}A_M) : M \in m_e(B)\}.$$

Proof. Suppose that

$$x \in \bigcap\{\kappa_M^{-1}(\mathrm{rad}A_M) : M \in m_e(B)\}$$

and I is an arbitrary closed maximal regular left (right) ideal of A. By Theorem 3.6, we can find $M \in m_e(B)$ and a closed maximal regular left (right, respectively) ideal J of A_M such that $I = \kappa_M^{-1}(J)$. Since

$$\kappa_M(x) \in \mathrm{rad}A_M \subset J,$$

then $x \in I$ for any closed maximal regular left (right, respectively) ideal I of A. Therefore $x \in \mathrm{rad}A$ and thus

$$\mathrm{rad}A \supseteq \bigcap\{\kappa_M^{-1}(\mathrm{rad}A_M) : M \in m_e(B)\}.$$

Suppose that $y \in \mathrm{rad}A$ and $M \in m_e(B)$. If J is an arbitrary closed maximal regular left (right) ideal of A_M, then $\kappa_M^{-1}(J)$ is a closed maximal regular left (right, respectively) ideal of A, by Lemma 3.5, and $\kappa_M(y) \in J$

for every closed maximal regular left (right, respectively) ideal J of A_M. Therefore $\kappa_M(y) \in \mathrm{rad}A_M$, if $M \in m_e(B)$, so that

$$\mathrm{rad}A \subseteq \bigcap\{\kappa_M^{-1}(\mathrm{rad}A_M) : M \in m_e(B)\},$$

which completes the proof.

3.3. The case when all closed maximal ideals of the center are extendible

In order to find the conditions for a topological algebra A in which $m(A) = m_e(A)$, we need the following results.

Proposition 3.8. *Let*[10] *A be a simplicial topological algebra over $\mathbb{C}$ and I a closed regular two-sided ideal of A. If*

$$\mathrm{cl}_A\Big(\bigcup_{M\in\, m_t(A)} M\Big) \subset \bigcup_{M\in\, M_t(A)} M, \tag{3}$$

then the hull $h(I)$ of I is compact in the hull-kernel topology.

Proof. The set $h(I) \neq \emptyset$, because A is a simplicial (with respect to two-sided ideals) topological algebra. Let $(F_\lambda)_{\lambda\in\Lambda}$ be an arbitrary set of nonempty closed subsets of $h(I)$, intersection of which is empty. Then

$$J = \sum_{\lambda\in\Lambda} k(F_\lambda)$$

is the smallest two-sided ideal of A, which contains each of ideals $k(F_\lambda)$. It means that every $j \in J$ is representable as a sum of elements $a_\lambda \in k(F_\lambda)$ with $\lambda \in \Lambda$, where only finite number of elements a_λ are different from θ_A. Since $F_\lambda \subset h(I)$ for each $\lambda \in \Lambda$, then $I \subset k(h(I)) \subset k(F_\lambda) \subset J$, because of which J is a regular two-sided ideal of A. Suppose that $\mathrm{cl}_A(J) \neq A$. Then $\mathrm{cl}_A(J)$ is a closed regular two-sided ideal of A and $I \subset \mathrm{cl}_A(J)$. Therefore there exists an ideal $M \in h(I)$ such that $\mathrm{cl}_A(J) \subset M$, because A is a simplicial (with respect to two-sided ideals) topological algebra. Since $k(F_\lambda) \subset \mathrm{cl}_A(J) \subset M$ for all $\lambda \in \Lambda$, then $M \in h(k(F_\lambda))$ for all $\lambda \in \Lambda$. The closedness of every F_λ in the hull-kernel topology means that $h(k(F_\lambda)) = F_\lambda$ for all $\lambda \in \Lambda$. Therefore

$$M \in \bigcap_{\lambda\in\Lambda} F_\lambda = \emptyset,$$

[10] For Banach algebras this result has been proved in [21], p. 373-374, and in [42], p. 23. The condition (3) holds, if A is a Q-algebra.

which is impossible. Hence $\mathrm{cl}_A(J) = A$.

Let now u be a regular unit for I in A, let a be any element of A and O an arbitrary neighbourhood of zero in A. Then we can find another neighbourhood O_1 of zero in A such that $O_1 a \subset O$. Since $\mathrm{cl}_A(J) = A$, then $(u + O_1) \cap J \neq \emptyset$ and we can find $n \in \mathbb{N}$, $o_1 \in O_1$, $\lambda_1, \lambda_2, ..., \lambda_n \in \Lambda$ and for every $\nu \in \{\lambda_1, \lambda_2, ..., \lambda_n\}$ an element $e_\nu \in k(F_\nu)$ such that $u + o_1 = e_{\lambda_1} + e_{\lambda_2} + ... + e_{\lambda_n}$. As every $k(F_{\lambda_i})$ is a two-sided ideal of A and $I \subset k(F_\lambda)$ for all $\lambda \in \Lambda$, then

$$a + o_1 a = (a - ua) + (u + o_1)a \in \sum_{i=1}^{n} k(F_{\lambda_i})$$

for each $a \in A$. Suppose now that

$$\bigcap_{\lambda \in S} F_\lambda \neq \emptyset$$

for every nonempty subset $S \subset \Lambda$. Then

$$\bigcap_{i=1}^{n} F_{\lambda_i} \neq \emptyset$$

and there exists an ideal $M_O \in h(I)$ such that

$$M_O \in \bigcap_{i=1}^{n} F_{\lambda_i}.$$

Since $M_O \in F_{\lambda_i} = h(k(F_{\lambda_i}))$ for each $i \in \{1, ..., n\}$, then $k(F_{\lambda_i}) \subset M_O$ for each $i \in \{1, ..., n\}$. Therefore $a + o_1 a \in M_O$. Thus

$$(a + O) \cap \Big(\bigcup_{M \in m_t(A)} M \Big) \neq \emptyset$$

for each $a \in A$ and each neighbourhood O of zero in A. Hence

$$A \subset \mathrm{cl}_A\Big(\bigcup_{M \in m_t(A)} M \Big) \subset \bigcup_{M \in M_t(A)} M,$$

but it is impossible. Thus there is a finite subset $S_0 \in \Lambda$ such that

$$\bigcap_{\lambda \in S_0} F_\lambda$$

is empty. It means that $h(I)$ is a compact subset of $m_t(A)$ in the hull-kernel topology.

Corollary 3.9. *If A is a unital simplicial (with respect to two-sided ideals) Hausdorff algebra over $\mathbb{C}$ and the condition* (3) *of Proposition* 3.8 *holds, then $m_t(A)$ is compact in the hull-kernel topology.*

Proof. Since A is a unital simplicial (with respect to two-sided ideals) Hausdorff algebra, then $\{\theta_A\}$ is a closed two-sided ideal of A and $h(\{\theta_A\}) = m_t(A)$. Hence $m_t(A)$ is compact in the hull-kernel topology, by Proposition 3.8.

Theorem 3.10. *Let*[11] *A be a unital strongly topologically semisimple simplicial Hausdorff algebra over $\mathbb{C}$, B a closed subalgebra of $Z(A)$ with the same unit e_A and $m(B)$ a Hausdorff space in the hull-kernel topology. If the condition* (3) *of Proposition* 3.8 *holds and A satisfies one of the conditions* a) - c) *of Theorem* 2.17, *then for every $M \in m(B)$ there exists $I \in m_k(A)$ (here $k = l$, $k = r$ or $k = t$) such that $M = I \cap B$ and therefore $m_e(B) = m(B)$.*

Proof. It is known, by Proposition 3.1, that $I \cap B \in m(B)$ for each $I \in m_k(A)$. Let $T : m_k(A) \to m(B)$ be a map, defined by $T(I) = I \cap B$ for each $I \in m_k(A)$. Let S be any closed subset of $m(B)$ and

$$E_S = T^{-1}(S) = \{I \in m_k(A) : T(I) \in S\}.$$

If $M_0 \in h(k(E_S))$, then

$$k(S) \subset \bigcap_{I \in E_S} (I \cap B) = k(E_S) \cap B \subset M_0 \cap B.$$

Therefore $M_0 \cap B \in h(k(S)) = S$ and $M_0 \in E_S$. Since always $E_S \subset h(k(E_S))$, then $E_S = h(k(E_S))$ which means that E_S is closed in the hull-kernel topology. Thus T is a continuous map from $m_k(A)$ into $m(B)$. The set $m_t(A)$ is compact in the hull-kernel topology, by Corollary 3.9. Now $T(m_t(A))$ is a closed subset of $m(B)$, because $m(B)$ is a Hausdorff space in the hull-kernel topology. As A is strongly topologically semisimple, then

$$k(T(m_t(A))) = \bigcap_{I \in m_t(A)} (I \cap B) = (k(m_t(A))) \cap B = \{\theta_A\},$$

because of which $T(m_t(A)) = h(k(T(m_t(A)))) = m(B)$. It means that T is an onto map. Therefore for every $M \in m(B)$ we can find an ideal

[11] A similar result for Banach algebras has been proved in [42], p. 85–86 and for spectral algebras in [41], Theorem 7.2.17.

$I_M \in m_t(A)$ such that $I_M \cap B = M$. Since every closed maximal two-sided ideal is also closed left (right) ideal of A and A is simplicial, then there exists $J \in m_k(A)$ such that $I_M \subset J$. Now $M = I_M \cap B \subset J \cap B$ and $e_A \notin J$. Thus $J \cap B \neq B$. Since $M \in m(B)$, then we have $M = J \cap B \subset m_e(B)$ for $J \in m_k(A)$, by Proposition 3.1. Hence $m_e(B) = m(B)$, in the present case.

From Theorem 3.6 and Theorem 3.10 follows the next result.

Theorem 3.11. *Let A be a unital strongly topologically semisimple simplicial (with respect to two-sided ideals) Hausdorff algebra* over $\mathbb{C}$, *B a closed subalgebra of $Z(A)$ with the same unit and $m(B)$ a Hausdorff space in the hull-kernel topology. If the condition* (3) *of Proposition* 3.8 *holds and A satisfies one of the conditions* a) - c) *of Theorem* 2.17, *then there exists a bijection*

$$\Lambda_k : \bigcup_{M \in m(B)} \{M\} \times m_k(A_M) \longrightarrow m_k(A),$$

where $k = l$, $k = r$ or $k = t$.

Remark. Note that Theorems 3.10 and 3.11 will remain true also in the cases of locally m-pseudoconvex algebras, of locally convex Waelbroeck algebras, of locally pseudoconvex Waelbroeck algebras, for which all elements are bounded or $m(A) \neq \emptyset$, and of Gelfand-Mazur algebras, for which $m(A) \neq \emptyset$.

3.4. Sectional representations

Let A be a unital topological algebra over $\mathbb{C}$ and B a subalgebra of $Z(A)$. Suppose that $m_e(B) \neq \emptyset$.

For every $M \in m_e(B)$, let $A_M = A/I(M)$, $\kappa_M : A \to A_M$ be the canonical homomorphism, $a \in A$, $a\hat{}(M) = \kappa_M(a)$ for each $M \in m_e(B)$ and

$$\Delta = \bigcup_{M \in m_e(B)} A_M.$$

Then $a\hat{}$ maps $m_e(B)$ into Δ.

Let now $\pi : \Delta \to m_e(B)$ be such mapping, which assigns to every $d \in \Delta$ the ideal $M \in m_e(B)$, for which $d \in A_M$, i.e., $d = \kappa_M(a)$ for some $a \in A$. It is easy to see that π is well defined. Indeed, if $d \in A_{M_1} \cap A_{M_2}$ for some $M_1, M_2 \in m_e(B)$, then $d = \kappa_{M_1}(a_1)$ (i.e., $\pi(d) = M_1$) and $d = \kappa_{M_2}(a_2)$ (i.e., $\pi(d) = M_2$) for some elements $a_1, a_2 \in A$. Consequently,

$$a_1 + I(M_1) = a_2 + I(M_2).$$

Since $a_1 = a_1 + \theta_A \in a_1 + I(M_1) = a_2 + I(M_2)$, then we can find an element $d_2 \in I(M_2)$ such that $a_1 = a_2 + d_2$ or $a_1 - a_2 = d_2 \in I(M_2)$.

Let now $c_1 \in I(M_1)$ be an arbitrary element. Then

$$a_1 + c_1 \in a_1 + I(M_1) = a_2 + I(M_2)$$

and we can find $c_2 \in I(M_2)$ such that $a_1 + c_1 = a_2 + c_2$. Since $a_1 - a_2 \in I(M_2)$ and $c_2 \in I(M_2)$, then $c_1 = c_2 - (a_1 - a_2) \in I(M_2)$. Thus $I(M_1) \subset I(M_2)$. Analogously, we get that $I(M_2) \subset I(M_1)$. Hence $I(M_1) = I(M_2)$, because of which

$$M_1 = I(M_1) \cap B = I(M_2) \cap B = M_2.$$

It means that π is well defined and $A_{M_1} \cap A_{M_2} \neq \emptyset$ if and only if $M_1 = M_2$.

