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DANIEL BLIXT

Hamiltonian analysis of covariant teleparallel theories of gravity

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Hamiltonian analysis of covariant teleparallel theories of gravity

UNIVERSITY OF TARTU
Press

This study was carried out at the University of Tartu.

The dissertation was admitted on 08.06.2021 in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics, and was allowed for defense by the Council of the Institute of Physics, University of Tartu.

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Defence: 17.08.2021, University of Tartu, Estonia

The author was supported by the University of Tartu ASTRA Project PER ASPERA. Part of the support came through the personal research funding project PUT790 (start-up grant), as well as PRG356 (team grant) PSG489 (start-up grant). Further support was given by the European Union through the European Regional Development Fund projects Center of Excellence TK133 (The Dark Side of the Universe). Additional financial support came from Archimedes Foundation through Kristjan Jaak and Dora Plus scholarships, and the CA-15117 Cost action CANTATA and the CA-16104 Cost action Gravitational waves, black holes and fundamental physics, and the Nordic Institute for Theoretical Physics NORDITA.

ISSN 1406-0647
ISBN 978-9949-03-658-5 (print)
ISBN 978-9949-03-659-2 (pdf)

University of Tartu Press
www.tyk.ee

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I D. Blixt, M. Hohmann, and C. Pfeifer "Hamiltonian and primary constraints of new general relativity" *Phys. Rev.* D **99**, 084025 (2019)
See Chapter 4, and arXiv:1811.11137 [INSPIRE] [ETIS]

II D. Blixt, M. Hohmann, and C. Pfeifer "On the gauge fixing in the Hamiltonian analysis of general teleparallel theories " *Universe* **5**, 6 (2019)
See Chapter 5, and arXiv:1905.01048 [INSPIRE] [ETIS]

III D. Blixt, M. Hohmann, M. Krššák and C. Pfeifer "Hamiltonian Analysis in New General Relativity." Submitted to:
Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity
See Chapter 6, and arXiv:1905.11919 [INSPIRE]

IV D. Blixt, M-J. Guzmán, M. Hohmann and C. Pfeifer "Review of the Hamiltonian analysis in teleparallel gravity. Accepted for publication in IJGMMP-D-20-00614
See Chapter 7, and arXiv:2012.09180 [INSPIRE]

Author's contribution

I, Daniel Blixt, have calculated and checked each and every equation in the papers. I wrote most of the manuscript for References I, II, III and a major part of IV. I was involved in all discussions and in the implementation of peer reviewers suggestions. A lot of credit goes to Manuel Hohmann and Christian Pfeifer in all of the References I, II, III, and IV for supervising, checking and deriving equations. They have been giving me fast feedback and made sure the scientific progress went in the right direction. For Reference III I would like to thank Martin Krššák for contributing with very valuable references and for making us improve the understanding of new general relativity. For Reference IV I am very grateful

In accordance with a tradition in theoretical physics, the authors are ordered alphabetically, except when a paper is published in conference proceedings.

for the contribution from María-José Guzmán. Her understanding of the Hamiltonian analysis in $f(\mathbb{T})$-gravity and of the Dirac-Bergmann algorithm as a whole have had a great impact on the this article.

I have presented the results of I by giving a talk at the conferences Teleparallel Gravity Workshop in Tartu (2018, Tartu) [ETIS], the Fifteenth Marcel Grossmann Meeting-MG15 (2018, Rome, Italy)-for which III also was presented, Tuorla-Tartu annual meeting 2018: The large scale properties of the universe as a whole (2018, Tuorla, Finland) [ETIS], Teleparallel Universes in Salamanca (2018, Salamanca, Spain). Paper I was also presented as poster presentations at the conference Gravity@Malta 2018 (2018, Valletta, Malta), and at the winter school 48^{th} Saas-Fee course: Black hole formation and growth (2018, Saas-Fee, Switzerland). I have presented the results of II by giving a talk at the conference Geometric Foundations of Gravity 2019 (2019, Tartu). I have presented the results of IV by giving a talk at the conferences THE FOURTH ZELDOVICH VIRTUAL MEETING (2020, fully online) and International Webinar on Recent Developments in Cosmology and Modified Gravity (RDCM-2021) (2021, fully online).

Chapter 1

Introduction

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1.1 Motivation

Currently it is widely accepted that there are four fundamental forces in nature-electromagnetic, weak, strong force, and gravitation. The first three forces are incorporated in the standard model of particle physics while gravity is described by the theory of general relativity. Gravity is the weakest among the fundamental forces of nature and will be the focus in this thesis. Even though the gravitational force is weak compared to the other forces it becomes very relevant in the presence of very massive objects, such as the Earth, Moon, Sun, galaxies, stars and black holes.

For consideration in experiments conducted on earth, Newtonian gravity is in most cases sufficient. However, in order to describe phenomena like the precession of Mercury's orbit around the sun, and the bending of light, by the gravity of the sun, corrections from general relativity are needed. The first successful experimental test of general relativity was performed in 1919 by Frank Watson Dyson and Arthur Stanley Eddington, which proved that, in contrast to Newtonian gravity, general relativity predicts the correct bending of light around the sun [1]. Furthermore, general relativity has predicted gravitational waves and black holes which were detected for the first time in the year 2015 [2] and 2019 [3], respectively.

Among these theories describing the four fundamental forces of physics, gravity is the only one which is not formulated as a quantum field theory. Applying the standard methods of quantization to gravity leads to a non-renormalizable theory [4] with the consequence that there is no unique way to quantize gravity (in our current understanding). Hence, the incorporation of gravity into the standard model of particle physics is an open question. Furthermore, at galactic, to cosmological scales, there are observations which have no conclusive explanation yet. These are dark matter and dark energy, which make up of 95% of our Universe and is yet to be detected, the tension in observations of the Hubble constant based on the standard model of cosmology on one hand [5], and observations from standard candles on the other hand [6]. Modified theories of gravity could be the most promising explanation for solving the aforementioned observational tension [7].

There is a zoo of modified gravity theories one could create in order to explain one or several issues which are confronting the modern view of theoretical physics, cosmology, and astrophysics. In this thesis modified theories of gravity are constructed from the building blocks of an action formulation of general relativity (discussed in section 2.1.3). The motivations are twofold. Firstly, these theories have a special case being general relativity, which has passed many experiments. Secondly, a few numbers of new parameters are introduced which makes it easier to falsify by observations and experiments.

Conventionally, general relativity is formulated in terms of the Levi-Civita connection with non-vanishing curvature, while having vanishing torsion and non-metricity. There are, however, alternative formulations for general relativity. Among these there is teleparallel equivalent to general relativity (TEGR for short), based on a connection with no curvature and no non-metricity, but non-zero torsion. In general, theories based on such a connection are called teleparallel theories of gravity and are studied in this thesis. In particular, the Hamiltonian analysis of their covariant formulation is considered for the first time.

1.2 Aim of the thesis

As mentioned in section 1.1 there are modified theories of gravity, which are motivated both from a fundamental and an observational point of view. Within the community of modified gravity there are different philosophies. Some part of the community aims to describe a unified theory of everything, which may deviate significantly from Einstein's theory of general relativity. An alternative philosophy is using the approach of extended theories of gravity, where the theory is formulated with general relativity as the starting point.

In the latter case most attention has been given to theories with the Einstein-Hilbert formulation of general relativity as the starting point. In this thesis, an alternative action for general relativity is instead considered as the base for modified theories of gravity. Namely, teleparallel theories of gravity, which are using the teleparallel equivalent to general relativity as its starting point.

The oldest, and most studied teleparallel modifications to general relativity are $f(\mathbb{T})$-gravity [8] and new general relativity [9]. Recently, many publications have been exploring observational consequences of teleparallel gravity in the context of cosmology and astrophysics. However, there are serious doubts that teleparallel theories of gravity (except those who are equivalent to extended theories of gravity based on the Einstein-Hilbert formulation of general relativity) are viable from a theoretical point of view. This is seen from the indications of strongly coupled fields in the aforementioned most basic teleparallel modifications of general relativity.

The main aim of this thesis is to get insights of the viability of teleparallel theories of gravity. Thus, it is in order to give a short description of what is meant by the word "viable" in this thesis. Firstly, it is required that the theory is free of ghost instabilities. Secondly, the theory should not contain strongly coupled fields since the appearance of those would create instabilities in the infrared limit. Strongly coupled field are found if there is a discrepancy of the number degrees of freedom at different orders around a specific background. If a strongly coupled field is found around a background the perturbations for this background becomes ill-defined [10]. This has a severe consequence, affecting the predictive power of gravity.

A powerful method for understanding the viability of a theory is by performing the Dirac-Bergmann algorithm (explained in section 3.1) which is used for the

Hamiltonian analysis. Before the initialization of this thesis the understanding of the Hamiltonian formulation of the most basic theories of teleparallel gravity (including the teleparallel equivalent to general relativity) was unclear. Furthermore, the Hamiltonians were never analyzed in the covariant formulation of teleparallel gravity, instead it was done in the so-called Weitzenböck gauge. Generically the Hamiltonian analysis answers the question of how many degrees of freedom propagate for the full nonlinear theory, and give insights on which symmetries are preserved or broken. Furthermore, the Hamiltonian analysis can reveal the appearance of ghosts.

In the finalization of this thesis, the role of the aforementioned gauge choice have been investigated in new general relativity I, and for more general teleparallel theories in II. Further, the Hamiltonian was written out in the irreducible decomposition of the torsion components in III in order to investigate if this can shed any light on the fundamental aspects of new general relativity. The literature on Hamiltonian formulation of teleparallel theories of gravity contains various approaches, with different conventions among different authors. Furthermore, the claims of some publications contradict others [11, 12]. This motivated the need for a review article IV. More than simply reviewing the current literature of the Hamiltonian analysis of teleparallel theories of gravity, the aim was to present an easy way of comparing the notations by different authors and to check the consistency among current publications. In order to do this in a compact way for both $f(\mathbb{T})$ and new general relativity, newly $f(\mathbb{T}_{\mathrm{NGR}})$ was considered. Furthermore, the review aimed to point out peculiarities in teleparallel theories of gravity as well as shedding some light on the viability of teleparallel theories of gravity. The aim of the overview article is to present the covariant Hamiltonian analysis for $f(\mathbb{T}_{\mathrm{NGR}})$ and to discuss the viability of teleparallel theories of gravity.

The aim of the thesis can be summarized in the following points:

- To investigate the role of the spin-connection in the Dirac-Bergmann algorithm of teleparallel theories of gravity.

- To review the literature on Hamiltonian analysis of teleparallel gravity and to check to what extent they are consistent.

- To create an easy way for the reader to compare different notations from various authors.

- To look at the Hamiltonian analysis of a parent theory in which the most studied teleparallel theories of gravity are included with a special choice of parameters.

- To perform the Hamiltonian analysis in the covariant formulation of teleparallel theories of gravity.

- To get further insights of the role irreducible components of the torsion scalar play in teleparallel theories of gravity.

- To draw conclusions about the viability of teleparallel theories of gravity.

1.3 Statements

As a result of this thesis, various statements can be made concerning teleparallel theories of gravity.

1. For very general classes of teleparallel theories (including $f(\mathbb{T}_{\text{NGR}})$) the number of degrees of freedom found from the Dirac-Bergmann algorithm is independent of the gauge choice of the spin-connection II.

2. In new general relativity, it is a necessary condition that the coefficient of the vector irreducible component of the torsion scalar is non-vanishing in order for the theory to describe gravity (see section 3.8).

3. In order for new general relativity to describe gravity and to, simultaneously, avoid ghost instabilities it is a necessary condition that the coefficient of the tensor irreducible component of the torsion scalar is non-vanishing (see section 3.8).

4. The propagation of a Kalb-Ramond field in new general relativity is related to the coefficient of the axial irreducible component of the torsion scalar in ghost-free new general relativity (see section 3.8).

5. For teleparallel theories of gravity the covariant Hamiltonian explicitly depends on the spin-connection (see section 3.5).

6. For teleparallel theories of gravity the covariant Hamiltonian has 6 primary constraints. They relate the conjugate momenta with respect to the Lorentz matrices with the conjugate momenta of the spatial tetrads (see Paper I, and II).

7. The conjugate momenta related to the Lorentz matrices (and thus related to the spin-connection) does not need to appear in the Hamiltonian except as a part of a Lagrange multiplier associated with primary constraints relating it with the conjugate momenta of the spatial tetrads II.

8. The literature considered in IV is completely consistent in the application of the Dirac-Bergmann algorithm up to and including the derivation of the primary Hamiltonian (canonical Hamiltonian + primary constraints).

9. There are many indications that both $f(\mathbb{T})$ gravity and new general relativity suffer from strongly coupled fields around physically motivated backgrounds (see section 3.8).

1.4 Structure of the thesis

Chapter 2 provides some mathematical prerequisites 2.1 for teleparallel gravity and in particular new general relativity 2.2 as well as $f(\mathbb{T})$ theories of gravity 2.3, and their parent theory, $f(\mathbb{T}_{\mathrm{NGR}})$ gravity 2.4. The main part of my thesis is covered in chapter 3. The first section 3.1 introduces the Dirac-Bergmann algorithm which is used in this thesis in the Hamiltonian analysis. In section 3.2 it is described how spacetime can be split in a 3+1 decomposition. Then, section 3.3 discusses different approaches for the Hamiltonian analysis of teleparallel gravity. The following sections present the results of my work.

In section 3.4 it is argued for how the structure of the Hamiltonian will differ in the Weitzenböck gauge contra how it looks like in its covariant formulation. From those arguments, it is concluded that the Dirac-Bergmann algorithm leads to the same answer regarding the numbers of degrees of freedom of the full nonlinear theory. Section 3.5 is dedicated to deriving the primary Hamiltonian for $f(\mathbb{T}_{\mathrm{NGR}})$-gravity, which is a new result. In section 3.6 the primary Hamiltonian for new general relativity is considered in the standard form as well as in terms of irreducible components of the torsion scalar. This sheds some new light on the fundamental meaning of the irreducible components of torsion. In section 3.7 the primary Hamiltonian for $f(\mathbb{T})$-gravity is considered.

Finally, section 3.8 is dedicated to the discussion of the viability of teleparallel theories of gravity. In order to reach conclusions for the viability the main results from perturbation theory around simple backgrounds is discussed as well. This is followed by the summary 3.8, acknowledgments, bibliography and a summary in Estonian. The next chapters are dedicated to the attached publications. In chapter 4 the Paper I "Hamiltonian and primary constraints of new general relativity" is presented. Paper II "On the Gauge Fixing in the Hamiltonian Analysis of General Teleparallel Theories" is presented in chapter 5. Then Paper III "Hamiltonian Analysis in New General Relativity" is presented in chapter 6. The last publication IV "Review of the Hamiltonian analysis in teleparallel gravity" is presented in chapter **??**. Finally, my curriculum vitae is presented in English and Estonian.

The convention used in this thesis are the following. Metrics assume the mostly negative Lorentzian signature. Greek indices $\mu, \nu, \rho, \ldots$ represents coordinate indices while capital latin indices $A, B, C, \ldots$ represents Lorentz indices. Minor latin indices $i, j, k, ...$ represents spatial coordinate indices. Units are chosen such that the velocity of light in vacuum equals unity: $c = 1$.

Chapter 2

Teleparallel gravity

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This chapter introduces modified teleparallel theories of gravity, i.e., teleparallel theories of gravity different from teleparallel equivalent to general relativity. Instead of going through the whole scope of possible teleparallel theories of gravity, this thesis focuses on the most popular modifications. They are named "new general relativity" and "$f(\mathbb{T})$-gravity" and are covered in section 2.2 and section 2.3, respectively. Some analysis can effectively be done for both theories by considering the parent theory "$f(\mathbb{T}_{\text{NGR}})$-gravity". Section 2.4 introduces the aforementioned theory. In chapter 3 the Hamiltonian analysis of $f(\mathbb{T}_{\text{NGR}}) - gravity$ will be presented and it will then be easy to relate the results from the publications to this result.

2.1 Mathematical preliminaries

In this subsection relevant mathematical preliminaries are introduced. In subsection 2.1.1 the notion of affine connection is introduced. Subsection 2.1.2 is giving a very short introduction to general relativity in the Einstein-Hilbert formulation. In subsection 2.1.3 different formulations of general relativity are presented, and the subsectionis also dedicated to explaining the relation between the different formulations.

2.1.1 The affine connection

Consider a metric $g_{\mu\nu}$ with Lorentzian signature, and a general linear connection $\Gamma^{\alpha}{}_{\mu\nu}$, generically more general than general relativity, which is considered in section 2.1.2. This connection is called the affine connection, and can be used to formulate theories of gravity. It defines the covariant derivative such that

$$\begin{aligned}\nabla_\lambda T^{\alpha\beta\ldots\gamma\delta}{}_{\mu\nu\ldots\rho\sigma} = \partial_\lambda T^{\alpha\beta\ldots\gamma\delta}{}_{\mu\nu\ldots\rho\sigma} + \Gamma^{\alpha}{}_{\kappa\lambda} T^{\kappa\beta\ldots.\gamma\delta}{}_{\mu\nu\ldots\rho\sigma} + \dots \\ +\Gamma^{\delta}{}_{\kappa\lambda} T^{\alpha\beta\ldots.\gamma\kappa}{}_{\mu\nu\ldots\rho\sigma} - \Gamma^{\kappa}{}_{\mu\lambda} T^{\alpha\beta\ldots\gamma\delta}{}_{\kappa\nu\ldots\rho\sigma} - \dots - \Gamma^{\kappa}{}_{\sigma\lambda} T^{\alpha\beta\ldots\gamma\delta}{}_{\mu\nu\ldots\rho\kappa}.\end{aligned} \tag{2.1}$$

A general geometrical framework defined by a metric and affine connection is characterized by a number of tensorial quantities, namely non-metricity, torsion, and curvature.

Non-metricity is defined by

$$Q_{\alpha\mu\nu} := \nabla_\alpha g_{\mu\nu}, \tag{2.2}$$

while torsion is defined by the antisymmetric part

$$T^{\alpha}{}_{\mu\nu} := 2\Gamma^{\alpha}{}_{[\nu\mu]}. \tag{2.3}$$

Curvature is defined by the Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} := 2\partial_{[\rho}\Gamma^{\mu}{}_{|\nu|\sigma]} + 2\Gamma^{\mu}{}_{\lambda[\rho}\Gamma^{\lambda}{}_{|\nu|\sigma]}. \tag{2.4}$$

There are various ways to contract the indices of the Riemann tensor. However, only the ones listed below will be used in this thesis, and hence, we omit to define other contractions. From this the Ricci tensor

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}, \tag{2.5}$$

and Ricci scalar

$$R := g^{\mu\nu} R_{\mu\nu} \tag{2.6}$$

are defined. The unique symmetric and metric compatible connection is the Levi-Civita connection

$$\begin{Bmatrix} \alpha \\ \mu\ \nu \end{Bmatrix} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda} \right). \tag{2.7}$$

The affine connection can be decomposed into

$$\Gamma^{\alpha}{}_{\mu\nu} = \begin{Bmatrix} \alpha \\ \mu\ \nu \end{Bmatrix} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}, \tag{2.8}$$

where

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2} T^{\alpha}{}_{\nu\mu} + T_{(\mu}{}^{\alpha}{}_{\nu)} \tag{2.9}$$

is called contortion, while

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu}{}^{\alpha}{}_{\nu)} \tag{2.10}$$

is the disformation. As a special case, choosing the connection to only depend on the Levi-Civita connection, or in other words to only posses curvature and not torsion nor nonmetricity, we denote these quantities by $\mathcal{R}^{\mu}{}_{\nu\rho\sigma}$, $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ instead. These quantities are central in the conventional description of general relativity.

Going to the special case of a connection with no curvature nor nonmetricity, equation 2.3 can be written out in more detail. Since the metric is symmetric it does not suffice to be used to express torsion. However, so-called tetrads or vielbeins which defines the metric by

$$g_{\mu\nu} = \eta_{AB} \theta^{A}{}_{\mu} \theta^{B}{}_{\nu} \tag{2.11}$$

can be used, with $\eta_{AB} = \mathrm{diag}\,(-1, 1, 1, 1)$ is the Minkowski metric. The inverse tetrads (cotetrads) are denoted by $e_A{}^{\mu}$. The tetrads and cotetrads satisfy the orthogonality condition

$$\theta^{A}{}_{\mu} e_{B}{}^{\mu} = \delta^{A}_{B}, \qquad \theta^{A}{}_{\mu} e_{A}{}^{\nu} = \delta^{\nu}_{\mu}. \tag{2.12}$$

The torsion components can be written explicitly in terms of tetrads in the following way:

$$T^A{}_{\mu\nu} = D_\mu \theta^A{}_\nu - D_\nu \theta^A{}_\mu = \partial_\mu \theta^A{}_\nu - \partial_\nu \theta^A{}_\mu + \omega^A{}_{B\mu}\theta^B{}_\nu - \omega^A{}_{B\nu}\theta^B{}_\mu, \tag{2.13}$$

where the spin-connection is defined as

$$\omega^A{}_{B\mu} = \Lambda^A{}_C \partial_\mu \Lambda_B{}^C, \tag{2.14}$$

with $\Lambda^A{}_B$ denoting Lorentz matrices. Since teleparallel gravity assumes a connection with zero curvature and nonmetricity these expressions will be very useful for the major part of this thesis.

2.1.2 General relativity

The most conventional action formulation of general relativity is through the so-called Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int \mathrm{d}^4x \sqrt{-g}\mathcal{R} + S_{\mathrm{M}}, \tag{2.15}$$

where $\sqrt{-g} := \sqrt{-\det g}$, $\kappa = 8\pi G$, and S_{M} is the matter action defined by. Varying this action gives rise to Eistein's field equations

$$G_{\alpha\beta} = 8\pi G \mathcal{T}_{\alpha\beta}, \tag{2.16}$$

where

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}, \tag{2.17}$$

and $\mathcal{T}_{\alpha\beta}$ is the stress-energy tensor. The matter action can be written as

$$S_{\mathrm{M}} = \int \mathrm{d}^4x L_M \sqrt{-g}, \tag{2.18}$$

and from this the stress energy tensor is defined as

$$\mathcal{T}_{\mu\nu} := -2\frac{\partial L_M}{\partial g^{\mu\nu}} + L_M g_{\mu\nu}. \tag{2.19}$$

The Hamiltonian analysis of general relativity will not be covered in this thesis, although there are many sources which cover this topic [13, 14]. Hamiltonian analysis of general relativity reveals that the nonlinear theory has two propagating degrees of freedom, which corresponds to a massless spin 2 field.

These two degrees of freedom are also expected to appear in linear perturbations. For later use consider the simplest case, being perturbations around Minkowski spacetime at lowest order. We define a perturbation of the metric by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}\left(h^2\right), \tag{2.20}$$

where $\eta_{\mu\nu} = \mathrm{diag}\left(-1, 1, 1, 1\right)$ is the Minkowski metric. A consistent definition for perturbations of the inverse metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}\left(h^2\right), \tag{2.21}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. With these perturbations the field equations become [15]

$$\Box h_{\mu\nu} + \partial_\nu\partial_\mu h - \partial_\nu\partial_\lambda h^\lambda{}_\nu - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\sigma\partial_\lambda h^{\lambda\sigma} = -16\pi G\mathcal{T}_{\mu\nu}. \tag{2.22}$$

2.1.3 The geometric trinity of gravity

The Einstein-Hilbert formulation is not unique in describing general relativity. This can be seen from the following observation. Consider the Ricci scalar of an affine connection

$$R = g^{\rho\nu}\left(\partial_\mu\Gamma^\mu{}_{\rho\nu} - \partial_\nu\Gamma^\mu{}_{\rho\mu} + \Gamma^\mu{}_{\sigma\mu}\Gamma^\sigma{}_{\rho\nu} - \Gamma^\mu{}_{\sigma\nu}\Gamma^\sigma{}_{\rho\mu}\right). \tag{2.23}$$

Using equation (2.8) this can be written as

$$R = \mathcal{R} + \mathbb{T} + \mathbb{Q} + Q_{\mu\nu\rho}T^{\rho\mu\nu} + Q_\mu T^\mu - \overline{Q}_\mu T^\mu + \mathcal{D}_\mu\left(Q^\mu - \overline{Q}^\mu - 2T^\mu\right), \tag{2.24}$$

where

$$\mathbb{T} := \frac{1}{4}T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} + \frac{1}{2}T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho - T^\rho{}_{\mu\rho}T^{\sigma\mu}{}_\sigma, \tag{2.25}$$

$$\mathbb{Q} := \frac{1}{4}Q_{\mu\nu\rho}Q^{\mu\nu\rho} - \frac{1}{2}Q_{\mu\nu\rho}Q^{\nu\mu\rho} - \frac{1}{4}Q_\mu Q^\mu + \frac{1}{2}Q_\mu\overline{Q}^\mu, \tag{2.26}$$

and traces

$$T_\mu := T^\rho{}_{\rho\mu}, \qquad Q_\mu := Q_{\mu\rho}{}^\rho, \qquad \overline{Q}_\mu = Q^\rho{}_{\rho\mu}, \tag{2.27}$$

and $\mathcal{D}_\mu$ is the covariant derivative with respect to the Levi-Civita connection.

The concept of teleparallelism is defined by $R^\mu{}_{\nu\rho\sigma} = 0$. This condition further implies $R = 0$. By imposing teleparallelism equation (2.23) can be written as [16]

$$\mathcal{R} = -\mathbb{T} - \mathbb{Q} - Q_{\mu\nu\rho}T^{\rho\mu\nu} - Q_\mu T^\mu + \overline{Q}_\mu T^\mu - \mathcal{D}_\mu\left(Q^\mu - \overline{Q}^\mu - 2T^\mu\right). \tag{2.28}$$

We can now insert this expression into equation (2.15). Note that the total derivative becomes a boundary term under the integral sign and can, hence, be dropped since it does not contribute to the field equations, although, these terms might become relevant for black hole entropies.

We will now look into two special cases within teleparallel gravity (without curvature). Firstly, let us assume that torsion $T^{A}{}_{\mu\nu}$ equals to zero. Then we get the symmetric teleparallel equivalent to general relativity

$$S_{\mathrm{STEGR}} = -\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\mathbb{Q} + S_{\mathrm{M}}. \tag{2.29}$$

In the second case we instead consider non-metricity $Q_{\mu\nu\rho}$ to be equal to zero while torsion is not. The action of teleparallel equivalent to general relativity is obtained in this case

$$S_{\mathrm{TEGR}} = -\frac{1}{2\kappa}\int \mathrm{d}^4x\theta\mathbb{T} + S_{\mathrm{M}}. \tag{2.30}$$

This action theory is the most basic of the theories considered in this thesis. Alternatively the action can be expressed as

$$S_{\mathrm{TEGR}} = -\frac{1}{4\kappa}\int \mathrm{d}^4x\theta T^{\rho\mu\nu}S_{\rho\mu\nu} + S_{\mathrm{M}}, \tag{2.31}$$

where the superpotential is defined as

$$S_{\rho\mu\nu} = \frac{1}{2}T_{\rho\mu\nu} + T_{[\mu\nu]\rho} + 2g_{\rho[\mu}T^{\sigma}{}_{\nu]\sigma}. \tag{2.32}$$

The action given by equation (2.30) is, for the scope of this thesis, the starting point in building modified teleparallel theories of gravity.

2.2 New general relativity

Changing the fixed parameters of equation (2.25) yields two parameter space[1] of new theories historically named "new general relativity"

$$S_{\mathbb{T}_{\mathrm{NGR}}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\theta\mathbb{T}_{\mathrm{NGR}} + S_{\mathrm{M}} \tag{2.33}$$

where

$$\mathbb{T}_{\mathrm{NGR}} = c_1 T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + c_2 T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} + c_3 T^{\rho}{}_{\mu\rho}T^{\sigma\mu}{}_{\sigma}.$$

[1] Note, that the theory is normalized such that only two of three constants become independent.

The torsion components can, alternatively, be decomposed into irreducible components [17]

$$T_{\mu\nu\rho} = \frac{2}{3}\left(t_{\mu\nu\rho} - t_{\mu\rho\nu}\right) + \frac{1}{3}\left(g_{\mu\nu}v_\rho - g_{\mu\rho}v_\nu\right) + \epsilon_{\mu\nu\rho\sigma}a^\sigma \tag{2.34}$$

where

$$v_\mu = T^\rho{}_{\rho\mu}, \tag{2.35}$$

$$a_\mu = -\frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma} \tag{2.36}$$

$$t_{\mu\nu\rho} = \frac{1}{2}\left(T_{\mu\nu\rho} - T_{\nu\mu\rho}\right) + \frac{1}{6}\left(g_{\mu\rho}v_\nu + g_{\nu\rho}v_\mu - 2g_{\mu\nu}v_\rho\right), \tag{2.37}$$

are the vector, axial, and tensor irreducible components of torsion. Torsion scalars can be expressed in these terms in the following way[2]

$$T_{\text{axi}} := a_\mu a^\mu = -\frac{1}{18}\left(T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} - 2T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho\right), \tag{2.38}$$

$$T_{\text{ten}} := t_{\rho\mu\nu}t^{\rho\mu\nu} = \frac{1}{2}\left(T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} + T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho\right) - \frac{1}{2}T^\mu{}_{\mu\rho}T_\nu{}^{\nu\rho}, \tag{2.39}$$

$$T_{\text{vec}} := v_\mu v^\mu = T^\mu{}_{\mu\rho}T_\nu{}^{\nu\rho}. \tag{2.40}$$

In this decomposition we can write

$$\mathbb{T}_{\text{NGR}} = c_{\text{axi}}T_{\text{axi}} + c_{\text{ten}}T_{\text{ten}} + c_{\text{vec}}T_{\text{vec}}, \tag{2.41}$$

with the relations

$$c_1 = -\frac{1}{18}c_{\text{axi}} + \frac{1}{2}c_{\text{ten}}, \qquad c_2 = \frac{1}{9}c_{\text{axi}} + \frac{1}{2}c_{\text{ten}}, \qquad c_3 = c_{\text{vec}} - \frac{1}{2}c_{\text{ten}}, \tag{2.42}$$

or inversely

$$c_{\text{axi}} = 6(c_2 - c_1), \qquad c_{\text{ten}} = \frac{2}{3}\left(2c_1 + c_2\right), \qquad c_{\text{vec}} = \frac{1}{3}\left(2c_1 + c_2 + 3c_3\right). \tag{2.43}$$

These irreducible components turn out to have a physical interpretation, where the tensor and vector parts are tightly related with the gravitational spin-2 field while the axial part is related to a Kalb-Ramond field (which is a pseudo-vector field). This will be discussed in more detail in section 3.6, and section 3.8.

[2] In [17] there is a sign mistake for the term T_{axi}.

2.3 $f(\mathbb{T})$-gravity

In the spirit of $f(\mathcal{R})$ theories of gravity, we can create $f(\mathbb{T})$-gravity

$$S_{f(\mathbb{T})} = \frac{1}{2\kappa}\int \mathrm{d}^4x\theta f\left(\mathbb{T}\right) + S_{\mathrm{M}}. \tag{2.44}$$

Which in its scalar-tensor representation becomes

$$S_{f(\mathbb{T})} = \frac{1}{2\kappa}\int \mathrm{d}^4x\theta\left(\phi\mathbb{T} - V(\phi)\right) + S_{\mathrm{M}}, \tag{2.45}$$

under the assumptions $f''(\mathbb{T}) \neq 0$ and $f'(\mathbb{T}) = \phi$. Here $'$ denotes derivative with respect to the torsion scalar.

Note that

$$f(\mathbb{T}) \neq f(\mathcal{R}) = f(-\mathbb{T} + B), \tag{2.46}$$

where B is defined as $B := 2\mathcal{D}_\mu T^\mu$, which can be realized from equation (2.28). It turns out that in contrast to $f(\mathcal{R})$, the scalar field ϕ in $f(\mathbb{T})$ cannot easily be identified with a healthy[3] scalar field. This peculiarity will be further discussed in section 3.7 and section 3.8.

2.4 $f(\mathbb{T}_{\mathrm{NGR}})$-gravity

In order to study new general relativity and $f(\mathbb{T})$-gravity simultaneously, it is possible to study a parent theory called $f(\mathbb{T}_{\mathrm{NGR}})$-gravity with its defining action being

$$S_{f(\mathbb{T}_{\mathrm{NGR}})} = \frac{1}{2\kappa}\int \mathrm{d}^4x\theta f\left(\mathbb{T}_{\mathrm{NGR}}\right) + S_{\mathrm{M}}. \tag{2.47}$$

This theory is a special case of $f(T_{\mathrm{axi}}, T_{\mathrm{ten}}, T_{\mathrm{vec}})$-gravity introduced in [17], however, the aim in this thesis is to study the simplest models of teleparallel gravity. The scalar-tensor representation becomes

$$S_{f(\mathbb{T}_{\mathrm{NGR}})} = \frac{1}{2\kappa}\int \mathrm{d}^4x\theta\left(\phi\mathbb{T}_{\mathrm{NGR}} - V(\phi)\right) + S_{\mathrm{M}}. \tag{2.48}$$

The Hamiltonian analysis of this theory was first approached by IV. One of the main findings in this thesis is the first steps towards an explicit covariant Hamiltonian formulation of $f(\mathbb{T}_{\mathrm{NGR}})$-gravity. This has not been explicitly performed in any other publications, even though important insights for how this would be done was pointed out in I, and II.

[3] It is meant that a field is healthy if it is not strongly coupled nor giving rise to ghost instabilities. See the discussion on viability of a theory in section 1.2.

Chapter 3

Hamiltonian analysis in teleparallel theories of gravity

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This chapter introduces the Hamiltonian analysis for $f(T_{\rm NGR})$-gravity. In order to reach to this point the Dirac-Bergmann algorithm for Hamiltonian analysis is shortly revised in section 3.1. The 3+1 decomposition of the fundamental fields are introduced in section 3.2, while different approaches to the Hamiltonian analysis in teleparallel gravity are discussed in section 3.3. In section 3.4 the role of the spin connection is discussed for the Hamiltonian analysis of general teleparallel theories of gravity. Finally, the Hamiltonian and primary constraints for $f(T_{\rm NGR})$-gravity is discussed as well as the implications for new general relativity and $f(\mathbb{T})$-gravity. Section 3.5 is dedicated to $f(\mathbb{T}_{\rm NGR})$-gravity. Since $f(T_{\rm NGR})$-gravity has many subcases which implies that the calculations of Poisson brackets has to be performed for very many special cases we will stop the algorithm after the derivation of the so-called primary Hamiltonian. In section section 3.6 the subcase new general relativity is discussed and in section 3.7 the subcase $f(\mathbb{T})$-gravity is considered. These are the two theories which have attracted most attention in the literature. Finally, the viability of teleparallel gravity is discussed in section 3.8.

3.1 Dirac-Bergmann algorithm

This section gives a brief introduction to the Dirac-Bergmann algorithm [18], on the example of simple mechanical systems. A more detailed description can be found in IV section 2 with an overview figure of the Dirac-Bergmann algorithm. The Dirac-Bergmann algorithm provides a method in going from the Lagrangian formalism to Hamiltonian formalism. It also includes a derivation of constraints as well as a classification of constraints which can be used to count the number of nonlinear degrees of freedom in the theory. The steps can be summarized as follows:

1. Let $q^k(\mathbf{x},t)$ be the canonical fields for which the Lagrangian is expressed as $L(q,\dot{q})$, where dots $\dot{q}$ are time derivatives: $\dot{q} := \frac{\mathrm{d}q}{\mathrm{d}t}$. The conjugate momenta are then defined as

$$\pi_k(q,\dot{q}) := \frac{\partial L}{\partial \dot{q}^k} = \tilde{\phi}(q,\dot{q}^k). \tag{3.1}$$

2. Next the relation between the velocities and momenta should be inverted such that it is possible to write everything in terms of the fields $q(\mathbf{x},t)$ and their conjugate momenta $\pi(\mathbf{x},t)$.

3. One then identifies, from the definition of the conjugate momenta, a potential number of primary constraints

$$\phi_\rho(q,p) = 0, \tag{3.2}$$

with ρ indexing the numbers of primary constraints. Simultaneously the following Legendre transformation is performed:

$$H_c(q,p) = \dot{q}^k\pi_k - L(q,\dot{q}). \tag{3.3}$$

4. Add the primary constraints to the canonical Hamiltonian with the help of Lagrange multipliers u^ρ

$$H_p = H_c + u^\rho \phi_\rho. \tag{3.4}$$

This is the primary Hamiltonian.

5. Demand that the constraints, in equation (3.4), are preserved in time. This can equivalently be written as demanding that the Poisson bracket between the constraints and the primary Hamiltonian H_p vanish on the constraint surface

$$\{H_p, \phi_\rho\} \overset{!}{\approx} 0, \tag{3.5}$$

where " $\approx''$ denotes weak equality (that is, only demanded on the constraint surface). This could lead to an identity or restrictions on the Lagrange multipliers u^ρ. If it turns out that this is not the case, i.e. the Poisson bracket yields something that is not trivial. Then this is identified as secondary constraints. They should then be added to the Hamiltonian and the equation is again calculated with the new constraints. This algorithm continues until the equation becomes trivial.

6. Finally, all constraints will be divided into first and second class constraints, where first class constraints have weakly vanishing Poisson brackets with all constraints, whereas they are called second class if not. The Dirac-Bergmann algorithm has now come to its end and with this information it is possible to count the number of degrees of freedom for the full non-linear theory. It is most straightforward to count them in phase-space where we start with $2k$ degrees of freedom. First class constraints then remove 2 degrees of freedom each, whereas second class constraints remove 1 each.

$$\#\text{degrees of freedom} = 2k - 2 \cdot \#F.C.C - \#S.C.C, \tag{3.6}$$

where $F.C.C$ and $S.C.C$ represents first and second class constrains.

The Dirac-Bergmann algorithm can be extended to field theories where the point particle q^k and their velocities $\dot{q}^k$ are replaced by fields and field velocities (for instance in this thesis point particles are replaced by tetrad fields $\theta^A{}_\mu(\mathbf{x}, t)$, and their field velocities $\dot{\theta}^A{}_\mu(\mathbf{x}, t)$). More details can be found in IV.

