

**PIECEWISE POLYNOMIAL COLLOCATION
METHODS FOR SOLVING WEAKLY SINGULAR
INTEGRO-DIFFERENTIAL EQUATIONS**

INGA PARTS



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Chapter 1

Introduction

Uniformly speaking, an integral equation is a functional equation where the unknown function appears under the integral sign and integro-differential equation is an equation that contains both derivatives and integrals of the unknown function. Systematic study of integral equations started from the works of Volterra [60], where he transformed an integral equation $\int_0^t \mathcal{K}(t, s)y(s)ds = g(t)$, $t \in [0, T]$ by differentiation with respect to t , into an integral equation of the form

$$y(t) = \int_0^t K(t, s)y(s)ds + f(t), \quad t \in [0, T] \quad (\text{VIE})$$

(later called Volterra integral equations of the second kind) and Fredholm [19], where he gave necessary and sufficient conditions for solvability of integral equations (later called Fredholm integral equations of the second kind) of the form

$$y(t) = \int_0^b K(t, s)y(s)ds + f(t), \quad t \in [0, b]. \quad (\text{FIE})$$

The main study objects of the present thesis are numerical methods for solving first order Volterra and Fredholm integro-differential equations of the form

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K(t, s)y(s)ds, \quad t \in [0, T] \quad (\text{VIDE})$$

and

$$y'(t) = p(t)y(t) + q(t) + \int_0^b K(t, s)y(s)ds, \quad t \in [0, b] \quad (\text{FIDE})$$

with appropriate initial or boundary conditions. In these equations K is called the kernel and assumed to be known. The functions f , p and q are also assumed to be known. The unknown function is denoted by y .

Both integral and integro-differential equations arise in many applications, for example in population dynamics [17, 18], identification of memory kernels in viscoelasticity and heat conduction [25, 26, 27], financial mathematics [34, 35]; we refer the reader for many additional examples to the monograph [11].

In many applications the kernels K of the integral operators are not smooth functions and may have an integrable singularity at the diagonal $t = s$ (see e.g. [6, 16, 55]). It turns out that in that case the solution to the corresponding integral or integro-differential equation is not a smooth function, its derivatives may have singularities only at 0 (in the cases of VIE and VIDE, see e.g. [8, 13, 14, 36]) or at both 0 and b (FIE and FIDE, see e.g. [46, 49, 55, 56, 57]). This property of weakly singular equations makes it much harder to construct effective numerical methods for solving them.

It is easy to see that integro-differential equations and integral equations are very closely related. There are two standard ways to relate them: 1) an integro-differential equation can be viewed as an integral equation with respect to the derivative of the unknown function; 2) by integration, the integro-differential equation can be transformed to an integral equation. In the view of such close relationship it is quite surprising that the standard theory of numerical methods for integral equations often, especially in the case of nonsmooth K, p and q , do not enable to get optimal results for numerical methods for integro-differential equations.

A standard method for solving integral and integro-differential equations is the collocation method, where one looks for an approximate solution in a finite dimensional space and determines the approximate solution by requiring that after substituting the approximate solution into the original equation, the equality would hold at certain points (so called collocation points).

The most comprehensive coverage of the theory of collocation methods for VIE and VIDE with extensive annotated list of references is [11], we

also refer to the monographs [4, 12, 33] for additional information on the numerical treatment of Volterra integral equations, to the papers [37, 38, 39, 40] for convergence and stability analysis of collocation methods with smooth splines for VIE and VIDE and to the survey papers by Baker [2, 3, 5] and Brunner [7, 10]. For the discussion of collocation methods (and many other methods) for solving Fredholm integral equations we refer to the monographs [1, 20, 32], comprehensive treatments of weakly singular Fredholm integral equations are [55, 57].

In this thesis we study piecewise polynomial collocation methods for solving weakly singular Volterra and Fredholm integro-differential equations, where the approximate solution is assumed to be piecewise polynomial function defined on a partition of the original interval. The general theory of such methods for weakly singular VIE in the case of special nonuniform grids is given in [13], where a connection between numerical methods for weakly singular Volterra and Fredholm equations is established. A comprehensive convergence analysis of similar methods for VIDE is presented in [14, 15], see also [9, 23, 51, 52, 54]. In the case of smooth kernels the numerical solution of FIDE is discussed in [22, 24, 31, 59]. The only paper known to the author of this thesis considering numerical methods of FIDE in the case of weakly singular kernels is [58]. This is in contrast to the number of works on weakly singular Fredholm integral equations, see e.g. [21, 28, 29, 41, 47, 50, 55, 57].

The main contributions of the present thesis to the numerical analysis of weakly singular VIE, VIDE and FIDE are as follows.

- 1) Convergence rate estimates for two different numerical methods for solving initial value problems of VIDE are established in terms of the length of the maximal subinterval of the underlying grid for unstructured grid sequences (Theorems 3.2.1 and 3.5.1). The first method considered (later Method 1) is based on viewing the VIDE as an integral equation for the derivative of the solution of VIDE, solving the equation for the derivative by a piecewise polynomial collocation method and finally determining the approximate solution of the original initial value problem of VIDE by integration. The second method (later Method 2) corresponds to integration of both sides of VIDE to obtain an integral equation for the solution of VIE, which is solved by a piecewise polynomial collocation method.
- 2) Uniform superconvergence results are obtained for Method 1 in the case of special nonuniform grids (or graded grids). Normally, if an

approximation of a function is obtained by integrating an approximation of the derivative at the function one would expect that the maximal error of the approximation of the function is of the same order as the error of the approximation of the derivative. It turns out that in the case of Method 1, by an appropriate choice of the collocation points it is possible to achieve much higher convergence rate for the approximate solution of VIDE. Theorem 3.4.1 characterizes the uniform superconvergence phenomenon for all values of the nonuniformity parameter of the grids. This result refines the results of [14, 15] about the convergence of Method 1 in the case of arbitrary collocation parameters and improves the superconvergence results of [53] obtained for a special class of weakly singular kernels.

- 3) A new local superconvergence result for piecewise polynomial collocation methods for solving VIE (Theorem 3.3.2). The optimal superconvergence rate at the collocation points in the case of graded grids was established in [13] for sufficiently large values of the nonuniformity parameter. Detailed technical analysis (Theorems 2.5.4 and 2.5.6) enable us to describe the superconvergence phenomenon for all values of the nonuniformity parameter and to establish the exact value after which the optimal convergence rate is achieved.
- 4) Characterization of (local) superconvergence for Method 2 (Theorem 3.6.1). In [14] an optimal superconvergence of Method 2 at the collocation points (under some additional assumptions) was established for sufficiently large values of the nonuniformity parameter of the graded grid. Numerical experiments showed that the optimal convergence rate is achieved for smaller value of the nonuniformity parameter than predicted in [14]. The result of Theorem 3.6.1, which was originally stated as a conjecture in [42], describes the dependence of the superconvergence phenomenon for all values of the nonuniformity parameter.
- 5) Convergence analysis of a piecewise polynomial collocation method in the case of special nonuniform grids (similar to Method 1 for VIDE) for initial and boundary value problems of FIDE. We study the regularity properties of a solution of a FIDE (Theorem 4.2.1), derive optimal global convergence error estimates and analyze the attainable order of convergence of numerical solutions for all values of the nonuniformity parameter of the underlying grid (Theorems 4.4.3 and 4.5.1).

- 6) Extensive numerical verification of the optimality of the theoretical results.

The thesis is organized as follows.

In Chapter 2 we provide definitions of weakly singular Volterra integral operators, relevant function spaces and various grids used for defining piecewise polynomial function spaces. We also prove several new approximation properties of piecewise polynomial interpolation operators which form the basis for analyzing numerical methods of solving VIE and VIDE in the next chapter.

Chapter 3 is devoted to studying piecewise polynomial collocation methods for solving initial value problems of first order linear weakly singular Volterra integro-differential equations. We introduce two different numerical methods and analyze the convergence of the methods both for unstructured and for graded grids.

In Chapter 4 we study initial and boundary value problems of weakly singular Fredholm integro-differential equations. After introducing relevant function spaces we prove the existence and uniqueness of a solution to the boundary value problem and describe a piecewise polynomial collocation method for solving the problem. We provide a complete convergence analysis of this method in the case of graded grids.

In Chapter 5 we discuss implementation details of the numerical methods described in the previous chapters, introduce test problems and verify the optimality of the theoretical results of Chapters 3 and 4 by extensive computational experiments.

Most of the results given in Chapters 3 - 5 are published in [30, 42, 43, 44, 45, 46], although the thesis contains also several new results, especially in Chapter 2. In some cases the results in this thesis are stated and proved in a more general form than in our published papers. When there are no cited references for a result in the text, the result is new.

Chapter 2

Function spaces and weakly singular Volterra integral operators

In order to discuss piecewise polynomial collocation methods for solving weakly singular Volterra integro-differential equations one has to define the class of weakly singular integral operators, introduce the relevant function spaces and various grids used for constructing an approximate solution. In this chapter we provide necessary definitions and analyze approximation properties of piecewise polynomial interpolation operators. In addition to quoting well-known results of other authors, this chapter contains also several original results, some of which have not been published before.

The most important new results are Theorem 2.5.4 and Theorem 2.5.6, which enable us to improve formerly known superconvergence results for weakly singular Volterra integral equations and to obtain new superconvergence results for various classes of integro-differential equations. Also worth mentioning is the new L^1 approximation result (Lemma 2.4.5) for piecewise polynomial interpolation on regular grids, which leads to new convergence results for corresponding numerical methods.

2.1. Notation and some basic results for linear operators

Throughout this work we denote by c, c_1, c_2, \dots real constants, which may be different at different places, by $\mathbb{N} = \{1, 2, \dots\}$ the set of all positive

integers and by $\mathbb{R} = (-\infty, \infty)$ the set of real numbers.

By $C^m(D)$ ($m \in \mathbb{N} \cup \{0\}$; $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$) we denote the set of continuous and m times continuously differentiable functions $x : D \rightarrow \mathbb{R}$.

By $C[a, b]$ we denote the Banach space of continuous functions $x : [a, b] \rightarrow \mathbb{R}$ with the norm

$$\|x\| = \|x\|_{C[a,b]} = \max_{t \in [a,b]} |x(t)|, \quad x \in C[a, b].$$

By $C^1[a, b]$ we denote the Banach space of continuously differentiable functions $x : [a, b] \rightarrow \mathbb{R}$ with the norm

$$\|x\|_{C^1[a,b]} = \|x\|_{C[a,b]} + \|x'\|_{C[a,b]}, \quad x \in C^1[a, b].$$

By $L^\infty(a, b)$ we denote the set of measurable functions $x : [a, b] \rightarrow \mathbb{R}$, such that

$$\inf_{\Omega \subset [a,b]: \mu(\Omega)=0} \sup_{t \in [a,b] \setminus \Omega} |x(t)| < \infty,$$

where $\mu(\Omega)$ is the Lebesgue measure of the set Ω . The set $L^\infty(a, b)$ is a Banach space with the norm

$$\|x\|_{L^\infty(a,b)} = \|x\|_\infty = \inf_{\Omega \subset [a,b]: \mu(\Omega)=0} \sup_{t \in [a,b] \setminus \Omega} |x(t)|, \quad x \in L^\infty(a, b).$$

Let X and Y be Banach spaces. By $\mathcal{L}(X, Y)$ we denote the Banach space of all linear continuous operators $A : X \rightarrow Y$ with the norm

$$\|A\| = \|A\|_{\mathcal{L}(X,Y)} = \sup_{x \in X, \|x\| \leq 1} \|Ax\| = \sup_{x \in X, x \neq 0} \frac{\|Ax\|}{\|x\|} \quad (A \in \mathcal{L}(X, Y)).$$

We use in this work the following well-known results from the theory of linear operators (see e.g. [1, 20, 32]).

Theorem 2.1.1. *Let X, Y, Z be normed spaces and let $A : X \rightarrow Y$ and $B : Y \rightarrow Z$ be bounded linear operators. Then the product $BA : X \rightarrow Z$ is compact if one of the two operators A or B is compact.*

Theorem 2.1.2. *Banach-Steinhaus theorem. Let $A : X \rightarrow Y$ be a bounded linear operator and let $A_n : X \rightarrow Y$ be a sequence of bounded linear operators from a Banach space X into a normed space Y . For pointwise convergence $A_n x \rightarrow Ax$, $n \rightarrow \infty$, for all $x \in X$ it is necessary and sufficient that $\|A_n\| \leq C$ for all $n \in \mathbb{N}$ with some constant C and that $A_n x \rightarrow Ax$, $n \rightarrow \infty$, for all $x \in U$ where U is some dense subset of X .*

Theorem 2.1.3. *Let X and Y be Banach spaces. If the operators $A, B \in \mathcal{L}(X, Y)$ are such that A is invertible ($A^{-1} \in \mathcal{L}(Y, X)$) and $\|B\| \|A^{-1}\| < 1$ then $A + B$ is invertible and the estimate*

$$\|(A + B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|B\|}$$

holds.

Theorem 2.1.4. *Let X, Z be normed spaces and let Y be a Banach space. Let A be a compact operator mapping X into Y and let $L_n : Y \rightarrow Z$ be a pointwise convergent sequence of bounded linear operators with limit operator $L : Y \rightarrow Z$. Then*

$$\|(L_n - L)A\| \rightarrow 0, \quad n \rightarrow \infty.$$

Theorem 2.1.5. Fredholm alternative. *Let X be a Banach space, and let $A \in \mathcal{L}(X, X)$ be a compact operator. Then the equation $x = Ax + g$, $g \in X$ has a unique solution $x \in X$ if and only if the homogeneous equation $z = Az$ has only the trivial solution $z = 0$. In such a case, the operator $I - A$ has a bounded inverse $(I - A)^{-1} \in \mathcal{L}(X, X)$.*

2.2. Spaces $C^{k,\mu}[0, T]$ and $\mathcal{W}^{k,\mu}(\Delta_T)$

Let $C^{k,\mu}[0, T]$, $k \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$ be defined as the collection of all continuous functions $x : [0, T] \rightarrow \mathbb{R}$, which are k times continuously differentiable in $(0, T]$ and such that the estimation

$$\left| x^{(i)}(t) \right| \leq c \begin{cases} 1 & \text{if } i < 1 - \mu, \\ 1 + |\log t| & \text{if } i = 1 - \mu, \\ t^{1-\mu-i} & \text{if } i > 1 - \mu \end{cases} \quad (2.1)$$

holds with a constant $c = c(x)$ for all $t \in (0, T]$ and $i = 0, 1, \dots, k$.

The set $\mathcal{W}^{k,\mu}(\Delta_T)$, with $k \in \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{R}$, $\mu < 1$ and

$$\Delta_T = \{(t, s) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq s < t\} \quad (2.2)$$

consists of continuous and k times continuously differentiable functions $K : \Delta_T \rightarrow \mathbb{R}$ satisfying

$$\left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j K(t, s) \right| \leq c \begin{cases} 1 & \text{if } \mu + i < 0, \\ 1 + |\log(t - s)| & \text{if } \mu + i = 0, \\ (t - s)^{-\mu-i} & \text{if } \mu + i > 0 \end{cases} \quad (2.3)$$

with a constant $c = c(K)$ for all $(t, s) \in \Delta_T$ and all integers $i, j \geq 0$ such that $i + j \leq k$.

The asymmetry of (2.3) with respect to t and s is only seeming; actually, using the equality

$$\frac{\partial}{\partial s} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) - \frac{\partial}{\partial t}$$

we can deduce from (2.3) the estimate

$$\left| \left(\frac{\partial}{\partial s} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j K(t, s) \right| \leq c \begin{cases} 1 & \text{if } \mu + i < 0, \\ 1 + |\log(t - s)| & \text{if } \mu + i = 0, \\ (t - s)^{-\mu - i} & \text{if } \mu + i > 0 \end{cases} \quad (2.4)$$

with a constant $c = c(K)$ for all $(t, s) \in \Delta_T$ and all integers $i, j \geq 0$ such that $i + j \leq k$.

Note that for any $k \in \mathbb{N} \cup \{0\}$ the function

$$K(t, s) = \kappa(t, s)(t - s)^{-\mu}, \quad \kappa \in C^k(\bar{\Delta}_T)$$

belongs to the space $\mathcal{W}^{k, \mu}(\Delta_T)$ for any $\mu < 1$ and functions of the form

$$K(t, s) = \kappa(t, s)(t - s)^p \log(t - s), \quad \kappa \in C^k(\bar{\Delta}_T)$$

belong to the space $\mathcal{W}^{k, -p}(\Delta_T)$ for any $p \in \{0, 1, \dots, k\}$. Here

$$\bar{\Delta}_T = \{(t, s) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq s \leq t\} \quad (2.5)$$

is the closure of the set Δ_T .

If $\mu < 0$ then according to (2.3) the functions $K \in \mathcal{W}^{k, \mu}(\Delta_T)$ are bounded for any $k \in \mathbb{N} \cup \{0\}$. Actually, we can deduce more from the conditions (2.3).

Lemma 2.2.1. *Assume that $K \in \mathcal{W}^{k, \mu}(\Delta_T)$ with $k \geq 1$ and $\mu < 0$. Then K can be extended to $\bar{\Delta}_T$ (defined by (2.5)) as a continuous function.*

Proof. Since for $\mu \leq -1$ we have $\mathcal{W}^{k, \mu}(\Delta_T) \subset \mathcal{W}^{k, -\frac{1}{2}}(\Delta_T)$ we consider only the case $\mu \in (-1, 0)$.

Fix $(t_1, s_1), (t_2, s_2) \in \Delta_T$. Without loss of generality we may assume that $s_1 \leq s_2$ (otherwise we may just swap the indexes 1 and 2). Using (2.3) and

(2.4) we get

$$\begin{aligned}
|K(t_1, s_1) - K(t_2, s_2)| &\leq |K(t_1, s_1) - K(t_2, s_1)| + |K(t_2, s_1) - K(t_2, s_2)| \\
&= \left| \int_{t_1}^{t_2} \frac{\partial}{\partial t} K(t, s_1) dt \right| + \left| \int_{s_1}^{s_2} \frac{\partial}{\partial s} K(t_2, s) ds \right| \\
&\leq c_1 \left| \int_{t_1}^{t_2} (t - s_1)^{-\mu-1} dt \right| + c_2 \int_{s_1}^{s_2} (t_2 - s)^{-\mu-1} ds \\
&\leq c_1 \left| \int_{t_1}^{t_2} (t - \min\{t_1, t_2\})^{-\mu-1} dt \right| + c_2 \int_{s_1}^{s_2} (s_2 - s)^{-\mu-1} ds \\
&\leq c_3 (|t_1 - t_2|^{-\mu} + (s_2 - s_1)^{-\mu}).
\end{aligned}$$

Thus, the function K is uniformly continuous on the set Δ_T and therefore can be extended as a continuous function to $\bar{\Delta}_T$. \diamond

Consider the integral operator S , defined by the formula

$$(Sx)(t) = \int_0^t K(t, s)x(s)ds, \quad 0 \leq t \leq T. \quad (2.6)$$

Definition 2.2.2. *Integral operator S defined by (2.6) is called weakly singular if the kernel K belongs to the space $\mathcal{W}^{k,\mu}(\Delta_T)$ for some $k \in \mathbb{N} \cup \{0\}$ and $\mu < 1$.*

Remark 2.2.3. It is customary to consider an integral operator to be weakly singular if its kernel $K(t, s)$ is not continuous on $\bar{\Delta}_T$ and has an integrable singularity at the diagonal $t = s$. If $\mu \geq 0$ then the kernel $K \in \mathcal{W}^{k,\mu}(\Delta_T)$ may be weakly singular in the usual sense. If $\mu < 0$ then the kernel $K \in \mathcal{W}^{k,\mu}(\Delta_T)$ is bounded on the set Δ_T but its derivatives of certain order may be unbounded at the diagonal $t = s$.

Since in case of most of the results of this thesis it is not essential to distinct the cases $\mu \geq 0$ and $\mu < 0$, we call integral operators with kernels $K \in \mathcal{W}^{k,\mu}(\Delta_T)$ weakly singular regardless of the value of μ .

Remark 2.2.4. If $K \in C(\bar{\Delta}_T)$ then $K \in \mathcal{W}^{0,\mu}(\Delta_T)$ for any $\mu < 1$, thus the class of weakly singular integral operators contains all integral operators with continuous kernels.

Remark 2.2.5. The class of kernels $\mathcal{W}^{k,\mu}(\Delta_T)$ allowed here is an adaption of a class of kernels originally introduced by Vainikko in his monograph [55] for n -dimensional weakly singular Fredholm integral equations. In the case $n = 1$ we will exploit the corresponding class of kernels in Chapter 4 for the numerical solution of Fredholm type integro-differential equations.

The proof of the following property of weakly singular integral operators can be found in the book [55] (Corollary 2.1).

Theorem 2.2.6. *Weakly singular integral operators are compact from $L^\infty(0, T)$ to $C[0, T]$ (and hence also from $L^\infty(0, T)$ to $L^\infty(0, T)$ and from $C[0, T]$ to $C[0, T]$).*

2.3. Grids and the spline spaces $\mathcal{S}_k^{(d)}(\Pi_N)$

For a given $N \in \mathbb{N}$ let

$$\Pi_N = \{t_0, t_1, \dots, t_N : 0 = t_0 < t_1 < \dots < t_N = T\} \quad (2.7)$$

be a partition (a mesh) of the interval $[0, T]$ (for ease of notation we suppress the index N in $t_n = t_n^{(N)}$ indicating the dependence of the grid points on N).

Later we will use various grids. A grid Π_N is called *regular*, if

$$\max_{n=1, \dots, N} (t_n - t_{n-1}) \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (2.8)$$

A sequence of partitions for $[0, T]$ is called *quasi-uniform* if there exists a constant Θ independent of N such that

$$\max_{n=1, \dots, N} (t_n - t_{n-1}) / \min_{n=1, \dots, N} (t_n - t_{n-1}) \leq \Theta, \quad n \in N. \quad (2.9)$$

We use the notation $\Pi_N = \Pi_{N, \Theta}$ for quasi-uniform grids. If $\Theta = 1$, then $\Pi_{N, 1}$ is a uniform grid. If the grid points of Π_N are given by

$$t_n = T \left(\frac{n}{N} \right)^r, \quad r \geq 1, \quad n = 0, 1, \dots, N, \quad (2.10)$$

then $\Pi_N = \Pi_N^r$ is called a *graded grid*. Here r is a parameter describing the nonuniformity of the grid Π_N . If $r = 1$ we get a uniform grid and if r increases the density of the grid points near 0 also increases.

Let us show that quasi-uniform and graded grids are regular. Denote

$$h_n = t_n - t_{n-1}, \quad n = 1, \dots, N; \quad (2.11)$$

$$h = \max_{n=1, \dots, N} (t_n - t_{n-1}). \quad (2.12)$$

Since

$$\min_{n=1, \dots, N} h_n \leq \frac{T}{N},$$

the inequality (2.9) gives us

$$h \leq \Theta \min_{n=1, \dots, N} h_n \leq \Theta \frac{T}{N}, \quad (2.13)$$

which proves the regularity of quasi-uniform grids.

In the case of graded grids

$$\begin{aligned} t_n - t_{n-1} &= T \left(\frac{n}{N} \right)^r - T \left(\frac{n-1}{N} \right)^r \\ &= TN^{-r} (n^r - (n-1)^r), \quad n = 1, \dots, N. \end{aligned}$$

Using Lagrange's mean-value theorem we get

$$TN^{-r} (n^r - (n-1)^r) \leq TrN^{-r} n^{r-1} \leq TrN^{-1}, \quad n = 1, \dots, N.$$

Therefore, for graded grids

$$h_n \leq h \leq TrN^{-1}, \quad n = 1, \dots, N, \quad (2.14)$$

and the convergence (2.8) holds. Thus, graded grids are also regular.

It is easy to see that graded grids Π_N^r with $r > 1$ are not quasi-uniform. Indeed, since

$$\frac{h_N}{h_1} = N^r - (N-1)^r \geq r(N-1)^{r-1} \rightarrow \infty \quad \text{as } N \rightarrow \infty$$

the condition (2.9) is not satisfied.

In this thesis we look for approximate solutions to integral and integro-differential equations in the form of piecewise polynomial functions. Such functions are called polynomial splines.

Definition 2.3.1. Let k and d be given integers satisfying $-1 \leq d \leq k-1$. We call

$$S_k^{(d)}(\Pi_N) = \{w : w|_{(t_{n-1}, t_n)} =: w_n \in \pi_k, \quad n = 1, \dots, N;$$

$$w_n^{(i)}(t_n) = w_{n+1}^{(i)}(t_n) \quad 0 \leq i \leq d, \quad n = 1, \dots, N-1\}$$

the space of (real) polynomial splines of degree k and of continuity class d . Here π_k denotes the set of polynomials of degree not exceeding k and $w|_{(t_{n-1}, t_n)}$ is the restriction of $w : [0, T] \rightarrow \mathbb{R}$ to the subinterval (t_{n-1}, t_n) .

Note that the elements of

$$S_k^{(-1)}(\Pi_N) = \{w : w|_{(t_{n-1}, t_n)} \in \pi_k, \quad n = 1, \dots, N\}$$

may have jump discontinuities at the interior grid points t_1, \dots, t_{N-1} .

The space $S_k^{(-1)}(\Pi_N)$ is thus the least smooth of the polynomial spline spaces, while $S_k^{(k-1)}(\Pi_N)$ is the smoothest of these spaces. The dimension of $S_k^{(d)}(\Pi_N)$ (see [12], Theorem 5.1.1) is given by

$$\dim S_k^{(d)}(\Pi_N) = N(k-d) + d + 1, \quad -1 \leq d \leq k-1.$$

We do not consider splines with continuity class $d \geq 1$ in this work.

2.4. Piecewise polynomial interpolation

We define $k \geq 1$ interpolation points in every subinterval $[t_{n-1}, t_n]$ ($n = 1, \dots, N$) of the grid Π_N by

$$t_{nj} = t_{n-1} + \eta_j h_n, \quad j = 1, \dots, k \quad (n = 1, \dots, N), \quad (2.15)$$

where $h_n = t_n - t_{n-1}$ and η_1, \dots, η_k are some fixed parameters (called *collocation parameters*) which do not depend on n and N and satisfy

$$0 \leq \eta_1 < \dots < \eta_k \leq 1. \quad (2.16)$$

We introduce an interpolation operator

$$P_N = P_N^{(k)} : C[0, T] \rightarrow S_{k-1}^{(-1)}(\Pi_N) \subset L^\infty(0, T)$$

which assigns to every continuous function $x : [0, T] \rightarrow \mathbb{R}$ its piecewise polynomial interpolation function which interpolates x at the points (2.15):

$$\begin{aligned} P_N x &\in S_{k-1}^{(-1)}(\Pi_N), \quad x \in C[0, T], \\ (P_N x)(t_{nj}) &= x(t_{nj}), \quad j = 1, \dots, k; \quad n = 1, \dots, N. \end{aligned} \quad (2.17)$$

If $\eta_1 = 0$, then by $(P_N x)(t_{n1})$ we mean the right limit $\lim_{t \rightarrow t_{n-1}, t > t_{n-1}} (P_N x)(t)$. If $\eta_k = 1$, then $(P_N x)(t_{nk})$ denotes the left limit $\lim_{t \rightarrow t_n, t < t_n} (P_N x)(t)$. Thus, $(P_N x)(t)$ is independently defined on every subinterval $[t_{n-1}, t_n]$, $n = 1, \dots, N$, and may be discontinuous at the interior grid points $t = t_n$, $n = 1, \dots, N - 1$. Note that in the case $\eta_1 = 0, \eta_k = 1$, function $P_N x$ is continuous on $[0, T]$.