We endowe the set $m_e(B)$ with the Gelfand topology τ, a subbase of neighbourhoods of $M_0 \in m_e(B)$ of which consists of sets

$$O(M_0) = \{M \in m_e(B) : |(\varphi_M - \varphi_{M_0})(b)| < \epsilon\},$$

where $\epsilon > 0$ and $b \in B$ vary, while φ_M denotes a nontrivial homomorphism $B \to \mathbb{C}$ such that $\ker\varphi_M = M$. On the algebra A_M we consider the quotient topology τ_M and on Δ the topology

$$\tau_\Delta = \{\pi^{-1}(U) : U \in \tau\}.$$

Then A_M is a topological algebra, $\pi[a\hat{}(M)] = M$ for each $M \in m_e(B)$, $(\Delta, \pi, m_e(B))$ is a fiber bundle (because π is a continuous open map from Δ onto $m_e(B)$) and $a\hat{} \in \Gamma(\pi)$ for each $a \in A$.

Next, we define a map $\Upsilon : A \to \Gamma(\pi)$ such that $\Upsilon(a) = a\hat{}$ for each $a \in A$. It is easy to see that Υ is a homomorphism and thus continuous if and only if it is continuous at θ_A. Let O be an arbitrary neighbourhood of zero in $\Upsilon(A)$. Then there exist $n \in \mathbb{N}$ and $O_1 \in B(\theta_{A_{M_1}}), ..., O_n \in B(\theta_{A_{M_n}})$ such that

$$\bigcap_{k=1}^{n} U_{O_k}(\hat{\theta_A}) \subset O,$$

where $U_{O_k}(\hat{\theta_A}) = \{f \in \Gamma(\pi) : f(M_k) \in O_k\}$. If we take

$$O_A = \bigcap_{k=1}^{n} \kappa_{M_k}^{-1}(O_k),$$

then O_A is a neighbourhood of zero in A. Let $a_0 \in O_A$. Since

$$\hat{a_0}(M_k) = \kappa_{M_k}(a_0) \subset \kappa_{M_k}(\kappa_{M_k}^{-1}(O_k)) = O_k$$

for every k, then

$$\Upsilon(a_0) = \hat{a_0} \in \bigcap_{k=1}^{n} \{f \in \Gamma(\pi) : f(M_k) \in O_k\} \subset O$$

for each $a_0 \in O_A$. Consequently, $\Upsilon(O_A) \subset O$ implies that Υ is continuous at θ_A. Therefore Υ is a continuous map and hence *a sectional representation of A.*

Proposition 3.12. *Let A be a topologically semisimple algebra which satisfies the conditions of Proposition* 3.7. *Then the map Υ is one-to-one.*

Proof. Since A is topologically semisimple, then its topological radical $\mathrm{rad}A = \{\theta_A\}$. Hence from

$$\ker \Upsilon = \bigcap \{\kappa_M^{-1}(\theta_M) : M \in m_e(B)\} \subseteq$$

$$\subseteq \bigcap \{\kappa_M^{-1}(\mathrm{rad}A_M) : M \in m_e(B)\} = \mathrm{rad}A = \{\theta_A\}$$

one obtains that Υ is one-to-one.

Theorem 3.13. *Let A be a topologically semisimple algebra over $\mathbb{C}$ with unit, having at least one closed maximal left* (*right*) *ideal. If one of the following statements is true*

a) *A is a locally A-pseudoconvex algebra;*

b) *A is a locally pseudoconvex Fréchet algebra;*

c) *A is an exponentially galbed algebra with bounded elements,*

then A can be considered as a subalgebra of the section algebra $\Gamma(\pi)$.

Proof. Since Υ is an one-to-one representation of A in $\Gamma(\pi)$, then we can consider A as a subalgebra of $\Gamma(\pi)$.

Remark. Note that Proposition 3.12 and Theorem 3.13 will remain true also in case of locally m-pseudoconvex algebras, of locally convex Waelbroeck Hausdorff algebras, of locally pseudoconvex Waelbroeck algebras, for which all elements are bounded or $m(A) \neq \emptyset$, and of Gelfand-Mazur algebras, for which $m(A) \neq \emptyset$.

3.5. Module representations

In the following, we try to describe representations of Gelfand-Mazur algebra, using techniques of module-algebras.

Let C be a topological algebra over $\mathbb{K}$ and A a topological C-algebra, which has the B-extension property (here B is the subalgebra of $Z(A)$).

Then every $\phi \in \hom B$ defines a $\Psi_\phi \in \hom_C(A, C)$ such that

$$\Psi_\phi\Big(\sum_{k=1}^n c_k b_k\Big) = \sum_{k=1}^n c_k \phi(b_k)$$

for each $n \in \mathbb{N}$, $c_1, ..., c_n \in C$ and $b_1, ..., b_n \in B$. Let $\hat{a}(\phi) = \Psi_\phi(a)$ for each $a \in A$ and $\phi \in \hom B$. Then $\hat{a}$ is a map from $\hom B$ into C. Next we show that $\hat{a}$ is a continuous map, if the set $\{\Psi_\phi : \phi \in \hom B\}$ is equicontinuous.

Let $a \in A$, $\phi_0 \in \hom B$ and O_C be any neighbourhood of zero in C. Then there is a neighbourhood O'_C of zero in C such that

$$O'_C + O'_C + O'_C \subset O_C.$$

Since $\{\Psi_\phi : \phi \in \hom B\}$ is equicontinuous, then there exists a balanced neighbourhood O_A of zero of A such that $\Psi_\phi(O_A) \subset O'_C$ for each $\phi \in \hom B$. As $C \otimes B$ is dense in A, then

$$(a + O_A) \cap (C \otimes B) \neq \emptyset.$$

Therefore there exists $n \in \mathbb{N}$, $c_1, ..., c_n \in C$ and $b_1, ..., b_n \in B$ such that

$$\sum_{k=1}^n c_k b_k \in a + O_A.$$

But then

$$\Big(\sum_{k=1}^n c_k b_k\Big) - a \in O_A.$$

Now there exists a neighbourhood O''_C of zero of C such that

$$\underbrace{O''_C + \ldots + O''_C}_{n} \subset O'_C.$$

Every $k \in \{1, \ldots, n\}$ defines a number $\epsilon_k > 0$ such that $c_k O_{\epsilon_k} \subset O''_C$ (here $O_{\epsilon_k} = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \epsilon_k\}$). Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then

$$O(\phi_0;\ b_1, \ldots, b_n,\ \epsilon\} =$$

$$= \{\phi \in \hom B\ :\ (\phi - \phi_0)(b_k) < \epsilon \ \text{ for each } k \in \{1, \ldots, n\}\}$$

is a neighbourhood of ϕ_0 in $\hom B$. If $\phi \in O(\phi_0;\ b_1, \ldots, b_n,\ \epsilon\}$, then

$$\hat{a}(\phi) - \hat{a}(\phi_0) = \Psi_\phi(a) - \Psi_{\phi_0}(a) =$$

$$= \Psi_\phi\Big(a - \sum_{k=1}^{n} c_k b_k\Big) + \sum_{k=1}^{n} c_k(\phi - \phi_0)(b_k) + \Psi_{\phi_0}\Big(\sum_{k=1}^{n} c_k b_k - a\Big) \subset$$

$$\subset \Psi_\phi(O_A) + c_1 O_{\epsilon_1} + \ldots + c_n O_{\epsilon_n} + \Psi_{\phi_0}(O_A) \subset O'_C + \underbrace{(O''_C + \ldots + O''_C)}_{n} + O'_C \subset$$

$$\subset O'_C + O'_C + O'_C \subset O_C.$$

We have shown that for each $a \in A$, $\phi_0 \in \mathrm{hom}B$ and any neighbourhood O_C of zero of C there exists a neighbourhood $O(\phi_0;\ b_1, \ldots, b_n,\ \epsilon\}$ of ϕ_0 in $\mathrm{hom}B$ such that from $\phi \in O(\phi_0;\ b_1, \ldots, b_n,\ \epsilon\}$ follows $\hat{a}(\phi) - \hat{a}(\phi_0) \subset O_C$. Hence $\hat{a}$ is a continuous map and thus $\hat{a} \in C(\mathrm{hom}B, C)$ for each $a \in A$.

Let now $\mathcal{A} : A \to C(\mathrm{hom}B, C)$ be a map defined by $\mathcal{A}(a) = \hat{a}$ for all $a \in A$ and let σ be any cover of $\mathrm{hom}B$. We endow $C(\mathrm{hom}B, C)$ with the topology, a subbase $\mathcal{B}$ of neighbourhoods of zero in which is

$$\{T(S, O) : S \in \sigma,\ O \text{ is a neighbourhood of zero in } C\},$$

where $T(S, O) = \{f \in C(\mathrm{hom}B,\ C) : f(S) \subset O\}$. Next, we show that $\mathcal{A}$ is a continuous map, if the set $\{\Psi_\phi : \phi \in \mathrm{hom}B\}$ is equicontinuous. Let O_C be a neighbourhood of zero in $C(\mathrm{hom}B, C)$. Then there exist $n \in \mathbb{N}$, $S_1, \ldots, S_n \in \sigma$ and neighbourhoods $O_1, \ldots, O_n$ of C such that

$$\bigcap_{k=1}^{n} T(S_k, O_k) \subset O_C.$$

Since $\{\Psi_\phi : \phi \in \mathrm{hom}B\}$ is equicontinuous, then for every $k \in \{1, \ldots, n\}$ there exists a neighbourhood U_k of zero in A such that $\Psi_\phi(U_k) \subset O_k$ for every $\phi \in \mathrm{hom}B$. Let

$$U = \bigcap_{k=1}^{n} U_k.$$

Then U is a neighbourhood of zero in A and $\Psi_\phi(U) \subset O_k$ for every $\phi \in \mathrm{hom}B$ and $k \in \{1, \ldots, n\}$.

Let now $k \in \{1, \ldots, n\}$, $a \in U$ and $\phi \in S_k$. Then

$$(\mathcal{A}(a))(\phi) = \hat{a}(\phi) = \Psi_\phi(a) \in O_k.$$

Thus

$$\mathcal{A}(U) \subset \bigcap_{k=1}^{n} T(S_k, O_k) \subset O_C$$

which means that $\mathcal{A}$ is continuous. It is easy to show that $\mathcal{A}$ is a homomorphism. Thus $\mathcal{A}$ is a module representation of A in $C(\mathrm{hom}B,\ \mathbb{C}; \sigma)$.

Herewith,

$$\ker\mathcal{A} = \{a \in A : \hat{a}(\phi) = \theta_C \text{ for each } \phi \in \text{ hom } B\} =$$

$$= \{a \in A : \Psi_\phi(a) = \theta_C \text{ for each } \phi \in \text{ hom } B\} = \bigcap_{\phi \,\in\, \text{hom}B} \ker\Psi_\phi.$$

Now we consider the case, when A is a topological C-algebra with the B-extension property, but for each $b \in B$, the map $b\tilde{}$, defined by $b\tilde{}(\phi) = \phi(b)$ for each $\phi \in \text{hom}B$, vanishes at infinity (i.e., for each $\varepsilon > 0$ there is a compact subset K_b of hom$B$ such that $b\tilde{}(\phi) < \varepsilon$, whenever $\phi \in \text{hom}B \setminus K_b$). Suppose that hom$B$ is not compact and the set $\{\Psi_\phi : \phi \in \text{hom}B\}$ is equicontinuous. Next, we show that every $\hat{a}$ vanishes at infinity (i.e., for every neighbourhood O of zero in C there is a compact subset K_O in hom$B$ such that $\hat{a}(\phi) \in O$, whenever $\phi \in \text{hom}B \setminus K_O$). For it, let O_C be a neighbourhood of zero of C. Then there is a neighbourhood O'_C of zero of C such that $O'_C + O'_C \subset O_C$. Similarily, as above, we can find a balanced neigbourhood O_A, a number $n \in \mathbb{N}$, a neighbourhood O''_C, elements $c_1, ..., c_n \in C$, $b_1, ..., b_n \in B$ and numbers $\epsilon_1, \ldots, \epsilon_k > 0$ such that $\Psi_\phi(O_A) \subset O'_C$ for each $\phi \in \text{hom}B$,

$$\Big(\sum_{k=1}^{n} c_k b_k\Big) - a \in O_A,$$

$$\underbrace{O''_C + \cdots + O''_C}_{n} \subset O'_C$$

and $c_k O_{\epsilon_k} \subset O''_C$ for each $k \in \{1, \ldots, n\}$. Since every $b_k\tilde{}$ vanishes at infinity, then every $k \in \{1, \ldots, n\}$ defines a compact set $K_k \subset \text{hom}B$ such that $\phi(b_k) = b_k\tilde{}(\phi) \subset O_{\epsilon_k}$ for each $\phi \in \text{hom}B \backslash K_k$. Let

$$K = \bigcup_{k=1}^{n} K_k.$$

Then K is compact subset of hom$B$ and hom$B$ is not compact. Thus $K \neq \text{hom}B$. Let now $\phi \in \text{hom}B\backslash K$. Then $\phi \in \text{hom}B\backslash K_k$ for each $k \in \{1, \ldots, n\}$. Therefore

$$\hat{a}(\phi) = \Psi_\phi(a) = \Psi_\phi\Big(\Big(\sum_{k=1}^{n} c_k b_k\Big) - a\Big) + \Psi_\phi\Big(\sum_{k=1}^{n} c_k b_k\Big) \subset$$

$$\subset \Psi_\phi(O_A) + \sum_{k=1}^{n} c_k \phi(b_k) \subset$$

$$\subset O'_C + c_1 O_{\epsilon_1} + \ldots + c_n O_{\epsilon_n} \subset O'_C + \underbrace{(O''_C + \ldots + O''_C)}_{n} \subset O'_C + O'_C \subset O_C.$$

Thus $\hat{a}(\phi) \subset O_C$, if $\phi \in \mathrm{hom}B \backslash K$, which means that $\hat{a} \in C_0(\mathrm{hom}B, C)$.

