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MASSLESS POINT PARTICLE MOTION FROM κ-POINCARÉ MODIFIED DISPERSION RELATION IN SPHERICAL SYMMETRY

Bachelor's thesis (6 ECTS)
Physics program

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Tartu 2020

Massless point particle motion from κ-Poincaré modified dispersion relation in spherical symmetry

There are several reasons to expect that the general theory of relativity is not a complete picture, and quantum theory of gravity is waiting around the corner to be discovered from observations. Models of quantum gravity phenomenology are trying to shed some light on the conundrum by providing us with predictions of observable effects, possibly caused by the quantum nature of gravity. We're studying the most general spherically symmetric time independent modified dispersion relation, later on to consider massless particles subject to the κ-Poincaré modification on curved space-time. We derive the Hamilton equations of motion in order to obtain corrections to general relativistic solutions for photon orbits around a Schwarzschild black hole, also for time delay and light deflection differential equations.

Keywords: Hamilton geometry, modified dispersion relations, quantum gravity phenomenology, spherical symmetry.

CERCS: P190 - Mathematical and general theoretical physics, classical mechanics, quantum mechanics, relativity, gravitation, statistical physics, thermodynamics.

Massitu punktosakese liikumine sfäärilise sümmeetriaga κ-Poincaré modifitseeritud dispersiooniseosest

On mitmeid põhjusi eeldada, et üldrelatiivsusteooria ei ole tervikpilt ning gravitatsiooni kvantkäsitlus on vaatlustest avastamist ootamas. Kvantgravitatsiooni fenomenoloogia mudelid soovivad sellele probleemile valgust heita, pakkudes meile ennustusi vaadeldavatest efektidest, mille võimalikuks allikaks on gravitatsiooni kvantolemus. Käesolevas töös uurime kõige üldisemat sfäärilise sümmeetriaga ajakoordinaadist sõltumatut modifitseeritud dispersiooniseost ning hiljem uurime massituid osakesi, mis alluvad kõverale aegruumile rakendatud κ-Poincaré modifikatsioonile. Me tuletame Hamiltoni võrrandid, et leida parandid üldrelatiivsusteooria lahenditele sfääriliste orbiitide jaoks ümber Schwarzschildi musta augu, lisaks leiame parandid valguse kõrvalekalde ja ajanihke diferentsiaalvõrranditele.

Märksõnad: Hamiltoni geomeetria, modifitseeritud dispersiooniseosed, kvantgravitatsiooni fenomenoloogia, sfääriline sümmeetria.

CERCS: P190 - Matemaatiline ja üldine teoreetiline füüsika, klassikaline mehaanika, kvantmehaanika, relatiivsus, gravitatsioon, statistiline füüsika, termodünaamika.

Contents

Conventions

1. Natural units, i.e. $c = G = \hbar = 1$, where c is the speed of light, G is the gravitational constant, and $\hbar$ is reduced Planck constant.
2. Metric signature is $(-, +, +, +)$.
3. Spacetime indices labelled with Greek letters take values of 0...3, for ordinary 3-dimensional physical space indices are labelled with Latin letters and take values of 1...3.
4. According to Einstein sum convention, sums are expressed as follows: $\sum_\mu a_\mu b^\mu = a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3$, same indices (upper and lower) are summed over.
5. Derivatives:
   (a) Differentiation with respect to an affine parameter, in our case proper time: $\frac{dx^\mu}{d\tau} = \dot{x}^\mu$
   (b) Partial differentiation with respect to spacetime coordinates: $\frac{\partial}{\partial x^\mu} = \partial_\mu$
   (c) Partial differentiation with respect to covariant momentum: $\frac{\partial}{\partial p_\mu} = \overline{\partial}^\mu$
   (d) Partial differentiation with respect to four-velocity: $\frac{\partial}{\partial \dot{x}^\mu} = \dot{\partial}_\mu$

Introduction

Since 1915 Einstein's general theory of relativity (GR) has remained a fundamental theory of gravity that is consistent with a vast amount of observations [1, 2]. The first success of GR was that the predicted value of Mercury's perihelion advance was compatible with measurements [1, 3]. Other successful tests, to name a few, include the bending of light around massive objects, gravitational redshift of Sirius B[1], and the Shapiro delay[2] [1, 2].

In the recent years, further remarkable observations have been made. On 14 September 2015 the Laser Interferometer Gravitational Wave Observatory (LIGO) pushed the boundaries of measurement sensitivity by detecting gravitational waves for the first time [4]. Just last year, on 10 April 2019 The Event Horizon Telescope Collaboration published the image of a black hole M87*, introducing a new tool to study gravity in extreme limits on mass scales that were so far inaccessible [5]. Both observations were yet again consistent with the predictions derived from GR made over the last centuries.

Regardless of the success stories, there are many reasons to expect that GR is not the fundamental theory of gravity [6, 7, 8]. The most prominent are

- the accelerated expansion of the universe, which requires the existence of the unexplained phenomenon called dark energy [9],
- unexplained features in galactic rotation curves and galaxy collisions demanding for the existence of dark matter [10],
- the incompatibility of general relativity with quantum mechanics and the search for a quantum theory of gravity [11, 12].

Most of the tests probe gravity in a weak-field regime at intermediate length and energy scales [6, 13]. Effects of quantum gravity are expected to become relevant around Planck length scales $L_P = \sqrt{\hbar G/c^3} \approx 1,6 \times 10^{-35}$ m or the Planck energy $E_P = \sqrt{\hbar c^5/G} \approx 1,2 \times 10^{19}$ GeV -

[1]White dwarf Sirius B is not affected by cosmological redshift due to its proximity to Earth $\sim 2,64pc$ [1].

[2]Light passing a nearby massive object arrives with a delay.

far out of reach for current accelerators [14]. Deviations from general relativity are expected to emerge where gravitational fields are strong, for example, in the vicinity of black holes [13, 15].

Other potential candidates for the manifestation of quantum gravity effects are propagation effects. When particles propagate through space-time, small quantum gravity effects may accumulate in a deviation from their trajectory predicted by GR. The pictorial idea is that particles with energies closer to Planck energy E_{PL} should interact stronger with the quantum nature of gravity as they probe smaller length scales. Distant gamma-ray bursts are good candidates for such observations [16, 17, 18].

Three fundamental forces are described by Standard Model of particle physics with a great success, the fourth fundamental force - gravity still remains a classical theory [19, 20]. Since up to today there is no fundamental theory of quantum gravity yet, phenomenological models are considered to search for detectable effects of the quantum nature of gravity [15, 21]. The idea is to observe propagating particles and look for manifestations of quantum gravitational interactions predicted by a phenomenological model [16, 21].

One intensively studied model in the literature is the possibility that particles are subject to a modified dispersion (energy-momentum) relation due to their interaction with the quantum nature of gravity [15, 22, 23]. Dispersion relations can be interpreted as level sets of Hamilton functions, which determine the motion of particles in the phase space [15, 24]. In the context of this thesis we consider a spherically symmetric stationary κ-Poincaré modified dispersion relation. The aim of the thesis is to derive the Hamilton equations of motion for particles subject to this modified dispersion relation. The equations of motion can later be used to find phase-space trajectories.

In chapter one we focus on the Lagrangian and Hamiltonian formulation of point particle motion. We explicitly show that both of the formalisms are equivalent and yield the same equations of motion. In chapter two we introduce the concept of dispersion relations, which can be interpreted as level sets of Hamilton functions. The importance of the latter two chapters becomes apparent when we move on to third chapter, which specializes in the first order modified dispersion relations inspired from quantum gravity phenomenology. With the help of perturbation theory, we derive the most general Hamilton equations of motion for an arbitrary modified dispersion relation, then we move on to consider the spherically symmetric stationary Hamiltonian. In fourth chapter we wish to derive some interesting observables for massless particles. We calculate circular orbits and differential equations for both, light deflection and time delays. Finally, we specify the equations further by considering κ-Poincaré modified Schwarzschild geometry. In the appendix we have included important aspects from perturbation theory used for calculations. We included lenghty derivations in the appendix, as well.

Chapter 1

Lagrangian and Hamiltonian formulation of point particle motion

The content of this chapter provides the tools from Lagrangian and Hamiltonian mechanics, which we need in chapter two to understand the meaning of a dispersion relation, and in chapters three and four to derive the observables from modified dispersion relations. In the next two sections we follow the papers [25, 26] when not specified otherwise.

1.1 Lagrangian approach

Equations of motion

We consider point particles whose motion is determined by a Lagrange function $L(x, \dot{x})$, where $x = x(\tau)$ and $\dot{x} = \dot{x}(\tau)$ denote generalized coordinates and generalized velocities respectively. The Lagrangian defines the action for the point particles

$$S[x] = \int_{\tau_i}^{\tau_f} d\tau L(x, \dot{x}). \tag{1.1}$$

The least action principle states that particles follow the trajectory which extremizes the action. This means that the variation of the action $\delta S[x]$ vanishes for all variations of the path $x(\tau) \to x(\tau) + \delta x(\tau)$. The end points of the path are fixed - their variations $\delta x(\tau_i)$ and $\delta x(\tau_f)$ vanish. It is well known that such trajectories which do extremize the action must satisfy the

Euler-Lagrange equations,

$$0 = \frac{d}{d\tau}\dot{\partial}_\mu L - \partial_\mu L = \dot{x}^\nu \partial_\nu \dot{\partial}_\mu L + \ddot{x}^\nu \dot{\partial}_\nu \dot{\partial}_\mu L - \partial_\mu L = \dot{x}^\nu \partial_\nu \dot{\partial}_\mu L + 2\ddot{x}^\nu g^L_{\mu\nu} - \partial_\mu L \tag{1.2}$$

$$= \ddot{x}^\sigma + \frac{1}{2} g^{L\mu\sigma}(\dot{x}^\nu \partial_\nu \dot{\partial}_\mu L - \partial_\mu L), \tag{1.3}$$

where one can introduce the abbreviation ${g^L}_{\mu\nu} = \frac{1}{2}\dot{\partial}_\nu \dot{\partial}_\mu L$, called Lagrange metric, and its inverse $g^{L\mu\nu}$ which satisfies $g^L_{\mu\sigma} g^{L\sigma\nu} = \delta^\nu_\mu$ [27].

Momentum conservation from Lagrangian

The Lagrangian unveils other important information about a system - conservation laws. If a particular coordinate $x^\mu(\tau)$ does not occur in the Lagrangian, it is called cyclic. For these the the Euler-Lagrange equation (1.2) becomes

$$\frac{d}{d\tau}\dot{\partial}_\mu L = 0, \tag{1.4}$$

and we see that the conserved quantity is

$$p_\mu = \dot{\partial}_\mu L, \tag{1.5}$$

which is called conjugate momentum. Conservation laws can also be derived from Noether's theorem.

1.2 Hamiltonian approach

Hamilton equations of motion

In the Lagrangian approach, the motion of a point particle is described in terms of its position and velocity. Alternatively, one can describe the motion of a particle in terms of its position and momentum. Hamiltonian function $H(x,p)$ and Lagrangian $L(x,\dot{x})$ are related through the Legendre transformation in the following way

$$H(x,p) = \dot{x}^\mu p_\mu - L(x,\dot{x}). \tag{1.6}$$

Instead of the particle's velocity, the Hamiltonian depends on the particle's momentum. In the last subsection we introduced conjugate momentum p_μ (1.8) which is to be inverted to obtain $\dot{x}^\mu(x,p)$, this in turn is plugged into equation (1.6). In the Hamiltonian approach, the dynamics of a system is given by a set of two first order partial differential equations

$$\dot{x}^\mu = \overline{\partial}^\mu H \tag{1.7}$$

$$\dot{p}_\mu = -\partial_\mu H. \tag{1.8}$$

The set of Hamilton equations of motion are also derived from the principle of least action, where we can express the Lagrangian $L(x, \dot{x})$ in the action (1.1) in terms of the Hamiltoninian as $S[x,p] = \int_{\tau_i}^{\tau_f} d\tau(\dot{x}^\mu p_\mu - H(x,p))$. Variation with respect to x yields (1.7), and variation with respect to p yields (1.8).

Equivalence between Hamiltonian and Lagrangian approach

Both, Lagrangian and Hamiltonian approach are equivalent. If we define conjugate momentum as $p_\mu = \dot{\partial}_\mu L$, Hamiltonian as $H = \dot{x}^\sigma(x,p)p_\sigma - L(x, \dot{x}(x,p))$, and the Euler Lagrange equations (1.2) hold, i.e. $\dot{p}_\mu = \partial_\mu L$, then

$$\bar{\partial}^\mu H = \dot{x}^\mu + p_\sigma \bar{\partial}^\mu \dot{x}^\sigma - \dot{\partial}_\sigma L\, \bar{\partial}^\mu \dot{x}^\sigma = \dot{x}^\mu \tag{1.9}$$

$$\partial_\mu H = p_\sigma \partial_\mu \dot{x}^\sigma - \partial_\mu L - \dot{\partial}_\sigma L\, \partial_\mu \dot{x}^\sigma = -\dot{p}_\mu\,. \tag{1.10}$$

The other way around, if the above Hamilton equations of motion hold, and Lagrangian can be defined as $L = \dot{x}^\sigma p_\sigma(x, \dot{x}) - H(x, p(x, \dot{x}))$, then

$$\dot{\partial}_\mu L = p_\mu + \dot{x}^\sigma \dot{\partial}_\mu p_\sigma - \bar{\partial}^\sigma H \dot{\partial}_\mu p_\sigma = p_\mu \tag{1.11}$$

$$\partial_\mu L = \dot{x}^\sigma \partial_\mu p_\sigma - \partial_\mu H - \bar{\partial}^\sigma H \partial_\mu p_\sigma = -\partial_\mu H = \dot{p}_\mu = \frac{d}{d\tau}\dot{\partial}_\mu L\,. \tag{1.12}$$

Hence indeed the formalism are equivalent. In particular momentum conservation $\dot{p}_\mu = 0$ is equivalent to $\partial_\mu L = 0$.

1.3 Relativistic particle motion

As an example for the Lagrangian and Hamiltonian description of point particle motion we consider a relativistic charged particle which interacts with electromagnetic four potential $A_\mu(x) = (\phi, \vec{A})$, where ϕ is scalar potential and $\vec{A}$ is vector potential. The introduction to relativistic particle motion in Lagrangian approach is based on the following sources: [28, 29]. The relativistic particle motion in Hamiltonian approach is based on [30].