Later we need the following result, which is stated as an easy observation in several papers (see e.g. [14, 15, 43], also cf [55] p.115).

Lemma 2.4.1. *Assume that the node points (2.15) with grid points (2.7) and parameters (2.16) are used. Let the operator P_N be determined by the conditions (2.17). Then*

$$\|P_N\|_{\mathcal{L}(C[0,T], L^\infty(0,T))} \leq c, \quad N \in \mathbb{N}, \quad (2.18)$$

where c is a constant which is independent of N .

In order to prove this lemma we introduce an operator $P_{a,b} : C[a, b] \rightarrow C[a, b]$ which to every continuous function $x \in C[a, b]$ assigns a function $P_{a,b}x \in C[a, b]$ by the formula

$$(P_{a,b}x)(t) = \sum_{i=1}^k x(\xi_i) \frac{(t - \xi_1) \dots (t - \xi_{i-1})(t - \xi_{i+1}) \dots (t - \xi_k)}{(\xi_i - \xi_1) \dots (\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1}) \dots (\xi_i - \xi_k)}, \quad t \in [a, b]. \quad (2.19)$$

Here $a, b \in \mathbb{R}$, $a < b$ and

$$\xi_i = a + \eta_i(b - a), \quad i = 1, \dots, k$$

with the collocation parameters η_1, \dots, η_k satisfying (2.16). For the operator $P_{a,b}$, the following lemma holds.

Lemma 2.4.2. *Let the operator $P_{a,b} : C[a, b] \rightarrow C[a, b]$ be determined by the formula (2.19). Then*

$$\|P_{a,b}\|_{\mathcal{L}(C[a,b], C[a,b])} \leq c, \quad (2.20)$$

and

$$\|P_{a,b}\|_{\mathcal{L}(C[a,b],C^1[a,b])} \leq c + \frac{c_1}{b-a}, \quad (2.21)$$

where c and c_1 are constants which are independent of a and b .

Proof. Let $x \in C[a, b]$. From formula (2.19) we get for every $t \in [a, b]$ the inequality

$$|(P_{a,b}x)(t)| \leq \left(\max_{i=1,\dots,k} |x(\xi_i)| \right) \sum_{i=1}^k \frac{|t - \xi_1| \dots |t - \xi_{i-1}| |t - \xi_{i+1}| \dots |t - \xi_k|}{|\xi_i - \xi_1| \dots |\xi_i - \xi_{i-1}| |\xi_i - \xi_{i+1}| \dots |\xi_i - \xi_k|}.$$

Here $|t - \xi_j| \leq b - a$ ($j = 1, \dots, k$, $j \neq i$) and the differences $|\xi_i - \xi_j|$ ($j = 1, \dots, k$, $j \neq i$) can be estimated as follows:

$$|\xi_i - \xi_j| \geq \min_{j=1,\dots,k,j \neq i} |\eta_i - \eta_j| (b - a), \quad i = 1, \dots, k.$$

Thus

$$\begin{aligned} |(P_{a,b}x)(t)| &\leq \left(\max_{i=1,\dots,k} |x(\xi_i)| \right) \sum_{i=1}^k \frac{(b-a)^{k-1}}{\left(\min_{j=1,\dots,k,j \neq i} |\eta_i - \eta_j| (b-a) \right)^{k-1}} \\ &\leq \left(\max_{i=1,\dots,k} |x(\xi_i)| \right) \sum_{i=1}^k \frac{1}{\left(\min_{j=1,\dots,k,j \neq i} |\eta_i - \eta_j| \right)^{k-1}}, \quad t \in [a, b]. \end{aligned} \quad (2.22)$$

Therefore

$$\|P_{a,b}x\|_{C[a,b]} \leq c \|x\|_{C[a,b]},$$

where

$$c = \sum_{i=1}^k \frac{1}{\left(\min_{j=1,\dots,k,j \neq i} |\eta_i - \eta_j| \right)^{k-1}}.$$

Since

$$\|P_{a,b}\|_{\mathcal{L}(C[a,b],C[a,b])} = \sup_{x \in C[a,b], \|x\|_{C[a,b]} \leq 1} \|P_{a,b}x\|_{C[a,b]},$$

the estimate (2.20) follows.

In order to prove (2.21) we note that for every $t \in [a, b]$ the inequality

$$\begin{aligned} &|[(P_{a,b}x)(t)]'| \\ &\leq \left(\max_{i=1,\dots,k} |x(\xi_i)| \right) \sum_{i=1}^k \frac{|[(t - \xi_1) \dots (t - \xi_{i-1})(t - \xi_{i+1}) \dots (t - \xi_k)]'|}{|\xi_i - \xi_1| \dots |\xi_i - \xi_{i-1}| |\xi_i - \xi_{i+1}| \dots |\xi_i - \xi_k|} \end{aligned}$$

holds. Since

$$|[(t - \xi_1) \dots (t - \xi_{i-1})(t - \xi_{i+1}) \dots (t - \xi_k)]'| \leq (k-1)(b-a)^{k-2},$$

we get

$$\begin{aligned} |[(P_{a,b}x)(t)]'| &\leq \left(\max_{i=1,\dots,k} |x(\xi_i)| \right) \sum_{i=1}^k \frac{(k-1)(b-a)^{k-2}}{\left(\min_{j=1,\dots,k,j \neq i} |\eta_i - \eta_j|(b-a) \right)^{k-1}} \\ &\leq \|x\|_{C[a,b]} \frac{k-1}{b-a} \sum_{i=1}^k \frac{1}{\left(\min_{j=1,\dots,k,j \neq i} |\eta_i - \eta_j| \right)^{k-1}}, \quad t \in [a,b]. \end{aligned}$$

Hence

$$\|P_{a,b}x\|_{C^1[a,b]} = \|P_{a,b}x\|_{C[a,b]} + \|(P_{a,b}x)'\|_{C[a,b]} \leq \left(c + \frac{c_1}{b-a} \right) \|x\|_{C[a,b]},$$

which proves the estimate (2.21). \diamond

Proof of Lemma 2.4.1. Fix an arbitrary function $x \in C[0, T]$ and denote

$$x_n = x|_{[t_{n-1}, t_n]}.$$

It follows from

$$(P_N x)(t) = (P_{t_{n-1}, t_n} x_n)(t), \quad t \in (t_{n-1}, t_n),$$

that

$$\begin{aligned} \|P_N x\|_{L^\infty(0, T)} &= \max_{n=1,\dots,N} \|P_N x\|_{L^\infty(t_{n-1}, t_n)} = \max_{n=1,\dots,N} \|P_{t_{n-1}, t_n} x_n\|_{C[t_{n-1}, t_n]} \\ &\leq \max_{n=1,\dots,N} \|P_{t_{n-1}, t_n}\|_{\mathcal{L}(C[t_{n-1}, t_n], C[t_{n-1}, t_n])} \|x\|_{C[t_{n-1}, t_n]} \\ &\leq c \|x\|_{C[0, T]}. \end{aligned}$$

Since

$$\|P_N\|_{\mathcal{L}(C[0, T], L^\infty(0, T))} = \sup_{x \in C[0, T], \|x\|_{C[0, T]} \leq 1} \|P_N x\|_{L^\infty(0, T)},$$

we get the estimate (2.18). \diamond

When studying the properties of piecewise polynomial interpolation operators one often needs error estimates on subintervals. Here we show that the estimates of [55] (p.116), proved for graded grids, hold for arbitrary subintervals.

Lemma 2.4.3. Let $x \in C^{k,\mu}[0, T]$, $k \in \mathbb{N}$, $\mu < 1$ and $0 \leq a < b \leq T$. Then for the operator $P_{a,b}$ defined by (2.19) the estimate

$$\max_{t \in [a,b]} |x(t) - (P_{a,b}x)(t)| \leq c(b-a)^k \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log b| & \text{if } k = 1 - \mu, \\ b^{1-\mu-k} & \text{if } k > 1 - \mu, \end{cases} \quad (2.23)$$

where c is a constant which is independent of a and b , holds.

Proof. Let ω be a polynomial of order $k-1$, then $P_{a,b}\omega = \omega$. With help of the inequality (2.20) we get

$$\begin{aligned} \|x - P_{a,b}x\|_{C[a,b]} &= \|x - \omega - P_{a,b}(x - \omega)\|_{C[a,b]} \\ &\leq (1 + \|P_{a,b}\|_{\mathcal{L}(C[a,b], C[a,b])}) \|x - \omega\|_{C[a,b]} \\ &\leq c \|x - \omega\|_{C[a,b]}, \end{aligned} \quad (2.24)$$

where the constant c is independent of a and b .

We fix ω as a Taylor polynomial for x at $t = b$:

$$\omega(t) = \sum_{i=0}^{k-1} \frac{x^{(i)}(b)}{i!} (t-b)^i, \quad t \in [a, b].$$

Since $x \in C^{k,\mu}[0, T]$, the integral form of the reminder term of the $(k-1)$ th order Taylor approximation of x at $t = b$ and the estimates (2.1) give us for all $t \in [a, b]$ the inequality

$$\begin{aligned} |x(t) - \omega(t)| &= \frac{1}{(k-1)!} \left| \int_t^b (s-t)^{k-1} x^{(k)}(s) ds \right| \\ &\leq c \int_t^b (s-t)^{k-1} \begin{cases} 1 & \text{if } k < 1 - \mu \\ 1 + |\log s| & \text{if } k = 1 - \mu \\ s^{1-\mu-k} & \text{if } k > 1 - \mu \end{cases} ds. \end{aligned} \quad (2.25)$$

If $k < 1 - \mu$ we have

$$\int_t^b (s-t)^{k-1} ds = \frac{(s-t)^k}{k} \Big|_{s=t}^{s=b} \leq c(b-t)^k \leq c(b-a)^k, \quad t \in [a, b]. \quad (2.26)$$

If $k = 1 - \mu$ then we can write

$$\int_t^b (s-t)^{k-1} (1 + |\log s|) ds \leq \int_t^b \frac{1}{\sqrt{s-t}} (s-t)^{k-\frac{1}{2}} (2 + |\log s|) ds, \quad t \in [a, b].$$

Since for a fixed $t \in [a, b]$ the function $x(s) = (s - t)^{k-\frac{1}{2}}(2 + |\log s|)$ is increasing on $[t, b]$, we obtain

$$\begin{aligned} \int_t^b \frac{1}{\sqrt{s-t}} (s-t)^{k-\frac{1}{2}} (2 + |\log s|) ds &\leq (b-t)^{k-\frac{1}{2}} (2 + |\log b|) \int_t^b \frac{1}{\sqrt{s-t}} ds \\ &\leq c(b-a)^k (1 + |\log b|), \quad t \in [a, b]. \end{aligned} \quad (2.27)$$

In the case $k > 1 - \mu$ we can use the equality

$$\int_t^b (s-t)^{k-1} s^{1-\mu-k} ds = \int_t^b (s-t)^{-\mu} (s-t)^{k-1+\mu} s^{1-\mu-k} ds.$$

Since for a fixed $t \in [a, b]$ the function $x(s) = (s-t)^{k-1+\mu} s^{1-\mu-k}$ is increasing on $[t, b]$, we get

$$\begin{aligned} \int_t^b (s-t)^{-\mu} (s-t)^{k-1+\mu} s^{1-\mu-k} ds &\leq (b-t)^{k-1+\mu} b^{1-\mu-k} \int_t^b (s-t)^{-\mu} ds \\ &\leq c(b-a)^k b^{1-\mu-k}, \quad t \in [a, b]. \end{aligned} \quad (2.28)$$

The inequalities (2.24) and (2.25) together with (2.26) if $k < 1 - \mu$, (2.27) if $k = 1 - \mu$ and (2.28) if $k > 1 - \mu$ give us the estimate (2.23). \diamond

Corollary 2.4.4. *Let $x \in C^{k,\mu}[0, T]$, $k \in \mathbb{N}$, $\mu < 1$ and assume that a grid Π_N and the interpolation points (2.15) are used. Then for the operator P_N defined by (2.17) the estimate*

$$\sup_{t \in (t_{n-1}, t_n)} |x(t) - (P_N x)(t)| \leq c h_n^k \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu, \\ t_n^{1-\mu-k} & \text{if } k > 1 - \mu, \end{cases} \quad (2.29)$$

where $h_n = t_n - t_{n-1}$ and c is a constant which is independent of n and N , holds.

Proof. Since for $t \in [t_{n-1}, t_n]$ we have $(P_N x)(t) = (P_{t_{n-1}, t_n} x)(t)$, the estimate (2.29) follows from (2.23) with $a = t_{n-1}$ and $b = t_n$. \diamond

2.4.1. Interpolation error estimates for regular grids

In this subsection we study the approximation properties of P_N without assuming anything about the structure of the underlying grid Π_N . The L^∞ estimate of the next lemma was originally proved by I. Parts, A. Pedas in [43], the L^1 estimate is new and enables us to show in the next chapter that some convergence results of piecewise polynomial collocation methods, that were previously proved only for uniform and quasi-uniform grids, also hold for all regular grids.

Lemma 2.4.5. *Let $x \in C^{k,\mu}[0, T]$, $k \in \mathbb{N}$, $\mu < 1$ and assume that a grid Π_N and the interpolation points (2.15) are used. Then for the operator P_N defined by (2.17) the estimates*

$$\|x - P_N x\|_\infty \leq c \begin{cases} h^k & \text{if } k < 1 - \mu, \\ h^k(1 + |\log h|) & \text{if } k = 1 - \mu, \\ h^{1-\mu} & \text{if } k > 1 - \mu \end{cases} \quad (2.30)$$

and

$$\int_0^T |x(s) - (P_N x)(s)| ds \leq c \begin{cases} h^k & \text{if } k < 2 - \mu, \\ h^k(1 + |\log h|) & \text{if } k = 2 - \mu, \\ h^{2-\mu} & \text{if } k > 2 - \mu \end{cases} \quad (2.31)$$

hold. Here h is defined by (2.12) and c is a positive constant which is independent of N and h .

Proof. We prove the estimations (2.30) and (2.31) separately.

(i) Proof of the uniform error estimates (2.30). Using Corollary 2.4.4 we get

$$\begin{aligned} \|x - P_N x\|_\infty &= \max_{n=1, \dots, N} \left(\sup_{t \in (t_{n-1}, t_n)} |x(t) - (P_N x)(t)| \right) \\ &\leq c \max_{n=1, \dots, N} h_n^k \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu, \\ t_n^{1-\mu-k} & \text{if } k > 1 - \mu, \end{cases} \end{aligned}$$

where $h_n = t_n - t_{n-1}$.

If $k < 1 - \mu$ then we estimate

$$h_n^k \leq h^k$$

and therefore

$$\|x - P_N x\|_\infty \leq c h^k.$$

In the case $k = 1 - \mu$, using the inequality $t_n \geq h_n$ and the fact that the function $x(s) = s^k(1 + |\log s|)$ is monotonically increasing for $s > 0$ we get

$$h_n^k(1 + |\log t_n|) \leq \left\{ \begin{array}{ll} h_n^k(1 + |\log h_n|) & \text{if } t_n \leq 1 \\ c h^k & \text{if } t_n > 1 \end{array} \right\} \leq c_1 h^k(1 + |\log h|).$$

Thus

$$\|x - P_N x\|_\infty \leq c h^k(1 + |\log h|).$$

If $k > 1 - \mu$ then the observation $t_n \geq h_n$ gives us

$$h_n^k t_n^{1-\mu-k} \leq h_n^k h_n^{1-\mu-k} \leq h^{1-\mu}.$$

Therefore

$$\|x - P_N x\|_\infty \leq c h^{1-\mu}.$$

This concludes the proof of (2.30).

(ii) Proof of L^1 estimates (2.31). It follows from Corollary 2.4.4 that

$$\begin{aligned} \int_0^T |x(s) - (P_N x)(s)| ds &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |x(s) - (P_N x)(s)| ds \\ &\leq c \sum_{n=1}^N h_n^{k+1} \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu, \\ t_n^{1-\mu-k} & \text{if } k > 1 - \mu. \end{cases} \end{aligned}$$

If $k < 1 - \mu$ then we get

$$\sum_{n=1}^N h_n^{k+1} \leq h^k \sum_{n=1}^N h_n \leq c h^k.$$

In the case $k = 1 - \mu$ we have

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1}(1 + |\log t_n|) &\leq h^k \sum_{n=1}^N h_n(1 + |\log t_n|) \\ &\leq c h^k \int_0^T (1 + |\log s|) ds \leq c_1 h^k. \end{aligned}$$

If $k > 1 - \mu$ and $k < 2 - \mu$ then we obtain

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1} t_n^{1-\mu-k} &\leq h^k \sum_{n=1}^N h_n t_n^{1-\mu-k} \\ &\leq h^k \int_0^T s^{1-\mu-k} ds \leq c h^k. \end{aligned}$$

Thus, if $k < 2 - \mu$ then the statement (2.31) holds. If $k \geq 2 - \mu$ then we can write

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1} t_n^{1-\mu-k} &= h_1^{k+1} t_1^{1-\mu-k} + \sum_{n=2}^N h_n^{k+1} t_n^{1-\mu-k} \\ &= t_1^{2-\mu} + \sum_{n=2}^N h_n^{k+1} t_n^{1-\mu-k}. \end{aligned}$$

Note that

$$t_1^{2-\mu} + \sum_{n=2}^N h_n^{k+1} t_n^{1-\mu-k} \leq t_2^{2-\mu} + \sum_{n=3}^N h_n^{k+1} t_n^{1-\mu-k}. \quad (2.32)$$

Indeed, since

$$\frac{t_1}{t_2} + \frac{h_2}{t_2} = 1$$

we get that

$$\left(\frac{t_1}{t_2}\right)^{2-\mu} + \left(\frac{h_2}{t_2}\right)^{k+1} \leq 1,$$

which gives us

$$t_1^{2-\mu} + h_2^{k+1} t_2^{1-\mu-k} \leq t_2^{2-\mu}.$$

Last inequality is equivalent to (2.32).

We choose n_0 so, that $h \leq t_{n_0} < 2h$. Using repeatedly the inequality (2.32), we obtain

$$\begin{aligned}
\sum_{n=1}^N h_n^{k+1} t_n^{1-\mu-k} &\leq t_{n_0}^{2-\mu} + \sum_{n=n_0+1}^N h_n^{k+1} t_n^{1-\mu-k} \\
&\leq (2h)^{2-\mu} + h^k \sum_{n=n_0+1}^N h_n t_n^{1-\mu-k} \\
&\leq (2h)^{2-\mu} + h^k \int_{t_{n_0}}^T s^{1-\mu-k} ds \\
&\leq (2h)^{2-\mu} + c h^k \begin{cases} 1 + |\log t_{n_0}| & \text{if } k = 2 - \mu \\ t_{n_0}^{2-\mu-k} & \text{if } k > 2 - \mu \end{cases} \\
&\leq (2h)^{2-\mu} + c_1 h^k \begin{cases} 1 + |\log h| & \text{if } k = 2 - \mu \\ h^{2-\mu-k} & \text{if } k > 2 - \mu \end{cases} \\
&\leq c_2 \begin{cases} h^k (1 + |\log h|) & \text{if } k = 2 - \mu, \\ h^{2-\mu} & \text{if } k > 2 - \mu. \end{cases}
\end{aligned}$$

So, the statement (2.31) holds true also for $k \geq 2 - \mu$. \diamond

Usually, when studying convergence properties of projection methods, the corresponding projectors are bounded and converge strongly to the identity operator on a suitable Banach space. Unfortunately there is no such "suitable" space for the operators P_N but the following result (which for graded grids is established in [14], Lemma 3.2) together with Lemma 2.4.1 still enables us to apply general convergence theorems to piecewise polynomial collocation methods for solving integral and integro-differential equations.

Lemma 2.4.6. *Let $S : L^\infty(0, T) \rightarrow C[0, T]$ be a linear compact operator. Suppose that the grid (see (2.7)) is regular i.e. satisfies (2.8). Then for the operator P_N defined by (2.17) we have*

$$\|S - P_N S\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (2.33)$$

Proof. From Lemma 2.4.5 we obtain for every $x \in C^1[0, T] \subset C^{1,0}[0, T]$ that

$$\|x - P_N x\|_\infty \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Since according to Lemma 2.4.1 the operators P_N are uniformly bounded and the set $C^1[0, T]$ is dense in $C[0, T]$, Banach-Steinhaus Theorem 2.1.2

gives us that for every $x \in C[0, T]$

$$\|x - P_N x\|_\infty \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Theorem 2.1.4 (with $A = S$, $L_n = P_N$ and $L = I$) yields now the convergence (2.33). Lemma 2.4.6 is proved. \diamond

2.4.2. Interpolation error estimates for graded grids

In the case of graded grids it is possible to reduce the interpolation error significantly by choosing an appropriate value for the nonuniformity parameter r . The following lemma characterizes the dependence of the interpolation error in L^∞ and L^1 norm on the parameter r . An analog for multidimensional case and slightly different nonuniform grids is formulated and proved in [55], the statement of the lemma with an outline of the proof can also be found in [14].

Lemma 2.4.7. *Let $x \in C^{k,\mu}[0, T]$, $k \in \mathbb{N}$, $\mu < 1$ and assume that a graded grid Π_N^r with $r \geq 1$ and the interpolation points (2.15) are used. Then for the operator P_N defined by (2.17) the following estimates hold:*

$$\|x - P_N x\|_\infty \leq c \begin{cases} N^{-r(1-\mu)} & \text{for } 1 \leq r < \frac{k}{1-\mu}, \\ N^{-k}(1 + \log N) & \text{for } r = \frac{k}{1-\mu} = 1, \\ N^{-k} & \text{for } r > \frac{k}{1-\mu} \text{ or } r = \frac{k}{1-\mu} > 1 \end{cases} \quad (2.34)$$

and

$$\int_0^T |x(s) - (P_N x)(s)| ds \leq c \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r < \frac{k}{2-\mu}, \\ N^{-k}(1 + \log N) & \text{for } r = \frac{k}{2-\mu}, \\ N^{-k} & \text{for } r > \frac{k}{2-\mu}. \end{cases} \quad (2.35)$$

Here c is a positive constant which is independent of N .

Proof. We prove the estimates (2.34) and (2.35) separately.

(i) Proof of the uniform error estimates (2.34). Using the inequality (2.29) we get

$$\|x - P_N x\|_\infty \leq c \max_{n=1, \dots, N} h_n^k \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu, \\ t_n^{1-\mu-k} & \text{if } k > 1 - \mu. \end{cases}$$

Note that for graded grids (see (2.10)) we have

$$\begin{aligned}
h_n^\beta &= (t_n - t_{n-1})^\beta = \left(T \left(\frac{n}{N} \right)^r - T \left(\frac{n-1}{N} \right)^r \right)^\beta \\
&= T^\beta N^{-r\beta} [n^r - (n-1)^r]^\beta \\
&\leq T^\beta r^\beta N^{-r\beta} n^{(r-1)\beta}, \quad r \geq 1, \beta \geq 0, n = 1, \dots, N
\end{aligned} \tag{2.36}$$

and

$$t_n^\beta = \left(T \left(\frac{n}{N} \right)^r \right)^\beta = T^\beta N^{-r\beta} n^{r\beta}, \quad r \geq 1, \beta \geq 0, n = 1, \dots, N. \tag{2.37}$$

If $k < 1 - \mu$ then, with the aid of (2.36), we have

$$h_n^k \leq T^k r^k N^{-rk} n^{(r-1)k} \leq c N^{-k}, \quad n = 1, \dots, N.$$

In the case $k = 1 - \mu$ we obtain for all $n = 1, \dots, N$ that

$$\begin{aligned}
h_n^k (1 + |\log t_n|) &\leq c N^{-rk} n^{(r-1)k} \left(1 + \left| \log \frac{n}{N} \right| \right) \\
&= c N^{-k} \left(\frac{n}{N} \right)^{(r-1)k} \left(1 + \left| \log \frac{n}{N} \right| \right) \\
&\leq c_1 \begin{cases} N^{-k} (1 + \log N) & \text{for } r = 1, \\ N^{-k} & \text{for } r > 1. \end{cases}
\end{aligned}$$

Similarly, in the case $k > 1 - \mu$ we get with the help of (2.36) and (2.37) that

$$\begin{aligned}
h_n^k t_n^{1-\mu-k} &\leq c N^{-rk} n^{(r-1)k} N^{-r(1-\mu-k)} n^{r(1-\mu-k)} \\
&= c N^{-r(1-\mu)} n^{r(1-\mu)-k} \\
&\leq c_1 \begin{cases} N^{-r(1-\mu)} & \text{for } 1 \leq r \leq \frac{k}{1-\mu}, \\ N^{-k} & \text{for } r > \frac{k}{1-\mu} \end{cases}
\end{aligned}$$

for $n = 1, \dots, N$.

Combining the results of the three cases we see, that the estimation (2.34) holds.

(ii) Proof of L^1 estimates (2.35) is very similar to the previous one. Applying the inequality (2.29) gives us

$$\begin{aligned} \int_0^T |x(s) - (P_N x)(s)| ds &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |x(s) - (P_N x)(s)| ds \leq \\ &\leq c \sum_{n=1}^N h_n^{k+1} \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu, \\ t_n^{1-\mu-k} & \text{if } k > 1 - \mu. \end{cases} \end{aligned}$$

Using (2.36) with $\beta = k + 1$ we get in the case $k < 1 - \mu$ that

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1} &\leq T^{k+1} r^{k+1} N^{-r(k+1)} \sum_{n=1}^N n^{(r-1)(k+1)} \\ &\leq c N^{-r(k+1)} N^{(r-1)(k+1)+1} = c N^{-k}. \end{aligned}$$

If $k = 1 - \mu$ then using (2.14) we have

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1} (1 + |\log t_n|) &\leq h^k \sum_{n=1}^N h_n (1 + |\log t_n|) \\ &\leq c N^{-k} \int_0^T (1 + |\log s|) ds \leq c_1 N^{-k}. \end{aligned}$$

Finally, in the case $k > 1 - \mu$ with help of (2.36) and (2.37) we obtain that

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1} t_n^{1-\mu-k} &\leq c N^{-r(k+1)} N^{-r(1-\mu-k)} \sum_{n=1}^N n^{(r-1)(k+1)} n^{r(1-\mu-k)} \\ &= c N^{-r(2-\mu)} \sum_{n=1}^N n^{r(2-\mu)-k-1}. \end{aligned}$$

Note that for $\alpha \in \mathbb{R}$, $\alpha \leq 0$ we have

$$\sum_{i=1}^N n^\alpha = 1 + \sum_{i=2}^N n^\alpha \leq 1 + \int_1^N s^\alpha ds \leq c \begin{cases} 1 & \text{if } \alpha < -1, \\ 1 + \log N & \text{if } \alpha \leq -1, \\ N^{\alpha+1} & \text{if } \alpha > -1 \end{cases}$$

and for $\alpha > 0$ the estimate

$$\sum_{i=1}^N n^\alpha \leq \sum_{i=1}^N N^\alpha = N^{\alpha+1}$$

holds, thus

$$\sum_{i=1}^N n^\alpha \leq c \begin{cases} 1 & \text{if } \alpha < -1, \\ 1 + \log N & \text{if } \alpha \leq 1, \\ N^{\alpha+1} & \text{if } \alpha > -1. \end{cases} \quad (2.38)$$

Using (2.38) with $\alpha = r(2 - \mu) - k - 1$ we get

$$\begin{aligned} \sum_{n=1}^N h_n^{k+1} t_n^{1-\mu-k} &\leq c N^{-r(2-\mu)} \begin{cases} 1 & \text{for } 1 \leq r < \frac{k}{2-\mu} \\ 1 + \log N & \text{for } r = \frac{k}{2-\mu} \\ N^{r(2-\mu)-k} & \text{for } r > \frac{k}{2-\mu} \end{cases} \\ &= c \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r < \frac{k}{2-\mu}, \\ N^{-k}(1 + \log N) & \text{for } r = \frac{k}{2-\mu}, \\ N^{-k} & \text{for } r > \frac{k}{2-\mu}. \end{cases} \end{aligned}$$

Combining the results of the three cases we get the estimation (2.35) of the lemma. Lemma 2.4.7 is proved. \diamond

2.5. Approximation of integrals and integral operators in the case of graded grids and special collocation parameters

Often numerical methods give approximations for the derivative of the solution of an equation such that the error at the interpolation points is much smaller than in the maximum norm over the whole interval. Next lemma shows that this may lead to a good uniform approximation of a solution itself.