In this particular case, $\mathcal{A}$ is a representation of A in $C_0(\mathrm{hom}B, A)$ (on $C_0(\mathrm{hom}B, A)$ is considered the topology, whose base of neighbourhoods of zero consists of sets $T(\mathrm{hom}B, O)$, where O is a neighbourhood of zero in C).

Chapter IV

Description of ideals in the algebra $C(X, A; \sigma)$ and its subalgebras

4.1. Properties of the algebra $C(X, A; \sigma)$

Let A be a topological algebra, X a topological space and σ a cover of X. We will find conditions for A, X and σ, for which $C(X, A; \sigma)$ is a locally pseudoconvex algebra, a locally m-pseudoconvex algebra, a Waelbroeck algebra or an exponentially galbed algebra.

Proposition 4.1. *Let X be a topological space, σ its cover and A a locally pseudoconvex algebra (locally m-pseudoconvex algebra) with jointly continuous multiplication. Then $C(X, A; \sigma)$ is also a locally pseudoconvex (locally m-pseudoconvex, respectively) algebra.*

Proof. Let X be a topological space, σ its cover and A a locally pseudoconvex (locally m-pseudoconvex) algebra with jointly continuous multiplication. Then A has a base

$$\mathcal{B}_A = \{U_\lambda : \lambda \in \Lambda\}$$

of neighbourhoods of zero, consisting of balanced pseudoconvex (balanced pseudoconvex and idempotent, respectively) sets. Let O be a neighbourhood of zero in $C(X, A; \sigma)$. Then there exist $n \in \mathbb{N}$, $S_1, ..., S_n \in \sigma$ and neighbourhoods $O_1, ..., O_n$ of zero of A such that

$$\bigcap_{k=1}^{n} T(S_k, O_k) \subset O.$$

For every k there exists a neighbourhood $U_{\lambda_k} \in \mathcal{B}_A$ such that $U_{\lambda_k} \subset O_k$. It is easy to see that

$$\Big\{\bigcap_{k=1}^{n} T(S_k, U_{\lambda_k}) : n \in \mathbb{N}, S_k \in \sigma, U_{\lambda_k} \in \mathcal{B}_A\Big\}$$

is such a base of neighbourhoods of zero of $C(X, A; \sigma)$, which consists of balanced pseudoconvex (and idempotent, if A is locally m-pseudoconvex) sets. Thus $C(X, A; \sigma)$ is a locally pseudoconvex (locally m-pseudoconvex, respectively) algebra.

Proposition 4.2. *Let X be a topological space, σ its cover and A a Waelbroeck algebra with jointly continuous multiplication. If $X \in \sigma$, then $C(X, A; \sigma)$ is also a Waelbroeck algebra.*

Proof. First,[12] we will show that

$$\mathrm{Qinv}C(X, A; \sigma) = C(X, \mathrm{Qinv}A; \sigma). \tag{4}$$

For it, let $f \in \mathrm{Qinv}C(X, A; \sigma)$. Then there exists an element $g \in C(X, A; \sigma)$ such that $f + g - fg = \theta_{C(X,A;\sigma)}$ or $f(x) + g(x) - f(x)g(x) = \theta_A$ for each $x \in X$. Thus $f(x) \in \mathrm{Qinv}A$ for each $x \in X$ which means that $f \in C(X, \mathrm{Qinv}A; \sigma)$. Therefore $\mathrm{Qinv}C(X, A; \sigma) \subset C(X, \mathrm{Qinv}A; \sigma)$. Let now $f \in C(X, \mathrm{Qinv}A; \sigma)$. Then from $f(x) \in \mathrm{Qinv}A$ for each $x \in X$ follows that $f(x)_q^{-1}$ exists in A for each $x \in X$. Let p_A^q denote the quasi-inversion in A and $h = p_A^q \circ f$. Since A is a Waelbroeck algebra, then p_A^q is a homeomorphism from $\mathrm{Qinv}A$ onto $\mathrm{Qinv}A$. Hence h is a continuous map on X and $\mathrm{cl}_A(h(S)) = \mathrm{cl}_A(p_A^q(f(S))) = p_A^q(\mathrm{cl}_A(f(S)))$ is compact in A for each $S \in \sigma$. Thus $h \in C(X, A; \sigma)$ is a quasi-inverse of f. Consequently, $f \in \mathrm{Qinv}C(X, A; \sigma)$. Therefore condition (4) is true.

Now from $X \in \sigma$ follows that $C(X,\mathrm{Qinv}A; \sigma) = T(X,\mathrm{Qinv}A)$ is an open subset of $C(X, A; \sigma)$. Hence $C(X, A; \sigma)$ is a Q-algebra. To show that the quasi-inversion is continuous in $C(X, A; \sigma)$, it is sufficient (see Lemma 1.9) to show that the quasi-inversion is continuous at $f = \theta$. For it, let $O(\theta)$ be a neighbourhood of θ in $\mathrm{Qinv}C(X, A; \sigma)$. Then $O(\theta) = U(\theta) \cap \mathrm{Qinv}C(X, A; \sigma)$ for some neighbourhood $U(\theta)$ of θ in $C(X, A; \sigma)$ and there exist $n \in \mathbb{N}$, $S_1, ..., S_n \in \sigma$ and neighbourhoods $O_1, ..., O_n$ of zero in A such that

$$\bigcap_{k=1}^{n} T(S_k, O_k) \subset U(\theta).$$

Since p_A^q is a homeomorphism from $\mathrm{Qinv}A$ onto $\mathrm{Qinv}A$, then

$$U_k = (p_A^q)^{-1}(O_k \cap \mathrm{Qinv}A)$$

is a neighbourhood of θ_A in $\mathrm{Qinv}A$. Therefore every $U_k = V_k \cap \mathrm{Qinv}A$ for some neighbourhood V_k of zero in A. Let now

$$W = \Big(\bigcap_{k=1}^{n} T(S_k, V_k)\Big) \cap \mathrm{Qinv}C(X, A; \sigma).$$

[12]The Proof of Proposition 4.2 in unital case is similar, using Lemma 1.8 instead of Lemma 1.9, because a unital topological algebra is a Waelbroeck algebra if and only if $\mathrm{Inv}A$ is open in A and the inversion is continuous in A (see, e.g., [21], Propositions 3.6.5 and 3.6.10).

Then W is a neighbourhood of θ in Qinv$C(X,A;\sigma)$. If $f \in W$, then from

$$f \in \bigcap_{k=1}^{n} T(S_k, V_k)$$

follows that $f(S_k) \in V_k$ for each $k \in \{1,...,n\}$. Therefore

$$f(x) \in V_k \cap \mathrm{Qinv}A = U_k,$$

by (4), and

$$f_q^{-1}(x) = p_A^q(f(x)) \in p_A^q(U_k) \subset O_k \cap \mathrm{Qinv}A$$

for each $k \in \{1,...,n\}$ and $x \in S_k$. Hence

$$f_q^{-1} \in \bigcap_{k=1}^{n} T(S_k, O_k)$$

or $f_q^{-1} \in O(\theta)$, whenever $f \in W$. Consequently, the quasi-inversion in $C(X,A;\sigma)$ is continuous and $C(X,A;\sigma)$ is a Waelbroeck algebra.

Proposition 4.3. *Let X be a topological space, σ its cover and A an exponentially galbed algebra with jointly continuous multiplication. Then $C(X,A;\sigma)$ is an exponentially galbed algebra.*

Proof. Let O be any neighbourhood of zero in $C(X,A;\sigma)$. Similarily, as in the proof of Proposition 4.1, we have

$$\bigcap_{k=1}^{n} T(S_k, O_k) \subset O$$

for some $n \in \mathbb{N}$, $S_1,...,S_n \in \sigma$ and neighbourhoods $O_1,...,O_n$ of zero in A. Since A is exponentially galbed, then we can find for every O_k a neighbourhood U_k of zero in A such that

$$\Big\{\sum_{l=0}^{m} \frac{a_l}{2^l} : a_0,...,a_m \in U_k\Big\} \subset O_k$$

for each $m \in \mathbb{N}$. Let

$$U = \bigcap_{k=1}^{n} T(S_k, U_k).$$

Then U is a neighbourhood of zero in $C(X,A;\sigma)$ and $f(x) \in U_k$ for each $f \in U$ and $x \in S_k$. Therefore

$$\Big(\sum_{l=0}^{m} \frac{f_l}{2^l}\Big)(x) = \sum_{l=0}^{m} \frac{f_l(x)}{2^l} \in O_k$$

for each $m \in \mathbb{N}$, $k \in \{1, ..., n\}$, $x \in S_k$ and $f_0, ..., f_m \in U$. Thus

$$\Big\{\sum_{l=0}^{m} \frac{f_l}{2^l} : f_0, ..., f_m \in U\Big\} \subset \bigcap_{k=1}^{n} T(S_k, O_k) \subset O$$

which means that $C(X, A; \sigma)$ is an exponentially galbed algebra.

To describe all closed maximal regular left (right or two-sided) ideals of $C(X, A; \sigma)$ and its subalgebras, we need the following results.

Lemma 4.4. *Let X be a completely regular Hausdorff space and σ a compact cover*[13] *of X, which is closed with respect to finite unions. Then every $\phi \in \hom C(X, \mathbb{K}; \sigma)$ defines an element $x_\phi \in X$ such that $\phi = \phi_{x_\phi}$, where $\phi_{x_\phi}(\alpha) = \alpha(x_\phi)$ for each $\alpha \in C(X, \mathbb{K}; \sigma)$.*

Proof. See [10], the proof of Theorem 2 v) in case of a compact cover.

Lemma 4.5. *Let A be a topological Hausdorff algebra and $a \in A\backslash\{\theta_A\}$. Then $\nu_a : \mathbb{K} \to A$, defined by $\nu_a(\lambda) = \lambda a$ for each $\lambda \in \mathbb{K}$, is a homeomorphism.*

Proof. It is clear that ν_A is a continuous bijection. To show the continuity of ν_a^{-1}, let O be a neighbourhood of zero in $\mathbb{K}$. Then there exists $\epsilon > 0$ such that

$$O_\epsilon = \{\lambda \in \mathbb{K} : |\lambda| \leqslant \epsilon\} \subset O.$$

If $\lambda_0 \in O_\epsilon\backslash\{0\}$, then $\lambda_0 a \neq \theta_A$. Since A is a Hausdorff space, then there exists a neighbourhood O_A of θ_A such that $\lambda_0 a \notin O_A$. Let V_A be a balanced neighbourhood of zero in A such that $V_A \subset O_A$. Now $O' = V_A \cap (\mathbb{K}a)$ is a neighbourhood of zero in $\mathbb{K}a$. If $\lambda a \in V_A$ and $|\lambda_0| \leqslant |\lambda|$, then $|\lambda_0\lambda^{-1}| \leqslant 1$ and $\lambda_0 a = (\lambda_0\lambda^{-1})\lambda a \subset V_A$, which is a contradiction. Therefore from $\lambda a \in O'$ follows $\lambda \in O_\epsilon \subset O$ which means that ν_a^{-1} is continuous. Consequently, ν_a is a homeomorphism.

4.2. Description of ideals in subalgebras of $C(X, A; \sigma)$

Let $\mathfrak{A}(X, A; \sigma)$ be a subalgebra of $C(X, A; \sigma)$, endowed with the subset topology.