Relativistic particle motion in Lagrangian approach

We want to describe particles moving through space-time with an arbitrary curvature. The action for a relativistic particle subject to an electromagnetic potential $A_\mu(x)$ is defined as

$$S[x] = \int_{d\tau_i}^{d\tau_f} d\tau \left(\frac{1}{2} m g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + q A_\mu(x)\dot{x}^\mu\right) = \int_{d\tau_i}^{d\tau_f} d\tau L_{\text{GR+EM}}(x, \dot{x}), \tag{1.13}$$

where m is the invariant mass parameter and q is the charge associated to a particle, $\dot{x}^\mu = \frac{\dot{x}^\mu}{d\tau}$ is the tangent vector to particle's curve $x^\mu(\tau)$, and $g_{\mu\nu}(x)$ is called the metric which describes the geometry of space-time. Again, we want to extremize the action, this yields the equations of motion

$$\ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}(x)\dot{x}^\rho \dot{x}^\sigma = \frac{q}{m} F^{\mu\nu}\dot{x}_\nu, \tag{1.14}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is an electromagnetic field tensor which is antisymmetric, and $\Gamma^\mu{}_{\sigma\rho}$ are Christoffel symbols which are expressed through metric the following way

$$\Gamma^\mu{}_{\rho\sigma}(x) = \frac{1}{2} g^{\mu\nu}(\partial_\sigma g_{\nu\rho} + \partial_\rho g_{\nu\sigma} - \partial_\nu g_{\rho\sigma}). \tag{1.15}$$

In the context of this thesis we study motion of massless particles, meaning that $q = 0$ and the equations of motion (1.14) then become

$$\ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma = 0, \tag{1.16}$$

which is also known as the geodesic equation. Solution to this equation is called a geodesic - generalization of an Euclidean space straight line. What an Euclidean straight line and a geodesic have in common is that both parallel transport the tangent vector $\dot{x}^\mu$ along the path which is also the "shortest" between two points. Whether in curved or flat space, the tangent vector is kept constant as we move along particle's path. Also, it's worth noting that geodesics are paths along which particles are freely-falling (unaccelerated).

Relativistic particle motion in Hamiltonian approach

The general relativistic Hamiltonian can be derived by performing a Legendre transformation on the Lagrangian (1.13) by taking $q = 0$. The factor of m is just a constant and doesn't affect the equations of motions derived from a Lagrangian, therefore we can take $L(x, \dot{x}) = \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$. The Hamiltonian becomes

$$H(x, p) = \dot{x}^\mu \dot{\partial}_\mu L - L = \dot{x}^\mu(p) p_\mu - \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu(p)\dot{x}^\nu(p), \tag{1.17}$$

where p_μ is covariant four-momentum, and $g^{\mu\nu}$ is contravariant metric, which satisfies the condition $g_{\mu\nu} g^{\mu\gamma} = \delta_\nu^\gamma$. We find momentum p_μ from the Lagrangian $L(x, \dot{x})$ and invert it to get the expression for $\dot{x}^\mu(x, p)$, which shall be plugged into (1.17) to eliminate the four-velocity dependence. Conjugate momentum is as follows:

$$p_\alpha = \dot{\partial}_\alpha L = g_{\alpha\mu}\dot{x}^\mu. \tag{1.18}$$

Hence, the four-velocity then becomes

$$\dot{x}^\mu = g^{\mu\gamma}p_\gamma. \tag{1.19}$$

We plug this into the Hamiltonian (1.17) where four-velocities appear. This yields the covariant expression for Hamiltonian

$$\begin{aligned} H(x,p) &= g^{\mu\gamma}p_\mu p_\gamma - \frac{1}{2}g_{\mu\nu}(g^{\mu\gamma}p_\gamma)(g^{\nu\alpha}p_\alpha) \\ &= \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu. \end{aligned} \tag{1.20}$$

The Hamilton equations of motion corresponding to the Hamiltonian are

$$\dot{x}^\mu = \overline{\partial}^\mu H = g^{\mu\alpha}p_\alpha \tag{1.21}$$

$$\dot{p}_\mu = -\partial_\mu H = -\frac{1}{2}\partial_\mu g^{\nu\sigma}p_\sigma p_\nu. \tag{1.22}$$

As we already saw in the section 1.2, the Hamiltonian and Lagrangian approaches to point particle mechanics are equivalent. This means that the Hamilton equations of motion (1.21) and (1.22) yield the geodesic equation (1.16). This can be seen as follows: firstly, we take τ derivative of (1.21) which gives us

$$\ddot{x}^\mu = \partial_\beta g^{\mu\alpha}\dot{x}^\beta p_\alpha + g^{\mu\alpha}\dot{p}_\alpha. \tag{1.23}$$

We notice $\dot{p}_\alpha$ appearing in the equation above, we replace it with (1.22), also we want to plug in the expression for p_α from equation (1.18) to obtain velocity dependence. We get

$$\ddot{x}^\mu = \partial_\beta g^{\mu\alpha}\dot{x}^\beta g_{\alpha\lambda}\dot{x}^\lambda - \frac{1}{2}g^{\mu\alpha}\partial_\alpha g^{\nu\sigma}p_\sigma p_\nu. \tag{1.24}$$

Partial derivatives of metric can be expressed in terms of Christoffel symbol in the following way

$$\partial_\mu g^{\kappa\lambda} = -g^{\nu\lambda}\Gamma^\kappa{}_{\mu\nu} - g^{\nu\kappa}\Gamma^\lambda{}_{\nu\mu}. \tag{1.25}$$

Equation (1.24) then becomes

$$\ddot{x}^\mu = (-g^{\nu\alpha}\Gamma^\mu{}_{\beta\nu} - g^{\nu\mu}\Gamma^\alpha{}_{\nu\beta})\dot{x}^\beta g_{\alpha\lambda}\dot{x}^\lambda - \frac{1}{2}g^{\mu\alpha}(-g^{\rho\sigma}\Gamma^\nu{}_{\alpha\rho} - \mathrm{g}^{\rho\nu}\Gamma^\sigma{}_{\rho\alpha})g_{\sigma\lambda}\dot{x}^\lambda g_{\nu\eta}\dot{x}^\eta \tag{1.26}$$

$$= (\Gamma^\mu{}_{\beta\lambda} - g^{\nu\mu}\Gamma_{\lambda\nu\beta})\dot{x}^\beta\dot{x}^\lambda - \frac{1}{2}g^{\mu\alpha}(-\Gamma_{\eta\alpha\lambda} - \Gamma_{\lambda\eta\alpha})\dot{x}^\lambda\dot{x}^\eta \tag{1.27}$$

$$= \Gamma^\mu{}_{\beta\lambda}\dot{x}^\beta\dot{x}^\lambda - \cancel{g^{\nu\mu}\Gamma_{\lambda\nu\beta}\dot{x}^\beta\dot{x}^\lambda} - \cancel{\frac{1}{2}g^{\nu\mu}(-\Gamma_{\lambda\nu\beta} - \Gamma_{\lambda\beta\nu})x^\beta\dot{x}^\lambda}, \tag{1.28}$$

where in the last step we changed some dummy indices, took into account the symmetry of metric $g^{\mu\nu} = g^{\nu\mu}$, and that Christoffel symbols are symmetric in the last two indices $\Gamma_{\lambda\nu\beta} = \Gamma_{\lambda\beta\nu}$. We saw explicitly the equivalence between the Hamiltonian and Lagrangian approach.

Chapter 2

Dispersion relations as Hamilton functions

Special relativistic energy-momentum relation as dispersion relation

Consider the massive Klein-Gordon equation (relativistic version of Schrödinger equation) in Minkowski spacetime $(\partial^\mu \partial_\mu - m^2)\phi = 0$, and a plane wave Ansatz $\phi \sim e^{-ik_0 t + i\vec{k}\cdot\vec{x}} = e^{ik_\mu x^\mu}$, then the wave covectors k_μ have to satisfy $\omega^2 = |\vec{k}|^2 + m^2$, which is called dispersion relation [31, 32]. A dispersion relation is a constraint which the plane wave ϕ has to satisfy in order to be physically viable. The quantity k_μ is usually called the wave covector or momentum. In the following we will use the notation p_μ since we are working on the level of point particles. By doing the replacement $\omega \to E$ and $\vec{k} \to \vec{p}$ in the dispersion relation $\omega(\vec{k})$, where we chose units such that $c = \hbar = 1$, we get the famous energy-momentum relation from special relativity as a result

$$E^2 = m^2 + |\vec{p}|^2, \tag{2.1}$$

where E is the energy, $\vec{p}$ is spatial momentum and m is the invariant mass parameter associated to a particle. It is also brought out in the literature that dispersion relations can depart from this well-known relation (2.1) [33, 34].

Dispersion relation and Hamilton functions

Energy-momentum relation (2.1) can be expressed alternatively with the help of inverse of the space-time metric g^{-1} and the four-momentum of a particle. The covariant expression of (2.1) then becomes

$$-\frac{1}{2}m^2 = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu, \tag{2.2}$$

where we divided both sides by $\frac{1}{2}$. Equation (2.2) becomes (2.1) when expressed with respect to an orthonormal frame of the metric. Moreover, the right hands side of the dispersion relation

(2.2) is precisely the Hamiltonian (1.20). It turns out that this can be used as general principle. Dispersion relations are level sets of Hamilton functions. Equation (2.1) generalizes to

$$H(x,p) = g^{\mu\nu} p_\mu p_\nu = -m^2, \tag{2.3}$$

where for a free falling particle we can forget about the factor $\frac{1}{2}$, because it doesn't affect the equations of motion, and we wish to follow the same convention as, for example, in the paper [15]. Dispersion relation, interpreted as Hamiltonian, determines motion of particles and geometry of phase space. Dispersion relation is closely intertwined to the space-time metric, Minkowski space-time is just one of the options. Choosing another type of metric changes the covariant dispersion relation (2.3) [22, 24]. In the context of this thesis we are interested in generalizations of the general relativistic dispersion relation. The next chapter is dedicated to modified dispersion relations inspired from quantum gravity phenomenology.

Chapter 3

First order modified dispersion relations

Quantum gravity effects are expected to become relevant around Planck energy scales ($E_{PL} = M_{PL}c^2 = \sqrt{\hbar c^5/G} \approx 10^{16}$ TeV) according to quantum gravity models [15, 21, 35]. One way quantum gravity effects may become detectable is when high energetic test particles probe space-time on Planck length scales ($L_{PL} = \sqrt{G\hbar/c^2} \approx 10^{-35}$ m), therefore they are expected to interact more with the quantum nature of gravity, causing deviations from general relativistic trajectories [16, 35]. Though, the described pictorial idea needs to be reformulated in terms of particle's four-momentum, because energy is an observer dependent quantity [16].

What some of the different quantum gravity models have in common is that they predict modified dispersion relations, usually in the following form $-m^2 = -E^2 + \vec{p}^2 + f(E, \vec{p}^2, E_{PL})$ [15, 36, 37, 38]. One interesting effect emerging from modified dispersion relations is energy dependent velocity ($v = \frac{\partial E}{\partial p} \neq 1$) of particles, and high energetic photons from gamma-ray bursts are an ideal testing ground for such models [15, 16, 18, 35]. Another place to look for hints of modified gravity is in the vicinity of massive objects where gravitational fields are strong [13, 15]. We will derive observables in the regime where space-time curvature can not be neglected. The aim of this chapter is to derive the Hamilton equations of motion for the most general spherically symmetric dispersion relation. In the last section we specify the dispersion relation by introducing the κ-Poincaré perturbation function.

3.1 Perturbed particle motion in general background

We consider Hamiltonian of the form

$$H(x,p) = H_{GR} + \epsilon h(x,p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu + \epsilon h(x,p). \tag{3.1}$$

The perturbation function $h(x,p)$ and metric inverse $g^{\mu\nu}(x)$ is right now left unspecified, and depending on the model one is interested in, different functions $h(x,p)$ can be chosen [15, 16]. Perturbation function $h(x,p)$ is multiplied by a small parameter ϵ (in the context of quantum gravity phenomenology identified with the Planck length L_{PL}), for our calculations we make power expansions in ϵ to first order (also see the appendix A). We consider first order modified curves to understand the effect of the perturbation function $h(x,p)$

$$(x^\mu(\tau), p_\mu(\tau)) = ({x_0}^\mu(\tau), p_{0\mu}(\tau)) + \epsilon({x_1}^\mu(\tau), p_{1\mu}(\tau))\,. \tag{3.2}$$

We find the Hamilton equations of motion corresponding to the perturbed Hamiltonian (3.1) expanded to the first order in ϵ as

$$\dot{x}^\mu = \dot{x}_0^\mu + \epsilon\dot{x}_1^\mu = \overline{\partial}^\mu H_{GR} + \epsilon\overline{\partial}^\mu h \tag{3.3}$$
$$= (g_0^{\mu\alpha} + \epsilon x_1^\sigma \partial_\sigma g_0^{\mu\alpha})(p_{0\alpha} + \epsilon p_{1\alpha}) + \epsilon\overline{\partial}^\mu h_0 \tag{3.4}$$
$$= {g_0}^{\mu\alpha} p_{0\alpha} + \epsilon({x_1}^\sigma \partial_\sigma {g_0}^{\mu\alpha} p_{0\alpha} + {g_0}^{\mu\alpha} p_{1\alpha} + \overline{\partial}^\mu h_0), \tag{3.5}$$

and

$$\dot{p}_\mu = \dot{p}_{0\mu} + \epsilon\dot{p}_{1\mu} = -\partial_\mu H_{GR} - \epsilon\partial_\mu h \tag{3.6}$$
$$= -\frac{1}{2}\partial_\mu(g_0^{\nu\sigma} + \epsilon x_1^\lambda \partial_\lambda g_0^{\nu\sigma})(p_{0\sigma} + \epsilon p_{1\sigma})(p_{0\nu} + \epsilon p_{1\nu}) - \epsilon\partial_\mu h_0 \tag{3.7}$$
$$= -\frac{1}{2}p_{0\sigma}p_{0\nu}\partial_\mu {g_0}^{\nu\sigma} - \frac{1}{2}\epsilon(2p_{0\sigma}p_{1\nu}\partial_\mu {g_0}^{\nu\sigma} + p_{0\sigma}p_{0\nu}{x_1}^\lambda \partial_\lambda\partial_\mu {g_0}^{\nu\sigma} + 2\partial_\mu h_0), \tag{3.8}$$

where $h_0 = h(x_0, p_0)$ and $g_0^{\mu\nu} = g_0^{\mu\nu}(x_0)$.