Lemma 2.5.1. *Let $x \in C^{k+1,\mu}[0, T]$, $w \in S_{k-1}^{(-1)}(\Pi_N^r)$, $k \in \mathbb{N}$, $\mu < 1$, $r \geq 1$ and the parameters η_1, \dots, η_k be chosen so that the quadrature approximation*

$$\int_0^1 \varphi(s) ds \approx \sum_{j=1}^k A_j \varphi(\eta_j), \quad 0 \leq \eta_1 < \dots < \eta_k \leq 1, \quad (2.39)$$

with appropriate weights $\{A_j\}$, is exact for all polynomials of degree k .

Then the estimate

$$\begin{aligned} & \max_{t \in [0, T]} \left| \int_0^t (x(s) - w(s)) ds \right| \\ & \leq c_1 \varepsilon'_N + c_2 \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu}, \\ N^{-k-1}(1 + \log N) & \text{for } r = \frac{k+1}{2-\mu}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu} \end{cases} \end{aligned}$$

holds. Here c_1 and c_2 are positive constants which are independent of N ,

$$\varepsilon'_N = \max_{n=1, \dots, N, j=1, \dots, k} |x(t_{nj}) - w(t_{nj})|$$

and the interpolation points $\{t_{nj}\}$ are given by (2.15).

Remark 2.5.2. The assumption of Lemma 2.5.1 about the parameters η_j ($j = 1, \dots, k$) is equivalent to the condition

$$\int_0^1 (s - \eta_1) \cdot (s - \eta_2) \cdot \dots \cdot (s - \eta_k) ds = 0.$$

Proof of Lemma 2.5.1. Fix $t \in [0, T]$, let $n \in \{1, \dots, N\}$ be such that $t \in [t_{n-1}, t_n]$. Then

$$\left| \int_0^t (x(s) - w(s)) ds \right| \leq \left| \int_0^{t_{n-1}} (x(s) - w(s)) ds \right| + \left| \int_{t_{n-1}}^t (x(s) - w(s)) ds \right|. \quad (2.40)$$

Consider the first term. Using the weights A_j ($j = 1, \dots, k$) from (2.39) we define an abstract operator $Q_{t_{n-1}}$ by

$$Q_{t_{n-1}} z = \sum_{i=1}^{n-1} \sum_{j=1}^k h_i A_j z(t_{ij}),$$

where h_i and t_{ij} are defined by (2.11) and (2.15), respectively. Since the quadrature approximation (2.39) is exact for all polynomials of order k we

have

$$\begin{aligned} \int_{t_{i-1}}^{t_i} w(s) ds &= h_i \int_0^1 w(t_{i-1} + \tau h_i) d\tau \\ &= h_i \sum_{j=1}^k A_j w(t_{i-1} + \eta_j h_i), \quad i = 1, \dots, n-1. \end{aligned}$$

Consequently

$$Q_{t_{n-1}} w = \int_0^{t_{n-1}} w(s) ds$$

and

$$\begin{aligned} \left| \int_0^{t_{n-1}} (x(s) - w(s)) ds \right| &= \left| \int_0^{t_{n-1}} x(s) ds - \int_0^{t_{n-1}} w(s) ds \right| = \left| \int_0^{t_{n-1}} x(s) ds - Q_{t_{n-1}} w \right| \\ &\leq \left| \int_0^{t_{n-1}} x(s) ds - Q_{t_{n-1}} x \right| + |Q_{t_{n-1}} x - Q_{t_{n-1}} w|. \end{aligned} \tag{2.41}$$

We choose $0 \leq \eta'_1 < \dots < \eta'_{k+1} \leq 1$ such that $\{\eta_1, \dots, \eta_k\} \subset \{\eta'_1, \dots, \eta'_{k+1}\}$ and define for each $N \in \mathbb{N}$ a piecewise polynomial function $\tilde{P}_N x \in S_k^{(-1)}(\Pi_N^r)$ by conditions

$$(\tilde{P}_N x)(t'_{nj}) = x(t'_{nj}), \quad j = 1, \dots, k+1, \quad n = 1, \dots, N,$$

where

$$t'_{nj} = t_{n-1} + \eta'_j h_n, \quad j = 1, \dots, k+1, \quad n = 1, \dots, N.$$

Since

$$Q_{t_{n-1}} x = \int_0^{t_{n-1}} (\tilde{P}_N x)(s) ds$$

we get

$$\begin{aligned}
\left| \int_0^{t_{n-1}} x(s) ds - Q_{t_{n-1}} x \right| &= \left| \int_0^{t_{n-1}} x(s) ds - \int_0^{t_{n-1}} (\tilde{P}_N x)(s) ds \right| \\
&= \left| \int_0^{t_{n-1}} (x(s) - (\tilde{P}_N x)(s)) ds \right| \\
&\leq \int_0^{t_{n-1}} |x(s) - (\tilde{P}_N x)(s)| ds.
\end{aligned}$$

Since $x \in C^{k+1, \mu}[0, T]$ we obtain from Lemma 2.4.7 (see (2.35)) the estimates

$$\int_0^{t_{n-1}} |x(s) - (\tilde{P}_N x)(s)| ds \leq c \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu}, \\ N^{-k-1}(1 + \log N) & \text{for } r = \frac{k+1}{2-\mu}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu}. \end{cases} \quad (2.42)$$

This together with the estimates

$$|Q_{t_{n-1}} x - Q_{t_{n-1}} w| \leq c \max_{i=1, \dots, N, j=1, \dots, k} |x(t_{ij}) - w(t_{ij})| = c \varepsilon'_N$$

and (2.41) gives us

$$\left| \int_0^{t_{n-1}} (x(s) - w(s)) ds \right| \leq c_1 \varepsilon'_N + c_2 \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu}, \\ N^{-k-1}(1 + \log N) & \text{for } r = \frac{k+1}{2-\mu}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu}. \end{cases} \quad (2.43)$$

It remains to estimate the last term in (2.40). We have

$$\left| \int_{t_{n-1}}^t (x(s) - w(s)) ds \right| \leq \int_{t_{n-1}}^{t_n} |x(s) - (P_N x)(s)| ds + \int_{t_{n-1}}^{t_n} |(P_N x)(s) - w(s)| ds, \quad (2.44)$$

where the operator P_N is defined by (2.17). Since $x \in C^{k+1, \mu}[0, T] \subset C^{k, \mu}[0, T]$ we have, using (2.29), that

$$\int_{t_{n-1}}^{t_n} |x(s) - (P_N x)(s)| ds \leq c h_n^{k+1} \begin{cases} 1 & \text{if } k < 1 - \mu, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu, \\ t_n^{1-\mu-k} & \text{if } k > 1 - \mu. \end{cases}$$

In the case $k < 1 - \mu$, using (2.36), we get

$$\begin{aligned} h_n^{k+1} &\leq T^{k+1} r^{k+1} N^{-r(k+1)} n^{(r-1)(k+1)} \\ &\leq c N^{-r(k+1)} N^{(r-1)(k+1)} = c N^{-k-1}. \end{aligned}$$

In the case $k = 1 - \mu$ we have with the help of (2.36) that

$$h_n^{k+1} (1 + |\log t_n|) \leq c N^{-k-1} \left(\frac{n}{N} \right)^{(r-1)(k+1)} \left(1 + \left| \log \frac{n}{N} \right| \right).$$

If $r = 1$ then $1 + \left| \log \frac{n}{N} \right| \leq 1 + \log N$ and

$$h_n^{k+1} (1 + |\log t_n|) \leq c N^{-k-1} (1 + \log N).$$

If $r > 1$ then boundedness of the function $f(s) = s^{(r-1)(k-1)} (1 + |\log s|)$, $s \in (0, 1]$ implies that

$$h_n^{k+1} (1 + |\log t_n|) \leq c N^{-k-1}.$$

In the case $k > 1 - \mu$, using (2.36) and (2.37), we obtain

$$\begin{aligned} h_n^{k+1} t_n^{1-\mu-k} &\leq T^{2-\mu} r^{k+1} N^{-r(2-\mu)} n^{r(2-\mu)-k-1} \\ &\leq c \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r \leq \frac{k+1}{2-\mu}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu}. \end{cases} \end{aligned}$$

We can summarize these cases as follows:

$$\int_{t_{n-1}}^{t_n} |x(s) - (P_N x)(s)| ds \leq c \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r \leq \frac{k+1}{2-\mu}, k > 1 - \mu, \\ N^{-k-1} (1 + \log N) & \text{for } r = \frac{k+1}{2-\mu} = 1, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu}. \end{cases} \quad (2.45)$$

In order to estimate the second addend on the right-hand side of (2.44) we note that

$$w(t) = (P_{t_{n-1}, t_n} w)(t), \quad t \in (t_{n-1}, t_n)$$

and

$$(P_N x)(t) = (P_{t_{n-1}, t_n} x)(t), \quad t \in (t_{n-1}, t_n),$$

where P_{t_{n-1}, t_n} is the operator defined by (2.19) with $a = t_{n-1}$ and $b = t_n$. Hence, using the estimate (2.22) we get

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} |(P_N x)(s) - w(s)| ds &= \int_{t_{n-1}}^{t_n} |(P_{t_{n-1}, t_n}(x - w))(s)| ds \\
&\leq h_n \max_{s \in [t_{n-1}, t_n]} |(P_{t_{n-1}, t_n}(x - w))(s)| \\
&\leq ch_n \max_{j=1, \dots, k} |x(t_{nj}) - w(t_{nj})| \\
&\leq c_1 N^{-1} \varepsilon'_N.
\end{aligned} \tag{2.46}$$

It follows from (2.44), (2.45) and (2.46) that the last term of (2.40) can be estimated as follows:

$$\begin{aligned}
\left| \int_{t_{n-1}}^t (x(s) - w(s)) ds \right| &\leq c N^{-1} \varepsilon'_N \\
+ c_1 \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r \leq \frac{k+1}{2-\mu}, k > 1 - \mu, \\ N^{-k-1}(1 + \log N) & \text{for } r = 1, k = 1 - \mu, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu}. \end{cases}
\end{aligned} \tag{2.47}$$

Combining the estimates (2.40), (2.43) and (2.47) we get the statement of the lemma. Lemma 2.5.1 is proved. \diamond

If we choose $w = P_N x$ in the previous lemma, then the error at the interpolation points t_{nj} is zero, hence we get the following result.

Corollary 2.5.3. *Let $x \in C^{k+1, \mu}[0, T]$, $k \in \mathbb{N}$, $\mu < 1$ and a graded grid Π_N^r , $r \geq 1$ be given. Assume that the parameters η_1, \dots, η_k are chosen so that the quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^k A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_k \leq 1$, with appropriate weights $\{A_j\}$, is exact for all polynomials of degree k and that the interpolation points (2.15) are used.*

Then for the operator P_N defined by (2.17) the estimate

$$\max_{t \in [0, T]} \left| \int_0^t (x(s) - (P_N x)(s)) \right| \leq c \begin{cases} N^{-r(2-\mu)} & \text{for } 1 \leq r \leq \frac{k+1}{2-\mu}, \\ N^{-k-1}(1 + \log N) & \text{for } r = \frac{k+1}{2-\mu}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu} \end{cases}$$

holds, where c is a positive constant which is independent of N .

Our next goal is to get sharp estimates for the L^∞ norm of the difference $Sx - SP_Nx$, $x \in C^{k+1,\mu}[0, T]$, where S is an Volterra integral operator with a nonsmooth kernel. Such estimates can be used for proving the superconvergence of approximate solutions to integral equations at the interpolation points.

First we consider integral operators with continuous kernels.

Theorem 2.5.4. *Assume:*

- (a) *The function $K \in C(\bar{\Delta}_T)$, where $\bar{\Delta}_T$ is defined by (2.5), is such that $\frac{\partial K}{\partial s}(t, \cdot) \in L^1(0, t)$ for each $t \in [0, T]$ and*

$$\sup_{t \in [0, T]} \int_0^t \left| \frac{\partial K}{\partial s}(t, s) \right| ds \leq \text{const.}$$

- (b) *The function $x \in C^{k+1,\mu_1}[0, T]$, $k \in \mathbb{N}$, $\mu_1 < 1$.*
- (c) *The parameters η_1, \dots, η_k are chosen so that the quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^k A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_k \leq 1$, with appropriate weights $\{A_j\}$, is exact for all polynomials of degree k .*
- (d) *A graded grid Π_N^r , $r \geq 1$ and the interpolation nodes (2.15) are used.*
- (e) *The operator P_N is defined by (2.17).*

Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the estimate

$$\max_{t \in [0, T]} \left| \int_0^t K(t, s) (x(s) - (P_N x)(s)) ds \right| \leq c \begin{cases} N^{-r(2-\mu_1)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu_1} \end{cases}$$

holds. Here c is a positive constant which is independent of N .

Proof. Denote

$$g(t) = \int_0^t (x(s) - (P_N x)(s)) ds, \quad t \in [0, T].$$

Integration by parts gives us

$$\begin{aligned} \int_0^t K(t,s)(x(s) - (P_N x)(s))ds &= K(t,s)g(s) \Big|_{s=0}^{s=t} - \int_0^t g(s) \frac{\partial K}{\partial s}(t,s)ds \\ &= K(t,t)g(t) - \int_0^t \frac{\partial K}{\partial s}(t,s)g(s)ds, \quad t \in [0, T]. \end{aligned}$$

Using the assumption (a) and Corollary 2.5.3 we get

$$\begin{aligned} \max_{t \in [0, T]} \left| \int_0^t K(t,s)(x(s) - (P_N x)(s))ds \right| &= \max_{t \in [0, T]} \left| K(t,t)g(t) - \int_0^t \frac{\partial K}{\partial s}(t,s)g(s)ds \right| \\ &\leq \max_{t \in [0, T]} |K(t,t)g(t)| + \sup_{t \in [0, T]} \int_0^t \left| \frac{\partial K}{\partial s}(t,s)g(s) \right| ds \\ &\leq c_1 \max_{t \in [0, T]} |g(t)| + \max_{s \in [0, T]} |g(s)| \sup_{t \in [0, T]} \int_0^t \left| \frac{\partial K}{\partial s}(t,s) \right| ds \\ &\leq c_2 \max_{t \in [0, T]} |g(t)| \\ &\leq c \begin{cases} N^{-r(2-\mu_1)} & \text{for } 1 \leq r \leq \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1} & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \end{aligned}$$

Theorem 2.5.4 is proved. \diamond

When the kernel of the Volterra integral operator is not continuous, then it is much more difficult to obtain good estimations. In order to shorten proof of the next theorem we prove first a technical result.

Proposition 2.5.5. *Let $\gamma < 0$ and β be real numbers, and let $N \geq 2$ be an integer. Then for all n satisfying $2 \leq n \leq N$ the estimates*

$$\sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq c \begin{cases} 1 & \text{if } \beta + \gamma < -1 \quad \text{and } \beta < 0, \\ N^\beta & \text{if } \beta \geq 0 \quad \text{and } \gamma < -1, \\ N^\beta \log N & \text{if } \beta \geq 0 \quad \text{and } \gamma = -1, \\ N^{\beta+\gamma+1} & \text{if } \beta + \gamma \geq -1 \quad \text{and } \gamma > -1 \end{cases} \quad (2.48)$$

and

$$\sum_{i=1}^{n-1} i^\beta \log\left(\frac{N}{i}\right) (n-i)^\gamma \leq c \begin{cases} \log N & \text{if } \beta + \gamma < -1 \text{ and } \beta \leq 0 \text{ or} \\ & \beta + \gamma = -1 \text{ and } \beta < 0, \\ (\log N)^2 & \text{if } \beta = 0 \text{ and } \gamma = -1, \\ N^\beta & \text{if } \beta > 0 \text{ and } \gamma < -1, \\ N^\beta \log N & \text{if } \beta > 0 \text{ and } \gamma = -1, \\ N^{\beta+\gamma+1} & \text{if } \beta + \gamma > -1 \text{ and } \gamma > -1 \end{cases} \quad (2.49)$$

hold, where c is a positive constant which does not depend on n and N .

Proof. We prove the estimates (2.48) and (2.49) separately.

(i) Proof of the estimate (2.48). In the case $\beta + \gamma < -1$ and $\beta < 0$ we have

$$i^\beta (n-i)^\gamma \leq \begin{cases} i^{\beta+\gamma} & \text{if } i \leq n-i, \\ (n-i)^{\beta+\gamma} & \text{if } i > n-i. \end{cases}$$

Hence

$$\sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq \sum_{i=1}^{n-1} \left(i^{\beta+\gamma} + (n-i)^{\beta+\gamma} \right) \leq 2 \sum_{i=1}^{\infty} i^{\beta+\gamma} = c.$$

In the case $\beta \geq 0$ and $\gamma < -1$ we have

$$\sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq n^\beta \sum_{i=1}^{n-1} (n-i)^\gamma \leq N^\beta \sum_{j=1}^{\infty} j^\gamma = cN^\beta.$$

If $\beta \geq 0$ and $\gamma = -1$, then

$$\sum_{i=1}^{n-1} i^\beta (n-i)^{-1} \leq n^\beta \sum_{i=1}^{n-1} (n-i)^{-1} \leq cN^\beta \log N.$$

In the case $\beta + \gamma \geq -1$ and $\gamma > -1$ we have

$$\begin{aligned} \sum_{i=1}^{n-1} i^\beta (n-i)^\gamma &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^\beta (n-i)^\gamma + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} i^\beta (n-i)^\gamma \\ &\leq \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^\beta \left(\frac{n}{2}\right)^\gamma + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} \max\left\{ \left(\frac{n}{2}\right)^\beta, n^\beta \right\} (n-i)^\gamma \\ &\leq \left(\frac{n}{2}\right)^\gamma \frac{n^{\beta+1}}{\beta+1} + \max\left\{ 1, \frac{1}{2^\beta} \right\} n^\beta \frac{n^{\gamma+1}}{\gamma+1} \leq cN^{\beta+\gamma+1}, \end{aligned}$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the integer part of $\frac{n}{2}$. Here, in the last line, we used the fact that for nonnegative increasing functions x we have

$$\sum_{i=p}^q x(i) \leq \int_p^{q+1} x(s) ds$$

and for nonnegative decreasing functions the estimate

$$\sum_{i=p}^q x(i) \leq \int_{p-1}^q x(s) ds$$

holds. The estimate (2.48) is proved.

(ii) Proof of the estimate (2.49). If $\beta + \gamma < -1$ and $\beta \leq 0$ or $\beta + \gamma = -1$ and $\beta < 0$ then with the help of (2.48) we get

$$\sum_{i=1}^{n-1} i^\beta \log\left(\frac{N}{i}\right) (n-i)^\gamma \leq \log N \sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq c \log N.$$

Indeed, if $\beta + \gamma < -1$, $\beta < 0$ then the first line of (2.48) gives us that

$$\sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq c;$$

if $\beta + \gamma < -1$, $\beta = 0$ then $\gamma < -1$ and the second line of (2.48) gives us that

$$\sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq c N^\beta = c;$$

if $\beta + \gamma = -1$, $\beta < 0$, then $\gamma > -1$ and the last line of (2.48) gives us

$$\sum_{i=1}^{n-1} i^\beta (n-i)^\gamma \leq c N^{\beta+\gamma+1} = c.$$

In the case $\beta = 0$ and $\gamma = -1$ we get

$$\begin{aligned} \sum_{i=1}^{n-1} i^\beta \log\left(\frac{N}{i}\right) (n-i)^\gamma &\leq \log N \sum_{i=1}^{n-1} (n-i)^{-1} = \log N \sum_{k=1}^{n-1} \frac{1}{k} \\ &\leq c \log N \log n \leq c (\log N)^2. \end{aligned}$$

In the case $\beta > 0$ and $\gamma \leq -1$ we get

$$\begin{aligned} \sum_{i=1}^{n-1} i^\beta \log \left(\frac{N}{i} \right) (n-i)^\gamma &\leq N^\beta \max_{j=1, \dots, n} \left[\left(\frac{j}{N} \right)^\beta \log \frac{N}{j} \right] \sum_{i=1}^{n-1} (n-i)^\gamma \\ &\leq N^\beta \sup_{\tau \in (0,1)} \left(\tau^\beta |\log \tau| \right) \sum_{i=1}^{n-1} (n-i)^\gamma \\ &\leq c \begin{cases} N^\beta & \text{if } \gamma < -1, \\ N^\beta \log N & \text{if } \gamma = -1. \end{cases} \end{aligned}$$

In the case $\beta + \gamma > -1$ and $\gamma > -1$ we divide proof into two part:

If $\beta > 0$ then

$$\begin{aligned} \sum_{i=1}^{n-1} i^\beta \log \left(\frac{N}{i} \right) (n-i)^\gamma &\leq N^\beta \sup_{\tau \in (0,1)} \left(\tau^\beta |\log \tau| \right) \sum_{i=1}^{n-1} (n-i)^\gamma \\ &\leq N^\beta c \frac{1}{\gamma+1} n^{\gamma+1} \leq c_1 N^{\beta+\gamma+1}. \end{aligned}$$

If $\beta \leq 0$ then

$$\begin{aligned} &\sum_{i=1}^{n-1} i^\beta \log \left(\frac{N}{i} \right) (n-i)^\gamma \\ &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^\beta \log \left(\frac{N}{i} \right) (n-i)^\gamma + \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} i^\beta \log \left(\frac{N}{i} \right) (n-i)^\gamma \\ &\leq \left(\frac{n}{2} \right)^\gamma N^{\beta+1} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{i}{N} \right)^\beta \log \left(\frac{N}{i} \right) \frac{1}{N} + \left(\frac{n}{2} \right)^\beta \log \left(\frac{N}{n/2} \right) \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} (n-i)^\gamma \\ &\leq \left(\frac{n}{2} \right)^\gamma N^{\beta+1} \int_0^1 \tau^\beta |\log \tau| d\tau + c \left(\frac{n}{N} \right)^{\beta+\gamma+1} \left(\log 2 + \log \left(\frac{N}{n} \right) \right) N^{\beta+\gamma+1} \\ &\leq c_1 N^{\beta+\gamma+1} + c_2 N^{\beta+\gamma+1} = c_3 N^{\beta+\gamma+1}. \end{aligned}$$

This concludes the proof of (2.49). Proposition 2.5.5 is proved. \diamond

Now we are ready to prove an analog of Theorem 2.5.4 for weakly singular integral operators.

Theorem 2.5.6. *Assume:*

- (a) *The kernel $K \in C(\Delta_T)$, where Δ_T is defined by (2.2), is differentiable with respect to s in Δ_T and satisfies for $i = 0$ and $i = 1$ the inequality*

$$\left| \left(\frac{\partial}{\partial s} \right)^i K(t, s) \right| \leq c \begin{cases} 1 + |\log(t-s)| & \text{if } \mu + i = 0 \\ |t-s|^{-\mu-i} & \text{if } \mu + i > 0 \end{cases}, \quad (t, s) \in \Delta_T,$$

with $\mu \in [0, 1)$.

- (b) *The function $x \in C^{k+1, \mu_1}[0, T]$, $k \in \mathbb{N}$, $\mu_1 < 1$.*
- (c) *The parameters η_1, \dots, η_k are chosen so that the quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^k A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_k \leq 1$, with appropriate weights $\{A_j\}$, is exact for all polynomials of degree k .*
- (d) *A graded grid Π_N^r , $r \geq 1$ and the interpolation nodes (2.15) are used.*
- (e) *The operator P_N is defined by (2.17).*

Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ for the quantity Ψ_N defined by

$$\Psi_N = \max_{t \in [0, T]} \left| \int_0^t K(t, s) (x(s) - (P_N x)(s)) ds \right|$$

the following estimates hold:

- (I) *If $\mu = 0$ then*

$$\Psi_N \leq c \begin{cases} N^{-r(2-\mu_1)}(1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \quad (2.50)$$

- (II) *If $\mu > 0$ then*

$$\Psi_N \leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu}(1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases} \quad (2.51)$$

Here c is a positive constant which is independent of N .

Proof. Fix t , let $n \in \{0, \dots, N-1\}$ be such that $t \in [t_n, t_{n+1}]$. Let us consider the cases $\mu = 0$ and $\mu > 0$ separately.