Lemma 4.6. *Let X be a topological space, σ its cover and A a topological algebra with jointly continuous multiplication. If*

$$\{f(x) : f \in \mathfrak{A}(X, A; \sigma)\} = A$$

[13] I.e., every $S \in \sigma$ is a compact subset of X.

for each $x \in X$, then

$$Z(\mathfrak{A}(X, A; \sigma)) = \mathfrak{A}(X, A; \sigma) \cap C(X, Z(A); \sigma).$$

Proof. Since $\mathfrak{A}(X, A; \sigma) \subset C(X, A; \sigma)$, then

$$Z(\mathfrak{A}(X, A; \sigma)) \supset Z(C(X, A; \sigma)) =$$

$$= C(X, Z(A); \sigma) \supset \mathfrak{A}(X; A; \sigma) \cap C(X, Z(A); \sigma).$$

Next, we show that $C(X, Z(A); \sigma) \supset Z(\mathfrak{A}(X, A; \sigma))$. For it, let $x \in X$ and $g \in Z(\mathfrak{A}(X, A; \sigma))$. Every $a \in A$ defines a function $f_a \in \mathfrak{A}(X, A; \sigma)$ such that $f_a(x) = a$. Since $gf_a = f_a g$ for each $a \in A$, then $(gf_a)(x) = (f_a g)(x)$ or $g(x)a = ag(x)$ for each $a \in A$. Thus $g(x) \in Z(A)$ and $g \in C(X, Z(A); \sigma)$, which implies that $C(X, Z(A); \sigma) \supset Z(\mathfrak{A}(X, A; \sigma))$. Hence

$$Z(\mathfrak{A}(X, A; \sigma)) \subset \mathfrak{A}(X, A; \sigma) \cap C(X, Z(A); \sigma)$$

and our proof is complete.

Lemma 4.7. *Let X be a topological space, σ its cover and A a topological algebra with jointly continuous multiplication. If*

$$\{f_a : a \in A, f_a(x) = a \text{ for each } x \in X\} \subset \mathfrak{A}(X, A; \sigma), \tag{5}$$

then

$$\mathcal{M} = \mathcal{M}_{x,J} = \{f \in \mathfrak{A}(X, A; \sigma) : f(x) \in J\}$$

is a closed maximal regular left (right or two-sided) ideal of $\mathfrak{A}(X, A; \sigma)$ for each $x \in X$ and a closed maximal regular left (right or two-sided, respectively) ideal J of A.

Proof. Let $x \in X$, $J \in m_l(A)$ ($J \in m_r(A)$ or $J \in m_t(A)$) and let u be a right (left or two-sided, respectively) regular unit for J. Then it is easy to see that f_u (here $f_u(x) = u$ for each $x \in X$) is a right (left or two-sided, respectively) regular unit for $\mathcal{M}_{x,J}$. Therefore $\mathcal{M}_{x,J}$ is a left (right or two-sided, respectively) regular ideal in $\mathfrak{A}(X, A; \sigma)$. If $f_0 \in \mathrm{cl}_{\mathfrak{A}(X,A;\sigma)}(\mathcal{M}_{x,J})$, then there is a net $(f_\lambda)_{\lambda \in \Lambda}$ of elements of $\mathcal{M}_{x,J}$, which converges to f_0. Let $\epsilon_x : \mathfrak{A}(X, A; \sigma) \to A$ be a homomorphism, defined by $\epsilon_x(f) = f(x)$ for each $f \in \mathfrak{A}(X, A; \sigma)$. Since $\epsilon_x(T(S, O)) \subset O$ for each neighbourhood O of zero in A and a set $S \in \sigma$ such that $x \in S$, then ϵ_x is continuous. Therefore $(\epsilon_x(f_\lambda))_{\lambda \in \Lambda}$ converges to $\epsilon_x(f_0) \in J$, because J is closed. Thus $f_0 \in \mathcal{M}_{x,J}$ which means that $\mathcal{M}_{x,J}$ is a closed ideal.

Next, we show that $\mathcal{M}_{x,J}$ is maximal. For it, let I be a left (right or two-sided, respectively) ideal of $\mathfrak{A}(X, A; \sigma)$ such that $\mathcal{M}_{x,J} \subset I$. Then

$\epsilon_x(\mathcal{M}_{x,J}) \subset \epsilon_x(I)$ and $\epsilon_x(\mathcal{M}_{x,J}) = J$, by the condition (5). If $\epsilon_x(I) = A$, then there exists an element $g \in I$ such that $u = \epsilon_x(g)$. Now

$$\epsilon_x(f_u - g) = f_u(x) - \epsilon_x(g) = \theta_A \in J$$

which means that $f_u - g \in \mathcal{M}_{x,J} \subset I$. Hence $f_u = (f_u - g) + g \in I$, which is not possible. Therefore $\epsilon_x(I) \neq A$ and $\epsilon_x(I)$ is a left (right or two-sided, respectively) ideal of A. Since J is a maximal left (right or two-sided, respectively) ideal in A, then $J = \epsilon_x(I)$. Because of it,

$$I \subset \epsilon_x^{-1}(\epsilon_x(I)) = \epsilon_x^{-1}(J) = \mathcal{M}_{x,J} \subset I.$$

Thus $I = \mathcal{M}_{x,J}$. Consequently, $\mathcal{M}_{x,J}$ is a closed maximal regular left (right or two-sided, respectively) ideal of $\mathfrak{A}(X, A; \sigma)$.

Next, we will consider only two classes of topological algebras A over $\mathbb{C}$:

I) A has a unit e_A;

II) A has no unit but there exists a nonzero idempotent $i_A \in Z(A)$.

Let X be a topological space, σ its cover and A a topological algebra over $\mathbb{C}$. For each $\alpha \in C(X, \mathbb{C})$ and $a \in A$, let αa be a map, defined by $(\alpha a)(x) = \alpha(x)a$ for all $x \in X$,

$$\mathfrak{A}_a(X, \mathbb{C}; \sigma) = \{\alpha \in C(X, \mathbb{C}) : \alpha a \in \mathfrak{A}(X, A; \sigma)\}$$

and $\mathfrak{A}(X, \mathbb{C}; \sigma) = \mathfrak{A}_{e_A}(X, \mathbb{C}; \sigma)$, if A is a unital algebra.

Lemma 4.8. *Let X be a topological space, σ its cover, closed with respect to finite unions, A a topological algebra over $\mathbb{C}$, $a \in A \backslash \{\theta_A\}$ and $\mathfrak{A}_a = \mathfrak{A}_a(X, \mathbb{C}; \sigma)a$. Then the map μ_a, defined by $\mu_a(\alpha) = \alpha a$ for each $\alpha \in \mathfrak{A}_a(X, \mathbb{C}; \sigma)$, is a topological isomorphism between $\mathfrak{A}_a(X, \mathbb{C}; \sigma)$ and $\mathfrak{A}_a$.*

Proof. It is clear that μ_a is a bijection between the sets $\mathfrak{A}_a(X, \mathbb{C}; \sigma)$ and $\mathfrak{A}_a$. Next, we show that both μ_a and μ_a^{-1} are continuous maps. For it, let O be a neighbourhood of zero in $\mathfrak{A}_a$. Then $O = O' \cap \mathfrak{A}_a$, where O' is a neighbourhood of zero in $\mathfrak{A}(X, A; \sigma)$. Now there exist $S \in \sigma$, neighbourhood O_A of zero of A and neighbourhood $O_\epsilon = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \epsilon\}$ of zero of $\mathbb{C}$ such that $(T(S, O_A) \cap \mathfrak{A}(X, A; \sigma)) \subset O'$ and $O_\epsilon a \subset O_A$. Since $T(S, O_\epsilon) \cap \mathfrak{A}_a(X,\mathbb{C}; \sigma)$ is a neighbourhood of zero in $\mathfrak{A}_a(X, \mathbb{C}; \sigma)$ and

$$\mu_a(T(S, O_\epsilon) \cap \mathfrak{A}_a(X, \mathbb{C}; \sigma)) \subset T(S, O_\epsilon a) \cap \mathfrak{A}_a \subset$$

$$\subset [T(S, O_\epsilon a) \cap \mathfrak{A}(X, A; \sigma)] \cap \mathfrak{A}_a \subset O' \cap \mathfrak{A}_a = O,$$

then μ_a is continuous.

Let now O be a neighbourhood of zero in $\mathfrak{A}_a(X,\mathbb{C};\sigma)$. Then there exist $S \in \sigma$ and $\epsilon > 0$ such that $(T(S,O_\epsilon) \cap \mathfrak{A}_a(X,\mathbb{C};\sigma)) \subset O$. Because $O_\epsilon a = \nu_a(O_\epsilon)$ and ν_a is a homeomorphism, by Lemma 4.5, then $O_\epsilon a$ is a neighbourhood of zero in $\nu_a(\mathbb{C})$. Hence there exists a neighbourhood O_A of zero in A such that $\nu_a(O_\epsilon) = O_A \cap \nu_a(\mathbb{C})$. Therefore $O' = T(S,O_A) \cap \mathfrak{A}_a$ is a neighbourhood of zero in $\mathfrak{A}_a$ and $\mu_a^{-1}(O') \subset T(S,O_\epsilon) \cap \mathfrak{A}_a(X,\mathbb{C};\sigma) \subset O$ which means that μ_a^{-1} is continuous.

Theorem 4.9. *Let X be a completely regular Hausdorff space, σ a compact cover of X, which is closed with respect to finite unions, A a topological algebra over $\mathbb{C}$ with jointly continuous multiplication, i_A a nonzero idempotent in $Z(A)$, $\mathfrak{A}(X,A;\sigma)$ a subalgebra of $C(X,A;\sigma)$ (with the same unit, if A is a unital algebra) and let the following conditions be true:*

1) *Either*

a) *A is a locally m-pseudoconvex Hausdorff algebra;*

b) *A is a locally pseudoconvex Waelbroeck Hausdorff algebra, $X \in \sigma$ and $\mathfrak{A}(X,A;\sigma)$ is a Q-algebra with $m(\mathfrak{A}(X,A;\sigma)) \neq \emptyset$;*

or

c) *A is an exponentially galbed Hausdorff algebra and all elements of $\mathfrak{A}(X,A;\sigma)$ are bounded.*

2) *$\{f_a : a \in A, f_a(x) = a$ for each $x \in X\} \subset \mathfrak{A}(X,A;\sigma)$;*

3) *every[14] $\phi \in \hom\mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)$ defines an element $x_\phi \in X$ such that $\phi = \phi_{x_\phi}$, where $\phi_{x_\phi}(\alpha) = \alpha(x_\phi)$ for each $\alpha \in \mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)$;*

4) *$\alpha f \in \mathfrak{A}(X,A;\sigma)$ for each $\alpha \in C(X,\mathbb{C})$ and $f \in \mathfrak{A}(X,A;\sigma)$;*

5) *If A is a nonunital algebra, then*

$$\mathrm{cl}_{\mathfrak{A}(X,A;\sigma)}(A \otimes \mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)i_A) = \mathfrak{A}(X,A;\sigma).$$

Then every closed maximal left (right or two-sided) ideal $\mathcal{M}$ of $\mathfrak{A}(X,A;\sigma)$ is representable in the form $\mathcal{M} = \mathcal{M}_{x,J}$ for some $x \in X$ and a closed maximal left (right or two-sided, respectively) ideal J of A.

Proof. We give the proof only for left ideals (the proof for right and two-sided ideals is similar) and separately for both unital and nonunital case.