3.2 3+1 decomposition

From the definition of the conjugate momenta, time derivatives have to be identified. In order to do this a 3+1 decomposition is useful. Geometrically, this can be obtained by splitting spacetime into 3 dimensional hypersurfaces of constant time slices and normal vectors to the aforementioned hypersurfaces. In order to parameterize this ADM variables are often adopted, where ADM stands for R. Arnowitt, S. Deser, and C.W. Misner [13]. These variables consist of the induced metric on the hypersurfaces of constant time slices γ_{ij} and the normal vector n^μ which has components written out in the new variables lapse α and shift β^i satisfying

$$n^0 = -\frac{1}{\alpha}, \qquad n^i = \frac{\beta^i}{\alpha}. \tag{3.7}$$

In these variables the metric components can be written as

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^i\beta^j\gamma_{ij} & \beta_i \\ \beta_j & \gamma_{ij} \end{bmatrix}, \qquad g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix}. \tag{3.8}$$

Teleparallel gravity is formulated in terms of tetrads $\theta^A{}_\mu$. They can be written in terms of ADM-variables in the following way

$$\theta^A{}_0 = \alpha n^A + \beta^i\theta^A{}_i, \tag{3.9}$$

$$\theta^A{}_i\theta^B{}_j\eta_{AB} = \gamma_{ij}. \tag{3.10}$$

The following relations are very useful:

$$\eta_{AB}n^A n^B = n_A n^A = -1, \qquad \eta_{AB}n^B\theta^A{}_i = n_A\theta^A{}_i = 0, \tag{3.11}$$

$$e_A{}^0 = -\frac{1}{\alpha}n_A, \qquad e_A{}^i = \theta_A{}^i + n_A\frac{\beta^i}{\alpha}, \tag{3.12}$$

where the short-hand notation $\theta_A{}^i = \eta_{AB}\gamma^{ij}\theta^B{}_j$ is used. These equations are sufficient to make a 3+1 decomposition of the Lagrangians considered in this thesis.

3.3 Different approaches to Hamiltonian analysis of teleparallel gravity

In the literature various formulations for teleparallel gravity can be found. The simplest is to work in the gauge where the spin connection does not appear. This is, however, not covariant under local Lorentz transformations [19]. Anyway, this formulation is very often used in the literature and section 3.4 is, hence, dedicated to discussing the implications of this formulation for the Hamiltonian analysis. The section presents the finding of
refteine for which the main conclusion is that the counting of the number of degrees of freedom is unaffected by this gauge choice. In the community a gauge where the spin connection vanish is normally referred to as the "Weitzenböck gauge". Therefore, we refer to this approach as "teleparallel gravity in the Weitzenböck gauge".

Another approach is to include the spin-connection. The minimal way to do this is by working in the formalism presented in section 2. In section 3.4 Hamiltonian analysis in this approach will be discussed and compared to the one done in the Weitzenböck gauge. Let us refer to this approach as "conventional formulation of teleparallel gravity".

Finally, another covariant approach is to demand flatness by Lagrange multipliers $\lambda_\mu{}^{\nu\rho\sigma} R^\mu{}_{\nu\rho\sigma}$ within the Poincaré gauge theory of gravity [20]. Due to the appearance of the Riemann curvature tensor, the spin connection can be treated as a dynamical variable a priory in contrast to the conventional formulation of teleparallel gravity, where no derivatives appear on the spin connection. This formalism has a priory very many degrees of freedom, since it is formulated with a connection both with torsion and curvature. Let us refer to this approach as "teleparallel gravity as a special case of Poincaré gauge gravity". Note that there is in principle an endless amount of ways to formulate teleparallel gravity as a special case of a more general theory, imposing constraints through Lagrange multipliers as was done in this case. Hence, there are no strong motivations for considering this formalism since it is more tedious to work in. However, it was considered in [19].

3.4 Implications of choosing the Weitzenböck gauge

In section 3.3 various formulations for teleparallel gravity were discussed. When it comes to the Hamiltonian analysis, there are a different amount of fields treated as being fundamental. Note that the two last approaches have an action formulation which is invariant under local Lorentz transformations, whereas the first one is not. Working in the Weitzenböck gauge, there are 16 fields to be treated as fundamental in the beginning. However, one may ask whether this gauge choice have any implications for the degrees of freedom, and if we lose anything by deriving Hamilton's equation. In II it is argued that this gauge choice does not affect the resulting number of degrees of freedom for a quite general class of teleparallel theories of gravity, although no explicit examples were done to prove this. In section 3.5 we will explicitly write down the Hamiltonian for $f(\mathbb{T}_{\text{NGR}})$ in the covariant formulation. Regarding Hamilton's equation the notion of "good" and "bad"[1] tetrads may play a similar role to what they do for the field equations [21].

From now on we shall only work in the conventional formulation of teleparallel gravity. In II this formulation was adopted for the following action:

$$S\left[\theta^A{}_\mu, \Lambda_C{}^D\right] = \frac{1}{2\kappa}\int \mathrm{d}x^4 \theta f\left(g_{\sigma\tau}, T^\rho{}_{\mu\nu}\right) + S_{\mathrm{M}}. \tag{3.13}$$

[1]Since the tetrads can be transformed under local Lorentz transformations one may wonder if there are any tetrads preferred over others. While the tetrads need to still solve the field equations they might impose restrictions in the fuction f in $f(\mathbb{T})$-gravity if spin connection is chosen to be zero. In [21] it is shown that what they define as "bad" tetrads impose restrictions on the form of $f(\mathbb{T})$ while they define "good" tetrads to be those which do not impose any restrictions on the form of $f(\mathbb{T})$.

Here it might not be very clear to the reader what is actually meant with $f(g_{\sigma\tau}, T^{\rho}{}_{\mu\nu})$, so it will be explained here. The meaning is the following. Take the torsion components $T^{\rho}{}_{\mu\nu}$ and imagine any contraction you can make of a function involving them, metrics, tetrads, and cotetrads. This involves very general theories of teleparallel gravity and in particular $f(\mathbb{T}_{\text{NGR}})$-gravity which is the most general theory considered in this thesis is included.

Next step is to make a 3+1 decomposition of the Lagrangian and write down the conjugate momenta. Let us a priory work in the covariant formulation of teleparallel gravity where the spin-connection is assumed to appear in the theory. The simplest way to do this is to work in the conventional formulation of teleparallel gravity. Hence, we will simply express the theory in terms of torsion and not consider the potential degrees of freedom that could appear from formulating the theory with vanishing curvature enforced by Lagrange multipliers. In this formulation the tetrads and Lorentz matrices are a priory considered dynamical. The Lorentz matrices have only 6 independent degrees of freedom. Introducing the auxiliary field

$$a_{AB} = \eta_{AC}\omega^{C}{}_{B0} = \eta_{C[A}\Lambda^{C}{}_{|D|}\partial_0 \left(\Lambda^{-1}\right)^{D}{}_{B]} \Leftrightarrow \partial_0\Lambda^{A}{}_{B} = a_{CD}\eta^{A[D}\Lambda^{C]}{}_{B}, \tag{3.14}$$

makes it easy to keep track of the Lorentz symmetry by the antisymmetry in its indices which results in 6 degrees of freedom. Our starting point is, hence, 16 degrees of freedom corresponding to tetrad degrees of freedom plus 6 degrees of freedom corresponding to the Lorentz matrices.

Next step is to identify field velocities in these fields so that we can define the conjugate momenta. All velocities appear in the torsion components, and in particular they appear only in the terms of the form

$$T^{A}{}_{0i} = -T^{A}{}_{i0} = \theta^{A}{}_{\mu}T^{\mu}{}_{0i} = \partial_0\theta^{A}{}_{i} - \partial_i\theta^{A}{}_{0} + \omega^{A}{}_{B0}\theta^{B}{}_{i} - \omega^{A}{}_{Bi}\theta^{B}{}_{0}. \tag{3.15}$$

The conjugate momenta of the tetrads are given by

$$\pi_{A}{}^{i} = \frac{\partial L}{\partial\partial_0\theta^{A}{}_{i}} = \frac{\partial L}{\partial T^{A}{}_{0i}}. \tag{3.16}$$

Note that $\pi_{A}{}^{0} = 0$ since $T^{A}{}_{00} = 0$, so time derivatives never appear on the temporal parts of the tetrads $\theta^{A}{}_{0}$.

The conjugate momenta of the Lorentz matrices are converted to conjugate momenta with respect to the auxiliary fields a_{AB}. To do this we first identify $\omega^{A}{}_{B0}\eta_{AC} = a_{CB}$. They are then expressed as

$$\hat{\pi}^{AB} = \frac{\partial L}{\partial a_{AB}} = -\pi_{C}{}^{i}\eta^{C[B}\theta^{A]}{}_{i}. \tag{3.17}$$

Two things are important to note here. Firstly the conjugate momenta $\hat{\pi}^{AB}$ is antisymmetric in its indices and have, hence, 6 linearly independent components. Secondly, this relation manifests 6 primary constraints, namely that the Lorentz matrix momenta and the tetrad momenta are linearly dependent. According to the Dirac-Bergmann algorithm they need to be added to the Hamiltonian. In II the relation between Hamiltonian teleparallel gravity theories in the Weitzenböck gauge is schematically compared with the Hamiltonian of conventional formulation of teleparallel gravity theories. By schematically deriving the Hamiltonian H of the conventional formulation and then performing a Lorentz transformation back to the Weitzenböck gauge $\tilde{H}$ it is argued that the conclusion of the number of degrees of freedom is unaffected. In particular the schematic expression for the Hamiltonian in the conventional is given by

$$H = \pi_A{}^i \partial_0 \theta^A{}_i + \hat{\pi}^{AB} a_{AB} + {}^{\hat{\pi}}\lambda_{AB} \hat{\pi}^{AB} - L + \text{primary constraints}, \quad (3.18)$$

where ${}^{\hat{\pi}}\lambda_{AB}$ are Lagrange multipliers corresponding to equation (3.17). By performing a Lorentz transformation of this Hamiltonian back to the Weitzenböck gauge (characterized by putting tildes over the transformed quantities) the spin-connection drops out. It is then claimed that the Poisson bracket $\{\tilde{\hat{\pi}}^{AB}, H\}$ which could alter the counting of the number of degrees of freedom vanish on the constraint surface. It is, thus, concluded the covariant formulation predicts the same number of degrees of freedom in the Dirac-Bergmann algorithm as the prediction in the Weitzenböck gauge.

3.5 Hamiltonian and primary constraints for $f(\mathbb{T}_{\mathrm{NGR}})$

In recent years there have been a lot of development in the Hamiltonian formulation of teleparallel equivalent to general relativity, $f(\mathbb{T})$-gravity, and specific examples for new general relativity (see IV and references therein). In order to adress all of these theories it is useful to study the parent theory $f(\mathbb{T}_{\mathrm{NGR}})$-gravity. In this section this will be done in the covariant formulation treating tetrads and the spin connection as fundamental variables. This has not been done before even though it was shown to be possible in II.

In equation (2.47) the action was presented as

$$S_{f(\mathbb{T}_{\mathrm{NGR}})} = \frac{1}{2\kappa} \int \mathrm{d}^4 x \theta f\left(\mathbb{T}_{\mathrm{NGR}}\right) + S_{\mathrm{M}}, \quad (3.19)$$

and its scalar-tensor representation is given by

$$S_{f(\mathbb{T}_{\mathrm{NGR}})} = \frac{1}{2\kappa} \int \mathrm{d}^4 x \theta \left(\phi \mathbb{T}_{\mathrm{NGR}} - V(\phi)\right) + S_{\mathrm{M}}. \quad (3.20)$$

Note that all velocities of this action are in $\mathbb{T}_{\text{NGR}}$, and that all tetrads have at most first order derivatives[2]. In the following coupling to matter will not be considered (for a discussion regarding coupling to matter see [22]). By the expressions derived in subsection 3.2 it is possible to split the action into parts quadratic, linear, and independent of $T^{A}{}_{i0}$ which itself is linear in velocities. The Lagrangian written in this way becomes

$$\begin{aligned} L_{f(\mathbb{T}_{\text{NGR}})} &= \frac{\sqrt{\gamma}}{2\alpha} M^{i\,j}_{A\,B} T^{A}{}_{0i} T^{B}{}_{0j} \\ &- \frac{\sqrt{\gamma}}{\alpha} T^{A}{}_{0i} T^{B}{}_{kl} \cdot \\ &\cdot \left[M^{i\,l}_{A\,B} \beta^{k} + \frac{\alpha\phi}{\kappa} \gamma^{il} \left(c_2 n_B \theta_A{}^{k} + c_3 n_A \theta_B{}^{k} \right) \right] + H_S, \end{aligned} \tag{3.21}$$

with

$$\begin{aligned} M^{i\,j}_{A\,B} = -\frac{\phi}{\kappa} \big(& 2c_1 \gamma^{ij} \eta_{AB} - (c_2 + c_3)\, n_A n_B \gamma^{ij} \\ & + c_2 \theta_A{}^{j} \theta_B{}^{i} + c_3 \theta_A{}^{i} \theta_B{}^{j} \big), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} H_S = & \frac{\sqrt{\gamma}}{\alpha} T^{A}{}_{ij} T^{B}{}_{kl} \beta^{i} \left[\frac{1}{2} M^{j\,l}_{A\,B} \beta^{k} + \frac{\alpha\phi}{\kappa} \gamma^{jl} \left(c_2 n_B \theta_A{}^{k} + c_3 n_A \theta_B{}^{k} \right) \right] \\ & + \frac{\alpha\sqrt{\gamma}}{2\kappa} {}^{3}\mathbb{T} - \frac{\theta V(\phi)}{2\kappa}, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} {}^{3}\mathbb{T} &= H_{AB}{}^{ijkl} T^{A}{}_{ij} T^{B}{}_{kl} \\ &= \phi \left(c_1 \eta_{AB} \gamma^{k[i} \gamma^{j]l} - c_2 \theta_B{}^{[i} \gamma^{j][k} \theta_A{}^{l]} - c_3 \theta_A{}^{[i} \gamma^{j][k} \theta_B{}^{l]} \right) T^{A}{}_{ij} T^{B}{}_{kl}. \end{aligned} \tag{3.24}$$

The conjugate momenta are given by equation (3.16) and (3.17). Using this action theory we get that

$$\begin{aligned} \pi_A{}^{i} &= \frac{\sqrt{\gamma}}{\alpha} M^{i\,j}_{A\,B} T^{B}{}_{0j} \\ &- \frac{\sqrt{\gamma}}{\alpha} T^{B}{}_{kl} \left[M^{i\,l}_{A\,B} \beta^{k} + \frac{\alpha\phi}{\kappa} \gamma^{il} \left(c_2 n_B \theta_A{}^{k} + c_3 n_A \theta_B{}^{k} \right) \right]. \end{aligned} \tag{3.25}$$

The Hamiltonian is schematically given by

$$H = \pi_A{}^{i} \dot{\theta}^{A}{}_{i} + \hat{\pi}^{AB} a_{AB} - L_{f(\mathbb{T}_{\text{NGR}})} - \lambda_{AB} \left(\hat{\pi}^{AB} + \pi_C{}^{i} \eta^{C[B} \theta^{A]}{}_{i} \right) - P.C, \tag{3.26}$$

[2] A crucial difference from Brans-Dicke theory (scalar-tensor representation of $f(\mathcal{R})$-gravity) where second order derivatives appear in $\mathcal{R}$ and integration by parts brings first order derivatives appearing on the scalar field as well.

where $P.C$ represent, additional theory specific primary constraints, which depends on the coefficients $c_1, c_2,$ and c_3, given by equations (3.33)-(3.36), presented later in this section. Firstly, it can be noted that the primary constraint can be used to rewrite equation (3.26) as

$$H = \pi_A{}^i \left(\dot{\theta}^A{}_i + \eta^{AB}\theta^C{}_i a_{CB}\right) - L_{f(\mathbb{T}_{\text{NGR}})} - \lambda_{AB}\left(\hat{\pi}^{AB} + \pi_C{}^i \eta^{C[B}\theta^{A]}{}_i\right) \\ - P.C. \tag{3.27}$$

Next step for the Hamiltonian analysis is to express velocities in variables canonical to the Hamiltonian formalism (fields, and conjugate momenta). For this we note that all velocities are in the terms of the form $T^A{}_{i0}$. We can rewrite equation (3.16) as

$$\begin{aligned} M^{i\,j}_{A\,B}\left(\dot{\theta}^B{}_j + \eta^{BD}a_{DC}\theta^C{}_j\right) &= \frac{\alpha}{\sqrt{\gamma}}\pi_A{}^i + M^{i\,j}_{A\,B}\left(\partial_j\theta^B{}_0 + \omega^B{}_{Cj}\theta^C{}_0\right) \\ &\quad + T^B{}_{kl}\left[M^{i\,l}_{A\,B}\beta^k + \frac{\alpha\phi}{\kappa}\gamma^{il}\left(c_2 n_B\theta_A{}^k + c_3 n_A\theta_B{}^k\right)\right] \\ &:= S_A{}^i. \end{aligned} \tag{3.28}$$

Note that velocities always appear in the combination $\left(\dot{\theta}^A{}_i + \eta^{AB}a_{CB}\theta^C{}_i\right)$, since the origin of all velocities in the action are $T^A{}_{0i}$. Hence, by finding the "Moore-Penrose pseudo inverse" of $M^{i\,j}_{A\,B}$ it is possible to invert all velocities and express them in canonical variables. Calculating the determinant of $M^{i\,j}_{A\,B}$, however, reveals that it may vanish and we hence need to find a way to invert this equation. This can be done by writing out the velocities and conjugate momenta into irreducible parts under the rotation group. Those are a vectorial part projected to the normal vector to hypersurfaces of constant time slices ${}^{\mathcal{V}}$, antisymmetric part ${}^{\mathcal{A}}$, symmetric trace free part ${}^{\mathcal{S}}$, and trace part ${}^{\mathcal{T}}$. The velocities of the tetrad are

$$\dot{\theta}^A{}_i = {}^{\mathcal{V}}\dot{\theta}_i n^A + {}^{\mathcal{A}}\dot{\theta}_{ji}\gamma^{kj}\theta^A{}_k + {}^{\mathcal{S}}\dot{\theta}_{ji}\gamma^{kj}\theta^A{}_k + {}^{\mathcal{T}}\dot{\theta}\theta^A{}_i, \tag{3.29}$$

in this decomposition, and the conjugate momenta is decomposed as

$$\pi_A{}^i = {}^{\mathcal{V}}\pi^i n_A + {}^{\mathcal{A}}\pi^{ji}\gamma_{kj}\theta_A{}^k + {}^{\mathcal{S}}\pi^{ji}\gamma_{kj}\theta_A{}^k + {}^{\mathcal{T}}\pi\theta_A{}^i. \tag{3.30}$$

As a new result, we write down the auxiliary field a_{AB} into irreducible components under the rotation group, which reads

$$a_{AB} = {}^{\mathcal{V}}a_{[A}n_{B]} + {}^{\mathcal{A}}a_{ij}\theta_A{}^i\theta_B{}^j, \tag{3.31}$$

which have vanishing symmetric parts since it is completely antisymmetric. Under this decomposition $M^{i\,j}_{A\,B}$ becomes

$$\begin{aligned} M^{i\,j}_{A\,B} = {}^{\mathcal{V}}M^{ij}n_A n_B + {}^{\mathcal{A}}M^{[ik][jl]}\theta^C{}_k\eta_{AC}\theta^D{}_l\eta_{BD} \\ + {}^{\mathcal{S}}M^{[ik][jl]}\theta^C{}_k\eta_{AC}\theta^D{}_l\eta_{BD} + {}^{\mathcal{T}}M\theta_A{}^i\theta_B{}^j. \end{aligned} \tag{3.32}$$

In IV equations (57)-(60) primary constraints are found to appear with the following choices of c_1, c_2, and c_3:

$$\mathcal{A}_{\mathcal{V}} = 2c_1 + c_2 + c_3 = 0 \implies {}^{\mathcal{V}}C^i = \frac{{}^{\mathcal{V}}\pi^i{}_{\mathcal{K}}}{\phi\sqrt{\gamma}} + c_3 T^B{}_{kl}\gamma^{il}\theta_B{}^k \approx 0, \tag{3.33}$$

$$\mathcal{A}_{\mathcal{A}} = 2c_1 - c_2 = 0 \implies {}^{\mathcal{A}}C^{ij} = \frac{{}^{\mathcal{A}}\pi^{ij}{}_{\mathcal{K}}}{\phi\sqrt{\gamma}} - c_2\gamma^{il}\gamma^{jk}T^A{}_{kl}n_A \approx 0, \tag{3.34}$$

$$\mathcal{A}_{\mathcal{S}} = 2c_1 + c_2 = 0 \implies {}^{\mathcal{S}}C^{ij} = \frac{{}^{\mathcal{S}}\pi^{ij}{}_{\mathcal{K}}}{\phi\sqrt{\gamma}} \approx 0, \tag{3.35}$$

$$\mathcal{A}_{\mathcal{T}} = 2c_1 + c_2 + 3c_3 = 0 \implies {}^{\mathcal{T}}C = \frac{{}^{\mathcal{T}}\pi_{\mathcal{K}}}{\phi\sqrt{\gamma}} \approx 0. \tag{3.36}$$

These equations are in general found by writing equation (3.28) in its irreducible decomposition. Introducing the index,

$$\mathcal{I} \in \{\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}\}, \tag{3.37}$$

one notes that

$$\mathcal{A}_{\mathcal{I}} = 0 \implies \det M^{i\,j}_{A\,B} = 0. \tag{3.38}$$

It is anyway possible to find the Moore-Penrose pseudo inverse of $M^{i\,j}_{A\,B}$ which is found in I, and II [3] to be

$$\begin{aligned} \left(M^{-1}\right)^{A\,C}_{i\,k} = \frac{\kappa}{4\phi}\mathcal{B}_{\mathcal{V}}n^A n^C\gamma_{ik} - \frac{\kappa}{4\phi}\mathcal{B}_{\mathcal{A}}\gamma^{r[s}\gamma^{m]n}\gamma_{kr}\gamma_{si}\theta^A{}_m\theta^C{}_n \\ - \frac{\kappa}{4\phi}\mathcal{B}_{\mathcal{S}}\left(\gamma^{r(s}\gamma^{m)n} - \frac{1}{3}\gamma^{sm}\gamma^{nr}\right)\gamma_{kr}\gamma_{si}\theta^A{}_m\theta^C{}_n - \frac{\kappa}{12\phi}\mathcal{B}_{\mathcal{T}}\theta^A{}_i\theta^C{}_k, \end{aligned} \tag{3.39}$$

with

$$\mathcal{B}_{\mathcal{I}} = \begin{cases} \frac{1}{\mathcal{A}_{\mathcal{I}}}, & \text{if } \mathcal{A}_{\mathcal{I}} \neq 0, \\ 0, & \text{if } \mathcal{A}_{\mathcal{I}} = 0. \end{cases} \tag{3.40}$$

[3]The definition of $M^{i\,j}_{A\,B}$ may slightly change by an overall multiple which do not change the form of the inverse expressed here.

Now it is possible to write the Hamiltonian 3.27 in canonical variables. Before doing this it turns out to be useful to make use of the following relation

$$\frac{\sqrt{\gamma}}{\alpha} M^{i\ j}_{\ A\ B} T^B{}_{0j} = \pi_A{}^i + \frac{\sqrt{\gamma}}{\alpha} T^B{}_{kl} \left[M^{i\ l}_{\ A\ B} \beta^k + \frac{\alpha\phi}{\kappa} \gamma^{il} \left(c_2 n_B \theta_A{}^k + c_3 n_A \theta_B{}^k \right) \right]. \tag{3.41}$$

As an intermediate result, we hence, obtain

$$\begin{aligned} H = & \frac{\pi_A{}^i}{2} \left(\dot{\theta}^A{}_i + \eta^{AB} \theta^C{}_i a_{CB} \right) \\ & + \frac{\phi\sqrt{\gamma}}{2\kappa} \left(\dot{\theta}^A{}_i + \eta^{AD} \theta^C{}_i a_{CD} \right) T^B{}_{kl} \gamma^{il} \left(c_2 n_B \theta_A{}^k + c_3 n_A \theta_B{}^k \right) \\ & - \tilde{H}_S - \lambda_{AB} \left(\hat{\pi}^{AB} + \pi_C{}^i \eta^{C[B} \theta^{A]}{}_i \right) - P.C, \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} \tilde{H}_S = & -\frac{\pi_A{}^i}{2} \left(\partial_i \theta^A{}_0 + \omega^A{}_{Bi} \theta^B{}_0 \right) - \frac{1}{2} \pi_A{}^i T^A{}_{ij} \beta^j \\ & + \frac{\sqrt{\gamma}}{2\alpha} T^A{}_{ij} T^B{}_{kl} \beta^k \left[M^{i\ l}_{\ A\ B} \beta^j + \frac{\alpha\phi}{\kappa} \gamma^{il} \left(c_2 n_A \theta_B{}^j + c_3 n_B \theta_A{}^j \right) \right] \\ & + H_S. \end{aligned} \tag{3.43}$$

By decomposing the first term of the Hamiltonian into irreducible parts it is found that

$$\begin{aligned} \pi_A{}^i \left(\dot{\theta}^A{}_i + \eta^{AB} \theta^C{}_i a_{CB} \right) = & -\left({}^{\mathcal{V}}\dot{\theta}_i + {}^{\mathcal{V}}a_i \right) {}^{\mathcal{V}}\pi^i \\ & + \left({}^{\mathcal{A}}\dot{\theta}_{ji} + {}^{\mathcal{A}}a_{ij} \right) {}^{\mathcal{A}}\pi^{ji} + {}^{\mathcal{S}}\dot{\theta}_{ji} \, {}^{\mathcal{S}}\pi^{ji} + {}^{\mathcal{T}}\dot{\theta} \, {}^{\mathcal{T}}\pi \\ = & \, \alpha \left(\mathcal{B}_{\mathcal{V}} \frac{{}^{\mathcal{V}}C_i \, {}^{\mathcal{V}}\pi^i}{4} - \frac{\mathcal{B}_{\mathcal{A}}}{4} {}^{\mathcal{A}}C_{ij} \, {}^{\mathcal{A}}\pi^{ij} \right. \\ & \left. - \frac{\mathcal{B}_{\mathcal{S}}}{4} {}^{\mathcal{S}}C_{ij} \, {}^{\mathcal{S}}\pi^{ij} - \frac{3\mathcal{B}_{\mathcal{T}}}{4} {}^{\mathcal{T}}C \, {}^{\mathcal{T}}\pi \right) \\ & + \pi_A{}^i \left(\partial_i \theta^A{}_0 + \omega^A{}_{Bi} \theta^B{}_0 \right) - \pi_A{}^i T^A{}_{ij} \beta^j, \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} & \left(\dot{\theta}^A{}_i + \eta^{AD} \theta^C{}_i a_{CD} \right) T^B{}_{kl} \gamma^{il} \left(c_2 n_B \theta_A{}^k + c_3 n_A \theta_B{}^k \right) = \\ & \left({}^{\mathcal{V}}\dot{\theta}^i + {}^{\mathcal{V}}a^i \right) c_3 T^A{}_{ij} \theta_A{}^j + \left({}^{\mathcal{A}}\dot{\theta}^{ij} + {}^{\mathcal{A}}a^{ji} \right) T^A{}_{ij} n_A = \\ & = -\frac{\alpha}{4} \mathcal{B}_{\mathcal{V}} c_3 T^A{}_{ij} \theta_A{}^j \, {}^{\mathcal{V}}C^i - \frac{\alpha}{4} \mathcal{B}_{\mathcal{A}} c_2 T^A{}_{ij} n_A \, {}^{\mathcal{A}}C^{ij} \\ & + \left(\partial_i \theta^A{}_0 + \omega^A{}_{Ci} \theta^C{}_0 \right) T^B{}_{kl} \gamma^{il} \left(c_2 n_B \theta_A{}^k + c_3 n_A \theta_B{}^k \right) \\ & + T^A{}_{ij} T^B{}_{kl} \beta^i \gamma^{jl} \left(c_2 n_B \theta_A{}^k + c_3 n_A \theta_B{}^k \right). \end{aligned} \tag{3.45}$$

Note the appearance of what could be primary constraints. If parameters chosen such that they are indeed primary constraints those terms are multiplied by $\mathcal{B}_{\mathcal{I}}$ which in that case is zero and they do not appear in the Hamiltonian (except as Lagrange multipliers). All terms of the Hamiltonian do now appear in its canonical variables and collecting everything gives us the Hamiltonian

$$\begin{aligned} H = & \frac{\alpha\sqrt{\gamma}\phi}{8\kappa}\left[\mathcal{B}_{\mathcal{V}}{}^{\mathcal{V}}C^{i\mathcal{V}}C_i - \mathcal{B}_{\mathcal{A}}{}^{\mathcal{A}}C_{ij}{}^{\mathcal{A}}C^{ij} - \mathcal{B}_{\mathcal{S}}{}^{\mathcal{S}}C_{ij}{}^{\mathcal{S}}C^{ij} - 3\mathcal{B}_{\mathcal{T}}{}^{\mathcal{T}}C^{\mathcal{T}}C\right] \\ & + \pi_A{}^i\left[\partial_i\theta^A{}_0 + \omega^A{}_{Ci}\theta^C{}_0 - T^A{}_{ij}\beta^j\right] - \frac{\alpha\sqrt{\gamma}}{2\kappa}{}^3\mathbb{T} + \frac{\alpha\sqrt{\gamma}V(\phi)}{2\kappa} \\ & - \lambda_{AB}\left(\hat{\pi}^{AB} + \pi_C{}^i\eta^{C[B}\theta^{A]}{}_i\right) - P.C. \end{aligned} \tag{3.46}$$

The Hamiltonian can then be written as Lagrange multipliers in lapse and shift after an integration by parts

$$\begin{aligned} H = \alpha & \left[\frac{\sqrt{\gamma}\phi}{8\kappa}\mathcal{B}_{\mathcal{V}}{}^{\mathcal{V}}C^{i\mathcal{V}}C_i - \frac{\sqrt{\gamma}\phi}{8\kappa}\mathcal{B}_{\mathcal{A}}{}^{\mathcal{A}}C_{ij}{}^{\mathcal{A}}C^{ij}\right. \\ & -\frac{\sqrt{\gamma}\phi}{8\kappa}\mathcal{B}_{\mathcal{S}}{}^{\mathcal{S}}C_{ij}{}^{\mathcal{S}}C^{ij} - \frac{3\sqrt{\gamma}\phi}{8\kappa}\mathcal{B}_{\mathcal{T}}{}^{\mathcal{T}}C^{\mathcal{T}}C \\ & \left.-\frac{\sqrt{\gamma}}{2\kappa}{}^3\mathbb{T} + \frac{\sqrt{\gamma}V(\phi)}{2\kappa} - n^A\partial_i\pi_A{}^i + \pi_A{}^i\omega^A{}_{Bi}n^B\right] \\ & + \beta^j\left[-\theta^A{}_j\partial_i\pi_A{}^i + \pi_A{}^i\omega^A{}_{Ci}\theta^C{}_j - \pi_A{}^iT^A{}_{ij}\right] \\ & - \lambda_{AB}\left(\hat{\pi}^{AB} + \pi_C{}^i\eta^{C[B}\theta^{A]}{}_i\right) - P.C + \partial_i\left(\pi_A{}^i\theta^A{}_0\right). \end{aligned} \tag{3.47}$$

From this expression it is easy to go into the subcases of new general relativity, $f(\mathbb{T})$-gravity and teleparallel equivalent to general relativity.

3.6 Hamiltonian analysis for new general relativity

The Hamiltonian for new general relativity can be obtained by choosing $\phi = 1$ and $V(\phi) = 0$ in equation (3.47):

$$\begin{aligned} H = \alpha & \left[\frac{\sqrt{\gamma}}{8\kappa}\mathcal{B}_{\mathcal{V}}{}^{\mathcal{V}}C^{i\mathcal{V}}C_i - \frac{\sqrt{\gamma}}{8\kappa}\mathcal{B}_{\mathcal{A}}{}^{\mathcal{A}}C_{ij}{}^{\mathcal{A}}C^{ij}\right. \\ & -\frac{\sqrt{\gamma}}{8\kappa}\mathcal{B}_{\mathcal{S}}{}^{\mathcal{S}}C_{ij}{}^{\mathcal{S}}C^{ij} - \frac{3\sqrt{\gamma}}{8\kappa}\mathcal{B}_{\mathcal{T}}{}^{\mathcal{T}}C^{\mathcal{T}}C \\ & \left.-\frac{\sqrt{\gamma}}{2\kappa}{}^3\mathbb{T} - n^A\partial_i\pi_A{}^i + \pi_A{}^i\omega^A{}_{Bi}n^B\right] \\ & \beta^j\left[-\theta^A{}_j\partial_i\pi_A{}^i + \pi_A{}^i\omega^A{}_{Ci}\theta^C{}_j - \pi_A{}^iT^A{}_{ij}\right] \\ & - \lambda_{AB}\left(\hat{\pi}^{AB} + \pi_C{}^i\eta^{C[B}\theta^{A]}{}_i\right) - P.C + \partial_i\left(\pi_A{}^i\theta^A{}_0\right). \end{aligned} \tag{3.48}$$

In I a list of different non-trivial theories of new general relativity have been listed and we display it here again:

Theory	Constraints	Location in figure 3.1
$A_I \neq 0\ \forall I \in \{\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}\}$	No constraints	white area
$A_\mathcal{V} = 0$	${}^\mathcal{V}C_i = 0$	red line
$A_\mathcal{A} = 0$	${}^\mathcal{A}C_{ji} = 0$	black line
$A_\mathcal{S} = 0$	${}^\mathcal{S}C_{ji} = 0$	green line
$A_\mathcal{T} = 0$	${}^\mathcal{T}C = 0$	blue line
$A_\mathcal{V} = A_\mathcal{A} = 0$	${}^\mathcal{V}C_i = {}^\mathcal{A}C_{ji} = 0$	turquoise point
$A_\mathcal{A} = A_\mathcal{S} = 0$	${}^\mathcal{A}C_{ji} = {}^\mathcal{S}C_{ji} = 0$	purple points (perimeter)[4]
$A_\mathcal{A} = A_\mathcal{T} = 0$	${}^\mathcal{A}C_{ji} = {}^\mathcal{T}C = 0$	orange point
$A_\mathcal{V} = A_\mathcal{S} = A_\mathcal{T} = 0$	${}^\mathcal{V}C_i = {}^\mathcal{S}C_{ji} = {}^\mathcal{T}C = 0$	gray point (center)

The same list applies to $f(\mathbb{T}_{\text{NGR}})$, where the scalar field ϕ needs to straightforwardly with reference to section 3.5 and was also displayed in IV. Paper I, section V also provides a plot of the different choices of theories within the parameter space of new general relativity. The same figure is presented in II, but in the parameters obtained from the irreducible decomposition of torsion. Since it is found in this thesis that the irreducible decomposition gives more physical insights, the figure from II is presented in this section.

In equation (2.41) new general relativity in irreducible components is obtained by rewriting the torsion scalar as

$$\mathbb{T}_{\text{NGR}} = c_{\text{axi}} T_{\text{axi}} + c_{\text{ten}} T_{\text{ten}} + c_{\text{vec}} T_{\text{vec}}, \tag{3.49}$$

with the relations (2.42). In terms of these new coefficients it is clear that III

$$\begin{aligned} \mathcal{A}_\mathcal{V} &\propto c_{\text{ten}} + c_{\text{vec}}, & \mathcal{A}_\mathcal{A} &\propto -4c_{\text{axi}} + 9c_{\text{ten}}, \\ \mathcal{A}_\mathcal{S} &\propto c_{\text{ten}}, & \mathcal{A}_\mathcal{T} &\propto c_{\text{vec}}. \end{aligned} \tag{3.50}$$

As shown in section 3.8 and [10, 23] among the parameters in new general relativity, it is necessary, as a condition to avoid instabilities, that $\mathcal{A}_\mathcal{V} = 0$. Furthermore, $\mathcal{A}_\mathcal{S} \neq 0$ and $\mathcal{A}_\mathcal{T} \neq 0$ are necessary conditions to have a theory of gravity. In the end this means that the (at this point) viable coefficients are fixed to be

$$c_{\text{axi}} = \frac{3}{2} + 18\epsilon, \qquad c_{\text{ten}} = \frac{2}{3}, \qquad c_{\text{vec}} = -\frac{2}{3}. \tag{3.51}$$

For $\epsilon = 0$ teleparallel equivalent to general relativity is recovered. In section 3.8 the viability of the case $\epsilon \neq 0$ is discussed. In Figure 3.1 the red line ${}^\mathcal{V}C = 0$

[4] This is actually only one point in the parameter space, since antipodal points on the perimeter correspond to the same theory.

