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ESTA KÄGO

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Natural vibrations of elastic stepped plates with cracks

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II J. Lellep, E. Kägo, Vibrations of elastic stretched strips with cracks, *International Journal of Mechanics*, 5(1) (2011), NAUN, 27–34.

III E. Kägo, J. Lellep, Free vibratons of plates on elastic foundation, *Procedia Engineering*, 57 (2013), Elsevier, 489-496.

IV E. Kägo, J. Lellep, Vibrations of a cracked anisotropic plate, *Proc. of the Int. Conf. on Optimization and Analysis of Structures* (accepted).

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Author's contribution

The author of this dissertation is responsible for majority of the research in all phases (including writing, simulation and preparation of images) of the papers I–VI. The solution procedure was developed in co-operation with the supervisor; the statement of the problem belongs to the supervisor.

1 Introduction

1.1 Historical review of literature

Generally speaking, all the materials used in the mechanical engineering and technology, can be divided into homogeneous and non-homogeneous materials. The most of non-homogeneous materials can be treated as composites, consisting of two or more ingredients incorporating various constituents have wide range of specific strength and others material moduli. Among various composites the unidirectionally reinforced materials seem to have the most favorable combination of high specific modulus and strength. Unidirectional composites can be modeled as anisotropic quasi-homogeneous materials having different moduli in the direction of fibers and in the transverse direction, respectively (see Jones [26], Daniel and Ishai [10]).

Due to the growing interest in non-destructive testing techniques and vibration monitoring of structures and machines there is an emerging demand for the vibration analysis of structural elements with flaws and cracks. It was already recognized long ago that the presence of surface flaws or intrinsic cracks in a machine element is a source of local flexibility, which in turn influences the dynamic behavior of the whole system. This leads to an important idea to model cracks as equivalent elastic springs, which was first used to quantify the relation between the applied load and the strain concentration in the vicinity of the crack tip by Irwin [24] in the early 1960s. Later Rice and Levy [71] extended the idea to rectangular plates with part-through cracks.

However, these initial works did not focus on the vibration analysis of the structures. At the same time the basic approaches to vibration analysis initially studied the isotropic plates without cracks and steps employing the classical theory with Kirchhoff-Love assumptions; see Leissa [38]. Subsequent studies have generalized the basic theories of vibration analysis to the anisotropic plates and the plates with cracks and steps.

The first study to combine vibration analysis with spring model of cracks was undertaken by Rizos *et al.* [72] in 1990. The authors developed a method to identify the crack location and magnitude in cantilever beams. Liang *et al.* [50], Dimarogonas [12] and Chondros *et al.* [8, 9] extended this approach to several cracked structures.

An important approach to identify crack size and location was introduced by Liang *et al.* [49]. The novel idea was to use the measurements of natural vibrations to detect cracks. This idea was later extended by Nandwana and Maiti [58]. De Rosa [11] investigated the influence of cracks on the natural vibrations of stepped beams with flexible ends. Prestressed beams with fixed ends were studied by Masoud *et al.* [54]. However, all previously mentioned methods of natural vibrations considered only a single crack. The effect of multiple cracks on the natural vibrations of uniform beams was studied by Lin *et al.* [52] making use of the transfer matrix method.

Zhu *et al.* [85] extended the work by Leissa [38] to the free vibration analysis of thin isotropic and anisotropic rectangular plates, while Gutierrez and Laura [16], Xiang and Wang [81], Li *et al.* [46] have modeled analytically an isotropic stepped plate with varying boundary conditions.

The well known Lévy method was used by Gorman *et al.* [14] for the vibration analysis of a plate with two opposite simply supported edges and arbitrary boundary conditions at the other two edges. The Navier method is used by Kant *et al.* [31] to analyze the vibration of a plate with all four edges simply supported. The Rayleigh-Ritz method was used by Nallim *et al.* [57] to model an anisotropic plate without steps and by Laura *et al.* [37] to model an anisotropic stepped plates. Ramamurti *et al.* [66] have applied the generalized Rayleigh-Ritz method to determine the natural frequency of cracked cantilevered plates.

Recently, more analytical and numerical methods for studying the vibration of anisotropic rectangular plates have been proposed. Anisotropic plates were studied by Huang *et al.* [23] and by Bui *et al.* [6] using an efficient mesh free-method. Exact solutions for free vibrations of rectangular plates are proposed by Wu *et al.* [80] using Bessel functions. Free vibrations of orthotropic rectangular plates are investigated by Jafari and Eftekhari [25] with mixed Ritz-differential quadrature method and Paiva *et al.* [63] using the boundary element method. A vibration analysis of thin rectangular plate has been undertaken by Li and Yuan [48]. The authors used a Green quasifunction method to obtain the solution of the free vibration problem of clamped thin plates. Wang *et al.* [78] used the discrete singular convolution algorithm to analyze free vibrations of a simply supported anisotropic rectangular plate. Natural frequencies for Lévy plate are studied by Park *et al.* [64] by using a harmonic response estimation method.

Further results on the application of finite element method for dynamic analysis of a thin rectangular plate with a crack have been obtained by Krawczuk *et al.* [34,35]. Liew *et al.* [51] employed the decomposition method to determine the vibration frequencies of cracked plates. An experimental study of natural frequencies of clamped rectangular plates with cracks has been described by Maruyama and Ichinomiya [53]. They investigated experimentally the effect of length, position and inclination angle of a crack on the natural frequencies. However, it should be noted that the work by Maruyama and Ichinomiya clarified the vibration of a plate with penetrating crack which is different from part-through cracks at the corners of the re-entrant parts of the steps. The latter is the focus of this dissertation.

In recent years the investigations of structures resting on elastic foundation have gained large importance. The vibration and bending problems of plates on elastic foundations are often faced in the practical engineering of structures. The examples of the plates resting on elastic foundations are road surfaces, airport runways, aircraft parking areas, railway tracks and building foundations. In this study we use the well known Winkler model [67], which was originally developed for the analysis of railroad tracks. The Winkler elastic model is the simplest example of the continuous elastic foundation.

The vibration analysis of thin structures on elastic foundations has been carried out by various researchers. Let us consider some of recent results and research methods in this area. Pengcheng and Peixiang [65] studied vibrations of the plates on elastic foundation using the multivariable spline element method. A few years later Cheung *et al.* [7] developed a finite strip method for the natural vibration analysis of stepped plate on an elastic foundation. At the same time Huang and Thambiratnam [22] proposed a procedure incorporating the finite strip method together with a spring system to analyze plate resting on elastic supports and elastic foundation.