I) A **is a unital algebra.** Let X, σ and A be as in the formulation of Theorem 4.9 and $B = \mathrm{cl}_{Z(\mathfrak{A}(X,A;\sigma))}(\mathfrak{A}(X,\mathbb{C};\sigma)e_A)$. Then B is a closed subalgebra of $Z(\mathfrak{A}(X,A;\sigma))$. The map μ_{e_A}, defined by $\mu_{e_A}(\alpha) = \alpha e_A$ for

[14] Here and in the sequel $i_A = e_A$ and $\mathfrak{A}_{i_A}(X,\mathbb{C};\sigma) = \mathfrak{A}(X,\mathbb{C};\sigma)$, if A is a unital algebra.

each $\alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma)$, is a topological isomorphism of $\mathfrak{A}(X, \mathbb{C}; \sigma)$ into B, by Lemma 4.8. Let τ denote the topology on $\mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))$, induced by the topology of $Z(\mathfrak{A}(X, A; \sigma))$. Since σ is closed with respect to finite unions, then every element of a base of neighbourhoods of zero in $\mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))$ has the form

$$U = \{\alpha e_A : \alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma),\ \alpha(S)e_A \subset O_{Z(A)} \cap \mathbb{C}e_A\}$$

for some $S \in \sigma$ and neighbourhood $O_{Z(A)}$ of zero in $Z(A)$, by Lemma 4.6. Since the map $\lambda \mapsto \lambda e_A$ is continuous (see Lemma 4.5), then there exists a number $\epsilon \in (0,1)$ such that $O_\epsilon e_A \subset O_{Z(A)} \cap \mathbb{C}e_A$, where $O_\epsilon = \{\lambda \in \mathbb{C} : |\lambda| < \epsilon\}$. It is easy to see that

$$\{T(S, O_\epsilon e_A) \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma)) : S \in \sigma, \epsilon > 0\}$$

is a base of neighbourhoods of zero in $\mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))$ in the topology τ. Because every $T(S, O_\epsilon e_A) \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))$ is an idempotent and absolutely convex set, then $(\mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma)), \tau)$ is a commutative locally m-convex algebra over $\mathbb{C}$. Hence B is a commutative locally m-convex algebra over $\mathbb{C}$ (if p is a homogeneous submultiplicative seminorm on $\mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))$, then (as in [30], Proposition 4, p. 129) its extension p' onto B is a homogenuous submultiplicative seminorm on B). Consequently, B is a commutative Gelfand-Mazur algebra, by Lemma 1.11. Therefore every $M \in m(B)$ defines a map $\psi_M \in \mathrm{hom}B$ such that $M = \ker\psi_M$. Since μ_{e_A} is a topological isomorphism from $\mathfrak{A}(X, \mathbb{C}; \sigma)$ into B (by Lemma 4.8) and $\mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))$ is dense in B, then $\psi_M \circ \mu_{e_A} \in \mathrm{hom}\mathfrak{A}(X, \mathbb{C}; \sigma)$. By the condition 3), there exists a unique element $x_0 \in X$ such that $\psi_M \circ \mu_{e_A} = \phi_{x_0}$, where $\phi_{x_0}(\alpha) = \alpha(x_0)$ for each $\alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma)$. Now

$$\mu_{e_A}(\ker\phi_{x_0}) = \mu_{e_A}(\overset{-1}{\mu_{e_A}}(\ker\psi_M)) =$$

$$= \ker\psi_M \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma)) = M \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma)).$$

Therefore

$$\mathrm{cl}_B(\mu_{e_A}(\ker\phi_{x_0})) = \mathrm{cl}_B(M \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{C}; \sigma))) = M,$$

by Lemma 1.10. Hence every M defines an element $x \in X$ such that

$$M = M_x = \mathrm{cl}_B(\{\alpha e_A : \alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma),\ \alpha(x) = 0\}).$$

For each $x \in X$ let ϵ_x be the map from $\mathfrak{A}(X, A; \sigma)$ into A, defined as in the proof of Lemma 4.7,

$$\kappa_{M_x} : \mathfrak{A}(X, A; \sigma) \to Y = \mathfrak{A}(X, A; \sigma)/I(M_x)$$

the quotient map and $\delta_x : Y \to A$ the map, defined by $\delta_x(\kappa_{M_x}(f)) = \epsilon_x(f)$ for each $x \in X$ and $f \in \mathfrak{A}(X, A; \sigma)$. To show that δ_x is well defined, we show that $\ker\epsilon_x = I(M_x)$. For it, let $f \in I(M_x)$. If we define the multiplication over A in $\mathfrak{A}(X, A; \sigma)$ by $(af)(x) = af(x)$ for each $x \in X$ and $a \in A$, then $M_x \subset \ker\epsilon_x$ (because ϵ_x is continuous and A is a Hausdorff space) and

$$\epsilon_x\Big(\sum_{k=1}^{n} a_k m_k\Big) = \theta_A$$

for each $n \in \mathbb{N}$, each $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in M_x$. Hence $\epsilon_x(I(M_x)) = \theta_A$ or $I(M_x) \subset \ker\epsilon_x$ for all $x \in X$.

Next, we show that $I(M_x) \supset \ker\epsilon_x$ for each $x \in X$. For it, let $x_0 \in X$ and $f \in \ker\epsilon_{x_0}$. Let $O(f)$ be any neighbourhood of f in $\mathfrak{A}(X, A; \sigma)$. Since σ is closed with respect to finite unions, then

$$f + (T(S_0, O_0) \cap \mathfrak{A}(X, A; \sigma)) \subset O(f)$$

for some $S_0 \in \sigma$ and a balanced neighbourhood O_0 of zero of A. Now there exists an open neighbourhood O' of zero such that $O' \subset O_0$. Then $X_{O'} = X \backslash f^{-1}(O')$ is closed in X and $x_0 \notin X_{O'}$, because $f(x_0) \in O'$. By assumption, X is a completely regular space. Therefore there exists $\alpha \in C(X, [0, 1]) \subset C(X, \mathbb{C})$ such that $\alpha(x_0) = 0$ and $\alpha(X_{O'}) = \{1\}$.

Let now $x \in S_0$. If $x \in X_{O'}$, then $(\alpha f - f)(x) = (\alpha(x) - 1)f(x) \in O_0$. If $x \notin X_{O'}$, then $x \in f^{-1}(O')$ and

$$(\alpha f - f)(x) = (\alpha(x) - 1)f(x) \in (\alpha(x) - 1)O' \subset O_0,$$

because $|\alpha(x) - 1| \leqslant 1$ and O_0 is a balanced set. Therefore

$$\alpha f - f \in T(S_0, O_0) \cap \mathfrak{A}(X, A; \sigma),$$

by the condition 4), and thus $\alpha f \in O(f)$. Since $\alpha f = f\alpha e_A \in I(M_{x_0})$, then $I(M_{x_0}) \cap O(f) \neq \emptyset$. Consequently, $f \in I(M_{x_0})$, which implies that $I(M_{x_0}) \supset \ker\epsilon_{x_0}$. Hence $I(M_{x_0}) = \ker\epsilon_{x_0}$. Since every $M \in m(B)$ defines an element $x \in X$ such that $M = M_x$ and $\ker\epsilon_x \neq \mathfrak{A}(X, A; \sigma)$, by the condition 2), then every closed maximal ideal in B is extendible.

It is easy to see that $C(X, A; \sigma)$ is a Hausdorff algebra, if A is a Hausdorff algebra. By the assumptions of Theorem 4.9 and Propositions 4.1, 4.2 and 4.3, we see that in the present case $C(X, A; \sigma)$ is a locally m-pseudoconvex Hausdorff algebra, a locally pseudoconvex Waelbroeck Hausdorff algebra or an exponentially galbed Hausdorff algebra. Now $\mathfrak{A}(X, A; \sigma)$, as a subalgebra of $C(X, A; \sigma)$, satisfies at least one of the conditions a) or c) of Theorem 2.17 or a condition of Corollary 2.19. Therefore (see the proof of Theorem 3.6) every closed maximal left ideal $\mathcal{M}$

of $\mathfrak{A}(X,A;\sigma)$ has the form $\mathcal{M} = \kappa_{M_x}^{-1}(\kappa_{M_x}(\mathcal{M}))$ for some $x \in X$ and $M_x \in m(B)$ (because every ideal of $m(B)$ is extendible). Hence

$$\mathcal{M} = \{f \in \mathfrak{A}(X,A;\sigma) : f(x) \in \epsilon_x(\mathcal{M})\}.$$

If $\epsilon_x(\mathcal{M}) = A$, then there exists an element $g \in \mathcal{M}$ such that $\epsilon_x(g) = e_A$. Now $hg \in \mathcal{M}$ for each $h \in \mathfrak{A}(X,A;\sigma)$. Therefore from

$$\epsilon_x(h) = h(x)e_A = (hg)(x) = \epsilon_x(hg)$$

follows $h - hg \in \ker\epsilon_x = I(M_x) \subset \mathcal{M}$, because $\kappa_{M_x}(\mathcal{M})$ is an ideal in Y, by Lemma 3.5. Thus $h \in \mathcal{M}$, which gives us a contradiction. Hence $\epsilon_x(\mathcal{M})$ is a left ideal in A. Let I be a left ideal in A such that $\epsilon_x(\mathcal{M}) \subset I$. Then

$$\mathcal{M} \subset \epsilon_x^{-1}(\epsilon_x(\mathcal{M})) \subset \epsilon_x^{-1}(I) \neq \mathfrak{A}(X,A;\sigma)$$

and $\epsilon_x^{-1}(I)$ is a left ideal in $\mathfrak{A}(X,A;\sigma)$. Since $\mathcal{M} \subset \epsilon_x^{-1}(I)$ and $\mathcal{M}$ is a maximal left ideal in $\mathfrak{A}(X,A;\sigma)$, then $\mathcal{M} = \epsilon_x^{-1}(I)$ or $\epsilon_x(\mathcal{M}) = I$. Consequently, $\epsilon_x(\mathcal{M})$ is a maximal left ideal in A.

Next, we prove that $\epsilon_x(\mathcal{M})$ is closed. For it, let a_0 be an arbitrary element of $\mathrm{cl}_A(\epsilon_x(\mathcal{M}))$. Then there exists a net $(m_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{M}$ such that $\epsilon_x(m_\lambda)$ converges to a_0. Let $\rho : A \to C(X,A;\sigma)$ be a map, defined by $(\rho(a))(x) = a$ for every $x \in X$ and $a \in A$. Then ρ is continuous (because ρ is linear and $\rho(O) \subset T(S,O)$ for each neighbourhood O of zero in A and $S \in \sigma$). Therefore $\rho(\epsilon_x(m_\lambda))$ converges to $\rho(a_0)$. Since

$$\epsilon_x[\rho(\epsilon_x(m_\lambda))] = (\rho(\epsilon_x(m_\lambda)))(x) = \epsilon_x(m_\lambda)$$

and δ_x is a one-to-one map, then $\kappa_{M_x}[\rho(\epsilon_x(m_\lambda))] = \kappa_{M_x}(m_\lambda)$ for each $\lambda \in \Lambda$. Thus

$$\rho(\epsilon_x(m_\lambda)) \in \kappa_{M_x}^{-1}[\kappa_{M_x}(\rho(\epsilon_x(m_\lambda)))] = \kappa_{M_x}^{-1}[\kappa_{M_x}(m_\lambda)] \in \mathcal{M}$$

for each $\lambda \in \Lambda$. Hence $\rho(a_0) \in \mathcal{M}$, because the ideal $\mathcal{M}$ is closed in $\mathfrak{A}(X,A;\sigma)$. Therefore $a_0 = \epsilon_x[\rho(a_0)] \in \epsilon_x(\mathcal{M})$. So we have proved that $\mathrm{cl}_A(\epsilon_x(\mathcal{M})) = \epsilon_x(\mathcal{M})$ which means that $J = \epsilon_x(\mathcal{M})$ is a closed maximal left ideal of A. Consequently, $\mathcal{M} = \mathcal{M}_{x,J}$ for some $x \in X$ and $J \in m_l(A)$.

II) *A* **has no unit, but there exists a nonzero idempotent** $i_A \in Z(A)$. Let X, A and σ be as in the formulation of Theorem 4.9, $i_A \in Z(A)$, μ a map from $C(X,A;\sigma)$ into $C(X,A;\sigma) \times \mathbb{C}$, defined by $\mu(f) = (f,0)$ for each $f \in C(X,A;\sigma)$, μ_{i_A} a map from $\mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)$ into $\mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)i_A$, defined by $\mu_{i_A}(\alpha) = \alpha i_A$ for each $\alpha \in \mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)$, and $B = \mathrm{cl}_{Z(\mathfrak{A}(X,A;\sigma))}(\mu_{i_A}(\mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)))$. Then μ is a homeomorphism,

$$\mu(B) = B \times \{0\} \subset Z(\mathfrak{A}(X,A;\sigma)) \times \mathbb{C} = Z(\mathfrak{A}(X,A;\sigma) \times \mathbb{C})$$

and $B \times \{0\}$ is a closed subalgebra of $Z(\mathfrak{A}(X, A; \sigma) \times \mathbb{C})$.