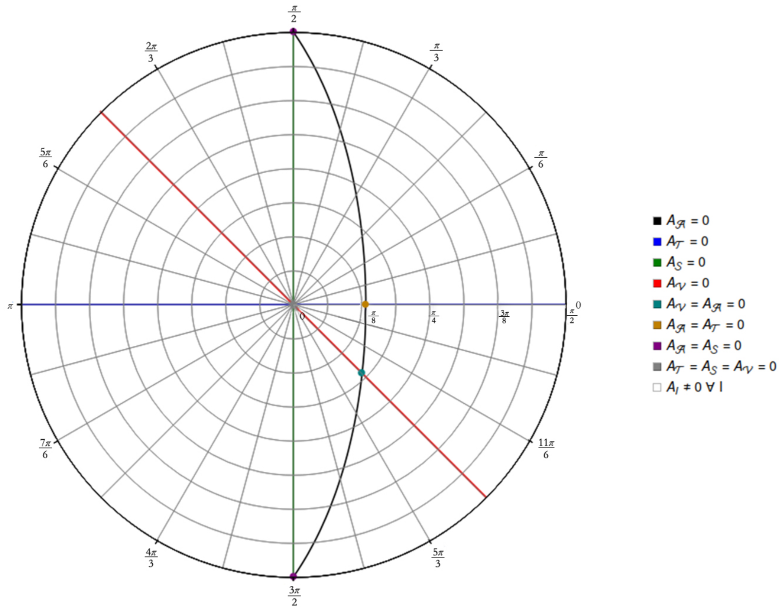

Figure 3.1: Visualization of the parameter space of new general relativity, colored by the occurrences of primary constraints. The radial axis shows the zenith angle θ, while the (circular) polar axis shows the azimuth angle ϕ, following the following definition of definition of the parameters: First the coefficients of the irreducible torsion components are normalized as: $\tilde{c}_{\text{axi}} = \frac{c_{\text{axi}}}{\sqrt{c_{\text{axi}}^2 + c_{\text{ten}}^2 + c_{\text{vec}}^2}}$, $\tilde{c}_{\text{ten}} = \frac{c_{\text{ten}}}{\sqrt{c_{\text{axi}}^2 + c_{\text{ten}}^2 + c_{\text{vec}}^2}}$, $\tilde{c}_{\text{vec}} = \frac{c_{\text{vec}}}{\sqrt{c_{\text{axi}}^2 + c_{\text{ten}}^2 + c_{\text{vec}}^2}}$. Then they are visualized in polar coordinates (θ, ϕ) on the unit sphere with $\tilde{c}_{\text{axi}} = \cos\theta$, $\tilde{c}_{\text{ten}} = \sin\theta\cos\phi$, and $\tilde{c}_{\text{vec}} = \sin\theta\sin\phi$.

various with ϵ except for the gray point at the origin $(\theta, \phi) = (0, 0)$ (or ${}^{\mathcal{T}}C = {}^{\mathcal{S}}C = {}^{\mathcal{V}}C = 0$) which is not describing a theory of gravity anyway. Furthermore, the point $\epsilon = 0$ corresponds to the turquoise point ${}^{\mathcal{V}}C = {}^{\mathcal{A}}C = 0$ describing teleparallel equivalent to general relativity.

3.7 Hamiltonian analysis for $f(\mathbb{T})$

The $f(\mathbb{T})$ limit of $f(\mathbb{T}_{\text{NGR}})$-gravity obtained by fixing the parameters in the following way

$$c_1 = \frac{1}{4}, \qquad c_2 = \frac{1}{2}, \qquad c_3 = -1. \tag{3.52}$$

This imposes the primary constraints

$$ {}^{\mathcal{V}}C^i = \frac{{}^{\mathcal{V}}\pi^i\kappa}{\phi\sqrt{\gamma}} - T^B{}_{kl}\gamma^{il}\theta_B{}^k \approx 0, \tag{3.53} $$

$$ {}^{\mathcal{A}}C^{ij} = \frac{{}^{\mathcal{A}}\pi^{ij}\kappa}{\phi\sqrt{\gamma}} - \frac{1}{2}\gamma^{il}\gamma^{jk}T^A{}_{kl}n_A \approx 0. \tag{3.54} $$

These are 3+3=6 primary constraints. In IV it was shown that these are equivalent to the alternative form

$$ C_{AB} = \pi_{[AB]} - \frac{\phi\sqrt{\gamma}}{\kappa}n^D S_{[A|D|B]} \approx 0, \tag{3.55} $$

where S_{ADB} is the superpotential introduced in equation (2.44). In this limit the primary Hamiltonian becomes

$$ \begin{aligned} H_{f(\mathbf{T})} = \alpha \Bigg[&-\frac{\sqrt{\gamma}\phi}{8\kappa}\mathcal{B}_{\mathcal{S}}{}^{\mathcal{S}}C_{ij}{}^{\mathcal{S}}C^{ij} - \frac{3\sqrt{\gamma}\phi}{8\kappa}\mathcal{B}_{\mathcal{T}}{}^{\mathcal{T}}C^{\mathcal{T}}C - \frac{\sqrt{\gamma}}{2\kappa}{}^3\mathbb{T} \\ &+\frac{\sqrt{\gamma}V(\phi)}{2\kappa} - n^A\partial_i\pi_A{}^i + \pi_A{}^i\omega^A{}_{Bi}n^B \Bigg] \\ &+ \beta^j\left[-\theta^A{}_j\partial_i\pi_A{}^i + \pi_A{}^i\omega^A{}_{Ci}\theta^C{}_j - \pi_A{}^iT^A{}_{ij}\right] \\ &- \lambda_{AB}\left(\hat{\pi}^{AB} + \pi_C{}^i\eta^{C[B}\theta^{A]}{}_i\right) - \tilde{\lambda}^{AB}C_{AB} + \partial_i\left(\pi_A{}^i\theta^A{}_0\right). \end{aligned} \tag{3.56} $$

This is the first time the Hamiltonian for the covariant formulation of $f(\mathbb{T})$-gravity is presented. From this it is possible to calculate the Poisson brackets among constraint, and it would provide an independent calculation from [11, 12] while providing an explicit verification (alternatively disproving) the claim of II.

3.8 Viability of teleparallel gravity

In this final section we will discuss the current state-of-art for teleparallel gravity and draw conclusions for the viability of teleparallel gravity. It turns out that most points towards that $f(\mathbb{T}_{\text{NGR}})$-gravity is not a viable (in the sense discussed in section 1.2) theory, except for the special case of teleparallel equivalent to general relativity. However, there are still some points that need further investigation before $f(\mathbb{T}_{\text{NGR}})$-gravity can be completely discarded as not viable.

To determine the viability of teleparallel theories of gravity the following questions should be answered

- How many degrees of freedom propagate at the nonlinear level for the given theory?

- Does the answer to the previous question depend on any gauge choices?

- Do the degrees of freedom at nonlinear level depend on the initial values of the fields?

- How many degrees of freedom appear on linear perturbations around physical backgrounds?

- Is there a discrepancy of the numbers of degrees of freedom between perturbations and the full nonlinear theory?

- Are the perturbative degrees of freedom depending on any gauge choice or choice of frame?

- Are there ghosts?

Many of these questions are connected. To address these it is insightful to start with the simplest choice for perturbations. At first order, tetrad perturbations can be written as

$$\theta^A{}_\mu := \delta^A{}_\mu + \eta^{AB}\delta^\nu{}_B B_{\nu\mu} = \delta^A{}_\mu + \eta^{AB}\delta^\nu{}_B \frac{h_{\nu\mu} + b_{\nu\mu}}{2}, \tag{3.57}$$

where $h_{\nu\mu} = h_{\mu\nu}$ are completely symmetric perturbations, while $b_{\nu\mu} = -b_{\mu\nu}$ are completely antisymmetric. The general structure of $f(\mathbb{T}_{\text{NGR}})$-gravity has quadratic terms of torsion components, which involve first order derivatives of the tetrad components, while the scalar field ϕ appears completely without derivatives. This means that at lowest order $B_{\nu\mu}$ must appear to be of second order, since derivatives of the background vanishes. Furthermore, at lowest order only the background of ϕ is contributing. As a consequence, no extra degrees of freedom appear in $f(\mathbb{T})$-gravity compared to teleparallel equivalent to general relativity. This answers the question of the most simple, physically motivated background for $f(\mathbb{T})$-gravity. Namely, *only two degrees of freedom propagate at lowest order for perturbations around Minkowski spacetime defined by* (3.57).[5]

Since the scalar field has trivial contributions at lowest order for the most simple perturbations the attention is now directed to new general relativity only. Since only the lowest order is considered the perturbations $B_{\nu\mu}$ appears at second order.

[5] In [24] an example, which contradict this statement is presented, and this is discussed further later in this section.

These terms can be grouped into those quadratic in completely symmetric, completely antisymmetric and mixed terms. Consider the form of the Lagrangian given by the torsion scalar in its irreducible components (2.41)

$$S_{\mathrm{NGR}} = \int \mathrm{d}^4 x \mathcal{L} + S_{\mathrm{matter}}, \tag{3.58}$$

decomposed as

$$\mathcal{L} = \mathcal{L}_{hh} + \mathcal{L}_{hb} + \mathcal{L}_{bb}. \tag{3.59}$$

Then

$$\begin{aligned} \mathcal{L}_{hh} = h^{\mu\nu} \Bigg(& \frac{3}{2} c_{\mathrm{ten}} \Box h_{\mu\nu} + \left(c_{\mathrm{vec}} - 2c_{\mathrm{ten}}\right) \partial^{\rho}\partial_{\mu} h_{\nu\rho} \\ & - \left(c_{\mathrm{vec}} - \frac{1}{2} c_{\mathrm{ten}}\right) \left(\partial_{\mu}\partial_{\nu} h + \eta_{\mu\nu}\partial_{\rho}\partial_{\sigma} h^{\rho\sigma}\right) \\ & + \left(c_{\mathrm{vec}} - \frac{1}{2} c_{\mathrm{ten}}\right) \eta_{\mu\nu} \Box h \Bigg), \end{aligned} \tag{3.60}$$

with $h := h_{\mu\nu}\eta^{\mu\nu}$,

$$\mathcal{L}_{hb} = 2\left(c_{\mathrm{ten}} + c_{\mathrm{vec}}\right) h^{\mu\nu} \partial_{\mu}\partial^{\rho} b_{\rho\nu}, \tag{3.61}$$

and

$$\mathcal{L}_{bb} = b^{\mu\nu} \left(\left(-\frac{4}{9} c_{\mathrm{axi}} - c_{\mathrm{vec}} \right) \partial_{\nu}\partial^{\rho} b_{\rho\mu} + \left(\frac{1}{2} c_{\mathrm{ten}} - \frac{2}{9} c_{\mathrm{axi}} \right) \Box b_{\mu\nu} \right). \tag{3.62}$$

Note that if and only if $\mathcal{A}_{\mathcal{V}} \propto c_{\mathrm{ten}} + c_{\mathrm{vec}} = 0$, then $\mathcal{L}_{hb} = 0$ in general. A theory consisting of mixed symmetric and antisymmetric modes is in general pathological [25] [6] Furthermore, if one would demand either c_{ten} or c_{vec} to be zero, then this would also hold for the other under the condition $c_{\mathrm{ten}} + c_{\mathrm{vec}} = 0$. Then $\mathcal{L}_{hh} = 0$ and all that is left is $\mathcal{L}_{bb}$. However, this term alone cannot describe anything that is just to call gravity [10].[7] Hence, the parameterization (3.51) is justified. Thus, the only possible viable theory admits the following form

$$\mathcal{L}_{hh} = h^{\mu\nu} \left(\Box h_{\mu\nu} - 2\partial^{\rho}\partial_{\mu} h_{\nu\rho} + \partial_{\mu}\partial_{\nu} h + \eta_{\mu\nu}\partial_{\rho}\partial_{\sigma} h^{\rho\sigma} + \eta_{\mu\nu} \Box h \right), \tag{3.63}$$

and

$$\mathcal{L}_{bb} = -4\epsilon \left(2\partial_{\nu}\partial^{\rho} b_{\rho\mu} + \Box b_{\mu\nu} \right) b^{\mu\nu}. \tag{3.64}$$

[6] In [26] it is claimed that ghosts and tachyons are only absent if the symmetric and antisymmetric fields are decoupled.

[7] In this case the theory would be described by a pseudo-vector field and the field equations would be completely antisymmetric. This is physically not close to anything describing gravity.

For teleparallel equivalent to general relativity, $\mathcal{L}_{bb} = 0$. While, $\mathcal{L}_{bb} \neq 0$ the perturbations corresponds to a Kalb-Ramond field and this is characterized by the value the coefficient c_{axi}. However, it was shown in [10] that continuing with these perturbations around Minkowski up to cubic order reveals a discrepancy in the number of degrees of freedom. Those the Kalb-Ramond field is strongly coupled and the perturbations are not valid. Thus, this theory loses its predictive power at perturbative level and is, hence, not considered viable. There is, however, still an open question. All these arguments were carried out considering Minkowski being the physical background. Though, our universe should strictly admit cosmological backgrounds, like Friedmann-Lemaître-Robertson-Walker spacetime or De Sitter. By considering cosmological backgrounds instead of the Minkowski background might turn out to avoid the problem of strong coupling.

Regarding the degrees of freedom at the nonlinear level new general relativity lacks a fully completed work on the Hamiltonian analysis. In [27] the new general relativity theory with $\mathcal{A}_{\mathcal{V}} \neq 0$, $\mathcal{A}_{\mathcal{A}} \neq 0$, $\mathcal{A}_{\mathcal{S}} \neq 0$, and $\mathcal{A}_{\mathcal{T}} \neq 0$ have a complete Hamiltonian analysis. The conclusion is that there are 8 propagating degrees of freedom. However, as already mentioned this theory is not viable. Though, the calculations performed there can be useful to investigate the one parameter theory with $\mathcal{A}_{\mathcal{V}} = 0$. The one-parameter theory was investigated in [28]. However, a strange behavior of the matrix of Poisson brackets among constraints was encountered where the rank depends on the field values, therefore it presents a variable rank. This peculiarity is also mentioned in IV and for related theories in [29, 30, 31, 32]. Since, the matrix of Poisson brackets depend on the field values, the viability might in principle depend on the initial values for the fields [28]. Field dependence in the nonlinear constraint structure is thought to be associated with the propagation of acausal degrees of freedom. However, the variable rank of the matrix of Poisson brackets among constraints is not necessarily implying the presence of acausal degrees of freedom, and it hence, need to be shown explicitly [32].

The Hamiltonian analysis of the aforementioned literature always assumed the Weitzenböck gauge (where the spin-connection is set to zero) in the beginning. Could this assumption have any impact on the conclusion for how many degrees of freedom present in the theory? In section 3.3 as well as in I and II it is argued that, *assuming the Weitzenböck gauge contra working in covariant formulation does not affect the conclusion for the number of degrees of freedom.* This statement is also valid for more general theories, including $f(\mathbb{T}_{\text{NGR}})$-gravity, and hence, also $f(\mathbb{T})$-gravity II.

Shifting focus to $f(\mathbb{T})$-gravity a quick literature study reveals contradicting statements regarding the nonlinear propagating degrees of freedom. In particular the conclusions are contradictory between [11] and [12]. In IV it was found that the Hamiltonian analysis of [11, 12] are consistent up to the point of writing out the primary Hamiltonian. Therefore, the contradictions have to appear in a later stage of the Dirac-Bergmann algorithm and indeed there were mistakes in [11] pointed out by [12]. The conclusion in [12] is that $f(\mathbb{T})$ gravity generically has five propagating degrees of freedom in the full nonlinear theory (compared to three degrees of freedom stated by [11]). Confirming this, however, lies beyond the scope of this thesis.

As argued above the $f(\mathbb{T})$-gravity does not reveal any additional degrees of freedom compared to general relativity at lowest order in perturbation theory. This is a discrepancy of three compared to the nonlinear degrees of freedom according to [12]. In general also other works point to more than two degrees of freedom IV. Comparing to perturbations given by (3.57) $f(\mathbb{T})$-gravity seems to suffer from strongly coupled fields. In [33] cosmological perturbations were considered with no additional degrees of freedom propagating at lowest order. Furthermore, a new mode was explicitly found in [34] at the fourth order perturbations around Minkowski confirming the claims of strongly coupled field(s) in $f(\mathbb{T})$-gravity. Interestingly, [24] considered perturbations around Lorentz rotated nontrivial Minkowski tetrads (still consistent with the field equations) and found extra modes propagating at lowest order. However, these modes do not correspond to the full degrees of freedom of a propagating field expected from the nonlinear analysis. Hence, the perturbative theory of $f(\mathbb{T})$-gravity still seems ill-defined.

In conclusion, as stated in section 1.3 *there are many indications that both $f(\mathbb{T})$-gravity and new general relativity, both suffer from strongly coupled fields around physically motivated backgrounds.* However, the finding of extra perturbative modes around nontrivial Minkowski backgrounds [24] raises the question if there is a possibility to find a background solution to the field equations which avoids strongly coupled fields. In particular, this has not been investigated in new general relativity. Furthermore, only perturbations around Minkowski have been considered for new general relativity, while the strong coupling problem might not appear for cosmological perturbations.

It should be emphasized that this thesis consider $f(\mathbb{T}_{\mathrm{NGR}})$-gravity, and its special cases $f(\mathbb{T})$-gravity, new general relativity, and teleparallel equivalent to general relativity. While teleparallel equivalent to general relativity is viable, all other theories of $f(\mathbb{T}_{\mathrm{NGR}})$-gravity seem to not be. Further modifications of $f(\mathbb{T}_{\mathrm{NGR}})$-gravity by introducing more degrees of freedom may in principle circumvent the strong coupling problem, while a more likely scenario is that they still suffer from these pathologies. However, teleparallel theories with the structure of teleparallel equivalent to general relativity [35] are not considered here, but they are more likely to be viable.

In general, there are a few directions for making a better understanding of the viability of teleparallel theories of gravity. One direction is to consider the Hamiltonian analysis for *covariant* teleparallel theories of gravity. This has never been done explicitly, and this could shed some light on how Lorentz symmetry is broken. Perturbations of physically motivated background at both lowest and higher order could give further insights of the viability of teleparallel gravity. The last, and probably the most promising, direction, would be to consider teleparallel theories based on constructions different from new general relativity and $f(\mathbb{T})$ such as the theory considered in [35].

Summary

The main focus of the thesis has been on the Hamiltonian analysis of teleparallel theories of gravity. In particular, new general relativity and $f(\mathbb{T})$-gravity are considered among their parent theory $f(\mathbb{T}_{\mathrm{NGR}})$-gravity. The main conclusion is that the viability of those theories is in serious doubt, although some clarifications are still needed. In IV $f(\mathbb{T}_{\mathrm{NGR}})$-gravity was considered, and their primary constraints were derived for the first time. Furthermore, it was argued in II that the conclusion of the number of degrees of freedom drawn from the Dirac-Bergmann algorithm is independent on gauge fixing of the spin-connection for general teleparallel theories of gravity including $f(\mathbb{T}_{\mathrm{NGR}})$-gravity. In I and III the special case of new general relativity is considered. The primary Hamiltonian is derived in I and nine distinct classes of theories are found. The arguments presented in II are made here for the special case of new general relativity.

Newly for this thesis physical interpretations of the irreducible parts of torsion are discussed. In III the Hamiltonian and primary constraints for new general relativity are derived, however, no clear realization of the physical interpretation is found there. From section 3.6 and 3.8 it is argued that the axial part of the Torsion scalar is tightly related with the appearance of the so-called "Kalb-Ramond" field while the other two parts are connected with gravity. The most novel finding in this thesis is an explicit expression for the covariant formulation of the primary Hamiltonian for $f(\mathbb{T}_{\mathrm{NGR}})$-gravity. This paves the way for a complete covariant Hamiltonian formulation of teleparallel gravity, which might resolve any doubt about the viability of the methods in previous works while giving new insights on the role of the spin connection in teleparallel theories of gravity.

Acknowledgments

First and foremost, I would like to thank my supervisor Manuel Hohmann as well as my colleague and friend Christian Pfeifer, who has spent a lot of time, energy, and enthusiasm for my PhD project. I would also like to thank the gravity group, Manuel, Christian, Laur, Martin, Sebastián, Ulbossyn, Kostas, Aneta, Margus, João, Jorge Tomi, Luca Priidik, and Lucy, as well as my friends from the Tartu observatory, Toni, Punya, Maarja and Teet, for great scientific discussions as well as for making my stay in Tartu very enjoyable. I would also like to thank Nisa, Suraj, Haruka, Kristjan, Mari, María-José, Nadja, Maggie, and Gerardo, for the good times. In particular, I would like to thank Laur Järv for being a great leader of the group. I am grateful to Manuel Hohmann, Christian Pfeifer, María-José Guzmán, and Sebastián Bahamonde for carefully reading this thesis manuscript and Ott Vilson for providing a very useful template. I would like to thank the people from the CANTATA network for many interesting conferences and discussions and very good times in various parts of Europe. Then I would also like to thank my flatmates, Merey, Kristjan, Kaarel, Eleni, Naza and Fidan for all the good times in Tartu, as well, I would like to thank Kapil, Keerthana, Arun, Sundeep, Qurat, Sharvari, Ankita, Shaina and Mainak for all the good times and this also goes to my friends from the Margarita group, Martin, Margarita, Abed, Umesh, Asif, Nuno, Wanyi (Smile), Orleanys, Lillit, Christian and Nadja. I would like to thank my family and friends from Sweden who have always been supporting me. For the final phase of my studies I would thank my girlfriend, Ruby Sengupta for giving me a lot of support.

My research was supported by the University of Tartu ASTRA Project PER ASPERA. Part of the support came through the personal research funding project PUT790 (start-up grant), as well as PRG356 (team grant) PSG489 (start-up grant). Further support was given by the European Union through the European Regional Development Fund projects Center of Excellence TK133 (The Dark Side of the Universe). Additional financial support came from Archimedes Foundation through Kristjan Jaak and Dora Plus scholarships, and the CA-15117 Cost action CANTATA and the CA-16104 Cost action Gravitational waves, black holes and fundamental physics, and the Nordic Institute for Theoretical Physics NORDITA as visiting PhD student and for traveling to their winter schools.

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Kokkuvõte (in Estonian)

Kovariantsete teleparalleelsete gravitatsiooniteooriate Hamiltoni analüüs

Väitekirja peamine fookus on teleparalleelsete gravitatsiooniteooriate Hamiltoni analüüs. Täpsemalt uuriti new general relativity (New General Relativity, Uus Üldrelatiivsusteooria) ja $f(\mathbb{T})$ gravitatsiooniteooriaid ning nende ühendit, $f(\mathbb{T}_{\text{NGR}})$ gravitatsiooniteooriat. Põhiline tulemus näitab, et nende kooskõla on tõsiselt kahtlustav, kuid tulemus vajab veel selgust. Artiklis IV uuriti $f(\mathbb{T}_{\text{NGR}})$ gravitatsiooniteooriat, ning esmakordselt tuletati selle teooria esimest järku sidemeid. Lisaks seletati artiklis II et Dirac-Bergmanni algoritmist tuletatud vabadusastmete arv ei sõltu spinni seostuse kalibratsiooni valikust üldistes teleparalleelsetes gravitatsiooniteooriates, seehulgas ka $f(\mathbb{T}_{\text{NGR}})$ teoorias. Artiklites I ja III uuriti new general relativity erijuhtu. Artiklis I tuletati esimest järku Hamiltoniaani ja leiti üheksa erinevat teooriateklassi. Siin rakendati artiklis II esitatud põhjendusi new general relativity erijuhule.

Käesolevad väitekirjas on lisaks kirjeldatud väände taandamatute komponentide interpretatsioon. Artiklis III tuletati new general relativity teooria Hamiltoniaani ja esimest järku sidemeid, kuid ei leidnud selget füüsikalist interpretatiooni. Peatükkides section 3.6 ja 3.8 seletati et väändeskalaari aksiaalosa on tihedalt seotud nii-nimetatud “Kalb-Ramond” välja ilmumisega, ning teised kaks osa on seotud gravitatsiooniga. Kõige uues tulemus selles väitekirjas on $f(\mathbb{T}_{\text{NGR}})$ teooria esimest järku Hamiltoniaani kovariantne kirjeldus. See tulemus avab teed teleparalleelse gravitatsiooni täielikule kovariantsele Hamiltoni kirjeldusele, mis võib taandada kahtlust varasemates töödes rakendatud meetodite kehtivuse kohta ning anda uusi perspektiive spinni seostuse rollile teleparalleelsetes gravitatsiooniteooriates.

Attached publications

I Hamiltonian and primary constraints of new general relativity 55

II On the Gauge Fixing in the Hamiltonian Analysis of General Teleparallel Theories 71

III Hamiltonian Analysis in New General Relativity 83

IV Review of the Hamiltonian analysis in teleparallel gravity 93

Chapter 4

Hamiltonian and primary constraints of new general relativity

I

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D. Blixt, M. Hohmann, and C. Pfeifer *Phys. Rev.* D **99**, 084025 (2019),
https://doi.org/10.1103/PhysRevD.99.084025,
arXiv:1811.11137 [inSpire] [ETIS]

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Hamiltonian and primary constraints of new general relativity

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(Received 29 December 2018; published 15 April 2019)

We derive the kinematic Hamiltonian for the so-called "new general relativity" class of teleparallel gravity theories, which is the most general class of theories whose Lagrangian is quadratic in the torsion tensor and does not contain parity violating terms. Our approach makes use of an explicit expression for the flat, in general, nonvanishing spin connection, which avoids the use of Lagrange multipliers, while keeping the theory invariant under local Lorentz transformations. We clarify the relation between the dynamics of the spin connection degrees of freedom and the tetrads. The terms constituting the Hamiltonian of the theory can be decomposed into irreducible parts under the rotation group. Using this, we demonstrate that there are nine different classes of theories, which are distinguished by the occurrence or nonoccurrence of certain primary constraints. We visualize these different classes and show that the decomposition into irreducible parts allows us to write the Hamiltonian in a common form for all nine classes, which reproduces the specific Hamiltonians of more restricted classes in which particular primary constraints appear.

DOI: 10.1103/PhysRevD.99.084025

I. INTRODUCTION

General relativity (GR) is usually formulated using the Levi-Civita connection induced by a pseudo-Riemannian metric. Alternatively, one may employ other connections, such as the flat connections used in teleparallel [1,2] or symmetric teleparallel gravity [3], in order to obtain sets of field equations equivalent to those of GR. In this work we consider teleparallel gravity, where the field variables are the 16 components of a tetrad (or vierbein), instead of the 10 components of a metric. Nowadays it is known that 6 components are related to local Lorentz transformations, while at most 10 encode the gravitational interaction. How many of them actually encode dynamical degrees of freedom (d.o.f.) of a teleparallel theory of gravity is not conclusively answered in general, and to gain insight into this question is one motivation for this work.

Large varieties of teleparallel theories of gravity have been constructed [4–6]. Since the building block of these theories is the torsion of the teleparallel connection and not the curvature of the Levi-Civita connection, second order derivatives of the fundamental fields do not appear in the Lagrangians, as long as no terms with additional derivatives on the torsion are introduced, and so no Gibbons-Hawking-York boundary term is required. In this way the teleparallel formulation allows for more freedom in the construction of gravity theories with second order derivative field equations than the metric approach. Moreover, teleparallel gravity theories can be understood as gauge theories with a Yang-Mills theorylike structure [7–9], which brings gravity closer to the standard model of particle physics, and might hence open a path to its unification with the other fundamental forces in physics. The other prominent reason to construct modified theories of gravity is to shed light on astrophysical observations which lack explanation within GR coupled to standard model matter only; the most famous ones being the dark matter and dark energy phenomena.

Before studying the phenomenology of modified teleparallel theories of gravity it is essential to identify those which are self-consistent, i.e., to understand the properties of their d.o.f. and if they contain ghosts. This can be done best in terms of a full-fledged Hamiltonian analysis in terms of the Dirac-Bergmann algorithm for constrained Hamiltonian systems. It is known that the teleparallel equivalent of general relativity (TEGR), which yields the same dynamics and solutions for the metric defined by the tetrads as general relativity and contains no additional d.o.f., is self-consistent and ghost-free [10–16]. The hope is that this is not the only contender of the class of healthy teleparallel theories of gravity in this sense. Because of the complexity in the calculation of the constraint algebra, the Hamiltonian analysis for modified theories of gravity is not done for all the models considered in the literature. With this work we aim to contribute to this goal.

One widely studied class of modified teleparallel theories of gravity are the $f(T)$ models. They are based on the

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Lagrangian T employed in TEGR, and can be thought of as the teleparallel counterpart of $f(R)$ theories considered in the metric formalism. While it is known that TEGR and GR are equivalent, this is in general not true for $f(T)$ and $f(R)$ theories. The Hamiltonian analysis of $f(T)$ theories has just recently been presented [15,17] with the conclusion that there are three propagating d.o.f., which differs from previous results [18,19]. Other, more general models are based on a Lagrangian that is a free function of the three parity even scalars that are quadratic in the torsion tensor and do not involve further fields than the tetrads [20]. Their Hamilton analysis is still missing, and, due to the generality of the model, could be very involved. However, among these general models, there are the new general relativity (NGR) models [21]: the most general class of teleparallel theory of gravity in four spacetime dimensions, whose Lagrangian is quadratic in the torsion tensor and contains only the tetrad and its first derivatives. This class is parametrized by three constant parameters appearing in the Lagrangian and contains TEGR for a special choice of the parameters.

Various work has been performed on NGR. Solar system constraints have been investigated [21] as well as the propagation and polarization modes of gravitational waves on a Minkowski spacetime background [22]. This analysis found that already on the linearized level, in general, NGR models predict more than two gravitational wave polarizations. However, it was also found that there exist NGR models different from TEGR with two gravitational wave polarizations. What remains open from the analysis of the linearized theory is if it differs from the full nonlinear theory. On the nonlinear level strongly coupled fields may appear, similar to what was pointed out in early attempts to formulate massive gravity theories [23]. A complete Hamiltonian analysis is needed in order to answer this question.

In this article we work towards the goal of a full Hamiltonian description of NGR. In particular, we derive the fully generic kinematic Hamiltonian for NGR, which is valid for any choice of the parameters appearing in the action. Further, we discuss the occurrence of primary constraints depending on the parameters of the theory. This analysis is an important cornerstone for further studies of NGR in its Hamiltonian formulation. Knowing the primary constraints, it is possible to calculate the successive Poisson brackets, and thus to derive the full constraint algebra, which implies the number of d.o.f. of the theory. In addition, it is the starting point to study the presence or absence of ghosts, and hence to test the viability of different theories within the NGR class. Further, the $3+1$ Hamiltonian formalism also leads to the initial value formulation of NGR, required for numerical calculations, such as the precise prediction of gravitational wave signatures.

Hamiltonian analyses of specific theories within the NGR class besides TEGR have been studied [24,25]. Additionally, this line of research extends to the Hamiltonian formulation of more general Poincaré gauge theories, where both torsion and curvature are present [26,27].

The main difference between the previous studies and the approach we present in this article lies in the method which is employed in order to implement the vanishing curvature of the teleparallel connection. Previous studies can mainly be divided into two groups, either assuming a vanishing spin connection (which is known as the Weitzenböck gauge) [10,15–17,25], or an arbitrary spin connection, whose curvature is then enforced to vanish by using Lagrange multipliers in the action functional [12,13]. Here we use a different ansatz, by allowing for a non-vanishing spin connection, as mandated by the covariant formulation of teleparallel gravity [1,2], which is obtained explicitly by applying a local Lorentz transformation to the vanishing Weitzenböck gauge spin connection. This spin connection is flat by construction, and we will show that it enters only as a gauge d.o.f.

The article is organized as follows: In Sec. II we present the Lagrangian for new general relativity. Then we write down the Lagrangian in $3+1$ decomposition and derive its conjugate momenta, and discuss the gauge fixing, in Sec. III. In Sec. IV we perform a decomposition into irreducible parts and find the possible primary constraints. Finally the kinematic Hamiltonian is written down in Sec. V, where we use the irreducible parts to write it in a block structure showing the most general expression. In Appendix we sketch how one can derive the Hamiltonian without fixing the gauge. Index conventions throughout this article are such that capital Latin indices $A, B, C, \dots$ are Lorentz indices running from 0 to 3, Greek indices $\mu, \nu, \rho, \dots$ are spacetime indices running from 0 to 3, and small Latin indices $i, j, k, \dots$ are spatial spacetime indices running from 1 to 3. A dot over a quantity always denotes derivative with respect to x^0 $\dot{X} = \partial_0 X$. The signature convention for the spacetime metric employed is $(-,+,+,+)$.

II. THE NEW GENERAL RELATIVITY LAGRANGIAN

Teleparallel theories of gravity are formulated in terms of tetrad fields θ^A, their duals e_A and a curvature-free spin connection $\omega^A{}_B$, which can at least locally be constructed out of local Lorentz transformations $\Lambda^A{}_B$. In local coordinates $(x^\mu, \mu = 0, \dots, 3)$ on spacetime they can be expressed as

$$\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu, \qquad e_A = e_A{}^\mu \partial_\mu,$$
$$\omega^A{}_B = \omega^A{}_{B\mu} \mathrm{d}x^\mu = \Lambda^A{}_C \mathrm{d}(\Lambda^{-1})^C{}_B = \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B \mathrm{d}x^\mu, \tag{1}$$

and satisfy

$$\theta^A(e_B) = \theta^A{}_\mu e_B{}^\mu = \delta^A_B, \qquad \theta^A{}_\mu e_A{}^\nu = \delta^\nu_\mu. \tag{2}$$

Implementing the flat teleparallel spin connection in this way has the advantage that it avoids the use of Lagrange multipliers as done in Refs. [12,28]. The spacetime metric $g_{\mu\nu}$, which is a fundamental field in other gravity theories such as GR, here becomes a derived quantity defined by

$$g_{\mu\nu} = \eta_{AB}\theta^A{}_\mu\theta^B{}_\nu, \qquad g^{\mu\nu} = \eta^{AB}e_A{}^\mu e_B{}^\nu. \tag{3}$$

The fundamental tensorial ingredient from which actions for the fields are built are the first covariant derivatives of the tetrad with respect to the covariant derivative defined by the spin connection

$$\begin{aligned} T^A = \mathcal{D}\theta^A &= (\partial_\mu\theta^A{}_\nu + \omega^A{}_{B\mu}\theta^B{}_\nu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \frac{1}{2}T^A{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \end{aligned} \tag{4}$$

which is nothing but the torsion of the connection. Using the covariant derivative $\mathcal{D}$ in the definition of the torsion ensures a covariant transformation behavior under local Lorentz transformations of the tetrad [1,2]. Changes of index types on tensors are performed by multiplication with tetrad components, for example, $T^\mu{}_{\rho\sigma} = T^A{}_{\rho\sigma}e_A{}^\mu$.

We now consider the most general Lagrange densities, in four spacetime dimensions, quadratic in torsion, which can be built from the components $T^A{}_{\mu\nu}$ of the torsion tensor and the tetrad alone, while not introducing further derivatives or parity violating terms. This class of theories can be parametrized in terms of three free parameters c_1, c_2, and c_3, and its Lagrangian is given by

$$\begin{aligned} L_{\mathrm{NGR}}[\theta,\Lambda] &= L_{\mathrm{NGR}}(\theta,\partial\theta,\Lambda,\partial\Lambda) \\ &= |\theta|(c_1T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} + c_2T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho + c_3T^\rho{}_{\mu\rho}T^{\sigma\mu}{}_\sigma) \\ &= |\theta|G_{\alpha\beta}{}^{\mu\nu\rho\sigma}T^\alpha{}_{\mu\nu}T^\beta{}_{\rho\sigma} = |\theta|G_{AB}{}^{\mu\nu\rho\sigma}T^A{}_{\mu\nu}T^B{}_{\rho\sigma}. \end{aligned} \tag{5}$$

In the last equality we introduced the convenient supermetric or constitutive tensor representation of the Lagrangian [8,9,15], where below the metric must be read as a function of the tetrads[1]

$$G_{AB}{}^{\mu\nu\rho\sigma} = c_1\eta_{AB}g^{\rho[\mu}g^{\nu]\sigma} - c_2e_B^{[\mu}g^{\nu][\rho}e_A^{\sigma]} - c_3e_A^{[\mu}g^{\nu][\rho}e_B^{\sigma]}. \tag{6}$$

Teleparallel theories of gravity with the action

$$S[\theta,\Lambda] = \int_M L_{\mathrm{NGR}}[\theta,\Lambda]\mathrm{d}^4x \tag{7}$$

are called NGR theories of gravity [21]. Choosing the parameters of the theory to be $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$, and $c_3 = -1$ the theory reduces to TEGR [4].

III. 3 + 1 DECOMPOSITION AND CONJUGATE MOMENTA

In order to derive the Hamilton formulation of the previously introduced NGR teleparallel theories we need to split spacetime into spatial hypersurfaces and a time direction before we derive the canonical momenta of the field variables. We introduce the $3+1$ decomposition in local coordinates (x^0, x^i), where the submanifolds $x^0 =$ const are the spatial hypersurfaces. As for the Hamiltonian formulation of general relativity, see, for example, the modern review [29] and references therein, the metric can be decomposed into the lapse function α, the shift vector β^i, and the metric on the spatial hypersurfaces h_{ij}

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^i\beta^jh_{ij} & \beta_i \\ \beta_i & h_{ij} \end{bmatrix}, \qquad g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & h^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix}. \tag{8}$$

Spatial indices $i, j, \ldots$ are raised and lowered with the components of the spatial metric h_{ij}, i.e., $\beta_i = \beta^jh_{ij}$.

In the teleparallel formulation of theories of gravity we need to apply the $3+1$ decomposition to the tetrad $\theta^A = \theta^A{}_0\mathrm{d}x^0 + \theta^A{}_i\mathrm{d}x^i$ and its dual $e_A = e_A{}^0\partial_0 + e_A{}^i\partial_i$ instead of to the metric. They can be further expanded into lapse and shift by writing

$$\theta^A{}_0 = \alpha\xi^A + \beta^i\theta^A{}_i, \tag{9}$$

where we introduced the components ξ^A of the normal vector n to the $x^0 =$ const hypersurfaces in the dual tetrad basis [10]

$$n = \xi^Ae_A, \qquad \xi^A = -\frac{1}{6}\epsilon^A{}_{BCD}\theta^B{}_i\theta^C{}_j\theta^D{}_k\epsilon^{ijk}. \tag{10}$$

Lowering and raising upper-case Latin indices with the Minkowski metric η_{AB}, the ξ^A satisfy

$$\eta_{AB}\xi^A\xi^B = \xi^A\xi_A = -1, \qquad \eta_{AB}\xi^A\theta^B{}_i = \xi_A\theta^A{}_i = 0. \tag{11}$$

The dual tetrads and the spatial metric can be expanded into lapse, shift, and spatial tetrads as

$$e_A{}^0 = -\frac{1}{\alpha}\xi_A, \qquad e_A{}^i = \theta_A{}^i + \xi_A\frac{\beta^i}{\alpha}, \qquad h_{ij} = \eta_{AB}\theta^A{}_i\theta^B{}_j. \tag{12}$$

[1]Alternatively, one may introduce the so-called axial, vector, tensor decomposition of the torsion, in which the NGR Lagrangian becomes $L = a_1T_{\mathrm{ax}} + a_2T_{\mathrm{tens}} + a_3T_{\mathrm{vec}}$ [20]. The coefficients translate as $c_1 = -\frac{1}{3}(a_1 + 2a_2)$, $c_2 = \frac{2}{3}(a_1 - a_2)$, and $c_3 = \frac{2}{3}(a_2 - a_3)$.