Hsu [20] investigated the eigenvalue problems of cracked hinged–hinged and cantilevers Bernoulli–Euler beams resting on elastic foundation using the differential quadrature method. A year later Hsu [21] proposed the differential quadrature method for the rectangular plate. Also the finite strip method was complemented by Hatami *et al.* [17]. Authors provided the exact finite strip method and used the method to solve the vibration and stability problems of the axially moving orthotropic plates on the elastic foundation.

Recently, Li and Yuan [47] adopted the quasi-Green's function method to analyze the free vibration of a clamped thin plate resting on the Winkler foundation. Motaghian *et al.* [55] proposed a novel mathematical approach to find the exact analytical solution of the problem of natural vibrations of plates resting on partially elastic foundation with certain boundary conditions. The natural vibrations of the cracked cantilever beam resting on elastic foundation is analyzed by Nassar *et al.* [59] using the differential quadrature method in the case when the beam is made of a functionally graded material.

1.2 Aim of the dissertation

The primary goal of this study is to develop a method for finding natural vibrations of elastic structures with steps and cracks. To the best of our knowledge this would be the first analytical method that has this feature. The goal is to have a method for both, for isotropic and anisotropic structures, with the cracks located at the re-entrant corners of the steps (see Fig. 1). It is worthwhile to mention that in the following we will use the term "crack" instead of a "crack-like defect" for the conciseness sake.

The secondary goal of the dissertation is the application of the proposed method to different plate structures in order to investigate the effect of cracks on the natural vibrations.

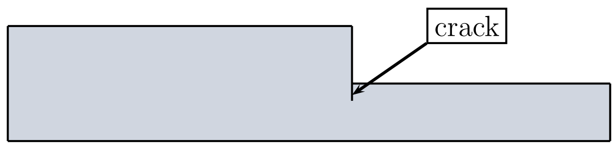


Figure 1: Side view of stepped plate with crack at re-entrant corner.

1.3 Structure of the dissertation

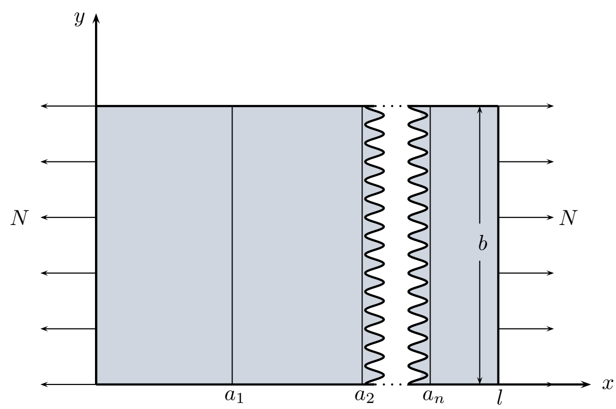
The dissertation has been organized as follows. Section 1 contains a historic background of vibration analysis, the aim and the structure of the dissertation. In Section 2 our method is described in detail and applied to isotropic plate strips that are stepped and have cracks. In Section 3 the method is extended to anisotropic plates for determination of frequencies of natural vibrations. Finally, Section 4 extends the approach to anisotropic plates resting on the elastic foundation.

2 Free vibrations of stepped plate strips with cracks

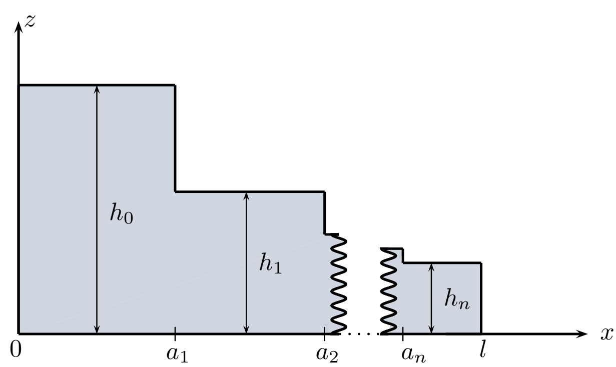
The objective of this section is to develop the basic concepts for the analysis of the free vibrations of plate strips. A review of the papers by Lellep and Kägo [39, 40] is presented herein.

2.1 Formulation of the problem

Let us consider natural vibrations of a plate strip subjected to the in-plane tension N (Fig. 2). Let the dimensions of the strip in x and y directions be l and b, respectively.



(a) Dimensions of the plate strip



(b) Side view of the stepped plate strip

Figure 2: Plate strip

It is assumed that the thickness $h(x, y) = h_j$ is piecewise constant for $x \in (a_j, a_{j+1})$, where $j = 0, \ldots, n$. The quantities a_j and h_j $(j = 0, \ldots, n)$ are given constants whereas $a_0 = 0$, $a_{n+1} = l$.

In the present study like in Rizos *et al.* [72], Chrondos *et al.* [9], Dimarogonas [12] the effect of crack propagation in the body during vibrations is disregarded.

The material of plates is considered as a linear elastic material. Both, homogeneous elastic plates and these made of non-homogeneous composite materials are studied.

The aim of this section is to elucidate the sensitivity of natural frequencies on the crack parameters and geometrical parameters of the plate strip.

2.2 Basic equations

In the present case of the plate it is reasonable to assume that the stress-strain state of the plate depends on the time t and coordinate x only, provided the stresses mean generalized stresses (bending moments and membrane forces). However, due to the tension N applied at the edge of the plate, in-plane forces have to be taken into account. If, moreover, the inertia of the rotation is not neglected the equilibrium equations of a plate element can be presented as (Reddy [70])

$$\begin{aligned} \frac{\partial M_x}{\partial x} &= Q_x \\ \frac{\partial Q_x}{\partial x} &= -N\frac{\partial^2 W}{\partial x^2} + \rho h_j \frac{\partial^2 W}{\partial t^2} - I_j \frac{\partial^4 W}{\partial x^2 \partial t^2} \end{aligned} \tag{1}$$

for $x \in (a_j, a_{j+1})$. In (1) $W = W(x,t)$ stands for the transverse deflection corresponding to the point with coordinate x at the middle plane of the plate whereas M_x is the bending moment and Q_x is the shear force. Herein

$$I_j = \frac{\rho h_j^3}{12} \tag{2}$$

where ρ stands for the density of the material. According to this theory the membrane force in the direction of the axis x $N_x = N$ in the present case.

Eliminating the shear force Q_x from (1) one easily obtains

$$\frac{\partial^2 M_x}{\partial x^2} + N\frac{\partial^2 W}{\partial x^2} = \rho h_j \frac{\partial^2 W}{\partial t^2} - I_j \frac{\partial^4 W}{\partial x^2 \partial t^2} \tag{3}$$

for $x \in (a_j, a_{j+1})$ where $j = 0, \ldots, n$.