If $M \in m(B \times \{0\})$, then $\mu^{-1}(M) \in m(B)$, because μ is a topological isomorphism. Similarily, as in the part I), we get that $\mu_{i_A}(\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma))$ and B are commutative Gelfand-Mazur algebras. Hence $\mu^{-1}(M) = \ker\psi_M$, where $\psi_M \in \hom B$. As in the part I) (using here i_A instead of e_A and $\mu^{-1}(M)$ instead of M), we see that $\psi_M \circ \mu_{i_A} \in \hom\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma)$ (because μ_{i_A} is a topological isomorphism from $\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma)$ into B, by Lemma 4.8, and $\mu_{i_A}(\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma))$ is dense in B). By the condition 3), there exists a unique element $x_0 \in X$ such that $\psi_M \circ \mu_{i_A} = \phi_{x_0}$, where $\phi_{x_0}(\alpha) = \alpha(x_0)$ for each $\alpha \in \mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma)$. Now

$$\mu_{i_A}(\ker\phi_{x_0}) = \mu_{i_A}(\mu_{i_A}^{-1}(\ker\psi_M)) =$$

$$= \ker\psi_M \cap \mu_{i_A}(\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma)) = \mu^{-1}(M) \cap \mu_{i_A}(\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma)).$$

Therefore

$$\mathrm{cl}_B[\mu_{i_A}(\ker\phi_{x_0})] = \mathrm{cl}_B[\mu^{-1}(M) \cap \mu_{i_A}(\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma))] = \mu^{-1}(M),$$

by Lemma 1.10, or

$$M = \mu[\mathrm{cl}_B(\mu_{i_A}[\ker\phi_{x_0}])] = \mathrm{cl}_{B\times\{0\}}[\mu \circ \mu_{i_A}(\ker\phi_{x_0})].$$

Hence

$$M = \mathrm{cl}_{B\times\{0\}}\{(\mu \circ \mu_{i_A})(\alpha) : \alpha \in \mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma),\ \alpha(x_0) = 0\} =$$

$$= \mathrm{cl}_{B\times\{0\}}\{(\alpha i_A, 0) : \alpha \in \mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma),\ \alpha(x_0) = 0\}.$$

Thus, for every $M \in m(B \times \{0\})$ there exists an element $x \in X$ such that

$$M = M_x = \mathrm{cl}_{B\times\{0\}}\{(\alpha i_A, 0) : \alpha \in \mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma), \alpha(x) = 0\}.$$

Let now $x \in X$ and ξ_x be a homomorphism from $\mathfrak{A}(X, A; \sigma) \times \mathbb{C}$ into $A \times \mathbb{C}$, defined by $\xi_x(f, \lambda) = (f(x), \lambda)$ for each $(f, \lambda) \in \mathfrak{A}(X, A; \sigma) \times \mathbb{C}$. Let κ_{M_x} be the quotient map from $\mathfrak{A}(X, A; \sigma) \times \mathbb{C}$ onto

$$Y = (\mathfrak{A}(X, A; \sigma) \times \mathbb{C})/I(M_x)$$

and δ_x the map from Y into $A \times \mathbb{C}$, defined by $\delta_x(\kappa_{M_x}((f, \lambda))) = \xi_x(f, \lambda)$ for each $x \in X$ and $(f, \lambda) \in \mathfrak{A}(X, A; \sigma) \times \mathbb{C}$. To show that δ_x is well defined, we show that $I(M_x) = \ker\xi_x$. For it, let $f \in I(M_x)$. Since

$$\xi_x(T(S, O_A) \times O_{\mathbb{C}})) \subset O_A \times O_{\mathbb{C}}$$

for every neighbourhood O_A of zero in A, every neighbourhood $O_{\mathbb{C}}$ of zero in $\mathbb{C}$ and every $S \in \sigma$ such that $x \in S$, then ξ_x is a continuous map. Therefore $M_x \subset \ker\xi_x$ and

$$\xi_x\Big(\sum_{k=1}^{n}(a_k,\lambda_k)(\alpha_k,0)\Big) = \Big(\sum_{k=1}^{n}[(a_k\alpha_k)(x)+(\lambda_k\alpha_k)(x)],0\Big) = (\theta_A,0)$$

for each $n \in \mathbb{N}$, $(a_1,\lambda_1),\dots,(a_n,\lambda_n) \in A \times \mathbb{C}$ and $\alpha_1,\dots,\alpha_n \in \mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)$ such that $\alpha_k(x) = 0$ for each $k \in \{1,\dots,n\}$. Hence

$$\Big\{\sum_{k=1}^{n}(a_k,\lambda_k)m_k : n \in \mathbb{N}, (a_1,\lambda_1),\dots,(a_n,\lambda_n) \in A \times \mathbb{C},$$

$$m_1,\dots,m_n \in M_x\Big\} \subset \ker\xi_x,$$

because ξ_x is continuous. Since $\ker\xi_x$ is a closed set in $\mathfrak{A}(X,A;\sigma) \times \mathbb{C}$, then $I(M_x) \subset \mathrm{cl}_A(\ker\xi_x) = \ker\xi_x$ for each $x \in X$.

To show that $\ker\xi_x \subset I(M_x)$ for each $x \in X$, let $x_0 \in X$, $(f,0) \in \ker\xi_{x_0}$ and $O((f,0))$ be any neighbourhood of $(f,0)$ in $\mathfrak{A}(X,A;\sigma) \times \mathbb{C}$. As σ is closed with respect to finite unions, then

$$(f + [T(S_0,O_0) \cap \mathfrak{A}(X,A;\sigma)]) \times \{0\} \subset O(f,0)$$

for some $S_0 \in \sigma$ and some neighbourhood O_0 of zero of A. Since addition is continuous in $C(X,A;\sigma)$, then there exist $S_1 \in \sigma$ and a balanced neighbourhood O_1 of zero in A such that

$$T(S_1,O_1) + T(S_1,O_1) \subset T(S_0,O_0).$$

Now $\omega_x : C(X,A;\sigma) \to A$, defined by $\omega_x(f) = f(x)$, is a continuous map for each $x \in X$ and $\omega_{x_0}(f) = \theta_A$. Therefore there exist $S_2 \in \sigma$ and a neighbourhood O_2 of zero in A such that $x_0 \in S_2$ and $O_2 \subset O_1$. Hence $\omega_{x_0}(T(S_2,O_2)) \subset O_1$. Because (by the condition 5))

$$\mathrm{cl}_{\mathfrak{A}(X,A;\sigma)}(A \otimes \mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)i_A) = \mathfrak{A}(X,A;\sigma),$$

then there exist $n \in \mathbb{N}$, $a_1,\dots,a_n \in A$ and $\alpha_1,\dots,\alpha_n \in \mathfrak{A}_{i_A}(X,\mathbb{C};\sigma)$ such that

$$F = \sum_{k=1}^{n} a_k\alpha_k i_A - f \in [T(S_1,O_1) \cap T(S_2,O_2)] \cap \mathfrak{A}(X,A;\sigma).$$

For each $\lambda \in \mathbb{C}$, let $\alpha_\lambda \in C(X,\mathbb{C})$ be a map such that $\alpha_\lambda(x) = \lambda$ for all $x \in X$. Moreover, for each $k \in \{1,...,n\}$ let $\beta_k = \alpha_k - \alpha_{\alpha_k(x_0)}$. Then

$\beta_k \in C(X,\mathbb{C})$ and $\beta_k i_A = \beta_k f_{i_A}$, where $f_{i_A}(x) = i_A$ for all $x \in X$. Since $f_{i_A} \in \mathfrak{A}(X, A; \sigma)$, by the condition 2), then $\beta_k f_{i_A} \in \mathfrak{A}(X, A; \sigma)$, by the condition 4). As $\beta_k(x_0) = 0$, then $(\beta_k i_A, 0) \in M_{x_0}$,

$$\Big(\sum_{k=1}^{n} a_k \alpha_{\alpha_k(x_0)} i_A\Big)(x) = \sum_{k=1}^{n} a_k \alpha_k(x_0) i_A = \omega_{x_0}(F+f) \in \omega_{x_0}(T(S_2, O_2)) \subset O_1$$

for each $x \in S_1$ and

$$\sum_{k=1}^{n} a_k \beta_k i_A - f = F - \sum_{k=1}^{n} a_k \alpha_{\alpha_k(x_0)} i_A \in$$

$$\in [T(S_1, O_1) + T(S_1, O_1)] \cap \mathfrak{A}(X, A; \sigma) \subset T(S_0, O_0) \cap \mathfrak{A}(X, A; \sigma).$$

Therefore from

$$\Big(\sum_{k=1}^{n} a_k \beta_k i_A, 0\Big) \in$$

$$\in [(f + [T(S_0, O_0) \cap \mathfrak{A}(X, A; \sigma)]) \times \{0\}] \cap I(M_{x_0}) \subset O(f, 0) \cap I(M_{x_0})$$

follows $O(f,0) \cap I(M_{x_0}) \neq \emptyset$. Consequently, $(f, 0) \in I(M_{x_0})$ implies $I(M_{x_0}) = \ker\xi_{x_0}$. Since every closed maximal ideal $M \in m(B \times \{0\})$ defines an element $x \in X$ such that $M = M_x$ and $\ker\xi_x \neq \mathfrak{A}(X, A; \sigma) \times \mathbb{C}$, then every closed maximal ideal in $B \times \{0\}$ is extendible in $\mathfrak{A}(X, A; \sigma) \times \mathbb{C}$.

It is easy to see that

$$A \otimes \mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma) i_A \subset I(\mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma) i_A) \subset I(B) \subset \mathfrak{A}(X, A; \sigma).$$

Thus

$$\mathrm{cl}_{\mathfrak{A}(X,A;\sigma)}(A \otimes \mathfrak{A}_{i_A}(X, \mathbb{C}; \sigma) i_A) = \mathfrak{A}(X, A; \sigma)$$

implies $I(B) = \mathfrak{A}(X, A; \sigma)$. Now, by Lemma 1.17, Propositions 2.7, 2.9, 4.1, 4.2 and 4.3, we see that $C(X, A; \sigma) \times \mathbb{C}$ is a locally A-pseudoconvex Hausdorff algebra, a locally pseudoconvex Waelbroeck Hausdorff algebra or an exponentially galbed Hausdorff algebra. Therefore $\mathfrak{A}(X, A; \sigma) \times \mathbb{C}$, as its subalgebra, satisfies the conditions a) or c) of Theorem 2.17 or Remark B. Hence every closed maximal left ideal I of $\mathfrak{A}(X, A; \sigma) \times \mathbb{C}$ has the form $I = \kappa_{M_x}^{-1}(\kappa_{M_x}(I))$ for some $x \in X$ and $M_x \in m(B \times \{0\})$ (because every ideal in $m(B \times \{0\})$ is extendible). Thus we have

$$I = \{(f, \lambda) \in \mathfrak{A}(X, A; \sigma) \times \mathbb{C} : (f(x), \lambda) \in \xi_x(I)\}.$$

If $\xi_x(I) = A \times \mathbb{C}$, then there exists an element $(g, 1) \in I$ such that $\xi_x(g, 1) = (\theta_A, 1)$. Now $(h, \nu)(g, 1) \in I$ for an arbitrary element (h, ν) of $\mathfrak{A}(X, A; \sigma) \times \mathbb{C}$. Thus from

$$\xi_x(h, \nu) = (h(x), \nu) = (h(x), \nu)(\theta_A, 1) = \xi_x(h, \nu)\xi_x(g, 1) = \xi_x[(h, \nu)(g, 1)]$$

follows

$$(h,\nu)-(h,\nu)(g,1)\in \ker\xi_x = I(M_x)\subset \kappa_{M_x}^{-1}(\kappa_{M_x}(I)) = I$$

(because $\kappa_{M_x}(I)$ is an ideal in Y, by Lemma 3.5). Therefore $(h,\nu)\in I$ and $I=\mathfrak{A}(X,A;\sigma)\times\mathbb{C}$, which gives us a contradiction with assumption that I is a left ideal in $\mathfrak{A}(X,A;\sigma)\times\mathbb{C}$. Hence $\xi_x(I)$ is a left ideal in $A\times\mathbb{C}$. Similarily, as in the part I), we can show that $\xi_x(I)$ is maximal.

Next, we prove that $\xi_x(I)$ is closed. For it, let (a_0,μ_0) be an arbitrary element of $\mathrm{cl}_A(\xi_x(I))$. Then there exists a net $(f_\lambda,\mu_\lambda)_{\lambda\in\Lambda}$ in I such that $(\xi_x(f_\lambda,\mu_\lambda))_{\lambda\in\Lambda}$ converges to (a_0,μ_0) in the product topology of $A\times\mathbb{C}$.