Observe the following possible source of confusion. The spatial components of the tetrad with noncanonical index positions are defined as $\theta_A{}^i = \eta_{AB} h^{ij} \theta^B{}_j \neq e_A{}^i = \theta_A{}^i + \xi_A \frac{\beta^i}{\alpha}$. This is related to the fact that in contrast to other approaches, such as the standard calculation for the Hamiltonian of GR, we do not expand tensors into components parallel or orthogonal to the spatial hypersurfaces, but parallel to the hypersurfaces or the time direction.

Inserting these expansions into the NGR Lagrangian we obtain the $3+1$ split of the theory

$$\begin{aligned} L_{\mathrm{NGR}}[\alpha, \beta^i, \theta^A{}_i, \Lambda^A{}_B] &= |\theta|(4G_{AB}{}^{i0j0} T^A{}_{i0} T^B{}_{j0} + 4G_{AB}{}^{ijk0} T^A{}_{ij} T^B{}_{k0} + G_{AB}{}^{ijkl} T^A{}_{ij} T^B{}_{kl}) \\ &= \frac{\sqrt{h}}{2\alpha} T^A{}_{i0} T^B{}_{j0} M^i{}_A{}^j{}_B + \frac{\sqrt{h}}{\alpha} T^A{}_{i0} T^B{}_{kl} [M^i{}_A{}^l{}_B \beta^k + 2\alpha h^{il} (c_2 \xi_B \theta_A{}^k + c_3 \xi_A \theta_B{}^k)] \\ &\quad + \frac{\sqrt{h}}{\alpha} T^A{}_{ij} T^B{}_{kl} \beta^i [\frac{1}{2} M^j{}_A{}^l{}_B \beta^k + 2\alpha h^{jl} (c_2 \xi_B \theta_A{}^k + c_3 \xi_A \theta_B{}^k)] + \alpha \sqrt{h} \cdot {}^3\mathbb{T}. \end{aligned} \tag{13}$$

The matrix $M^i{}_A{}^j{}_B$ is a map from 3×4 matrices to their duals, i.e., 4×3 matrices, and will play an important role when we express the velocities of the tetrads in terms of the canonical momenta and vice versa. It can be written in the form

$$\begin{aligned} M^i{}_A{}^j{}_B &= 8\alpha^2 G_{AB}{}^{i0j0} \\ &= -2(2c_1 h^{ij} \eta_{AB} - (c_2 + c_3) \xi_A \xi_B h^{ij} \\ &\quad + c_2 \theta_A{}^j \theta_B{}^i + c_3 \theta_A{}^i \theta_B{}^j). \end{aligned} \tag{14}$$

The purely intrinsic torsion scalar on the $x^0 = \mathrm{const}$ hypersurface is given by

$$\begin{aligned} {}^3\mathbb{T} \equiv{} & c_1 \eta_{AB} T^A{}_{ij} T^B{}_{kl} h^{ik} h^{jl} + c_2 \theta_A{}^i \theta_B{}^j T^A{}_{kj} T^B{}_{li} h^{kl} \\ & + c_3 \theta_A{}^i \theta_B{}^j h^{kl} T^A{}_{ki} T^B{}_{lj} = H_{AB}{}^{ijkl} T^A{}_{ij} T^B{}_{kl}, \end{aligned} \tag{15}$$

where the spatial supermetric is

$$H_{AB}{}^{ijkl} = c_1 \eta_{AB} h^{k[i} h^{j]l} - c_2 \theta_B{}^{[i} h^{j][k} \theta^{l]}{}_A - c_3 \theta_A^{[i} h^{j][k} \theta^{l]}{}_B. \tag{16}$$

In the $3+1$ decomposed form (13) it is not difficult to derive the canonical momenta of the tetrads $\theta^A{}_\mu$ and the Lorentz transformations $\Lambda^A{}_B$ which generate the spin connection. Time derivatives on the variables of the theory only appear in torsion terms $T^A{}_{0i}$ and never act on $\theta^A{}_0$, due to the antisymmetry of the torsion tensor in its lower indices, nor on the lapse α and the shift β. Hence the canonical momenta of lapse and shift are, not surprisingly, all identically zero,

$$\pi_\alpha = \frac{\partial L_{\mathrm{NGR}}}{\partial \dot{\alpha}} = 0, \qquad \pi_{\beta^i} = \frac{\partial L_{\mathrm{NGR}}}{\partial \dot{\beta}^i} = 0. \tag{17}$$

The canonical momenta of the spatial tetrad components are given by

$$\begin{aligned} \frac{\alpha}{\sqrt{h}} \pi_A{}^i = \frac{\alpha}{\sqrt{h}} \frac{\partial L_{\mathrm{NGR}}}{\partial \dot{\theta}^A{}_i} &= T^B{}_{0j} M^i{}_A{}^j{}_B \\ &\quad + T^B{}_{kl} [M^i{}_A{}^k{}_B \beta^l + 2\alpha h^{ik} (c_2 \xi_B \theta_A{}^l + c_3 \xi_A \theta_B{}^l)], \end{aligned} \tag{18}$$

while the momenta for the connection generating Lorentz transformations turn out to be completely determined from the momenta of the tetrad.

To see this first observe that the Lorentz group is six-dimensional and therefore not all components of the $\Lambda^A{}_B$ are independent of each other. To reflect this during the derivation of the corresponding momenta we introduce the auxiliary antisymmetric field a_{AB} in the following way:

$$\begin{aligned} a_{AB} \coloneqq \eta_{AC} \omega^C{}_{B0} = \eta_{C[A} \Lambda^C{}_{|D|} (\dot{\Lambda^{-1}})^D{}_{B]} \Leftrightarrow \dot{\Lambda}^A{}_B \\ = a_{MN} \eta^{A[N} \Lambda^{M]}{}_B. \end{aligned} \tag{19}$$

The independent components of the momenta of the Lorentz matrices are then given by

$$\hat{\pi}^{AB} = \frac{\partial L_{\mathrm{NGR}}}{\partial a_{AB}}, \tag{20}$$

and satisfy

$$\hat{\pi}^{AB} = -\pi_C{}^i \eta^{C[B} \theta^{A]}{}_i, \tag{21}$$

which can easily be realized from

$$\begin{aligned} \frac{\partial L_{\mathrm{NGR}}}{\partial a_{MN}} = \frac{\partial L_{\mathrm{NGR}}}{\partial \dot{\Lambda}^A{}_B} \frac{\partial \dot{\Lambda}^A{}_B}{\partial a_{MN}} &= \frac{\partial L_{\mathrm{NGR}}}{\partial T^C{}_{0k}} \frac{\partial T^C{}_{0k}}{\partial \dot{\Lambda}^A{}_B} \frac{\partial \dot{\Lambda}^A{}_B}{\partial a_{MN}} \\ &= -\frac{\partial L_{\mathrm{NGR}}}{\partial T^C{}_{0k}} \frac{\partial T^C{}_{0k}}{\partial \dot{\theta}^A{}_i} [\theta^D{}_i (\Lambda^{-1})^B{}_D] \frac{\partial \dot{\Lambda}^A{}_B}{\partial a_{MN}} \end{aligned} \tag{22}$$

$$= -\frac{\partial L_{\mathrm{NGR}}}{\partial \dot{\theta}^A{}_i} [\theta^D{}_i (\Lambda^{-1})^B{}_D] \eta^{A[N} \Lambda^{M]}{}_B. \tag{23}$$

The fact that the momenta $\hat{\pi}$ are not independent of the momenta π demonstrates that the $\Lambda^A{}_B$ are not independent, but only gauge d.o.f.

In the following, we introduce new field variables $(\tilde{\alpha}, \tilde{\beta}^i, \tilde{\theta}^A{}_i, \tilde{\Lambda}^A{}_B)$, where $\tilde{\theta}^A{}_i(\theta, \Lambda) \coloneqq \theta^B{}_i(\Lambda^{-1})^A{}_B$ is the so-called Weitzenböck tetrad and all other fields are not changed: $\tilde{\alpha} = \alpha$, $\tilde{\beta}^i = \beta^i$, and $\tilde{\Lambda}^A{}_B = \Lambda^A{}_B$. Using the inverse of this definition $\theta^B{}_i = \tilde{\theta}^A{}_i\Lambda^B{}_A$ to express the Lagrangian (5) in terms of the Weitzenböck tetrad yields that $\tilde{L}_{\mathrm{NGR}}[\alpha, \beta^i, \tilde{\theta}^A{}_i, \Lambda^A{}_B] \coloneqq L_{\mathrm{NGR}}[\alpha, \beta^i, \theta^A{}_i(\tilde{\theta}, \Lambda), \Lambda^A{}_B]$ is independent of Λ, respectively, $\tilde{\Lambda}$. The α and β^i momenta are not affected by this field redefinition at all. For the momenta in the new frame we find the transformation behavior

$$\begin{aligned} \tilde{\pi}_A{}^i &= \frac{\partial \tilde{L}_{\mathrm{NGR}}}{\partial \dot{\tilde{\theta}}^A{}_i} = \pi_B{}^i\Lambda^B{}_A, \\ \hat{\tilde{\pi}}^{MN} &= \frac{\partial \tilde{L}_{\mathrm{NGR}}}{\partial a_{MN}} = \pi_A{}^j\eta^{A[N}\theta^{M]}{}_j + \hat{\pi}^{MN}, \end{aligned} \tag{24}$$

with inverse transformation

$$\begin{aligned} \pi_A{}^i &= \tilde{\pi}_B{}^i(\Lambda^{-1})^B{}_A, \\ \hat{\pi}^{MN} &= \hat{\tilde{\pi}}^{MN} - \tilde{\pi}_B{}^j(\Lambda^{-1})^B{}_A\eta^{A[N}\Lambda^{M]}{}_C\tilde{\theta}^C{}_j. \end{aligned} \tag{25}$$

Applying the constraint (21) to the second part of the transformation (24) shows that in the Weitzenböck gauge the momenta of the Lorentz transformations all vanish, $\hat{\tilde{\pi}}^{AB} = 0$.

This reproduces the well-known fact that in teleparallel gravity the spin connection represents pure gauge d.o.f. [1,2]. Therefore, without loss of generality, we can set the spin connection coefficients to zero and work in the so-called Weitzenböck gauge, in which the connection coefficients of the spin connection vanish identically.

The Hamiltonian in the Weitzenböck gauge is then given by the Legendre transform of the Lagrangian where we have to add the primary constraints we already discovered, Eqs. (17) and (21) with Lagrange multipliers ${}^{\tilde{\alpha}}\lambda$, ${}^{\tilde{\beta}}\lambda^i$, and ${}^{\hat{\tilde{\pi}}}\lambda$

$$\begin{aligned} &\tilde{H}_{\mathrm{NGR}}[{}^{\tilde{\alpha}}\lambda, {}^{\tilde{\beta}}\lambda^i, {}^{\hat{\tilde{\pi}}}\lambda_{AB}, \tilde{\alpha}, \tilde{\pi}_\alpha, \tilde{\beta}^i, \tilde{\pi}_{\beta^i}, \tilde{\theta}^A{}_i, \tilde{\pi}_A{}^i, \tilde{\Lambda}^A{}_B, \hat{\tilde{\pi}}^{AB}] \\ &\quad = \tilde{\pi}_\alpha\dot{\tilde{\alpha}} + \tilde{\pi}_{\beta^i}\dot{\tilde{\beta}}^i + \tilde{\pi}_A{}^i\dot{\tilde{\theta}}^A{}_i + \hat{\tilde{\pi}}^{AB}\tilde{a}_{AB} + {}^{\tilde{\alpha}}\lambda\tilde{\pi}_\alpha + {}^{\tilde{\beta}}\lambda^i\tilde{\pi}_{\beta^i} \\ &\qquad + {}^{\hat{\tilde{\pi}}}\lambda_{AB}\hat{\tilde{\pi}}^{AB} - \tilde{L}_{\mathrm{NGR}}[\tilde{\alpha}, \tilde{\beta}^i, \tilde{\theta}_i, \tilde{\Lambda}]. \end{aligned} \tag{26}$$

The term $\hat{\tilde{\pi}}^{AB}\tilde{a}_{AB}$ is identical to the term one would use naively in terms of the canonical variables $\frac{\partial \tilde{L}_{\mathrm{NGR}}}{\partial \dot{\tilde{\Lambda}}^A{}_B}\dot{\tilde{\Lambda}}^A{}_B$, as can easily be seen from the definition of the auxiliary variable a_{AB} in Eq. (19). As mentioned $\tilde{\alpha} = \alpha$, $\tilde{\beta}^i = \beta^i$, and $\tilde{\Lambda}^A{}_B = \Lambda^A{}_B$; $\tilde{L}_{\mathrm{NGR}}[\tilde{\alpha}, \tilde{\beta}^i, \tilde{\theta}_i, \tilde{\Lambda}]$ is independent of Λ. Therefore, on shell, where the constraint $\hat{\tilde{\pi}}^{AB} = 0$ is implemented, the gauge fixed Hamiltonian does neither depend on Λ nor on $\hat{\tilde{\pi}}^{AB}$. Moreover the evolution of the constraints is preserved since their Poisson bracket with the Hamiltonian vanishes $\{\tilde{\pi}_\alpha, \tilde{H}\} \approx 0$, $\{\tilde{\pi}_{\beta^i}, \tilde{H}\} \approx 0$, $\{\hat{\tilde{\pi}}^{AB}, \tilde{H}\} \approx 0$ on the constraint surface $\tilde{\pi}_\alpha = \tilde{\pi}_{\beta^i} = \hat{\tilde{\pi}}^{AB} = 0$.

These findings on the level of canonical momenta demonstrate that we do not need to include the variables $\tilde{\pi}_\alpha, \tilde{\pi}_{\beta^i}, \Lambda$ and $\hat{\pi}$ in the Hamiltonian and again justify the approach in Ref. [30]. In the following we will work in the Weitzenböck gauge and omit the tilde from $\tilde{\theta}, \tilde{\pi}, \hat{\tilde{\pi}}$ for readability.

IV. INVERTING THE MOMENTUM-VELOCITY RELATION

One essential step in the reformulation of a physical field theory from its Lagrangian to its Hamiltonian description is to invert the relation between the momenta and the velocities, to express the latter in terms of the former. For NGR this amounts to inverting Eq. (18). To do so we rewrite the equation as a linear map from the space of 4×3 matrices to the space of 3×4 matrices

$$S_A{}^i = M^i{}_A{}^j{}_B\dot{\theta}^B{}_j, \tag{27}$$

with a source term $S_A{}^i$, which only depends on the momenta, the fields, and their spatial derivatives,

$$\begin{aligned} &S_A{}^i[\alpha, \beta, \theta^A{}_i, \pi_A{}^i] \\ &\quad = \frac{\alpha}{\sqrt{h}}\pi_A{}^i + [D_k(\alpha\xi^B + \beta^m\theta^B{}_m) - T^B{}_{kl}\beta^l]M^i{}_A{}^k{}_B \\ &\qquad - 2\alpha T^B{}_{kl}h^{ik}(c_2\xi_B\theta_A{}^l + c_3\xi_A\theta_B{}^l), \end{aligned} \tag{28}$$

where D_i is the Levi-Civita covariant derivative of the hypersurface metric h_{ij}. By inverting this equation we can reexpress the field velocities in terms of the canonical variables: the fields themselves and their momenta.

To explicitly invert Eq. (27) we decompose the velocities of the spatial tetrads into irreducible parts with respect to the rotation group. It turns out that in this decomposition the matrix M has a block diagonal structure which can be inverted block by block. Since for certain combinations of the c_1, c_2, c_3 parameters of the theory some blocks become identically zero, we employ the Moore-Penrose pseudoinverse of a matrix [15] to display the inverse in a closed form for all choices of the parameters. This then carries over when we display the Hamiltonian.

The irreducible decomposition with respect to the rotation group amounts in defining a vectorial ($\mathcal{V}$), antisymmetric ($\mathcal{A}$), symmetric trace-free ($\mathcal{S}$), and trace ($\mathcal{T}$) part of the tetrad velocities and their momenta:

$$\dot{\theta}^A{}_i = {}^{\mathcal{V}}\dot{\theta}_i\xi^A + {}^{\mathcal{A}}\dot{\theta}_{ji}h^{kj}\theta^A{}_k + {}^{\mathcal{S}}\dot{\theta}_{ji}h^{kj}\theta^A{}_k + {}^{\mathcal{T}}\dot{\theta}\theta^A{}_i, \tag{29}$$

$$\pi_A{}^i = {}^{\mathcal{V}}\pi^i\xi_A + {}^{\mathcal{A}}\pi^{ji}h_{kj}\theta_A{}^k + {}^{\mathcal{S}}\pi^{ji}h_{kj}\theta_A{}^k + {}^{\mathcal{T}}\pi\theta_A{}^i. \tag{30}$$

Decomposing ${S_A}^i$ into the same irreducible parts and using the explicit form of M, see Eq. (14), yields

$$\begin{aligned} {}^{\mathcal{V}}S^i &= -\xi^A {S_A}^i \\ &= {}^{\mathcal{V}}\pi^i \frac{\alpha}{\sqrt{h}} - 2\alpha c_3 {T^B}_{kl} h^{ik} {\theta_B}^l + 2(2c_1+c_2+c_3)[D_k(\alpha\xi^B + \beta^m {\theta^B}_m) - {T^B}_{kl}\beta^l]\xi_B h^{ik} \\ &= -2^{V}\dot{\theta}_j h^{ij}(2c_1+c_2+c_3), \end{aligned} \tag{31}$$

for the vector part,

$$\begin{aligned} {}^{\mathcal{A}}S^{mp} &= {\theta^A}_i h^{i[m} {S_A}^{p]} \\ &= {}^{\mathcal{A}}\pi^{mp} \frac{\alpha}{\sqrt{h}} - 2\alpha c_2 h^{lm} h^{pk} {T^B}_{kl} \xi_B - 2(2c_1-c_2)[D_k(\alpha\xi^B + \beta^s {\theta^B}_s) - {T^B}_{kl}\beta^l] {\theta_B}^{[m} h^{p]k} \\ &= -2^{\mathcal{A}}\dot{\theta}^{mp}(2c_1-c_2) \end{aligned} \tag{32}$$

for the antisymmetric part,

$$\begin{aligned} {}^{\mathcal{S}}S^{mp} &= {\theta^A}_q h^{q(m} {S_A}^{p)} - \frac{1}{3}{\theta^A}_i {S_A}^i h^{mp} \\ &= {}^{\mathcal{S}}\pi^{mp} \frac{\alpha}{\sqrt{h}} - 2(2c_1+c_2)[D_k(\alpha\xi^B + \beta^s {\theta^B}_s) - {T^B}_{kl}\beta^l]\left({\theta_B}^{(m} h^{p)k} - \frac{1}{3} h^{pm} {\theta_B}^k\right) \\ &= -2^{\mathcal{S}}\dot{\theta}^{mp}(2c_1+c_2) \end{aligned} \tag{33}$$

for the trace-free symmetric part, and

$$\begin{aligned} {}^{\mathcal{T}}S &= \frac{1}{3}{\theta^A}_i {S_A}^i \\ &= {}^{\mathcal{T}}\pi \frac{\alpha}{\sqrt{h}} - \frac{2}{3}(2c_1+c_2+3c_3)[D_k(\alpha\xi^B\beta^m {\theta^B}_m) - {T^B}_{kl}\beta^l]{\theta_B}^k \\ &= -2^{T}\dot{\theta}(2c_1+c_2+3c_3) \end{aligned} \tag{34}$$

for the trace part.

These equations are easily solved for the velocities in terms of their dual momenta in case the coefficients

$$A_{\mathcal{V}} = 2c_1+c_2+c_3, \qquad A_{\mathcal{A}} = 2c_1 - c_2,$$
$$A_{\mathcal{S}} = 2c_1+c_2, \quad \text{and} \quad A_{\mathcal{T}} = 2c_1+c_2+3c_3 \tag{35}$$

are all nonvanishing. In case one or more of these coefficients vanish they induce primary constraints:

$$A_{\mathcal{V}} = 0 \Rightarrow {}^{\mathcal{V}}C^i \coloneqq \frac{{}^{\mathcal{V}}\pi^i}{\sqrt{h}} - 2c_3 {T^B}_{kl} h^{ik} {\theta_B}^l = 0, \tag{36}$$

$$A_{\mathcal{A}} = 0 \Rightarrow {}^{\mathcal{A}}C^{ij} \coloneqq \frac{{}^{\mathcal{A}}\pi^{ij}}{\sqrt{h}} - 2c_2 h^{li} h^{jk} {T^B}_{kl}\xi_B = 0, \tag{37}$$

$$A_{\mathcal{S}} = 0 \Rightarrow {}^{\mathcal{S}}C^{ij} \coloneqq \frac{{}^{\mathcal{S}}\pi^{ij}}{\sqrt{h}} = 0, \tag{38}$$

$$A_{\mathcal{T}} = 0 \Rightarrow {}^{\mathcal{T}}C \coloneqq \frac{{}^{\mathcal{T}}\pi}{\sqrt{h}} = 0. \tag{39}$$

Observe that ${}^{\mathcal{V}}C^i$ correspond to 3 constraints, ${}^{\mathcal{A}}C^{mp}$ to 3 (since it is antisymmetric in its indices), ${}^{\mathcal{S}}C^{mp}$ to 5 (since it is symmetric in its indices, but does not contain the trace part), and ${}^{\mathcal{T}}C$ corresponds to 1 constraint. For any choice of the parameters c_1, c_2, c_3 we either can invert the appearing velocities of the tetrads in terms of the tetrads and their momenta, or we obtain a constraint from the Lagrangian, which must be implemented in the Hamiltonian later by a Lagrange multiplier.

The Moore-Penrose pseudoinverse of the matrix M in the irreducible decomposition of the rotation group we employed is given by the inverse of the separate blocks if the coefficient in front of the block $A_{\mathcal{V}}, A_{\mathcal{A}}, A_{\mathcal{S}}$, or $A_{\mathcal{T}}$ is nonvanishing. In case one of the coefficients is vanishing the block in the inverse matrix is simply a block of zeros. For completeness we display M and its Moore-Penrose pseudoinverse explicitly. Expanding M itself into the irreducible parts basis

$$\begin{aligned} {M^i}_A{}^j{}_B &= {}^{\mathcal{V}}M^{ij}\xi_A\xi_B + {}^{\mathcal{A}}M^{[ir][js]}{\theta^C}_r \eta_{AC} {\theta^D}_s \eta_{BD} \\ &\quad + {}^{\mathcal{S}}M^{(ir)(js)}{\theta^C}_r \eta_{AC} {\theta^D}_s \eta_{BD} + {}^{\mathcal{T}}M {\theta_A}^i {\theta_B}^j \end{aligned} \tag{40}$$

yields

$$\begin{aligned}M^i{}_A{}^j{}_B &= 2A_{\mathcal{V}}\xi_A\xi_B h^{ij} - 2A_{\mathcal{A}} h^{i[j}h^{s]r}\theta^C{}_r\eta_{AC}\theta^D{}_s\eta_{BD} \\ &\quad - 2A_{\mathcal{S}}\left(h^{i(j}h^{s)r} - \frac{1}{3}h^{ir}h^{js}\right)\theta^C{}_r\eta_{AC}\theta^D{}_s\eta_{BD} \\ &\quad - \frac{2}{3}A_{\mathcal{T}}\theta_A{}^i\theta_B{}^j. \end{aligned} \tag{41}$$

By using the identity $\eta^{AB} + \xi^A\xi^B = \theta^A{}_i\theta^B{}_j h^{ij}$ one may check that this representation of M is indeed identical to its definition (14). Its pseudoinverse is

$$\begin{aligned}(M^{-1})^{A}_{i}{}^{C}_{k} &= \frac{1}{2}B_{\mathcal{V}}\xi^A\xi^C h_{ik} - \frac{1}{2}B_{\mathcal{A}} h^{r[s}h^{m]n}h_{kr}h_{si}\theta^A{}_m\theta^C{}_n \\ &\quad - \frac{1}{2}B_{\mathcal{S}}\left(h^{r(s}h^{m)n} - \frac{1}{3}h^{sm}h^{nr}\right)h_{kr}h_{si}\theta^A{}_m\theta^C{}_n \\ &\quad - \frac{1}{6}B_{\mathcal{T}}\theta^A{}_i\theta^C{}_k, \end{aligned} \tag{42}$$

where the different blocks are implemented by defining $(I = \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T})$ $B_I = \{ {}^{0 \text{ for } A_I = 0}_{\frac{1}{A_I} \text{ for } A_I \neq 0}$.

V. THE NGR HAMILTONIAN

To obtain the Hamiltonian from the Lagrangian we use its definition as Legendre transform omitting the variables Λ and $\hat{\pi}$, as discussed below Eq. (26). We display the dependencies on the remaining variables explicitly for clarification, and the square brackets shall indicate that the function may depend on the spatial derivatives of the fields,

$$\begin{aligned}H[\alpha, \beta^i, \theta^A{}_j, \pi_A{}^k] &= \dot{\theta}^A{}_i[\alpha, \beta^i, \theta^A{}_j, \pi_A{}^k]\pi_A{}^i \\ &\quad - L[\alpha, \beta^i, \theta^A{}_j, \dot{\theta}^A{}_k[\alpha, \beta^r, \theta^A{}_s, \pi_A{}^m]]. \end{aligned} \tag{43}$$

We will suppress these dependencies in the brackets from now on for the sake of readability. Moreover, we comment on how to remove the gauge fixing, i.e., how to reintroduce the dependence on Λ and $\hat{\pi}$ at the end of this section. A sketch on how the calculations would be carried out without gauge fixing is made in Appendix.

To derive the Hamiltonian explicitly we can first use the source expression S, defined in Eq. (28), to simplify the Lagrangian. This can be done by expanding the $T^A{}_{i0}$ terms in Eq. (5) into the time derivatives of the tetrad and combining them with the M matrices to the source term whenever possible. By their definition, they can then be expanded in terms of the momenta and spatial derivatives acting on the fields. As an intermediate result the Hamiltonian becomes

$$\begin{aligned}H &= \frac{1}{2}\dot{\theta}^A{}_i\pi_A{}^i - \sqrt{h}T^B{}_{jk}\dot{\theta}^A{}_i h^{ij}[c_2\xi_B\theta_A{}^k + c_3\xi_A\theta_B{}^k] \\ &\quad + \frac{1}{2}\pi_A{}^i D_i(\alpha\xi^A + \beta^j\theta^A{}_j) + \sqrt{h}T^B{}_{kl}D_i(\alpha\xi^A + \beta^j\theta^A{}_j)h^{ik}[c_2\xi_B\theta_A{}^l + c_3\xi_A\theta_B{}^l] \\ &\quad - \frac{1}{2}\pi_B{}^j T^B{}_{jk}\beta^k - \sqrt{h}T^A{}_{ij}T^B{}_{kl}\beta^k h^{il}[c_2\xi_A\theta_B{}^j + c_3\xi_B\theta_A{}^j] - \alpha\sqrt{h}\cdot {}^3\mathbb{T}. \end{aligned} \tag{44}$$

To eliminate the remaining velocities we expand them into the $\mathcal{V}$, $\mathcal{A}$, $\mathcal{S}$, $\mathcal{T}$ decomposition we introduced in the previous section and replace them according to Eqs. (31) to (34).

Expanding the first term in the irreducible decomposition yields

$$\dot{\theta}^A{}_i\pi_A{}^i = -{}^{\mathcal{V}}\dot{\theta}_i{}^{\mathcal{V}}\pi^i + {}^{\mathcal{A}}\dot{\theta}_{ji}{}^{\mathcal{A}}\pi^{ji} + {}^{\mathcal{S}}\dot{\theta}_{ji}{}^{\mathcal{S}}\pi^{ji} + 3{}^{\mathcal{T}}\dot{\theta}{}^{\mathcal{T}}\pi \tag{45}$$

$$= \alpha\left(\frac{{}^{\mathcal{V}}C_i{}^{\mathcal{V}}\pi^i}{2A_{\mathcal{V}}} - \frac{{}^{\mathcal{A}}C_{ij}{}^{\mathcal{A}}\pi^{ij}}{2A_{\mathcal{A}}} - \frac{{}^{\mathcal{S}}C_{ij}{}^{\mathcal{S}}\pi^{ij}}{2A_{\mathcal{S}}} - \frac{3{}^{\mathcal{T}}C^{\mathcal{T}}\pi}{2A_{\mathcal{T}}}\right) + \pi_A{}^i D_i(\alpha\xi^A + \beta^m\theta^A{}_m) - \pi_A{}^i T^A{}_{im}\beta^m, \tag{46}$$

while for the second we find

$$\begin{aligned}\sqrt{h}T^B{}_{jk}\dot{\theta}^A{}_i h^{ij}[c_2\xi_B\theta_A{}^k + c_3\xi_A\theta_B{}^k] &= c_2 T^B{}_{jk}{}^{\mathcal{A}}\dot{\theta}_{mi}h^{km}h^{ij} - c_3 T^B{}_{jk}{}^{\mathcal{V}}\dot{\theta}_i h^{ij}\theta_B{}^k \\ &= \frac{\alpha}{2A_{\mathcal{A}}}c_2\xi_B T^B{}_{jk}{}^{\mathcal{A}}C^{jk} + \frac{\alpha}{2A_{\mathcal{V}}}c_3\theta_B{}^k T^B{}_{jk}{}^{\mathcal{V}}C^j \\ &\quad - [D_i(\alpha\xi^C + \beta^m\theta^C{}_m) - T^C{}_{im}\beta^m]T^B{}_{jk}h^{ki}[c_2\xi_B\theta_C{}^j + c_3\xi_C\theta_B{}^j]. \end{aligned} \tag{47}$$

Inserting the expressions (45) and (47) into Eq. (44) finally yields the kinematic Hamilton density of the NGR teleparallel theories of gravity,

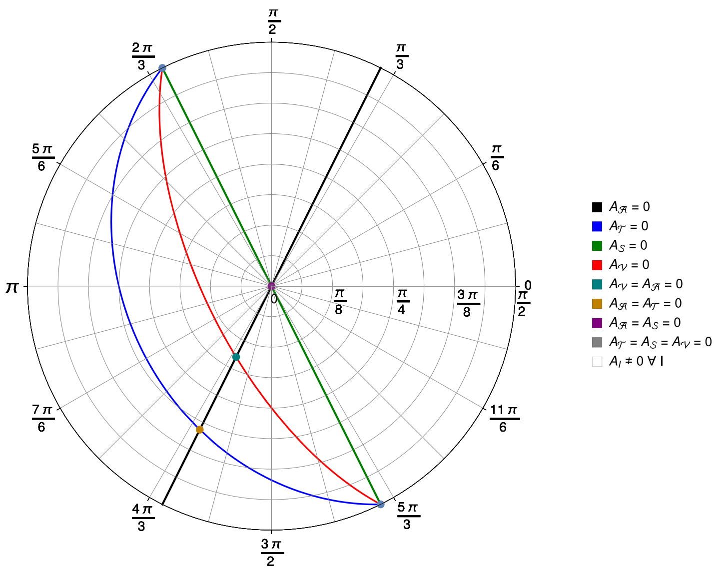

FIG. 1. Visualization of the parameter space of new general relativity, colored by the occurrences of primary constraints. The radial axis shows the zenith angle θ, while the (circular) polar axis shows the azimuth angle ϕ, following the definition (50).

$$H = \alpha\sqrt{h}\left(\frac{{}^{\mathcal{V}}C_i {}^{\mathcal{V}}C^i}{4A_{\mathcal{V}}} - \frac{{}^{\mathcal{A}}C_{ij} {}^{\mathcal{A}}C^{ij}}{4A_{\mathcal{A}}} - \frac{{}^{\mathcal{S}}C_{ij} {}^{\mathcal{S}}C^{ij}}{4A_{\mathcal{S}}} - \frac{3{}^{\mathcal{T}}C {}^{\mathcal{T}}C}{4A_{\mathcal{T}}} - {}^3\mathbb{T} - \frac{\xi^A D_i \pi_A{}^i}{\sqrt{h}}\right) - \beta^k (T^A{}_{jk}\pi_A{}^j + \theta^A{}_k D_i \pi_A{}^i)$$
$$+ D_i[\pi_A{}^i(\alpha\xi^A + \beta^j\theta^A{}_j)], \tag{48}$$

which we here display in terms of the constraints (36) to (39), as this is the most convenient expression. Observe that, even though we use the irreducible $\mathcal{V}$, $\mathcal{A}$, $\mathcal{S}$, $\mathcal{T}$ decomposition of the fields to display the Hamiltonian, since in this form the dependence on the parameters c_i becomes most clear, the canonical variables on which the Hamiltonian depends are $\{\alpha, \beta^i, \theta^A{}_j, \pi_A{}^k\}$. As in general relativity we immediately see that we deal with a pure constraint Hamiltonian up to boundary terms. Lapse α and shift β have vanishing momenta, $\pi_\alpha = 0$ and $\pi_{\beta_i} = 0$, and appear only as Lagrange multipliers. To obtain the dynamically equivalent Hamiltonian to the Lagrangian (5) we need to add possible further nontrivial constraints via Lagrange multipliers. To find all constraints it is necessary to calculate the Poisson brackets between all primary constraints, check if they are first class, and, in case they are not, add possible secondary constraints. This algorithm has to be continued until a closed constraint algebra is obtained [31].

From our analysis in Sec. IV we conclude that the NGR theories of gravity decay into nine subclasses depending on the choice of the parameters c_1, c_2, and c_3, which correspond to the appearance of different primary class constraints, in addition to the lapse and shift constraints arising from the diffeomorphism invariance of the action. We have visualized these classes in Fig. 1, which we constructed as follows. We started from the assumption that at least one of the parameters c_1, c_2, c_3 is nonvanishing, since otherwise the Lagrangian would be trivial, and introduced normalized parameters

$$\tilde{c}_i = \frac{c_i}{\sqrt{c_1^2 + c_2^2 + c_3^2}}, \tag{49}$$

for $i = 1, 2, 3$. One easily checks that the constraint classes we found only depend on these normalized parameters. We then introduced polar coordinates (θ, ϕ) on the unit sphere to express the parameters as

$$\tilde{c}_1 = \sin\theta\cos\phi, \qquad \tilde{c}_2 = \sin\theta\sin\phi, \qquad \tilde{c}_3 = \cos\theta. \tag{50}$$

Since the same constraints appear for antipodal points on the parameter sphere, we restrict ourselves to the hemisphere $\tilde{c}_3 \geq 0$, and hence $0 \leq \theta \leq \frac{\pi}{2}$; this is equivalent to identifying antipodal points on the sphere and working with the projective sphere instead, provided that we also identify antipodal points on the equator $\tilde{c}_3 = 0$. We then considered (θ, ϕ) as polar coordinates on the plane in order to draw the diagram shown in Fig. 1. Note that antipodal points on the perimeter, such as the two gray points for the most constrained case, are identified with each other, since they describe the same class of theories. To summarize, we find the following constraints:

Theory	Constraints	Location in Fig. 1
$A_I \neq 0 \; \forall \, I \in \{\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}\}$	No constraints	white area
$A_\mathcal{V} = 0$	${}^\mathcal{V}C_i = 0$	red line
$A_\mathcal{A} = 0$	${}^\mathcal{A}C_{ji} = 0$	black line
$A_\mathcal{S} = 0$	${}^\mathcal{S}C_{ji} = 0$	green line
$A_\mathcal{T} = 0$	${}^\mathcal{T}C = 0$	blue line
$A_\mathcal{V} = A_\mathcal{A} = 0$	${}^\mathcal{V}C_i = {}^\mathcal{A}C_{ji} = 0$	turquoise point
$A_\mathcal{A} = A_\mathcal{S} = 0$	${}^\mathcal{A}C_{ji} = {}^\mathcal{S}C_{ji} = 0$	purple point (center)
$A_\mathcal{A} = A_\mathcal{T} = 0$	${}^\mathcal{A}C_{ji} = {}^\mathcal{T}C = 0$	orange point
$A_\mathcal{V} = A_\mathcal{S} = A_\mathcal{T} = 0$	${}^\mathcal{V}C_i = {}^\mathcal{S}C_{ji} = {}^\mathcal{T}C = 0$	gray points (perimeter)[a]

[a]This is actually only one point in the parameter space, since antipodal points on the perimeter correspond to the same theory.