I Case $\mu = 0$. If $n = 0$ then with the help of Corollary 2.4.4 we obtain for $x \in C^{k+1, \mu_1}[0, T] \subset C^{k, \mu_1}[0, T]$ that

$$\begin{aligned} & \left| \int_0^t K(t, s)(x(s) - (P_N x)(s)) ds \right| \\ & \leq c \left(\sup_{s \in (0, t_1)} |x(s) - (P_N x)(s)| \right) \int_0^{t_1} (1 + |\log(t_1 - s)|) ds \\ & \leq c t_1^{k+1} (1 + |\log t_1|) \begin{cases} 1 & \text{if } k < 1 - \mu_1, \\ 1 + |\log t_1| & \text{if } k = 1 - \mu_1, \\ t_1^{1 - \mu_1 - k} & \text{if } k > 1 - \mu_1. \end{cases} \end{aligned}$$

Using (2.37) and taking into account that $r \geq 1$ we get

$$\begin{aligned} & \left| \int_0^t K(t, s)(x(s) - (P_N x)(s)) ds \right| \\ & \leq c \begin{cases} N^{-r(k+1)}(1 + \log N) & \text{if } k < 1 - \mu_1 \\ N^{-r(k+1)}(1 + (\log N)^2) & \text{if } k = 1 - \mu_1 \\ N^{-r(2-\mu_1)}(1 + \log N) & \text{if } k > 1 - \mu_1 \end{cases} \\ & \leq c \begin{cases} N^{-r(2-\mu_1)}(1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \quad (2.52) \end{aligned}$$

If $n = 1, \dots, N-1$ then we have

$$\begin{aligned} & \left| \int_0^t K(t, s)(x(s) - (P_N x)(s)) ds \right| \\ & \leq \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, s)(x(s) - (P_N x)(s)) ds \right| + \int_{t_{n-1}}^t |K(t, s)(x(s) - (P_N x)(s))| ds. \end{aligned} \quad (2.53)$$

To estimate the first term, we write it in the form

$$\begin{aligned}
& \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, s) (x(s) - (P_N x)(s)) ds \right| \\
& \leq \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (K(t, s) - K(t, t_i)) (x(s) - (P_N x)(s)) ds \right| \\
& \quad + \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (P_N x)(s)) ds \right|. \quad (2.54)
\end{aligned}$$

Lagrange's mean-value theorem and the assumption (a) give us for all $s \in [t_{i-1}, t_i]$ ($i = 1, \dots, n-1$) that

$$\begin{aligned}
|K(t, s) - K(t, t_i)| &= \left| \frac{\partial K}{\partial s}(t, \xi)(s - t_i) \right| \leq c|t - \xi|^{-1}|s - t_i| \\
&\leq c_1 h_i |t - t_i|^{-1}, \quad \xi \in [s, t_i].
\end{aligned}$$

Using the assumption $x \in C^{k+1, \mu_1}[0, T] \subset C^{k, \mu_1}[0, T]$ and the inequality (2.29) we get

$$\begin{aligned}
& \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (K(t, s) - K(t, t_i)) (x(s) - (P_N x)(s)) ds \right| \\
& \leq c \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-1} \begin{cases} 1 & \text{if } k < 1 - \mu_1, \\ 1 + |\log t_i| & \text{if } k = 1 - \mu_1, \\ t_i^{1 - \mu_1 - k} & \text{if } k > 1 - \mu_1. \end{cases}
\end{aligned}$$

Since $t \in [t_n, t_{n+1}]$ and $t_n - t_i \geq (n-i)h_i$ ($i = 1, \dots, n-1$) we have

$$|t - t_i|^{-1} \leq (n-i)^{-1} h_i^{-1}, \quad i = 1, \dots, n-1. \quad (2.55)$$

In the case $k < 1 - \mu_1$, according to (2.36), we have

$$\begin{aligned}
\sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-1} &\leq \sum_{i=1}^{n-1} h_i^{k+2} (n-i)^{-1} h_i^{-1} = \sum_{i=1}^{n-1} h_i^{k+1} (n-i)^{-1} \\
&\leq c_1 N^{-r(k+1)} \sum_{i=1}^{n-1} i^{(r-1)(k+1)} (n-i)^{-1}.
\end{aligned}$$

Due to Proposition 2.5.5 (see (2.48)₃ i.e. the case $\beta = (r-1)(k+1) \geq 0$ and $\gamma = -1$) we get

$$\sum_{i=1}^{n-1} i^{(r-1)(k+1)} (n-i)^{-1} \leq c_2 N^{(r-1)(k+1)} (1 + \log N).$$

Thus in the case $k < 1 - \mu_1$ we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-1} &\leq c_3 N^{-r(k+1) + (r-1)(k+1)} (1 + \log N) \\ &= c_3 N^{-k-1} (1 + \log N). \end{aligned}$$

Analogously, in the case $k = 1 - \mu_1$ we have

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-1} (1 + |\log t_i|) &\leq c \sum_{i=1}^{n-1} h_i^{k+2} (n-i)^{-1} h_i^{-1} \left(1 + \log \left(\frac{N}{i} \right) \right) \\ &\leq c \sum_{i=1}^{n-1} h_i^{k+1} (n-i)^{-1} + c \sum_{i=1}^{n-1} h_i^{k+1} (n-i)^{-1} \log \left(\frac{N}{i} \right) \\ &\leq c_1 N^{-k-1} (1 + \log N) \\ &\quad + c_2 N^{-r(k+1)} \sum_{i=1}^{n-1} i^{(r-1)(k+1)} \log \left(\frac{N}{i} \right) (n-i)^{-1}. \end{aligned}$$

Due to Proposition 2.5.5 (see (2.49)₂ and (2.49)₄) we get

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-1} (1 + |\log t_i|) &\leq c_1 N^{-k-1} (1 + \log N) \\ &\quad + c_2 N^{-r(k+1)} \begin{cases} 1 + (\log N)^2 & \text{for } r = 1 \\ N^{(r-1)(k+1)} (1 + \log N) & \text{for } r > 1 \end{cases} \\ &\leq c_3 \begin{cases} N^{-k-1} (1 + (\log N)^2) & \text{for } r = 1, \\ N^{-k-1} (1 + \log N) & \text{for } r > 1. \end{cases} \end{aligned}$$

In the case $k > 1 - \mu_1$, using (2.36), (2.37) and Proposition 2.5.5 (see (2.48)₁ and (2.48)₃), we obtain

$$\begin{aligned}
\sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-1} t_i^{1-\mu_1-k} &\leq \sum_{i=1}^{n-1} h_i^{k+1} (n-i)^{-1} t_i^{1-\mu_1-k} \\
&\leq c N^{-r(2-\mu_1)} \sum_{i=1}^{n-1} i^{r(2-\mu_1)-k-1} (n-i)^{-1} \\
&\leq c_1 N^{-r(2-\mu_1)} \begin{cases} 1 & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1} \\ N^{r(2-\mu_1)-k-1} (1 + \log N) & \text{for } r \geq \frac{k+1}{2-\mu_1} \end{cases} \\
&= c_1 \begin{cases} N^{-r(2-\mu_1)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + \log N) & \text{for } r \geq \frac{k+1}{2-\mu_1}. \end{cases}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (K(t, s) - K(t, t_i)) (x(s) - (P_N x)(s)) ds \right| \\
&\leq c \begin{cases} N^{-r(2-\mu_1)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1} = 1, \\ N^{-k-1} (1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1} \text{ or } r = \frac{k+1}{2-\mu_1} > 1. \end{cases}
\end{aligned}$$

In order to estimate the second term on the right-hand side of (2.54) we choose $0 \leq \eta'_1 < \dots < \eta'_{k+1} \leq 1$ such that $\{\eta_1, \dots, \eta_k\} \subset \{\eta'_1, \dots, \eta'_{k+1}\}$ and define for each $N \in \mathbb{N}$ a piecewise polynomial function $\tilde{P}_N x \in S_k^{(-1)}(\Pi_N^r)$ by conditions

$$(\tilde{P}_N x)(t'_{nj}) = x(t'_{nj}), \quad j = 1, \dots, k+1, \quad n = 1, \dots, N,$$

where

$$t'_{nj} = t_{n-1} + \eta'_j h_n, \quad j = 1, \dots, k+1, \quad n = 1, \dots, N.$$

Since $(\tilde{P}_N x)(t_{ij}) = (P_N x)(t_{ij})$ ($i = 1, \dots, N$, $j = 1, \dots, k$) and η_j are nodes of a quadrature formula that is exact for all polynomials of order k ,

we have

$$\int_{t_{i-1}}^{t_i} ((\tilde{P}_N x)(s) - (P_N x)(s)) ds = 0,$$

hence

$$\begin{aligned} & \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (P_N x)(s)) ds \right| \\ &= \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (\tilde{P}_N x)(s)) ds \right|. \end{aligned}$$

If $\mu = 0$ then with the help of Corollary 2.4.4 we get

$$\begin{aligned} & \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (\tilde{P}_N x)(s)) ds \right| \\ & \leq \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (1 + |\log(t - t_i)|) ds \left(\sup_{s \in (t_{i-1}, t_i)} |x(s) - (\tilde{P}_N x)(s)| \right) \\ & \leq c \sum_{i=1}^{n-1} h_i (1 + |\log(t - t_i)|) h_i^{k+1} \begin{cases} 1 & \text{if } k < -\mu_1, \\ 1 + |\log t_i| & \text{if } k = -\mu_1, \\ t_i^{-\mu_1 - k} & \text{if } k > -\mu_1. \end{cases} \end{aligned}$$

In the case $k < -\mu_1$, using (2.55), (2.36) and Proposition 2.5.5 (see (2.48)₄), we get

$$\begin{aligned} & \sum_{i=1}^{n-1} h_i^{k+2} (1 + |\log(t - t_i)|) \leq c \sum_{i=1}^{n-1} h_i^{k+2} (t - t_i)^{-\frac{1}{2}} \\ & \leq c \sum_{i=1}^{n-1} h_i^{k+\frac{3}{2}} (n - i)^{-\frac{1}{2}} \\ & \leq c_1 N^{-r(k+\frac{3}{2})} \sum_{i=1}^{n-1} i^{(r-1)(k+\frac{3}{2})} (n - i)^{-\frac{1}{2}} \\ & \leq c_2 N^{-r(k+\frac{3}{2})} N^{(r-1)(k+\frac{3}{2}) - \frac{1}{2} + 1} \\ & = c_2 N^{-k-1}. \end{aligned}$$

In the case $k = -\mu_1$, with the help of (2.55), (2.36), (2.37) and Proposition 2.5.5 (see (2.49)₅), we have

$$\begin{aligned}
\sum_{i=1}^{n-1} h_i^{k+2} (1 + |\log(t - t_i)|) (1 + |\log t_i|) &\leq c \sum_{i=1}^{n-1} h_i^{k+2} (t - t_i)^{-\frac{1}{2}} (1 + |\log t_i|) \\
&\leq c_1 \sum_{i=1}^{n-1} h_i^{k+\frac{3}{2}} (n-i)^{-\frac{1}{2}} \left(1 + \log \left(\frac{N}{i} \right) \right) \\
&= c_1 \sum_{i=1}^{n-1} h_i^{k+\frac{3}{2}} (n-i)^{-\frac{1}{2}} + c_1 \sum_{i=1}^{n-1} h_i^{k+\frac{3}{2}} (n-i)^{-\frac{1}{2}} \log \left(\frac{N}{i} \right) \\
&\leq c_2 N^{-k-1} + c_3 N^{-r(k+\frac{3}{2})} \sum_{i=1}^{n-1} i^{(r-1)(k+\frac{3}{2})} \log \left(\frac{N}{i} \right) (n-i)^{-\frac{1}{2}} \\
&\leq c_2 N^{-k-1} + c_4 N^{-r(k+\frac{3}{2})} N^{(r-1)(k+\frac{3}{2})-\frac{1}{2}+1} = c_5 N^{-k-1}.
\end{aligned}$$

Consider now the case $k > -\mu_1$. Using the inequalities $t_1 = h_1 \leq h_n \leq t - t_i \leq T$ we get

$$|\log(t - t_i)| \leq \max\{|\log t_1|, |\log T|\} \leq \left| \log \frac{T}{N^r} \right| + |\log T| \leq c(1 + \log N).$$

Thus

$$\sum_{i=1}^{n-1} h_i^{k+2} (1 + |\log(t - t_i)|) t_i^{-\mu_1 - k} \leq c_1 (1 + \log N) \sum_{i=1}^{n-1} h_i^{k+2} t_i^{-\mu_1 - k}.$$

With the help of (2.36) and (2.37) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-1} h_i^{k+2} (1 + |\log(t - t_i)|) t_i^{-\mu_1 - k} \\
&\leq c_1 (1 + \log N) N^{-r(2-\mu_1)} \sum_{i=1}^{n-1} i^{r(2-\mu_1) - k - 2} \\
&\leq c_2 (1 + \log N) N^{-r(2-\mu_1)} \begin{cases} 1 & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1} \\ 1 + \log N & \text{for } r = \frac{k+1}{2-\mu_1} \\ N^{r(2-\mu_1) - k - 1} & \text{for } r > \frac{k+1}{2-\mu_1} \end{cases} \\
&= c_2 \begin{cases} N^{-r(2-\mu_1)} (1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-r(2-\mu_1)} (1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases}
\end{aligned}$$

Summarizing these cases we get

$$\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (\tilde{P}_N x)(s)) ds \right| \leq c \begin{cases} N^{-r(2-\mu_1)}(1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-r(2-\mu_1)}(1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1} \end{cases}$$

and we have shown that the first term of (2.53) can be estimated as follows:

$$\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, s) (x(s) - (P_N x)(s)) ds \right| \leq c \begin{cases} N^{-r(2-\mu_1)}(1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \quad (2.56)$$

It remains to estimate the second term of (2.53), which we can write in the form

$$\begin{aligned} & \int_{t_{n-1}}^t |K(t, s) (x(s) - (P_N x)(s))| ds \\ &= \int_{t_{n-1}}^{t_n} |K(t, s) (x(s) - (P_N x)(s))| ds + \int_{t_n}^t |K(t, s) (x(s) - (P_N x)(s))| ds. \end{aligned} \quad (2.57)$$

Using the assumptions (a) and (b) and inequality (2.29), we get

$$\begin{aligned} & \int_{t_{n-1}}^{t_n} |K(t, s) (x(s) - (P_N x)(s))| ds \\ & \leq c \left(\sup_{s \in (t_{n-1}, t_n)} |x(s) - (P_N x)(s)| \right) \int_{t_{n-1}}^{t_n} (1 + |\log(t_n - s)|) ds \\ & \leq c_1 h_n^{k+1} (1 + |\log h_n|) \begin{cases} 1 & \text{if } k < 1 - \mu_1, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu_1, \\ t_n^{1-\mu_1-k} & \text{if } k > 1 - \mu_1. \end{cases} \end{aligned}$$

With the help of (2.36) we have in the case $k < 1 - \mu_1$ that

$$h_n^{k+1}(1 + |\log h_n|) \leq c N^{-k-1}(1 + \log N).$$

Using (2.36) and (2.37) we obtain in the case $k = 1 - \mu_1$ that

$$\begin{aligned} h_n^{k+1}(1 + |\log t_n|)(1 + |\log h_n|) &\leq c N^{-k-1} \left(1 + \left|\log \frac{n}{N}\right|\right) (1 + \log N) \\ &\leq c N^{-k-1}(1 + \log N)(1 + \log N) \leq c_1 N^{-k-1}(1 + (\log N)^2) \end{aligned}$$

and in the case $k > 1 - \mu_1$ that

$$\begin{aligned} h_n^{k+1} t_n^{1-\mu_1-k} (1 + |\log h_n|) &\leq c N^{-r(2-\mu_1)} n^{r(2-\mu_1)-k-1} (1 + \log N) \\ &\leq c \begin{cases} N^{-r(2-\mu_1)} (1 + \log N) & \text{for } 1 \leq r \leq \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \end{aligned}$$

Summarizing these cases we get

$$\begin{aligned} \int_{t_{n-1}}^{t_n} |K(t, s)(x(s) - (P_N x)(s))| ds \\ \leq c \begin{cases} N^{-r(2-\mu_1)} (1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \end{aligned}$$

Analogously (instead of n we have here $n + 1$) we obtain

$$\begin{aligned} \int_{t_n}^t |K(t, s)(x(s) - (P_N x)(s))| ds \\ \leq c \left(\sup_{s \in (t_n, t_{n+1})} |x(s) - (P_N x)(s)| \right) \int_{t_n}^{t_{n+1}} (1 + |\log(t_{n+1} - s)|) ds \\ \leq c h_{n+1}^{k+1} (1 + |\log h_{n+1}|) \begin{cases} 1 & \text{if } k < 1 - \mu_1 \\ 1 + |\log t_{n+1}| & \text{if } k = 1 - \mu_1 \\ t_{n+1}^{1-\mu_1-k} & \text{if } k > 1 - \mu_1 \end{cases} \\ \leq c \begin{cases} N^{-r(2-\mu_1)} (1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1} (1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \end{aligned}$$

Therefore for the second term of (2.53) we get the estimation

$$\int_{t_{n-1}}^t |K(t, s)(x(s) - (P_N x)(s))| ds \leq c \begin{cases} N^{-r(2-\mu_1)}(1 + \log N) & \text{for } 1 \leq r < \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + (\log N)^2) & \text{for } r = \frac{k+1}{2-\mu_1}, \\ N^{-k-1}(1 + \log N) & \text{for } r > \frac{k+1}{2-\mu_1}. \end{cases} \quad (2.58)$$

It follows from (2.52), (2.56) and (2.58) that the estimation (2.50) of Theorem 2.5.6 holds.

II Case $\mu > 0$. If $n = 0$ then it follows from Corollary 2.4.4 that

$$\left| \int_0^t K(t, s)(x(s) - (P_N x)(s)) ds \right| \leq c \left(\sup_{s \in (0, t_1)} |x(s) - (P_N x)(s)| \right) \int_0^{t_1} |t_1 - s|^{-\mu} ds \leq c t_1^{k+1-\mu} \begin{cases} 1 & \text{if } k < 1 - \mu_1, \\ 1 + |\log t_1| & \text{if } k = 1 - \mu_1, \\ t_1^{1-\mu_1-k} & \text{if } k > 1 - \mu_1. \end{cases}$$

Using (2.37) and the condition $r \geq 1$ we get

$$\left| \int_0^t K(t, s)(x(s) - (P_N x)(s)) ds \right| \leq c \begin{cases} N^{-r(k+1-\mu)} & \text{if } k < 1 - \mu_1 \\ N^{-r(k+1-\mu)}(1 + \log N) & \text{if } k = 1 - \mu_1 \\ N^{-r(2-\mu_1-\mu)} & \text{if } k > 1 - \mu_1 \end{cases} \leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu}(1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases} \quad (2.59)$$

If $n = 1, \dots, N - 1$ then we use the same estimates (2.53) and (2.54) as in case $\mu = 0$. We start with the first term of (2.54). Using the assumptions

(a), (b) and the inequality (2.29) we get

$$\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (K(t, s) - K(t, t_i)) (x(s) - (P_N x)(s)) ds \right|$$

$$\leq c \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-\mu-1} \begin{cases} 1 & \text{if } k < 1 - \mu_1, \\ 1 + |\log t_i| & \text{if } k = 1 - \mu_1, \\ t_i^{1-\mu_1-k} & \text{if } k > 1 - \mu_1. \end{cases}$$

Since $t \in [t_n, t_{n+1}]$, $t_n - t_i \geq (n - i)h_i$ ($i = 1, \dots, n - 1$) and $-\mu - 1 < 0$ we have

$$|t - t_i|^{-\mu-1} \leq (n - i)^{-\mu-1} h_i^{-\mu-1}. \quad (2.60)$$

Considering the case $k < 1 - \mu_1$ we get, with the aid of (2.36) and Proposition 2.5.5 (see (2.48)₂), that

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-\mu-1} &\leq \sum_{i=1}^{n-1} h_i^{k+2} (n - i)^{-\mu-1} h_i^{-\mu-1} \\ &\leq \sum_{i=1}^{n-1} h_i^{k+1-\mu} (n - i)^{-\mu-1} \leq c N^{-r(k+1-\mu)} \sum_{i=1}^{n-1} i^{(r-1)(k+1-\mu)} (n - i)^{-\mu-1} \\ &\leq c_1 N^{-r(k+1-\mu)} N^{(r-1)(k+1-\mu)} = c_1 N^{-k-1+\mu}. \end{aligned}$$

If $k = 1 - \mu_1$, then with the help of (2.60), (2.36) and Proposition 2.5.5 (see (2.49)₁ and (2.49)₅), we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-\mu-1} (1 + |\log t_i|) &\leq c \sum_{i=1}^{n-1} h_i^{k+2} (n - i)^{-\mu-1} h_i^{-\mu-1} \left(1 + \log \left(\frac{N}{i} \right) \right) \\ &\leq c \sum_{i=1}^{n-1} h_i^{k+1-\mu} (n - i)^{-\mu-1} + c \sum_{i=1}^{n-1} h_i^{k+1-\mu} (n - i)^{-\mu-1} \log \left(\frac{N}{i} \right) \\ &\leq c_1 N^{-k-1+\mu} + c_2 N^{-r(k+1-\mu)} \sum_{i=1}^{n-1} i^{(r-1)(k+1-\mu)} \log \left(\frac{N}{i} \right) (n - i)^{-\mu-1} \\ &\leq c_1 N^{-k-1+\mu} + c_3 N^{-r(k+1-\mu)} \begin{cases} 1 + \log N & \text{for } r = 1 \\ N^{(r-1)(k+1-\mu)} & \text{for } r > 1 \end{cases} \\ &\leq c_4 \begin{cases} N^{-k-1+\mu} (1 + \log N) & \text{for } r = 1, \\ N^{-k-1+\mu} & \text{for } r > 1. \end{cases} \end{aligned}$$

In the case $k > 1 - \mu_1$, using (2.60), (2.36), (2.37) and Proposition 2.5.5 (see (2.48)₁ and (2.48)₂), we get

$$\begin{aligned}
\sum_{i=1}^{n-1} h_i^{k+2} |t - t_i|^{-\mu-1} t_i^{1-\mu_1-k} &\leq \sum_{i=1}^{n-1} h_i^{k+1-\mu} (n-i)^{-\mu-1} t_i^{1-\mu_1-k} \\
&\leq c N^{-r(2-\mu-\mu_1)} \sum_{i=1}^{n-1} i^{r(2-\mu-\mu_1)-k-1+\mu} (n-i)^{-\mu-1} \\
&\leq c_1 N^{-r(2-\mu-\mu_1)} \begin{cases} 1 & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1} \\ N^{-r(2-\mu-\mu_1)-k-1+\mu} & \text{for } r \geq \frac{k+1-\mu}{2-\mu-\mu_1} \end{cases} \\
&= c_1 \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu} & \text{for } r \geq \frac{k+1-\mu}{2-\mu-\mu_1}. \end{cases}
\end{aligned}$$

Summarizing these cases gives us

$$\begin{aligned}
&\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (K(t, s) - K(t, t_i)) (x(s) - (P_N x)(s)) ds \right| \\
&\leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu} (1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases} \quad (2.61)
\end{aligned}$$

For the second term on the right-hand side of (2.54), using the same ideas as in the case $\mu = 0$, we have

$$\begin{aligned}
&\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (P_N x)(s)) ds \right| \\
&= \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) (x(s) - (\tilde{P}_N x)(s)) ds \right| \\
&\leq c \sum_{i=1}^{n-1} h_i |t - t_i|^{-\mu} \left(\sup_{s \in (t_{i-1}, t_i)} |x(s) - (\tilde{P}_N x)(s)| \right) \\
&\leq c_1 \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} \begin{cases} 1 & \text{if } k < -\mu_1, \\ 1 + |\log t_i| & \text{if } k = -\mu_1, \\ t_i^{-\mu_1-k} & \text{if } k > -\mu_1. \end{cases}
\end{aligned}$$

If $k < -\mu_1$, then using (2.36) and Proposition 2.5.5 (see (2.48)₄), we get

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} &\leq c N^{-r(k+2-\mu)} \sum_{i=1}^{n-1} i^{(r-1)(k+2-\mu)} (n-i)^{-\mu} \\ &\leq c_1 N^{-r(k+2-\mu)} N^{(r-1)(k+2-\mu)-\mu+1} = c_1 N^{-k-1}. \end{aligned}$$

In the case $k = -\mu_1$, with the aid of (2.36) and Proposition 2.5.5 (see (2.49)₅), we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} (1 + |\log t_i|) &\leq c \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} \left(1 + \log \left(\frac{N}{i} \right) \right) \\ &= c \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} + c \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} \log \left(\frac{N}{i} \right) \\ &\leq c_1 N^{-k-1} + c_2 N^{-r(k+2-\mu)} \sum_{i=1}^{n-1} i^{(r-1)(k+2-\mu)} \log \left(\frac{N}{i} \right) (n-i)^{-\mu} \\ &\leq c_1 N^{-k-1} + c_2 N^{-r(k+2-\mu)} N^{(r-1)(k+2-\mu)-\mu+1} = c_3 N^{-k-1}. \end{aligned}$$

Considering the case $k > -\mu_1$ we get with the help of (2.37), (2.36) and Proposition 2.5.5 (see (2.48)₁ and (2.48)₄) that

$$\begin{aligned} \sum_{i=1}^{n-1} h_i^{k+2-\mu} (n-i)^{-\mu} t_i^{-\mu_1-k} &\leq N^{-r(2-\mu-\mu_1)} \sum_{i=1}^{n-1} i^{r(2-\mu-\mu_1)-k-2+\mu} (n-i)^{-\mu} \\ &= c N^{-r(2-\mu-\mu_1)} \left\{ \begin{array}{ll} 1 & \text{for } 1 \leq r < \frac{k+1}{2-\mu-\mu_1} \\ N^{r(2-\mu-\mu_1)-k-1} & \text{for } r \geq \frac{k+1}{2-\mu-\mu_1} \end{array} \right\} \\ &\leq c \left\{ \begin{array}{ll} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu-\mu_1}, \\ N^{-k-1} & \text{for } r \geq \frac{k+1}{2-\mu-\mu_1}. \end{array} \right. \end{aligned}$$

Summarizing these cases we have shown that

$$\left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, t_i) \left(x(s) - (\tilde{P}_N x)(s) \right) ds \right| \leq c \left\{ \begin{array}{ll} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1}{2-\mu-\mu_1}, \\ N^{-k-1} & \text{for } r \geq \frac{k+1}{2-\mu-\mu_1}. \end{array} \right.$$

The last inequality with (2.61) and (2.54) gives us the following estimate for the first term of (2.53):

$$\begin{aligned} & \left| \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} K(t, s)(x(s) - (P_N x)(s)) ds \right| \\ & \leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu}(1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases} \end{aligned} \quad (2.62)$$

It remains to estimate the second term of (2.53), which we can write in the form (2.57). Using the assumptions (a) and (b) and the inequality (2.29), we get

$$\begin{aligned} & \int_{t_{n-1}}^{t_n} |K(t, s)(x(s) - (P_N x)(s))| ds \\ & \leq c \left(\sup_{s \in (t_{n-1}, t_n)} |x(s) - (P_N x)(s)| \right) \int_{t_{n-1}}^{t_n} |t_n - s|^{-\mu} ds \\ & \leq c h_n^{k+1-\mu} \begin{cases} 1 & \text{if } k < 1 - \mu_1, \\ 1 + |\log t_n| & \text{if } k = 1 - \mu_1, \\ t_n^{1-\mu_1-k} & \text{if } k > 1 - \mu_1. \end{cases} \end{aligned}$$

Here, with the help of (2.36) and (2.37), we get in the case $k < 1 - \mu_1$ that

$$h_n^{k+1-\mu} \leq c N^{-r(k+1-\mu)} n^{(r-1)(k+1-\mu)} \leq c N^{-k-1+\mu}$$

and in the case $k = 1 - \mu_1$ that

$$\begin{aligned} h_n^{k+1-\mu}(1 + |\log t_n|) & \leq c N^{-r(k+1-\mu)} n^{(r-1)(k+1-\mu)} \left(1 + \left| \log \frac{n}{N} \right| \right) \\ & = c N^{-k-1+\mu} \left(\frac{n}{N} \right)^{(r-1)(k+1-\mu)} \left(1 + \left| \log \frac{n}{N} \right| \right) \\ & \leq c_1 \begin{cases} N^{-k-1+\mu}(1 + \log N) & \text{for } r = 1, \\ N^{-k-1+\mu} & \text{for } r > 1. \end{cases} \end{aligned}$$

If $k > 1 - \mu_1$ then

$$\begin{aligned} h_n^{k+1-\mu} t_n^{1-\mu_1-k} & \leq c N^{-r(2-\mu-\mu_1)} n^{r(2-\mu-\mu_1)-k-1+\mu} \\ & \leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r \leq \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1}. \end{cases} \end{aligned}$$

Summarizing these cases we get

$$\int_{t_{n-1}}^{t_n} |K(t, s)(x(s) - (P_N x)(s))| ds$$

$$\leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu}(1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases}$$

Analogously (instead of n we have here $n+1$) we obtain

$$\int_{t_n}^t |K(t, s)(x(s) - (P_N x)(s))| ds$$

$$\leq c \left(\sup_{s \in (t_n, t_{n+1})} |x(s) - (P_N x)(s)| \right) \int_{t_n}^{t_{n+1}} |t_{n+1} - s|^{-\mu} ds$$

$$\leq c h_{n+1}^{k+1-\mu} \begin{cases} 1 & \text{if } k < 1 - \mu_1 \\ 1 + |\log t_{n+1}| & \text{if } k = 1 - \mu_1 \\ t_{n+1}^{1-\mu_1-k} & \text{if } k > 1 - \mu_1 \end{cases}$$

$$\leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu}(1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases}$$

Therefore the second term of (2.53) can be estimated as

$$\int_{t_{n-1}}^t |K(t, s)(x(s) - (P_N x)(s))| ds$$

$$\leq c \begin{cases} N^{-r(2-\mu-\mu_1)} & \text{for } 1 \leq r < \frac{k+1-\mu}{2-\mu-\mu_1}, \\ N^{-k-1+\mu}(1 + \log N) & \text{for } r = \frac{k+1-\mu}{2-\mu-\mu_1} = 1, \\ N^{-k-1+\mu} & \text{for } r > \frac{k+1-\mu}{2-\mu-\mu_1} \text{ or } r = \frac{k+1-\mu}{2-\mu-\mu_1} > 1. \end{cases} \quad (2.63)$$

Finally, it follows from (2.59), (2.53), (2.62) and (2.63) that the estimation (2.51) of Theorem 2.5.6 holds. \diamond

Chapter 3

Collocation approximations for linear Volterra integro-differential equations

In this chapter we study mainly numerical methods for Volterra integro-differential equations, although some attention is also paid to similar methods for Volterra integral equations. Most of the convergence results of this chapter are new, refining previously known ones by H. Brunner, A. Pedaş and G. Vainikko [14, 15], I. Parts and A. Pedaş [43, 45], R. Kangro and I. Parts [30].