It is well known that (Reddy [70], Soedel [74])

$$M_x = -D_j \frac{\partial^2 W}{\partial x^2} \tag{4}$$

where $D_j = Eh_j^3/[12(1-\nu^2)]$; $j = 0, \ldots, n$.

Substituting (4) in (3) yields

$$D_j \frac{\partial^4 W}{\partial x^4} - N \frac{\partial^2 W}{\partial x^2} = -\rho h_j \frac{\partial^2 W}{\partial t^2} + I_j \frac{\partial^4 W}{\partial x^2 \partial t^2} \tag{5}$$

for $x \in (a_j, a_{j+1})$ where $j = 0, \ldots, n$. Here W is the transverse deflection,

$$I_j = \frac{\rho h_j^3}{12},\ D_j = \frac{E h_j^3}{12(1-\nu^2)}, \tag{6}$$

E and ν are elastic moduli and ρ is density of the material. The equation (5) will be considered as the equation of motion for the segment (a_j, a_{j+1}); $j = 0, \ldots, n$. It can be solved accounting for appropriate boundary conditions.

In the case of a free edge of a strip the boundary conditions are

$$\frac{\partial^2 W}{\partial x^2} = 0,\ \frac{\partial^3 W}{\partial x^3} = 0, \tag{7}$$

for the case of the clamped edge

$$W = 0,\ \frac{\partial W}{\partial x} = 0, \tag{8}$$

and for the simply supported edge

$$\frac{\partial W}{\partial x} = 0,\ \frac{\partial^2 W}{\partial x^2} = 0. \tag{9}$$

Let at the initial moment

$$\frac{\partial W}{\partial t} = 0,\ W = \varphi(x) \tag{10}$$

where φ is a given function.

2.3 Solution of the equation of motion

It is reasonable to look for the general solution of (5) in the form

$$W(x,t) = w_j(x) T(t) \tag{11}$$

for $x \in (a_j, a_{j+1})$ where $j = 0, \ldots, n$.

Differentiating (11) with respect to variables x, t and substituting in (5) one easily obtains

$$D_j w_j^{IV} T - N w_j'' T = -\rho h_j w_j \ddot{T} + I_j w_j'' \ddot{T} \tag{12}$$

for $x \in (a_j, a_{j+1})$ where $j = 0, \ldots, n$. Here prims denote the differentiation with respect to the coordinate x and dots with respect to time t.

Separating variables in (11) yields

$$D_j w_j^{IV} - (I_j \omega^2 + N) w_j'' + \rho h_j \omega^2 w_j = 0 \tag{13}$$

for $j = 0, \ldots, n$ and

$$\ddot{T} + \omega^2 T = 0 \tag{14}$$

where ω stands for the frequency of natural vibrations. Evidently, the solution of (14) which satisfies according to (10) initial conditions $T(0) = d$, $\dot{T}(0) = 0$ has the form

$$T = d \cos \omega t \tag{15}$$

where d is a constant.

The equation (13) is a linear fourth order ordinary equation with respect to the variable w_j. The characteristic equation corresponding to (13) is

$$D_j r_j^4 - (I_j \omega^2 + N) r_j^2 + \rho h_j \omega^2 = 0 \tag{16}$$

From (16) one easily obtains the roots

$$r_j = \pm \sqrt{\frac{I_j \omega^2 + N}{2D_j} \pm \sqrt{\frac{(I_j \omega^2 + N)^2}{4D_j^2} - \frac{\rho h_j}{D_j}}}. \tag{17}$$

Introducing the notation

$$\begin{aligned} (r_j^2)_1 &= -\lambda_j^2 \\ (r_j^2)_2 &= \mu_j^2 \end{aligned} \tag{18}$$

one can present the general solution of (13) as

$$\begin{aligned} w_j(x) \quad &= \quad A_{1j} \cos \lambda_j x + A_{2j} \sin \lambda_j x + \\ &\quad\ + A_{3j} \sinh \mu_j x + A_{4j} \cosh \mu_j x \end{aligned} \tag{19}$$

which holds good for $x \in (a_j, a_{j+1})$, $j = 0, \ldots, n$. Here $A_{1j}, \ldots, A_{4j}$ stand for unknown constants of integration. These will be determined from boundary conditions and requirements on the continuity of displacements and generalized stresses.

However, it appears that the quantity W' cannot be continuous at $x = a_j$ according to the model of distributed line springs developed by Rice and Levy [71]; Dimarogonas [12], Chondros *et al.* [8].

2.4 Local compliance of the plate strip

Let us consider the influence of the crack located at the cross section $x = a$ on the stress-strain state of the strip in the vicinity of the crack. For the conciseness sake we shall study the case when $n = 1$ and thus in the adjacent segments to the crack the thickness equals to h_0 and h_1, respectively. Let $h = min(h_0, h_1)$.

According to the distributed line spring method the slope of the deflection has a jump

$$\Theta = w'(a + 0) - w'(a - 0) \tag{20}$$

at the cross section $x = a$. The angle Θ can be treated as a generalized displacement corresponding to the generalized stress M_x. Thus

$$\Theta = CM_x(a) \tag{21}$$

or

$$C = \frac{\partial\Theta}{\partial M_x(a)} \tag{22}$$

where C is the local compliance due to the crack. It is known in the linear elastic fracture mechanics that (see Anderson [1], Broberg [4])

$$\Theta = \frac{\partial U_T}{\partial M_x(a)} \tag{23}$$

where U_T is the extra strain energy caused by the crack. Combining (21)–(23) one obtains

$$C = \frac{\partial^2 U_T}{\partial M_x^2(a)}. \tag{24}$$

According to the concept of the distributed line spring $1/C = K$, where K stands for the stress intensity coefficient. It is known in the fracture mechanics that (see Anderson [1])

$$K_M = \sigma_M \sqrt{\pi c} F_M \left(\frac{c}{h}\right). \tag{25}$$

In (25) c is the crack depth and

$$\sigma_M = \frac{6M_x(a)}{bh^2}, \tag{26}$$

provided the element involving the cross section $x = a$ is loaded by the bending moment M_x only. Here the function F_M is to be approximated on the basis of experimental data [76].

If the element is loaded by the axial tension N then the stress intensity coefficient

$$K_N = \sigma_N \sqrt{\pi c} F_N \left(\frac{c}{h}\right) \tag{27}$$

where

$$\sigma_N = \frac{N}{bh}. \tag{28}$$

In the case of a combined loading the stress intensity coefficient

$$K_T = K_M + K_N. \tag{29}$$

Note that (29) holds good under the condition that (25)–(28) refer to the common mode of fracture (see Anderson [1] and Broek [5]).