In order to understand the d.o.f. and derive the full Hamiltonian of the theory, we would need to calculate the Poisson brackets and deduce whether they are first or second class constraints and if more constraints appear (secondary, tertiary, etc). For teleparallel equivalence to general relativity this has already been done in Refs. [10–12,14–16,28,30] and it was found that the dynamical equivalent Hamiltonian to TEGR can be expressed with the help of two sets of Lagrange multipliers, ${}^\mathcal{V}\lambda^i$ and ${}^\mathcal{A}\lambda^{ij}$, as

$$\begin{aligned} H_{\text{TEGR}} = {} & \sqrt{h}({}^\mathcal{V}\lambda^i {}^\mathcal{V}C_i + {}^\mathcal{A}\lambda^{ij} {}^\mathcal{A}C_{ij}) + D_i[\pi_A{}^i(\alpha\xi^A + \beta^j\theta^A{}_j)] \\ & - \alpha\sqrt{h}\left(\frac{1}{4}{}^\mathcal{S}C_{ij}{}^\mathcal{S}C^{ij} - \frac{3}{8}{}^\mathcal{T}C^\mathcal{T}C + {}^3\mathbb{T} + \frac{\xi^A D_i \pi_A{}^i}{\sqrt{h}}\right) \\ & - \beta^k(T^A{}_{jk}\pi_A{}^j + \theta^A{}_k D_i \pi_A{}^i). \end{aligned} \tag{51}$$

In the future we aim to derive the dynamically equivalent Hamiltonians for all nine classes we identified among the NGR theories of gravity. By introducing additional Lagrange multipliers ${}^\mathcal{S}\lambda^{ij}$ and ${}^\mathcal{T}\lambda^{ij}$ in the short-hand notation

$${}^\mathcal{V}H = \begin{cases} \alpha\sqrt{h}\frac{{}^\mathcal{V}C_i {}^\mathcal{V}C^i}{4A_\mathcal{V}} & \text{for } {}^\mathcal{V}A \neq 0 \\ \sqrt{h}{}^\mathcal{V}\lambda_i {}^\mathcal{V}C^i & \text{for } {}^\mathcal{V}A = 0, \end{cases} \qquad {}^\mathcal{A}H = \begin{cases} -\alpha\sqrt{h}\frac{{}^\mathcal{A}C_{ij} {}^\mathcal{A}C^{ij}}{4A_\mathcal{A}} & \text{for } {}^\mathcal{A}A \neq 0 \\ \sqrt{h}{}^\mathcal{A}\lambda_{ij} {}^\mathcal{A}C^{ij} & \text{for } {}^\mathcal{A}A = 0, \end{cases} \tag{52}$$

$${}^\mathcal{S}H = \begin{cases} -\alpha\sqrt{h}\frac{{}^\mathcal{S}C_{ij} {}^\mathcal{S}C^{ij}}{4A_\mathcal{S}} & \text{for } {}^\mathcal{S}A \neq 0 \\ \sqrt{h}{}^\mathcal{S}\lambda_{ij} {}^\mathcal{S}C^{ij} & \text{for } {}^\mathcal{S}A = 0, \end{cases} \qquad {}^\mathcal{T}H = \begin{cases} -\alpha\sqrt{h}\frac{3{}^\mathcal{T}C_{ij} {}^\mathcal{T}C^{ij}}{4A_\mathcal{T}} & \text{for } {}^\mathcal{T}A \neq 0 \\ \sqrt{h}{}^\mathcal{T}\lambda_{ij} {}^\mathcal{T}C^{ij} & \text{for } {}^\mathcal{T}A = 0, \end{cases} \tag{53}$$

we can display a first step towards the dynamical Hamiltonians

$$\begin{aligned} H = {} & ({}^\mathcal{V}H + {}^\mathcal{A}H + {}^\mathcal{S}H + {}^\mathcal{T}H) - \alpha(\sqrt{h}{}^3\mathbb{T} - \xi^A D_i \pi_A{}^i) - \beta^k(T^A{}_{jk}\pi_A{}^j + \theta^A{}_k D_i \pi_A{}^i) + D_i[\pi_A{}^i(\alpha\xi^A + \beta^j\theta^A{}_j)] \\ & + \text{secondary-, tertiary-, … constraints}. \end{aligned} \tag{54}$$

However, the list of secondary-, tertiary-, … constraints, which have to be added in addition, has to be investigated separately for the nine classes we derived. Even within a single class there may appear different constraint algebras. For example, in the class with all A_I being nonzero, the Poisson bracket of the Hamilton constraint with itself in general generates new constraints since the Poisson brackets of the Hamiltonian and momenta constraints do not form a closed algebra. However, for particular values of the parameters the terms which cause this behavior are absent from the action, thus allowing the Poisson brackets to close [25]. Because of the lengthiness of the calculations even in seemingly simple cases such as TEGR [11] we present these studies in separate articles. Another potential issue that must receive attention is the possible bifurcation of constraints, i.e., the situation where the closing or non-closing of the Poisson brackets depends on the particular values of the fields, as found in previous studies [32], which we plan to investigate in detail in further work.

Before we conclude this article we like to add one more remark on the gauge fixing. The Hamiltonian we obtained is derived in the Weitzenböck gauge. To remove the gauge fixing and to reintroduce the variables Λ and $\hat{\pi}$, which we removed in the course of the discussion in Sec. III, the following two steps have to be performed. First replace the Levi-Civita covariant derivatives D_i in Eq. (54) by a total covariant derivative $\mathfrak{D}_i$ which also acts on the Lorentz indices of the objects appearing,

$$D_i \pi_A{}^j \rightarrow \mathfrak{D}_i \pi_A{}^j = D_i \pi_A{}^j - \omega^B{}_{Ai} \pi_B{}^j, \tag{55}$$

and, second, add the constraint (21) with the help of a Lagrange multiplier. The result is a gauge invariant Hamiltonian depending on the field variables α, β^i, $\theta^A{}_i, \pi_A{}^i$, and $\Lambda^A{}_B$ as well as $\hat{\pi}^{AB}$.

VI. CONCLUSION

We have derived a closed form for the kinematic Hamiltonian of new general relativity theories of gravity, starting from its Lagrangian formulation including the teleparallel spin connection. The latter we implemented explicitly in terms of local Lorentz transformations, thus avoiding the need for Lagrange multipliers in the action. We found that the canonical momenta for the spin connection are not independent and can fully be expressed in terms of the momenta for the tetrad. Further, only the 12 spatial components of the tetrads have nonvanishing momenta, while the 4 temporal components can be expressed in terms of the ADM variables lapse and shift, whose momenta vanish identically. We have shown that it is not possible to invert the relation between the time derivatives of the spatial tetrad components and their conjugate momenta, which results in the appearance of up to four types of further primary constraints, depending on the choice of parameters defining the theory. We find that the family of NGR theories is divided into nine different classes, which are distinguished by the presence or absence of these primary constraints. We visualized the locations of these nine classes in the parameter space of the theory, and identified a prototype of a dynamically equivalent Hamiltonian for the different classes, which serves as a starting point for the continuation towards a complete systematic Hamiltonian analysis of NGR.

Our results invite further investigations in various directions. The most logical next step is the calculation of the Poisson brackets for all possible constraints. This will show under which circumstances the constraint algebra closes, and under which circumstances additional constraints must be included, and finally lead to the full, dynamical Hamiltonian. It should be noted that the calculation of the Poisson brackets is straightforward, although it can be very lengthy, even in the case of TEGR [11]. Naively, the unconstrained case would be the easiest, since it involves the least number of constraints to calculate Poisson brackets with. However, the Poisson brackets do not form a closed algebra, hence are not first class, except for special cases [25], and thus generate further secondary constraints. Another class of new general relativity theories of particular interest besides general relativity is the one where only the vector constraint $A_{\mathcal{V}} = 0$ is imposed. It has been argued that this constraint is necessary in order to avoid the appearance of ghosts at the linearized level [33,34]. The constraint algebra has been worked out for this case, and it turns out that also in this case the constraints are not first class, so that secondary constraints appear [24].

An important result which we expect from the aforementioned further work on the constraint algebra is the number of d.o.f. for general parameters of new general relativity. A hint towards the existence of further d.o.f. compared to TEGR comes from comparing the d.o.f. in new general relativity with the number of polarization modes of gravitational waves in the Newman-Penrose formalism [22]. This result gives a lower bound of the number of d.o.f., since the polarization modes which appear in the linearized theory must come from the fundamental d.o.f. in the complete nonlinear theory. Once the full Hamiltonian is derived, it can be compared with the propagators presented in Ref. [35]. Results for a systematic categorization of theoretical pathologies (tachyons and ghosts) in a large class of theories including NGR was recently presented in Ref. [36]. Future work could consist of confirming their results using the Hamiltonian analysis and getting guidance in which theories are mostly motivated and perform the full-fledged Hamiltonian analysis in these cases.

The full dynamical Hamiltonian would also be useful for further tests of NGR with observations, in particular considering gravitational waves. The results we presented here show that the vicinity of TEGR in the parameter space, which is known to be compatible with post-Newtonian observations in the solar system [21], is composed out of different classes of possible constraint algebras. Studying their Hamiltonian dynamics one may expect new results on the generation of gravitational waves in these theories, from which tighter bounds on the NGR parameters would be obtained.

ACKNOWLEDGMENTS

The authors are grateful to Martin Krššák for numerous discussions and pointing out references, and to María José

Guzmán for commenting on a previous version of this article. They were supported by the Estonian Research Council through the Institutional Research Funding Project No. IUT02-27 and the Personal Research Funding Project No. PUT790 (start-up project), as well as by the European Regional Development Fund through the Center of Excellence TK133 "The Dark Side of the Universe".

APPENDIX: HAMILTONIAN ANALYSIS WITHOUT GAUGE FIXING

Looking at Eq. (18) and noting that the conjugate momenta are related to each other via an algebraic equation (21) it at first seems like it is impossible to solve the velocities for momenta. However, there is a way to attack this problem and successfully derive the Hamiltonian. First, we note that Eq. (27) before fixing the gauge becomes

$$\begin{aligned} S_A{}^i &= M^i{}_A{}^j{}_B(\dot\theta^B{}_j - (\Lambda^{-1})^D{}_C\theta^C{}_j\dot\Lambda^B{}_D) \\ &= M^i{}_A{}^j{}_B\Lambda^B{}_D\partial_0(\theta^C{}_j(\Lambda^{-1})^D{}_C), \end{aligned} \tag{A1}$$

with

$$\begin{aligned} &S_A{}^i[\alpha,\beta,\theta^A{}_i,\pi_A{}^i] \\ &= \frac{\alpha}{\sqrt{h}}\pi_A{}^i + [\Lambda^B{}_D D_k[(\alpha\xi^C + \beta^m\theta^C{}_m)(\Lambda^{-1})^D{}_C] \\ &\quad - T^B{}_{kl}\beta^l]M^i{}_A{}^k{}_B - 2\alpha T^B{}_{kl}h^{ik}(c_2\xi_B\theta_A{}^l + c_3\xi_A\theta_B{}^l). \end{aligned} \tag{A2}$$

In the Lagrangian, velocities only appear from terms of the structure

$$\begin{aligned} T^B{}_{0j} &= \Lambda^B{}_D\partial_0(\theta^C{}_j(\Lambda^{-1})^D{}_C) \\ &\quad - \Lambda^B{}_D D_j[(\alpha\xi^C + \beta^m\theta^C{}_m)(\Lambda^{-1})^D{}_C]. \end{aligned} \tag{A3}$$

Hence, the velocities in the Lagrangian appear exactly as in Eq. (A1). This means that we can get rid of all velocities and express them in terms of conjugate momenta by applying $(M^{-1})^{A}_{i}{}^{C}_{k}$ on both sides of Eq. (A1), where we have used the same decomposition of the Weitzenböck tetrad $\dot{\overset{\bullet}{\theta}}{}^A{}_i = \partial_0(\theta^B{}_i(\Lambda^{-1})^A{}_B)$ as in Eq. (29) into irreducible parts.

Second, we need to write down the Hamiltonian together with its primary constraints. The algebraic relation between the conjugate momenta is a primary constraint and needs to be added. The Hamiltonian is then by definition

$$\begin{aligned} H &= \pi_A{}^i\dot\theta^A{}_i + \hat\pi^{AB}a_{AB} - L(\theta^A{}_i,\pi_A{}^i) \\ &\quad - {}^\pi\lambda^A{}_B(\hat\pi^B{}_A + \pi_A{}^i\eta^{B[N}\theta^{M]}{}_i), \end{aligned} \tag{A4}$$

which is the gauge independent correspondence to Eq. (26). Using the equation imposed by the Lagrange multiplier to express all conjugate momenta solely in the conjugate momenta with respect to the spatial tetrad field $\pi_A{}^i$ we get that the Hamiltonian is of the form

$$\begin{aligned} H &= \pi_A{}^i\Lambda^A{}_B\partial_0(\theta^C{}_i(\Lambda^{-1})^B{}_C) - L[\alpha,\beta,\theta^A{}_i,\pi_A{}^i,\Lambda^A{}_B] \\ &\quad - {}^\pi\lambda^A{}_B(\hat\pi^B{}_A + \pi_A{}^i\eta^{B[N}\theta^{M]}{}_i). \end{aligned} \tag{A5}$$

From this we can see that the Hamiltonian can be expressed in canonical variables without gauge fixing. By using Eq. (A1) we get

$$\begin{aligned} &H[\alpha,\beta,\theta^A{}_i,\pi_A{}^i,\Lambda^A{}_B,\hat\pi^B{}_A] \\ &= \pi_A{}^i(M^{-1})^{A}_{i}{}^{C}_{k}S_C{}^k[\alpha,\beta,\theta^A{}_i,\pi_A{}^i] - L[\alpha,\beta,\theta^A{}_i,\pi_A{}^i,\Lambda^A{}_B] \\ &\quad - {}^\pi\lambda^A{}_B(\hat\pi^B{}_A + \pi_A{}^i\eta^{B[N}\theta^{M]}{}_i). \end{aligned} \tag{A6}$$

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Chapter 5

On the Gauge Fixing in the Hamiltonian Analysis of General Teleparallel Theories

II

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D. Blixt, M. Hohmann, and C. Pfeifer *Universe* **5**, 6 (2019),
https://www.mdpi.com/2218-1997/5/6/143,
arXiv:1905.01048 [inSpire] [ETIS]

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Communication

On the Gauge Fixing in the Hamiltonian Analysis of General Teleparallel Theories †

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† This Paper Is Based on the Talk at the Teleparallel Universes in Salamanca, Salamanca, Spain, 26–28 November 2018.

Received: 2 May 2019; Accepted: 5 June 2019; Published: 10 June 2019

Abstract: The covariant formulation of teleparallel gravity theories must include the spin connection, which has 6 degrees of freedom. One can, however, always choose a gauge such that the spin connection is put to zero. In principle this gauge may affect counting of degrees of freedom in the Hamiltonian analysis. We show for general teleparallel theories of gravity, that fixing the gauge such that the spin connection vanishes in fact does not affect the counting of degrees of freedom. This manifests in the fact that the momenta of the Lorentz transformations which generate the spin connection are fully determined by the momenta of the tetrads.

Keywords: teleparallel gravity; Weitzenböck gauge; Hamiltonian analysis

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1. Introduction

General relativity (GR) has successfully passed a huge amount of experimental tests, which probe the nature of gravity, up to today. Despite this success there are still many open questions associated with our understanding of gravity. Firstly, general relativity is highly non-renormalizable, so it cannot be formulated as a quantum field theory in the same way as it is done for the other fundamental forces, and thus can not directly be embedded into the standard model of particle physics. Secondly, there is strong evidence for inflation. To describe this, one is led to either introduce an extra field (like the inflaton) in the early universe or modify the laws of gravity. The latter gives a better fit to the data [1]. Thirdly, there are tensions in cosmological data, such as the value of the Hubble constant [2,3], which need to be explained. Furthermore, the standard model of cosmology is based on the ΛCDM model, whose main ingredients are cold dark matter particles and a cosmological constant as dark energy, to explain the dark sector of our universe. However, also this model faces some issues, where the biggest issue probably is the smallness of the cosmological constant.

In order to deal with the aforementioned issues, modified theories of gravity have been studied. Most are based on the formulation of general relativity in terms of the Levi-Civita connection, which is induced by a spacetime metric. However, general relativity has other equivalent formulations, based on connections that are not induced by the metric. One of these is called "symmetric teleparallel equivalent of general relativity" (STEGR) and uses a flat (no curvature) and torsion free connection with non-metricity ($\nabla g_{\mu\nu} \neq 0$). Another is called "teleparallel equivalent of general relativity" (TEGR) and employs a flat metric compatible connection with torsion. The Lagrangian of STEGR is given by the so-called non-metricity scalar Q, while the Lagrangian of TEGR by the so-called torsion scalar T. These reformulations of Einstein's theory of general relativity are sometimes referred as "the geometrical trinity" [4].

Due to the experimental success of general relativity we need to formulate modified theories of gravity such that they are compatible with experimental tests on solar system scales. That is, they should not deviate too much from general relativity on these scales. Since general relativity can equivalently be formulated in different geometries, we have the freedom to choose which geometry we want to formulate modified theories of gravity in. After modifying general relativity, the modified theories will in general be in-equivalent.

For example, popular modifications of general relativity are to consider functions of the defining Lagrangian. In the three different formulations this amounts to consider as Lagrangian either $f(R)$, where R is the Ricci scalar of the Levi-Civita connection, $f(T)$ or $f(Q)$, which lead to non-equivalent theories. The reason for this is that they differ by a boundary term, which can no longer be completely neglected when a function is acting on the original GR, STEGR or TEGR Lagrangian.

In this work we will consider the Hamiltonian analysis of modified theories of gravity in the teleparallel framework. The Hamiltonian analysis gives the number of degrees of freedoms in a theory. However, in the so-called $f(T)$ theories of gravity disputing results have been found for this number. Where it was claimed in [5,6] that the theory has 5 degrees of freedom. More recent work, on the contrary, found that $f(T)$ has 3 degrees of freedom [7]. The aforementioned works were, however, done in a gauge where the spin connection is put to zero, which is not the covariant formulation of teleparallel gravity [8,9]. We show in this work, for general covariant teleparallel theories, that the spin connection momenta are determined by the tetrad momenta .

In Section 2 we display the most general teleparallel gravity theories we consider in this article. Section 3 is devoted to derive the conjugate momenta, and to show that the gauge fixing does not affect the counting of numbers of degrees of freedom. A concrete example is provided in Section 4 with an explicit expression for the Hamiltonian. Finally, discussion and concluding remarks are made in Section 5.

We use the following conventions. Greek indices μ, ν, ρ... denotes global coordinate indices which are raised and lowered with the metric $g_{\mu\nu}$, capital Latin indices denotes Lorentz indices raised and lowered with the Minkowski metric η_{AB}, and small Latin indices are spatial indices and 0 denotes the temporal index. The Minkowski metric η_{AB} is taken to be diag $(-1, 1, 1, 1)$. Brackets [] denote dependence on the explicit variables and their derivatives.

2. Generalized Theories of Teleparallel Gravity

The fundamental variables for teleparallel gravity theories are the tetrads (or vierbeins) θ^A, and for the covariant formulation a curvature-free spin-connection $\omega^A{}_B$ is needed [8,9]. In local coordinates these variables can be expressed as

$$\begin{aligned} &\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu, \quad e_A = e_A{}^\mu \partial_\mu, \\ &\omega^A{}_B = \omega^A{}_{B\mu}\left[\Lambda^C{}_D\right] \mathrm{d}x^\mu = \Lambda^A{}_C \mathrm{d}\left(\Lambda^{-1}\right)^C{}_B = \Lambda^A{}_C \partial_\mu \left(\Lambda^{-1}\right)^C{}_B \mathrm{d}x^\mu, \end{aligned} \tag{1}$$

where $\Lambda^A{}_B$ are Lorentz matrices. Any Lorentzian metric can be expressed in terms of tetrads by the following relations

$$g_{\mu\nu} = g_{\mu\nu}\left(\theta^A{}_\mu\right) = \eta_{AB}\theta^A{}_\mu \theta^B{}_\nu, \quad g^{\mu\nu} = \eta^{AB} e_A{}^\mu e_B{}^\nu. \tag{2}$$

The torsion components expressed in tetrad fields and the spin connection are

$$T^\rho{}_{\mu\nu} = e_A{}^\rho T^A{}_{\mu\nu}[\theta^A{}_\mu, \mathrm{d}\theta^A{}_\mu, \Lambda^A{}_B, \mathrm{d}\Lambda^A{}_B] = 2e_A{}^\rho\left(\partial_{[\mu}\theta^A{}_{\nu]} + \omega^A{}_{B[\mu}\theta^B{}_{\nu]}\right) \tag{3}$$

We can write a generic action made from the Torsion components $T^\rho{}_{\mu\nu}$ and the metric (which depend on the tetrad fields) as

$$S\left[\theta^A{}_\mu, \Lambda_C{}^D\right] = \int \mathrm{d}^4x L\left[\theta^A{}_\mu, \Lambda_C{}^D\right] = \int \mathrm{d}^4x\ |\theta| f(g_{\sigma\tau}, T^\rho{}_{\mu\nu}), \tag{4}$$

where $|\theta| := \det(\theta^A{}_\mu)$ which is the normal volume element ($\sqrt{-g}$ in metric formalism). This is the most general teleparallel gravity theory in 4 dimensions without introducing extra fields, and without breaking local Lorentz invariance, with all derivatives being of first order and coming from the torsion components, and includes the theories discussed in [10]. The analysis can easily be extended to higher dimensions. In order to derive the conjugate momenta and make a canonical Legendre transformation to the Hamiltonian, we make use of the 3 + 1 decomposition analogous to [11]. In this decomposition we have

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^i\beta^j h_{ij} & \beta_i \\ \beta_j & h_{ij} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & h^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix}. \tag{5}$$

The indices $i, j, \ldots$ are spatial and run from 1 to 3 and are raised and lowered with the induced metric h_{ij}, i.e., $\beta_i = \beta^j h_{ij}$. For the tetrad fields (which are canonical variables for teleparallel gravity theories) we have

$$\theta^A{}_0 = \alpha\xi^A + \beta^i\theta^A{}_i, \tag{6}$$

where ξ^A are components of the normal vector n to the $x^0 =$ const hypersurfaces in the dual tetrad basis [12]

$$n = \xi^A e_A, \quad \xi^A = -\frac{1}{6}\epsilon^A{}_{BCD}\theta^B{}_i\theta^C{}_j\theta^D{}_k\epsilon^{ijk}. \tag{7}$$

The components ξ^A further satisfy

$$\eta_{AB}\xi^A\xi^B = \xi^A\xi_A = -1, \quad \eta_{AB}\xi^A\theta^B{}_i = \xi_A\theta^A{}_i = 0. \tag{8}$$

Furthermore, the dual tetrads and the induced metric can be expressed as

$$e_A{}^0 = -\frac{1}{\alpha}\xi_A, \quad e_A{}^i = \theta_A{}^i + \xi_A\frac{\beta^i}{\alpha}, \quad h_{ij} = \eta_{AB}\theta^A{}_i\theta^B{}_j. \tag{9}$$

For readability we sometimes suppress metrics which raises or lowers indices, even when indices are at non-canonical positions. For example $\theta_A{}^i = \eta_{AB}h^{ij}\theta^B{}_j \neq e_A{}^i = \theta_A{}^i + \xi_A\frac{\beta^i}{\alpha}$.

3. Conjugate Momenta

To derive the conjugate momenta we note that time derivatives always appear in $T^\rho{}_{0i} = -T^\rho{}_{i0} = e_A{}^\rho T^A{}_{0i}$ due to the antisymmetric property of the torsion components $T^\rho{}_{00} = 0$. Time derivatives act on tetrad fields $\theta^A{}_i$ and Lorentz matrices $\Lambda^A{}_B$ and explicitly it reads

$$T^A{}_{0i} = \partial_0\theta^A{}_i + \Lambda^A{}_C\partial_0\left(\Lambda^{-1}\right)^C{}_B\theta^B{}_i - \partial_i\theta^A{}_0 - \Lambda^A{}_C\partial_i\left(\Lambda^{-1}\right)^C{}_B\theta^B{}_0. \tag{10}$$

One immediately finds that time derivatives never act on temporal tetrads ($\theta^A{}_0$) nor lapse and shifts (α, β). They only act on the spatial tetrads $\theta^A{}_i$ and Lorentz matrices $\Lambda^A{}_B$. Hence, the conjugate momenta only need to be defined for these variables. The conjugate momenta with respect to the spatial tetrad fields are defined by

$$\pi_A{}^i := \frac{\partial L}{\partial\partial_0\theta^A{}_i} = |\theta|\frac{\partial f}{\partial T^\mu{}_{0j}}\frac{\partial T^\mu{}_{0j}}{\partial\partial_0\theta^A{}_i} = |\theta|e_A{}^\mu\frac{\partial f}{\partial T^\mu{}_{0i}}. \tag{11}$$

Since the Lorentz matrices only have 6 independent components, we introduce an auxiliary antisymmetric field which preserves the Lorentz symmetries and thus also those of the spin connection

$$a_{AB} := \eta_{AC}\omega^{C}{}_{B0} = \eta_{C[A}\Lambda^{C}{}_{|D|}\partial_0\left(\Lambda^{-1}\right)^{D}{}_{B]} \Leftrightarrow \partial_0\Lambda^{A}{}_{B} = a_{CD}\eta^{A[D}\Lambda^{C]}{}_{B}. \tag{12}$$

The conjugate momenta of the independent components of the Lorentz matrices are hence represented by

$$\hat{\pi}^{AB} := \frac{\partial L}{\partial a_{AB}} = |\theta|\frac{\partial f}{\partial T^{\mu}{}_{0i}}\frac{\partial T^{\mu}{}_{0i}}{\partial a_{AB}} = -\pi_{C}{}^{i}\eta^{C[B}\theta^{A]}{}_{i}. \tag{13}$$

This can be realized from

$$\begin{aligned}\frac{\partial L}{\partial a_{AB}} &= \frac{\partial L}{\partial\partial_0\Lambda^{C}{}_{D}}\frac{\partial\partial_0\Lambda^{C}{}_{D}}{\partial a_{AB}} = \frac{\partial L}{\partial T^{\mu}{}_{0i}}\frac{\partial T^{\mu}{}_{0i}}{\partial\partial_0\Lambda^{C}{}_{D}}\frac{\partial\partial_0\Lambda^{C}{}_{D}}{\partial a_{AB}} \\ &= -\frac{\partial L}{\partial T^{\mu}{}_{0i}}\frac{\partial T^{\mu}{}_{0i}}{\partial\partial_0\theta^{C}{}_{j}}\left[\theta^{D}{}_{j}\left(\Lambda^{-1}\right)^{F}{}_{D}\right]\frac{\partial\partial_0\Lambda^{C}{}_{F}}{\partial a_{AB}} \\ &= -|\theta|\frac{\partial f}{\partial T^{\mu}{}_{0i}}e_{C}{}^{\mu}\left[\theta^{D}{}_{i}\left(\Lambda^{-1}\right)^{F}{}_{D}\right]\eta^{C[B}\Lambda^{A]}{}_{F}.\end{aligned} \tag{14}$$

The conjugate momenta $\pi^{A}{}_{i}$ and $\hat{\pi}^{AB}$ are hence manifestly algebraically related to each other. This means that we need to add Equation (13) as a Lagrange multiplier. Furthermore, it can be cumbersome to express the velocities into their conjugate momenta, but for new general relativity it has been shown how this can be done [11]. To simplify we perform a transformation in which the spin connection vanishes and show that this transformation in this gauge is consistent with the constraints in the covariant formulation. This transformation is done by introducing new field variables $(\tilde{\alpha}, \tilde{\beta}^{i}, \tilde{\theta}^{A}{}_{i}, \tilde{\Lambda}^{A}{}_{B})$ so that $\tilde{\theta}^{A}{}_{i} = \theta^{B}{}_{i}\left(\Lambda^{-1}\right)^{A}{}_{B}$, $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$, and $\tilde{\Lambda}^{A}{}_{B} = \Lambda^{A}{}_{B}$. It follows that $\tilde{a}_{AB} = a_{AB}$, $\tilde{g}_{\mu\nu} = g_{\mu\nu}$, $|\tilde{\theta}| = |\theta|$ and that $\tilde{T}^{\rho}{}_{\mu\nu} = \tilde{e}_{A}{}^{\rho}\partial_{[\mu}\tilde{\theta}^{A}{}_{\nu]}$. Furthermore,

$$\tilde{L} = |\tilde{\theta}|\tilde{f}(g_{\sigma\tau}, T^{\rho}{}_{\mu\nu}) = |\tilde{\theta}|f(\tilde{g}_{\sigma\tau}, \tilde{T}^{\rho}{}_{\mu\nu}) = |\theta|f(g_{\sigma\tau}, \tilde{T}^{\rho}{}_{\mu\nu}), \tag{15}$$

which manifestly is independent of the Lorentz matrices $\Lambda_{A}{}^{B}$. From this transformation we find that the conjugate momenta transforms as

$$\begin{aligned}\tilde{\pi}_{A}{}^{i} &= \frac{\partial\tilde{L}}{\partial\partial_0\tilde{\theta}^{A}{}_{i}} = \pi_{B}{}^{i}\Lambda^{B}{}_{A}, \\ \hat{\tilde{\pi}}^{AB} &= \frac{\partial\tilde{L}}{\partial a_{AB}} = \pi_{C}{}^{i}\eta^{C[B}\theta^{A]}{}_{i} + \hat{\pi}^{AB}.\end{aligned} \tag{16}$$

Inverting these formulas gives

$$\begin{aligned}\pi_{A}{}^{i} &= \tilde{\pi}_{B}{}^{i}\left(\Lambda^{-1}\right)^{B}{}_{A}, \\ \hat{\pi}^{AB} &= \hat{\tilde{\pi}}^{AB} - \tilde{\pi}_{D}{}^{i}\left(\Lambda^{-1}\right)^{D}{}_{C}\eta^{C[B}\Lambda^{A]}{}_{E}\tilde{\theta}^{E}{}_{i}.\end{aligned} \tag{17}$$

Applying Equation (13) to Equation (16) shows that $\hat{\tilde{\pi}}^{AB} = 0$ in the Weitzenböck gauge, and they are hence pure gauge degrees of freedom as expected from [8,9]. A vital point is now to show that the gauge fixing is imposed consistently with the constraints. Hence, we need to show that $\{\hat{\tilde{\pi}}^{AB}, \tilde{H}\} \approx 0$. The transformed Hamiltonian is defined as

$$\tilde{H} = \tilde{\pi}_{A}{}^{i}\partial_0\tilde{\theta}^{A}{}_{i} + \hat{\tilde{\pi}}^{AB}\tilde{a}_{AB} + {}^{\hat{\tilde{\pi}}}\lambda_{AB}\hat{\tilde{\pi}}^{AB} - \tilde{L} + \text{primary constraints}, \tag{18}$$

where primary constraints need to be added (which differ from different theories). Looking at the transformation behaviors of each term it is hence clear that $\{\hat{\tilde{\pi}}^{AB}, H\} \approx 0$. The gauge fixing is hence consistent with the constraints and can not in any way affect the counting of degrees of freedom for teleparallel gravity theories.

4. New General Relativity

One interesting class of teleparallel gravity theories is the so-called "new general relativity" theory introduced in [13]. In this section we derive the Hamiltonian for "new general relativity" as was done in [11]. In this section we work in the Weitzenböck gauge motivated by the preceding sections. Furthermore, we drop all $\tilde{}$ for readability. Assume that we want a teleparallel theory defined by Equation (4) and only consider terms quadratic in the torsion components $T^\rho{}_{\mu\nu}$ without introducing parity violating terms. Then the action looks like

$$S_{\mathrm{NGR}} = \int \mathrm{d}^4x|\theta| \left(c_1 T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + c_2 T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho + c_3 T^\rho{}_{\mu\rho} T^{\sigma\mu}{}_\sigma\right). \tag{19}$$

After a 3 + 1 decomposition it is found that

$$\begin{aligned} L_{\mathrm{NGR}} = {} & \frac{\sqrt{h}}{2\alpha} T^A{}_{i0} T^B{}_{j0} M^{i\,j}_{A\,B} + \frac{\sqrt{h}}{\alpha} T^A{}_{i0} T^B{}_{kl} \left[M^{i\,l}_{A\,B} \beta^k + 2\alpha h^{il} \left(c_2 \xi_B \theta_A{}^k + c_3 \xi_A \theta_B{}^k \right) \right] \\ & + \frac{\sqrt{h}}{\alpha} T^A{}_{ij} T^B{}_{kl} \beta^i \left[\frac{1}{2} M^{j\,l}_{A\,B} \beta^k + 2\alpha h^{jl} \left(c_2 \xi_B \theta_A{}^k + c_3 \xi_A \theta_B{}^k \right) \right] + \alpha \sqrt{h}\, {}^3\mathbb{T}. \end{aligned} \tag{20}$$

Here

$$M^{i\,j}_{A\,B} := -2(2c_1 h^{ij} \eta_{AB} - (c_2 + c_3) \xi_A \xi_B h^{ij} + c_2 \theta_A{}^j \theta_B{}^i + c_3 \theta_A{}^i \theta_B{}^j), \tag{21}$$

and

$${}^3\mathbb{T} := c_1 \eta_{AB} T^A{}_{ij} T^B{}_{kl} h^{ik} h^{jl} + c_2 \theta_A{}^i \theta_B{}^j T^A{}_{jk} T^B{}_{il} h^{kl} + c_3 \theta_A{}^i \theta_B{}^j h^{kl} T^A{}_{ik} T^B{}_{jl}. \tag{22}$$

The theory is covariant and the spatial derivatives can all be replaced (simultaneously) by the Levi-Civita covariant derivative D_i associated with the induced metric such that $D_i h_{jk} = 0$. Derivatives on the temporal parts of the tetrads ($\theta^A{}_0$) generally do not appear and hence the conjugate momenta for new general relativity are

$$\frac{\alpha}{\sqrt{h}} \pi_A{}^i = \frac{\alpha}{\sqrt{h}} \frac{L_{\mathrm{NGR}}}{\partial \partial_0 \theta^A{}_i} = T^B{}_{0j} M^{i\,j}_{A\,B} + T^B{}_{kl} \left[M^{i\,k}_{A\,B} \beta^l + 2\alpha h^{ik} \left(c_2 \xi_B \theta_A{}^l + c_3 \xi_A \theta_B{}^l \right) \right]. \tag{23}$$

We can now define

$$S_A{}^i = \frac{\alpha}{\sqrt{h}} \pi_A{}^i + \left[D_k \left(\alpha \xi^B + \beta^m \theta^B{}_m \right) - T^B{}_{kl} \beta^l \right] M^{i\,k}_{A\,B} - 2\alpha T^B{}_{kl} h^{ik} \left(c_2 \xi_B \theta_A{}^l + c_3 \xi_A \theta_B{}^l \right), \tag{24}$$

so that $S_A{}^i$ is independent of velocities and Equation (23) can equivalently be written as

$$S_A{}^i = \partial_0 \theta^B{}_j M^{i\,j}_{A\,B}. \tag{25}$$

The remaining task is then to invert the $M^{i\,j}_{A\,B}$ and solve for $\partial_0 \theta^B{}_j$. This is a rather non-trivial task, and hence, we refer to [11] for details. Here we simply write out the possible primary constraints and the expression for the Hamiltonian. Existence, or non-existence of primary constraints depend on the specific values of c_1, c_2, c_3, related to the irreducible components under the rotation group into vectorial, antisymmetric, symmetric (but trace-free), and trace parts $(\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T})$. We define

$$A_{\mathcal{V}} = 2c_1 + c_2 + c_3, \quad A_{\mathcal{A}} = 2c_1 - c_2, \quad A_{\mathcal{S}} = 2c_1 + c_2, \quad A_{\mathcal{T}} = 2c_1 + c_2 + 3c_3, \tag{26}$$

and an index $\mathcal{I} = \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}$. Putting any of the $A_{\mathcal{I}} = 0$ gives rise to primary constraints.

$$A_{\mathcal{V}} = 0 \implies {}^{\mathcal{V}}C^i := S_A{}^i \xi^A = 0, \tag{27}$$

$$A_{\mathcal{A}} = 0 \implies {}^{\mathcal{A}}C^{ij} := S_A{}^k \theta^A{}_{[j} h_{i]k} = 0, \tag{28}$$

$$A_{\mathcal{S}} = 0 \implies {}^{\mathcal{S}}C^{ij} := S_A{}^k \theta^A{}_{(j} h_{i)k} - \frac{1}{3} S_A{}^k \theta^A{}_k h_{ij} = 0, \tag{29}$$

$$A_{\mathcal{T}} = 0 \implies {}^{\mathcal{T}}C := S_A{}^i \theta^A{}_i = 0. \tag{30}$$

The important thing to note is that if any of these primary constraints are imposed, they need to be added as Lagrange multipliers in the Hamiltonian. For new general relativity, the expression for the Hamiltonian is

$$\begin{aligned} H = \alpha\sqrt{h} \left(B_{\mathcal{V}} \frac{{}^{\mathcal{V}}C_i {}^{\mathcal{V}}C^i}{4} - B_{\mathcal{A}} \frac{{}^{\mathcal{A}}C_{ij} {}^{\mathcal{A}}C^{ij}}{4} - B_{\mathcal{S}} \frac{{}^{\mathcal{S}}C_{ij} {}^{\mathcal{S}}C^{ij}}{4} - B_{\mathcal{T}} \frac{3 {}^{\mathcal{T}}C {}^{\mathcal{T}}C}{4} - {}^3\mathbb{T} - \frac{\xi^A D_i \pi_A{}^i}{\sqrt{h}} \right) \\ - \beta^k \left(T^A{}_{jk} \pi_A{}^j + \theta^A{}_k D_i \pi_A{}^i \right) + D_i \left[\pi_A{}^i \left(\alpha \xi^A + \beta^j \theta^A{}_j \right) \right], \end{aligned} \tag{31}$$

where

$$B_{\mathcal{I}} = \begin{cases} \frac{1}{A_{\mathcal{I}}} & \text{if } A_{\mathcal{I}} \neq 0 \\ 0 & \text{if } A_{\mathcal{I}} = 0. \end{cases} \tag{32}$$

This is, however, not the final Hamiltonian. As mentioned before, Lagrange multipliers related to primary constraints need to be added. Furthermore, the analysis might further provide secondary, tertiary, etc., constraints after the evaluation of the Poisson brackets. This also needs to be added.

5. Discussion

We showed that for a very general class of teleparallel gravity theories one is allowed to fix the gauge such that the spin connection vanishes without affecting the counting of degrees of freedom in the theory. This significantly simplifies the Hamiltonian analysis of teleparallel gravity theories, assuring that the result does not differ from the covariant formulation. Furthermore, this justifies previous work where this gauge choice has been implemented in the analysis. Since the Hamiltonian analysis tends to be very cumbersome, it is highly suggestive to use this result and put the spin connection to zero in theories covered by this analysis. If one looks at more general teleparallel gravity theories (for example addition of extra fields or more dimensions, as they are discussed in the literature [14–16]) one can follow the same approach in order to figure out if the gauge fixing affects the counting of degrees of freedom.