3.1. Equation and spline collocation methods

Consider the initial-value problem for the linear integro-differential equation,

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K(t,s)y(s)ds, \quad t \in [0, T], \quad (3.1)$$

with a given initial condition

$$y(0) = y_0, \quad y_0 \in \mathbb{R}. \quad (3.2)$$

Here p, q and K are given functions, y_0 is a given number and y is an unknown function.

The regularity of the solution of the problem $\{(3.1),(3.2)\}$ is described in the following theorem.

Theorem 3.1.1. [14] *Let $p, q \in C^{m,\nu}[0, T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, $y_0 \in \mathbb{R}$. Then the Cauchy problem $\{(3.1),(3.2)\}$ has a unique solution $y \in C^{m+1,\nu-1}[0, T]$.*

Following [14] we construct two different piecewise polynomial collocation methods for the numerical solution of problem $\{(3.1),(3.2)\}$. Our approach is based on two reformulations of $\{(3.1),(3.2)\}$.

First we introduce a new unknown function

$$z = y'. \quad (3.3)$$

Then, using (3.3) and (3.2), equation (3.1) may be rewritten as a linear integral equation of the second kind with respect to z :

$$z(t) = f_1(t) + p(t) \int_0^t z(s) ds + \int_0^t K(t, s) \left(\int_0^s z(\tau) d\tau \right) ds, \quad (3.4)$$

where

$$f_1(t) = q(t) + y_0 p(t) + y_0 \int_0^t K(t, s) ds, \quad t \in [0, T]. \quad (3.5)$$

Method 1. We look for an approximation v to the solution z of equation (3.4) in $S_{m-1}^{(-1)}(\Pi_N)$, $m, N \in \mathbb{N}$. We determine $v = v^{(N)} \in S_{m-1}^{(-1)}(\Pi_N)$ ($m \geq 1$) by the collocation method from the following conditions:

$$v_n(t_{nj}) = f_1(t_{nj}) + p(t_{nj}) \int_0^{t_{nj}} v(s) ds + \int_0^{t_{nj}} K(t_{nj}, s) \left(\int_0^s v(\tau) d\tau \right) ds, \\ j = 1, \dots, m; \quad n = 1, \dots, N. \quad (3.6)$$

Here $v_n = v|_{(t_{n-1}, t_n)}$ is the restriction of v to (t_{n-1}, t_n) , $n = 1, \dots, N$, the function f_1 is defined by (3.5) and the collocation points $\{t_{nj}\}$ are given by

$$t_{nj} = t_{n-1} + \eta_j h_n, \quad j = 1, \dots, m \quad (n = 1, \dots, N), \quad (3.7)$$

where $\{t_n\}$ are the nodes of Π_N and η_1, \dots, η_m are some fixed collocation parameters which do not depend on N and satisfy $0 \leq \eta_1 < \dots < \eta_m \leq 1$.

Having determined the approximation v for $z = y'$, we can determine also the approximation u for y , the solution of the Cauchy problem $\{(3.1),(3.2)\}$, setting

$$u(t) = y_0 + \int_0^t v(s)ds, \quad t \in [0, T]. \quad (3.8)$$

The second reformulation of the problem $\{(3.1),(3.2)\}$ is based on integration of both sides of (3.1) over $(0, t)$. Using this and (3.2), the equation (3.1) may be rewritten as a linear Volterra integral equation with respect to y :

$$y(t) = f_2(t) + \int_0^t K_2(t, s)y(s)ds, \quad t \in [0, T], \quad (3.9)$$

where

$$f_2(t) = y_0 + \int_0^t q(s)ds, \quad K_2(t, s) = p(s) + \int_s^t K(\tau, s)d\tau, \quad t \in [0, t], \quad s \in [0, t]. \quad (3.10)$$

Method 2. We look for an approximate solution u of equation (3.9) in $S_m^{(-1)}(\Pi_N)$, $m, N \in \mathbb{N}$: this approximation $u = u^{(N)} \in S_m^{-1}(\Pi_N)$ will be determined by the collocation method from the following conditions

$$u_n(t_{nj}) = f_2(t_{nj}) + \int_0^{t_{nj}} K_2(t_{nj}, s)u(s)ds, \quad j = 1, \dots, m+1; \quad n = 1, \dots, N. \quad (3.11)$$

Here, f_2 and K_2 are defined in (3.10), $u_n = u|_{(t_{n-1}, t_n)}$ ($n = 1, \dots, N$) is the restriction of $u \in S_m^{(-1)}(\Pi_N)$ to (t_{n-1}, t_n) , and the collocation points $\{t_{nj}\}$ are given by

$$t_{nj} = t_{n-1} + \eta_j h_n, \quad j = 1, \dots, m+1 \quad (n = 1, \dots, N), \quad (3.12)$$

where $\{t_n\}$ are the nodes of Π_N and $0 \leq \eta_1 < \dots < \eta_{m+1} \leq 1$ is a fixed system of parameters which does not depend on n and N .

Remark 3.1.2. Actually Method 1 is equivalent to the approach, known also as the *direct method*, where we look for an approximate solution

$u = u^{(N)} \in S_m^{(0)}(\Pi_N)$ to the solution y of equation (3.1) and require that

$$u'_n(t_{nj}) = p(t_{nj})u_n(t_{nj}) + q(t_{nj}) + \int_0^{t_{nj}} K(t_{nj}, s)u(s)ds, \\ j = 1, \dots, m, \quad n = 1, \dots, N, \quad (3.13)$$

as well as

$$u(0) = y_0$$

hold. Here $u_n = u|_{[t_{n-1}, t_n]}$ ($n = 1, \dots, N$) is the restriction of $u \in S_m^{(0)}(\Pi_N)$ to $[t_{n-1}, t_n]$, and the collocation points $\{t_{nj}\}$ are given by (3.7).

Indeed, since, in the case of Method 1, the solution u determined by (3.6)-(3.8) belongs to the space $S_m^{(0)}(\Pi_N)$ and satisfies equations (3.13), the direct method and Method 1 are equivalent.

Remark 3.1.3. It is known that in the case of sufficiently smooth p, q and K the approximate solution u to the initial value problem $\{(3.1), (3.2)\}$ found by Method 1 satisfies the estimates (see [11], Theorem 3.2.3)

$$\|u^{(i)} - y^{(i)}\|_\infty \leq ch^m, \quad i = 0, 1$$

and approximate solution u found by Method 2 satisfies the estimate (see [11], Theorem 2.2.3)

$$\|u - y\|_\infty \leq ch^{m+1},$$

where $h = \max_{n=1, \dots, N} (t_n - t_{n-1})$.

We characterize the convergence behaviour of those methods in the case of weakly singular equations. We also show that by choosing suitable graded grids it is possible to achieve the same convergence rates in the case of weakly singular equations as in the smooth case.

3.2. Convergence results for Method 1 in the case of arbitrary collocation parameters

First we show that it is possible to obtain convergence rate estimates for Method 1 in terms of the maximal stepsize h of the grid Π_N without imposing any constraints to the structure of the grid. The following result improves the corresponding estimates in [43, 45].

Theorem 3.2.1. *Let $p, q \in C^{m,\nu}[0, T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, and assume that a regular grid Π_N and the collocation points (3.7) are used.*

Then, for all sufficiently large $N \in \mathbb{N}$ and for every choice of parameters $0 \leq \eta_1 < \dots < \eta_m \leq 1$ equations (3.8) and (3.6) determine unique approximations $u \in S_m^{(0)}(\Pi_N)$ and $v \in S_{m-1}^{(-1)}(\Pi_N)$ (with $v|_{(t_{n-1}, t_n)} = \left(u|_{(t_{n-1}, t_n)}\right)'$, $n = 1, \dots, N$) to the solution y of the Cauchy problem {(3.1), (3.2)} and its derivative y' , respectively. If $\eta_1 = 0$, $\eta_m = 1$, then actually $u \in S_m^{(1)}(\Pi_N)$ and $v = u' \in S_{m-1}^{(0)}(\Pi_N)$. The following error estimates hold for $i = 0$ and $i = 1$:

$$\|u^{(i)} - y^{(i)}\|_\infty \leq c \begin{cases} h^m & \text{if } m < 2 - \nu - i, \\ h^m(1 + |\log h|) & \text{if } m = 2 - \nu - i, \\ h^{2-\nu-i} & \text{if } m > 2 - \nu - i. \end{cases} \quad (3.14)$$

Here h is defined by (2.12) and c is a constant which is independent of N and h .

Proof. Recall, that using equality (3.3) and initial condition (3.2), the Cauchy problem {(3.1), (3.2)} is equivalent by the equation (3.4):

$$z(t) = f_1(t) + p(t) \int_0^t z(s) ds + \int_0^t K(t, s) \left(\int_0^s z(\tau) d\tau \right) ds,$$

where f_1 is given by (3.5). Let

$$(S_1 z)(t) = p(t) \int_0^t z(s) ds + \int_0^t K(t, s) \left(\int_0^s z(\tau) d\tau \right) ds, \quad t \in [0, T].$$

Introduce an integral operator S_0 by the formula

$$(S_0 z)(t) = \int_0^t z(s) ds, \quad t \in [0, T]. \quad (3.15)$$

Evidently, operator S_0 is compact from $L^\infty(0, T)$ to $C[0, T]$.

Define an integral operator $S : C[0, T] \rightarrow C[0, T]$ with weakly singular kernel by

$$(Sx)(t) = \int_0^t K(t, s)x(s) ds, \quad t \in [0, T].$$

According to Theorem 2.2.6 the operator $S : C[0, T] \rightarrow C[0, T]$ is compact.

Let

$$(Mx)(t) = p(t)x(t), \quad t \in [0, T].$$

Evidently $M \in \mathcal{L}(C[0, T], C[0, T])$ and we can rewrite the operator S_1 in the form

$$S_1 = MS_0 + SS_0. \quad (3.16)$$

Theorem 2.1.1 gives us that the operator $S_1 : L^\infty(0, T) \rightarrow C[0, T]$ is compact. Therefore the operator S_1 is compact from $L^\infty(0, T)$ to $L^\infty(0, T)$ as well. We rewrite function f_1 , given by formula (3.5), in the form

$$f_1 = q + y_0p + y_0S_1.$$

Since $S \in \mathcal{L}(C[0, T], C[0, T])$ and $y_0 \in \mathbb{R}$, $p, q \in C^{m, \nu}[0, T]$, we get that $f_1 \in C[0, T] \subset L^\infty(0, T)$.

Equation (3.4) can be presented in the form

$$z = f_1 + S_1z. \quad (3.17)$$

Since $S_1 : L^\infty(0, T) \rightarrow L^\infty(0, T)$ is compact, $f_1 \in L^\infty(0, T)$ and the homogeneous equation $z = S_1z$ has only the trivial solution in $L^\infty(0, T)$, the equation (3.17) has (see Theorem 2.1.5) a unique solution

$$z = (I - S_1)^{-1}f_1,$$

where I is the identity transformation and

$$(I - S_1)^{-1} \in \mathcal{L}(L^\infty(0, T), L^\infty(0, T)).$$

Moreover, $z = y' \in C^{m, \nu}[0, T]$ by Theorem 3.1.1.

Further, the conditions (3.6) are equivalent to the operator equation representation

$$v = P_N f_1 + P_N S_1 v, \quad (3.18)$$

with P_N defined by (2.17) with $k = m$ in Section 2.4.

It follows from Lemma 2.4.6 that $\|S_1 - P_N S_1\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \rightarrow 0$ as $N \rightarrow \infty$. From this and from the boundedness of $(I - S_1)^{-1}$ in $L^\infty(0, T)$, we obtain with help of Theorem 2.1.3, that $I - P_N S_1$ is invertible in $L^\infty(0, T)$ for all sufficiently large N , say $N \geq N_0$, and the norms of $(I - P_N S_1)^{-1}$ are uniformly bounded in N :

$$\|(I - P_N S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \leq c, \quad N \geq N_0. \quad (3.19)$$

Indeed, since $I - P_N S_1, S_1 - P_N S_1 \in \mathcal{L}(L^\infty(0, T), L^\infty(0, T))$, $(I - S_1)^{-1} \in \mathcal{L}(L^\infty(0, T), L^\infty(0, T))$ and for sufficiently large N

$$\|S_1 - P_N S_1\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \|(I - S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} < 1,$$

we obtain by Theorem 2.1.3 that for sufficiently large N , there exist $(I - P_N S_1)^{-1} \in \mathcal{L}(L^\infty(0, T), L^\infty(0, T))$ and the estimate

$$\begin{aligned} & \|(I - P_N S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \\ & \leq \frac{\|(I - S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))}}{1 - \|(I - S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \|(S_1 - P_N S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))}} \end{aligned}$$

holds. As $N \rightarrow \infty$, the right-hand part of this estimation converges to $\|(I - S_1)^{-1}\|$, which gives us that the estimation (3.19) holds. Thus, for $N \geq N_0$ equation (3.18) has a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N)$ ($v \in S_{m-1}^{(0)}$ if $\eta_1 = 0, \eta_m = 1$). With the help of (3.17) and (3.18) we get

$$\begin{aligned} (I - P_N S_1)(v - z) &= P_N f_1 - (I - P_N S_1)z \\ &= P_N(f_1 + S_1 z) - z = P_N z - z \end{aligned}$$

and it follows that

$$v - z = (I - P_N S_1)^{-1}(P_N z - z), \quad N \geq N_0. \quad (3.20)$$

Using inequality (3.19), we get

$$\|v - z\|_\infty \leq c \|P_N z - z\|_\infty, \quad N \geq N_0, \quad (3.21)$$

with a constant c which is independent of N . Applying Lemma 2.4.5 (see (2.30) with $k = m, \mu = \nu$) we obtain the estimate (3.14) with $i = 1, z = y', v = u'$.

Using the equality

$$(I - P_N S_1)^{-1} = I + (I - P_N S_1)^{-1} P_N S_1, \quad N \geq N_0,$$

we rewrite (3.20) in the form

$$\begin{aligned} v - z &= (I + (I - P_N S_1)^{-1} P_N S_1)(P_N z - z) \\ &= P_N z - z + (I - P_N S_1)^{-1} P_N S_1 (P_N z - z). \end{aligned}$$

Hence we get for $t \in [0, T]$ that (see (3.15), (3.16))

$$\begin{aligned}
|u(t) - y(t)| &= \left| \int_0^t (v(s) - z(s)) ds \right| \\
&\leq \left| \int_0^t (P_N z - z)(s) ds \right| + \|(I - P_N S_1)^{-1} P_N S_1 (P_N z - z)\|_\infty \int_0^t ds \\
&\leq |(S_0(P_N z - z))(t)| + c \|S_0(P_N z - z)\|_{C[0, T]} \\
&\leq c_1 \|S_0(P_N z - z)\|_{C[0, T]} \\
&\leq c_1 \int_0^T |P_N z(s) - z(s)| ds, \quad t \in [0, T],
\end{aligned}$$

where

$$\begin{aligned}
c &= T \|(I - P_N S_1)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \|P_N\|_{\mathcal{L}(C[0, T], L^\infty(0, T))} \\
&\quad \cdot \|M + S\|_{\mathcal{L}(C[0, T], C[0, T])}
\end{aligned}$$

and $c_1 = 1 + c$. Applying Lemma 2.4.5 (see (2.31) with $k = m$, $\mu = \nu$) we obtain the estimate (3.14) with $i = 0$ for $u - y$. \diamond

Since in the case of a quasi-uniform mesh, $h \leq \frac{c}{N}$ (see (2.13)), Theorem 3.2.1 yields the following result, obtained by I. Parts and A. Pedas in [45].

Corollary 3.2.2. *Suppose that conditions of Theorem 3.2.1 are satisfied. Assume additionally that the grid $\Pi_N = \Pi_{N, \Theta}$ is quasi-uniform (i.e. satisfies (2.9)).*

Then, for all sufficiently large $N \in \mathbb{N}$, in the notation of Theorem 3.2.1, the following error estimates hold for $i = 0$ and $i = 1$:

$$\|u^{(i)} - y^{(i)}\|_\infty \leq c \begin{cases} N^{-m} & \text{if } m < 2 - \nu - i, \\ N^{-m}(1 + \log N) & \text{if } m = 2 - \nu - i, \\ N^{-(2-\nu-i)} & \text{if } m > 2 - \nu - i. \end{cases} \quad (3.22)$$

Here c is a constant not depending on N .

Similar results for a more general weakly singular integro-differential equations of the form

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K_1(t, s)y(s)ds + \int_0^t K_2(t, s)y'(s)ds, \quad t \in [0, T], \quad (3.23)$$

with a given initial condition (3.2) are established by I. Parts and A. Pedas in [44], see also [48]. In particular, if $p, q \in C^{m, \nu}[0, T]$, $K_1, K_2 \in \mathcal{W}^{m, \nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, then the Cauchy problem $\{(3.23), (3.2)\}$ has a unique solution $y \in C^{m+1, \nu-1}[0, T]$. Moreover, denoting $z = y'$ and using (3.2), equation (3.23) may be rewritten as a linear Volterra integral equation of the second kind with respect to z ,

$$z(t) = f(t) + \int_0^t K_1(t, s) \int_0^s z(\tau)d\tau ds + \int_0^t (p(t) + K_2(t, s))z(s)ds, \quad t \in [0, T], \quad (3.24)$$

where

$$f(t) = q(t) + y_0 p(t) + y_0 \int_0^t K_1(t, s)ds, \quad t \in [0, T]. \quad (3.25)$$

Now we can employ (3.24) in the construction of numerical solutions for (3.23) by Method 1: for given $m, N \in \mathbb{N}$ we find an approximation v to the solution z of equation (3.24) determining it from the conditions

$$\begin{aligned} v &\in S_{m-1}^{(-1)}(\Pi_N), \\ v(t_{nj}) &= f(t_{nj}) + \int_0^{t_{nj}} K_1(t_{nj}, s) \left(\int_0^s v(\tau)d\tau \right) ds \\ &+ \int_0^{t_{nj}} (p(t_{nj}) + K_2(t_{nj}, s))v(s)ds, \quad j = 1, \dots, m; \quad n = 1, \dots, N. \end{aligned} \quad (3.26)$$

Here f and $\{t_{nj}\}$ are given by the formulas (3.25) and (3.7), respectively. Having determined the approximation v for z , we can also determine the approximation u for y , the solution of the initial-value problem $\{(3.23), (3.2)\}$, setting

$$u(t) = y_0 + \int_0^t v(s)ds, \quad t \in [0, T]. \quad (3.27)$$

The attainable order of the convergence of this method in the case of quasi-uniform grid can be characterized by the following theorem.

Theorem 3.2.3. [44] *Let $p, q \in C^{m,\nu}[0, T]$, $K_1 \in \mathcal{W}^{m,\nu}(\Delta_T)$, $K_2 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, and assume that the underlying grid $\Pi_N = \Pi_{N,\Theta}$ is quasi-uniform (i.e. satisfies (2.9)).*

Then, for all sufficiently large $N \in \mathbb{N}$, say $N \geq N_0$, and for every choice of parameters $0 \leq \eta_1 < \dots < \eta_m \leq 1$ equations (3.27) and (3.26) determine unique approximations $u \in S_m^{(0)}(\Pi_N)$ and $v \in S_{m-1}^{(-1)}(\Pi_N)$ (with $v|_{(t_{n-1}, t_n)} = (u|_{(t_{n-1}, t_n)})'$, $n = 1, \dots, N$) to the solution y of the Cauchy problem $\{(3.23), (3.2)\}$ and its derivative y' , respectively. If $\eta_1 = 0$, $\eta_m = 1$, then actually $u \in S_m^{(1)}(\Pi_N)$ and $v = u' \in S_{m-1}^{(0)}(\Pi_N)$. If $N \geq N_0$, then the estimate (3.22) hold for $i = 0$ and $i = 1$.

The estimate (3.22) shows that the convergence rate of the method may, in the case of unbounded kernels of the Volterra integral operators (i.e. if $\nu \geq 0$), be quite slow. The next theorem, by H. Brunner, A. Pedas and G. Vainikko, shows that it is possible to improve the convergence rate by using nonuniform grids.

Theorem 3.2.4. [15] *Suppose that conditions of Theorem 3.2.1 are satisfied. Assume additionally that the grid $\Pi_N = \Pi_N^r$ ($r \geq 1$) is a graded grid (i.e. satisfies (2.10)).*

Then there exists an $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the conditions (3.6) and the formula (3.8) determine unique approximation $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ to y' and $u \in S_m^{(0)}(\Pi_N^r)$ to y ($v \in S_{m-1}^{(0)}(\Pi_N^r)$ and $u \in S_m^{(1)}(\Pi_N^r)$ if $\eta_1 = 0$ and $\eta_m = 1$), where y is the exact solution of the problem $\{(3.1), (3.2)\}$. Moreover, for $N \geq N_0$ the error estimates

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m}{2-\nu} \\ N^{-m}(1 + \log N) & \text{for } r = \frac{m}{2-\nu} \\ N^{-m} & \text{for } r > \frac{m}{2-\nu} \end{cases}$$

and

$$\|v - y'\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r < \frac{m}{1-\nu} \\ N^{-m}(1 + \log N) & \text{for } r = \frac{m}{1-\nu} = 1 \\ N^{-m} & \text{for } r > \frac{m}{1-\nu} \text{ or } r = \frac{m}{1-\nu} > 1 \end{cases}$$

hold. Here c is a positive constant which is independent of N .

The results of numerical experiments (see Chapter 5) show that the estimates of the previous theorem can not be improved unless additional

assumptions about collocation parameters are made. Our next aim is to show that specially chosen collocation parameters lead to a higher convergence rate (so called superconvergence phenomenon). As equation (3.4) is equivalent to Volterra integral equation of the second kind with respect to the derivative $z = y'$ of the solution of problem $\{(3.1), (3.2)\}$ we start by studying superconvergence properties of collocation methods for solving Volterra integral equations in the next section.

3.3. Superconvergence results for Volterra integral equations

Both Method 1 and Method 2 are based on solving certain Volterra integral equations by piecewise polynomial collocation methods. It is known that often in the case of such methods the maximal error of the approximate solution at the collocation points converges to zero more quickly than the maximal error over the whole interval, provided that some additional restrictions on the choice of the collocation parameters are imposed. The following lemma together with Theorems 2.5.4 and 2.5.6 enables us to prove such superconvergence results for Volterra integral equations of the form

$$y(t) = \int_0^t K(t, s)y(s)ds + f(t), \quad t \in [0, T] \quad (3.28)$$

under various assumptions about the kernel K .

Lemma 3.3.1. *Assume:*

- (a) *Integral operator S , defined by (2.6) is compact from $L^\infty(0, T)$ to $C[0, T]$.*
- (b) *The function $f \in C[0, T]$.*
- (c) *The collocation points $\{t_{nj}\}$ ($n = 1, \dots, N, j = 1, \dots, k$) (see (3.7)) are generated by the grid points (2.10) and by the parameters η_1, \dots, η_k satisfying (2.16).*

Then for all sufficiently large $N \in \mathbb{N}$ and for every choice of parameters

$0 \leq \eta_1 < \dots < \eta_k \leq 1$ the collocation conditions

$$u_n(t_{nj}) = \int_0^t K(t_{nj}, s)u(t_{nj})ds + f(t_{nj}), \quad j = 1, \dots, k, \quad n = 1, \dots, N, \quad (3.29)$$

where $u_n = u|_{(t_{n-1}, t_n)}$ ($n = 1, \dots, N$), determine a unique approximation $u \in S_{k-1}^{(-1)}(\Pi_N^r)$ to y , the solution of equation (3.28). For the quantity δ_N defined by

$$\delta_N = \max_{n=1, \dots, N, j=1, \dots, k} |u_n(t_{nj}) - y(t_{nj})| \quad (3.30)$$

and the interpolation operator P_N defined by (2.17) the estimate

$$\delta_N \leq \max_{t \in [0, T]} \left| \int_0^t K(t, s)(y(s) - (P_N y)(s))ds \right|$$

holds.

Proof. We can consider the equation (3.28) as the equation

$$y = Sy + f, \quad (3.31)$$

with the operator S , defined by (2.6). The collocation conditions (3.29) can be written in the form

$$u = P_N Su + P_N f. \quad (3.32)$$

For the solutions of equations (3.31) and (3.32) we have

$$u - P_N y = P_N S(u - y) = P_N S(u - P_N y) + P_N S(P_N y - y)$$

i.e.

$$(I - P_N S)(u - P_N y) = P_N S(P_N y - y).$$

As $I - S$ is invertible in $L^\infty(0, T)$ (see Theorem 2.1.5) and $\|P_N S - S\| \xrightarrow{N \rightarrow \infty} 0$ according to Lemma 2.4.6, $I - P_N S$ is also invertible for sufficiently large N , say $N \geq N_0$, and the norms of $(I - P_N S)^{-1}$ are uniformly bounded in N (cf the proof of (3.19)):

$$\|(I - P_N S)^{-1}\|_{\mathcal{L}(L^\infty(0, T), L^\infty(0, T))} \leq c, \quad N \geq N_0.$$

Therefore

$$u - P_N y = (I - P_N S)^{-1} P_N S (P_N y - y), \quad N \geq N_0,$$

implying that

$$\|u - P_N y\|_{L^\infty(0,T)} \leq c_1 \|S(P_N y - y)\|_{L^\infty(0,T)},$$

where $c_1 = c \|P_N\|_{\mathcal{L}(C[0,T], L^\infty(0,T))}$ is independent of N (see Lemma 2.4.1).

As

$$\|S(P_N y - y)\|_{L^\infty(0,T)} = \max_{t \in [0,T]} \left| \int_0^t K(t,s) (y(s) - (P_N y)(s)) ds \right|$$

and

$$\delta_N = \max_{n=1, \dots, N, j=1, \dots, k} |u_n(t_{nj}) - (P_N y)(t_{nj})| \leq \|u - P_N y\|_{L^\infty(0,T)}$$

the statement of Lemma 3.3.1 follows. \diamond

As an example of the power of the technical results of Section 2.5 we prove the following theorem.

Theorem 3.3.2. *Let $f \in C^{m+1,\nu}[0,T]$, $K \in \mathcal{W}^{m+1,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$. Let the collocation points (3.7) be generated by the grid points of a graded grid Π_N^r , $r \geq 1$ and by the knots η_j , $j = 1, \dots, m$ of a quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^m A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_m \leq 1$, with appropriate weights $\{A_j\}$, which is exact for all polynomials of degree m .*

Then for all sufficiently large $N \in \mathbb{N}$ the collocation conditions (3.29) determine a unique approximation $u \in S_{m-1}^{(-1)}(\Pi_N^r)$ to y , the solution of equation (3.28). For the quantity δ_N defined by (3.30) the following error estimates hold:

(i) *If $\nu < 0$ then*

$$\delta_N \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m+1}{2-\nu}, \\ N^{-m-1}(1 + \log N) & \text{for } r = \frac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \frac{m+1}{2-\nu}. \end{cases} \quad (3.33)$$

(ii) If $\nu = 0$ then

$$\delta_N \leq c \begin{cases} N^{-r(2-\nu)}(1 + \log N) & \text{for } 1 \leq r < \frac{m+1}{2-\nu}, \\ N^{-m-1}(1 + (\log N)^2) & \text{for } r = \frac{m+1}{2-\nu}, \\ N^{-m-1}(1 + \log N) & \text{for } r > \frac{m+1}{2-\nu}. \end{cases} \quad (3.34)$$

(iii) If $\nu > 0$ then

$$\delta_N \leq c \begin{cases} N^{-r(2-2\nu)} & \text{for } 1 \leq r < \frac{m+1-\nu}{2-2\nu}, \\ N^{-m-1+\nu}(1 + \log N) & \text{for } r = \frac{m+1-\nu}{2-2\nu} = 1, \\ N^{-m-1+\nu} & \text{for } r > \frac{m+1-\nu}{2-2\nu} \text{ or } r = \frac{m+1-\nu}{2-2\nu} > 1. \end{cases} \quad (3.35)$$

Here c is a positive constant which is independent of N .