In the present case this requirement is fulfilled, K_M and K_N regard to the first mode of the fracture. It was shown in the previous studies (Lellep *et al.* [43], [41], [45]) that in the case of loading by the moment

$$K_M = \frac{E'h^2b}{72\pi f(s)} \tag{30}$$

where $E' = E$ for plane stress state and $E' = E/(1-\nu^2)$ in the case plane deformation state.

Here $s = c/h$ and the compliance

$$C = \frac{72\pi}{E'h^2b}\int_0^s sF_M^2(s)ds \tag{31}$$

whereas

$$f(s) = \int_0^s sF_M^2(s)ds. \tag{32}$$

The function F_M was taken in the studies by Dimarogonas [12]; Rizos *et al.* [72] as

$$F_M = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.8s^4. \tag{33}$$

According to the handbook by Tada *et al.* [76] the function F_N can be approximated as

$$F_N = 1.122 - 0.23s + 10.55s^2 - 21.71s^3 + 30.38s^4. \tag{34}$$

2.5 Determination of natural frequencies

In the case when the plate has a unique step the deflected shape of the plate can be presented according to (19) as

$$w(x) = A_1 \sin \lambda_0 x + A_2 \cos \lambda_0 x + A_3 \sinh \mu_0 x + A_4 \cosh \mu_0 x \tag{35}$$

for $x \in [0, a]$ and as

$$w(x) = B_1 \sin \lambda_1 x + B_2 \cos \lambda_1 x + B_3 \sinh \mu_1 x + B_4 \cosh \mu_1 x \tag{36}$$

for $x \in [a, l]$.

Arbitrary constants A_i, B_i $(i = 1, \dots, 4)$ have to meet boundary requirements and intermediate conditions at $x = a$. The latters can be presented as (Lellep, Roots [43])

$$\begin{aligned} w(a-0) &= w(a+0) \\ w'(a-0) &= w'(a+0) - pw''(a+0) \\ h_0^3 w''(a-0) &= h_1^3 w''(a+0) \\ h_0^3 w'''(a-0) &= h_1^3 w'''(a+0) \end{aligned} \tag{37}$$

where according to (29), (30)

$$p = \frac{Eh^3}{12(1-\nu^2)K_T}. \tag{38}$$

It is worthwhile to mention that the third and the fourth equality in (37) express the continuity of the bending moment and the shear force, respectively, when passing the step at $x = a$. It is known from the solid mechanics that these quantities must be continuous (Soedel [74]).

Boundary conditions (8) at $x = 0$ admit to eliminate from (35) the unknown constants

$$\begin{aligned} A_4 &= -A_2, \\ A_3 &= -A_1 \frac{\lambda_0}{\mu_0}. \end{aligned} \tag{39}$$

The intermediate conditions (37) with boundary requirements (8) at $x = l$ lead to the system of six equations which will be presented in the matrix form. The continuity of the deflection leads to the equation

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}^\top \times \begin{bmatrix} \sin\lambda_0 a - \frac{\lambda_0}{\mu_0}\sinh\mu_0 a \\ \cos\lambda_0 a - \cosh\mu_0 a \\ -\sin\lambda_1 a \\ -\cos\lambda_1 a \\ -\sinh\mu_1 a \\ -\cosh\mu_1 a \end{bmatrix} = 0. \tag{40}$$

According to the second relation in (37) one has

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}^\top \times \begin{bmatrix} \lambda_0(\cos\lambda_0 a - \cosh\mu_0 a) \\ -\lambda_0\sin\lambda_0 a - \mu_0\sinh\mu_0 a) \\ \lambda_1(p\lambda_1\sin\lambda_1\alpha - \cos\lambda_1\alpha) \\ \lambda_1(\sin\lambda_1\alpha - p\lambda_1\cos\lambda_1\alpha) \\ -\mu_1(\cosh\mu_1\alpha + p\mu_1\sinh\mu_1\alpha) \\ -\mu_1(\sinh\mu_1\alpha + p\mu_1\cosh\mu_1\alpha) \end{bmatrix} = 0. \tag{41}$$

The continuity requirements imposed on the bending moment and the shear force, respectively, lead to the equations

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}^\top \times \begin{bmatrix} -h_0^3\lambda_0(\lambda_0\sin\lambda_0 a - \mu_0\sinh\mu_0 a) \\ -h_0^3(\lambda_0^2\cos\lambda_0 a - \mu_0^2\cosh\mu_0 a) \\ h_1^3\lambda_1^2\sin\lambda_1 a \\ h_1^3\lambda_1^2\cos\lambda_1 a \\ -h_1^3\mu_1^2\sinh\mu_1 a \\ -h_1^3\mu_1^2\cosh\mu_1 a \end{bmatrix} = 0 \tag{42}$$

and

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}^\top \times \begin{bmatrix} -h_0^3\lambda_0(\lambda_0^2\cos\lambda_0 a - \mu_0^2\cosh\mu_0 a) \\ h_0^3(\lambda_0^3\sin\lambda_0 a - \mu_0^3\sinh\mu_0 a) \\ h_1^3\lambda_1^3\cos\lambda_1 a \\ -h_1^3\lambda_1^3\sin\lambda_1 a \\ -h_1^3\mu_1^3\cosh\mu_1 a \\ -h_1^3\mu_1^3\sinh\mu_1 a \end{bmatrix} = 0. \tag{43}$$

The boundary conditions (7) can be expressed as

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}^\top \times \begin{bmatrix} 0 \\ 0 \\ -\lambda_1^2\sin\lambda_1 l \\ -\lambda_1^2\cos\lambda_1 l \\ \mu^2\sinh\mu_1 l \\ \mu^2\cosh\mu_1 l \end{bmatrix} = 0 \tag{44}$$

and

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}^\top \times \begin{bmatrix} 0 \\ 0 \\ -\lambda_1^3\sin\lambda_1 l \\ -\lambda_1^3\cos\lambda_1 l \\ \mu^3\sinh\mu_1 l \\ \mu^3\cosh\mu_1 l \end{bmatrix} = 0. \tag{45}$$

The system (40)–(45) is a linear homogeneous system of algebraic equations. It has a non-trivial solution only in the case, if its determinant Δ equals to zero. The equation $\Delta = 0$ is solved up to the end numerically. Let us consider some examples. The results regarding to cantilever plate strips are presented in the Fig. 3 and Fig. 4. Here the natural frequency is plotted versus the location of the step $a = \alpha l$. It can be seen from Fig. 3 and Fig. 4 that the frequency of a cracked structure is always less than that of a strip without defects.