Author Contributions: Writing—original draft preparation: D.B.; conceptualization, methodology, validation, formal analysis, investigation, writing—review and editing: D.B., M.H., and C.P.; supervision: M.H., and C.P.; project administration, funding acquisition: M.H.

Funding: This work was supported by the Estonian Research Council through the Personal Research Funding project PRG356 and by the European Regional Development Fund through the Center of Excellence TK133 "The Dark Side of the Universe".

Acknowledgments: The authors would like to thank María José Guzmán and Martin Krššak for fruitful discussions.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

GR	General relativity
STEGR	Symmetric teleparallel equivalent of general relativity
TEGR	Teleparallel equivalent of general relativity

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Chapter 6

Hamiltonian Analysis in New General Relativity

III

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D. Blixt, M. Hohmann,M. Krššák, and C. Pfeifer,
Submitted to:
Proceedings of the Fifteenth Marcel Grossman Meeting on General Relativity ,
arXiv:1905.11919 [inSpire]

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Hamiltonian Analysis In New General Relativity

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It is known that one can formulate an action in teleparallel gravity which is equivalent to general relativity, up to a boundary term. In this geometry we have vanishing curvature, and non-vanishing torsion. The action is constructed by three different contractions of torsion with specific coefficients. By allowing these coefficients to be arbitrary we get the theory which is called "new general relativity". In this note, the Lagrangian for new general relativity is written down in ADM-variables. In order to write down the Hamiltonian we need to invert the velocities to canonical variables. However, the inversion depends on the specific combination of constraints satisfied by the theory (which depends on the coefficients in the Lagrangian). It is found that one can combine these constraints in 9 different ways to obtain non-trivial theories, each with a different inversion formula.

Keywords: Teleparallel gravity; New general relativity; ADM-variables.

1. Conventions

Greek indices denote global coordinate indices running from 0 to 3, small Latin indices are spatial coordinate indices running from 1 to 3, whereas capital Latin indices denote Lorentz indices running from 0 to 3. We are always dealing with Lorentzian metrics. Sign convention for the Minkowski metric is $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1)$.

2. Introduction

Gravity is conventionally described with the Levi-Civita connection which is induced by a pseudo-Riemannian metric. This means that the covariant derivative of the metric is zero, and the connection is torsion-free but has curvature. However, there are equivalent theories to general relativity [1]. We will focus on teleparallel gravity [2] where we have vanishing curvature, but non-vanishing torsion.

In particular we will perform the Hamiltonian analysis of "new general relativity" (NGR)[a]. For discussions of certain issues with these theories see [4–6] Previous work on the Hamiltonian analysis on teleparallel gravity theories have been performed in [6–18]. However, the full Hamiltonian analysis of NGR has not been performed. NGR is described by the following action:

$$S_{\mathrm{NGR}} = m_{Pl}^2 \int |\theta| \left(a_1 T^{\mu}{}_{\nu\rho} T_{\mu}{}^{\nu\rho} + a_2 T^{\mu}{}_{\nu\rho} T^{\rho\nu}{}_{\mu} + a_3 T^{\mu}{}_{\rho\mu} T^{\nu\rho}{}_{\nu} \right) \mathrm{d}^4 x, \tag{1}$$

[a] With NGR, we refer to the more general three-parameter teleparallel gravity in contrast to the special one-parameter teleparallel gravity theory which NGR originally referred to [3].

where m_{Pl} is the Planck mass, $T^{\mu}_{\ \nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho}$ is the torsion component with $\Gamma^{\mu}{}_{\nu\rho} = e_A^{\ \mu}\partial_\rho\theta^A_{\ \nu} + e_A^{\ \mu}\left(\Lambda^{-1}\right)^A_{\ D}\partial_\rho\Lambda^D_{\ B}\theta^B_{\ \nu}$, with θ being the tetrad, e its inverse and Λ is a Lorentz matrix. Global spacetime indices are raised and lowered with $g_{\mu\nu} = \theta^A_{\ \mu}\theta^B_{\ \nu}\eta_{AB}$, while Lorentz indices are raised and lowered with η_{AB}. A theory equivalent to general relativity is obtained by setting $a_1 = \frac{1}{4},\ a_2 = \frac{1}{2}$, and $a_3 = -1$.

Alternatively, the NGR action can be written down in the so-called axial, vector, and tensor decomposition[19]. Then

$$S_{\mathrm{NGR}} = m_{\mathrm{Pl}}^2 \int |\theta| \left(c_1 T_{\mathrm{ax}} + c_2 T_{\mathrm{ten}} + c_3 T_{\mathrm{vec}}\right), \tag{2}$$

with $a_1 = -\frac{1}{18}c_1 + \frac{1}{2}c_2,\ a_2 = \frac{1}{9}c_1 + \frac{1}{2}c_2,\ a_3 = c_3 - \frac{1}{2}c_2$, and

$$\begin{aligned} T_{\mathrm{vec}} &= T^{\rho}{}_{\rho\mu}T_{\nu}{}^{\nu\mu}, \\ T_{\mathrm{ax}} &= -\frac{1}{18}\left(T_{\rho\mu\nu}T^{\rho\mu\nu} - 2T_{\rho\mu\nu}T^{\mu\rho\nu}\right), \\ T_{\mathrm{ten}} &= \frac{1}{2}\left(T_{\rho\mu\nu}T^{\rho\mu\nu} + T_{\rho\mu\nu}T^{\mu\rho\nu}\right) - \frac{1}{2}T^{\rho}{}_{\rho\mu}T_{\nu}{}^{\nu\mu}. \end{aligned} \tag{3}$$

3. Method

In order to go from the Lagrangian to the Hamiltonian analysis we need to identify the velocities, derive the conjugate momenta and express everything in canonical variables. We may decompose the torsion scalar in the ADM variables[18] lapse α, shift β^i and the spatial components of the tetrad $\theta^A{}_i$:

$$\begin{aligned} \mathbb{T} = &\frac{1}{2\alpha^2}T^A_{\ i0}T^B_{\ j0}M^{i\ j}_{\ A\ B} \\ &+ \frac{1}{\alpha^2}T^A_{\ i0}T^B_{\ kl}\left[M^{i\ l}_{\ A\ B}\beta^k + 2\alpha a_2 h^{il}\xi_B\theta_A^{\ k} + 2\alpha a_3 h^{il}\xi_A\theta_B^{\ k}\right] \\ &+ \frac{1}{\alpha^2}T^A_{\ ij}T^B_{\ kl}\left[\frac{1}{2}M^{\ i\ k}_{A\ B}\beta^j\beta^l + 2\alpha a_2 h^{jl}\xi_A\theta_B^{\ i}\beta^k + 2\alpha a_3 h^{jl}\xi_A\theta_B^{\ k}\beta^i\right] + {}^3\mathbb{T}, \end{aligned} \tag{4}$$

where $h_{ij} = \theta^A{}_i\theta^B{}_j\eta_{AB}$ is the induced metric, which is used to raise and lower spatial indices, $\xi^A = -\frac{1}{6}\epsilon^A{}_{BCD}\theta^B{}_i\theta^C{}_j\theta^D{}_k\epsilon^{ijk}$,

$$M^{i\ j}_{\ A\ B} = -2a_1 h^{ij}\eta_{AB} + (a_2 + a_3)\xi_A\xi_B h^{ij} - a_2\theta_A^{\ j}\theta_B^{\ i} - a_3\theta_A^{\ i}\theta_B^{\ j}, \tag{5}$$

and

$$\begin{aligned} {}^3\mathbb{T} \equiv\ &a_1\eta_{AB}T^A_{\ ij}T^B_{\ kl}h^{ik}h^{jl} + a_2\eta_{AC}\theta^C_{\ m}h^{im}\eta_{BD}\theta^D_{\ p}h^{jp}T^A_{\ kj}T^B_{\ li}h^{kl} \\ &+ a_3\eta_{AC}\theta^C_{\ m}h^{im}\eta_{BD}\theta^D_{\ p}h^{jp}h^{kl}T^A_{\ ki}T^B_{\ lj}. \end{aligned} \tag{6}$$

Without any loss of generality[7] we can restrict ourselves to the Weitzenböck gauge for which the torsion components are expressed as $T^A{}_{\mu\nu} = \partial_\nu\theta^A{}_\mu - \partial_\mu\theta^A{}_\nu$, and hence the conjugate momenta become,

$$\alpha\frac{\pi^i_A}{\sqrt{h}} = T^B_{\ j0}M^{i\ j}_{\ A\ B} + T^B_{\ kl}\left[M^{i\ l}_{\ A\ B}\beta^k + 2\alpha a_2 h^{il}\xi_B\theta_A^{\ k} + 2\alpha a_3 h^{il}\xi_A\theta_B^{\ k}\right]. \tag{7}$$

The velocities can now be inverted and expressed in canonical variables using

$$S^i_A = \dot{\theta}^B{}_j M^i{}_A{}^j{}_B, \tag{8}$$

with

$$\begin{aligned} S^i_A = & D_j \left(\alpha \xi^B + \beta^m \theta^B{}_m\right) M^i{}_A{}^j{}_B \\ & - T^B_{kl} \left[M^i{}_A{}^l{}_B \beta^k + 2\alpha a_2 h^{il} \xi_B \theta_A{}^k + 2\alpha a_3 h^{il} \xi_A \theta_B{}^k\right] + \alpha \frac{\pi^i_A}{\sqrt{h}}, \end{aligned} \tag{9}$$

where D_i is the Levi-Civita covariant derivative with respect to the induced metric. However, M in equation (8) is singular for certain combinations of parameters of the theory and can hence only be inverted by the Moore-Penrose pseudo-inverse matrix [12]. This is apparent if one decomposes the equation into irreducible representations of the rotation group, which generates the following constraints,

$$2a_1 + a_2 + a_3 =: {}^{\mathcal{V}}A = 0 \implies {}^{\mathcal{V}}C^i := S^i_A \xi^A = 0, \tag{10}$$

$$2a_1 - a_2 =: {}^{\mathcal{A}}A = 0 \implies {}^{\mathcal{A}}C_{ij} := S^k_A \theta^A{}_{[j} h_{i]k} = 0, \tag{11}$$

$$2a_1 + a_2 =: {}^{\mathcal{S}}A = 0 \implies {}^{\mathcal{S}}C_{ij} := S^k_A \theta^A{}_{(j} h_{i)k} - \frac{1}{3} S^k_A \theta^A{}_k h_{ij} = 0, \tag{12}$$

$$2a_1 + a_2 + 3a_3 =: {}^{\mathcal{T}}A = 0 \implies {}^{\mathcal{T}}C := S^i_A \theta^A{}_i = 0. \tag{13}$$

These are primary constraints, since these constrain both the tetrad field and their conjugate momenta, which also can be decomposed into irreducible parts. In the axial, vector, tensor decomposition we have that

$$c_2 + c_3 = {}^{\mathcal{V}}A = 0, \tag{14}$$

$$-\frac{2}{9}c_1 + \frac{1}{2}c_2 = {}^{\mathcal{A}}A = 0, \tag{15}$$

$$\frac{3}{2}c_2 = {}^{\mathcal{S}}A = 0, \tag{16}$$

$$3c_3 = {}^{\mathcal{T}}A = 0. \tag{17}$$

In this language the primary constraints get some further geometrical meaning. Equations (14) and (15) together imposes the teleparallel equivalent to general relativity and impose invariance of the Lagrangian under pure tetrad local Lorentz transformations [9]. This is, however, not more apparent from the axial, vector, tensor decomposition we made. What is more interesting are the constraints imposed by equations (16) and (17). In this decomposition of the torsion scalar they exactly correspond to putting T_{ten} and T_{vec} to zero respectively.

4. Results

Different combinations of (10)-(13) yield 9 non-trivial classes of theories:

Theory	Constraints	Location in figure 1
$A_I \neq 0 \ \forall I \in \{\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}\}$	No constraints	white area
$A_{\mathcal{V}} = 0$	${}^{\mathcal{V}}C_i = 0$	red line
$A_{\mathcal{A}} = 0$	${}^{\mathcal{A}}C_{ji} = 0$	black line
$A_{\mathcal{S}} = 0$	${}^{\mathcal{S}}C_{ji} = 0$	vertical green line
$A_{\mathcal{T}} = 0$	${}^{\mathcal{T}}C = 0$	horizontal blue line
$A_{\mathcal{V}} = A_{\mathcal{A}} = 0$	${}^{\mathcal{V}}C_i = {}^{\mathcal{A}}C_{ji} = 0$	turquoise point
$A_{\mathcal{A}} = A_{\mathcal{S}} = 0$	${}^{\mathcal{A}}C_{ji} = {}^{\mathcal{S}}C_{ji} = 0$	purple points (perimeter)
$A_{\mathcal{A}} = A_{\mathcal{T}} = 0$	${}^{\mathcal{A}}C_{ji} = {}^{\mathcal{T}}C = 0$	orange point
$A_{\mathcal{V}} = A_{\mathcal{S}} = A_{\mathcal{T}} = 0$	${}^{\mathcal{V}}C_i = {}^{\mathcal{S}}C_{ji} = {}^{\mathcal{T}}C = 0$	gray point (center)

Any other solutions would be trivial ($c_1 = c_2 = c_3 = 0$). Excluding these trivial solutions we can normalize our parameters to

$$\tilde{c}_i = \frac{c_i}{\sqrt{c_1^2 + c_2^2 + c_3^2}}, \tag{18}$$

for $i = 1, 2, 3$, which means that we can make a 2-dimensional plot to visualize these theories in the normalized parameter-space. This can be nicely visualized in polar coordinates (θ, ϕ) on the unit sphere with

$$\tilde{c}_1 = \cos\theta, \quad \tilde{c}_2 = \sin\theta\cos\phi, \quad \tilde{c}_3 = \sin\theta\sin\phi. \tag{19}$$

Every pair of antipodal points on the sphere corresponds to a ray in the 3-dimensional parameter space, whose elements describe the same theory. Hence, it suffices to display only the upper half sphere $\tilde{c}_1 \geq 0$, which is done in figure 1. However, note that points on the equator $\tilde{c}_1 = 0$ still appear twice, and both copies should be identified with each other. This applies in particular to the two purple points in figure 1, both describing the class of theories defined by pure vector torsion $\tilde{c}_1 = \tilde{c}_2 = 0$. The Hamiltonian is found to always appear with four Lagrange multipliers (linearity in lapse and shifts) with,

$$H = \alpha\mathcal{H}\left(\theta, M^{-1}\right) + \beta^k\mathcal{H}_k\left(\theta, M^{-1}\right) + D_i\left[\left(\alpha\xi^A + \beta^j\theta^A_{\ j}\right)\pi^i_A\right], \tag{20}$$

in the unconstrained case[7].

5. Discussion

One can distinguish 9 different classes of NGR theories by the presence or absence of primary constraints appearing in their Hamiltonian formulation. What remains to be determined is how many secondary constraints are induced by demanding closure of the constraint algebra. Some considerations in this direction have been studied in[6,18], however, our work invites for further investigation. The theories satisfying $A_I \neq 0, \forall\ I \in \{\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}\}$ can be parameterized by two free parameters (and a global rescaling of the Lagrangian, fixing the value of the Planck mass, which does not affect the presence or absence of primary constraints). Models which exhibit one primary constraint $A_I = 0$ have one free parameter left, while for those with

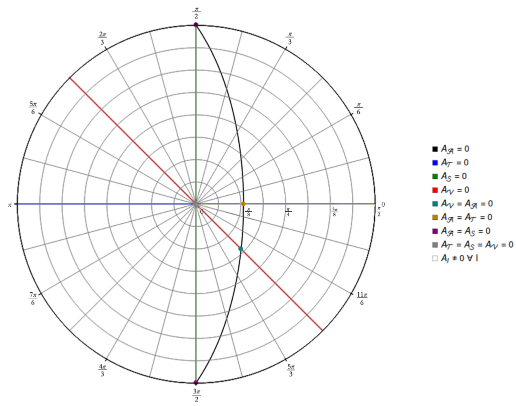

Fig. 1. Visualization of the parameter space of new general relativity in coordinates reflecting the axial, vector, tensor decomposition of the Lagrangian, colored by the occurrences of primary constraints. The radial axis shows the zenith angle θ, while the (circular) polar axis shows the azimuth angle ϕ, following the definition (19).

more primary constraints all parameters are fixed. The free parameters might affect the vanishing, or non-vanishing of certain Poisson brackets, which therefore have to be calculated in order to obtain the number of degrees of freedom.

The number of degrees of freedom can be compared with polarization modes in gravitational waves [20]. Furthermore, it can be compared with the linear level in order to find out if the theories are strongly coupled. One may extend this analysis to $f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec})$ [19] or include parity violating terms.

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Chapter 7

Review of the Hamiltonian analysis in teleparallel teleparallel gravity

IV

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D. Blixt, M-J. Guzmán, M. Hohmann, and C. Pfeifer
Accepted for publication in IJGMMP-D-20-00614 ,
arXiv:2012.09180 [inSpire]

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International Journal of Geometric Methods in Modern Physics

Review of the Hamiltonian analysis in teleparallel gravity

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Received (Day Month Year)
Revised (Day Month Year)

We review different approaches to the Hamiltonian analysis of teleparallel theories of gravity. In particular the Hamiltonian analysis for $f(\mathbb{T})$ theories led to disputed results in the literature. The aim of this review is to relate the different notations and assumptions in the different approaches in a comprehensive way, so that they can be compared more easily. To do this we present the primary constraints of the $f(\mathbb{T}_{\mathrm{NGR}})$ gravity class of theories for the first time. The particular cases studied in the literature, $f(\mathbb{T})$ gravity and new general relativity, are contained in this parent theory. We compare their Hamiltonian analyses done by different authors in the literature among each other by relating them to our analysis of $f(\mathbb{T}_{\mathrm{NGR}})$ in detail.

Keywords: teleparallel gravity; Hamiltonian formalism.

1. Introduction

In the recent years the geometric foundations of general relativity (GR) have been reassessed, and it has been highlighted that its commonly known formulation in terms of the curvature of spacetime is not unique. Equivalently GR can be formulated in terms of a flat, metric compatible connection with torsion, called the teleparallel equivalent to general relativity (TEGR) or in terms of a flat, torsion free connection that is not metric compatible, called the symmetric teleparallel equivalent to general relativity (STEGR) [1].

From there on, numerous modified theories of gravity have been constructed to overcome the shortcomings of GR such as not explaining the dark matter and dark energy phenomenology, not being consistently quantizable and predicting singularities [2–10].

To understand the properties of the theories of gravity beyond GR based on its teleparallel or symmetric teleparallel formulation, it is crucial to have a proper understanding of their canonical structure. Applying the Hamiltonian formalism to these theories allows a nonperturbative counting of the physical degrees of freedom, it states the well-posedness of the Cauchy problem, and can shed some light on canonical quantization.

In this article we investigate and review the canonical structure of the most famous teleparallel generalizations of general relativity, so-called $f(\mathbb{T})$ theories [11] and new general relativity (NGR) [12]. Both kinds can be studied collectively by considering $f(\mathbb{T}_{\mathrm{NGR}})$ teleparallel theories of gravity.

In the literature there are several approaches to the Hamiltonian analysis of TEGR [13–24], of $f(\mathbb{T})$ gravity [25–27] and NGR and special cases of it [28–34], in which slightly different definitions of the Lagrangians, the canonical momenta as well as the primary and secondary constraints, and many different notations for all appearing quantities are in use. The approaches differ in how they perform the canonical analysis of the theories, i.e. employing an ADM decomposition or not (whose necessity one can already discuss on the level of GR [35]), using tensor components or differential forms and how the gauge freedom encoded in the spin connection is taken into account.

In particular for $f(\mathbb{T})$ gravity these studies come to different result on the number of physical degrees of freedom. It is believed that extra degrees of freedom should appear from the breaking of local Lorentz invariance, although the machinery on how this works is complex and requires special care to be taken in its application. There is evidence that some partial or total violation of Lorentz invariance occurs for some circumstances that remain to be studied. The constraint algebra of $f(\mathbb{T})$ gravity is very involved and the matrix of Poisson brackets among constraints presents a variable rank. As a consequence, the number of d.o.f. might not be uniquely defined independent of the field configuration, therefore the disagreement about its number. More details on the current status of the discussion will be presented in Subsections 7.1 and 7.2. The evidence points most probably to 5 d.o.f. in the most general case [25,27], for Minkowski and Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes there is contradictory claims whether it should be 3 d.o.f. [26] or 2 d.o.f. as in GR [27], and some far-fetched cases could give 4 d.o.f. or even zero [27]. Despite all controversy, $f(\mathbb{T})$ gravity seems an intriguing toy model to build, hopefully, more healthy modified teleparallel gravities.

Less work has been committed to new general relativity. The name of this theory was introduced in [12] as a one-parameter theory agreeing with solar system tests of gravity. However, in this review we refer to new general relativity as the most general parity even teleparallel gravity theory quadratic in the torsion components. In particular the one-parameter theory is the most general of the NGR-theories different from TEGR whilst avoiding pathologies of mixing symmetric and antisymmetric perturbations which is nicely shown in [36] (see [37] for earlier works, and [38] regarding the pathology in general). In [28] it is found that this one-parameter theory has a non-deterministic evolution for certain intial values. In [33] the Hamiltonian analysis of the NGR-theory with minimal amount of primary constraints was carried out and a special case of this theory was studied in [39]. They found that the constraint algebra close without any introduction of secondary constraints (except for the Hamiltonian and momenta constraints generic for any teleparallel theory of gravity). Furthermore, the Hamiltonian analysis for general NGR have been partly carried out in [27, 30, 31, 34].

The disagreement in the conclusions which are drawn from the canonical analysis of modified teleparallel theories of gravity by different authors motivates us to presents a comparison of the approaches, and how they can be translated into each other. To do this we present the primary constraints of the parent class of theories, $f(\mathbb{T}_{\mathrm{NGR}})$ gravity, which includes $f(\mathbb{T})$ gravity and NGR as special cases, as reference. We then compare the existing approaches in the literature against our findings and discuss how they are related among each other. We hope this work will simplify the comparison between different approaches to the canonical analysis of modified teleparallel theories of gravity, and thus enable the community to come to a definite answer on the number and nature of the degrees of freedom in these theories.

This work is organized as follows. We perform an introduction to the Dirac-Bergmann algorithm in Sec. 2. In Sec. 3 we introduce the covariant formulation of the teleparallel formalism and TEGR. Some important points to consider in the Hamiltonian analysis are discussed in Sec. 4. In Sec. 5 we introduce $f(\mathbb{T}_{\mathrm{NGR}})$ gravity, its primary constraints and their classification. We make a compilation on the different notation and primary constraints for $f(\mathbb{T})$ gravity and NGR that can be found in the literature in Sec. 6. A discussion on the difficulties in applying the Dirac-Bergmann algorithm can be found in Sec. 7. Finally our outlook and conclusions are in Sec. 8.

2. Dirac-Bergmann algorithm for Hamiltonian analysis

Most theories of physical interest are gauge systems: the equations of motion do not determine all the dynamical variables, since there are relations among them that leave the state of the system unaltered. Such situation translates as a constrained Hamiltonian system; in this picture the canonical variables are not all independent. All gauge systems can be regarded as constrained Hamiltonian systems, but not all constraints from a Hamiltonian system arise from a gauge invariance. Hamiltonian systems with constraints can be studied through the Dirac-Bergmann[a] algorithm. In what follows, we will review this method [41–43] and put emphasis on some of its peculiarities [44–46]. The main steps of the Dirac-Bergmann algorithm are highlighted throughout the text in concordance with the notation introduced in Fig. 1.

[a] Also called Rosenfeld-Dirac-Bergmann algorithm, see [40] and the discussion on Rosenfeld's contribution in [35]

In the following, we will introduce the Lagrangian formulation of a theory with finite degrees of freedom, that is a finite number of coordinates depending on time $q^i = q^i(t)$, which define the state of the system. However, later we will study gravitational theories, on which the fields depend on the space-time coordinates. Each field represents, strictly speaking, infinite degrees of freedom. The counting convention therefore is that each field component is called a single physical degree of freedom.

We give a brief introduction to the Lagrange formalism in section 2.1. The canonical momenta and primary constraints are defined in section 2.2. The Dirac-Bergmann algorithm and determining the constraint surface is laid out in section 2.3. Finally, we define the notions of first and second class constraints in section 2.4. The algorithm is summarized in figure 1.

2.1. *Lagrangian formalism*

Let us consider a finite-dimensional system with an n-dimensional configuration space Q spanned by the coordinates $q^i, i = 1, \ldots, n$. The system is described by a Lagrangian $L(q(t), \dot{q}(t))$ without explicit dependence on time t, which defines the action $S = \int_{t_i}^{t_f} dt L(q, \dot{q})$. The variational principle, i.e. the requirement of the action to be stationary

$$\delta S = \int_{t_i}^{t_f} dt \frac{\delta L}{\delta q^i(t)} \delta q^i(t) = 0 \tag{1}$$

under variations vanishing at the endpoints t_i, t_f, yields that physical trajectories $q^i(t)$ must satisfy the Euler-Lagrange equations

$$\frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = 0. \tag{2}$$

Expanding the time derivative in (2), we obtain

$$\left(\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j} \right) \ddot{q}^j + \left(\frac{\partial^2 L}{\partial \dot{q}^k \partial q^j} \right) \dot{q}^j - \frac{\partial L}{\partial q^k} = 0, \tag{3}$$

$$V_k - W_{kj} \ddot{q}^j = 0, \tag{4}$$

where $V_k = \frac{\partial L}{\partial q^k} - \left(\frac{\partial^2 L}{\dot{q}^k \dot{q}^j} \right) \dot{q}^j$ and $W_{kj} = \left(\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j} \right)$. The object W_{kj} is the Hessian of L with respect to the velocities $\dot{q}^k$, and it has an important role. If the rank of the Hessian $\mathrm{Rank}(W_{ij}) = r = n$, then all accelerations $\ddot{q}^i$ can be solved in terms of q^i and $\dot{q}^i$. For a constrained physical system the Hessian has non-maximal rank, that is $\mathrm{Rank}(W_{ij}) = r < n$. This means that not all accelerations $\ddot{q}^i$ can be uniquely determined in terms of q^i and $\dot{q}^i$. In the Hamiltonian picture, this implies the existence of primary constraints.

We now recall the steps of the Dirac-Bergmann algorithm needed to obtain the constrained Hamiltonian formulation of the dynamics of the physical system described by the Lagrangian L.

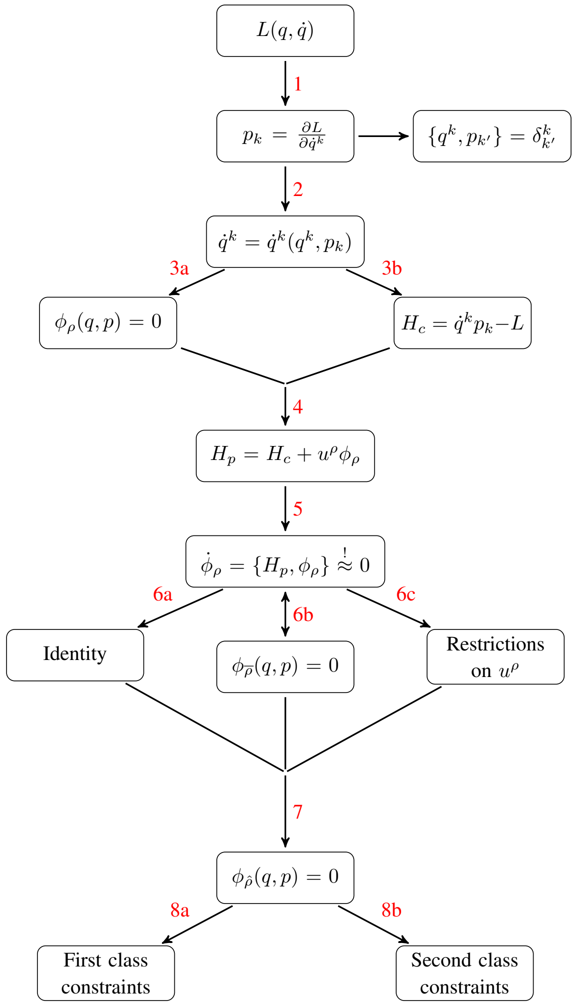

Fig. 1. Dirac-Bergmann algorithm

2.2. *Canonical momenta and primary constraints*

Step 1. For going to the Hamiltonian formalism of a dynamical system, we start with the definition of the canonical momenta as functions of $(q, \dot{q})$

$$p_k(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^k} = \tilde{\phi}_k(q, \dot{q}), \qquad k = 1, \ldots, n. \tag{5}$$

The fact that the Hessian $W_{ij} = \frac{\partial p_i}{\partial \dot{q}^j}$ has non-maximal rank r implies that it has a non-trivial kernel of dimension $n-r$. This kernel is spanned by $n-r$ vectors with components l^k_ρ such that

$$l^k{}_\rho W_{kj} = 0, \qquad \rho = 1, \ldots, n-r. \tag{6}$$

Step 2. When the rank of W_{kj} is $r < n$ (also that det$(W_{kj}) = 0$), we can solve r velocities in terms of the momenta and positions from Eq.(5),

$$\dot{q}^{\hat{i}} = \mathcal{F}^{\hat{i}}(q, p_\alpha, \dot{q}^{\bar{i}}), \tag{7}$$

where, without loss of generality, it can be assumed that $\hat{i} = 1, \ldots, r$, label the solvable velocities, and $\bar{i} = r+1, \ldots, n$ are the velocities that can not be solved. We also assume that the index α can take r values. If we substitute (7) in (5), we obtain

$$p_k = \tilde{\phi}_k(q, \dot{q}^{\hat{i}}, \dot{q}^{\bar{i}}) = \tilde{\phi}_k(q, \mathcal{F}^{\hat{i}}(q, p_\alpha, \dot{q}^{\bar{i}}), \dot{q}^{\bar{i}}) = \hat{\phi}_k(q, p_\alpha, \dot{q}^{\bar{i}}) \tag{8}$$

The functions $\hat{\phi}_k(q, p_\alpha, \dot{q}^{\bar{i}})$ cannot depend on $\dot{q}^{\bar{i}}$ any longer, otherwise it would be possible to solve for more of the velocities.

Step 3a. For $k = 1, \ldots, r$ the equations $p_k = \hat{\phi}_k$, (8), are trivially satisfied, from the definition of the momenta and the assumption of being able to express the velocities $\dot{q}^{\hat{i}}$ as functions of q and p, while for $k = r, \ldots, n$ one obtains $n-r$ non-trivial relations $p_\rho = \hat{\phi}_\rho(q, p_\alpha)$ that relate coordinates and momenta. These give rise to the so called primary constraints

$$\phi_\rho(q, p) = p_\rho - \hat{\phi}_\rho = 0, \quad \rho = 1, \ldots, n-r. \tag{9}$$

An important notion for the Hamiltonian formalism is "weak equality" (denoted by the symbol "$\approx$") which is an equality on the constraint surface. The symbol "$\stackrel{!}{\approx}$" will denote that weak equality is imposed, which means that the equality of both sides of the equations is not necessarily implied by the previously found constraints, but must be imposed as an additional condition in order to obtain the final constraint surface.

Step 3b. The canonical Hamiltonian can be obtained in terms of the momenta as

$$H_c = \dot{q}^i p_i - L(q, \dot{q}) \tag{10}$$

which can be proved to depend only on q^i and p_i, not in the velocities $\dot{q}^i$. This Hamiltonian does not encode a priori the primary constraints and thus does not describe the same dynamical system as the Lagrangian. To keep the predictions in the transition from the Lagrangian to the Hamiltonian formulation of the dynamics of the system unaltered one needs to add the primary constraints with the help of Lagrange multipliers $H_c \longrightarrow H_c + u^\rho \phi_\rho(q,p)$.

Step 4. We consider the primary Hamiltonian

$$H_p(q,p) = H_c + u^\rho \phi_\rho(q,p) \tag{11}$$

where the u^ρ are Lagrange multipliers (arbitrary functions); they ensure the primary constraints from the beginning. It can be shown that H_p generates the time evolution of the physical system through the Poisson brackets (PB) in the following way. For any function F in the phase space, it is

$$\dot{F} = \{F, H_p\} \approx \{F, H_c\} + u^\rho \{F, \phi_\rho\}, \tag{12}$$

where the term $\{F, u^\rho\}\phi_\rho$ is dropped since it vanishes weakly, and the PB between two functions $F(p,q)$ and $G(p,q)$ is defined as

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}. \tag{13}$$

This definition allows to compute the time evolution of any primary constraint (12), whose outcome has many different branches, as we explain in the next section.

2.3. *Dirac-Bergmann algorithm and determination of constraint surface*

Step 5. In order for the physical system to be consistent, the evolution on time $\dot{\phi}_\rho$ of primary constraints should be zero. This imposes the condition

$$\dot{\phi}_\rho = \{\phi_\rho, H\} + u^\sigma \{\phi_\rho, \phi_\sigma\} \overset{!}{\approx} 0. \tag{14}$$

If we define $h_\rho = \{\phi_\rho, H\}$ and $C_{\rho\sigma} = \{\phi_\rho, \phi_\sigma\}$, then (14) can have two outcomes, depending if $\det(C)$ is weakly zero or not.

Step 6a. If $\det(C) \approx 0$, then the multipliers are not uniquely determined, and (14) is only solvable if the h_ρ satisfy the conditions

$$\omega^\rho_\alpha h_\rho \overset{!}{\approx} 0, \tag{15}$$

where ω^ρ_α are $p-m$ linearly independent vectors spanning the kernel of C, which has rank m.

Step 6b. These conditions can be fulfilled like in the previous step, or lead to a certain number s' of new constraints

$$\phi_{\overline{\rho}} \approx 0, \quad \overline{\rho} = n - r + 1, \ldots, n - r + s' \tag{16}$$

called secondary constraints.

Step 6c. If $\det(C) \not\approx 0$, Eq.(14) is an inhomogeneous system of linear equations with solutions

$$u^\sigma \approx -(C^{-1})^{\rho\sigma} h_\rho, \tag{17}$$

and the Hamilton equations of motion for a function $F(p,q)$ become

$$\dot{F} \approx \{F, H_c\} - \{F, \phi_\rho\}(C^{-1})^{\rho\sigma}\{\phi_\sigma, H_c\}, \tag{18}$$

which do not contain arbitrary multipliers, that is, they are fully determined.

Unlike primary constraints, secondary constraints have been derived from the equations of motion. The procedure of vanishing the time evolution should be iterated with secondary constraints, which could give rise to tertiary constraints and so on, until no more constraints appear. In most cases of physical relevance, the algorithm terminates at the stage of secondary constraints, but it is not hard to build pathological examples on which there is an infinite tower of constraints, or the conditions of time consistency give rise to physically inequivalent branches [43]. We could also face the unlikely case on which the consistency conditions are incompatible with each other, then it is said that the Hamiltonian system is inconsistent, and the algorithm is terminated [44]

Step 7. When no more constraints appear, we are left with a hypersurface defined by

$$\phi_\rho \approx 0, \ \ (\rho = 1, \dots, n-r), \tag{19}$$

$$\phi_{\overline{\rho}} \approx 0, \ \ (\overline{\rho} = n-r+1, \dots, n-r+s). \tag{20}$$

The first set $\{\phi_\rho\}$ contains all p primary constraints, while the set $\{\phi_{\overline{\rho}}\}$ contains s secondary, tertiary, etc. constraints. By using a common notation for all constraints as $\phi_{\hat{\rho}}$, with $\hat{\rho} = 1, \dots, n-r+s$, we can define the matrix of constraints as

$$C_{\hat{\rho}\rho} = \{\phi_{\hat{\rho}}, \phi_\rho\}. \tag{21}$$

If $\omega^{\hat{\rho}}_\alpha$ span the left kernel of $C_{\hat{\rho}\rho}$, then the conditions $\omega^{\hat{\rho}}_\alpha\{\phi_{\hat{\rho}}, H_c\} \approx 0$ are satisfied. Also for the multipliers, the equations

$$\{\phi_{\overline{\rho}}, H_c\} + \{\phi_{\overline{\rho}}, \phi_\rho\}u^\rho \approx 0 \tag{22}$$

are fulfilled. Note that the weak equalities are defined with respect to the final constraint hypersurface of all constraints.

2.4. *First and second class constraints*

Solving the multiplier functions from (22) leads to the definition of first and second class constraints. If the rank of the matrix $C_{\hat{\rho}\rho}$ is $n-r$, then all multipliers are fixed, but if its rank is $k < n-r$, there are $n-r-k$ linearly independent solutions of the equation

$$C_{\hat{\rho}\rho}V^\rho_\alpha = \{\phi_{\hat{\rho}}, \phi_\rho\}V^\rho_\alpha \approx 0, \tag{23}$$

which is the homogeneous part of (22). Notice that V^ρ_α span the right kernel of $C_{\hat{\rho}\rho}$. With all this, the most general solution of (22) is a sum of a particular solution U^ρ and a linear combination of the solutions of the homogeneous part, that is

$$u^\rho = U^\rho + v^\alpha V^\rho_\alpha \tag{24}$$

where the coefficients v^α are arbitrary.

It is important to keep in mind that the rank of $C_{\hat{\rho}\rho}$ can be variable, and in such case it can still give rise to different, but consistent physical evolution. This is not always the case, and some counterexamples can be found at [43]. This feature seems to be crucial for modified teleparallel gravities.

Step 8a. We define a function $\mathcal{F}(p,q)$ in the phase-space to be first class if the PB with all constraints in the theory vanishes,

$$\{\mathcal{F}(p,q), \phi_{\hat{\rho}}\} \approx 0. \tag{25}$$

Step 8b. If a phase-space function is not first class, it is called to be second class.