3.2 Particle motion in general spherically symmetric background

We consider a general spherically symmetric stationary (independent of t coordinate) Hamiltonian, see [15]

$$H(x,p) = H(r, p_t, p_r, w) \quad w^2 = p_\theta^2 + \frac{1}{\sin^2\theta}p_\phi^2. \tag{3.9}$$

The conservation laws that emerge from this choice of Hamiltonian make calculations significantly easier. As we already discussed in the chapter 2, physical trajectories of test particles are determined by dispersion relation

$$H = -m^2, \tag{3.10}$$

where m is an invariant mass parameter associated to a particle, and the Hamilton equations of motion. The equations of motion for each of the phase space coordinate are

$$\dot{t} = \bar{\partial}^t H \qquad \dot{p}_t = -\partial_t H \tag{3.11}$$

$$\dot{r} = \bar{\partial}^r H \qquad \dot{p}_r = -\partial_r H \tag{3.12}$$

$$\dot{\theta} = \bar{\partial}^\theta H \qquad \dot{p}_\theta = -\partial_\theta H \tag{3.13}$$

$$\dot{\phi} = \bar{\partial}^\phi H \qquad \dot{p}_\phi = -\partial_\phi H. \tag{3.14}$$

Constants of motion

From spherical symmetry, the equations of motion corresponding to θ and ϕ coordinate can be immediately solved without having to specify Hamiltonian (3.9) further. Equations of motion for θ coordinate (3.13) are

$$\dot{\theta} = \bar{\partial}^\theta H = \frac{\partial H}{\partial w}\frac{1}{2w}\bar{\partial}^\theta w^2 = \frac{\partial H}{\partial w}\frac{1}{w}p_\theta \tag{3.15}$$

$$\dot{p}_\theta = -\partial_\theta H = -\frac{\partial H}{\partial w}\frac{1}{2w}\partial_\theta w^2 = \frac{\partial H}{\partial w}\frac{1}{w}\frac{\cos\theta}{\sin^3\theta}p_\phi^2\,. \tag{3.16}$$

Without loss of generality, the equations above can be solved by setting $\theta = \frac{\pi}{2}$ and $p_\theta = 0$, implying that $w^2 = p_\theta^2 + \frac{1}{\sin^2\theta}p_\phi^2 \overset{\theta\to\frac{\pi}{2},\, p_\theta\to 0}{=} p_\phi^2$. We will see that these solutions directly lead to the conservation of the direction of angular momentum, meaning that orbital motion is planar [28]. One can choose the orientation of the spherical coordinates such that their equatorial plane agrees with the orbital plane, hence the solution $\theta = \frac{\pi}{2}$, which is also the angle between the angular momentum vector and the plane in which the particle propagates.

The equations of motion corresponding to ϕ-coordinate (3.14) become

$$\dot{\phi} = \bar{\partial}^\phi H = \frac{\partial H}{\partial w}\bar{\partial}^\phi w = \frac{\partial H}{\partial w} \tag{3.17}$$

$$\dot{p}_\phi = -\partial_\phi H = 0, \tag{3.18}$$

where we already set $w = p_\phi$. The second equation (3.18) yields a constant of motion, we call it angular momentum. The equation is solved by setting $p_\phi = \mathcal{L}$, where $\mathcal{L} = const$.

Another equation of motion immediately follows from the second equation of (3.11)

$$\dot{p}_t = -\partial_t H = 0\,, \tag{3.19}$$

which is solved by setting $p_t = \mathcal{E}$, where $\mathcal{E} = const$. We call the constant of motion $\mathcal{E}$ energy.

Spherically symmetric modified dispersion relation and equations of motion

Having presented these general findings, we move on to consider a modified dispersion relation in the form of

$$H(r, p_t, p_r, w) = H_{GR}(r, p_t, p_r, w) + \epsilon h(r, p_t, p_r, w) \tag{3.20}$$

$$= \frac{1}{2}\left(-a(r)p_t^2 + b(r)p_r^2 + \frac{1}{r^2}w^2\right) + \epsilon h(r, p_t, p_r, w), \tag{3.21}$$

where H_{GR} is a general relativistic Hamiltonian and $h(x, p)$ is a perturbation function. The expressions for functions $a(r)$ and $b(r)$ are left open for now, later on we proceed to study perturbations around the Schwarzschild solution, the most general spherically symmetric solution of the Einstein vacuum equations [39]. We will set $a(r) = b(r)^{-1}$, and finally, we specify even further by setting $a(r) = \left(1 - \frac{r_s}{r}\right)^{-1}$.

As mentioned earlier, we're looking for curves of the type (3.2), more precisely with what we found earlier we are left with

$$t = t_0 + \epsilon t_1 \qquad p_t = \mathcal{E}_0 + \epsilon \mathcal{E}_1 \tag{3.22}$$

$$r = r_0 + \epsilon r_1 \qquad p_r = p_{0r} + \epsilon p_{1r} \tag{3.23}$$

$$\theta = \frac{\pi}{2} \qquad p_\theta = 0 \tag{3.24}$$

$$\phi = \phi_0 + \epsilon \phi_1 \qquad p_\phi = \mathcal{L}_0 + \epsilon \mathcal{L}_1 \tag{3.25}$$

One can calculate the remaining Hamilton equations of motion (3.11)-(3.14) directly from the perturbed Hamiltonian (3.21) and expand the results to first order in the perturbation parameter ϵ. Alternatively, one can exploit the general Hamilton equations of motion (3.5), (3.8), and expand the terms following Einstein sum convention.

Firstly, we find velocity equations for the three remaining space-time coordinates. Hamilton equation of motion for t motion becomes

$$\dot{t} = \dot{t}_0 + \epsilon\dot{t}_1 = \overline{\partial}^t H_{GR} + \epsilon\overline{\partial}^t h = g_0^{t\nu} p_{0\nu} + \epsilon\left(x_1^\alpha \partial_\alpha g_0^{t\nu} p_{0\nu} + g_0^{t\nu} p_{1\nu} + \overline{\partial}^t h_0\right) \tag{3.26}$$

$$= -a_0\mathcal{E}_0 + \epsilon\left(\bar{\partial}^t h_0 - a_0' r_1 \mathcal{E}_0 - a_0\mathcal{E}_1\right). \tag{3.27}$$

Velocity equation corresponding to ϕ equation is

$$\dot{\phi} = \dot{\phi}_0 + \epsilon\dot{\phi}_1 = \overline{\partial}^\phi H_{GR} + \epsilon\overline{\partial}^\phi h = g_0^{\phi\phi} p_{0\phi} + \epsilon\left(r_1\partial_r g_0^{\phi\phi} p_{0\phi} + g_0^{\phi\phi} p_{1\phi} + \overline{\partial}^\phi h_0\right) \tag{3.28}$$

$$= \frac{\mathcal{L}_0}{r_0^2} + \epsilon\left(\frac{\mathcal{L}_1}{r_0^2} - 2\frac{r_1}{r_0^3}\mathcal{L}_0 + \frac{\partial h_0}{\partial w}\right). \tag{3.29}$$

Last but not least, the $\dot{r}$ equation becomes

$$\dot{r} = \dot{r}_0 + \epsilon\dot{r}_1 = \overline{\partial}^r H_{GR} + \epsilon\overline{\partial}^r h = g_0^{rr} p_{0r} + \epsilon\left(r_1\partial_r g_0^{rr} p_{0r} + g_0^{rr} p_{1r} + \overline{\partial}^r h\right) \tag{3.30}$$

$$= b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \bar{\partial}^r h_0). \tag{3.31}$$

Finally, we calculate the Hamilton equation of motion for the phase space momentum coordinate p_r, we get

$$\dot{p}_r = \dot{p}_{0r} + \epsilon\dot{p}_{1r} = -\partial_r H_{GR} - \epsilon\partial_r h \tag{3.32}$$

$$= -\frac{1}{2}p_{0\alpha}p_{0\beta}\partial_r g_0^{\alpha\beta} - \frac{1}{2}\epsilon\left(2p_{0\alpha}p_{1\beta}\partial_r g_0^{\alpha\beta} + p_{0\alpha}p_{0\beta}x_1^\nu\partial_\nu\partial_r g_0^{\alpha\beta} + 2\partial_r h_0\right). \tag{3.33}$$

The above expression can be expanded by following Einstein sum convention, this procedure yields

$$\begin{aligned}\dot{p}_r = &\frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2}p_{0r}^2 b_0' + \frac{\mathcal{L}_0^2}{r_0^3} \\ &+ \frac{1}{2}\epsilon\Big(2\mathcal{E}_0 a_0'\mathcal{E}_1 + \mathcal{E}_0^2 a_0'' r_1 - p_{0r}^2 b_0'' r_1 - 2p_{0r}b_0' p_{1r} - 6\frac{\mathcal{L}_0^2}{r_0^4}r_1 + \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} - 2\partial_r h_0\Big).\end{aligned} \tag{3.34}$$

Throughout the calculations we have set $a_0 = a(r_0)$ and $b_0 = b(r_0)$, the same applies to their first and second order derivatives respect to r. Also, the zeroth order of the perturbation function is denoted as follows: $h_0 = h(r_0, p_{0t} = \mathcal{E}_0, p_{0r}, w_0 = \phi_0 = \mathcal{L}_0)$.

We have all the differential equations needed to describe dynamics of test particles. The dispersion relation induced by (3.21) for particles on trajectories solving the Hamilton equations of motion can be later on used to make some important substitutions. We shall already expand the dispersion relation (3.21) to the first order into perturbation parameter ϵ to get

$$-m^2 = m_0^2 + \epsilon m_0 m_1 = \frac{1}{2}\left(-a_0\mathcal{E}_0^2 + b_0 p_{0r}^2 + \frac{1}{r_0^2}\mathcal{L}_0^2\right) + \frac{1}{2}\epsilon\Big(- a_0'\mathcal{E}_0^2 r_1 - 2\mathcal{E}_0 a_0 \mathcal{E}_1 + p_{0r}^2 b_0' r_1 + 2p_{0r}b_0 p_{1r} - 2\frac{\mathcal{L}_0^2}{r_0^3}r_1 + 2\frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + 2h_0\Big), \tag{3.35}$$

where we have regrouped zeroth and first order terms.

In this section we have established all important equations needed to calculate observables from the κ-Poincaré perturbation in the next paragraph.

3.3 κ-Poincaré modification

Following [15], we now study the influence of the κ-Poincaré dispersion relation on particle motion in spherical symmetry.

Modified dispersion relations are one approach to describe effectively the interaction of fundamental particles with the quantum nature of gravity. One of the most studied modified dispersion relations is the κ-Poincaré dispersion relation, which emerges from quantum deformations of the Poincaré algebra [40, 41]. In this framework, without violating the principle of relativity, the second invariant is the Planck energy E_{PL} (or length L_{PL}) scale in addition to the speed of light c [42]. To preserve these scales, Lorentz transformations are modified, which in turn gives rise to modified energy-momentum (dispersion) relations [35, 43].

Particles with κ-Poincaré symmetry can be interpreted as propagating on flat Minkowski space-time. In order to describe quantum gravity effects on generally curved space-times, the κ-Poincaré model can be adjusted in a way that a flat space-time κ-Poincaré dispersion relation can be restored in a local frame.

We already discussed in the chapter 2 that dispersion relations can be interpreted as the level sets of the Hamilton function. κ-Poincaré dispersion relation is as follows:

$$H_\kappa(x,p) = -\frac{4}{\ell^2}\sinh\left(\frac{\ell}{2}p_t\right)^2 + e^{\ell p_t}\vec{p}^2, \tag{3.36}$$

where ℓ is the deformation parameter, which is the inverse of Planck energy E_{PL}. If $\ell \to 0$, the dispersion relation above reduces to that of a special relativistic one $H(x,p) = -p_t^2 + \vec{p}^2 = \eta^{\mu\nu}p_\mu p_\nu$. The point particle motion determined by the Hamiltonian

(3.36) is argued to be the semiclassical limit of quantum gravity. In the context of this thesis we are interested in situations where curvature of space-time has to be taken into account. In the literature κ-deformed Hamiltonian (3.36) has been implemented on general curved space-times, ensuring the local κ-Poincaré invariance, likewise GR is locally Lorentz invariant.

The Hamiltonian of interest is defined as

$$H_Z(x,p) \equiv -\frac{4}{\ell^2}\sinh\left(\frac{\ell}{2}Z(p)\right) + e^{\ell Z(p)}(g^{\mu\nu}(x)p_\mu p_\nu + Z(p)^2), \tag{3.37}$$

where $Z(p) = Z^\mu(x)p_\mu$ is the timelike vector field action on the momentum, with the constraint that $g_{\mu\nu}(x)Z^\mu(x)Z^\nu(x) = -1$. Performing power-series expansion on (3.37) with respect to the deformation parameter ℓ yields

$$\begin{aligned} H_Z(x,p) &= g^{\mu\nu}p_\mu p_\nu + \ell Z(p)(g^{\mu\nu}p_\nu p_\nu + Z(p)^2) + \mathcal{O}(\ell^2) && (3.38)\\ &= H_{GR} + \ell h(x,p) + \mathcal{O}(\ell^2). && (3.39)\end{aligned}$$

The Hamiltonian (3.1) now looks exactly like the Hamiltonian above, except that the deformation parameter ℓ is denoted with ϵ.

For simplicity, we consider a spherically symmetric static Hamiltonian. The vector field then takes the form of

$$Z(p) = Z^\mu(x)p_\mu = c(r)p_t + d(r)p_r, \tag{3.40}$$

satisfying the constraint

$$g_{\mu\nu}Z^\mu(x)Z^\nu(x) = -\frac{c(r)^2}{a(r)} + \frac{d(r)^2}{b(r)} = -1, \tag{3.41}$$

where the components of the GR metric can be found from the inverse metric used in the Hamiltonian (3.21). Plugging (3.40) into (3.38) yields the most generally spherically symmetric κ-deformed Hamiltonian, where the perturbation function $h(x,p)$ becomes

$$h(r,p_t,p_r,w) = (c(r)p_t + d(r)p_r)\left(-a(r)p_t^2 + b(r)p_r^2 + \frac{1}{r^2}w^2 + (c(r)p_t + d(r)p_r)^2\right). \tag{3.42}$$

We will derive all observables for the specific choice $c = \sqrt{a(r)}$, $d = 0$, where it can be easily seen that the constraint (3.41) is satisfied. This choice is done for simplicity, so that the observables can be analyzed well. Our choice of functions simplifies the perturbation function

(3.42) resulting in

$$h(r, p_t, p_r, w) = \sqrt{a(r)}p_t \left(\cancel{-a(r)p_t^2} + b(r)p_r^2 + \frac{1}{r^2}w^2 + \cancel{a(r)p_t^2} \right) \tag{3.43}$$

Now that we have specified the form of the perturbation function, we proceed to calculate some observables in the next section.