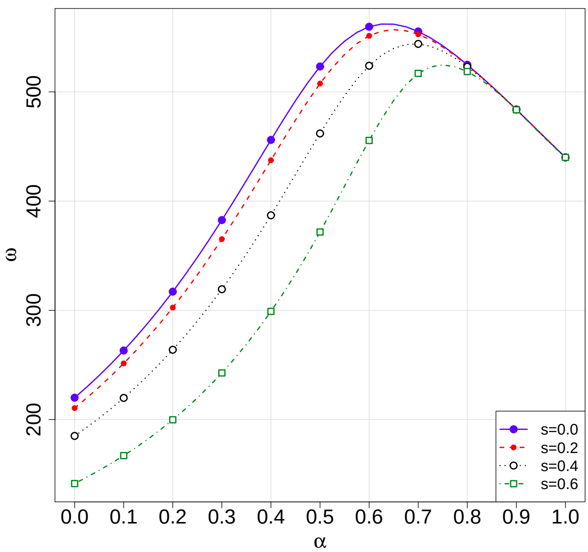


Figure 3: Natural frequencies of stretched strips; $\gamma = 0.5$.

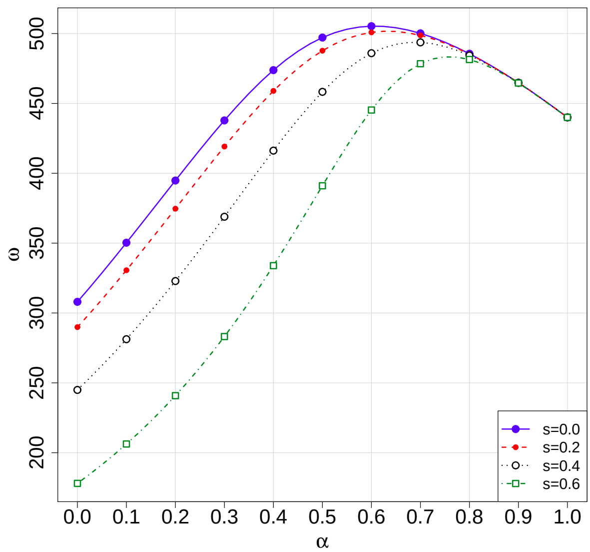


Figure 4: Natural frequencies of strips; $\gamma = 0.7$.

3 Free vibrations of stepped plate with cracks

The aim of this section is to determine the eigenfrequencies of the plate and to study the sensitivity of free vibrations on the crack location and depth. This section is based on the papers Kägo and Lellep [29, 30].

3.1 Statement of the problem and governing equations

Consider a thin stepped rectangular plate of anisotropic material, as shown in Fig. 5. The plate is simply supported at all edges and the plate has width b, length l. Assume that the thickness $h = h_j$ for $x \in (a_j, a_{j+1})$, where $j = 0, \ldots, n$. Let us introduce the notation

$$h_j = \gamma_j h_0, \tag{46}$$

where $0 < \gamma_j \leq 1$. The parameters h_j, a_j, γ_j will be treated as given constants.

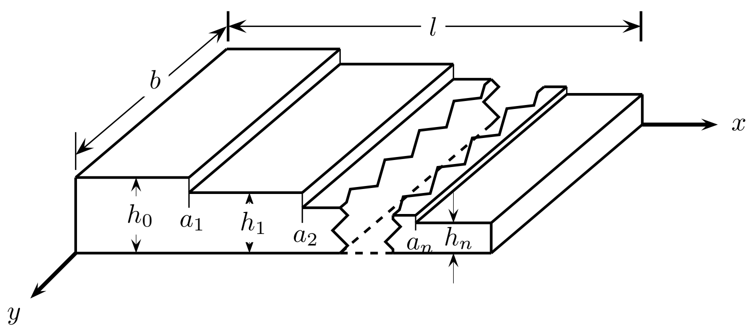


Figure 5: Stepped plate with crack

It is assumed, that the plate has part-through cracks [19] at the corners of the re-entrant parts of the steps. Hence according parameters for the cracks are the crack position

$$a_j = \alpha_j l, \tag{47}$$

where $0 \leq \alpha_j \leq 1$ and the crack length

$$c_j = s_j h_j, \tag{48}$$

where $0 \leq s_j < 1$.

The differential equation of the vibration of the anisotropic plate is (Reddy [69])

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = q - I_0\frac{\partial^2 w}{\partial t^2}. \tag{49}$$

In (49) $w(x, y, t)$ is the deflection of the plate, D_{ij} stand for flexural stiffness coefficients, $I_0 = \rho h b$ is the moment of inertia, ρ is the density of the material and q is the load intensity. Since we study free vibrations of the plate one has to take $q = 0$.

The function $w(x, y, t)$ is presented in the form

$$w(x, y, t) = X(x) \sin\left(\frac{k\pi y}{b}\right) \cos(\omega t). \tag{50}$$

Substituting (50) into (49) we obtain

$$D_{11} X^{IV} - 2\left(D_{12} + 2D_{66}\right)\left(\frac{k\pi}{b}\right)^2 X'' + D_{22}\left(\frac{k\pi}{b}\right)^4 - I_0\omega^2 = 0. \tag{51}$$

If the material is an unidirectionally reinforced fiber composite then the constants D_{ij} take the form

$$D_{11} = \frac{E_1 h^3}{12(1 - \nu_{12}\nu_{21})}, \tag{52}$$

$$D_{22} = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})}, \tag{53}$$

$$D_{12} + 2D_{66} = \frac{\nu_{12} E_2 h^3}{12(1 - \nu_{12}\nu_{21})} + \frac{2G_{12} h^3}{12}. \tag{54}$$

Here E_1 and E_2 are Young's moduli, ν_{12} and ν_{21} are Poisson's ratios, h_i are the plate thicknesses, where $i = 0, \ldots, n$, and G_{12} is shear modulus (Herakovich [18]). Hence, solving the characteristic equation of (51) the solution of the fourth order equation can be expressed as

$$X_j(x) = A_{1j} \sin\lambda_j x + A_{2j} \cos\lambda_j x + A_{3j} \sinh\mu_j x + A_{4j} \cosh\mu_j x \tag{55}$$

for $x \in (a_j, a_{j+1})$, $j = 0, \ldots, n$. Here $A_{1j}, \ldots, A_{4j}$ are integration constants, which are to be determined using the boundary and continuity conditions.

However, it appears that the quantity X' cannot be continuous at $x = a_j$ according to the model of distributed line springs developed by Rice and Levy [71]; Dimarogonas [12], Chondros *et al.* [8].