Remark 3.3.3. The best published superconvergence result for weakly singular Volterra integral equations can be found in [13], Theorem 2.3, where nonlinear weakly singular equations are considered. In the particular case of linear equations our result improves it in the following ways:

- 1) the superconvergence phenomenon is described in Theorem 3.3.2 for all values of the nonuniformity parameter r while the result of [13] is stated only for sufficiently large values of r ;
- 2) our result guarantees the maximal convergence rate for smaller values of r . For example, if $m = 2$ and $\nu = 1/2$ then according to Theorem 3.3.2 the maximal convergence rate $h^{m+1-\nu} = h^{2.5}$ holds for $r \geq \frac{m+1-\nu}{2-2\nu} = 2.5$ while the corresponding conditions from [13] are $r > \frac{m}{1-\nu}$ and $r > \frac{m+1-\nu}{2-\nu}$ which are equivalent to $r > 4$.

Proof of Theorem 3.3.2. Under the assumptions of the theorem the solution y of equation (3.28) satisfies $y \in C^{m+1,\nu}[0, T]$ (see [13], Theorem 2.1).

Introduce an integral operator S by the formula

$$(Sy)(t) = \int_0^t K(t, s)y(s)ds, \quad t \in [0, T].$$

Note that $f \in C[0, T]$ and S is compact as an operator from $L^\infty(0, T)$ to $C[0, T]$ (see Theorem 2.2.6). Applying Lemma 3.3.1 we have

$$\delta_N \leq \max_{t \in [0, T]} \left| \int_0^t K(t, s)(y(s) - (P_N y)(s))ds \right|.$$

If $\nu < 0$ then $K \in C(\bar{\Delta}_T)$ (see Lemma 2.2.1) and $\frac{\partial K}{\partial s}(t, s)$ is uniformly integrable, since according to the estimate (2.4) with $\mu = \nu$, $i = 1$, and $j = 0$ we have

$$\left| \frac{\partial K}{\partial s}(t, s) \right| \leq c \begin{cases} 1 & \text{if } \nu < -1, \\ 1 + |\log(t-s)| & \text{if } \nu = -1, \\ (t-s)^{-\nu-1} & \text{if } \nu > -1, \end{cases}$$

thus the assumptions of Theorem 2.5.4 are satisfied with $k = m$ and $\mu_1 = \nu$. Applying Theorem 2.5.4 we get the estimate (3.33).

In the case $\nu \geq 0$ we use Theorem 2.5.6 with $k = m$ and $\mu = \mu_1 = \nu$ to obtain the estimates (3.34) and (3.35). \diamond

3.4. Superconvergence for Method 1

In this section we show that by a careful choice of the collocation parameters η_j it is possible, assuming a little more regularity of functions p, q and K , to improve the convergence rate of the approximate solution of problem $\{(3.1), (3.2)\}$ found by Method 1. Namely we prove the following result (originally proved for $\nu \in (-\infty, 1) \setminus \mathbf{Z}$ by R. Kangro, I. Parts in [30]), which improves the results of T. Tang [53], where in the case $0 < \nu < 1$ and $r = \frac{m+1-\nu}{2-\nu}$ the convergence rate $cN^{-m-1+\nu}$ is proved.

Theorem 3.4.1. *Assume that $p, q \in C^{m+1, \nu}[0, T]$, $K \in \mathcal{W}^{m+1, \nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$. Let the collocation points (3.7) be generated by the grid points of a graded grid Π_N^r , $r \geq 1$ and by the knots η_j , $j = 1, \dots, m$ of a quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^m A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_m \leq 1$, with appropriate weights $\{A_j\}$, which is exact for all polynomials of degree m .*

Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the conditions (3.6) and the formula (3.8) determine unique approximation $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ to y' and $u \in S_m^{(0)}(\Pi_N^r)$ to y , where y is the exact solution of problem $\{(3.1), (3.2)\}$. Moreover, for all $N \geq N_0$ the error estimates

$$\max_{n=1, \dots, N, j=1, \dots, m} |v_n(t_{nj}) - y'(t_{nj})| \leq c e_N(m, \nu, r) \quad (3.36)$$

and

$$\|u - y\|_\infty \leq c e_N(m, \nu, r) \quad (3.37)$$

hold. Here $v_n = v|_{(t_{n-1}, t_n)}$ ($n = 1, \dots, N$), c is a positive constant which is independent of N and

$$e_N(m, \nu, r) = \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m+1}{2-\nu}, \\ N^{-m-1}(1 + \log N) & \text{for } r = \frac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \frac{m+1}{2-\nu}. \end{cases}$$

Proof. We know that the Cauchy problem $\{(3.1), (3.2)\}$ is equivalent to the equation (3.4) which we can write in the form

$$z(t) = f_1(t) + \int_0^t \tilde{K}(t, s)z(s)ds, \quad t \in [0, T],$$

where

$$f_1(t) = q(t) + y_0p(t) + y_0 \int_0^t K(t, s)ds, \quad t \in [0, T],$$

and

$$\tilde{K}(t, s) = p(t) + \int_s^t K(t, \tau)d\tau, \quad (t, s) \in \bar{\Delta}_T.$$

Let P_N be defined by (2.17), with $k = m$, then the collocation conditions (3.6) can be written as

$$v = P_N f_1 + P_N \tilde{S}v,$$

where the integral operator \tilde{S} is given by formula

$$(\tilde{S}x)(t) = \int_0^t \tilde{K}(t, s)x(s)ds, \quad t \in [0, T].$$

Since \tilde{S} is a compact operator from $L^\infty(0, T)$ to $C[0, T]$ (see Remark 2.2.4 and Theorem 2.2.6) and $f_1 \in C[0, T]$ we get with the help of Lemma 3.3.1 that

$$|v_n(t_{nj}) - z(t_{nj})| \leq c \max_{t \in [0, T]} \left| \int_0^t \tilde{K}(t, s)(z(s) - (P_N z)(s))ds \right|.$$

Denote

$$\hat{K}(t, s) = \int_s^t K(t, \tau) d\tau$$

then

$$\tilde{K}(t, s) = p(s) + \hat{K}(t, s).$$

Since $p \in C^{m+1, \nu}[0, T] \subset C[0, T]$ and $\hat{K} \in \mathcal{W}^{m+1, \nu-1}(\Delta_T)$ (see Lemma 2.4 in [14]) the function \tilde{K} is continuous on $\tilde{\Delta}_T$ (see Lemma 2.2.1) and

$$\left| \frac{\partial \tilde{K}}{\partial s}(t, s) \right| = | -K(t, s) | \leq c \begin{cases} 1 & \text{if } \nu < 0, \\ 1 + |\log(t-s)| & \text{if } \nu = 0, \\ (t-s)^{-\nu} & \text{if } \nu > 0, \end{cases}$$

the assumptions of Theorem 2.5.4 with $k = m$ and $\mu_1 = \nu$ are satisfied and we get the estimation (3.36) of Theorem 3.4.1, since $z(t) = y'(t)$.

In order to prove the estimation (3.37) fix $t \in [0, T]$ and let $n \in \{1, \dots, N\}$ be such that $t \in [t_{n-1}, t_n]$. Using initial condition $y(0) = y_0$ and equation (3.8) we obtain

$$|u(t) - y(t)| = \left| \int_0^t (v(s) - z(s)) ds \right|.$$

Since $v \in S_m^{(-1)}(\Pi_N^r)$ and according to Theorem 3.1.1, $z = y' \in C^{m+1, \nu}[0, T]$ we get from Lemma 2.5.1 (with $k = m$, $\mu = \nu$) that

$$\left| \int_0^t (v(s) - z(s)) ds \right| \leq c_1 \max_{n=1, \dots, N, j=1, \dots, m} |v_n(t_{nj}) - z(t_{nj})| + c_2 e_N(m, \nu, r).$$

Using the estimation (3.36) it follows that

$$|u(t) - y(t)| \leq c e_N(m, \nu, r).$$

Theorem 3.4.1 is proved. \diamond

Next we turn our attention to studying the convergence properties of Method 2.

3.5. Global convergence results for Method 2

Similarly to Method 1 we start our analysis assuming only regularity of the grid. The following result by I. Parts, A. Pedas in [43] was originally

published without a full proof of the error estimate for the derivative of the approximate solution.

Theorem 3.5.1. *Let $p, q \in C^{m,\nu}[0, T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, and assume that the collocation points (3.12) with a regular grid Π_N are used.*

Then, for all sufficiently large $N \in \mathbb{N}$ and for every choice of parameters $0 \leq \eta_1 < \dots < \eta_{m+1} \leq 1$ the collocation conditions (3.11) determine a unique approximation $u \in S_m^{(-1)}(\Pi_N)$ (if $\eta_1 = 0$, $\eta_{m+1} = 1$, then $u \in S_m^{(0)}(\Pi_N)$) to the solution y of the Cauchy problem $\{(3.1), (3.2)\}$. The following error estimate holds:

$$\|u - y\|_\infty \leq c \begin{cases} h^{m+1} & \text{if } m < 1 - \nu, \\ h^{m+1}(1 + |\log h|) & \text{if } m = 1 - \nu, \\ h^{2-\nu} & \text{if } m > 1 - \nu, \end{cases} \quad (3.38)$$

where h is defined by (2.12). If the grid $\Pi_N = \Pi_{N,\Theta}$ is quasi-uniform, then

$$\|u' - y'\|_\infty \leq c \begin{cases} h^m & \text{if } m < 1 - \nu, \\ h^m(1 + |\log h|) & \text{if } m = 1 - \nu, \\ h^{1-\nu} & \text{if } m > 1 - \nu. \end{cases} \quad (3.39)$$

Here h is defined by (2.12), c is a constant which is independent of N and $u' \in S_{m-1}^{(-1)}(\Pi_{N,\Theta})$ is defined by differentiating the function u on the each subinterval of the grid $\Pi_{N,\Theta}$ separately.

Proof. We know (see Section 3.1) that the problem $\{(3.1), (3.2)\}$ is equivalent to the equation (3.9) which we rewrite in the form

$$y = f_2 + S_2 y,$$

where the integral operator S_2 is given by the formula

$$(S_2 x)(t) = \int_0^t K_2(t, s)x(s)ds$$

and f_2, K_2 are given by (3.10).

Note that $f_2 \in C[0, T]$ and that S_2 is compact as an operator from $L^\infty(0, T)$ to $C[0, T]$ (see Theorem 2.2.6). Repeating now the arguments of the proof of Theorem 3.2.1 starting from the equation (3.17) with y instead of z, u

instead of v , S_2 instead of S_1 and $m + 1$ instead of m we get the estimate (see (3.21))

$$\|u - y\|_\infty \leq c \|P_N y - y\|_\infty, \quad N \geq N_0,$$

with the operator P_N defined by (2.17) ($k = m + 1$) and a constant c which is independent of N . Applying the estimation (2.30) of Lemma 2.4.5 with $k = m + 1$ and $\mu = \nu - 1$ we obtain the estimation (3.38) of Theorem 3.5.1.

In order to prove the estimation (3.39) fix $t \in [0, T]$, and let $n \in \{1, \dots, N\}$ be such that $t \in [t_{n-1}, t_n]$. Fix ω as a Taylor polynomial for y at $t = t_n$

$$\omega(t) = \sum_{i=0}^m \frac{y^{(i)}(t_n)}{i!} (t - t_n)^i, \quad t \in [t_{n-1}, t_n],$$

and let the operator P_{t_{n-1}, t_n} be defined by (2.19) with $k = m + 1$, $a = t_{n-1}$ and $b = t_n$. Using the notation $u_n = u|_{(t_{n-1}, t_n)}$ and extending the polynomial u_n to the closed interval $[t_{n-1}, t_n]$ we have

$$\begin{aligned} \|u'_n - y'\|_{C[t_{n-1}, t_n]} &= \|(u_n - \omega)' + \omega' - y'\|_{C[t_{n-1}, t_n]} \\ &= \|(P_{t_{n-1}, t_n}(u_n - \omega))' + \omega' - y'\|_{C[t_{n-1}, t_n]} \\ &\leq \|(P_{t_{n-1}, t_n}(u_n - \omega))'\|_{C[t_{n-1}, t_n]} + \|\omega' - y'\|_{C[t_{n-1}, t_n]} \\ &\leq \|P_{t_{n-1}, t_n}(u_n - \omega)\|_{C^1[t_{n-1}, t_n]} + \|\omega' - y'\|_{C[t_{n-1}, t_n]} \\ &\leq \|P_{t_{n-1}, t_n}\|_{\mathcal{L}(C[t_{n-1}, t_n], C^1[t_{n-1}, t_n])} \|u_n - \omega\|_{C[t_{n-1}, t_n]} \\ &\quad + \|\omega' - y'\|_{C[t_{n-1}, t_n]} \\ &\leq \|P_{t_{n-1}, t_n}\|_{\mathcal{L}(C[t_{n-1}, t_n], C^1[t_{n-1}, t_n])} (\|u_n - y\|_{C[t_{n-1}, t_n]} \\ &\quad + \|y - \omega\|_{C[t_{n-1}, t_n]}) + \|\omega' - y'\|_{C[t_{n-1}, t_n]}. \end{aligned} \quad (3.40)$$

Since the grid $\Pi_N = \Pi_{N, \Theta}$ is assumed to be quasi-uniform we get with help of estimation (2.21) of Lemma 2.4.2 that

$$\|P_{t_{n-1}, t_n}\|_{\mathcal{L}(C[t_{n-1}, t_n], C^1[t_{n-1}, t_n])} \leq c + \frac{c_1}{h_n} \leq c + \frac{c_2}{h} \leq \frac{c_3}{h}. \quad (3.41)$$

For the term $\|u_n - y\|_{C[t_{n-1}, t_n]}$ we can use the estimation (3.38). Further,

since $t \in [t_{n-1}, t_n]$ and $y \in C^{m+1, \nu-1}[0, T]$,

$$\begin{aligned} |y(t) - \omega(t)| &= \frac{1}{m!} \left| \int_t^{t_n} (s-t)^m y^{(m+1)}(s) ds \right| \\ &\leq c \int_t^{t_n} (s-t)^m \left\{ \begin{array}{ll} 1 & \text{if } m < 1 - \nu \\ 1 + |\log s| & \text{if } m = 1 - \nu \\ s^{1-\nu-m} & \text{if } m > 1 - \nu \end{array} \right\} ds \\ &\leq c \int_t^{t_n} (s-t)^m \left\{ \begin{array}{ll} 1 & \text{if } m < 1 - \nu \\ 1 + |\log(s-t)| & \text{if } m = 1 - \nu \\ (s-t)^{1-\nu-m} & \text{if } m > 1 - \nu \end{array} \right\} ds. \end{aligned}$$

If $m < 1 - \nu$ then we have

$$\begin{aligned} \int_t^{t_n} (s-t)^m ds &= \frac{(s-t)^{m+1}}{m+1} \Big|_{s=t}^{s=t_n} \leq c(t_n - t)^{m+1} \leq c(t_n - t_{n-1})^{m+1} \\ &= ch_n^{m+1} \leq ch^{m+1}. \end{aligned}$$

In the case $m = 1 - \nu$ we get

$$\int_t^{t_n} (s-t)^m (1 + |\log(s-t)|) ds \leq ch_n^{m+1} (1 + |\log h_n|) \leq ch^{m+1} (1 + |\log h|).$$

If $m > 1 - \nu$ then

$$\begin{aligned} \int_t^{t_n} (s-t)^{1-\nu} ds &= \frac{(s-t)^{2-\nu}}{2-\nu} \Big|_{s=t}^{s=t_n} \leq c(t_n - t)^{2-\nu} \leq c(t_n - t_{n-1})^{2-\nu} \\ &= ch_n^{2-\nu} \leq ch^{2-\nu}. \end{aligned}$$

Therefore

$$\|y - \omega\|_{C[t_{n-1}, t_n]} \leq c \begin{cases} h^{m+1} & \text{if } m < 1 - \nu, \\ h^{m+1} (1 + |\log h|) & \text{if } m = 1 - \nu, \\ h^{2-\nu} & \text{if } m > 1 - \nu. \end{cases} \quad (3.42)$$

Analogously

$$\begin{aligned} |\omega'(t) - y'(t)| &= \frac{1}{(m-1)!} \left| \int_t^{t_n} (s-t)^{m-1} y^{(m+1)}(s) ds \right| \\ &\leq c \int_t^{t_n} (s-t)^{m-1} \left\{ \begin{array}{ll} 1 & \text{if } m < 1 - \nu \\ 1 + |\log s| & \text{if } m = 1 - \nu \\ s^{1-\nu-m} & \text{if } m > 1 - \nu \end{array} \right\} ds \end{aligned}$$

and

$$\|\omega' - y'\|_{C[t_{n-1}, t_n]} \leq c \begin{cases} h^m & \text{if } m < 1 - \nu, \\ h^m(1 + |\log h|) & \text{if } m = 1 - \nu, \\ h^{1-\nu} & \text{if } m > 1 - \nu. \end{cases} \quad (3.43)$$

Thus the estimation (3.39) of the theorem follows from (3.40) by (3.41), (3.38), (3.42) and (3.43). \diamond

For comparison, we quote the result of H. Brunner, A. Pedas and G. Vainikko about the error estimates in the case of nonuniform grids.

Theorem 3.5.2. [14] *Suppose that conditions of Theorem 3.5.1 are satisfied. Assume additionally that the grid $\Pi_N = \Pi_N^r$ ($r \geq 1$) is graded grid (i.e. satisfies (2.10)).*

Then for all sufficiently large $N \in \mathbb{N}$ and for every choice of parameters $0 \leq \eta_1 < \dots < \eta_{m+1} \leq 1$ the collocation conditions (3.11) determine a unique approximation $u \in S_m^{(-1)}(\Pi_N^r)$ to y , the solution of the Cauchy problem {(3.1), (3.2)}. Then the error estimates

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m+1}{2-\nu} \\ N^{-m-1}(1 + \log N) & \text{for } r = \frac{m+1}{2-\nu} = 1 \\ N^{-m-1} & \text{for } r > \frac{m+1}{2-\nu} \text{ or } r = \frac{m+1}{2-\nu} > 1 \end{cases}$$

and

$$\|u' - y'\|_{\varepsilon, \infty} \leq c_\varepsilon \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r \leq \frac{m}{1-\nu} \\ N^{-m} & \text{for } r > \frac{m}{1-\nu} \end{cases}$$

hold. Here the constants c and c_ε are independent of N , $u' \in S_{m-1}^{(-1)}(\Pi_N^r)$ is defined by differentiating the function u on the each subinterval of the grid Π_N^r separately and

$$\|u' - y'\|_{\varepsilon, \infty} = \max_{j=1, \dots, N} \left(\max_{t \in [t_{j-1}, t_j] \cap [\varepsilon, T]} |u'_j(t) - y'(t)| \right), \quad 0 < \varepsilon < T.$$

Moreover if $r = 1$ then we additionally have

$$\|u' - y'\|_\infty \leq c \begin{cases} N^{-m} & \text{if } m < 1 - \nu, \\ N^{-m}(1 + \log N) & \text{if } m = 1 - \nu, \\ N^{-(1-\nu)} & \text{if } m > 1 - \nu, \end{cases}$$

where the constant c is independent of N .

3.6. Superconvergence for Method 2

If we use collocation parameters corresponding to a quadrature formula that is exact for all polynomials of order $m + 1$, it is possible to get superconvergence at the collocation points for Volterra integro-differential equation. The following result has originally been stated as a conjecture in [42] and is a refinement of the corresponding result of [13] (see Theorem 2.3).

Theorem 3.6.1. *Assume that $p, q \in C^{m+1,\nu}[0, T]$, $K \in \mathcal{W}^{m+1,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$ and the collocation points (3.12) are generated by the grid points of a graded grid Π_N^r , $r \geq 1$ and by the knots η_j , $j = 1, \dots, m + 1$ of a quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^{m+1} A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_{m+1} \leq 1$, with appropriate weights $\{A_j\}$, which is exact for all polynomials of degree $m + 1$.*

Then, there exists an $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ and for every choice of parameters $0 \leq \eta_1 < \dots < \eta_{m+1} \leq 1$, the collocation conditions (3.11) determine a unique approximation $u \in S_m^{(-1)}(\Pi_N^r)$ (if $\eta_1 = 0$, $\eta_{m+1} = 1$, then $u \in S_m^{(0)}(\Pi_N^r)$) to the solution y of the Cauchy problem $\{(3.1), (3.2)\}$. The approximate solution u satisfies the estimate

$$\delta_N \leq c \begin{cases} N^{-r(3-\nu)} & \text{for } 1 \leq r < \frac{m+2}{3-\nu}, \\ N^{-m-2}(1 + \log N) & \text{for } r = \frac{m+2}{3-\nu}, \\ N^{-m-2} & \text{for } r > \frac{m+2}{3-\nu}, \end{cases} \quad (3.44)$$

where the constant c is independent of N and

$$\delta_N = \max_{n=1, \dots, N, j=1, \dots, m+1} |u_n(t_{nj}) - y(t_{nj})|, \quad u_n = u|_{(t_{n-1}, t_n)}, \quad n = 1, \dots, N.$$

Proof. Recall that the initial value problem $\{(3.1), (3.2)\}$ is equivalent to the equation (3.9)

$$y(t) = f_2(t) + \int_0^t K_2(t, s)y(s)ds, \quad t \in [0, T],$$

where

$$f_2(t) = y_0 + \int_0^t q(s)ds, \quad K_2(t, s) = p(s) + \int_s^t K(\tau, s)d\tau, \quad t \in [0, T], \quad s \in [0, t].$$

Introduce an integral operator S_2 by the formula

$$(S_2x)(t) = \int_0^t K_2(t, s)x(s)ds, \quad t \in [0, T].$$

Denote

$$\tilde{K}(t, s) = \int_s^t K(\tau, s)d\tau$$

then

$$K_2(t, s) = p(s) + \tilde{K}(t, s).$$

Since $p \in C^{m+1, \nu}[0, T]$ and $\tilde{K} \in \mathcal{W}^{m+1, \nu-1}(\Delta_T)$ (see Lemma 2.4 in [14]) the function K_2 is continuous on $\bar{\Delta}_T$ (see Lemma 2.2.1), therefore S_2 is a compact operator from $L^\infty(0, T)$ to $C[0, T]$. As $f_2 \in C[0, T]$, we get from Lemma 3.3.1 that

$$\delta_N \leq \max_{t \in [0, T]} \left| \int_0^t K(t, s)(y(s) - (P_N y)(s))ds \right|,$$

where P_N is defined by (2.17). Using the estimate (2.4) with $k = m+1$, $\mu = \nu - 1$, $i = 1$, $j = 0$ and $K = \tilde{K}$ we get

$$\begin{aligned} \left| \frac{\partial}{\partial s} K_2(t, s) \right| &\leq |p'(s)| + \left| \frac{\partial}{\partial s} \tilde{K}(t, s) \right| \\ &\leq c_1 \left\{ \begin{array}{ll} 1 & \text{if } \nu < 0 \\ 1 + |\log s| & \text{if } \nu = 0 \\ s^{-\nu} & \text{if } \nu > 0 \end{array} \right\} + c_2 \\ &\quad + c_3 \left\{ \begin{array}{ll} 1 & \text{if } \nu < 0, \\ 1 + |\log(t-s)| & \text{if } \nu = 0, \\ (t-s)^{-\nu} & \text{if } \nu > 0, \end{array} \right. \end{aligned}$$

hence the assumption (a) of Theorem 2.5.4 is satisfied. From Theorem 3.1.1 we get that the solution y of equation (3.9) belongs to the space $C^{m+2, \nu-1}[0, T]$. Applying Theorem 2.5.4 with $k = m+1$ and $\mu_1 = \nu - 1$ gives us the estimation (3.44) of Theorem 3.6.1. \diamond

Chapter 4

Collocation approximations for linear Fredholm integro- differential equations with weakly singular kernels

In this chapter we study the initial or boundary-value problems of linear integro-differential equations of the form

$$\begin{aligned} y'(t) &= p(t)y(t) + q(t) + \int_0^b K(t,s)y(s)ds, \quad t \in [0, b], \\ \alpha y(0) + \beta y(b) &= \gamma. \end{aligned} \quad (4.1)$$

Since the integral operator in (4.1) is of Fredholm type the first equation in (4.1) is called Fredholm integro-differential equation. Note that a special case of the problem (4.1), with $\alpha = 1$, $\beta = 0$ and $K(t, s) = 0$ for $s > t$, is the initial-value problem for a Volterra integro-differential equation, so (4.1) is a generalization of problem $\{(3.1), (3.2)\}$. From the perspective of constructing numerical methods the main difference between Volterra and Fredholm integral operators is the fact, that weakly singular Fredholm integral operators usually introduce singularities at both endpoints of the interval $[0, b]$ while corresponding Volterra operators introduce singularities only at $t = 0$. Therefore, in order to achieve the optimal rate of convergence it is necessary to use different nonuniform grids in the Fredholm case. Surprisingly piecewise polynomial collocation methods for solving FIDE have

not been studied before. The main results of this chapter are published in [46].

4.1. Definitions of function spaces

In this chapter $C_F^{m,\nu}[0, b]$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$ denotes the collection of all continuous functions $x : [0, b] \rightarrow \mathbb{R}$, which are m times continuously differentiable in $(0, b)$ such that the estimation

$$|x^{(i)}(t)| \leq c \begin{cases} 1 & \text{if } i < 1 - \nu, \\ 1 + |\log \varrho(t)| & \text{if } i = 1 - \nu, \\ \varrho(t)^{1-\nu-i} & \text{if } i > 1 - \nu \end{cases}$$

holds with $\varrho(t) = \min\{t, b - t\}$, $0 < t < b$, and with a constant $c = c(x)$ for all $t \in (0, b)$ and $i = 1, \dots, m$. Equipped with the norm

$$\|x\|_{m,\nu} = \max_{t \in [0,b]} |x(t)| + \sum_{i=1}^m \sup_{t \in (0,b)} (\omega_{i+\nu-1}(t) |x^{(i)}(t)|), \quad x \in C_F^{m,\nu}[0, b],$$

$C_F^{m,\nu}[0, b]$ is a Banach space. Here

$$\omega_\lambda(t) = \begin{cases} 1 & \text{for } \lambda < 0, \\ (1 + |\log \varrho(t)|)^{-1} & \text{for } \lambda = 0, \\ \varrho(t)^\lambda & \text{for } \lambda > 0 \end{cases}$$

with $t \in (0, b)$. It is easy to see that if $\mu < \nu < 1$ then $C_F^{m,\mu}[0, b] \subset C_F^{m,\nu}[0, b]$ and $\|x\|_{m,\nu} \leq c \|x\|_{m,\mu}$ for $x \in C_F^{m,\mu}[0, b]$, with a constant $c > 0$. Notice also that $C^m[0, b] \subset C_F^{m,\nu}[0, b]$, $m \in \mathbb{N}$, $\nu < 1$.