Chapter 4

Observables from modified dispersion relations

4.1 Spherical orbits

For the existence of spherical orbits the condition $\dot{r} = 0$ has to be satisfied [44]. Spherical orbits are characterized by setting $r = const. = r_0 + \epsilon r_1$, which immediately follows from the condition, which the Hamilton equation of motion $\dot{r}$ has to satisfy. We will calculate the spherical orbits from the modified dispersion relation to find the correction r_1 to the GR value r_0. Spherical orbits then determine the shadow of a black hole [45]. If a photon reaches too close to a black hole, it is not able to escape, causing a darker area [46]. Though, for realistic shadows one needs to consider rotating black holes, for example, Kerr black holes [45, 47].

4.1.1 Spherical orbits in general spherically symmetric background

Radius of circular orbits can be derived by setting radial velocity $\dot{r}$ to zero. The Hamilton equation of motion corresponding to radial velocity (3.31) then becomes

$$0 = b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \bar{\partial}^r h_0) \tag{4.1}$$

The zeroth and first order equation can be solved for p_{0r} and p_{1r} assuming that $b_0 \neq 0$.

$$p_{0r} = 0 \tag{4.2}$$

$$p_{1r} = -\frac{1}{b_0}\bar{\partial}^r h_0 \tag{4.3}$$

First order momentum p_{1r} can be said to balance radial momentum contribution caused by $\bar{\partial}^r h_0$ so that the total radial velocity vanishes at each order. We found the zeroth and first order momentum (p_{0r} and p_{1r}) from the Hamilton equation of motion for radial velocity (3.31). Now, with the help of Hamilton equation of motion (3.34) and momentum equations (4.2), (4.3) we can derive zeroth and first order radius for spherical orbits.

From the equation (4.2) we can immediately see that $\dot{p}_{0r} = 0$, and it can be easily shown that also $\dot{p}_{1r} = 0$ by applying the τ derivative to equation (4.3)

$$\dot{p}_{1r} = \frac{b_0'}{b_0^2}\dot{r}_0\bar{\partial}^r h_0 - \frac{1}{b_0}(\partial_r\bar{\partial}^r h_0\,\dot{r}_0 + \bar{\partial}^t\bar{\partial}^r h_0\,\dot{p}_{0t} + \bar{\partial}^r\bar{\partial}^r h_0\,\dot{p}_{0r} + \bar{\partial}^\theta\bar{\partial}^r h_0\,\dot{p}_{0\theta} + \bar{\partial}^\phi\bar{\partial}^r h_0\,\dot{p}_{0\phi}) = 0. \tag{4.4}$$

This means that Hamilton equation of motion (3.34) reduces to zero, $\dot{p}_r = 0$, and plugging it into the equation (3.34) yields

$$0 = \dot{p}_{0r} + \epsilon\dot{p}_{1r} = \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\epsilon\left(2\mathcal{E}_0 a_0'\mathcal{E}_1 + \mathcal{E}_0^2 a_0'' r_1 - 6\frac{\mathcal{L}_0^2}{r_0^4}r_1 + \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} - 2\partial_r h_0\right). \tag{4.5}$$

By solving the zeroth and first order dispersion relation (3.35), we can replace in the equation (4.5) either $\mathcal{L}_0$ and $\mathcal{L}_1$ or $\mathcal{E}_0$ and $\mathcal{E}_1$. We choose the latter two, which yields for the zeroth order energy solution the known expression

$$\mathcal{E}_0 = \pm\frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}}{r_0\sqrt{a_0}}, \tag{4.6}$$

and the first order energy solution is

$$\mathcal{E}_1 = \pm\frac{1}{\sqrt{a_0}}\left(\frac{-\mathcal{L}_0^2 r_1 + \mathcal{L}_0\mathcal{L}_1 r_0 + (h_0 + 2m_0 m_1)r_0^3}{r_0^2\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}} - \frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}\,a_0'}{2r_0 a_0}r_1\right). \tag{4.7}$$

For sake of readability, the derivation of $\mathcal{E}_0$ and $\mathcal{E}_1$ can be found in the appendix B.1. Now that we have expressed $\mathcal{E}_0$ and $\mathcal{E}_1$ in terms of $\mathcal{L}_0$ and $\mathcal{L}_1$, we can plug (4.6) and (4.7) into equation (4.5). The zeroth order equation of (4.5) yields

$$m_0^2\frac{a_0'}{a_0} + \frac{\mathcal{L}_0^2}{2r_0^2}\left(\frac{2}{r_0} + \frac{a_0'}{a_0}\right) = 0. \tag{4.8}$$

To find the first order radius correction r_1, we analyze the first order equation (4.5), plug in expressions for zeroth and first order energy, (4.6) and (4.7) respectively. As a result, we get

$$r_1 = \frac{2\left(r_0^3(\partial_r h_0 - \frac{a_0'}{a_0}(h_0 + 2m_0 m_1)) - \mathcal{L}_0\mathcal{L}_1(2 + \frac{a_0'}{a_0}r_0)\right)}{r_0^3\mathcal{L}_0^2(\frac{a_0''}{a_0 r_0^2} - \frac{a_0'^2}{a_0^2 r_0^2} - \frac{2a_0'}{r_0^3 a_0} - \frac{6}{r_0^4}) - 2m_0^2 r_0^3(\frac{a_0'^2}{a_0^2} - \frac{a_0''}{a_0})}. \tag{4.9}$$

The complete derivation of (4.8) and (4.9) can be found in the appendix B.2. The first order correction r_1 depends on the choice of the function a_0, the choice of perturbation function h_0, and the type of particles we're observing (whether massive $m \neq 0$ or massless $m = 0$).

4.1.2 Spherical orbits from modified Schwarzschild geometry

We are analyzing massless particles (photons, for example) from modified Schwarzschild geometry. We set $a_0 = \left(1 - \frac{r_s}{r_0}\right)^{-1}$, where r_s is the Schwarzschild radius and r_0 is the zeroth order radius. For massless particles $m_0 = 0$ and $m_1 = 0$. Assuming that $\mathcal{L}_0 \neq 0$, we find that the zeroth order equation (4.8) takes the following form

$$\frac{2}{r_0} + \frac{a_0'}{a_0} = 0. \tag{4.10}$$

After plugging in expressions for a_0 and $a_0' = -\frac{1}{(1-\frac{r_s}{r_0})^2}\frac{r_s}{r_0^2}$, the equation immediately yields the general relativistic solution

$$r_0 = \frac{3}{2}r_s, \tag{4.11}$$

which only depends on Schwarzschild radius.

For r_1 correction we will use the equation (4.9). We can calculate $a_0 = \left(1 - \frac{r_s}{r_0}\right)^{-1}$ and its first and second order derivatives knowing the zeroth order solution $r_0 = \frac{3}{2}r_s$. The first order correction becomes

$$r_1 = \frac{9r_s^3}{16\mathcal{L}_0^2}(4h_0 + 3r_s\partial_r h_0). \tag{4.12}$$

The intermediate steps are shown in the appendix B.3. Correction r_1 can be further specified for an arbitrary modification h_0. In the next subsection apply κ-Poincaré modification.

4.1.3 κ-Poincaré modification to spherical orbits

We are considering a spherically symmetric κ-Poincaré perturbation (3.43). For Schwarzschild geometry, where $a_0 = \frac{1}{b_0}$, this becomes

$$h(r, p_t, p_r, w) = \sqrt{a(r)}p_t\left(a(r)^{-1}p_r^2 + \frac{1}{r^2}w^2\right). \tag{4.13}$$

We need to find h_0 and $\partial_r h_0$ as they appear in the equation (4.12). First let's find h_0. In the Hamiltonian (3.21) the perturbation function $h(r, p_t, p_r, p_\phi)$ is multiplied by ϵ, therefore the power expansion up to first order respect to ϵ only leaves us with the zeroth order

$$h_0 = h(r_0, p_{0t} = \mathcal{E}_0, p_{0r}, w_0 = \mathcal{L}_0) = \sqrt{a_0}\mathcal{E}_0 \left(a_0^{-1} p_{0r}^2 + \frac{1}{r_0^2}\mathcal{L}_0^2 \right) \tag{4.14}$$

From zeroth order circular orbits for massless particles, we found that $p_{0r} = 0$ (4.2) and $r_0 = \frac{3}{2} r_S$ (4.11). The latter yields that $a_0 = a(r_0) = \left(1 - \frac{r_s}{r_0}\right)^{-1} = 3$. We wish to express energy $\mathcal{E}_0$ in terms of angular momentum $\mathcal{L}_0$ in the equation (4.14). We use equation (4.6), which becomes $\mathcal{E}_0 = \pm \frac{2\mathcal{L}_0}{3\sqrt{3}r_s}$ after we plug in the values of r_0 and $a(r_0)$. Finally, the zeroth order perturbation function h_0 in equation (4.14) becomes

$$h_0 = \pm \frac{8}{27} \frac{\mathcal{L}_0^3}{r_s^3}. \tag{4.15}$$

The last thing left to do is to calculate $\partial_r h_0$, we take partial derivative from (4.14) respect to r

$$\partial_r h_0 = \mathcal{E}_0 \mathcal{L}_0^2 \left(\frac{1}{2\sqrt{a_0}} a_0' \frac{1}{r_0^2} - \sqrt{a_0} \frac{2}{r_0^3} \right) \tag{4.16}$$

$$= \mp \frac{16}{27} \frac{\mathcal{L}_0^3}{r_s^4}, \tag{4.17}$$

where on the last step we plugged in the GR value of radius r_0, values of a_0 and a_0' at r_0, and the $\mathcal{E}_0$ expression which only depends on $\mathcal{L}_0$. Now the radius correction r_1 can be evaluated. We plug in h_0 from equation (4.15) and the partial derivative $\partial_r h_0$ from (4.17) into (4.12) to get

$$r_1 = \pm \frac{\mathcal{L}_0}{3}. \tag{4.18}$$

For the sake of summarizing results

$$r = r_0 + \epsilon r_1 = \frac{3}{2} r_s \pm \epsilon \frac{\mathcal{L}_0}{3} \tag{4.19}$$

We choose the $+$ sign, since we want that zeroths order particles have positive energy $\mathcal{E}_0$. Radius of a photon sphere depends on the angular momentum $\mathcal{L}_0$ of the particles, which can also be translated into the energy $\mathcal{E}_0$ of the particles. Photons with different energies/angular momenta form a thick shell around the Schwarzschild black hole when κ-Poincaré modification is introduced. In GR case all photons orbit at the same altitude around the black hole. The difference between our result (4.19) and earlier result from [15], where $r_1 = \frac{\mathcal{L}_0}{6}$ comes from our choice of Hamiltonians. Further observables are immediately influenced by the modified photon sphere - gravitational lensing [48] and the shadow of black holes, for more information see [15] and references therein.

4.2 Time delays

A light pulse will travel a longer distance in three-space between an emitter and a mirror object when a massive object is brought near a path of the pulse, causing an arrival delay [49]. Possible time delays in time of flight are very good observables to detect deviations from the general relativistic dispersion relations. In the context of cosmology, the time of arrival of high energetic gamma rays and neutrinos from gamma ray bursts is observed, as the quantum gravity effects might accumulate during the propagation of a particle [16, 18].

Apart from the one way travel time of high energetic particles, radar experiments are considered in the measurement of the Shapiro delay (gravitational time delay) and Lunar Laser ranging [50, 51]. The light pulse is sent from an emitter at a position x_e to a mirror at position x_m, where light pulse gets reflected back. The Shapiro delay is a correction to the Newtonian time of flight, and in Lunar Laser ranging the travel time of a light pulse from Earth to the Moon and back is measured with high precision.

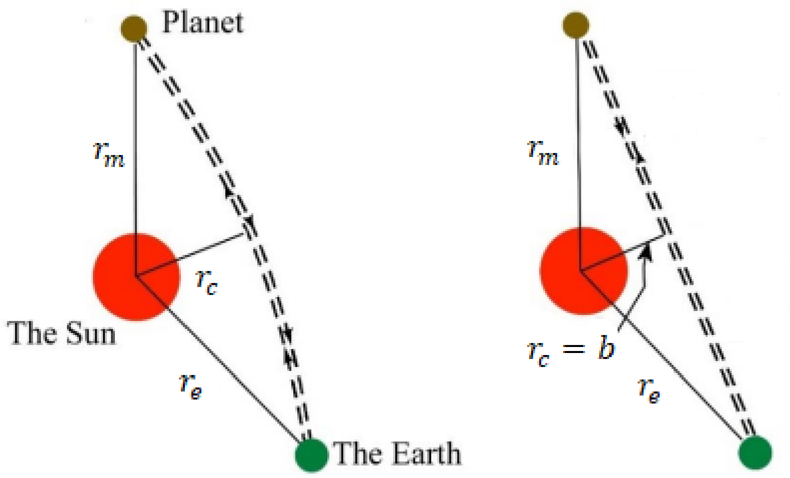

Figure 4.1: The radar experiment. The source of the image is [1]. The displayed image has been slightly modified.

The figure 4.1 is an illustration of the radar experiment. A light beam is sent from the emitter (Earth) at a radial distance r_e from the Sun to a mirror object (a planet, Moon, etc) at a radial distance r_m from the Sun. Light pulse gets reflected back from the mirror object and is sent back to the emitter. The closest point of the encounter between the light pulse and the Sun is denoted as r_c. The left image illustrates the actual curved path taken by a light pulse predicted by GR. The right image is the path taken in the Newtonian limit, where the closest point of the encounter is $r_c = b$, b is the impact parameter.

Having obtained the general Hamilton equations of motion, we will now derive the differential equation for radar time of flight of a light signal subject to a spherically symmetric and stationary modified dispersion relation. In a radar experiment the light pulse propagates on a phase space curve $C(\tau) = (x(\tau), p(\tau))$, which satisfies the massless dispersion relation $H(x(\tau), p(\tau)) = 0$, and the Hamilton equations of motion.