3.2 Additional flexibility due to the crack

The weakening effect of cracks, flaws, notches and different types of defects was recognized a long ago. The relationship between the additional compliance C and the stress intensity coefficient K of a cracked beam was caught by Irwin [24]. On the other hand, the stress intensity coefficient is coupled with the energy release rate computed for the infinitesimal change of the crack length.

In the case of anisotropic bodies with eventual fracture modes I and II the energy release rate is (see Nikpour [61], Nikpur and Dimarogonas [62])

$$G = -\frac{A_{22}}{2}\,\mathrm{Im}\left(\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right)K_I^2 + \frac{A_{11}}{2}\,\mathrm{Im}(\mu_1+\mu_2)K_{II}^2 \\ + A_{11}\,\mathrm{Im}(\mu_1\mu_2)K_I K_{II}. \tag{56}$$

In (56) A_{11}, A_{22}, A_{12} stand for elastic constants for an anisotropic material (see Herakovich [18], Reddy [69])

$$\begin{aligned} A_{11} &= \frac{1}{E_1}\left(1-\frac{E_2}{E_1}\nu_{12}^2\right), \\ A_{22} &= \frac{1}{E_2}\left(1-\nu_{23}^2\right), \\ A_{12} &= -\frac{\nu_{12}}{E_1}\left(1+\nu_{23}\right). \end{aligned}$$

Here

$$\mathrm{Im}(\mu) = y$$

if the complex number $\mu = x + iy$.

The complex numbers μ_1, μ_2 in (56) are the roots of the characteristic equation

$$A_{11}\mu^4 - 2A_{16}\mu^3 + (2A_{12}+A_{66})\mu^2 - 2A_{26}\mu + A_{22} = 0. \tag{57}$$

In the following we shall confine to the unidirectionally reinforced composites. Moreover, let us confine to the case of cracks of the first mode. Let the stress intensity coefficient and the bending moment applied to the crack at $x = a_j$ be K_j and M_j, respectively. It is well known in the linear elastic fracture mechanics that for isotropic materials

$$K_j = \frac{6M_j}{bh^2}\sqrt{\pi c_j}F \tag{58}$$

where

$$h = \min(h_{j-1}, h_j)$$

Following the paper [3] we can state that in the case of the orthotropic materials

$$K_j = \frac{6M_j}{bh_j^2}\sqrt{\pi c_j}F(s_j)Y(\xi), \tag{59}$$

provided $h_j < h_{j-1}$ and

$$Y(\xi) = 1 + 0.1(\xi - 1) - 0.016(\xi - 1)^2 + 0.002(\xi - 1)^3. \tag{60}$$

Here

$$\xi = \frac{\sqrt{E_{11}E_{22}}}{2G_{12}} - \nu_{12}\sqrt{\frac{E_{22}}{E_{11}}}$$

and F is a shape function which must be determined on the basis of experimental data. It is known in the fracture mechanics that the energy release rate; the generalized force P and the compliance C are coupled as (see Broek [5])

$$G = \frac{P^2}{2b}\frac{dC}{dc}. \tag{61}$$

Specifying the last formula for the crack located at $x = a_j$ and taking into account that the generalized force $P = M_j$ one obtains

$$\frac{M_j^2}{2b}\frac{dC_j}{dc_j} = \frac{36M_j^2 A}{h_j^4 b}\pi c_j F^2(s_j)Y^2(\xi) \tag{62}$$

where

$$A = \frac{A_{22}}{2}\,\mathrm{Im}\left(\frac{\mu_1 + \mu_2}{\mu_1\mu_2}\right). \tag{63}$$

The solution of (62) which meets the natural initial condition ($C_j = 0$ when $s_j = 0$) can be presented as

$$C_j = \frac{72\pi}{h_j^2}f(s_j)AY^2 \tag{64}$$

where

$$\begin{aligned} f(s_j) = {} & 1.8624s_j^2 - 3.95s_j^3 + 16.375s_j^4 - 37.226s_j^5 + 76.81s_j^6 \\ & - 126.9s_j^7 + 172.5s_j^8 - 143.97s_j^9 + 66.56s_j^{10} \end{aligned} \tag{65}$$

It was shown in the previous section that the slope discontinuity at $x = a_j$

$$\Theta_j = \frac{\partial w}{\partial x}(a_{j+0}, t) - \frac{\partial w}{\partial x}(a_{j-0}, t) \tag{66}$$

can be calculated as

$$\Theta_j = C_j M_j \tag{67}$$

where C_j is given by (64) and $M_j = M_x(a_j, t)$. Since M_x is continuous with respect to x one can select

$$M_j = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right)\bigg|_{x=a_{j+0}}. \tag{68}$$

3.3 Determination of eigenfrequencies

In the case when the plate has an unique step the deflected shape of the plate can be presented according to (55) as

$$w(x) = A_1 \sin \lambda_0 x + A_2 \cos \lambda_0 x + A_3 \sinh \mu_0 x + A_4 \cosh \mu_0 x \qquad (69)$$

for $x \in [0, a]$ and as

$$w(x) = B_1 \sin \lambda_1 x + B_2 \cos \lambda_1 x + B_3 \sinh \mu_1 x + B_4 \cosh \mu_1 x \qquad (70)$$

for $x \in [a, l]$.

Arbitrary constants A_i, B_i $(i = 1, \ldots, 4)$ have to meet boundary requirements and intermediate conditions at $x = a$. The latters can be presented as (Lellep and Kägo [40])

$$\begin{aligned} w(a-0) &= w(a+0) \\ w'(a-0) &= w'(a+0) - pw''(a+0) \\ h_0^3 w''(a-0) &= h_1^3 w''(a+0) \\ h_0^3 w'''(a-0) &= h_1^3 w'''(a+0) \end{aligned} \qquad (71)$$

where p is the stiffness parameter defined according to (64)–(68).

It is worthwhile to mention that the third and the fourth equality in (71) express the continuity of the bending moment and the shear force, respectively, when passing the step at $x = a$. It is known from the solid mechanics that these quantities must be continuous (Soedel [74]).