For each $a\in A$, let $f_a\in C(X,A;\sigma)$ be again a map, defined by $f_a(x)=a$ for each $x\in X$, and $\rho: A\times\mathbb{C}\to C(X,A;\sigma)\times\mathbb{C}$ a map such that $\rho(a,\lambda)=(f_a,\lambda)$ for each $a\in A$ and $\lambda\in\mathbb{C}$. We show that ρ is continuous. For it, let O be a neighbourhood of zero of $C(X,A;\sigma)\times\mathbb{C}$. Then there exist neighbourhoods O_1 of zero of $C(X,A;\sigma)$ and O_2 of zero in $\mathbb{C}$ such that $O_1\times O_2\subset O$. Now we can find $S\in\sigma$ (because σ is closed with respect to finite unions) and a neighbourhood O_1' of zero of A such that $T(S,O_1')\subset O_1$. Obviously,

$$\rho(O_1'\times O_2)\subset T(S,O_1')\times O_2\subset O_1\times O_2\subset O$$

which means that ρ is a continuous map. Therefore $\rho(\xi_x(f_\lambda,\mu_\lambda))$ converges to $\rho(a_0,\mu_0)$. Since from

$$\xi_x[\rho(\xi_x(f_\lambda,\mu_\lambda))] = \xi_x[\rho(f_\lambda(x),\mu_\lambda)] = \xi_x(f_{f_\lambda(x)},\mu_\lambda) =$$

$$= (f_\lambda(x),\mu_\lambda) = \xi_x(f_\lambda,\mu_\lambda)$$

follows

$$\delta_x(\kappa_{M_x}[\rho(\xi_x(f_\lambda,\mu_\lambda))]) = \delta_x(\kappa_{M_x}[(f_\lambda,\mu_\lambda)])$$

and δ_x is a one-to-one map, then $\kappa_{M_x}[\rho(\xi_x(f_\lambda,\mu_\lambda))] = \kappa_{M_x}[(f_\lambda,\mu_\lambda)]$ for each $\lambda\in\Lambda$. Consequently,

$$\rho(\xi_x(f_\lambda,\mu_\lambda))\in \kappa_{M_x}^{-1}[\kappa_{M_x}(\rho(\xi_x(f_\lambda,\mu_\lambda)))] = \kappa_{M_x}^{-1}[\kappa_{M_x}(f_\lambda,\mu_\lambda)]\in I$$

for each $\lambda\in\Lambda$. As I is a closed set in $\mathfrak{A}(X,A;\sigma)\times\mathbb{C}$, then $\rho(a_0,\mu_0)\in I$. Therefore $(a_0,\mu_0) = \xi_x[\rho(a_0,\mu_0)] \in \xi_x(I)$. So we have proved that $\mathrm{cl}_A(\xi_x(I)) = \xi_x(I)$ which means that $J_I=\xi_x(I)$ is a closed maximal left ideal of $A\times\mathbb{C}$. Hence every $I\in m_l(\mathfrak{A}(X,A;\sigma)\times\mathbb{C})$ has the form

$$I = I_{x,J_I} = \{(f,\lambda)\in\mathfrak{A}(X,A;\sigma)\times\mathbb{C} : (f(x),\lambda)\in J_I\}$$

for some $x\in X$ and $J_I\in m_l(A\times\mathbb{C})$.

Let now $\mathcal{M} \in m_l(\mathfrak{A}(X, A; \sigma))$. Then $\mathcal{M} \times \{0\} \in m_l(\mathfrak{A}(X, A; \sigma) \times \{0\})$. By Theorem 2.12, there exists an ideal $I_{\mathbb{C}} \in m_l(\mathfrak{A}(X, A; \sigma) \times \mathbb{C})$ such that $I_{\mathbb{C}} \not\subset \mathfrak{A}(X, A; \sigma) \times \{0\}$ and

$$I_{\mathbb{C}} \cap (\mathfrak{A}(X, A; \sigma) \times \{0\}) = \mathcal{M} \times \{0\}.$$

Now there exist $x \in X$ and $J_{\mathbb{C}} \in m_l(A \times \mathbb{C})$ such that

$$I_{\mathbb{C}} = \{(f, \lambda) \in \mathfrak{A}(X, A; \sigma) \times \mathbb{C} : (f(x), \lambda) \in J_{\mathbb{C}}\}.$$

Clearly, $J_{\mathbb{C}} \not\subset (A \times \{0\})$. If $J_{\mathbb{C}} \supset (A \times \{0\})$, then $J_{\mathbb{C}} = A \times \{0\}$, by Lemma 2.11, which implies that $I_{\mathbb{C}} = \mathfrak{A}(X, A; \sigma) \times \{0\}$, but it is not possible. Therefore

$$J \times \{0\} = J_{\mathbb{C}} \cap (A \times \{0\}) \in m_l(A \times \{0\}),$$

by Theorem 2.12, and $J \in m_l(A)$, because the map μ is a topological isomorphism from A onto $A \times \{0\}$. Consequently,

$$\mathcal{M} \times \{0\} = I_{\mathbb{C}} \cap (\mathfrak{A}(X, A; \sigma) \times \{0\}) =$$

$$= \{(f, 0) \in \mathfrak{A}(X, A; \sigma) \times \{0\} : (f(x), 0) \in J_{\mathbb{C}} \cap (A \times \{0\})\} =$$

$$= \{(f, 0) \in \mathfrak{A}(X, A; \sigma) \times \{0\} : f(x) \in J\}.$$

Thus $\mathcal{M} = \{f \in \mathfrak{A}(X, A; \sigma) : f(x) \in J\}$ or $\mathcal{M} = \mathcal{M}_{x,J}$ for some $x \in X$ and $J \in m_l(A)$.

Lemma 4.10. *Let all the conditions of Theorem* 4.9 *be fulfilled. Then the map Ω from $m_k(\mathfrak{A}(X, A; \sigma))$ into $X \times m_k(A)$, defined by*

$$\Omega(\mathcal{M}_{x,J}) = (x, J)$$

for each $x \in X$ and $J \in m_k(A)$, is a bijection (here $k = l$, $k = r$ or $k = t$).

Proof. It is clear that Ω maps $m_k(\mathfrak{A}(X, A; \sigma))$ onto $X \times m_k(A)$, by Lemma 4.7 and Theorem 4.9. If now $\Omega(\mathcal{M}_{x_1,J_1}) = \Omega(\mathcal{M}_{x_2,J_2})$, then from $(x_1, J_1) = (x_2, J_2)$ follows $x_1 = x_2$ and $J_1 = J_2$. Hence $\mathcal{M}_{x_1,J_1} = \mathcal{M}_{x_2,J_2}$ and thus Ω is a bijection.

Corollary 4.11. *Let all the conditions of Theorem* 4.9 *be fulfilled. Then*

a) $\mathrm{rad}\mathfrak{A}(X, A; \sigma) = \mathfrak{A}(X, A; \sigma) \bigcap C(X, \mathrm{rad}A; \sigma)$;

b) *$\mathfrak{A}(X, A; \sigma)$ is topologically semisimple if and only if A is topologically semisimple;*

c) *if A is a commutative algebra, then every $\Phi \in \mathrm{hom}\mathfrak{A}(X, A; \sigma)$ defines $x \in X$ and $\phi \in \mathrm{hom}A$ such that $\Phi = \phi \circ \epsilon_x$.*

Proof. a) Let $g \in \operatorname{rad}\mathfrak{A}(X, A; \sigma)$. Then $g \in \mathfrak{A}(X, A; \sigma)$ and

$$g \in \bigcap_{\mathcal{M} \in m_l(\mathfrak{A}(X,A;\sigma))} \mathcal{M}.$$

Therefore $g(x) \in J$ for all $x \in X$ and $J \in m_l(A)$, by Theorem 4.9. It means that $g \in C(X, \operatorname{rad}A; \sigma)$. Thus $g \in \mathfrak{A}(X, A; \sigma) \bigcap C(X, \operatorname{rad}A; \sigma)$.

Let now $g \in \mathfrak{A}(X, A; \sigma) \bigcap C(X, \operatorname{rad}A; \sigma)$. Then $g(x) \in J$ for all $x \in X$ and $J \in m_l(A)$. Thus

$$g \in \bigcap_{(x,J) \in X \times m_l(A)} \mathcal{M}_{x,J} = \operatorname{rad}\mathfrak{A}(X, A; \sigma),$$

by Theorem 4.9.

b) follows immediately from a).

c) Let A be a commutative topological algebra and $\Phi \in \operatorname{hom}\mathfrak{A}(X, A; \sigma)$. Then $\ker\Phi$ is a closed maximal two-sided ideal of $\mathfrak{A}(X, A; \sigma)$ (see, e.g., [36], p. 68). Thus $\ker\Phi = \mathcal{M}_{x,J}$ for some $x \in X$ and $J \in m(A)$, by Theorem 4.9.

The algebra A in the present Corollary is a commutative Gelfand-Mazur algebra in both cases a) and b) of Theorem 4.9, by Lemma 1.11. To show that A is also a Gelfand-Mazur algebra in case c) of Theorem 4.9, we show that every element in A is bounded. For it, let a_0 be an arbitrary element of A and $f_0 \in \mathfrak{A}(X, A; \sigma)$ be a function such that $f_0(x) = a_0$ for each $x \in X$. Since all elements in $\mathfrak{A}(X, A; \sigma)$ are bounded, then for every neighbourhood of zero O and $S \in \sigma$ there exist numbers $\gamma > 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$\left\{ \left(\frac{f_0}{\lambda} \right)^n : n \in \mathbb{N} \right\} \subset \gamma(T(S, O) \cap \mathfrak{A}(X, A; \sigma)).$$

Therefore

$$\left(\frac{a_0}{\lambda} \right)^n \subset \gamma O$$

for each $n \in \mathbb{N}$. Hence a_0 is bounded in A. Consequently, (see Lemma 1.11), A is a commutative Gelfand-Mazur algebra in case c), as well. Thus there exists a map $\phi \in \operatorname{hom}A$ such that $J = \ker\phi$. Now from $\epsilon_x(f) \in \ker\phi$ follows $f \in \ker(\phi \circ \epsilon_x)$ for each $f \in \ker\Phi$. Hence $\ker\Phi \subset \ker(\phi \circ \epsilon_x)$. Since $\ker(\phi \circ \epsilon_x)$ is a two-sided ideal and $\ker\Phi$ is a maximal two-sided ideal in $\mathfrak{A}(X, A; \sigma)$, then $\ker\Phi = \ker(\phi \circ \epsilon_x)$ and therefore $\Phi = \phi \circ \epsilon_x$, by Lemma 1.7.

Proposition 4.12. *Let all the conditions of Theorem* 4.9 *be fulfilled and* $\phi_x \mapsto x$ *be a continuous map from* $\operatorname{hom}\mathfrak{A}(X, A; \sigma)$ *onto* X. *If, in addition,* A *is a commutative unital algebra, for which* $\operatorname{hom}A$ *is locally equicontinuous, then* $\operatorname{hom}\mathfrak{A}(X, A; \sigma)$ *and* $X \times \operatorname{hom}A$ *are homeomorphic.*

Proof. By Corollary 4.11, every $\Phi \in \hom\mathfrak{A}(X, A; \sigma)$ is representable in the form $\Phi = \Phi_{(x,\phi)} = \phi \circ \epsilon_x$ for some $x \in X$ and $\phi \in \hom A$. Moreover, $\phi \circ \epsilon_x \in \hom\mathfrak{A}(X, A; \sigma)$ for each $x \in X$ and $\phi \in \hom A$, because ϵ_x is a continuous homomorphism from $\mathfrak{A}(X, A; \sigma)$ onto A. Therefore Ω, defined by

$$\Omega(\Phi) = \Omega(\Phi_{(x,\ \phi)}) = (x, \phi)$$

for each $x \in X$ and $\phi \in \hom A$ maps $\hom\mathfrak{A}(X, A; \sigma)$ onto $X \times \hom A$. If $\Omega(\Phi_{(x,\phi)}) = \Omega(\Phi_{(x',\phi')})$, then $(x, \phi) = (x', \phi')$. Hence $\Phi_{(x,\phi)} = \Phi_{(x',\phi')}$ which means that Ω is a bijection.

Next, we show that Ω is continuous. For it, let $(\Phi_i)_{i\in I} = (\Phi_{(x_i,\phi_i)})_{i\in I}$ be a net in $\hom\mathfrak{A}(X, A; \sigma)$, which converges to $\Phi_0 = \Phi_{(x_0,\phi_0)}$ in the topology of $\hom\mathfrak{A}(X, A; \sigma)$. Then the net $(\Phi_i(f))_{i\in I}$ converges to $\Phi_0(f)$ for each $f \in \mathfrak{A}(X, A; \sigma)$. Since $f_a \in \mathfrak{A}(X, A; \sigma)$ for each $a \in A$, by the condition 2) of Theorem 4.9, and

$$\Phi_i(f_a) = (\phi_i \circ \epsilon_{x_i})(f_a) = \phi_i(\epsilon_{x_i}(f_a)) = \phi_i(f_a(x_i)) = \phi_i(a)$$

for each $i \in I \cup \{0\}$, then $(\phi_i(a))_{i\in I}$ converges to $\phi_0(a)$ for each $a \in A$. Hence $(\phi_i)_{i\in I}$ converges to ϕ_0 in the topology of $\hom A$.