4.2.1 Time delay in general spherically symmetric background

In spherical symmetry the position of the emitter and the observer can be labelled as $x_e = (t_e, r_e, \frac{\pi}{2}, \phi_e)$, and $x_m = (t_m, r_m, \frac{\pi}{2}, \phi_m)$ respectively. Moreover, We assume the emitter and the mirror are spatially at rest, i.e. they propagate on a worldlines $\gamma_e(\tau) = (t_e(\tau), r_e, \theta_e, \phi_e)$ and $\gamma_m(\tau) = (t_m(\tau), r_m, \theta_m, \phi_m)$. For our modified dispersion relation setting, the phase space curve of the light ray is

$$C(\tau) = (t_0(\tau) + \epsilon t_1(\tau), r_0(\tau) + \epsilon r_1(\tau), \tfrac{\pi}{2}, \phi_0(\tau) + \epsilon\phi_1(\tau), \mathcal{E}_0 + \epsilon\mathcal{E}_1, p_{0r}(\tau) + \epsilon p_{0r}(\tau), 0, \mathcal{L}_0 + \epsilon\mathcal{L}_1)\,. \tag{4.20}$$

The differential equation, which is to be solved in order to find time delays, is obtained from the velocity equations (3.27) and (3.31) in the following way [52]

$$\frac{\dot{t}}{\dot{r}} = \frac{dt/d\tau}{dr/d\tau} = \frac{dt}{dr}. \tag{4.21}$$

This differential equation can be solved to obtain time of flight of a radar signal

$$T = 2\left(\int_{r_c}^{r_e} dr\frac{dt}{dr} + \int_{r_c}^{r_m} dr\frac{dt}{dr}\right), \tag{4.22}$$

where r_c is the radial coordinate of the point of closest encounter to the central mass object of the light ray trajectory. At the closest point radial velocity is $\dot{r} = 0$ and thus the Hamilton equation (3.31) yields $p_{0r} = 0$ and $p_{1r} = -\frac{1}{b_{0c}}\bar{\partial}^r h_{0c}$. The zeroth order value of the closest encounter r-coordinate is determined from the zeroth order dispersion relation (3.35) by

$$\frac{\mathcal{L}_0^2}{\mathcal{E}_0^2} = r_{0c}^2 a(r_{0c}) = r_{0c}^2 a_{0c}\,, \tag{4.23}$$

where we also set $m = 0$. The quantity $\frac{\mathcal{L}_0}{\mathcal{E}_0}$ is called impact parameter [52]. We find the first order correction of the value of the closest encounter from the first order dispersion relation

$$r_{1c} = \frac{2r_{0c}(\mathcal{L}_0\mathcal{L}_1 + r_{0c}^2 h_0 - \frac{\mathcal{L}_0^2}{\mathcal{E}_0}\mathcal{E}_1)}{2\mathcal{L}_0^2 + \mathcal{E}_0^2 r_{0c}^3 a_{0c}'}, \tag{4.24}$$

where we also also used the equation (4.23).

For the thesis we only derive the influence of the perturbation $h(x,p)$ on dt/dr. For the sake of clarity, we display velocity equations (3.27) and (3.31)

$$\dot{t} = -a_0\mathcal{E}_0 + \epsilon(\overline{\partial}^t h_0 - a_0' r_1 \mathcal{E}_0 - a_0\mathcal{E}_1) \tag{4.25}$$

$$\dot{r} = b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0). \tag{4.26}$$

Dividing both equations as shown in the equation (4.21) yields

$$\frac{dt}{dr} = \frac{-a_0\mathcal{E}_0 + \epsilon(\overline{\partial}^t h_0 - a_0' r_1 \mathcal{E}_0 - a_0\mathcal{E}_1)}{b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0)}. \tag{4.27}$$

Using the dispersion relation (3.35) we can replace p_{0r} and p_{1r} in the equation above. The first order dispersion relation yields

$$p_{0r} = \pm\sqrt{\frac{a_0}{b_0}}\sqrt{1 - \frac{\mathcal{L}_0^2}{\mathcal{E}_0^2}\frac{1}{a_0 r_0^2}}, \tag{4.28}$$

and the first order dispersion relation gives us p_{1r}

$$p_{1r} = \frac{1}{p_{0r}b_0}\left(\mathcal{E}_0\mathcal{E}_1 a_0 - \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} - h_0\right) + \frac{r_1}{p_{0r}b_0}\left(\frac{a_0'\mathcal{E}_0^2}{2} + \frac{\mathcal{L}_0^2}{r_0^3} - \frac{1}{2}p_{0r}^2 b_0'\right). \tag{4.29}$$

Derivation of p_{0r} and p_{1r} is shown in the appendix C.1. Now that we have expressions for p_{0r} and p_{1r}, the differential equation (4.27) becomes

$$\begin{aligned}\frac{dt}{dr} = -\frac{a_0\mathcal{E}_0}{b_0 p_{0r}} + \epsilon\Bigg(&\frac{b_0 p_{0r}\overline{\partial}^t h_0 + a_0\mathcal{E}_0\left(-\frac{h_0}{p_{0r}} + \overline{\partial}^r h_0\right)}{b_0^2 p_{0r}^2} + \frac{a_0}{b_0 p_{0r}}\left(-\mathcal{E}_1 + \frac{\mathcal{E}_0^2\mathcal{E}_1 a_0}{b_0 p_{0r}^2} - \frac{\mathcal{E}_0\mathcal{L}_0\mathcal{L}_1}{r_0^2 b_0 p_{0r}^2}\right)\Bigg) \\ &+ \epsilon\frac{\mathcal{E}_0}{b_0 p_{0r}}r_1\left(-a_0' + \frac{a_0 b_0'}{b_0} + \frac{a_0}{p_{0r}^2 b_0}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2}p_{0r}^2 b_0'\right)\right),\end{aligned} \tag{4.30}$$

where we expanded the equation (4.27) using the formula shown in the appendix A.4, plugged in p_{1r} from (4.29). We grouped together terms with h_0 and its partial derivatives, also terms with and without r_1. The intermediate steps up until the (4.37) can be found in the appendix C.2.

4.2.2 Time delay from modified Schwarzschild geomtery

We choose the Schwarzschild geometry, therefore we can express b_0 and b_0' in terms of a_0 and a_0', more specifically $b_0 = \frac{1}{a_0}$, and $b_0' = -\frac{a_0'}{a_0^2}$. After the replacements, the differential equation (4.37) becomes

$$\frac{dt}{dr} = -\frac{a_0^2\mathcal{E}_0}{p_{0r}} + \epsilon\left(\frac{a_0 p_{0r}\overline{\partial}^t h_0 + a_0^3\mathcal{E}_0\left(-\frac{1}{p_{0r}}h_0 + \overline{\partial}^r h_0\right)}{p_{0r}^2} + \frac{a_0^2}{p_{0r}}\left(-\mathcal{E}_1 + \frac{\mathcal{E}_0^2\mathcal{E}_1 a_0^2}{p_{0r}^2} - \frac{\mathcal{E}_0\mathcal{L}_0\mathcal{L}_1}{r_0^2 p_{0r}^2}a_0\right)\right)$$
$$+ \epsilon\frac{\mathcal{E}_0 a_0}{p_{0r}}r_1\left(-2a_0' + \frac{a_0^2}{p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2}p_{0r}^2\frac{a_0'}{a_0^2}\right)\right), \tag{4.31}$$

where $a_0 = \left(1 - \frac{r_s}{r_0}\right)^{-1}$. Perturbation function h_0 and its partial derivatives appear in the differential equation, in the next section we evaluate these considering κ-Poincaré modification. In the Schwarzschild geometry p_{0r} in the equation (4.28) becomes

$$p_{0r} = \pm a_0\sqrt{1 - \frac{\mathcal{L}_0^2}{\mathcal{E}_0^2}\frac{1}{a_0 r_0^2}}. \tag{4.32}$$

4.2.3 κ-Poincaré modification to Shapiro time delay

We consider a perturbation (3.43)

$$h = \sqrt{a(r)}p_t\left(+b(r)p_r^2 + \frac{1}{r^2}w^2\right), \tag{4.33}$$

and the zeroth order h_0 was given by (4.14)

$$h_0 = \sqrt{a(r_0)}\mathcal{E}_0\left(a(r_0)^{-1}p_{0r}^2 + \frac{1}{r_0^2}\mathcal{L}_0^2\right). \tag{4.34}$$

Next we calculate the needed partial derivatives of h_0. The first one we find is $\overline{\partial}^r h_0$

$$\overline{\partial}^r h_0 = \frac{2}{\sqrt{a_0}}\mathcal{E}_0 p_{0r} \tag{4.35}$$

And finally we calculate $\overline{\partial}^t h_0$, which gives us

$$\overline{\partial}^t h_0 = \sqrt{a_0}\left(\frac{1}{a_0}p_{0r}^2 + \frac{1}{r_0^2}\mathcal{L}_0^2\right) \tag{4.36}$$

We plug h_0 and its partial derivatives, namely (4.34)-(4.36) into the time delay differential equation (4.31), multiply terms in the brackets, and rearrange terms with $\frac{1}{p_{0r}}$ and r_1. We arrive to the final result

$$\begin{aligned}\frac{dt}{dr} = -\frac{a_0^2\mathcal{E}_0}{p_{0r}} &+ \epsilon\left(\sqrt{a_0}p_{0r} + \frac{1}{p_{0r}}\left(\sqrt{a_0^3}\frac{\mathcal{L}_0^2}{r_0^2} - \mathcal{E}_1 a_0^2 + \sqrt{a_0^5\mathcal{E}_0^2}\right)\right)\\ &- \epsilon\frac{1}{p_{0r}^3}\left(\frac{\sqrt{a_0^7}\mathcal{E}_0\mathcal{L}_0^2}{r_0^2} + \frac{\mathcal{E}_0 a_0^3\mathcal{L}_0\mathcal{L}_1}{r_0^2} - a_0^4\mathcal{E}_0^2\mathcal{E}_1\right)\\ &+ \epsilon\frac{\epsilon_0 a_0}{p_{0r}}r_1\left(-2a_0' + \frac{a_0^2}{p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2}p_{0r}^2 + \frac{a_0'}{a_0}\right)\right)\end{aligned} \tag{4.37}$$

Intermediate steps from (4.31) to (4.37) which were not shown in this section, can be found in the appendix C.3. One can also plug in all the remaining quantities a_0, a_0', p_{0r}. Solving this differential equation is unfortunately out of the scope of this thesis.

The κ-Poincaré perturbation causes a complicated dependence on $\mathcal{L}$ and $\mathcal{E}$ in the first order dt/dr equation. Photons with different energy and angular momentum will experience different time delay correction from their GR value.

4.3 Deflection of light

The deflection angle $\delta\phi$ of a light pulse passing by a massive object is the deviation of its trajectory from a straight line, on which it would propagate in empty space [39]. We assume that the light pulse propagates from infinitely far away towards the source of the gravitational field, reaching the minimal or closest distance r_c, and then propagates away to infinite distance. The deflection angle in this process is the deviation of the light trajectories ϕ coordinate from the value π after passing to infinite distance. The figure 4.2 illustrates the aforementioned scenario.

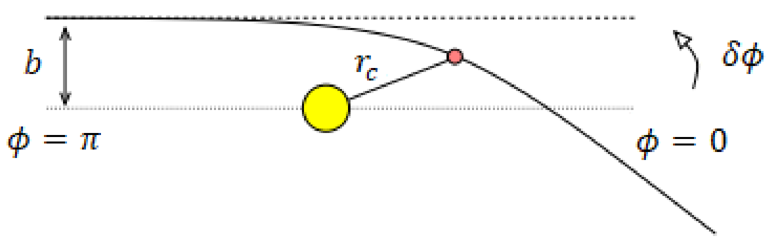

Figure 4.2: Bending of light around a massive object. Source of the image is [29]. The displayed image has been slightly modified.

One way to study the geometry produced by a massive object is by the means of light deflection [46]. Black holes cause the largest space-time curvatures accessible to observations, thus providing an ideal testing ground for different phenomenological models and gravity theories. An interesting effect accompanying the bending of light is the gravitational lensing caused by very massive objects, such as black holes or entire galaxies. We'll study the effects of a modified dispersion relation on light deflection.

4.3.1 Deflection of light in general spherically symmetric background

Having obtained the general Hamilton equations of motion, we will now derive the differential equation for deflection of light subject to a spherically symmetric and stationary modified dispersion relation. Light coming from a distant source propagates on a phase space curve $C(\tau) = (x(\tau), p(\tau))$, which satisfies the massless dispersion relation $H(x(\tau), p(\tau)) = 0$, and the Hamilton equations of motion. The light curve we are considering is

$$C(\tau) = (t_0(\tau)+\epsilon t_1(\tau), r_0(\tau)+\epsilon r_1(\tau), \tfrac{\pi}{2}, \phi_0(\tau)+\epsilon\phi_1(\tau), \mathcal{E}_0+\epsilon\mathcal{E}_1, p_{0r}(\tau)+\epsilon p_{0r}(\tau), 0, \mathcal{L}_0+\epsilon\mathcal{L}_1), \tag{4.38}$$

satisfying the modified dispersion relation for massless particles. Similarly to time delays, the differential equation, which is to be solved in order to find the deflection of light, is obtained from the velocity equations (3.29) and (3.31), see [39]

$$\frac{\dot{\phi}}{\dot{r}} = \frac{d\phi/d\tau}{dr/d\tau} = \frac{d\phi}{dr}. \tag{4.39}$$

The deflection angle $\delta\phi$ can be obtained from

$$\pi + \delta\phi = \int_0^{\phi+\delta\phi} d\phi = 2\int_{r_c}^{\infty} dr\frac{d\phi}{dr}\,, \tag{4.40}$$

where $r_c = r_{0c} + r_{1c}$ is the radial coordinate of the point of closest encounter of the light ray to the central mass. (Same as in the time delay section.) Since $r_c = r_{0c} + \epsilon r_{1c}$ is a minimum in the curve $r(t)$, we again have that $\dot{r} = 0$ and thus $p_{0r} = 0$ and $p_{1r} = -\frac{1}{b_{0c}}\bar{\partial}^r h_{0c}$. The zeroth and first order value of the closest encounter r-coordinate is given by the equations (4.23) and (4.24).