The set $\mathcal{W}^{m,\nu}(D)$, with $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$ and

$$D = \{(t, s) : 0 \leq t \leq b, 0 \leq s \leq b, t \neq s\},$$

consists of all m times continuously differentiable functions $K : D \rightarrow \mathbb{R}$ satisfying

$$\left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j K(t, s) \right| \leq c \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log |t - s|| & \text{if } \nu + i = 0, \\ |t - s|^{-\nu-i} & \text{if } \nu + i > 0 \end{cases} \quad (4.2)$$

with a constant $c = c(K)$ for all $(t, s) \in D$ and all nonnegative integers i and j such that $i + j \leq m$.

Taking $i = j = 0$, the condition (4.2) yields that $K \in \mathcal{W}^{m,\nu}(D)$ may possess a weak singularity at $t = s$ for $0 \leq \nu < 1$. If $\nu < 0$, then K itself is bounded on D , but its derivatives may be singular at $t = s$. Often the kernel K of problem (4.1) has the form

$$K = K_\nu(t, s) = \kappa(t, s)|t - s|^{-\nu}, \quad 0 < \nu < 1,$$

or

$$K = K_0(t, s) = \kappa(t, s) \log |t - s|,$$

where $\kappa \in C^m(\bar{D})$, with $m \in \mathbb{N}$ and $\bar{D} = [0, b] \times [0, b]$. Clearly, $K_\nu \in \mathcal{W}^{m,\nu}(D)$ for $0 < \nu < 1$ and $K_0 \in \mathcal{W}^{m,0}(D)$.

4.2. Smoothness of the solution

The regularity of the solution of problem (4.1) is described in the following theorem.

Theorem 4.2.1. *Let $p, q \in C_F^{m,\nu}[0, b]$, $K \in \mathcal{W}^{m,\nu}(D)$, $m \in \mathbb{N}$, $\nu < 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$. Moreover, assume that the homogeneous problem*

$$y'(t) = p(t)y(t) + \int_0^b K(t, s)y(s)ds, \quad \alpha y(0) + \beta y(b) = 0, \quad (4.3)$$

corresponding to the problem (4.1), has in the set $\{y : y \in C[0, b], y' \in L^\infty(0, b)\}$ only the trivial solution $y = 0$.

Then the problem (4.1) has a unique solution $y \in C_F^{m+1,\nu-1}[0, b]$.

Proof. If $\alpha + \beta \neq 0$ and $z \in L^\infty(0, b)$, then the problem

$$y'(t) = z(t), \quad \alpha y(0) + \beta y(b) = \gamma$$

has a unique solution

$$y(t) = (Jz)(t) + \frac{\gamma}{\alpha + \beta}, \quad t \in [0, b], \quad (4.4)$$

where

$$\begin{aligned} (Jz)(t) &= \int_0^t z(s)ds - \frac{\beta}{\alpha + \beta} \int_0^b z(s)ds \\ &= \int_0^b \kappa(t-s)z(s)ds, \quad t \in [0, b], \end{aligned} \quad (4.5)$$

with

$$\kappa(\tau) = \begin{cases} -\frac{\beta}{\alpha+\beta}, & \text{if } -b \leq \tau < 0, \\ 1 - \frac{\beta}{\alpha+\beta}, & \text{if } 0 < \tau \leq b. \end{cases}$$

It follows from (4.5) and the expression of the norm $\|\cdot\|_{m,\nu}$ that $J \in \mathcal{L}(C_F^{m,\nu}[0, b], C_F^{m+1,\nu-1}[0, b])$. It is proved in [57] (Proposition 1.3) that the operator J is compact from $C_F^{m,\nu}[0, b]$ to $C_F^{m,\nu}[0, b]$ (see [57]). Further, we can write (4.1) in the form

$$y'(t) = (Ay)(t) + (Ty)(t) + q(t), \quad t \in [0, b], \quad \alpha y(0) + \beta y(b) = \gamma,$$

where

$$(Ay)(t) = p(t)y(t), \quad (Ty)(t) = \int_0^b K(t, s)y(s)ds, \quad t \in [0, b]. \quad (4.6)$$

Therefore, if y is a solution of problem (4.1), then it can be presented in the form (4.4), where z is the solution of the equation

$$z = T_1 z + f_1, \quad (4.7)$$

with $T_1 = (A + T)J$ (see (4.5)-(4.6)) and

$$f_1(t) = q(t) + \frac{\gamma}{\alpha + \beta}p(t) + \frac{\gamma}{\alpha + \beta} \int_0^b K(t, s)ds, \quad t \in [0, b]. \quad (4.8)$$

Next we show, that T_1 is compact as an operator from $C_F^{m,\nu}[0, b]$ into $C_F^{m,\nu}[0, b]$ and $f_1 \in C_F^{m,\nu}[0, b]$.

Indeed, as $K \in \mathcal{W}^{m,\nu}(D)$ we have $T \in \mathcal{L}(C_F^{m,\nu}[0, b], C_F^{m,\nu}[0, b])$ (see [55], Lemma 3.4). If $p, z \in C_F^{m,\nu}[0, b]$ then (cf. [14], Lemma 2.1)

$$\|pz\|_{m,\nu} \leq c\|p\|_{m,\nu}\|z\|_{m,\nu}.$$

Therefore, $A \in \mathcal{L}(C_F^{m,\nu}[0, b], C_F^{m,\nu}[0, b])$. This together with the compactness of $J \in \mathcal{L}(C_F^{m,\nu}[0, b], C_F^{m,\nu}[0, b])$ implies that $T_1 : C_F^{m,\nu}[0, b] \rightarrow C_F^{m,\nu}[0, b]$ is linear and compact. Since $1 \in C_F^{m,\nu}[0, b]$ then $T_1 1 \in C_F^{m,\nu}[0, b]$. This together with $q, p \in C_F^{m,\nu}[0, b]$ yields $f_1 \in C_F^{m,\nu}[0, b]$.

Further, since the problem (4.3) has only the trivial solution, the equation

$$z = T_1 z$$

has only the trivial solution $z = 0$ in $L^\infty(0, b)$ and therefore also in $C_F^{m,\nu}[0, b] \subset L^\infty(0, b)$. Thus, by the Fredholm alternative (see Theorem 2.1.5), $I - T_1$ has a bounded inverse $(I - T_1)^{-1} : C_F^{m,\nu}[0, b] \rightarrow C_F^{m,\nu}[0, b]$ and equation (4.7) has a unique solution

$$z = (I - T_1)^{-1} f_1 \in C_F^{m,\nu}[0, b],$$

where I is the identity mapping. This, in turn, implies that the problem (4.1) has a unique solution y and

$$y = Jz + \frac{\gamma}{\alpha + \beta} \in C_F^{m+1,\nu-1}[0, b].$$

If $\alpha + \beta = 0$ then we introduce a new unknown function

$$\tilde{y}(t) = e^{-t}y(t)$$

and note that $y(t)$ is a solution of (4.1) if and only if $\tilde{y}(t)$ solves the problem

$$\begin{aligned} \tilde{y}'(t) &= \tilde{p}(t)\tilde{y}(t) + \tilde{q}(t) + \int_0^b \tilde{K}(t, s)\tilde{y}(s)ds, \quad t \in [0, b], \\ \alpha\tilde{y}(0) + \tilde{\beta}\tilde{y}(b) &= \gamma, \end{aligned} \quad (4.9)$$

where $\tilde{y}(t) = e^{-t}y(t)$, $\tilde{p}(t) = p(t) - 1$, $\tilde{q}(t) = e^{-t}q(t)$, $\tilde{K}(t, s) = e^{-(t-s)}K(t, s)$ and $\tilde{\beta} = e^b\beta$.

It is easy to check that $\tilde{p}, \tilde{q} \in C_F^{m,\nu}[0, b]$, $\tilde{K} \in \mathcal{W}^{m,\nu}(D)$, $m \in \mathbb{N}, \nu \in \mathbb{R}, \nu < 1$. Since $\alpha + \tilde{\beta} \neq 0$ the argument of the first part of the proof gives us that $\tilde{y} \in C_F^{m+1,\nu-1}[0, b]$, hence $y \in C_F^{m+1,\nu-1}[0, b]$. \diamond

Remark 4.2.2. Since the problem (4.1) with $\alpha = -\beta \neq 0$ can be transformed to an equivalent problem (4.9), where $\alpha + \tilde{\beta} \neq 0$, we consider further only the case $\alpha + \beta \neq 0$.

4.3. Piecewise polynomial interpolation

For $N \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geq 1$, let

$$\square_N^r = \{t_0, t_1, \dots, t_{2N} : 0 = t_0 < t_1 < \dots < t_{2N} = b\}$$

be a partition (a graded grid) of the interval $[0, b]$ given by the grid points

$$\begin{aligned} t_n &= \frac{b}{2} \left(\frac{n}{N} \right)^r, \quad n = 0, 1, \dots, N, \\ t_{N+n} &= b - t_{N-n}, \quad n = 1, \dots, N. \end{aligned} \quad (4.10)$$

Here the real number $r \in [1, \infty)$ characterizes the nonuniformity of the grid Γ_N^r : for $r > 1$ the points (4.10) are more densely clustered near the endpoints of the interval $[0, b]$.

Denote

$$h_n = t_n - t_{n-1}, \quad n = 1, \dots, 2N.$$

It is easy to see (cf (2.14)) that

$$h_n \leq \frac{b}{2} r N^{-1}, \quad n = 1, \dots, 2N.$$

For given integers $m \geq 0$ and $-1 \leq d \leq m - 1$, let $S_m^{(d)}(\Gamma_N^r)$ be the spline space of piecewise polynomial functions on the grid Γ_N^r :

$$\begin{aligned} S_m^{(d)}(\Gamma_N^r) &= \{w : w|_{(t_{n-1}, t_n)} =: w_n \in \pi_m, \quad n = 1, \dots, 2N; \\ &w_n^{(i)}(t_n) = w_{n+1}^{(i)}(t_n) \quad 0 \leq i \leq d, \quad n = 1, \dots, 2N - 1\}. \end{aligned}$$

Here $w|_{(t_{n-1}, t_n)}$ ($n = 1, \dots, 2N$) is the restriction of $w : [0, b] \rightarrow \mathbb{R}$ to the subinterval $(t_{n-1}, t_n) \subset [0, b]$ and π_m denotes the set of polynomials of degree not exceeding m . Note that the elements of $S_m^{(-1)}(\Gamma_N^r)$ may have jump discontinuities at the interior points t_1, \dots, t_{2N-1} of the grid Γ_N^r .

In every subinterval $[t_{n-1}, t_n]$ ($n = 1, \dots, 2N$), we introduce $m \geq 1$ interpolation points

$$t_{nj} = t_{n-1} + \eta_j h_n, \quad j = 1, \dots, m \quad (n = 1, \dots, 2N), \quad (4.11)$$

where η_1, \dots, η_m are some fixed parameters which do not depend on n and N and satisfy the conditions

$$0 \leq \eta_1 < \dots < \eta_m \leq 1. \quad (4.12)$$

We introduce an interpolation operator $P_N = P_N^{(m)} : C[0, b] \rightarrow S_{m-1}^{(-1)}(\Gamma_N^r) \subset L^\infty(0, b)$ which assigns to every continuous function $x : [0, b] \rightarrow \mathbb{R}$ its

piecewise polynomial interpolation function which interpolates x at the points (4.11):

$$\begin{aligned} P_N x &\in S_{m-1}^{(-1)}(\Pi_N^r), \quad x \in C[0, b], \\ (P_N x)(t_{nj}) &= x(t_{nj}), \quad j = 1, \dots, m; \quad n = 1, \dots, 2N. \end{aligned} \quad (4.13)$$

If $\eta_1 = 0$, then by $(P_N x)(t_{n1})$ we mean the right limit $\lim_{t \rightarrow t_{n-1}, t > t_{n-1}} (P_N x)(t)$. If $\eta_m = 1$, then $(P_N x)(t_{nm})$ denotes the left limit $\lim_{t \rightarrow t_n, t < t_n} (P_N x)(t)$. Thus, $(P_N x)(t)$ is independently defined on every subinterval $[t_{n-1}, t_n]$, $n = 1, \dots, 2N$, and may be discontinuous at the interior grid points $t = t_n$, $n = 1, \dots, 2N - 1$. Note that in the case $\eta_1 = 0, \eta_m = 1$, function $P_N x$ is continuous on $[0, b]$.

From [55], pp. 115 - 119, we obtain the following Lemmas 4.3.1-4.3.3 (cf. also [14]).

Lemma 4.3.1. *Let the interpolation nodes (4.11) with grid points (4.10) and parameters (4.12) be used. Then $P_N \in \mathcal{L}(C[0, b], L^\infty(0, b))$ and*

$$\|P_N\|_{\mathcal{L}(C[0, b], L^\infty(0, b))} \leq c, \quad N \in \mathbb{N},$$

with a positive constant c which is independent of N .

Lemma 4.3.2. *Let $x \in C_F^{m, \nu}[0, b]$, $m \in \mathbb{N}$, $\nu < 1$, and assume that a graded grid Π_N^r with $r \geq 1$ and the interpolation points (4.11) are used. Then for the operator P_N defined by (4.13) the following estimates hold:*

$$\begin{aligned} \|x - P_N x\|_\infty &\leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r < \frac{m}{1-\nu}, \\ N^{-m}(1 + \log N) & \text{for } r = \frac{m}{1-\nu} = 1, \\ N^{-m} & \text{for } r > \frac{m}{1-\nu} \text{ or } r = \frac{m}{1-\nu} > 1; \end{cases} \\ \int_0^b |x(s) - (P_N x)(s)| ds &\leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m}{2-\nu}, \\ N^{-m}(1 + \log N) & \text{for } r = \frac{m}{2-\nu}, \\ N^{-m} & \text{for } r > \frac{m}{2-\nu}. \end{cases} \end{aligned}$$

Here c is a positive constant not depending on N .

Lemma 4.3.3. *Let the conditions of Lemma 4.3.2 be fulfilled. Then*

$$\sup_{t \in (t_{n-1}, t_n)} |x(t) - (P_N x)(t)| \leq c h_n^m \begin{cases} 1 & \text{if } m < 1 - \nu, \\ 1 + |\log t_n| & \text{if } m = 1 - \nu, \\ t_n^{1-\nu-m} & \text{if } m > 1 - \nu, \end{cases}$$

for $n = 1, \dots, N$, and

$$\sup_{t \in (t_{n-1}, t_n)} |x(t) - (P_N x)(t)| \leq c h_n^m \begin{cases} 1 & \text{if } m < 1 - \nu, \\ 1 + |\log(b - t_{n-1})| & \text{if } m = 1 - \nu, \\ (b - t_{n-1})^{1-\nu-m} & \text{if } m > 1 - \nu, \end{cases}$$

for $n = N + 1, \dots, 2N$. Here $h_n = t_n - t_{n-1}$ and c is a positive constant which is independent of n and N .

Remark 4.3.4. The properties of the interpolation operator P_N defined by (4.11) - (4.13) are closely related to similar operators studied in Chapter 2. Indeed, let us define an operator $\tilde{P}_N : C[0, \frac{b}{2}] \rightarrow L^\infty(0, \frac{b}{2})$ by the conditions

$$\begin{aligned} \tilde{P}_N z|_{(t_{n-1}, t_n)} &\in \pi_{m-1}, \quad z \in C[0, \frac{b}{2}], \\ (\tilde{P}_N z)(t'_{nj}) &= z(t'_{nj}), \quad j = 1, \dots, m; \quad n = 1, \dots, N, \end{aligned} \tag{4.14}$$

where

$$t'_{nj} = t_{n-1} + \eta'_j h_n, \quad j = 1, \dots, m \quad (n = 1, \dots, N)$$

with

$$\eta'_j = 1 - \eta_{m+1-j}, \quad j = 1, \dots, m. \tag{4.15}$$

Then for $x \in C[0, b]$, using the notation

$$g(t) = x(b - t), \quad t \in [0, \frac{b}{2}],$$

we have

$$(P_N x)(t) = (\tilde{P}_N g)(b - t), \quad t \in (\frac{b}{2}, b].$$

Since

$$\|x - P_N x\|_\infty = \max \left(\|(x - P_N x)|_{(0, \frac{b}{2})}\|_\infty, \|g - \tilde{P}_N g\|_\infty \right)$$

and

$$\int_0^b |x(t) - (P_N x)(t)| dt = \int_0^{\frac{b}{2}} |x(t) - (P_N x)(t)| dt + \int_0^{\frac{b}{2}} |g(t) - (\tilde{P}_N g)(t)| dt,$$

the Lemmas 4.3.1, 4.3.2 and 4.3.3 follow directly from Lemmas 2.4.1, 2.4.7 and Corollary 2.4.4, respectively.

4.4. Collocation method

Assume $\alpha + \beta \neq 0$, then the problem (4.1) is equivalent to the problem $\{(4.4),(4.7)\}$:

$$\begin{aligned} y(t) &= \int_0^t z(s)ds - \frac{\beta}{\alpha + \beta} \int_0^b z(s)ds + \frac{\gamma}{\alpha + \beta}, \quad t \in [0, b], \\ z(t) &= p(t) \left(\int_0^t z(s)ds - \frac{\beta}{\alpha + \beta} \int_0^b z(s)ds \right) \\ &\quad + \int_0^b K(t, s) \left(\int_0^s z(\tau)d\tau - \frac{\beta}{\alpha + \beta} \int_0^b z(\tau)d\tau \right) ds + f_1(t), \end{aligned}$$

where f_1 is given by the formula (4.8). In order to solve the problem (4.1) we construct a collocation method for the numerical solution of the problem $\{(4.4),(4.7)\}$.

We look for an approximate solution $u = u^{(N)}$, $N \in \mathbb{N}$, of (4.1) in the form

$$u(t) = \int_0^t v(s)ds - \frac{\beta}{\alpha + \beta} \int_0^b v(s)ds + \frac{\gamma}{\alpha + \beta}, \quad t \in [0, b], \quad (4.16)$$

where v satisfies the conditions

$$\begin{aligned} v &\in S_{m-1}^{(-1)}(\Pi_N^r), \quad m \in \mathbb{N}, \\ v_n(t_{nj}) &= p(t_{nj}) \left(\int_0^{t_{nj}} v(s)ds - \frac{\beta}{\alpha + \beta} \int_0^b v(s)ds \right) \\ &\quad + \int_0^b K(t_{nj}, s) \left(\int_0^s v(\tau)d\tau - \frac{\beta}{\alpha + \beta} \int_0^b v(\tau)d\tau \right) ds + f_1(t_{nj}), \\ &j = 1, \dots, m; \quad n = 1, \dots, 2N. \end{aligned} \quad (4.17)$$

Here $v_n = v|_{(t_{n-1}, t_n)}$ is the restriction of v to (t_{n-1}, t_n) , $n = 1, \dots, 2N$, and $f_1, \{t_{nj}\}$ are given by the formulas (4.8) and (4.11), respectively.

Remark 4.4.1. As $v \in S_{m-1}^{(-1)}(\Pi_N^r)$, the function u (see (4.16)) belongs to $S_m^{(0)}(\Pi_N) \subset C[0, b]$. If $\eta_1 = 0$ and $\eta_m = 1$ (see (4.12)), then $v \in S_{m-1}^{(0)}(\Pi_N^r) \subset C[0, b]$ and $u \in S_m^{(1)}(\Pi_N^r) \subset C^1[0, b]$.

Remark 4.4.2. The conditions (4.17) have the operator equation representation

$$v = P_N T_1 v + P_N f_1, \quad T_1 = (A + T)J, \quad (4.18)$$

with A, T, J and P_N defined in Sections 4.2 and 4.3, respectively.

Theorem 4.4.3. *Let the conditions of the Theorem 4.2.1 be fulfilled and a graded grid Γ_N^r , $r \geq 1$ with collocation points (4.11) be used. Assume additionally that $\alpha + \beta \neq 0$.*

Then there exists an $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the conditions (4.17) and the formula (4.16) determine unique approximation $v \in S_{m-1}^{(-1)}(\Gamma_N^r)$ to y' and $u \in S_m^{(0)}(\Gamma_N^r)$ to y ($v \in S_{m-1}^{(0)}(\Gamma_N^r)$ and $u \in S_m^{(1)}(\Gamma_N^r)$ if $\eta_1 = 0$ and $\eta_m = 1$), where y is the exact solution of problem (4.1). Moreover, for $N \geq N_0$ the error estimates

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m}{2-\nu}, \\ N^{-m}(1 + \log N) & \text{for } r = \frac{m}{2-\nu}, \\ N^{-m} & \text{for } r > \frac{m}{2-\nu} \end{cases} \quad (4.19)$$

and

$$\|v - y'\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r < \frac{m}{1-\nu}, \\ N^{-m}(1 + \log N) & \text{for } r = \frac{m}{1-\nu} = 1, \\ N^{-m} & \text{for } r > \frac{m}{1-\nu} \text{ or } r = \frac{m}{1-\nu} > 1 \end{cases} \quad (4.20)$$

hold. Here c is a positive constant not depending on N .

Proof. Due to the assumptions of Theorem 4.2.1, $f_1 \in C[0, b]$ and $T_1 = (A+T)J$ is compact as an operator from $L^\infty(0, b)$ to $C[0, b]$ and to $L^\infty(0, b)$, too. Since equation $z = T_1 z$ has in $L^\infty(0, b)$ only the trivial solution $z = 0$, then (see Theorem 2.1.5) there exists an inverse operator $(I - T_1)^{-1} \in \mathcal{L}(L^\infty(0, b), L^\infty(0, b))$ and the equation (4.7) has a unique solution

$$z = (I - T_1)^{-1} f_1 \in L^\infty(0, b).$$

By Theorem 4.2.1, $y' = z \in C_F^{m,\nu}[0, b]$. A standard discussion (see the proof of Theorem 3.2.1) together with Lemmas 4.3.1 and 4.3.2 yields that there exists a number $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ the operator $(I - P_N T_1)$ is invertible in $L^\infty(0, b)$ and

$$\|(I - P_N T_1)^{-1}\|_{\mathcal{L}(L^\infty(0, b), L^\infty(0, b))} \leq c, \quad N \geq N_0. \quad (4.21)$$

Thus, equation (4.18) possesses a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ for $N \geq N_0$ and

$$\|z - v\|_\infty \leq c\|z - P_N z\|_\infty, \quad N \geq N_0,$$

where $z = y' \in C_F^{m,\nu}[0, b]$ is the solution of the equation (4.7). This together with Lemma 4.3.2 yields the estimate (4.20).

Further, using v , we find for $N \geq N_0$ an approximation u for y in the form (4.16). It follows from (4.4), (4.5) and (4.16) that

$$\begin{aligned} y(t) - u(t) &= (J(z - v))(t) \\ &= \int_0^t (z(s) - v(s)) ds - \frac{\beta}{\alpha + \beta} \int_0^b (z(s) - v(s)) ds, \quad t \in [0, b]. \end{aligned} \quad (4.22)$$

Therefore

$$\max_{t \in [0, b]} |y(t) - u(t)| \leq \left(1 + \left| \frac{\beta}{\alpha + \beta} \right| \right) \int_0^b |z(s) - v(s)| ds. \quad (4.23)$$

Since $T_1 = (A + T)J$, $(I - P_N T_1)(z - v) = z - P_N z$ and

$$(I - P_N T_1)^{-1} = I + (I - P_N T_1)^{-1} P_N T_1, \quad N \geq N_0,$$

we get from (4.21) and Lemma 4.3.1 the estimate

$$\begin{aligned} |z(s) - v(s)| &\leq |z(s) - (P_N z)(s)| \\ &+ c \int_0^b |z(t) - (P_N z)(t)| dt, \quad s \in [0, b], \quad N \geq N_0. \end{aligned} \quad (4.24)$$

Now it follows from (4.22) - (4.24) that

$$\|y - u\|_\infty \leq c \int_0^b |z(t) - (P_N z)(t)| dt, \quad N \geq N_0.$$

This together with $z \in C_F^{m,\nu}[0, b]$ and Lemma 4.3.2 yields the estimate (4.19). \diamond

4.5. Superconvergence phenomenon

It follows from Theorem 4.4.3 that by using the method $\{(4.16), (4.17)\}$, one can reach a convergence order

$$\|y - u\|_\infty \leq cN^{-m}, \quad \|y' - v\|_\infty \leq cN^{-m} \quad (4.25)$$

for sufficiently large values of the grid parameter r and for every choice of collocation parameters η_1, \dots, η_m satisfying the condition (4.12). Since $u \in S_m^{(0)}(\Pi_N^r)$, the first estimate of (4.25) is not of optimal order. In the following we show that by a careful choice of the collocation parameters (4.12) it is possible, assuming a little more regularity of functions p, q and K , to prove a superconvergence result for the values of v at the collocation points and improve the convergence rate of u in the maximum norm.

Theorem 4.5.1. *Let $p, q \in C_F^{m+1, \nu}[0, b]$, $K \in \mathcal{W}^{m+1, \nu}(D)$, $m \in \mathbb{N}$, $\nu < 1$; $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha + \beta \neq 0$, and assume that problem (4.3) has in the set $\{y \in C[0, b] : y' \in L^\infty(0, b)\}$ only the trivial solution $y = 0$. Moreover, let the collocation points (4.11) be generated by the grid points of a graded grid Π_N^r , $r \geq 1$ and by the knots η_j , $j = 1, \dots, m$ of a quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^m A_j \varphi(\eta_j)$, $0 \leq \eta_1 < \dots < \eta_m \leq 1$, with appropriate weights $\{A_j\}$, which is exact for all polynomials of degree m .*

Then there exists an $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the conditions (4.17) and the formula (4.16) determine unique approximation $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ to y' and $u \in S_m^{(0)}(\Pi_N^r)$ to y , where y is the exact solution of problem (4.1). Moreover, for all $N \geq N_0$ the error estimates

$$\max_{n=1, \dots, 2N, j=1, \dots, m} |v_n(t_{nj}) - y'(t_{nj})| \leq c e_N(m, \nu, r) \quad (4.26)$$

and

$$\|u - y\|_\infty \leq c e_N(m, \nu, r) \quad (4.27)$$

hold. Here $v_n = v|_{(t_{n-1}, t_n)}$ ($n = 1, \dots, 2N$), c is a positive constant not depending on N and

$$e_N(m, \nu, r) = \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{m+1}{2-\nu}, \\ N^{-m-1}(1 + \log N) & \text{for } r = \frac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \frac{m+1}{2-\nu}. \end{cases}$$

Proof. We know from the proof of Theorem 4.4.3 that equation (4.18) has a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ for $N \geq N_0$. We have for it and $z = y'$,

the solution of equation (4.7), that

$$(I - P_N T_1)(v - P_N z) = P_N T_1(P_N z - z). \quad (4.28)$$

As $I - P_N T_1$ is invertible in $L^\infty(0, b)$ for $N \geq N_0$, we obtain from (4.19), (4.28) and Lemma 4.3.1 the estimate

$$\|P_N z - v\|_\infty \leq c \|T_1(z - P_N z)\|_\infty, \quad N \geq N_0. \quad (4.29)$$

Since $T_1 = (A + T)J$ and

$$(P_N z)(t_{nj}) = z(t_{nj}), \quad j = 1, \dots, m; \quad n = 1, \dots, 2N,$$

it follows from (4.5), (4.6) and (4.29) that

$$|z(t_{nj}) - v_n(t_{nj})| \leq \|P_N z - v\|_\infty \leq c \max_{t \in [0, b]} \left| \int_0^t (z(s) - (P_N z)(s)) ds \right|, \\ j = 1, \dots, m; \quad n = 1, \dots, 2N, \quad N \geq N_0. \quad (4.30)$$

According to Theorem 4.2.1 we have $z \in C_F^{m+1, \nu}[0, b]$. Since $z|_{[0, \frac{b}{2}]}$ and the collocation parameters η_1, \dots, η_m satisfy the assumptions of Corollary 2.5.3 with $k = m$, $\mu = \nu$ and $T = \frac{b}{2}$ we get immediately

$$\max_{t \in [0, \frac{b}{2}]} \left| \int_0^t (z(s) - (P_N z)(s)) ds \right| \leq c e_N(m, \nu, r). \quad (4.31)$$

If $t \in (\frac{b}{2}, b]$ we write

$$\int_0^t (z(s) - (P_N z)(s)) ds = \left(\int_0^{\frac{b}{2}} + \int_{\frac{b}{2}}^b - \int_t^b \right) (z(s) - (P_N z)(s)) ds \\ = \int_0^{\frac{b}{2}} (z(s) - (P_N z)(s)) ds + \int_0^{\frac{b}{2}} (g(s) - (\tilde{P}_N g)(s)) ds \\ - \int_0^{b-t} (g(s) - (\tilde{P}_N g)(s)) ds,$$

where

$$g(s) = z(b - s), \quad s \in [0, \frac{b}{2}]$$

and the operator \tilde{P}_N is defined by (4.14).