For any $x \in X$, let $\phi_x : \mathfrak{A}(X, \mathbb{C}; \sigma) \to \mathbb{C}$ be a map, defined by $\phi_x(\alpha) = \alpha(x)$ for each $\alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma)$. Then the net $(\Phi_i(\alpha e_A))_{i\in I}$ converges to $(\Phi_0(\alpha e_A))$ for each $\alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma)$. Since

$$\Phi_{(x,\phi)}(\alpha e_A) = \alpha(x)\phi(e_A) = \alpha(x) = \phi_x(\alpha)$$

for each $x \in X$ and $\phi \in \hom A$, then $(\phi_{x_i}(\alpha))_{i\in I}$ converges to $\phi_{x_0}(\alpha)$ for each $\alpha \in \mathfrak{A}(X, \mathbb{C}; \sigma)$. Hence $(\phi_{x_i})_{i\in I}$ converges to ϕ_{x_0} in the topology of $\hom\mathfrak{A}(X, \mathbb{C}; \sigma)$. By assumption, $\phi_x \mapsto x$ is a continuous map from $\hom\mathfrak{A}(X, A; \sigma)$ onto X. Thus $(x_i)_{i\in I}$ converges to x_0 in the topology of X and therefore $(x_i, \phi_i)_{i\in I}$ converges to (x_0, ϕ_0) in the product topology on $X\times \hom A$. It means that Ω is continuous.

Now we show that Ω^{-1} is continuous. First, we show that for any $x_0 \in X$ and $\phi_0 \in \hom A$ and any neighbourhood $O(\Phi_{(x_0,\phi_0)})$ of $\Phi_{(x_0,\phi_0)}$ there exists a neighbourhood $O((x_0, \phi_0))$ of (x_0, ϕ_0) such that

$$\Omega^{-1}(O((x_0, \phi_0))) \subset O(\Phi_{(x_0,\phi_0)}).$$

For that, it is enough to show that for each $\epsilon > 0$ and $f \in \mathfrak{A}(X, A; \sigma)$ there exists a neighbourhood $O((x_0, \phi_0))$ such that

$$|\Phi_{(x,\phi)}(f) - \Phi_{(x_0,\phi_0)}(f)| < \epsilon,$$

whenever $(x, \phi) \in O((x_0, \phi_0))$.

Since $\hom A$ is locally equicontinuous, then every $\phi_0 \in \hom A$ has an equicontinuous neighbourhood $O_e(\phi_0)$ of ϕ_0 in the topology of $\hom A$. Therefore, for every $x_0 \in X$, $f \in \mathfrak{A}(X, A; \sigma)$ and $\epsilon > 0$, there exists a neighbourhood $O(f(x_0))$ of $f(x_0)$ such that

$$|\phi(a) - \phi(f(x_0))| = |\phi(a - f(x_0))| < \frac{\epsilon}{2}$$

for each $a \in O(f(x_0))$ and $\phi \in O_e(\phi_0)$. Since f is continuous, then there is a neighbourhood $O(x_0)$ of x_0 such that $f(x) \in O(f(x_0))$, whenever $x \in O(x_0)$. Hence

$$|\phi(f(x) - f(x_0))| < \frac{\epsilon}{2},$$

whenever $\phi \in O_e(\phi_0)$ and $x \in O(x_0)$. Let now

$$O(\phi_0) = O_e(\phi_0) \bigcap \left\{\phi \in \hom\mathfrak{A}(X, A; \sigma) : \ |(\phi - \phi_0)(f(x_0))| < \frac{\epsilon}{2}\right\}.$$

Since $O(x_0) \times O(\phi_0)$ is a neighbourhood of (x_0, ϕ_0) in the product topology of $X \times \hom A$ and

$$\begin{aligned} |\Phi_{(x,\phi)}(f) - \Phi_{(x_0,\phi_0)}(f)| &= |\phi(f(x)) - \phi_0(f(x_0))| \leqslant \\ &\leqslant |\phi(f(x) - f(x_0))| + |(\phi - \phi_0)(f(x_0))| < \epsilon \end{aligned}$$

for each $f \in \mathfrak{A}(X, A; \sigma)$, whenever $(x, \phi) \in O(x_0) \times O(\phi_0)$, then Ω^{-1} is continuous. So we have showed that Ω is a homeomorphism.

Corollary 4.13. *Let X be a completely regular Hausdorff space, σ be a compact cover of X, which is closed with respect to finite unions, A a unital topological algebra over $\mathbb{C}$ with jointly continuous multiplication and let one of the following conditions be true:*

1) *A is a locally m-pseudoconvex Hausdorff algebra;*

2) *A is a locally pseudoconvex Waelbroeck Hausdorff algebra, $X \in \sigma$ and $m(C(X, A; \sigma)) \neq \emptyset$;*

3) *A is an exponentially galbed Hausdorff algebra and every element in $C(X, A; \sigma)$ is bounded.*

Then

a) $\operatorname{rad}C(X, A; \sigma) = C(X, \operatorname{rad}A; \sigma)$*;*

b) *$C(X, A; \sigma)$ is topologically semisimple if and only if A is topologically semisimple;*

c) *if A is a commutative algebra, then every $\Phi \in \hom C(X, A; \sigma)$ defines $x \in X$ and $\phi \in \hom A$ such that $\Phi = \phi \circ \epsilon_x$.*

In addition, if A is a commutative algebra, for which $\mathrm{hom}A$ *is locally equicontinuous, then* $\mathrm{hom}C(X, A;\sigma)$ *and* $X \times \mathrm{hom}A$ *are homeomorphic.*

Proof. All the conditions of Theorem 4.9 have been fulfilled, by Lemma 4.4 and Proposition 4.2. Moreover, $\phi_x \mapsto x$ is a continuous map from $\mathrm{hom}C(X, A;\sigma)$ onto X, by Theorem 2 v) in [10]. Therefore the assumptions of Proposition 4.12 are true and $\mathrm{hom}C(X, A;\sigma)$ and $X \times \mathrm{hom}A$ are homeomorphic, by Proposition 4.12.

Corollary 4.14. *Let X be a completely regular Hausdorff space, σ a compact cover of X, which is closed with respect to finite unions, A a unital topological algebra over $\mathbb{C}$ with jointly continuous multiplication and let one of the following conditions be true:*

1) *A is a locally m-pseudoconvex Hausdorff algebra;*

2) *A is a locally pseudoconvex Waelbroeck Hausdorff algebra, $X \in \sigma$ and $m(C(X, A;\sigma)) \neq \emptyset$;*

3) *A is an exponentially galbed Hausdorff algebra and every element in $C(X, A;\sigma)$ is bounded.*

Then

a) *every closed maximal left (right or two-sided) ideal $\mathcal{M}$ of $C(X, A;\sigma)$ is representable in the form*

$$\mathcal{M} = \mathcal{M}_{x,J} = \{f \in C(X, A;\sigma) : f(x) \in J\}$$

for some $x \in X$ and a closed maximal left (right or two-sided, respectively) ideal J of A;

b) *$\mathcal{M}_{x,J}$ is a closed maximal left (right or two-sided) ideal of $C(X, A;\sigma)$ for each $x \in X$ and each closed maximal left (right or two-sided, respectively) ideal J of A.*

Proof. It is easy to see that the conditions of Lemma 4.7 and Theorem 4.9 are fulfilled, by Proposition 4.2 and Lemma 4.4. Thus the statements a) and b) are true.

Corollary 4.15. *Let X be a completely regular Hausdorff space, σ a compact cover of X, which is closed with respect to finite unions, A a topological algebra over $\mathbb{C}$ with jointly continuous multiplication, i_A a nonzero idempotent in $Z(A)$ and let one of the following conditions be true:*

1) *A is a locally m-pseudoconvex Hausdorff algebra;*

2) *A is a locally pseudoconvex Waelbroeck Hausdorff algebra, $X \in \sigma$ and $m(C(X, A;\sigma)) \neq \emptyset$;*

3) *A is an exponentially galbed Hausdorff algebra and every element in*

$C(X, A; \sigma)$ *is bounded.*

If $cl_{C(X,A;\sigma)}(A \otimes C(X, \mathbb{C}; \sigma) i_A) = C(X, A; \sigma)$, *then*

a) *every closed maximal regular left (right or two-sided) ideal* $\mathcal{M}$ *of* $C(X, A; \sigma)$ *is representable in the form*

$$\mathcal{M} = \mathcal{M}_{x,J} = \{f \in C(X, A; \sigma) : f(x) \in J\}$$

for some $x \in X$ *and* $J \in m_l(A)$ ($J \in m_r(A)$ *or* $J \in m_t(A)$, *respectively*);

b) $\mathcal{M}_{x,J}$ *is a closed maximal regular left (right or two-sided) ideal of* $C(X, A; \sigma)$ *for each* $x \in X$ *and* $J \in m_l(A)$ ($J \in m_r(A)$ *or* $J \in m_t(A)$, *respectively*).

Proof. It is easy to see that the conditions of Lemma 4.7 and Theorem 4.9 are fulfilled, by Proposition 4.2 and Lemma 4.4. Thus the statements a) and b) are true.

Remark. If $\mathrm{cl}_A(Ai_A) = A$, then

$$\mathrm{cl}_{C(X,A;\sigma)}(A \otimes C(X, \mathbb{C}; \sigma) i_A) = C(X, A; \sigma)$$

is fulfilled in the following cases (see [11], Theorem 1, p. 27):

a) X is completely regular Hausdorff space and A is a locally convex Hausdorff algebra;

b) X is a completely regular Hausdorff space and A is a Hausdorff algebra, which has the approximation property (see [36], p. 313–314 and 445–446);

c) the cover dimension or the Lebesque dimension (see [37], p. 8–9) $\dim X$ of X is finite and A is a Hausdorff algebra;

d) X is a completely regular Hausdorff space, the Lebesque dimension $\dim S$ is finite for each element $S \in \sigma$ and A is a Hausdorff algebra.

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GELFAND-MAZURI ALGEBRATE EHITUS

Kokkuvõte

Algebra struktuuri kirjeldamise seisukohalt on oluline osata kirjeldada vaadeldava algebra ideaale. Teades ideaalide kirjeldust, saab teha üldisemaid järeldusi algebra enda ehituse kohta.

Algebras ei ole üldjuhul teada, millal algebra A maksimaalse pärisideaali M ühisosa algebra A alamalgebraga B osutub algebras B maksimaalseks pärisideaaliks. Väitekirja teises peatükis saadud tulemused annavad võimaluse kolmanda peatüki alguses näidata, et teatud Gelfand-Mazuri algebrate klassides osutub $M \cap B$ alati algebra A tsentri kinnise alamalgebra B kinniseks maksimaalseks regulaarseks pärisideaaliks algebra A iga kinnise maksimaalse regulaarse ideaali M korral.

Sageli osutub lihtsamaks kirjeldada kinniseid ideaale topoloogilistes algebrates või nende konkreetsetes alamalgebrates. Senini oli kõigi maksimaalsete ideaalide kirjeldusi tsentri alamalgebra ideaalide kaudu saadud vaid ühikuga Banachi algebrate korral.

Selles töös on leitud meetod teatud topoloogiliste algebrate (nn Gelfand-Mazuri algebrate) kõigi kinniste maksimaalsete regulaarsete (ühe- või kahepoolsete) ideaalide kirjeldamiseks juhul, kui on teada kinniste maksimaalsete regulaarsete ideaalide kirjeldus uuritava algebra tsentri mingis kinnises alamalgebras. Seejuures ei sõltu kirjelduse üldskeem sellest, millises tsentri kinnises alamalgebras on ideaalide kirjeldus teada. Käesoleva töö kolmandas peatükis saadud tulemused annavad võimaluse ühiku olemasolu ja Banachi algebraks oleku nõuetest loobuda.

Mõnikord on võimalik uuritava algebra A tsentrisse sisestada alamalgebrana mingit (meile juba tuntumat) algebrat B. Käesoleva väitekirja kolmandas peatükis saadud tulemused võimaldavad antud situatsioonis kirjeldada ka algebra A kõiki kinniseid maksimaalseid regulaarseid (ühe- või kahepoolseid) ideaale juhul, kui on teada sellise algebra B kõigi kinniste maksimaalsete regulaarsete (ühe- või kahepoolsete) ideaalide kirjeldus.

Kolmanda peatüki lõpus näidatakse, et Gelfand-Mazuri algebrat on teatud tingimustel võimalik vaadelda kui teatud liiki lõikekujutuste algebra alamalgebrat või teatud liiki vektorväärtustega funktsioonide algebrat. Seetõttu osutub vajalikuks teada ideaalide kirjeldusi vektorväärtustega funktsioonide algebrates või lõikekujutuste algebrates.

Neljas peatükk on pühendatud vektorväärtustega funktsioonide algebra $C(X, A)$ ning tema alamalgebrate kõigi kinniste maksimaalsete regulaarsete ideaalide kirjeldamisele juhul, kui A on teatud omadustega Gelfand-Mazuri algebra.

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