For the thesis we only derive the influence of the perturbation $h(x,p)$ on $d\phi/dr$. For the sake of clarity, we display velocity equations (3.29) and (3.31),

$$\dot{\phi} = \frac{\mathcal{L}_0^2}{r_0^2} + \epsilon\left(\frac{\mathcal{L}_1}{r_0^2} - 2\frac{r_1}{r_0^3}\mathcal{L}_0 + \frac{\partial h_0}{\partial w}\right) \tag{4.41}$$

$$\dot{r} = b_0 p_{0r0} + \epsilon(b_0' r_1 o_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0) \tag{4.42}$$

We plug the velocity equations above into the differential equation (4.39) to get

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{t}} = \frac{\frac{\mathcal{L}_0^2}{r_0^2} + \epsilon\left(\frac{\mathcal{L}_1}{r_0^2} - 2\frac{r_1}{r_0^3}\mathcal{L}_0 + \frac{\partial h}{\partial w}\right)}{b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0)} \tag{4.43}$$

We have already found expressions for p_{0r} and p_{1r} from the modified dispersion relation, equations (4.28) and (4.29) respectively. Again, we see that the right side of the differential equation (4.43) is in the form of a fraction. We can use the formula in the appendix (A.4) to expand the differential equation to the first order in ϵ. After that we plug in the momentum p_{1r}, we get

$$\begin{aligned}\frac{d\phi}{dr} = \frac{\mathcal{L}_0}{r_0^2 b_0 p_{0r}} &+ \epsilon\left(\frac{b_0 p_{0r}\frac{\partial h_0}{\partial w} - \frac{\mathcal{L}_0}{r_0^2}\left(-\frac{h_0}{p_{0r}} + \overline{\partial}^r h_0\right)}{b_0^2 p_{0r}^2} + \frac{1}{b_0 p_{0r}}\left(\frac{\mathcal{L}_1}{r_0^2} - \frac{\mathcal{L}_0}{r_0^2}\frac{\mathcal{E}_0\mathcal{E}_1 a_0}{b_0 p_{0r}} + \frac{\mathcal{L}_0^2\mathcal{L}_1}{r_0^4 b_0 p_{0r}^2}\right)\right) \\ &+ \epsilon\frac{\mathcal{L}_0}{r_0^2}\frac{r_1}{b_0 p_{0r}}\left(-\frac{2}{r_0} - \frac{b_0'}{b_0} - \frac{1}{b_0 p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2}p_{0r}^2 b_0'\right)\right),\end{aligned} \tag{4.44}$$

where we regrouped terms with h_0 and its partial derivatives, also terms with and without r_1. All the intermediate steps from (4.43) to the equation above are shown in the appendix D.1.

4.3.2 Deflection of light from modified Schwarzschild geometry

In the Schwarzschild geometry we have that $b_0 = \frac{1}{a_0}$ and $b_0' = -\frac{a_0'}{a_0^2}$. We plug these into (4.44), which then becomes

$$\frac{d\phi}{dr} = \frac{\mathcal{L}_0 a_0}{r_0^2 p_{0r}} + \epsilon \left(\frac{a_0 p_{0r} \frac{\partial h_0}{\partial w} - \frac{\mathcal{L}_0}{r_0^2} \left(-\frac{h_0}{p_{0r}} + \overline{\partial}^r h_0 \right)}{p_{0r}^2} + \frac{a_0}{p_{0r}} \left(\frac{\mathcal{L}_1}{r_0^2} - \frac{\mathcal{L}_0}{r_0^2} \frac{\mathcal{E}_0 \mathcal{E}_1 a_0^2}{p_{0r}^2} + \frac{\mathcal{L}_0^2 \mathcal{L}_1 a_0}{r_0^4 p_{0r}^2} \right) \right)$$
$$+ \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{a_0}{p_{0r}} r_1 \left(-\frac{2}{r_0} + \frac{a_0'}{a_0} - \frac{a_0}{p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2} \mathcal{E}_0^2 a_0' + \frac{1}{2} p_{0r}^2 \frac{a_0'}{a_0} \right) \right), \tag{4.45}$$

where $a_0 = \left(1 - \frac{r_s}{r_0}\right)^{-1}$. The perturbation function h_0 and its partial derivatives appear in the differential equation. In the next section we evaluate these considering κ-Poincaré modification.

4.3.3 κ-Poincaré modification to deflection of light

We consider a perturbation (3.43)

$$h = \sqrt{a(r)} p_t \left(+b(r) p_r^2 + \frac{1}{r^2} w^2 \right), \tag{4.46}$$

and the zeroth order h_0 was given by (4.14)

$$h_0 = \sqrt{a(r_0)} \mathcal{E}_0 \left(a(r_0)^{-1} p_{0r}^2 + \frac{1}{r_0^2} \mathcal{L}_0^2 \right), \tag{4.47}$$

where we set $b_0 = \frac{1}{a_0}$ due to the choice of Schwarzschild geometry. Next, we find $\overline{\partial}^r h_0$

$$\overline{\partial}^r h_0 = 2 \frac{1}{\sqrt{a_0}} \mathcal{E}_0. \tag{4.48}$$

Last but not least, we see that $\frac{\partial h_0}{\partial w}$ becomes

$$\frac{\partial h_0}{\partial w} = 2 \sqrt{a_0} \mathcal{E}_0 \frac{\mathcal{L}_0}{r_0^2}. \tag{4.49}$$

We plug the results above (4.47)-(4.49) into (4.45) to get the final result

$$\begin{aligned}\frac{d\phi}{dr} = \frac{\mathcal{L}_0 a_0}{r_0^2 p_{0r}} + \epsilon \left(\frac{1}{r_0^2 p_{0r}} \left(\sqrt{a_0^3} \mathcal{E}_0 \mathcal{L}_0 + a_0 \mathcal{L}_1 \right) + \frac{\mathcal{L}_0}{r_0^2 p_{0r}^3} \left(\frac{\sqrt{a_0^5} \mathcal{E}_0 \mathcal{L}_0^2}{r_0^2} - \mathcal{E}_0 \mathcal{E}_1 a_0^3 + \frac{\mathcal{L}_0 \mathcal{L}_1}{r_0^2} a_0^2 \right) \right) \\ + \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{a_0}{p_{0r}} r_1 \left(-\frac{2}{r_0} + \frac{a_0'}{a_0} - \frac{a_0}{p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2} \mathcal{E}_0^2 a_0' + \frac{1}{2} p_{0r}^2 \frac{a_0'}{a_0} \right) \right)\end{aligned} \tag{4.50}$$

Intermediate steps from the equation (4.45) to the equation above are shown in the appendix D.2.

The κ-Poincaré perturbation causes that $d\phi/dr$ obtains a complicated dependence on $\mathcal{E}$ and $\mathcal{L}$, hence photons of different energy and angular momentum will experience different deflection correction from their GR value. However, already on the GR level, the light deflection depends on $\mathcal{E}_0$ and $\mathcal{L}_0$.

Conclusion

We began with a small introduction on Lagrangian and Hamiltonian formalism. We saw that both formalisms for describing point particle motion are equivalent for they yield the same equations of motion. The well-known special relativistic energy-momentum relation (dispersion relation) can be expressed covariantly with the help of space-time metric. The covariant representation of the dispersion relation was exactly the relativistic Hamiltonian. It's a general principle that dispersion relations can be interpreted as level sets of Hamilton functions. The Hamilton functions, on the other hand, determine the motion of test particles.

Dispersion relations are relevant in the context of quantum gravity phenomenology. Particles with higher energy are expected to interact stronger with the quantum nature of gravity. These quantum gravitational interactions are effectively encoded into modified dispersion relations predicted by some of the quantum gravity models. Hamilton equations of motion determine the perturbed motion of particles which then can be tested by observations. One interesting effect implying from modified dispersion relations is energy-dependent propagation velocity. High energetic gamma-ray bursts are an ideal testing ground for modified dispersion relations.

Another place to look for manifestations of quantum gravity is where gravitational field is strong, for example, in the vicinity of black holes. One of the most studied models is the κ-Poincaré dispersion relation emerging from quantum deformations of the Poincaré algebra. Without violating the principle of relativity, another invariant scale besides the speed of light is introduced - the Planck energy. Lorentz transformation is modified to preserve these scales, giving rise to a modified energy-momentum (dispersion) relation. Particles with κ-Poincaré symmetry can be interpreted as propagating on flat space-time. In the literature κ-deformed Hamiltonian has been implemented on space-time with an arbitrary curvature because a lot of observable effects are expected to become apparent in the strong field regime. The constructed Hamiltonian takes locally the form of a κ-Poincaré dispersion relation. Local κ-Poincaré invariance is ensured in a similar way that general relativity is locally Lorentz invariant.

Initially we considered the most general spherically symmetric and time independent modified dispersion relation. Depending on the model one is interested in, other perturbation functions can be chosen. We derived the most general Hamilton equations of motion in spherical symmetry for arbitrarily curved space-time. We then moved on to derive circular orbits around a Schwarzschild black hole for massless particles. From the generalization of the κ-Poincaré dispersion relation we found that the correction term to GR solution is angular momentum dependent, which in turn can be translated to energy. Photons with different angular momentum/energy propagate on different altitudes, forming a thick shell around a black hole. Last but not least, we derived the differential equations for both effects, light deflection and time delay. The first order correction terms from κ-Poincaré dispersion relation in Schwarzschild background attained a complex dependence on both, angular momentum and energy, hence photons with different angular momentum and energy will experience different deflection and time delay corrections.

Then future prospects are to finish the calculation of the time delay and the deflection angle. We encountered mathematical difficulties while evaluating the time delay and light deflection integrals. Solving these is one task for an ongoing research project. Another plan is to derive observations from axially symmetric modified dispersion relations, since all realistic black holes are rotating.

Acknowledgements

I would like to express my sincere gratitude to my supervisors Christian Pfeifer and Manuel Hohmann for being supportive and patient. Working alongside them was very inspiring, the discussions we had were insightful and encouraging. I would also like to thank my professors and fellow students who have made the university experience one of a kind. Last but not least, I would like to thank my family and friends for their support.

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Appendix A

Perturbation theory

We study the κ-Poincaré modified dispersion relation and the equations of motion derived from it by using the well known perturbation theory. The perturbative approach is also chosen in the article [15], which we used as one of the reference points for our calculations. In our case and in the article [15] the equations derived from the Hamiltonian (3.37) are impossible to solve analytically, and perturbation theory turns out to be a good approximation method.

Perturbation theory is based on Taylor expansion [53]. In general, the Taylor expansion of a function f around the point x is

$$f(x+\epsilon y)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k f}{d\epsilon^k}\bigg|_{\epsilon=0}\epsilon^k = f(x)+\frac{d}{d\epsilon}f(x+\epsilon y)\bigg|_{\epsilon=0}\epsilon+\frac{1}{2}\frac{d^2}{d\epsilon^2}f(x+\epsilon y)\bigg|_{\epsilon=0}\epsilon^2+... \tag{A.1}$$

$$= f(x)+y^\mu\partial_\mu f(x)\epsilon+\frac{1}{2}y^\mu y^\nu\partial_\mu\partial_\nu f(x)\epsilon^2+..., \tag{A.2}$$

where ϵ is a small parameter, $f(x)$ is the so-called zeroth order solution (in our case, general relativistic solution), and the terms with ϵ of first and higher powers are the first and higher order corrections to the well-known zeroth order solution.

Another useful formula is the Taylor expansion of a function which is in the form of a fraction. The Taylor expansion of such function is

$$\frac{x+\epsilon y}{v+\epsilon w}=\frac{x}{v+\epsilon w}+\epsilon\frac{y}{v+\epsilon w}=\frac{x}{v}\frac{1}{(1+\epsilon\frac{w}{v})}+\epsilon\frac{y}{v}+\mathcal{O}(\epsilon^2)=\frac{x}{v}\left(1-\epsilon\frac{w}{v}\right)+\epsilon\frac{y}{v}+\mathcal{O}(\epsilon^2) \tag{A.3}$$

$$=\frac{x}{v}+\epsilon\left(\frac{y}{v}-\frac{xw}{v^2}\right)+\mathcal{O}(\epsilon^2). \tag{A.4}$$

Appendix B

Derivations for circular photon orbits

B.1 $\mathcal{E}_0$ and $\mathcal{E}_1$ from modified dispersion relation

We wish to derive expressions for $\mathcal{E}_0$ and $\mathcal{E}_1$ which appeared in the equations (4.6) and (4.7). We start off by displaying the dispersion relation (3.35), where we already set p_{0r} and p_{1r} equal to zero,

$$-m^2 = -m_0^2 - \epsilon 2 m_0 m_1 = \frac{1}{2}\left(-a_0\mathcal{E}_0^2 + \frac{1}{r_0^2}\mathcal{L}_0^2\right) + \epsilon\left(-\frac{1}{2}a_0'\mathcal{E}_0^2 r_1 - \mathcal{E}_0 a_0 \mathcal{E}_1 - \frac{\mathcal{L}_0^2}{r_0^3}r_1 + \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + h_0\right). \tag{B.1}$$

The zeroth order equation is

$$-m_0^2 = \frac{1}{2}\left(-a_0\mathcal{E}_0^2 + \frac{1}{r_0^2}\mathcal{L}_0^2\right). \tag{B.2}$$

We rearrange the terms in the equation in a way that the term with $\mathcal{E}_0^2$ is on the left side and the rest is on the right side of the equation. We get

$$a_0\mathcal{E}_0^2 = \frac{1}{r_0^2}\mathcal{L}_0^2 + m_0^2. \tag{B.3}$$

We divide both sides by a_0 assuming that it is not equal to zero, and we take a square root of both sides. It can be easily seen that the final result is the following:

$$\mathcal{E}_0 = \pm\frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}}{r_0\sqrt{a_0}}. \tag{B.4}$$

To find the first order energy correction $\mathcal{E}_1$, we analyze the first order dispersion relation (B.1),

$$-2m_0m_1 = -\frac{1}{2}a_0'\mathcal{E}_0^2 r_1 - \mathcal{E}_0 a_0 \mathcal{E}_1 - \frac{\mathcal{L}_0^2}{r_0^3}r_1 + \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + h_0. \tag{B.5}$$

We take terms with $\mathcal{E}_1$ to the left side of the equation and the rest to the right side of the equation and we divide both sides by $\mathcal{E}_0 a_0$. We get

$$\mathcal{E}_1 = -\frac{1}{2}\frac{a_0'}{a_0}\mathcal{E}_0 r_1 + \frac{1}{\mathcal{E}_0 a_0}\left(-\frac{\mathcal{L}_0^2}{r_0^3}r_1 + \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + h_0 + 2m_0m_1\right). \tag{B.6}$$

We plug in the zeroth order energy (B.4) where ever we see $\mathcal{E}_0$ appearing, this yields

$$\mathcal{E}_1 = -\frac{1}{2}\frac{a_0'}{a_0}\left(\pm\frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2r_0^2}}{r_0\sqrt{a_0}}r_1\right) + \frac{1}{a_0}\frac{r_0\sqrt{a_0}}{(\pm\sqrt{\mathcal{L}_0^2 + 2m_0^2r_0^2})}\left(-\frac{\mathcal{L}_0^2}{r_0^3}r_1 + \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + h_0 + 2m_0m_1\right). \tag{B.7}$$