The obtained system is a linear homogeneous system of algebraic equations. It has a non-trivial solution only in the case, if its determinant Δ equals to zero. The equation $\Delta = 0$ is resolved up to the end numerically. In the following figures some examples are shown. The results for simply supported plates are presented in Fig. 6 and Fig. 7. Fig. 6 corresponds to the case $h_1 = 0.5h_0$ whereas Fig. 7 is associated with $h_1 = 0.7h_0$

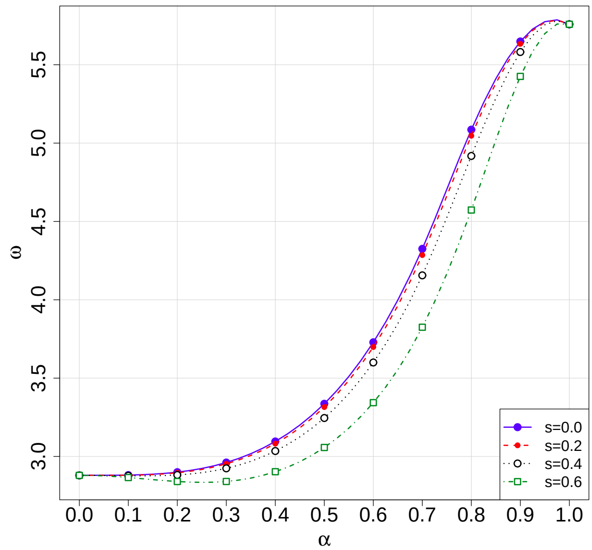


Figure 6: Natural frequencies of an anisotropic plate; $\gamma = 0.5$.

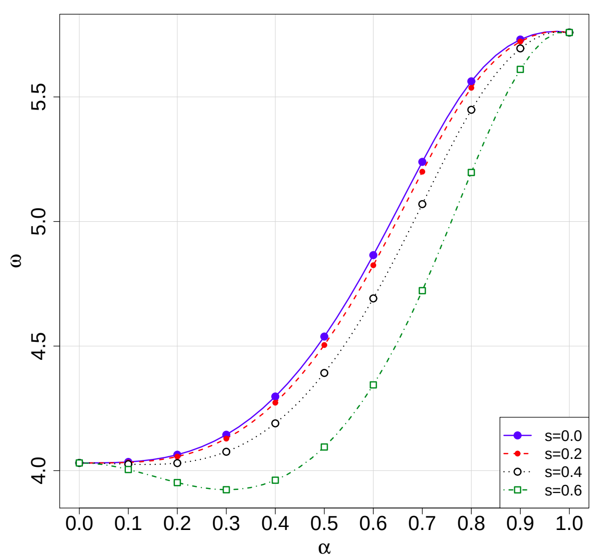


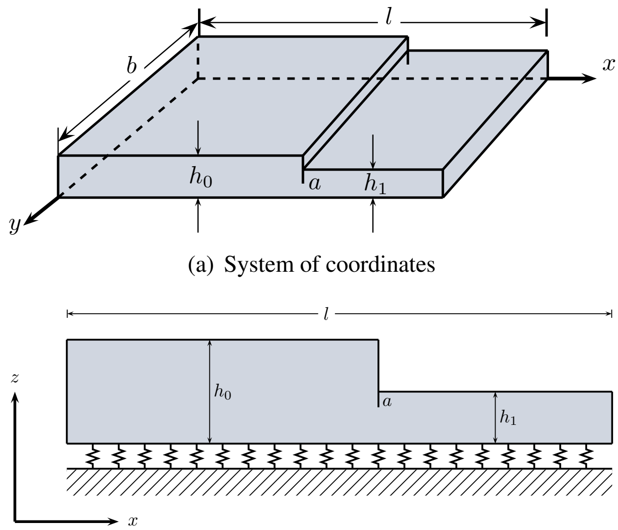
Figure 7: Natural frequencies of an anisotropic plate; $\gamma = 0.7$.

4 Natural vibrations of stepped plate with cracks on elastic foundation

The aim of this section is to consider the effect of the elastic foundation on the eigenfrequencies of the plate. Additionally, we study the sensitivity of free vibrations on the crack location and depth. This section is based on the papers by Kägo and Lellep [27, 28].

4.1 Formulation of the problem

We consider an elastic stepped rectangular plate (Fig. 8) made of anisotropic material with dimensions and crack locations like in previous section.



(a) System of coordinates

(b) Influence of the foundation

Figure 8: Geometry of the plate

4.2 Equations of motion

According to the classical thin plate theory given by Reddy [69] the differential equation of the free vibration of thin plates on the Winkler foundation (Fig. 8) can be expressed as follows [47]

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + \kappa w - \omega^2 \rho w = 0, \qquad (72)$$

where $w(x, y, t)$ denotes the function of the mode shape, κ is the elasticity coefficient of the foundation, ω is the natural frequency, ρ is the mass density.

The transverse displacement w can be written in the form

$$w(x, y, t) = X(x) \sin\left(\frac{k\pi y}{b}\right) e^{\lambda t}, \tag{73}$$

where k is the wave number of the kth mode in y-direction ($k = 1, 2, \dots$), b is the width of the plate, λ is a complex number.

Substituting (73) into (72) one obtains

$$\begin{aligned} D_{11} X^{IV} - 2\left(D_{12} + 2D_{66}\right)\left(\frac{k\pi}{b}\right)^2 X'' + \\ + \left(D_{22}\left(\frac{k\pi}{b}\right)^4 + \kappa - \omega^2 \rho\right) X = 0. \end{aligned} \tag{74}$$

The solution of equation (74) can be expressed as

$$X_j(x) = A_{1j} \sin \lambda_j x + A_{2j} \cos \lambda_j x + A_{3j} \sinh \mu_j x + A_{4j} \cosh \mu_j x \tag{75}$$

for $x \in (a_j, a_{j+1})$, $j = 0, \dots, n$. Here $A_{1j}, \dots, A_{4j}$ are integration constants, which are to be determined using the boundary and continuity conditions [40].

However, it appears that the quantity X' cannot be continuous at $x = a_j$ according to the model of distributed line springs developed by Chondros *et al.* [8], Rice and Levy [71], Dimarogonas [12].

The crack emanating from the re-entrant corner of the plate affects the vibrational behavior of the structure. The influence of cracks on vibrations of the plate was described in the previous section.

Some examples are presented in Fig. 9 and Fig. 10. These results regard to anisotropic plates on elastic foundation, which are clamped at two opposite edges and other two are free. Fig. 9 corresponds to the case $h_1 = 0.5h_0$ and Fig. 10 is associated with $h_1 = 0.7h_0$. The length of the crack is the same in the both cases.

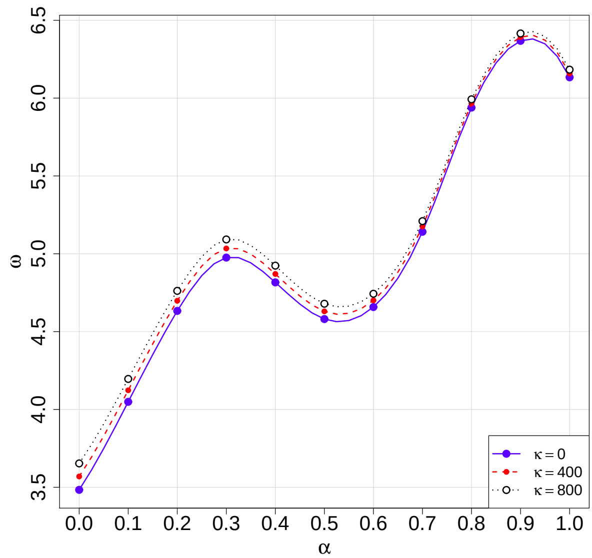


Figure 9: Natural frequencies of a stepped plate on elastic foundation; $\gamma = 0.5$, $s = 0.6$.