Since the PB satisfy the Jacobi identity, it is possible to prove that the PB of two first class constraints is itself first class. It is convenient to reformulate a theory in terms of its maximal number of independent first class and second class constraints. Let us assume that the maximal number of first class constraints is obtained after building some linear combination, which we will denote as $\Phi_I, I = 1,\ldots,l$, and the remaining set of second class constraints is χ_A. Then, to make sure that the maximum number of Φ_I has been found, it is convenient to build the PB matrix of second class constraints

$$\Delta_{AB} = \{\chi_A, \chi_B\} \tag{26}$$

and check that it has non-vanishing determinant. After this, we make sure that Δ_{AB} has an inverse, and that the Lagrange multipliers for second class constraints can be solved univocally. After all this procedure, it is possible to count the physical degrees of freedom of the theory through the formula

$$\text{Number of d.o.f.} = \text{Number of } (p,q) - \text{Number of f.c.c.} - \frac{1}{2}(\text{Number of s.c.c.}). \tag{27}$$

In summary the procedure of the Dirac-Bergmann algorithm has two main goals: firstly finding all primary, secondary, tertiary, ..., constraints from the definition of the canonical momenta and the time evolution of the system, and secondly grouping them into first and second class constraints.

3. Covariant formulation of teleparallel gravity

In the following a covariant formulation for teleparallel gravity will be introduced and notation and conventions for this article will be fixed. Greek letters $\mu,\nu,\rho,\ldots$ denote spacetime indices, Lorentz tangent space indices are denoted by the first letters of the Latin alphabet $A,B,C,\ldots$, and their spatial part is denoted with hats $\hat{A},\hat{B},\hat{C},\ldots$. The sign convention for the Minkowski metric is the mostly negative one: $\eta_{AB} = \mathrm{diag}(1,-1,-1,-1)$. The torsion components are defined as

$$T^A{}_{\mu\nu} = \partial_\mu\theta^A{}_\nu - \partial_\nu\theta^A{}_\mu + \omega^A{}_{B\mu}\theta^B{}_\nu - \omega^A{}_{B\nu}\theta^B{}_\mu, \tag{28}$$

where $\theta^A{}_\mu$ are tetrad components and $\omega^A{}_{B\mu}$ are components of the spin connection defined as

$$\omega^A{}_{B\mu} = -\left(\Lambda^{-1}\right)^C{}_B\partial_\mu\Lambda_C{}^A, \tag{29}$$

where $\Lambda_C{}^A$ are Lorentz matrices. Taking into account this spin connection and applying Lorentz transformations simultaneously to spin connection and tetrad only, yields that the torsion tensor transform covariantly and the formulation of teleparallel gravity, in this sense, satisfy local Lorentz invariance [47, 48].

Cotetrads are denoted by $e_A{}^\mu$ and the following relations are satisfied

$$\eta_{AB} = g_{\mu\nu} e_A{}^\mu e_B{}^\nu, \tag{30}$$

$$g_{\mu\nu} = \eta_{AB} \theta^A{}_\mu \theta^B{}_\nu. \tag{31}$$

Lorentz indices and spacetime indices can be transformed between each other by contraction with a tetrad resp. cotetrad in the obvious correct way, i.e a spacetime index $^\mu$ becomes a Lorentz index ^{A} through contraction with a tetrad $\theta^A{}_\mu$, while a spacetime index $_\mu$ becomes a Lorentz index $_A$ through contraction with an inverse tetrad $e_A{}^\mu$. Lorentz indices are raised and lowered with the Minkowski metric, while spacetime indices are raised and lowered with the spacetime metric.

The teleparallel equivalent to general relativity (TEGR) is obtained from rewriting the classical Einstein-Hilbert action of general relativity in the teleparallel geometric language, and yields the action

$$S = \int \mathrm{d}^4 x L_{\mathrm{TEGR}} + S_{\mathrm{M}}\,, \tag{32}$$

where the Lagrangian is given by

$$\begin{aligned} L_{\mathrm{TEGR}} &= \frac{1}{2\kappa}\theta\mathbb{T} = \frac{1}{4\kappa}\theta T^{\rho\mu\nu} S_{\rho\mu\nu} & (33)\\ &= \frac{1}{2\kappa}\theta\left(\frac{1}{4}T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} - \frac{1}{2}T^\rho{}_{\mu\nu}T^{\mu\nu}{}_\rho - T^\rho{}_{\mu\rho}T^{\sigma\mu}{}_\sigma\right), & (34) \end{aligned}$$

$\kappa = \frac{8\pi G}{c^4}$, θ is the determinant of the tetrad and $S_{\rho\mu\nu}$ is the so called superpotential

$$S_{\rho\mu\nu} = \frac{1}{2}T_{\rho\mu\nu} + T_{[\nu\mu]\rho} + 2g_{\rho[\mu}T^\sigma{}_{\nu]\sigma}\,. \tag{35}$$

The scalar $\mathbb{T}$ is called the canonical torsion scalar. It is related to the Ricci scalar $\mathring{R}$ of the Levi-Civita connection of the metric generated by the tetrad by a total derivative term

$$\mathbb{T} = \frac{1}{\sqrt{\det g}}\partial_\mu(\sqrt{\det g}\,T_\nu{}^{\nu\mu}) - \mathring{R}\,, \tag{36}$$

which is used to prove the dynamical equivalence between TEGR and GR [22, 49].

4. Aspects of Hamiltonian analysis of teleparallel gravity

We now state relevant, preliminary steps which are necessary for the Hamiltonian analysis of teleparallel gravity theories. First, we discuss different possible choices for the fundamental fields and gauge choices in section 4.1, then in section 4.2 we discuss how to establish the proper split of the teleparallel geometry into space and time components.

4.1. *Fundamental fields and gauge fixing*

Before starting the Dirac-Bergmann algorithm one should consider another step, which could be labeled as "step 0" in the diagram in figure 1. This step comprises of identifying the fundamental fields q^i, which constitute the dynamical variables of the theory. Here, it is desirable to reduce the number of constrained variables as much as possible, in order to reduce the dimension of the Hessian and the number of Poisson brackets between constraints to be calculated. In the ideal case, one may parametrize the physical phase space using only independent variables, and obtain a system without constraints. Even if this is not always possible, one may usually reduce the number of constrained variables by a change of parametrization. Nevertheless, this reparametrization possibly comes at the cost of a more complicated Lagrangian.

As an example, one may consider general relativity. In the usual metric formulation, the fundamental fields, which correspond to the generalized coordinates q^i in section 2, are the ten independent components of the metric. However, it is possible to formulate GR purely in the tetrad formalism. If the tetrad components are used instead of the metric ones, there are six more fundamental fields to be treated in the Hamiltonian analysis. In consequence, there are as well six additional primary constraints corresponding to generators of local Lorentz transformations. They reflect the arbitrariness in the choice of the tetrad for a given metric. Hence, choosing the metric components as fundamental variables reduces the number of constrained variables, compared to the tetrad formulation.

In teleparallel gravity the metric components do not suffice as the fundamental fields, due to the formulation of the Lagrangian in terms of the torsion tensor, which cannot be obtained from the metric alone. The most common variables chosen in the covariant formulation of teleparallel gravity displayed in the previous section, which is manifestly invariant under local Lorentz transformations, are the tetrad θ and the flat Lorentz spin connection ω. The latter may further be parametrized by finite Lorentz transformations. However, it follows from the Lorentz invariance of the teleparallel gravity action in the covariant formulation that the canonical momenta of the spin connection are related to the momenta of the tetrad, revealing that these spin connection degrees of freedom are not independent [32]. As a consequence, one may choose a different parametrization, in order to reduce the number of constrained field variables. These different parametrizations give rise to several approaches on how to consider the spin connection as dynamical field in teleparallel gravity; see Ref. [50] for an extended discussion. In summary, we have the following choices for our fundamental fields:

(1) The most straightforward approach is to consider both, the 16 tetrad components and the 24 components of a Lorentz spin connection, as fundamental variables. However, since the spin connection components are not independent of each other, but constrained by the flatness condition, in addition a set $\lambda_\mu{}^{\nu\rho\sigma}$ of Lagrange multipliers is required, which enforce the vanishing of the curvature. The full teleparallel action then takes the symbolic form [51]

$$S_{\mathrm{TG}} = S_{\mathrm{TG}}(\theta, \omega) - \int \mathrm{d}^4x\, \theta\, \lambda_\mu{}^{\nu\rho\sigma} R^\mu{}_{\nu\rho\sigma}\,. \tag{37}$$

(2) Alternatively, one may implement the flatness of the spin connection by exploiting the relation (29) in order to express the spin connection in terms of a local Lorentz transformation. In this case, the fundamental variables become the tetrad components and the components of the Lorentz matrices that parametrize the inertial spin connection, so that the action takes the structure $S_{\text{TG}} = S_{\text{TG}}(\theta, \Lambda)$.
(3) Finally, one may use the aforementioned approach as a starting point to further reduce the number of fundamental variables, based on the aforementioned observation that the canonical momenta of the spin connection are linearly dependent on the tetrad momenta. This is a consequence of the fact that the spin connection is a pure gauge degree of freedom, so that the Lorentz matrices introduced above enter the action only in the combination $\tilde{\theta}^A{}_\mu = \theta^B{}_\mu \Lambda_B{}^A$. Hence, one may replace the tetrad θ in the action by $\tilde{\theta}$, and obtains an action which is independent of Λ; schematically, $S_{\text{TG}} = S_{\text{TG}}(\theta, \Lambda) = S_{\text{TG}}(\tilde{\theta})$. This leaves only the 16 components of the tetrad $\tilde{\theta}$ as fundamental field variables, and is formally equivalent to imposing the Weitzenböck gauge $\omega^A{}_{B\mu} = 0$ (or alternatively $\Lambda_B{}^A = \text{const}$).

Clearly, while all three approaches lead to the same physical phase space, and thus equivalent results, the latter approach introduces the smallest number of fundamental field variables. In the following, we will therefore assume that the Weitzenböck gauge is imposed, leaving the tetrad as only fundamental variable, and not consider the other two approaches. Also we will drop the tilde in the notation and simply denote the tetrad by θ.

4.2. $3+1$ *decomposition*

The Hamiltonian formalism requires a Legendre transformation from the set of fields and their velocities to the fields and their conjugate momenta. To invert the velocity momentum relations it is very convenient to employ a 3+1 decomposition of spacetime. Geometrically this mean that we split the 4-dimensional spacetime into 3-dimensional hypersurfaces and a time direction. The geometry of such a foliated spacetime can be described in two ways. The first is by introducing adapted coordinates $(x^0, x^i), i = 1, 2, 3$, where x^i denote intrinsic coordinates on each hypersurface, and the time coordinate x^0 labels the hypersurfaces of the foliation, such that each hypersurface is given by setting $x^0 = \text{const}$. These coordinates define a coordinate basis ∂_μ of the tangent space, with ∂_i being tangent to each hypersurface, and an extrinsic basis vector field ∂_0. Alternatively, instead of the coordinate vector field ∂_0 associated to time coordinate, given a metric $g_{\mu\nu}$ one may use the normal vector n to the hypersurfaces as additional extrinsic reference direction. Both descriptions are related by the so called lapse function α and shift vector $\beta = \beta^i \partial_i$, by expanding the x^0 tangent direction as

$$\partial_0 = \alpha n^\mu \partial_\mu + \beta^i \partial_i. \tag{38}$$

From this we can read off that the component of the normal vector to hypersurfaces of constant time slices reads

$$n^0 = \frac{1}{\alpha}, \qquad n^i = -\frac{\beta^i}{\alpha}. \tag{39}$$

Introducing a negative definite intrinsic metric $\gamma = \gamma_{ij} dx^i dx^j$ on the hypersurfaces the metric can be written as

$$g_{\mu\nu} = \begin{bmatrix} \alpha^2 + \beta^i \beta^j \gamma_{ij} & \beta_i \\ \beta_i & \gamma_{ij} \end{bmatrix} . \tag{40}$$

The variables $\alpha, \beta_i, \gamma_{ij}$ are normally referred as "ADM-variables", named after R. Arnowitt, S. Deser, and C.W. Misner [52]. It is not difficult to find a tetrad for which the ADM decomposition of the metric is reproduced. For instance

$$\theta^A{}_\mu = \begin{bmatrix} \alpha & 0 \\ \beta^i \theta^{\hat{A}}{}_i & \theta^{\hat{A}}{}_i \end{bmatrix} , \tag{41}$$

yields that $g_{\mu\nu} = \theta^A{}_\mu \theta^B{}_\nu \eta_{AB}$ has the form (40), where $\gamma_{ij} = \theta^{\hat{A}}{}_i \theta^{\hat{B}}{}_j \eta_{AB}$. However, a few things must be remarked about this tetrad. First, note that it requires a $3+1$ split into space and time components not only for the spacetime indices, but also for the Lorentz indices. Further, its form is not invariant under (global or local) Lorentz transformations. This implies that the choice of this tetrad explicitly introduces a gauge condition, by imposing certain components of the tetrad to vanish. Fixing the gauge does not go without consequence in teleparallel gravity theories, since in general such theories are not invariant under (local) Lorentz transformations of the tetrad alone. It follows that by imposing gauge conditions on the tetrad, one must allow for a non-vanishing spin connection, in order to restore the Lorentz invariance under simultaneous transformations of the tetrad and the spin connection. The alternative approach, which we favor here, is to impose a gauge condition on the spin connection only, such as the Weitzenböck gauge $\omega^A{}_{B\mu} \equiv 0$, and to keep all 16 components of the tetrad as unrestricted dynamical variables. Also in this case an ADM decomposition (40) can be achieved. The crucial insight is that the spatial metric components $\gamma_{ij} = \eta_{AB} \theta^A{}_i \theta^B{}_j$, and in consequence the Lorentz components n^A of the unit normal vector, can be fully expressed in terms of the spatial tetrad components $\theta^A{}_i$, and do not involve the time components $\theta^A{}_0$. To express the latter, one realizes that $(n^A, \theta^A{}_i)$ are a linearly independent set of four Lorentz vectors, and hence form a basis of the Minkowski space. One may thus express the Lorentz vector $\theta^A{}_0$ in this basis as

$$\theta^A{}_0 = \alpha n^A + \beta^i \theta^A{}_i , \tag{42}$$

thereby defining the lapse α and shift β^i as the coefficients of $\theta^A{}_0$ with respect to this basis. One finds that the metric is indeed of the form (40), while keeping 16 independent dynamical variables $(\alpha, \beta^i, \theta^A{}_i)$ without imposing gauge conditions on them.

5. Canonical momenta and primary constraints in $f(\mathbb{T}_{\mathrm{NGR}})$

In the literature of the Hamiltonian analysis of modified teleparallel theories of gravity, the focus lies on two different classes of theories: new general relativity (NGR) [12], and $f(\mathbb{T})$ gravity [11]. To compare the different approaches in an efficient way later, we newly present here the derivation of the momenta for $f(\mathbb{T}_{\mathrm{NGR}})$ teleparallel theories of gravity, from which different primary constraints in different subclasses of the theory arise.

$f(\mathbb{T}_{\text{NGR}})$ gravity is a special case of the $f(T_{\text{ax}}, T_{\text{ten}}, T_{\text{vec}})$ gravity theories which were introduced in [53]. Phenomenologically this theory has not yet been studied in full detail, but some results about gravitational waves [54, 55] and its post-Newtonian limit [56] are known.

The starting point of the analysis is the Lagrange density

$$L_{f(\mathbb{T}_{\text{NGR}})} = \frac{1}{2\kappa}\theta f(\mathbb{T}_{\text{NGR}})\,, \tag{43}$$

where

$$\mathbb{T}_{\text{NGR}} = H_{ABC}T^{ABC}\,, \tag{44}$$

with

$$H_{ABC} = c_1 T_{ABC} + \frac{c_2}{2}(T_{CBA} - T_{BCA}) + \frac{c_3}{2}(\eta_{AB}T^D{}_{DC} - \eta_{AC}T^D{}_{DB})\,. \tag{45}$$

We call H_{ABC} the NGR induction tensor, in resemblance with induction tensors in electrodynamics [57], or the NGR superpotential. By fixing the three parameters c_1, c_2, c_3 to the specific values

$$c_1 = \frac{1}{4}\,, \quad c_2 = \frac{1}{2}\,, \quad c_3 = -1, \tag{46}$$

the NGR induction tensor becomes proportional to the usual superpotential (35), as one finds $H_{ABC} = \frac{1}{2}S_{ABC}$, and the Lagrangian (43) becomes the standard $f(\mathbb{T})$ gravity Lagrangian.

To perform the Hamiltonian analysis of $f(\mathbb{T}_{\text{NGR}})$ gravity a Legendre transform is performed, in order to obtain a mathematically equivalent formulation of $f(\mathbb{T}_{\text{NGR}})$ gravity. This is done with help of a scalar field $\phi(\mathbb{T}_{\text{NGR}})$ in the form

$$L_{\text{f}} = \frac{1}{2\kappa}\theta(\phi\mathbb{T}_{\text{NGR}} - V(\phi))\,. \tag{47}$$

The fundamental variables of the theory are now the tetrad $\theta^A{}_\mu$ as well as the scalar field ϕ. This theory reduces to NGR by setting $\phi = 1$ and $V(\phi) = 0$, and it reduces to $f(\mathbb{T})$ gravity for the TEGR choice of parameters (46).

The canonical momenta of these fields are identified as

$$\pi_A{}^\mu = \frac{\partial L}{\partial_0\theta^A{}_\mu} = \frac{\partial L}{\partial T^A{}_{0\mu}} = \frac{2}{\kappa}\theta\phi H_A{}^{0\mu}\,, \tag{48}$$

$$\frac{\kappa}{2}\pi_\phi = \frac{\partial L}{\partial\partial_0\phi} = 0\,, \tag{49}$$

The momentum equations yield immediately five trivial universal primary constraints

$$\pi_A{}^0 = 0 \text{ and } \pi_\phi = 0\,, \tag{50}$$

where the first four come from the index symmetries of H_{ABC}.

For further analysis we consider the momenta π_{AB} with pure lowered Lorentz indices. These are related to the momenta $\pi_A{}^\mu$ with the canonical index positions as

$$\begin{aligned}\frac{\kappa}{2}\pi_{AB} &= \frac{\kappa}{2}\pi_A{}^\mu\theta^C{}_\mu\eta_{BC} = \phi\theta H_A{}^0{}_B = \phi\frac{\theta}{\alpha}n_\mu H_A{}^\mu{}_B \\ &= \phi\sqrt{\gamma}n^D H_{ADB}\,,\end{aligned} \tag{51}$$

where we used that in a coordinate basis we have $n_0 = \alpha$ and $n_i = 0$.

It satisfies $n^B\pi_{AB} = 0$, which represents the four constraints

$$\pi_A{}^0 = 0. \tag{52}$$

Further, we can decompose the remaining 12 components with respect to the three dimensional rotational group on the equal time hypersurface, also called $\mathcal{VAST}$ decomposition, with help of the projectors $\Xi^A_B = \delta^A_B - n_B n^A$ as

(1) a $\mathcal{V}$ectorial part

$$\begin{aligned}\frac{\kappa}{2\phi\sqrt{\gamma}}n^A\pi_{AB} &= n^A n^D H_{ADB} \\ &= \frac{1}{2}(2c_1+c_2)n^A n^D T_{ADB} + \frac{c_3}{2}(T^A{}_{AB} - n_B n^D T^A{}_{AD}) \\ &= \frac{1}{2}(2c_1+c_2+c_3)n^A n^D T_{ADB} + \frac{c_3}{2}T^Q{}_{PD}\Xi^D_B\Xi^P_Q\,;\end{aligned} \tag{53}$$

(2) an $\mathcal{A}$ntisymmetric part

$$\begin{aligned}\frac{\kappa}{2\phi\sqrt{\gamma}}\Xi^C_{[A}\Xi^D_{B]}\pi_{CD} &= \frac{\kappa}{2\phi\sqrt{\gamma}}(\hat{\pi}_{[AB]} - n_{[A}n^C\hat{\pi}_{|C|B]}) \\ &= \left(\frac{1}{2}(2c_1-c_2)n^D T_{PDQ} + \frac{c_2}{2}n^D T_{DPQ}\right)\Xi^P_{[A}\Xi^Q_{B]}\,;\end{aligned} \tag{54}$$

(3) a $\mathcal{S}$ymmetric trace free part

$$\begin{aligned}&\frac{\kappa}{2\phi\sqrt{\gamma}}(\Xi^C_{(A}\Xi^D_{B)}\pi_{CD} - \frac{1}{3}(\eta_{AB} - n_A n_B)\pi^C{}_C) \\ &= \frac{1}{2}(2c_1+c_2)(\Xi^C_{(A}\Xi^D_{B)} - \frac{1}{3}(\eta_{AB} - n_A n_B)\eta^{CD})T_{CQD}n^Q\,;\end{aligned} \tag{55}$$

(4) a $\mathcal{T}$race part

$$\begin{aligned}\frac{\kappa}{2\phi\sqrt{\gamma}}\eta^{AB}\Xi^C_A\Xi^D_B\pi_{CD} &= \frac{\kappa}{2\phi\sqrt{\gamma}}\pi^A{}_A \\ &= \frac{1}{2}(2c_1+c_2+3c_3)n^D T^A{}_{DA}\,.\end{aligned} \tag{56}$$

From these relations we can identify possible constraints as

$$
\begin{aligned}
&{}^{\mathcal{V}}\mathcal{A} = 2c_1 + c_2 + c_3 = 0 \\
\Longrightarrow\ &{}^{\mathcal{V}}C_B = \frac{\kappa}{2\phi\sqrt{\gamma}} n^A \pi_{AB} + \frac{c_3}{2} T^Q{}_{PD} \Xi^D_B \Xi^P_Q \approx 0\,, && (57) \\
&{}^{\mathcal{A}}\mathcal{A} = 2c_1 - c_2 = 0 \\
\Longrightarrow\ &{}^{\mathcal{A}}C_{AB} = \frac{1}{2}\Xi^C_{[A}\Xi^D_{B]}\left(\frac{\kappa}{\phi\sqrt{\gamma}}\pi_{CD} - c_2 n^E T_{ECD}\right) \approx 0\,, && (58) \\
&{}^{\mathcal{S}}\mathcal{A} = 2c_1 + c_2 = 0 \\
\Longrightarrow\ &{}^{\mathcal{S}}C_{AB} = \frac{\kappa}{2\phi\sqrt{\gamma}}\left(\Xi^C_{(A}\Xi^D_{B)} - \frac{1}{3}(\eta_{AB} - n_A n_B)\eta^{CD}\right)\pi_{CD} \approx 0\,, && (59) \\
&{}^{\mathcal{T}}\mathcal{A} = 2c_1 + c_2 + 3c_3 = 0 \\
\Longrightarrow\ &{}^{\mathcal{T}}C = \frac{\kappa}{2\phi\sqrt{\gamma}}\pi^A{}_A \approx 0\,. && (60)
\end{aligned}
$$

Which of the relations ${}^{\mathcal{I}}C = 0$, $\mathcal{I} = \mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{T}$, imposes a primary constraint on the canonical momenta depends on the choice of values for the parameters c_1, c_2, c_3. Setting the scalar field $\phi = 1$ in these equations one obtains the corresponding relations for the NGR class of theories.

For the $f(\mathbb{T})$ choice (46) the vector and antisymmetric parts impose primary constraints, while the trace and the symmetric trace free parts represent invertible equations which relate the time derivatives of the tetrad to the momenta. The general analysis for all $f(\mathbb{T}_{\mathrm{NGR}})$ classes gives the following additional constraints to the universal ones (50): No

	Theory parameter combinations				#
${}^{\mathcal{I}}\mathcal{A}$	$2c_1+c_2+c_3$	$2c_1-c_2$	$2c_1+c_2$	$2c_1+c_2+3c_3$	
Constraints if ${}^{\mathcal{I}}\mathcal{A}=0$	${}^{\mathcal{V}}C_B \approx 0$	${}^{\mathcal{A}}C_{AB} \approx 0$	${}^{\mathcal{S}}C_{AB} \approx 0$	${}^{\mathcal{T}}C \approx 0$	
Case 1	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0
Case 2	0	$\neq 0$	$\neq 0$	$\neq 0$	3
Case 3	$\neq 0$	0	$\neq 0$	$\neq 0$	3
Case 4	$\neq 0$	$\neq 0$	0	$\neq 0$	5
Case 5	$\neq 0$	$\neq 0$	$\neq 0$	0	1
Case 6	0	0	$\neq 0$	$\neq 0$	6
Case 7	$\neq 0$	0	0	$\neq 0$	8
Case 8	$\neq 0$	0	$\neq 0$	0	4
Case 9	0	$\neq 0$	0	0	9

Table 1. All possible non-trivial combinations of primary constraints in $f(\mathbb{T}_{\mathrm{NGR}})$. The last column denoted # contains the number of independent primary constraints which are incurred in case that the given combinations of theory parameters vanish.

more possibilities to set combinations of the parameters c_1, c_2 and c_3 to zero without fixing all of them to zero exist. This classifies all possible primary constraints in the $f(\mathbb{T}_{\rm NGR})$ class of theories.

As a final remark of this section we would like to display the constraints for the class of theories defined by the relation $(2c_1+c_2+c_3) = (2c_1-c_2) = 0$, which contains basically TEGR and $f(\mathbb{T})$ gravity, in the following compact form

$$C_{AB} = \pi_{[AB]} - \frac{1}{\kappa}\phi\sqrt{\gamma}n^D S_{[A|D|B]} \approx 0. \tag{61}$$

In TEGR ($\phi = 1$) these constraints represent the freedom of applying local Lorentz transformations only to the tetrad alone without considering a spin connection. An important remark is that such constraints slightly differ from the Lorentz constraints obtained in the tetrad formulation of GR [58, 59]. [b] Lorentz constraints from tetradic GR and TEGR are different since in TEGR there is an additional term, represented by the second term depending on the superpotential S in (61). This essential difference has not been noticed enough in the literature (probably only mentioned in [23] and [61]), but it can be understood by considering that the TEGR Lagrangian is pseudo-invariant under local Lorentz transformations. That is, the Lagrangian is modified by a four-divergence once we perform such transformations, which is integrated out once in the action. This fact has a great importance for $f(\mathbb{T})$ gravity, where the four-divergence is not integrated out and the Lorentz symmetry of the tetrads alone is partially or totally broken [61].

6. Dictionary relating the analysis from different authors

Using the general form of the primary constraints in $f(\mathbb{T}_{\rm NGR})$ theories introduced in the previous section, we can now relate the primary constraints obtained by other approaches and the more specific $f(\mathbb{T})$ and NGR classes of theories. We briefly summarize the key aspects of the discussed approaches in section 6.1. A detailed discussion is then given in section 6.2.

6.1. *Summary of methods used by different authors*

In Table 2 we summarize the notation used by different authors in the Hamiltonian formalism for TEGR, regarding the use of indices and definition of fields. Some of the formalisms introduced here have been applied for NGR and $f(\mathbb{T})$ gravity cases. The heading of each column denotes the surname of the first one of the authors (in alphabetic order), that represents a groups of independent references that encompass similar notation and analysis. We enumerate the different groups as follows

[b] However, it is possible to consider tetrad-based formulation of GR in the same sense as it has been done for TEGR. This was considered in [60].
[c] This reference essentially uses the same notation as Maluf.
[d] This reference essentially uses the same notation as Blagojevic.
[e] For the auxiliary field related to the spin connection the notion of its conjugate momenta is $\hat{\pi}^{AB}$.

	Blagojević	Blixt	Ferraro	Maluf	Okołów
TEGR $f(\mathbb{T})$ NGR	[16] [27] [33][d]	- - [30]	[23] [26] -	[19, 22] [25][c] -	[21, 39] - [29]
Spacetime indices	$\mu, \nu, \rho \dots$	$\mu, \nu, \rho \dots$	$\mu, \nu, \rho \dots$	$\mu, \nu, \rho \dots$	$\mu, \nu, \rho \dots$
Lorentz indices	$i, j, k \dots$	$A, B, C \dots$	$a, b, c \dots$	$a, b, c \dots$	$A, B, C \dots$
Time index	0	0	0	0	0
Spatial indices	-	$i, j, k \dots$	$i, j, k \dots$	$i, j, k \dots$	$i, j, k \dots$
Tetrad	ϑ^i	θ^A	E^a	e^a	$\boldsymbol{\theta}^A$
Cotetrad	e_i	e_A	e_a	e_a	-
det of tetrad	ϑ	$\lvert\theta\rvert$	E	e	-
Metric sign η_{00}	$+1$	-1	$+1$	-1	-1
Lorentz 3+1	Yes	No	No	No	No
Spin connection	ω^{ij}	$\omega^A{}_B$	0	0	0
Tetrad momenta	$\pi_i{}^\mu$	$\pi_A{}^i$ [e]	Π_a^μ	$\Pi^{a\mu}$	p_A
Scalar momenta	π_ϕ	-	π	-	-
NGR Coefficients	h_1, h_2, h_3	c_1, c_2, c_3	-	A, B, C	a_1, a_2, a_3
Lapse	N	α	-	-	N
Shift	N^α	β^i	-	-	$\vec{N}$
Induced metric	implicit	h_{ij}	-	-	q_{ij}
Normal vector	n_i	n^A	-	-	ξ^A

Table 2. Dictionary for notation of various authors (only the first author is given).

(1) **Blagojević et al.** (section 6.2.1): In [16] by M. Blagojević and I. A. Nikolic the full Hamiltonian analysis for TEGR was performed. The Hamiltonian analysis and constraint algebra for NGR has been calculated in detail by P. Mitric at [33], but only for the case with the least amount of constraints. This analysis has been applied for $f(\mathbb{T})$ gravity in [27] by M. Blagojević and J. M. Nester.

(2) **Blixt et al.** (section 6.2.2): In [30, 32] by D. Blixt, M. Hohmann, C. Pfeifer the Hamiltonian analysis for NGR has been introduced and primary constraints have been found for all possible cases. It was proved that the formalism is independent of the Weitzenböck choice in the connection. These two articles use a notation in tensor components; the same calculation is performed using differential form notation in [31], by M. Hohmann.

(3) **Ferraro et al.** (section 6.2.3): In [23] by R. Ferraro and M.J. Guzmán the Hamiltonian formalism for TEGR has been introduced in a premetric approach. For $f(\mathbb{T})$ gravity it has been studied in [26], some guidelines for $f(\mathbb{T})$ in the Einstein frame in [62], and the role of the pseudoinvariance in the Hamiltonian formalism in [61], by the same authors. The same notation has been used for the classification of primary constraints in NGR in [34] by M.J. Guzmán and Sh. Khaled-Ibraheem.

(4) **Maluf et al.** (section 6.2.4): There are several works of J. W. Maluf, J.F. da Rocha-Neto, A.A. Sousa and S. C. Ulhoa. on the Hamiltonian formalism of TEGR, some of them are: [13] Hamiltonian analysis of TEGR with gauge fixing conditions, with a time gauge [15], considering a null surface and no time gauge condition [14], without any condition of gauge fixing, [19] analysis of unimodular teleparallel gravity [20], and a review on TEGR which includes a summary on Hamiltonian formalism of TEGR [22]. The formalism introduced by J. W. Maluf et al. has been used for $f(\mathbb{T})$ gravity in the work of [25] by M. Li, R.X. Miao and Y.G. Miao.

(5) **Okołów** (section 6.2.5): Calculations are performed using the language of differential forms. In [29] by A. Okolów, a simple subcase of NGR is studied, where ${}^{\mathcal{I}}\mathcal{A} \neq 0$ for all $\mathcal{I}$ so that none of the primary constraints in equations (57)-(60). TEGR is considered in the articles [21, 39], by the same author. The Hamiltonian, primary constraints and their PB are calculated for all cases.

6.2. *Dictionary of primary constraints*

We now provide a detailed discussion and comparison of the primary constraints derived by different authors. These are Blagojević et al. in section 6.2.1, Blixt et al. in section 6.2.2, Ferraro et al. in section 6.2.3, Maluf et al. in section 6.2.4 and Okołów et al. in section 6.2.5.

6.2.1. *Blagojević*

In [16] the Hamiltonian analysis of new general relativity, [63], has been presented. In [17] the gauge symmetries of teleparallel gravity in the framework of Poincaré gauge gravity has been investigated. Finally, in [27] the Hamiltonian analysis of $f(\mathbb{T})$ gravity, [11], has been studied. The results of the investigations [16] and [27] are reproduced by our analysis in Section 5 with help of the following relations.

First, for the analysis of the constraints of $f(\mathbb{T})$ gravity the parameter choice (46) has to be employed. Under this condition the induction tensor (45) is directly related to the usual superpotential (35), $H_{ABC} = \frac{1}{2}S_{ABC}$.

Second, in [16] and [27] the following notation for the identification of the constraints from the irreducible decomposition (53) to (55) is employed. In the local cotetrad basis the normal vector to the hypersurfaces of the foliation of spacetime can be expanded as $n = n^A e_A$. This can be used to expand every vector V as

$$V = V^A e_A = V_\perp + \bar{V}\,, \tag{62}$$

where, with help of the projector $\Xi^A_B = (\delta^A_B - n_B n^A)$, we can write

$$V_\perp = V^A n_A\, n, \quad \bar{V} = V - V_\perp = V^B \Xi^A_B e_B = V^{\bar{A}} e_A\,, \tag{63}$$

i.e. barred indices are projected indices. Moreover $\sqrt{\gamma} = \frac{\theta}{\alpha} = J$ and $a_0 = \frac{1}{2\kappa}$. In addition in [16] and [27] the momenta with lower Lorentz indices are denoted by $\hat{\pi}_{AB}$.

With this translation it is easy to see the relations (53) to (56) become

$$n^A \hat{\pi}_{AB} = -2a_0 \phi J T^{\bar{A}}{}_{\bar{A}\bar{B}}\,, \tag{64}$$

$$\hat{\pi}_{[\bar{A}\bar{B}]} = a_0 \phi J T_{\perp \bar{A}\bar{B}}\,, \tag{65}$$

$$(\hat{\pi}_{(\bar{A}\bar{B})} - \frac{1}{3}(\eta_{AB} - n_A n_B)\hat{\pi}^C{}_C) = 2a_0\phi J \left(T_{(\bar{A}|\perp|\bar{B})} - \tfrac{(\eta_{AB}-n_A n_B)}{3} T^{\bar{C}}{}_{\perp\bar{C}}\right)\,, \tag{66}$$

$$\hat{\pi}^A{}_A = 4a_0\phi J T^{\bar{A}}{}_{\bar{A}\perp}\,. \tag{67}$$

The six equations (64) and (65) are primary constraints, since the right hand side of these relations do not contain any time derivatives of the tetrad, while the six equations (67) and (66) relate momenta and time derivatives of the tetrad. In a compact way the constraints can be captured with help of equation (51) or (61) as

$$C_{AB} := \hat{\pi}_{[AB]} - 4a_0\phi J H_{[A|\perp|B]} = \hat{\pi}_{[AB]} - 2a_0\phi J S_{[A|\perp|B]} = 0\,. \tag{68}$$

6.2.2. *Blixt*

In [30] the primary constraints for new general relativity are expressed in the irreducible decomposition under the rotation group. This also goes for the conjugate momenta which are decomposed as:

$$\pi_A{}^i = {}^{\mathcal{V}}\pi^i n_A + {}^{\mathcal{A}}\pi^{ji}\gamma_{kj}\theta_A{}^k + {}^{\mathcal{S}}\pi^{ji}\gamma_{kj}\theta_A{}^k + {}^{\mathcal{T}}\pi\theta_A{}^i. \tag{69}$$

This leads to primary constraints consistent with equation (53)-(56). They are expressed in spatial components instead of Lorentz components and explicitly written out in lapse and shift. Coupling to matter is not considered in their work and their coefficients $c_1,\ c_2,\ c_3$ differs from those in equation (45) by a factor $\frac{1}{2\kappa}$. Starting with the vector constraint satisfied in theories with $2c_1 + c_2 + c_3 = 0$ and $\phi = 1$ which brings (53) to

$$\frac{\kappa}{2\phi\sqrt{\gamma}} n^A \hat{\pi}_{AB} = \frac{c_3}{2} T^Q{}_{PD}\Xi^D_B \Xi^P_Q\,. \tag{70}$$

Multiplying this equation with $\theta^B{}_j\gamma^{ij}$ yields[f]

$$\begin{aligned}
0 &= \frac{\kappa}{2\sqrt{\gamma}} n^A \pi_A{}^\mu \theta^C{}_\mu \eta_{BC} \theta^B{}_j \gamma^{ij} - \frac{c_3}{2} T^Q{}_{PB} \theta^B{}_j \gamma^{ij} \left(\delta^P_Q - n_Q n^P\right) \\
&= \frac{\kappa}{2\sqrt{\gamma}} n^A \pi_A{}^k \theta^C{}_k \eta_{BC} \theta^B{}_j \gamma^{ij} - \frac{c_3}{2} T^P{}_{PB} \theta^B{}_j \gamma^{ij} + \frac{c_3}{2} T^Q{}_{PB} \theta^B{}_j \gamma^{ij} n_Q n^P \\
&= \frac{\kappa}{2\sqrt{\gamma}} {}^{\mathcal{V}}\pi^i + \frac{c_3}{2} T^B{}_{kl} \theta_B{}^l \gamma^{ik}.
\end{aligned} \tag{71}$$

The antisymmetric constraints are obtained for theories with $2c_1 + c_2 = 0$ and $\phi = 1$ which brings equation (54) to

$$\frac{\kappa}{2\phi\sqrt{\gamma}} \Xi^C_{[A}\Xi^D_{B]} \hat{\pi}_{CD} \theta^A{}_i \theta^B{}_j = \frac{c_2}{2} n^D T_{DPQ} \Xi^P_{[A}\Xi^Q_{B]} \theta^A{}_m \theta^B{}_p \gamma^{im}\gamma^{jp} \,. \tag{72}$$

Further we multiply this with $\theta^A{}_m\theta^B{}_p\gamma^{im}\gamma^{jp}$ so that

$$\begin{aligned}
0 &= \frac{\kappa}{4\sqrt{\gamma}} \pi_A{}^k \theta^E{}_k \eta_{BE} \theta^A{}_m \theta^B{}_p \gamma^{im}\gamma^{jp} - \frac{\kappa}{4\sqrt{\gamma}} \pi_B{}^k \theta^E{}_k \eta_{AE} \theta^A{}_m \theta^B{}_p \gamma^{im}\gamma^{jp} \\
&- \frac{c_2}{4} n_D \left(T^D{}_{AB} - T^D{}_{BA}\right) \theta^A{}_m \theta^B{}_p \gamma^{im}\gamma^{jp} \\
&= \frac{\kappa}{2\sqrt{\gamma}} {}^{\mathcal{A}}\pi^{ji} - \frac{c_2}{2} \gamma^{ik}\gamma^{jl} T^B{}_{kl} n_B.
\end{aligned} \tag{73}$$

If the theory satisfies $2c_1 - c_2 = 0$ and $\phi = 1$ equation (55) reduces to

$$\frac{\kappa}{2\sqrt{\gamma}} (\Xi^C_{(A}\Xi^D_{B)} \hat{\pi}_{CD} - \frac{1}{3}(\eta_{AB} - n_A n_B) \hat{\pi}^C{}_C) = 0 \,. \tag{74}$$

Similar to the antisymmetric constraints we multiply this with $\theta^A{}_m\theta^B{}_p\gamma^{im}\gamma^{jp}$ and get

$$\begin{aligned}
0 &= \frac{\kappa}{4\sqrt{\gamma}} \pi_A{}^k \theta^E{}_k \eta_{BE} \theta^A{}_m \theta^B{}_p \gamma^{im}\gamma^{jp} + \frac{\kappa}{4\sqrt{\gamma}} \pi_B{}^k \theta^E{}_k \eta_{AE} \theta^A{}_m \theta^B{}_p \gamma^{im}\gamma^{jp} \\
&- \frac{\kappa}{6\sqrt{\gamma}} \pi_C{}^C \gamma^{ij} \\
&= \frac{\kappa \pi^{(ji)}}{2\sqrt{\gamma}} - \frac{\kappa\, {}^{\mathcal{T}}\pi}{2\sqrt{\gamma}} \gamma^{ij} \\
&= \frac{\kappa}{2} \cdot \frac{{}^{\mathcal{S}}\pi^{ij}}{\sqrt{\gamma}}.
\end{aligned} \tag{75}$$

If $2c_1 + c_2 + 3c_3$ and $\phi = 1$ are satisfied equation (56) reads

$$\frac{\kappa}{2\phi\sqrt{\gamma}} \eta^{AB} \Xi^C_A \Xi^D_B \hat{\pi}_{CD} = \frac{\kappa}{2\phi\sqrt{\gamma}} \hat{\pi}^A{}_A = 0 \,. \tag{76}$$

[f] Note that the relative sign of this constraint differ from [30]. They have a different sign convention but equation (69) is defined in the same way. When the irreducible vector part then is contracted with n^A, this particular term will have the opposite sign compared to those in [30]. Note that we could have defined ${}^{\mathcal{V}}\pi^i$ with the opposite sign to make the expression look like the one in [30].