We notice the global $\pm$ sign, and both terms on the right side are multiplied by a factor of $\frac{1}{\sqrt{a_0}}$, we bring it in front of the brackets. Also, in the second term the common denominator is r_0^3. We arrive to the final result

$$\mathcal{E}_1 = \pm\frac{1}{\sqrt{a_0}}\left(\frac{-\mathcal{L}_0^2 r_1 + \mathcal{L}_0\mathcal{L}_1 r_0 + (h_0 + 2m_0m_1)r_0^3}{r_0^2\sqrt{\mathcal{L}_0^2 + 2m_0^2r_0^2}} - \frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2r_0^2}\; a_0'}{2r_0a_0}r_1\right). \tag{B.8}$$

B.2 General expressions for r_0 and r_1

We have already derived expressions for $\mathcal{E}_0$ and $\mathcal{E}_1$ which was needed to find the zeroth order radius r_0 and first order correction r_1. The next step is to use the remaining Hamilton equation of motion (4.5)

$$0 = \dot{p}_{0r} + \epsilon\dot{p}_{1r} = \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\epsilon\left(2\mathcal{E}_0 a_0'\mathcal{E}_1 + \mathcal{E}_0^2 a_0'' r_1 - 6\frac{\mathcal{L}_0^2}{r_0^4}r_1 + \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} - 2\partial_r h_0\right). \tag{B.9}$$

The zeroth order equation is

$$0 = \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{\mathcal{L}_0^2}{r_0^3}. \tag{B.10}$$

We wish to replace $\mathcal{E}_0$ in the equation above by using the expression for $\mathcal{E}_0$ (B.4). We get

$$0 = \frac{1}{2}\frac{\mathcal{L}_0^2 + 2m_0^2r_0^2}{r_0^2a_0}a_0' + \frac{\mathcal{L}_0^2}{r_0^3}. \tag{B.11}$$

After regrouping similar terms we arrive to the final result

$$0 = m_0^2 \frac{a_0'}{a_0} + \frac{\mathcal{L}_0^2}{2r_0^2}\left(\frac{a_0'}{a_0} + \frac{2}{r_0}\right), \tag{B.12}$$

which is exactly the same as (4.8). After we have specified the function a_0, this equation can be solved for zeroth order radius solution r_0. First order correction r_1 can be found using the first order equation from (B.9),

$$0 = 2\mathcal{E}_0 a_0' \mathcal{E}_1 + \mathcal{E}_0^2 a_0'' r_1 - 6\frac{\mathcal{L}_0^2}{r_0^4} r_1 + \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} - 2\partial_r h_0. \tag{B.13}$$

We take terms with r_1 to the left side of the equation, and divide both sides by the factor $\mathcal{E}_0^2 a_0'' - 6\frac{\mathcal{L}_0^2}{r_0^4}$. This gives us

$$r_1 = \frac{1}{\mathcal{E}_0^2 a_0'' - 6\frac{\mathcal{L}_0^2}{r_0^4}}\left(-2\mathcal{E}_0\mathcal{E}_1 a_0' - \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} + 2\partial_r h_0\right). \tag{B.14}$$

We plug in $\mathcal{E}_0$ and $\mathcal{E}_1$ from the equations (B.4) and (B.6) respectively, yielding

$$\begin{aligned} r_1 = \frac{1}{\frac{\mathcal{L}_0^2 + 2m_0^2 r_0^2}{r_0^2 a_0} a_0'' - 6\frac{\mathcal{L}_0^2}{r_0^4}}\Bigg[-2\left(\pm\frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}}{r_0\sqrt{a_0}}\right) a_0' \Bigg(\pm\frac{1}{\sqrt{a_0}}\Bigg(-\frac{\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}}{2r_0}\frac{a_0'}{a_0} r_1 + \\ + \frac{-\mathcal{L}_0^2 r_1 + \mathcal{L}_0\mathcal{L}_1 r_0 + (h_0 + 2m_0 m_1) r_0^3}{r_0^2\sqrt{\mathcal{L}_0^2 + 2m_0^2 r_0^2}}\Bigg)\Bigg) - \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} + 2\partial h_0\Bigg]. \end{aligned} \tag{B.15}$$

After multiplying all terms in the square brackets, we get

$$r_1 = \frac{1}{\frac{\mathcal{L}_0^2 + 2m_0^2 r_0^2}{r_0^2 a_0} a_0'' - 6\frac{\mathcal{L}_0^2}{r_0^4}}\left(\frac{\mathcal{L}_0^2 + 2m_0^2 r_0^2}{r_0^2 a_0^2} a_0'^2 r_1 + \frac{-\mathcal{L}_0^2 r_1 + \mathcal{L}_0\mathcal{L}_1 r_0 + (h_0 + 2m_0 m_1) r_0^3}{r_0^3 a_0} a_0' - \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} + 2\partial_r h_0\right). \tag{B.16}$$

We notice that there are terms with r_1 on the both side of the equation. After regrouping terms multiplied with r_1, the equation then becomes

$$\begin{aligned} \left(\frac{\mathcal{L}_0^2 + 2m_0^2 r_0^2}{r_0^2 a_0} a_0'' - 6\frac{\mathcal{L}_0^2}{r_4} - \frac{\mathcal{L}_0^2 + 2m_0^2 r_0^2}{r_0^2 a_0^2} a_0'^2 - \frac{\mathcal{L}_0^2 a_0'}{r_0^3 a_0}\right) r_1 = -2\frac{\mathcal{L}_0\mathcal{L}_1 r_0 + (h_0 + 2m_0 m_1) r_0^3}{r_0^3 a_0} a_0' - \frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3} + \\ + 2\partial_r h_0. \end{aligned} \tag{B.17}$$

On the left side of the equation we group together terms multiplied with $\mathcal{L}_0^2$ and m_0^2. On the right side we choose the common denominator to be r_0^3. We get

$$\left[\mathcal{L}_0^2\left(\frac{a_0''}{r_0^2 a_0}-\frac{a_0'^2}{r_0^2 a_0^2}-\frac{2a_0'}{r_0^3 a_0}-6\frac{\mathcal{L}_0^2}{r_0^4}\right)+2m_0^2\left(\frac{a_0''}{a_0}-\frac{a_0'^2}{a_0^2}\right)\right]r_1=-2\frac{\mathcal{L}_0\mathcal{L}_1 r_0+(h_0+2m_0m_1)r_0^3}{r_0^3 a_0}a_0'-\\ -\frac{4\mathcal{L}_0\mathcal{L}_1}{r_0^3}+\frac{2\partial_r h_0 r_0^3}{r_0^3}. \tag{B.18}$$

At this point it can be seen that the final result is

$$r_1=\frac{2\left(r_0^3(\partial_r h_0-\frac{a_0'}{a_0}(h_0+2m_0m_1))-\mathcal{L}_0\mathcal{L}_1(2+\frac{a_0'}{a_0}r_0)\right)}{r_0^3\mathcal{L}_0^2(\frac{a_0''}{a_0r_0^2}-\frac{a_0'^2}{a_0^2r_0^2}-\frac{2a_0'}{r_0^3a_0}-\frac{6}{r_0^4})-2m_0^2r_0^3(\frac{a_0'^2}{a_0^2}-\frac{a_0''}{a_0})}, \tag{B.19}$$

which is the r_1 correction displayed in the equation (4.9).

B.3 κ-Poincaré modified photon orbits from spherically symmetric background

The expression (B.19) can be further specified by taking $a_0=a(r_0)=\left(1-\frac{r_s}{r_0}\right)^{-1}$, where $r_0=\frac{3}{2}r_s$. As we are considering massless particles, we take $m=m_0+\epsilon m_1=0$. The values of a_0, its first and second order derivatives at $r_0=\frac{3}{2}r_s$ are the following

$$a_0=\frac{1}{1-\frac{r_s}{r_0}} \qquad\Longrightarrow\qquad a_0\left(r_0=\frac{3}{2}r_s\right)=3 \tag{B.20}$$

$$a_0'=-\frac{1}{(1-\frac{r_s}{r_0})^2}\frac{r_s}{r_0^2} \qquad\Longrightarrow\qquad a_0'\left(r_0=\frac{3}{2}r_s\right)=\frac{-4}{r_s} \tag{B.21}$$

$$a_0''=\frac{2}{\left(1-\frac{r_s}{r_0}\right)^3}\left(\frac{r_s}{r_0^2}\right)^2+2\frac{1}{\left(1-\frac{r_s}{r_0}\right)^2}\left(\frac{r_s}{r_0^3}\right) \qquad\Longrightarrow\qquad a_0''\left(r_0=\frac{3}{2}r_s\right)=\frac{16}{r_s^2}. \tag{B.22}$$

After we plug in $m_0=0$, a_0 and its derivatives into the first order radius correction (B.19), we get

$$r_1=\frac{2\left(\partial_r h_0+\frac{4}{3r_s}h_0\right)}{\mathcal{L}_0^2\left(\frac{16}{r_s^2}\frac{1}{3}\frac{4}{9r_s^2}-\frac{16}{r_s^2}\frac{1}{9}\frac{4}{9r_s^2}+2\frac{4}{r_s}\frac{8}{27r_s^3}\frac{1}{3}-6\frac{16}{81r_s^4}\right)}=\frac{9r_s^3}{16\mathcal{L}_0^2}(4h_0+3r_s\partial_r h_0), \tag{B.23}$$

which is the result we displayed in the equation (4.12).

Appendix C

Derivations for time delays

C.1 p_{0r} and p_{1r} from modified dispersion relation

We start off by displaying the dispersion relation equation (3.35) for massless $m = 0$ particles

$$0 = -a_0\mathcal{E}_0^2+b_0p_{0r}^2+\frac{1}{r_0^2}\mathcal{L}_0^2+\epsilon\left(-a_0'\mathcal{E}_0^2r_1 - 2\mathcal{E}_0\mathcal{E}_1a_0 + p_{0r}^2b_0'r_1 + 2p_{0r}p_{1r}b_0 - 2\frac{\mathcal{L}_0^2}{r_0^3}r_1 + 2\frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + 2h_0\right). \tag{C.1}$$

From zeroth order dispersion relation we find that p_{0r} takes the following form

$$b_0p_{0r}^2 = a_0\mathcal{E}_0^2 - \frac{1}{r_0^2}\mathcal{L}_0^2 \quad \implies p_{0r} = \pm\frac{\sqrt{a_0\mathcal{E}_0^2 - \frac{1}{r_0^2}\mathcal{L}_0^2}}{\sqrt{b_0}}. \tag{C.2}$$

By pulling $\mathcal{E}_0$ and $\sqrt{a_0}$ out of the square root, it is easy to see that we arrive to the expression

$$p_{0r} = \pm\mathcal{E}_0\sqrt{\frac{a_0}{b_0}}\sqrt{a - \frac{\mathcal{L}_0^2}{\mathcal{E}_0^2}\frac{1}{a_0r_0^2}}, \tag{C.3}$$

which is the same as (4.28). First order momentum p_{1r} can be found using the first order dispersion relation

$$2p_{0r}p_{1r}b_0 = a_0'\mathcal{E}_0^2r_1 - 2\mathcal{E}_0\mathcal{E}_1a_0 + p_{0r}^2b_0'r_1 - 2\frac{\mathcal{L}_0^2}{r_0^3}r_1 + 2\frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} + 2h_0, \tag{C.4}$$

where we already took the term with p_{1r} in the equation (C.1) to the left side of the equation and changed the signs. We divide both sides by a factor of $2p_{0r}p_{1r}b_0$ to find that

$$p_{1r} = \frac{a_0'\mathcal{E}_0^2 r_1 + 2\mathcal{E}_0\mathcal{E}_1 a_0 - p_{0r}^2 b_0' r_1 + 2\frac{\mathcal{L}_0^2}{r_0^3} r_1 - 2\frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} - 2h_0}{2p_{0r}b_0} \tag{C.5}$$

$$= \frac{1}{p_{0r}b_0}\left(\mathcal{E}_0\mathcal{E}_1 a_0 - \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} - h_0\right) + \frac{r_1}{p_{0r}b_0}\left(\frac{a_0'\mathcal{E}_0^2}{2} + \frac{\mathcal{L}_0^2}{r_0^3} - \frac{1}{2}p_{0r}^2 b_0'\right), \tag{C.6}$$

where in the final step we regrouped terms with and without r_1. The above result is identical to that of we wanted to show (4.29).

C.2 General expression for dt/dr

Starting point of the derivation is equation (4.27) which is displayed as (C.7). There are terms with ϵ in the denominator, as we see in the equation (C.7). To get rid of those, we can perform a Taylor expansion on the fraction given with the formula (A.4) to obtain

$$\frac{dt}{dr} = \frac{-a_0\mathcal{E}_0 + \epsilon(\overline{\partial}^t h_0 - a_0' r_1 \mathcal{E}_0 - a_0\mathcal{E}_1)}{b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0)} \tag{C.7}$$

$$= -\frac{a_0\mathcal{E}_0}{b_0p_{0r}} + \epsilon\left(\frac{\overline{\partial}^t h_0 - a_0' r_1\mathcal{E}_0 - a_0\mathcal{E}_1}{b_0p_{0r}} + \frac{a_0\mathcal{E}_0(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0)}{b_0^2 p_{0r}^2}\right). \tag{C.8}$$

We plug in the p_{1r} expression from the equation (4.29), which gives us

$$\frac{dt}{dr} = -\frac{a_0\mathcal{E}_0}{b_0p_{0r}} + \epsilon\Bigg(\frac{\underline{\overline{\partial}^t h_0} - \underline{a_0' r_1\mathcal{E}_0} - a_0\mathcal{E}_1}{b_0p_{0r}} \tag{C.9}$$

$$+ \frac{a_0\mathcal{E}_0\left(\underline{b_0' r_1 p_{0r}} + b_0\left(\frac{1}{b_0p_{0r}}\left(\mathcal{E}_0\mathcal{E}_1 a_0 - \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} - \underline{h_0}\right) + \underline{\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2}p_{0r}^2 b_0'\right)\frac{r_1}{p_{0r}b_0}}\right) + \underline{\overline{\partial}^r h_0}\right)}{b_0^2 p_{0r}^2}\Bigg). \tag{C.10}$$

To make it easier to follow, similar terms have been underlined with different colors. We group those together and see that the final result is the following

$$\frac{dt}{dr} = -\frac{a_0\mathcal{E}_0}{b_0 p_{0r}} + \epsilon\left(\frac{b_0 p_{0r}\overline{\partial}^t h_0 + a_0\mathcal{E}_0\left(-\frac{h_0}{p_{0r}} + \overline{\partial}^r h_0\right)}{b_0^2 p_{0r}^2} + \frac{a_0}{b_0 p_{0r}}\left(-\mathcal{E}_1 + \frac{\mathcal{E}_0^2\mathcal{E}_1 a_0}{b_0 p_{0r}^2} - \frac{\mathcal{E}_0\mathcal{L}_0\mathcal{L}_1}{r_0^2 b_0 p_{0r}^2}\right)\right) \tag{C.11}$$

$$+\epsilon\frac{\mathcal{E}_0}{b_0 p_{0r}} r_1\left(-a_0' + \frac{a_0 b_0'}{b_0} + \frac{a_0}{p_{0r}^2 b_0}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2}p_{0r}^2 b_0'\right)\right), \tag{C.12}$$

it is the same expression as (4.30) which we wanted to show.