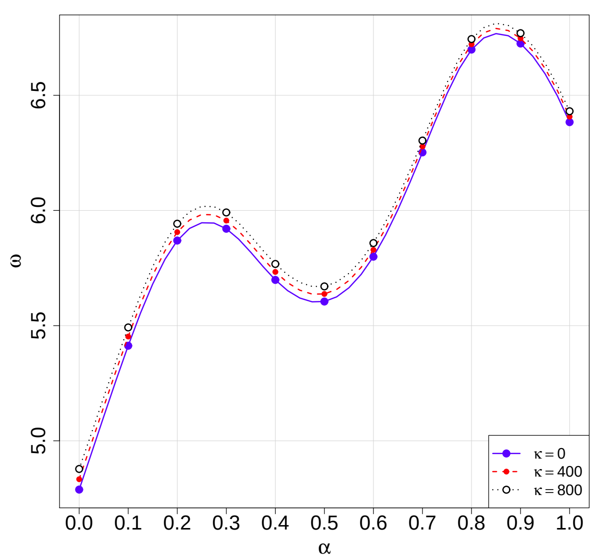


Figure 10: Natural frequencies of a stepped plate on elastic foundation; $\gamma = 0.7$, $s = 0.6$

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Summary

In the current dissertation free vibrations of elastic plates and plate strips are studied. It is assumed that the plates are weakened by crack-like defects and have piecewise constant thickness. The cracks are considered as stationary surface cracks which are located at the re-entrant corners of the steps and which have not fully penetrated the plate thickness.

By combining the theory of elastic plates and the theory of the linear elastic fracture mechanics a new method for determining the natural frequencies of elastic structures is developed in the dissertation.

The dissertation is based on the six papers of the author (three of these are published during the last three years). The dissertation consists of the review paper of the obtained results, the copies of the papers, the list of literature and CV of the author.

The review paper consists of the historical review of the literature (Section 1) and of three main sections.

In the second section of the paper natural frequencies of plate strips subjected to the axial tension are studied. The material of the plate strips is assumed to be a pure elastic material and the hypothesis of Kirchhoff are assumed to hold good. A refined version of the classical bending theory is employed.

The influence of cracks on the vibrational characteristics is taken into account according to the model of distributed line springs. The latter uses the stress intensity coefficient known in the elastic fracture mechanics.

In the subsequent sections the developed method is used for determination of the natural frequency of free vibrations of anisotropic plates with and without an elastic foundation. The influence of geometrical and material parameters on the vibration of the plates resting on the elastic foundation has been analyzed.

This solution can be used in nondestructive testing.

Kokkuvõte

Pragudega elastsete astmeliste plaatide omavõnkumised

Käesolevas dissertatsioonis vaadeldakse isotroopsete ribade ja anisotroopsete plaatide omavõnkumisi. Vaatluse all olevad plaadid on tükiti konstantse paksusega ning nõrgestatud pragudega.

Antud dissertatsioon põhineb autori kuuel teaduslikul publikatsioonil. Kolm neist publikatsioonidest on avaldatud kolme viimase aasta jooksul.

Väitekiri koosneb kokkuvõtvast ülevaateartiklist, publikatsioonide koopiatest, kirjanduse ülevaatest ja autori elulookirjeldusest. Ülevaateartikkel omakorda koosneb ajaloolisest kirjanduse ülevaatest ja kolmest põhiosast koos kokkuvõtvate järeldustega.

Esimeses põhiosas vaadeldakse plaadi ribade vabavõnkumisi erinevate rajatingimuste korral, juhul kui plaadi ribale mõjub telgsuunaline tõmbejõud. Eeldame, et plaadi ribad on elastsest materjalist ja kehtivad Kirchhoffi hüpoteesid.

Teises põhiosas on vaatluse all anisotroopsed plaadid ning kolmandas põhiosas anisotroopsed plaadid elastsel alusel. Mõlemal juhul vaadeldakse ülesannet erinevate rajatingimuste korral.

Plaatide ja plaadi ribade geomeetrilisteks iseärasusteks on astmelisus ja paksuse muutumise kohtades asuvad praod. Praod on stabiilsed ja konstantse pikkusega. Prao mõju plaadi omavõnkumisele võetakse arvesse lokaalse järeleandlikkuse koefitsiendi abil ning pinge intensiivsuse koefitsiendiga, mis tuleneb purunemismehaanikast.

Kõikidel juhtudel on omavõnkumiste uurimiseks kasutatud analüütilist meetodit, mis põhineb klassikalisel plaatide teooria ning purunemismehaanika võrranditel ja kriteeriumitel.

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Publications

Curriculum vitae

Name:	Esta Kägo
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Education

1997 - 2000	Võru Kreutzwald Gymnasium
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Research work

The main field of reasearch is studing of vibrations of elastic cracked plates and plate strips with analytical-numerical methods. Results have been published in 3 papers and other 3 are submitted for publication. Results have been presented at the five conferences:

"The Third Finnish-Estonian Mathematical Colloquium", Tartu, Estonia (2009),

The Int. Conf. "Mathematical Models for Engineering Science", Puerto de la Cruz, Spain (2010),

The Int. Conf. "Optimization and Analysis of Structures", Tartu, Estonia (2011),

17th Int. Conf. "Mechanics Of Composite Materials", Riga, Latvia (2012),

11th Int. Conf. "Modern Building Materials, Structures and Techniques", Vilnius, Lithuania (2013).

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Haridus

1997 - 2000	Võru Kreutzwaldi Gümnaasium
2002 - 2005	Tartu Ülikool, matemaatika-informaatikateaduskond, loodusteaduse bakalaureuse kraad matemaatikas
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Teaduslik tegevus

Peamine uurimisvaldkond on elastsete pragudega plaatide ja plaadi ribade võnkumiste uurimine analüütilis-numbriliste meetoditega. Tulemused on publitseeritud kolmes teadusartiklis, lisaks on kolm artiklit esitatud publitseerimiseks. Tulemustega on esinetud viiel teaduskonverentsil:

The Third Finnish-Estonian Mathematical Colloquium, Tartu, Eesti (2009),

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