Expanding this notion from the definition gives

$$
\begin{aligned}
0 &= \frac{\kappa}{2\sqrt{\gamma}} \pi_A{}^\mu \theta^C{}_\mu \eta^{AB} \eta_{BC} = \frac{\kappa}{2} \frac{\pi_A{}^A}{\sqrt{\gamma}} = \frac{\kappa}{2} \frac{\pi_\mu{}^\mu}{\sqrt{\gamma}} = \frac{\kappa}{2} \frac{\pi_i{}^i}{\sqrt{\gamma}} \\
&= \frac{3\kappa}{2} \frac{{}^{\mathcal{T}}\pi}{\sqrt{\gamma}},
\end{aligned}
\tag{77}
$$

All of the above constraints can now easily be seen to be consistent with [30].

6.2.3. *Ferraro*

We will present the primary constraints for $f(\mathbb{T})$ gravity as presented in [26]. The Jordan frame representation of $f(\mathbb{T})$ gravity has been taken as a starting point, with the Lagrangian

$$
L_{\mathrm{f}} = \theta[\phi \mathbb{T} - V(\phi)]. \tag{78}
$$

This differs from (47) only in the gravitational constant factor $\frac{1}{2\kappa}$, which appears in the action instead of the Lagrangian. The torsion scalar is rewritten as

$$
\mathbb{T} = \theta \partial_\mu \theta^A{}_\nu \partial_\rho \theta^B{}_\lambda e_C{}^\mu e_E{}^\nu e_D{}^\rho e_F{}^\lambda \chi_{AB}{}^{CEDF}, \tag{79}
$$

where the object $\chi_{AB}{}^{CEDF}$ is the constitutive tensor, a mathematical object depending only on the components of the Minkowski metric and Kronecker deltas. Although in [26] it was only considered the generalization of TEGR, it is possible to write this object for the most general NGR case as [34]

$$
\chi_{AB}{}^{CEDF} = 4c_1 \eta_{AB} \eta^{C[D} \eta^{F]E} - 4c_2 \delta_A^{[D} \eta^{F][C} \delta_B^{E]} + 4c_3 \delta_A^{[C} \eta^{E][D} \delta_B^{F]}, \tag{80}
$$

where the particular TEGR case is obtained for the values of the c_i presented in Eq. (46).

The canonical momenta are defined as

$$
\pi_A{}^\mu = \frac{\partial L}{\partial \partial_0 \theta^A{}_\mu} = \phi \theta \partial_\rho \theta^B{}_\lambda e_C{}^0 e_E{}^\mu e_D{}^\rho e_F{}^\lambda \chi_{AB}{}^{CEDF}, \tag{81}
$$

where we remark that the ADM decomposition has not been used for the tetrad. The following trivial constraints appear

$$
C_A = \pi_A{}^0 \approx 0. \tag{82}
$$

These can be extracted from (81) by noticing that for $\mu = 0$, it appears the pair $e_C{}^0 e_E{}^0$ which is symmetric in CE, but is multiplied by the constitutive tensor $\chi_{AB}{}^{CEDF}$ which is antisymmetric on such indices. These constraints are equivalent to (52).

In Ref. [26] it has been proposed an alternative way of obtaining these primary constraints in terms of the kernel of the Hessian matrix $C_{AB}{}^{EF} = e_C{}^0 e_D{}^0 \chi_{AB}{}^{CEDF}$. The canonical momenta in (81) is rewritten in terms of $C_{AB}{}^{EF}$ as

$$
\pi_A{}^\mu \theta^E{}_\mu = \theta C_{AB}{}^{EF} e_F{}^\lambda \dot{\theta}^B{}_\lambda + \theta \partial_i \theta^B{}_\lambda e_C{}^0 e_D{}^i e_F{}^\lambda \chi_{AB}{}^{CEDF}. \tag{83}
$$

By noticing that $e_E{}^0 \delta_G^A$ lies in the kernel of the Hessian, that is

$$
e_E{}^0 \delta_G^A C_{AB}{}^{EF} = e_E{}^0 e_C{}^0 e_D{}^0 \delta_G^A \chi_{AB}{}^{CEDF} = 0, \tag{84}
$$

we obtain the primary constraints $C_A \approx 0$. Notice that in Ref. [26] uppercase Latin indices have been used to define the superindices (which denote pairs of Lorentz indices). In order to avoid confusion, we have omitted their use in this review.

Lorentz constraints can be obtained by an additional set of vectors in the kernel given by $2\delta^A_{[G}\eta_{H]E}$. We can prove that these lie in the kernel by calculating

$$2\delta^A_{[G}\eta_{H]E}C_{AB}{}^{EF} = 2\delta^A_{[G}\eta_{H]E}e_C{}^0 e_D{}^0 \chi_{AB}{}^{CEDF}. \tag{85}$$

We obtain the following contraction of the constitutive tensor

$$\delta^A_{[G}\eta_{H]E}\chi_{AB}{}^{CEDF} = \eta_{E[H}\chi_{G]B}{}^{CEDF} = -2\delta^{CDF}_{GHB}. \tag{86}$$

The triple totally antisymmetrized Kronecker delta δ^{GHF}_{CAB} has been defined in [23, Equation (A5)] as

$$-\delta^{GHF}_{CAB} = \delta^H_{[A}\delta^F_{C]}\delta^G_B + \delta^G_{[A}\delta^H_{C]}\delta^F_B + \delta^F_{[A}\delta^G_{C]}\delta^H_B. \tag{87}$$

Here notice that the triple antisymmetrization has not been defined with the conventional $1/3$ factor. However, antisymmetrization of pairs of indices is defined as usual, that is

$$V_{[AB]} = \frac{1}{2}\left(V_{AB} - V_{BA}\right). \tag{88}$$

Taking into account these subtleties, it is found the primary constraints

$$C_{AB} = 2\eta_{E[B}\pi_{A]}{}^i\theta^E{}_i + 4\theta\partial_i\theta^C{}_j\left(e_{[B}{}^0 e_{A]}{}^i e_C{}^j + e_{[B}{}^i e_{A]}{}^j e_C{}^0 + e_{[B}{}^j e_{A]}{}^0 e_C{}^i\right). \tag{89}$$

These constraints can alternatively be written as

$$C_{AB} = \pi_{AB} - \pi_{BA} - 2\phi\theta[\theta_A{}^i\theta_B{}^j T^0{}_{ij} - (\theta_A{}^i\theta_B{}^0 - \theta_B{}^i\theta_A{}^0)T^j{}_{ij}]. \tag{90}$$

This expression is found to be consistent with equation (61), as we will explicitly demonstrate in the next section 6.2.4.

6.2.4. *Maluf*

In this section we will present the primary constraints of the Hamiltonian formalism in $f(\mathbb{T})$ gravity performed by Li, Miao, Miao [25]. Their analysis heavily relies on the Hamiltonian approach to TEGR done by W. Maluf [22], so the name of this subsection. The authors [25] start from the following action

$$S = -\int d^4x\theta f(\mathbb{T}) \tag{91}$$

where $G = \frac{1}{16\pi}$ was taken. After passing to the equivalent scalar-torsion form (47) and considering the form of the torsion scalar $\mathbb{T} = T_{ABC}\Sigma^{ABC} = \frac{1}{2}T_{ABC}S^{ABC}$[g] identifies

[g] In [22] there is an overall factor $k = \frac{1}{2c\kappa}$, also the matter is rescaled compared to our convention by a factor $\frac{1}{c}$.

the Σ^{ABC} as half the superpotential S^{ABC}. Note that in the convention of this article this action is recovered by choosing $\kappa = 1$. We find that the momentum is

$$\pi^{A\mu} = \frac{\partial L}{\partial \partial_0 e_{A\mu}} = -4\phi\theta\Sigma^{A0\mu}, \tag{92}$$

which is consistent with equation (61)[h]. In this expression notice that the first index in $\Sigma^{A0\mu}$ is Lorentzian and the next two are spacetime ones. Notice that it is considered $\Sigma^{ABC} = H^{ABC}$ like in (45), but using the TEGR coefficients. After removing the trivial primary constraints $\pi^{A0} \approx 0$ from (92), and considering that the object $S^{A0i} = S^{ABC} e_B{}^0 e_C{}^i$ remains, one can obtain the following primary constraints

$$C^{AB} = \pi^{AB} - \pi^{BA} + 2\phi\theta\left[\theta^{Am}\theta^{Bj}T^0{}_{mj} - (\theta^{Am}\theta^{B0} - \theta^{Bm}\theta^{A0})T^j{}_{mj}\right] \tag{93}$$

In the works of Maluf et al. [22] it can be found another form for these constraints (for TEGR, but they can be easily generalized for $f(\mathbb{T})$): [i]

$$C^{AB} = -C^{BA} = \pi^{[AB]} = \pi^{AB} - \pi^{BA} + 4k\phi\theta(\Sigma^{A0B} - \Sigma^{B0A}). \tag{94}$$

We can prove that the term $\Sigma^{A0B} - \Sigma^{B0A}$ gives indeed the torsion components appearing in (93) as follows.

$$\Sigma^{A0B} - \Sigma^{B0A} = \frac{1}{2}\left(T^0{}_{\mu\nu}\theta^{A\mu}\theta^{B\nu} - \theta^{A0}T^{\mu}_{\ \mu}{}^{B} + \theta^{B0}T^{\mu}_{\ \mu}{}^{A}\right) \tag{95}$$

The first term in the previous expression correspond to

$$T^0{}_{\mu\nu}\theta^{A\mu}\theta^{B\nu} = T^0{}_{0i}\theta^{Bi}\theta^{A0} + T^0{}_{i0}\theta^{B0}\theta^{Ai} + T^0{}_{ij}\theta^{Ai}\theta^{Bj}, \tag{96}$$

while the remaining terms can be worked as

$$\begin{aligned} -\theta^{A0}T^{\mu}_{\ \mu}{}^{B} + \theta^{B0}T^{\mu}_{\ \mu}{}^{A} &= -\theta^{A0}\theta^{B\nu}T^0{}_{0\nu} + \theta^{B0}\theta^{A\nu}T^0{}_{0\nu} - \theta^{A0}\theta^{B\nu}T^i{}_{i\nu} + \theta^{B0}\theta^{A\nu}T^i{}_{i\nu} \\ &= -\theta^{A0}\theta^{Bi}T^0{}_{0i} + \theta^{B0}\theta^{Ai}T^0{}_{0i} - \theta^{A0}\theta^{B0}T^k{}_{k0} - \theta^{A0}\theta^{Bi}T^k{}_{ki} \\ &\quad + \theta^{B0}\theta^{A0}T^k{}_{k0} + \theta^{B0}\theta^{Ai}T^k{}_{ki}. \end{aligned} \tag{97}$$

By combining (96) and (97) in (95) we recover the form (93) for the Lorentz constraints.

6.2.5. *Okołów*

The action is written in the form

$$S = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}\theta^A \wedge \star\left(\sum_{i=1}^{3} a_i\, \mathrm{d}\theta^{(i)}_A\right) = \int_{\mathcal{M}} \mathrm{d}t \wedge L_\perp\,, \tag{98}$$

where $\star$ is the Hodge star of the spacetime metric g, $\mathcal{M} = \mathbb{R}\times\Sigma$ is the spacetime manifold and $L_\perp$ is a differential three-form. It depends on three constants $a_{1,2,3}$ which determine a

[h]Note that [25] as well as [22] use the opposite metric sign convention $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ which must be taken into account when contracting n^D in equation (61)

[i]Note that [22] has a different convention for antisymmetrization brackets compared to this article. Their convention is $\pi^{[AB]} = \pi^{AB} - \pi^{BA}$.

particular choice of the NGR Lagrangian. The three terms appearing in the action are the tensor, vector and axial torsion components

$$\mathrm{d}\theta_A^{(1)} = \mathrm{d}\theta_A - \mathrm{d}\theta_A^{(2)} - \mathrm{d}\theta_A^{(3)}\,, \tag{99a}$$

$$\mathrm{d}\theta_A^{(2)} = \frac{1}{3}\theta_A \wedge (e_B \lrcorner \mathrm{d}\theta^B)\,, \tag{99b}$$

$$\mathrm{d}\theta_A^{(3)} = \frac{1}{3}e_A \lrcorner (\theta_B \wedge \mathrm{d}\theta^B)\,, \tag{99c}$$

where the Weitzenböck gauge is assumed, so that no spin connection appears. Writing the torsion in the tetrad basis in the form

$$\mathrm{d}\theta_A = T_A = \frac{1}{2}T_{ABC}\theta^B \wedge \theta^C\,, \tag{100}$$

the three terms are related to the torsion components by

$$\mathrm{d}\theta_A^{(1)} = \frac{1}{2}\left(T_{ABC} - \frac{2}{3}\eta_{AB}T^D{}_{DC} - T_{[ABC]}\right)\theta^B \wedge \theta^C\,, \tag{101a}$$

$$\mathrm{d}\theta_A^{(2)} = \frac{1}{3}\eta_{AB}T^D{}_{DC}\theta^B \wedge \theta^C\,, \tag{101b}$$

$$\mathrm{d}\theta_A^{(3)} = \frac{1}{2}T_{[ABC]}\theta^B \wedge \theta^C\,. \tag{101c}$$

Further using the expressions for the Hodge star given by

$$\star 1 = \frac{1}{4!}\epsilon_{ABCD}\theta^A \wedge \theta^B \wedge \theta^C \wedge \theta^D = \theta\, \mathrm{d}^4x\,, \tag{102a}$$

$$\star(\theta^A \wedge \theta^B) = \frac{1}{2}\epsilon^{ABCD}\theta_C \wedge \theta_D\,, \tag{102b}$$

from which follows the inner product

$$\theta^A \wedge \theta^B \wedge \star(\theta^C \wedge \theta^D) = 2\eta^{A[C}\eta^{D]B} \star 1\,, \tag{103}$$

the three terms in the action are found to be

$$\mathrm{d}\theta^A \wedge \star\mathrm{d}\theta_A^{(1)} = \frac{1}{3}(T^{ABC}T_{ABC} + T^{ABC}T_{CBA} - T_A{}^{AC}T^B{}_{BC}) \star 1\,, \tag{104a}$$

$$\mathrm{d}\theta^A \wedge \star\mathrm{d}\theta_A^{(2)} = \frac{1}{3}T_A{}^{AC}T^B{}_{BC} \star 1\,, \tag{104b}$$

$$\mathrm{d}\theta^A \wedge \star\mathrm{d}\theta_A^{(3)} = \frac{1}{6}T^{ABC}(T_{ABC} - 2T_{CBA}) \star 1\,. \tag{104c}$$

Hence, by comparing with the general Lagrangian (45), one finds that the constants $a_{1,2,3}$ are related to $c_{1,2,3}$ by

$$c_1 = \frac{1}{6}(2a_1 + a_3)\,, \quad c_2 = \frac{1}{3}(a_1 - a_3)\,, \quad c_3 = \frac{1}{3}(a_2 - a_1)\,, \tag{105}$$

or equivalently,

$$a_1 = 2c_1 + c_2\,, \quad a_2 = 2c_1 + c_2 + 3c_3\,, \quad a_3 = 2c_1 - 2c_2\,, \tag{106}$$

where in addition the convention $\kappa \equiv -1$ for the value of the gravitational constant and the sign of the action are used. Setting $a_1 = a_2 = a_3 = 1$ then yields the toy model studied in [29], while for $a_1 = 1, a_2 = -2, a_3 = -1/2$ one obtains TEGR [21, 39]. They are related to the constants in [31] by $a_1 = -2C_T, a_2 = -2C_V, a_3 = -2C_A$.

The tetrad one-form is then split in the form

$$\theta^A = \theta^A{}_\mu \mathrm{d}x^\mu = \theta^A{}_0 \mathrm{d}t + \theta^A{}_i \mathrm{d}x^i = \theta^A_\perp \mathrm{d}t + \vec{\theta}^A \,. \tag{107}$$

Time derivatives are defined as

$$\dot{\theta}^A = \mathcal{L}_{\partial_t} \theta^A \,. \tag{108}$$

One finds that only time derivatives of the spatial tetrad components $\vec{\theta}^A$ appear in the Lagrangian $L_\perp$, but not of the time components $\theta^A_\perp$. Thus, the canonical momenta are introduced only for the spatial tetrad components, since they would be vanishing for the time components. They are defined as the differential two-forms π_A by

$$\delta_{\dot{\theta}} L_\perp = \delta \dot{\vec{\theta}}^A \wedge \pi_A \,, \tag{109}$$

which are related to the momenta $\pi_A{}^\mu$ in the definition (48) by

$$\pi_A{}^\mu \, \mathrm{d}^4x = \mathrm{d}t \wedge \mathrm{d}x^\mu \wedge \pi_A \,. \tag{110}$$

Explicitly, they are given by

$$\begin{aligned}\pi_A = \frac{1}{3\alpha}\Big\{&(2a_1 + a_2) * \left[\dot{\vec{\theta}}_A - \mathrm{d}(\alpha n_A) - \mathcal{L}_{\vec{\beta}} \vec{\theta}_A\right] \\ &+ (a_1 - a_2)\vec{\theta}_B \wedge * \left(\dot{\vec{\theta}}^B \wedge \vec{\theta}_A + E^B{}_A\right) + (a_3 - a_1)\vec{\theta}_A \wedge * \left(\dot{\vec{\theta}}^B \wedge \vec{\theta}_B + E^B{}_B\right)\Big\}\end{aligned} \tag{111}$$

for general NGR [31], using the Hodge star $*$ of the induced metric γ on the spatial hypersurfaces, as well as the abbreviation

$$E^B{}_A = -\mathrm{d}(\alpha n^B) \wedge \vec{\theta}_A + \alpha n_A \mathrm{d}\vec{\theta}^B - (\mathcal{L}_{\vec{\beta}} \vec{\theta}^B) \wedge \vec{\theta}_A \,. \tag{112}$$

Rewriting this in the parameters $c_{1,2,3}$, we thus have

$$\begin{aligned}\pi_A = \frac{1}{\alpha}\Big\{&(2c_1 + c_2 + c_3) * \left[\dot{\vec{\theta}}_A - \mathrm{d}(\alpha n_A) - \mathcal{L}_{\vec{\beta}} \vec{\theta}_A\right] \\ &- c_3 \vec{\theta}_B \wedge * \left(\dot{\vec{\theta}}^B \wedge \vec{\theta}_A + E^B{}_A\right) - c_2 \vec{\theta}_A \wedge * \left(\dot{\vec{\theta}}^B \wedge \vec{\theta}_B + E^B{}_B\right)\Big\}.\end{aligned} \tag{113}$$

With the particular choices of the three constant parameters one thus obtains

$$\pi_A = \frac{1}{\alpha} * \left[\dot{\vec{\theta}}_A - \mathrm{d}(\alpha n_A) - \mathcal{L}_{\vec{\beta}} \vec{\theta}_A\right] \tag{114}$$

for the toy model studied in [29], as well as

$$\pi_A = \frac{1}{\alpha}\left[\vec{\theta}_B \wedge * \left(\dot{\vec{\theta}}^B \wedge \vec{\theta}_A + E^B{}_A\right) - \frac{1}{2}\vec{\theta}_A \wedge * \left(\dot{\vec{\theta}}^B \wedge \vec{\theta}_B + E^B{}_B\right)\right] \tag{115}$$

for TEGR [21, 39]. Depending on the choice of the constant parameters a_i, the following primary constraints may appear, which are given by

$$ {}^{\mathcal{V}}C_A = * {}^{\mathcal{V}}\pi_A + \frac{1}{3}(a_1 - a_2) n_A \vec{\theta}^{\sharp}_B \lrcorner \mathrm{d}\vec{\theta}^B \,, \tag{116a} $$

$$ {}^{\mathcal{A}}C_A = * {}^{\mathcal{A}}\pi_A + \frac{1}{3}(a_3 - a_1) \vec{\theta}^{\sharp}_A \lrcorner (\vec{\theta}^B \wedge \mathrm{d}n_B) \,, \tag{116b} $$

$$ {}^{\mathcal{S}}C_A = * {}^{\mathcal{S}}\pi_A \,, \tag{116c} $$

$$ {}^{\mathcal{T}}C_A = * {}^{\mathcal{T}}\pi_A \,, \tag{116d} $$

where we made use of the musical isomorphism, which for a one-form τ_A yields the vector field

$$ \tau^{\sharp}_A = \gamma^{-1}(\cdot, \tau_A) \tag{117} $$

and the irreducible decomposition introduced in section 5, which acts on one-forms as

$$ {}^{\mathcal{V}}\tau_A = -n_A n^B \tau_B \,, \tag{118a} $$

$$ {}^{\mathcal{A}}\tau_A = \frac{1}{2}\left[\gamma^{-1}(\vec{\theta}_A, \vec{\theta}^B)\tau_B - \gamma^{-1}(\vec{\theta}_A, \tau_B)\vec{\theta}^B\right] \,, \tag{118b} $$

$$ {}^{\mathcal{S}}\tau_A = \frac{1}{2}\left[\gamma^{-1}(\vec{\theta}_A, \vec{\theta}^B)\tau_B + \gamma^{-1}(\vec{\theta}_A, \tau_B)\vec{\theta}^B\right] - \frac{1}{3}\gamma^{-1}(\vec{\theta}^B, \tau_B)\vec{\theta}_A \,, \tag{118c} $$

$$ {}^{\mathcal{T}}\tau_A = \frac{1}{3}\gamma^{-1}(\vec{\theta}^B, \tau_B)\vec{\theta}_A \,, \tag{118d} $$

is extended to the momentum two-forms as $* {}^{\bullet}\pi_A = {}^{\bullet}(*\pi_A)$.

7. Discussion

An essential point in the discussion of the Hamiltonian formalism of modified teleparallel gravities is the correct implementation of the Dirac-Bergmann algorithm. Two crucial steps in the algorithm that should be taken with care are: (i) that the Hessian could have variable rank, which we discuss in section 7.1, and (ii) that the matrix of PB among constraints has variable rank, as discussed in section 7.2. An important conclusion one can draw from the Hamiltonian analysis is the number of degrees of freedom which, when compared with the outcome of perturbation theory, may reveal eventual strongly coupled fields; this will be discussed in section 7.3.

7.1. *The Hessian in modified teleparallel gravities*

A caveat of concern in the proper application of the Dirac-Bergmann algorithm occurs when we are in presence of a Hessian that can have variable rank once evaluated in the constraint surface [64]. The Hessian for our classical finite-dimensional system

$$ W_{ij} = \frac{\partial^2 L(q^k, \dot{q}^k)}{\partial \dot{q}^i \partial \dot{q}^j}, \tag{119} $$

can be generalized for a field theory dependent on the tetrad as the following tensor

$$W_{AB}{}^{\mu\nu} = \frac{\partial^2 L(\theta^C{}_\lambda, \dot{\theta}^C{}_\lambda)}{\partial\partial_0\theta^A{}_\mu \partial\partial_0\theta^B{}_\nu}, \tag{120}$$

whose expression has mixed Lorentz and spacetime indices. The full Hessian for NGR was presented in [34] as

$$W_{AB}{}^{\mu\nu} = \theta e_C{}^0 e_D{}^0 e_E{}^\mu e_F{}^\nu \chi_{AB}{}^{CEDF} = e_E{}^\mu e_F{}^\nu \tilde{W}_{AB}{}^{EF}, \tag{121}$$

where the expression $\tilde{W}_{AB}{}^{EF} = e_C{}^0 e_D{}^0 \chi_{AB}{}^{CEDF}$ has been explicitly written in matrix form in [34, Appendix A]. We notice that the NGR Hessian is linearly dependent on a quartic combination of cotetrad components $e_A{}^\mu$. It is reasonable to assume that in the NGR case such expression does not vanish on the constraint surface. This is because no dynamical terms for the tetrad or cotetrad appear there, therefore no constraints can show up. Therefore, the rank of the Hessian in NGR remains constant.

We notice that the Hessian (121) for NGR can be easily extended to the $f(\mathbb{T}_{\text{NGR}})$ gravity case by following equation (47) and realizing that the $f(\mathbb{T}_{\text{NGR}})$ Hessian is just the NGR Hessian multiplied by the auxiliary scalar field ϕ. For $f(\mathbb{T}_{\text{NGR}})$ the fundamental fields are the tetrads and the scalar field ϕ, however the Lagrangian does not contain any derivatives of ϕ. Hence the components of the Hessian emerging through the extra scalar field are zero, since $\partial_{\dot{\phi}} L = 0$. Therefore, the nonvanishing components of the $f(\mathbb{T}_{\text{NGR}})$ Hessian are obtained from (121) by multiplication with ϕ.

Note that ϕ is a field whose value in principle can be zero. This can impose a vanishing Hessian for $f(\mathbb{T}_{\text{NGR}})$ as well, which would also be the case for $f(\mathbb{T})$ gravity. This unlikely case might need an special considerations in the Dirac-Bergmann algorithm [64].

7.2. *Matrix of PB among constraints with variable rank*

There are several indications that modified teleparallel theories suffer from variable rank on their matrix of PB among constraints. Firstly, it was found that in the one-parameter teleparallel gravity model [28] a field-dependent PB among constraints exists, which would change its value for certain tetrad configurations. The authors suggest that this could be a generic feature of teleparallel theories. Later, for Poincaré gauge gravity similar results were found [65–68]. For $f(\mathbb{T})$ gravity, such findings appear throughout the literature but probably their impact has not been stressed enough, as for instance in [25–27]. There is an ongoing discussion in the literature regarding the physical number of degrees of freedom in $f(\mathbb{T})$, and although a better understanding on the constraint structure is needed, it is believed that in the most general case there are five degrees of freedom. A controversial point is what happens when the scalar field ϕ does not depend on the spatial hypersurface coordinates, for example $\phi = \phi(x^0)$ only. In this case the PB of the Lorentz constraints is weakly zero, and there is some apparent recovery of the invariance of the theory under pure tetrad Lorentz transformations. However, there could still be room for other pieces of the algebra to be different from zero, since some Lorentz constraints could become second class due to noncommuting PB with additional primary constraints [26,62]. Other evidence pointing towards three d.o.f. in this subcase comes from perturbative analysis [69, 70].

The results do not immediately contradict each other, since more than one jump in the rank of the PB matrix among constraints could be possible. Further work to clarify the aforementioned discussion is motivated in order to resolve controversial points stated in the literature for $f(\mathbb{T})$.

7.3. *Possible conclusions from perturbation theory*

Although the Hamiltonian analysis provides solutions for the complete non-linear theory, perturbation theory becomes very important not only for doing practical calculation, but also for understanding the properties of the fundamental physical fields. In particular applying perturbation theory around different backgrounds might reveal an issue with strongly coupled fields, which invalidates the perturbation theory around this particular background. Another important piece of information perturbation theory might give us is a consistency check of the Hamiltonian analysis. We do not expect more modes at the perturbative level than at the nonlinear level.

To summarize the work that has been done for the main teleparallel gravity theories discussed in the literature (that is TEGR, $f(\mathbb{T})$ and NGR) we collect results for perturbations around several backgrounds. In the end of the section we list the main conclusions in a compact way firstly for linear perturbations and then for higher order.

Starting with TEGR, the second order (lowest order) perturbations around a Minkowski background is consistent with linearized general relativity, see Ref. [71] in Section 4.6. These perturbations are known to be those of a massless spin-2 field and consists of two degrees of freedom. Furthermore, many works agree, through a Hamiltonian analysis, that the full nonlinear theory propagate two degrees of freedom [16, 19, 23]. Hence, it is expected that exactly two degrees of freedom propagate for TEGR in any background, which is explicitly observed for Minkowski and flat FLRW backgrounds [72].

For $f(\mathbb{T})$ the lowest order perturbations are very reminiscent to those of TEGR around diagonal tetrads for Minkowski spacetime [72–75] and, hence, two modes propagate around these backgrounds. However, it was found in [70] that there are non-diagonal tetrads representing Minkowski spacetime, still consistent with the vanishing spin connection, which exhibit non-trivial dynamics. In [69] hints towards extra modes at 4th order around Minkowski backgrounds were found, which indicates the presence of strongly coupled fields. Cosmological perturbations around a diagonal tetrad for flat FLRW cosmology do not exhibit additional modes [72] , and the same result is obtained when considering the spin connection [75, 76]. The Hamiltonian analysis suggest that more than two degrees of freedom propagate at the full nonlinear level [25–27] even though their conclusions for the case $\partial_i\phi = 0$ are inconsistent (the claims are that in this case there would be two [27] or three of them [26]). There is evidence that for the most general case $\partial_i\phi \neq 0$ the number of propagating degrees of freedom is five.

For the one-parameter teleparallel gravity, which is a particular subcase of NGR, it was found in [36] that, by considering cubic interactions, the gauge symmetry in the linear theory required to avoid ghostly modes [71] cannot be extended to higher orders. So far, up to our knowledge, there are no works studying perturbations for NGR around backgrounds other than Minkowski. The Hamiltonian analysis of the one-parameter teleparallel gravity has only been presented in [28] (references therein present full calculations) where their claim is that the number and type of constraints depends on special values of the dynamical variables, and "conditional bifurcations" of the constraint algorithm appear.

Below we summarize the conclusions for the aforementioned theories around Minkowski, FLRW, and general backgrounds:

(1) Around Minkowski spacetime, TEGR recovers linearized general relativity as expected [71]. The propagating degrees of freedom for $f(\mathbb{T})$ is disputed [75]. In addition to the massless spin-2 field, the one-parameter parameter family of NGR propagates a massless Kalb-Ramond field [36, 71].
(2) Around flat FLRW spacetime, $f(\mathbb{T})$ does not seem to propagate any additional degrees of freedom, whilst the TEGR limit is consistent with general relativity [72]. To our knowledge no work have considered perturbations around FLRW backgrounds for NGR.
(3) TEGR is expected to have 2 degrees of freedom around general backgrounds, since the Hamiltonian analysis consistently gives 2 propagating degrees of freedom at the non-linear level [16, 19, 23]. To our knowledge perturbations around general backgrounds have not been considered in the literature for teleparallel theories of gravity.

Furthermore, the conclusion for higher order perturbations are listed below:

(1) TEGR is expected to propagate 2 degrees of freedom at all orders since the Hamiltonian analysis gives the consistent value of 2 degrees of freedom at the non-linear level [16, 19, 23].
(2) Higher order perturbations for $f(\mathbb{T})$ around Minkowski backgrounds were considered in [70] and they found indications of extra strongly coupled modes appearing at higher order.
(3) Cubic interactions of the one-parameter family of NGR-theories around Minkowski backgrounds were considered in [36]. They found the Kalb-Ramond field to be strongly coupled around those backgrounds.

8. Summary and Outlook

We have provided a review over the current understanding of the Hamiltonian analysis in teleparallel theories of gravity. Firstly we have derived the primary constraints for $f(\mathbb{T}_{\mathrm{NGR}})$ gravity, from which the derivation of the primary constraints for $f(\mathbb{T})$ and new general relativity is straightforward. Taking this theory as reference, we present a table from which the reader can compare five different classes of notation and conventions [22, 26, 27, 29, 30]. We show that the primary constraints in these works are all consistent among them, and with the primary constraints we derived for $f(\mathbb{T}_{\mathrm{NGR}})$. This also holds for [25, 31, 34] which have been using similar notation as the aforementioned references. We have also provided valuable discussion on important aspects relevant for the Hamiltonian analysis in tetrad-based teleparallel theories of gravity, as for instance the different fundamental fields considered, the choice of the spin connection, and the necessity of the ADM decomposition.

We also discuss possible difficulties and misinterpretations in the application of the Dirac-Bergmann algorithm regarding the change in the rank of the Hessian matrix and of the matrix of PB among constraints, both of them appearing for some particular subcases of modified teleparallel gravities.

We observe that our parent theory $f(\mathbb{T}_{\rm NGR})$ possess $16+1$ canonical variables represented by the components of the tetrad $\theta^{A}{}_{\mu}$ and an auxiliar scalar field ϕ. The theory is universally endowed with $4+1$ primary constraints $\pi_{A}{}^{0}\approx 0$ and $\pi_{\phi}\approx 0$, the five of them being also present in $f(\mathbb{T})$ gravity. Meanwhile, for TEGR and NGR the auxiliary scalar field ϕ does not appear, the theories possess only 16 canonical variables, and only the first four constraints apply. We have performed a decomposition in the remaining 12 components of the momenta obtaining nine possible cases with different number of *if*-constraints, in agreement with nine combinations of vanishing some of the coefficients ${}^{\mathcal{I}}\mathcal{A}$ (presented in table 1). Notice that the structure of cases and their number of primary constraints is identical for both $f(\mathbb{T}_{\rm NGR})$ and NGR, but due to the appearance of ϕ in the primary constraints of $f(\mathbb{T}_{\rm NGR})$, it could be expected that the constraint structure of this theory is much more intricate than that of NGR. Finally, for the particular case ${}^{\mathcal{V}}\mathcal{A}=0$, ${}^{\mathcal{A}}\mathcal{A}=0$ we obtain 6 primary constraints, which appear in both $f(\mathbb{T})$ and TEGR, being associated with local Lorentz transformations. A good understanding of the structure of the Lorentz constraints is crucial for the comprehension of Lorentz violation in non-linear modifications of TEGR, and henceforth in the study of their viability.

We expect that this review will pave the way for future work on the Hamiltonian analysis for teleparallel theories of gravity. A good understanding of the Hamiltonian and constraint structure of TEGR is expected if committed to the task of proposing a teleparallel-based approach to canonical quantum gravity. The outcome of the Dirac-Bergmann algorithm on the counting of degrees of freedom is also a relevant issue for $f(\mathbb{T})$ gravity, on which there are disputing conclusions even for the simplest Minkowski and FLRW spacetimes. We have shown that the primary constraints for $f(\mathbb{T})$ gravity are consistent throughout the literature, therefore the differences on the outcomes must lie in the calculations of the PB, as pointed out in [27]. Furthermore, the full analysis for new general relativity have only been done for the ghost-free one parameter teleparallel gravity theory in [28] which does not provide many calculation details. With a better understanding for $f(\mathbb{T})$ and new general relativity (which are the most simple teleparallel theories) will give us more insights for more general modified teleparallel gravity theories. Still, Hamilton's equations have not yet been derived for any teleparallel gravity theory. And the primary Hamiltonian have not been written down without going off from the Weitzenböck gauge either. Finally, some considerations on Hessian and matrix of PB among constraints with variable rank might be necessary to take into account in this kind of models, for which the guidelines proposed in [64] can be helpful.

Acknowledgments

The authors are grateful to R. Ferraro, A. Golovnev, J. M. Nester, and an anonymous referee for helpful discussion and reference suggestions. D.B. was supported by the University of Tartu ASTRA Project PER ASPERA, financed by the European Regional Development Fund. M.J.G. was funded by FONDECYT-ANID postdoctoral grant 3190531. M.H. and C.P. were supported by the Estonian Ministry for Education and Science through the Personal Research Funding Grants PRG356 and PSG489. D.B., M.H. and C.P. were supported by the European Regional Development Fund through the Center of Excellence TK133 "The Dark Side of the Universe".

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