C.3 κ-Poincaré modified dt/dr equation from spherically symmetric background

We start off with displaying the equation (4.31)

$$\frac{dt}{dr} = -\frac{a_0^2\mathcal{E}_0}{p_{0r}} + \epsilon\left(\frac{a_0 p_{0r}\overline{\partial}^t h_0 + a_0^3\mathcal{E}_0\left(-\frac{1}{p_{0r}}h_0 + \overline{\partial}^r h_0\right)}{p_{0r}^2} + \frac{a_0^2}{p_{0r}}\left(-\mathcal{E}_1 + \frac{\mathcal{E}_0^2\mathcal{E}_1 a_0^2}{p_{0r}^2} - \frac{\mathcal{E}_0\mathcal{L}_0\mathcal{L}_1}{r_0^2 p_{0r}^2}a_0\right)\right) \tag{C.13}$$

$$+\epsilon\frac{\mathcal{E}_0 a_0}{p_{0r}} r_1\left(-2a_0' + \frac{a_0^2}{p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2}p_{0r}^2\frac{a_0'}{a_0^2}\right)\right). \tag{C.14}$$

We plug in h_0 and its partial derivatives from the equations (4.34)-(4.36) to get

$$\frac{dt}{dr} = -\frac{a_0^2\mathcal{E}_0^2}{p_{0r0}} + \epsilon\left(\frac{1}{p_{0r}^2}\left(a_0 p_{0r}\sqrt{a_0}\left(\frac{p_{0r}^2}{a_0} + \frac{\mathcal{L}_0^2}{r_0^2}\right) + a_0^3\mathcal{E}_0\left(-\frac{\sqrt{a_0}}{p_{0r}}\mathcal{E}_0\left(\frac{p_{0r}^2}{a_0} + \frac{\mathcal{L}_0^2}{r_0^2}\right) + \frac{2}{\sqrt{a_0}}\mathcal{E}_0 p_{0r}\right)\right)\right) \tag{C.15}$$

$$+\epsilon\frac{\mathcal{E}_0}{a_{\ 0}} p_{0r} r_1\left(-2a_0' + \frac{a_0^2}{p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2}p_{0r}^2\frac{a_0'}{a_0}\right)\right) \tag{C.16}$$

$$+\epsilon\frac{a_0^2}{p_{0r}}\left(-\mathcal{E}_1 + \frac{a_0^2\mathcal{E}_0^2\mathcal{E}_1}{p_{0r}^2} - \frac{\mathcal{E}_1\mathcal{L}_0\mathcal{L}_1}{r_0^2 p_{0r}^2}a_0\right). \tag{C.17}$$

After multiplying all terms in the brackets, the equation above becomes

$$\frac{dt}{dr} = -\frac{a_0^2\mathcal{E}_0}{p_{0r}} + \epsilon\left(\sqrt{a_0}p_{0r} + \sqrt{a_0^3}\frac{\mathcal{L}_0^2}{r_0^2 p_{0r}} + \mathcal{E}_0^2\left(-\frac{\sqrt{a_0^5}}{p_{0r}} - \frac{\sqrt{a_0^7}\mathcal{L}_0^2}{r_0^2 p_{0r}^3}\right) + \frac{2\sqrt{a_0}}{p_{0r}}\right) \tag{C.18}$$

$$+ \epsilon\frac{\mathcal{E}_0 a_0}{p_{0r}} r_1\left(\frac{a_0^2}{p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2}p_{0r}^2\frac{a_0'}{a_0}\right)\right) \tag{C.19}$$

$$+ \epsilon\left(-\mathcal{E}_1 + \frac{a_0^2\mathcal{E}_0^2\mathcal{E}_1}{p_{0r}^2} - \frac{\mathcal{E}_0\mathcal{L}_0\mathcal{L}_1}{p_{0r}^2 r_0^2}a_0\right) \tag{C.20}$$

$$= -\frac{a_0^2\mathcal{E}_0}{p_{0r}} + \epsilon\left(\sqrt{a_0}p_{0r} + \frac{1}{p_{0r}}\left(\sqrt{a_0^3}\frac{\mathcal{L}_0^2}{r_0^2} - \mathcal{E}_1 a_0^2 + \sqrt{a_0^5\mathcal{E}_0^2}\right)\right) \tag{C.21}$$

$$- \epsilon\frac{1}{p_{0r}^3}\left(\frac{\sqrt{a_0^7}\mathcal{E}_0\mathcal{L}_0^2}{r_0^2} + \frac{\mathcal{E}_0 a_0^3\mathcal{L}_0\mathcal{L}_a}{r_0^2} - a_0^4\mathcal{E}_0^2\mathcal{E}_1\right) \tag{C.22}$$

$$+ \epsilon\frac{\epsilon_0 a_0}{p_{0r}} r_1\left(-2a_0' + \frac{a_0^2}{p_{0r}^2}\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2}p_{0r}^2 + \frac{a_0'}{a_0}\right)\right), \tag{C.23}$$

where in the last step we grouped together terms with $\frac{1}{p_{0r}}$, $\frac{1}{p_{0r}^3}$, and r_1. We arrived to the same result which was displayed in the equation (4.37).

Appendix D

Derivations for deflections of light

D.1 General expression for $d\phi/dr$

The differential equation (4.43) of interest is the following

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{t}} = \frac{\frac{\mathcal{L}_0^2}{r_0^2} + \epsilon\left(\frac{\mathcal{L}_1}{r_0^2} - 2\frac{r_1}{r_0^3}\mathcal{L}_0 + \frac{\partial h}{\partial w}\right)}{b_0 p_{0r} + \epsilon(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0)}. \tag{D.1}$$

We perform a Taylor expansion on the differential equation above using the formula in the appendix A.4, this yields

$$\frac{d\phi}{dr} = \frac{\mathcal{L}_0}{r_0^2 b_0 p_{0r}} + \epsilon\left(\frac{\frac{\mathcal{L}_1}{r_0^2} - 2\frac{r_1}{r_0^3}\mathcal{L}_0 + \frac{\partial h_0}{\partial w}}{b_0 p_{0r}} - \frac{\frac{\mathcal{L}_0}{r_0^2}(b_0' r_1 p_{0r} + b_0 p_{1r} + \overline{\partial}^r h_0)}{b_0^2 p_{0r}^2}\right). \tag{D.2}$$

Next, we plug in p_{1r} from (4.29) which we derived from the dispersion relation. We get

$$\frac{d\phi}{dr} = \frac{\mathcal{L}_0}{r_0^2 b_0 p_{0r}} + \epsilon\left(\frac{\frac{\mathcal{L}_1}{r_0^2} - \underline{-2\frac{r_1}{r_0^3}\mathcal{L}_0} + \underline{\frac{\partial h_0}{\partial w}}}{b_0 p_{0r}}\right. \tag{D.3}$$

$$\left. - \frac{\frac{\mathcal{L}_0}{r_0^2}\left(\underline{b_0' r_1 p_{0r}} + b_0\left(\frac{1}{b_0 p_{0r}}\left(\mathcal{E}_0\mathcal{E}_1 a_0 - \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} - \underline{h_0}\right) + \underline{\left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2}p_{0r}^2 b_0'\right)\frac{r_1}{p_{0r}b_0}}\right) + \underline{\overline{\partial}^r h_0}\right)}{b_0^2 p_{0r}^2}\right). \tag{D.4}$$

To make it easier to follow, similar terms have been underlined with different colors. We group those together and see that the final result is the following

$$\frac{d\phi}{dr} = \frac{\mathcal{L}_0}{r_0^2 b_0 p_{0r}} + \epsilon \left(\frac{b_0 p_{0r} \frac{\partial h_0}{\partial w} - \frac{\mathcal{L}_0}{r_0^2}\left(-\frac{h_0}{p_{0r}} + \overline{\partial}^r h_0\right)}{b_0^2 p_{0r}^2} + \frac{1}{b_0 p_{0r}} \left(\frac{\mathcal{L}_1}{r_0^2} - \frac{\mathcal{L}_0}{r_0^2} \frac{\mathcal{E}_0 \mathcal{E}_1 a_0}{b_0 p_{0r}} + \frac{\mathcal{L}_0^2 \mathcal{L}_1}{r_0^4 b_0 p_{0r}^2} \right) \right)$$
$$+ \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{r_1}{b_0 p_{0r}} \left(-\frac{2}{r_0} - \frac{b_0'}{b_0} - \frac{1}{b_0 p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' - \frac{1}{2} p_{0r}^2 b_0' \right) \right), \tag{D.5}$$

which is exactly the equation (4.44).

D.2 κ-Poincaré modified $d\phi/dr$ equation from spherically symmetric background

We start calculating the final steps for the $d\phi/dr$ equation from the (4.45), which is as follows:

$$\frac{d\phi}{dr} = \frac{\mathcal{L}_0 a_0}{r_0^2 p_{0r}} + \epsilon \left(\frac{a_0 p_{0r} \frac{\partial h_0}{\partial w} - \frac{\mathcal{L}_0}{r_0^2}\left(-\frac{h_0}{p_{0r}} + \overline{\partial}^r h_0\right)}{p_{0r}^2} + \frac{a_0}{p_{0r}} \left(\frac{\mathcal{L}_1}{r_0^2} - \frac{\mathcal{L}_0}{r_0^2} \frac{\mathcal{E}_0 \mathcal{E}_1 a_0^2}{p_{0r}^2} + \frac{\mathcal{L}_0^2 \mathcal{L}_1 a_0}{r_0^4 p_{0r}^2} \right) \right)$$
$$+ \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{a_0}{p_{0r}} r_1 \left(-\frac{2}{r_0} + \frac{a_0'}{a_0} - \frac{a_0}{p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2} p_{0r}^2 \frac{a_0'}{a_0} \right) \right), \tag{D.6}$$

We plug in the perturbation function h_0 and its partial derivatives from the equations (4.47)-(4.49). We get

$$\frac{d\phi}{dr} = \frac{\mathcal{L}_0 a_0}{r_0^2 p_{0r}} + \epsilon \left(\frac{2 a_0 p_{0r} \sqrt{a_0} \frac{\mathcal{L}_0}{r_0^2} - a_0^2 \frac{\mathcal{L}_0}{r_0^2} \left(-\frac{1}{p_{0r}} \left(\sqrt{a_0} \mathcal{E}_0 \left(\frac{p_{0r}^2}{a_0} + \frac{\mathcal{L}_0^2}{r_0^2} \right) \right) + 2 \frac{1}{\sqrt{a_0}} \mathcal{E}_0 p_{0r} \right)}{p_{0r}^2} \right)$$
$$+ \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{a_0}{p_{0r}} r_1 \left(-\frac{2}{r_0} + \frac{a_0'}{a_0} - \frac{a_0}{p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2} p_{0r}^2 \frac{a_0'}{a_0} \right) \right) \tag{D.7}$$
$$+ \epsilon \frac{a_0}{p_{0r}} \left(\frac{\mathcal{L}_1}{r_0^2} - \frac{\mathcal{L}_0}{r_0^2} \frac{\mathcal{E}_0 \mathcal{E}_1 a_0^2}{p_{0r}^2} + \frac{\mathcal{L}_0^2 \mathcal{L}_1 a_0}{r_0^4 p_{0r}^2} \right)$$

After multiplying all the terms in the brackets the differential equation above becomes

$$\begin{aligned}\frac{d\phi}{dr} = \frac{\mathcal{L}_0 a_0}{r_0^2 p_{0r}} &+ \epsilon \left(\cancel{\frac{2\sqrt{a_0^3}\mathcal{E}_0\mathcal{L}_0}{r_0^2 p_{0r}}} + \frac{\sqrt{a_0^3}\mathcal{E}_0\mathcal{L}_0}{r_0^2 p_{0r}} + \frac{\sqrt{a_0^5}\mathcal{E}_0\mathcal{L}_0^3}{r_0^4 p_{0r}^3} - \cancel{\frac{2\sqrt{a_0^3}\mathcal{L}_0\mathcal{E}_0}{r_0^2 p_{0r}}} \right) \\ &+ \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{a_0}{p_{0r}} r_1 \left(-\frac{2}{r_0} + \frac{a_0'}{a_0} - \frac{a_0}{p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2} p_{0r}^2 \frac{a_0'}{a_0} \right) \right) \\ &+ \epsilon \left(\frac{\mathcal{L}_1}{r_0^2} - \frac{\mathcal{L}_0}{r_0^2} \frac{\mathcal{E}_0\mathcal{E}_1 a_0^2}{p_{0r}^2} + \frac{\mathcal{L}_0^2\mathcal{L}_1 a_0}{r_0^4 p_{0r}^2} \right)\end{aligned} \tag{D.8}$$

We regroup terms with $\frac{1}{p_{0r}}$ and $\frac{1}{p_{0r}^3}$, the final result is

$$\begin{aligned}\frac{d\phi}{dr} = \frac{\mathcal{L}_0 a_0}{r_0^2 p_{0r}} &+ \epsilon \left(\frac{1}{r_0^2 p_{0r}} \left(\sqrt{a_0^3}\mathcal{E}_0\mathcal{L}_0 + a_0\mathcal{L}_1 \right) + \frac{\mathcal{L}_0}{r_0^2 p_{0r}^3} \left(\frac{\sqrt{a_0^5}\mathcal{E}_0\mathcal{L}_0^2}{r_0^2} - \mathcal{E}_0\mathcal{E}_1 a_0^3 + \frac{\mathcal{L}_0\mathcal{L}_1}{r_0^2} a_0^2 \right) \right) \\ &+ \epsilon \frac{\mathcal{L}_0}{r_0^2} \frac{a_0}{p_{0r}} r_1 \left(-\frac{2}{r_0} + \frac{a_0'}{a_0} - \frac{a_0}{p_{0r}^2} \left(\frac{\mathcal{L}_0^2}{r_0^3} + \frac{1}{2}\mathcal{E}_0^2 a_0' + \frac{1}{2} p_{0r}^2 \frac{a_0'}{a_0} \right) \right),\end{aligned} \tag{D.9}$$

and it is exactly the equation (4.50).

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