Since for any polynomial $\omega \in \pi_m$ we have

$$\begin{aligned} \sum_{j=1}^m A_j \omega(\eta_j) &= \int_0^1 \omega(s) ds = \int_0^1 \omega(1-s) ds \\ &= \sum_{j=1}^m A_j \omega(1-\eta_j) = \sum_{j=1}^m A_{m+1-j} \omega(\eta'_j), \end{aligned}$$

the parameters $\eta'_j = 1-\eta_{m+1-j}$ ($j = 1, \dots, m$) and the function g satisfy the assumptions of Corollary 2.5.3 with $k = m$, $\mu = \nu$ and $T = \frac{b}{2}$. Therefore

$$\begin{aligned} &\max_{t \in [\frac{b}{2}, b]} \left| \int_0^t (z(s) - (P_N z)(s)) ds \right| \\ &\leq \max_{t \in [0, \frac{b}{2}]} \left| \int_0^t (z(s) - (P_N z)(s)) ds \right| + 2 \max_{t \in [0, \frac{b}{2}]} \left| \int_0^t (g(s) - (\tilde{P}_N g)(s)) ds \right| \\ &\leq c e_N(m, \nu, r). \end{aligned}$$

This together with (4.31) and (4.30) proves the estimate (4.26).

Using (4.22) and $y' = z$ we obtain for every $t \in [0, b]$ that

$$\begin{aligned} |y(t) - u(t)| &\leq \left| \int_0^t (z(s) - v(s)) ds \right| + \left| \frac{\beta}{\alpha + \beta} \right| \left| \int_0^b (z(s) - v(s)) ds \right| \\ &\leq c \max_{t \in [0, b]} \left| \int_0^t (z(s) - v(s)) ds \right|. \end{aligned} \quad (4.32)$$

Using Lemma 2.5.1 with $k = m$, $\mu = \nu$, $T = \frac{b}{2}$, $x = z|_{[0, \frac{b}{2}]}$, $w = v|_{[0, \frac{b}{2}]}$ and the estimate (4.26) we get

$$\max_{t \in [0, \frac{b}{2}]} \left| \int_0^t (z(s) - v(s)) ds \right| \leq c e_N(m, \nu, r). \quad (4.33)$$

For $t \in (\frac{b}{2}, b]$ we again write

$$\begin{aligned} \int_0^t (z(s) - v(s)) ds &= \left(\int_0^{\frac{b}{2}} + \int_{\frac{b}{2}}^b - \int_t^b \right) (z(s) - v(s)) ds \\ &= \int_0^{\frac{b}{2}} (z(s) - v(s)) ds + \int_0^{\frac{b}{2}} (g(s) - \tilde{v}(s)) ds - \int_0^{b-t} (g(s) - \tilde{v}(s)) ds, \end{aligned}$$

where

$$g(s) = z(b-s), \quad s \in [0, \frac{b}{2}]$$

and

$$\tilde{v}(s) = v(b-s), \quad s \in [0, \frac{b}{2}].$$

Applying Lemma 2.5.1 with $k = m$, $\mu = \nu$, $T = \frac{b}{2}$, $x = g$, $w = \tilde{v}$, the collocation parameters (4.15) and the estimate (4.26), we get

$$\max_{t \in [\frac{b}{2}, b]} \left| \int_0^t (z(s) - v(s)) ds \right| \leq c e_N(m, \nu, r). \quad (4.34)$$

The estimates (4.32), (4.33) and (4.34) give us (4.27). Theorem 4.5.1 is proved. \diamond

Chapter 5

Numerical experiments

In this chapter we discuss implementation details of the numerical methods described in the previous chapters and analyze results obtained by extensive computational experiments.

5.1. Introduction

We have described the collocation methods for Volterra and Fredholm integro-differential equations in Sections 3.1 and 4.4, respectively. The collocation conditions form a system of equations whose exact form is determined by the choice of a basis in the polynomial spline space. For example, for a $v \in S_{m-1}^{(-1)}(\Pi_N)$ in each subinterval (t_{n-1}, t_n) , $n = 1, \dots, N$ we may use the Lagrange fundamental polynomial representation

$$v_n(t_{n-1} + \tau h_n) = \sum_{\ell=1}^m Y_\ell^{(n)} l_\ell(\tau), \quad \tau \in (0, 1),$$

where $v_n = v|_{(t_{n-1}, t_n)}$ and

$$l_\ell(\tau) = \prod_{r=1, r \neq \ell}^m \frac{\tau - \eta_r}{\eta_\ell - \eta_r}, \quad \tau \in [0, 1], \quad \ell = 1, \dots, m, \quad 0 \leq \eta_1 < \dots < \eta_m \leq 1,$$

or simple polynomial representation

$$v_n(t_{n-1} + \tau h_n) = \sum_{\ell=1}^m Y_\ell^{(n)} \tau^{\ell-1}, \quad \tau \in (0, 1).$$

Here $Y_\ell^{(n)}$, $\ell = 1, \dots, m$ are unknown coefficients, that have to be determined from the collocation conditions.

In the following we give more appropriate form to the collocation conditions for numerical implementation.

5.2. Numerical solution of Volterra integro-differential equations

In this section we transform the collocation conditions of Method 1 and Method 2 (see Section 3.1) into concrete system of linear algebraic equations, discuss a parallelization possibility of the solution process, introduce test equations and analyze the numerical results of both Method 1 and 2.

5.2.1. Method 1 for solving Volterra integro-differential equations

At first we write the collocation conditions (3.6) for $v \in S_{m-1}^{(-1)}(\Pi_N)$ in the form

$$\begin{aligned}
v_n(t_{nj}) &= p(t_{nj}) + p(t_{nj}) \left[y_0 + \int_0^{t_{n-1}} v(s) ds + \int_{t_{n-1}}^{t_{nj}} v(s) ds \right] \\
&+ \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} K(t_{nj}, s) \left[y_0 + \int_0^s v(\tau) d\tau \right] ds \\
&+ \int_{t_{n-1}}^{t_{nj}} K(t_{nj}, s) \left[y_0 + \int_0^s v(\tau) d\tau \right] ds, \\
j &= 1, \dots, m; \quad n = 1, \dots, N,
\end{aligned} \tag{5.1}$$

where $\{t_n\}$ are the nodes of Π_N (see (2.7)) and the set of collocation points $\{t_{nj}\}$ is given by (3.7). Since (see (3.8))

$$u(t_{n-1}) = y_0 + \int_0^{t_{n-1}} v(\tau) d\tau$$

we get after the change of variables

$$s = t_{k-1} + xh_k \quad \text{for } s \in [t_{k-1}, t_k]$$

and

$$s = t_{n-1} + xh_n \quad \text{for } s \in [t_{n-1}, t_{nj}]$$

from (5.1)

$$\begin{aligned} v_n(t_{nj}) &= q(t_{nj}) + p(t_{nj}) \left[u(t_{n-1}) + h_n \int_0^{\eta_j} v(t_{n-1} + xh_n) dx \right] \\ &\quad + \sum_{k=1}^{n-1} h_k \int_0^1 K(t_{nj}, t_{k-1} + xh_k) \left[u(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1} + xh_k} v(\tau) d\tau \right] dx \\ &\quad + h_n \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) \left[u(t_{n-1}) + \int_{t_{n-1}}^{t_{n-1} + xh_n} v(\tau) d\tau \right] dx, \\ j &= 1, \dots, m; \quad n = 1, \dots, N. \end{aligned} \tag{5.2}$$

Using again the change of variables

$$\tau = t_{k-1} + sh_k \quad (\tau = t_{n-1} + sh_n)$$

we write the conditions (5.2) in the form

$$\begin{aligned} v_n(t_{nj}) &= q(t_{nj}) + p(t_{nj}) \left[u(t_{n-1}) + h_n \int_0^{\eta_j} v(t_{n-1} + xh_n) dx \right] \\ &\quad + \sum_{k=1}^{n-1} h_k \int_0^1 K(t_{nj}, t_{k-1} + xh_k) \left[u(t_{k-1}) + h_k \int_0^x v(t_{k-1} + sh_k) ds \right] dx \\ &\quad + h_n \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) \left[u(t_{n-1}) + h_n \int_0^x v(t_{n-1} + sh_n) ds \right] dx, \\ j &= 1, \dots, m; \quad n = 1, \dots, N \end{aligned}$$

or

$$\begin{aligned}
v_n(t_{nj}) &= q(t_{nj}) + p(t_{nj})u(t_{n-1}) + p(t_{nj})h_n \int_0^{\eta_j} v(t_{n-1} + xh_n)dx \\
&+ \sum_{k=1}^{n-1} h_k u(t_{k-1}) \int_0^1 K(t_{nj}, t_{k-1} + xh_k)dx \\
&+ \sum_{k=1}^{n-1} h_k^2 \int_0^1 K(t_{nj}, t_{k-1} + xh_k) \left(\int_0^x v(t_{k-1} + sh_k)ds \right) dx \\
&+ h_n u(t_{n-1}) \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n)dx \\
&+ h_n^2 \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) \left(\int_0^x v(t_{n-1} + sh_n)ds \right) dx, \\
j &= 1, \dots, m; \quad n = 1, \dots, N.
\end{aligned} \tag{5.3}$$

We choose a basis $\{\varphi_1, \dots, \varphi_m\}$ in the space of polynomials of order $m-1$:

$$\varphi_\ell \in \pi_{m-1}, \quad \ell = 1, \dots, m.$$

Then

$$v_n(t_{n-1} + \tau h_n) = \sum_{\ell=1}^m Y_\ell^{(n)} \varphi_\ell(\tau), \quad \tau \in (0, 1),$$

where $Y_\ell^{(n)}$ ($\ell = 1, \dots, m$) are unknown coefficients, that have to be determined. Thus

$$\int_0^x v_n(t_{n-1} + \tau h_n) d\tau = \sum_{\ell=1}^m Y_\ell^{(n)} \beta_\ell(x), \quad x \in [0, 1]$$

where

$$\beta_\ell(x) = \int_0^x \varphi_\ell(\tau) d\tau, \quad x \in [0, 1], \quad \ell = 1, \dots, m.$$

This together with (5.3) yields

$$\begin{aligned}
\sum_{\ell=1}^m Y_{\ell}^{(n)} \varphi_{\ell}(\eta_j) &= q(t_{nj}) + p(t_{nj})u(t_{n-1}) + p(t_{nj})h_n \sum_{\ell=1}^m Y_{\ell}^{(n)} \beta_{\ell}(\eta_j) \\
&+ \sum_{k=1}^{n-1} h_k u(t_{k-1}) \int_0^1 K(t_{nj}, t_{k-1} + xh_k) dx \\
&+ \sum_{k=1}^{n-1} h_k^2 \sum_{\ell=1}^m Y_{\ell}^{(k)} \int_0^1 K(t_{nj}, t_{k-1} + xh_k) \beta_{\ell}(\eta_j) dx \\
&+ h_n u(t_{n-1}) \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) dx \\
&+ h_n^2 \sum_{\ell=1}^m Y_{\ell}^{(n)} \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) \beta_{\ell}(x) dx, \\
j &= 1, \dots, m; \quad n = 1, \dots, N,
\end{aligned}$$

where

$$u(t_{n-1}) = y_0 + \sum_{k=1}^{n-1} h_k \sum_{\ell=1}^m Y_{\ell}^{(n)} \beta_{\ell}(1).$$

We have obtained a system of linear algebraic equations for finding quantities $Y_1^{(n)}, \dots, Y_m^{(n)}$ ($n = 1, \dots, N$):

$$A^{(n)} Y^{(n)} = F^{(n)}, \quad n = 1, \dots, N, \quad (5.4)$$

where

$$\begin{aligned}
A^{(n)} &= (a_{j\ell}^{(n)}), \quad j, \ell = 1, \dots, m; \quad n = 1, \dots, N, \\
Y^{(n)} &= (Y_1^{(n)}, \dots, Y_m^{(n)})^T, \quad n = 1, \dots, N, \\
F^{(n)} &= (f_1^{(n)}, \dots, f_m^{(n)})^T, \quad n = 1, \dots, N,
\end{aligned}$$

with

$$\begin{aligned}
a_{j\ell}^{(n)} &= \varphi_{\ell}(\eta_j) - p(t_{nj})h_n \beta_{\ell}(\eta_j) - h_n^2 \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) \beta_{\ell}(x) dx, \\
j, \ell &= 1, \dots, m, \quad n = 1, \dots, N
\end{aligned}$$

and

$$\begin{aligned}
f_j^{(n)} &= q(t_{nj}) + p(t_{nj})u(t_{n-1}) + \sum_{k=1}^{n-1} h_k u(t_{k-1}) \int_0^1 K(t_{nj}, t_{k-1} + xh_k) dx \\
&+ \sum_{k=1}^{n-1} h_k^2 \sum_{\ell=1}^m Y_\ell^{(k)} \int_0^1 K(t_{nj}, t_{k-1} + xh_k) \beta_\ell(\eta_j) dx \\
&+ h_n u(t_{n-1}) \int_0^{\eta_j} K(t_{nj}, t_{n-1} + xh_n) dx, \quad j = 1, \dots, m, \quad n = 1, \dots, N.
\end{aligned}$$

System (5.4) can be solved step by step:

- 1) we solve system (5.4) for $n = 1$ to find the quantities $Y_1^{(1)}, \dots, Y_m^{(1)}$ and compute $u(t_1)$;
- 2) using the known values $Y_1^{(1)}, \dots, Y_m^{(1)}, u(t_0) = y_0, u(t_1)$, we solve system (5.4) for $n = 2$ and find the quantities $Y_1^{(2)}, \dots, Y_m^{(2)}, u(t_2)$;
- 3) generally, for $i \leq N$, using the known values

$$Y_1^{(1)}, \dots, Y_m^{(1)}, \dots, Y_1^{(i-1)}, \dots, Y_m^{(i-1)}, u(t_0), u(t_1), \dots, u(t_{i-1}),$$

we find the quantities $Y_1^{(i)}, \dots, Y_m^{(i)}$ from (5.4) and compute $u(t_i)$.

The values $u(t_i)$ should be computed recursively by

$$u(t_0) = y_0, \quad u(t_i) = u(t_{i-1}) + h_i \sum_{\ell=1}^m Y_\ell^{(i)} \beta_\ell(1)$$

and stored in an array.

5.2.2. Method 2 for solving Volterra integro-differential equations

We write conditions (3.11) in the form

$$\begin{aligned}
u_n(t_{nj}) &= f_2(t_{nj}) + \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} K_2(t_{nj}, s) u(s) ds + \int_{t_{n-1}}^{t_{nj}} K_2(t_{nj}, s) u(s) ds \\
& \quad j = 1, \dots, m+1; \quad n = 1, \dots, N, \quad (5.5)
\end{aligned}$$

where

$$f_2(t_{nj}) = y_0 + \int_0^{t_{nj}} q(s)ds, \quad K_2(t_{nj}, s) = p(s) + \int_s^{t_{nj}} K(\tau, s)d\tau,$$

and $\{t_n\}$ are the nodes of Π_N (see (2.7)) and the set of collocation points $\{t_{nj}\}$ is given by (3.12).

Similarly to the Method 1, using the change of variables

$$s = t_{k-1} + xh_k \quad (s = t_{n-1} + xh_n),$$

we get from (5.5)

$$\begin{aligned} u_n(t_{nj}) &= f_2(t_{nj}) + \sum_{k=1}^{n-1} h_k \int_0^1 K_2(t_{nj}, t_{k-1} + xh_k)u(t_{k-1} + xh_k)dx \\ &\quad + h_n \int_0^{\eta_j} K_2(t_{nj}, t_{n-1} + xh_n)u(t_{n-1} + xh_n)dx \\ j &= 1, \dots, m+1; \quad n = 1, \dots, N. \end{aligned} \quad (5.6)$$

Choose a polynomial basis $\{\varphi_1, \dots, \varphi_{m+1}\}$ with

$$\varphi_\ell \in \pi_m, \quad \ell = 1, \dots, m+1.$$

Then the restriction of the collocation solution $u \in S_m^{(-1)}(\Pi_N)$ to the subinterval (t_{n-1}, t_n) ($n = 1, \dots, N$) can be written as

$$u_n(t_{n-1} + \tau h_n) = \sum_{\ell=1}^{m+1} Y_\ell^{(n)} \varphi_\ell(\tau), \quad \tau \in (0, 1),$$

where $Y_1^{(n)}, \dots, Y_{m+1}^{(n)}$ are unknown coefficients, that have to be determined from the collocation conditions (5.6). The conditions (5.6) can be written as follows

$$\begin{aligned} \sum_{\ell=1}^{m+1} Y_\ell^{(n)} \varphi_\ell(\eta_j) &= f_2(t_{nj}) + \sum_{k=1}^{n-1} h_k \sum_{\ell=1}^{m+1} Y_\ell^{(k)} \int_0^1 K_2(t_{nj}, t_{k-1} + xh_k) \varphi_\ell(x) dx \\ &\quad + h_n \sum_{\ell=1}^{m+1} Y_\ell^{(n)} \int_0^{\eta_j} K_2(t_{nj}, t_{n-1} + xh_n) \varphi_\ell(x) dx \\ j &= 1, \dots, m+1; \quad n = 1, \dots, N. \end{aligned}$$

This is a system of linear algebraic equations for finding quantities $Y_1^{(n)}, \dots, Y_{m+1}^{(n)}$ ($n = 1, \dots, N$):

$$A^{(n)}Y^{(n)} = F^{(n)}, \quad n = 1, \dots, N, \quad (5.7)$$

where

$$A^{(n)} = \left(a_{j\ell}^{(n)} \right), \quad j, \ell = 1, \dots, m+1; \quad n = 1, \dots, N,$$

$$Y^{(n)} = \left(Y_1^{(n)}, \dots, Y_{m+1}^{(n)} \right)^T, \quad n = 1, \dots, N,$$

$$F^{(n)} = \left(f_1^{(n)}, \dots, f_{m+1}^{(n)} \right)^T, \quad n = 1, \dots, N,$$

with

$$a_{j\ell}^{(n)} = \varphi_\ell(\eta_j) - h_n \int_0^{\eta_j} K_2(t_{nj}, t_{n-1} + xh_n) \varphi_\ell(x) dx,$$

$$j, \ell = 1, \dots, m+1, \quad n = 1, \dots, N$$

and

$$f_j^{(n)} = f_2(t_{nj}) + \sum_{k=1}^{n-1} h_k \sum_{\ell=1}^{m+1} Y_\ell^{(k)} \int_0^1 K_2(t_{nj}, t_{k-1} + xh_k) \varphi_\ell(x) dx,$$

$$j = 1, \dots, m+1, \quad n = 1, \dots, N.$$

Analogously to Method 1, we can solve it step by step.

5.2.3. Parallelization possibilities

Solving the systems (5.4) and (5.7) step by step, we can see that the approximate solution, found on previous subintervals, contributes to the term $F^{(n)}$, the number of addends of which is growing quite fast (quadratically in n). Most of the time is expended for calculation of this term. But the calculations of terms $F_1^{(n)}, \dots, F_m^{(n)}$ ($F_1^{(n)}, \dots, F_{m+1}^{(n)}$ in the case of Method 2) needed for finding qualities $Y_1^{(n)}, \dots, Y_m^{(n)}$ ($Y_1^{(n)}, \dots, Y_{m+1}^{(n)}$) in n th subinterval, can be done in parallel.

A possible scheme for parallelization of computation is following:

- one processor (called master) coordinates the solution process and solves the system of linear algebraic equations for each subinterval;
- other processors (called slaves) are calculating the terms $F_j^{(n)}$.

Using a parallelized version of the numerical methods, we can reduce the solution time quite significantly. For illustration, we have presented Table 5.1, of computation times in the case of a test equation of ordinary and parallel implementation of Method 1 for solving linear Volterra integro-differential equation. Parallel version of the program is written in Fortran95 with MPI (The Message Passing Interface), the experiment was performed in the computer laboratory of SUN's workstations of Tartu University.

Table 5.1: Efficiency of parallelization

N computers	512	speedup	1024	speedup	2048	speedup
1	2.7		10.8		43.3	
2	2.7	(1.0)	10.3	(1.0)	40.9	(1.1)
3	1.4	(1.9)	5.3	(2.0)	20.8	(2.1)
4	9.6E-1	(2.8)	3.6	(3.0)	14.0	(3.1)
5	7.7E-1	(3.5)	2.8	(3.9)	10.7	(4.0)
6	6.4E-1	(4.2)	2.2	(4.9)	8.5	(5.1)
7	5.6E-1	(4.8)	1.9	(5.7)	7.2	(6.0)
8	5.3E-1	(5.1)	1.7	(6.4)	6.3	(6.9)
error	4.8E-8		1.2E-8		3.0E-9	

In the first column is the number of computers, which we have used for computation. Following columns contain the time of computation in case of various numbers of subintervals and the relative speedup compared to ordinary (not parallel) program. The last row shows the error of the approximate solution. For the computation times and the errors we have used the notation $a 10^b = aEb$. We see from Table 5.1, that in case of large number of subintervals we can achieve an essential speedup. Relative speedup is almost proportional to the number of slave processors we have used.

5.2.4. Test problem of Volterra type

For numerical verification of theoretical results we consider the following linear Volterra integro-differential equation

$$y'(t) = -y(t) + q_\nu(t) + \int_0^t K_\nu(t, s)y(s)ds, \quad 0 \leq t \leq 1, \quad (5.8)$$

where

$$K_\nu(t, s) = \begin{cases} -(t-s)^{-\nu} & \text{if } \nu \neq 0, \\ -\log(t-s) & \text{if } \nu = 0 \end{cases}$$

and

$$q_\nu(t) = \begin{cases} (2-\nu)t^{1-\nu} + t^{2-\nu} + t^{3-2\nu}\gamma_1 & \text{if } \nu \neq 0, \\ \frac{1}{3}t^3(\log t)^2 - (\frac{13}{18}t^3 - t^2 - 2t)\log t + t + t^3\gamma_2 & \text{if } \nu = 0, \end{cases}$$

$$\gamma_1 = \int_0^1 (1-x)^{-\nu} x^{2-\nu} dx, \quad \gamma_2 = \int_0^1 x^2 \log x \log(1-x) dx,$$

with the initial condition $y(0) = 0$. The choice of the function q_ν corresponds to the exact solution $y(t) = t^{2-\nu}$ in the case $\nu \neq 0$ and $y(t) = t^2 \log t$ in the case $\nu = 0$. The equation (5.8) is an equation of type (3.1) with $p(t) = -1$ and for any $\nu \in (-\infty, 1)$ the assumptions $p, q_\nu \in C^{m,\nu}(0, 1]$ and $K_\nu \in \mathcal{W}^{m,\nu}(\Delta_T)$ hold with arbitrary $m \in \mathbb{N}$.

Problem (5.8) is solved numerically by Method 1 ($\{(3.6), (3.8)\}$) and by Method 2 (3.11) in the case $m = 2$ for $\nu = -\frac{1}{4}, 0, \frac{1}{2}, \frac{9}{10}$. In order to model the errors $\|u - y\|_\infty$ and $\|u' - y'\|_\infty$ the quantities

$$\varepsilon_N = \{\max |u(\tau_{jk}) - y(\tau_{jk})| : k = 1, \dots, 9; j = 1, \dots, N\}$$

and

$$\varepsilon'_N = \{\max |u'(\tau_{jk}) - y'(\tau_{jk})| : k = 1, \dots, 9; j = 1, \dots, N\}$$

with

$$\tau_{jk} = t_{j-1} + k \frac{t_j - t_{j-1}}{10}, \quad k = 1, \dots, 9, \quad j = 1, \dots, N$$

are used. The ratios of the actual errors $\varrho_N = \frac{\varepsilon_{N/2}}{\varepsilon_N}$ and $\varrho'_N = \frac{\varepsilon'_{N/2}}{\varepsilon'_N}$ are presented in the columns with headings in the form $\varrho(x_r)$ and $\varrho'(x'_r)$, where x_r and x'_r are real numbers (that are independent of N) corresponding to similar ratios of the error estimates. In order to save space we have presented numerical results only for $N = 4, 32, 256, 1024$ although the computations were performed for all values $N = 2^j$, $j = 1, \dots, 10$.

5.2.5. Numerical results for Method 1

Let us first consider Method 1 with collocation parameters $\eta_1 = \frac{1}{4}$ and $\eta_2 = \frac{3}{4}$. Since these parameters do not satisfy the assumptions of Theorem 3.4.1,

we get from Theorem 3.2.4 in the case of our test equation the following error estimates:

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{2}{2-\nu}, \\ N^{-2}(1 + \log N) & \text{for } r = \frac{2}{2-\nu}, \\ N^{-2} & \text{for } r > \frac{2}{2-\nu} \end{cases} \quad (5.9)$$

and

$$\|u' - y'\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r < \frac{2}{1-\nu}, \\ N^{-2}(1 + \log N) & \text{for } r = \frac{2}{1-\nu} = 1, \\ N^{-2} & \text{for } r > \frac{2}{1-\nu} \text{ or } r = \frac{2}{1-\nu} > 1. \end{cases} \quad (5.10)$$

Therefore the ratios of the error estimates (for $N = \frac{N'}{2}$ and $N = N'$) are as follows:

$$x_r = \begin{cases} 2^{r(2-\nu)} & \text{for } 1 \leq r < \frac{2}{2-\nu}, \\ 4 & \text{for } r > \frac{2}{2-\nu} \end{cases}$$

and

$$x'_r = \begin{cases} 2^{r(1-\nu)} & \text{for } 1 \leq r < \frac{2}{1-\nu}, \\ 4 & \text{for } r \geq \frac{2}{1-\nu}. \end{cases}$$

As we can see from Table 5.2, the observed errors of u' behave exactly according to the right-hand side of the estimate (5.10) starting from $N = 256$ (in most cases even from $N = 32$). For example for $r = 1$ we have $x'_1 \approx 2.4$ if $\nu = -\frac{1}{4}$, $x'_1 = 2$ if $\nu = 0$, $x'_1 \approx 1.4$ if $\nu = \frac{1}{2}$, $x'_1 \approx 1.1$ if $\nu = \frac{9}{10}$, which are exactly equal to the observed ratios of actual errors. We may conclude, that in the case of the test equations, the estimate (5.10) corresponds to the leading term of the error $\|u' - y'\|_\infty$, which is dominating even for small values of N .

The observed errors of u are also in good agreement with the theoretical estimates of Theorem 3.2.4 except in cases when r is close to the value $\frac{2}{2-\nu}$, after which the maximal theoretical convergence rate is achieved. If r is close to the critical value then the observed convergence rate is smaller than the one predicted by the error estimate (5.9) but converges slowly to the theoretical value. To get a better picture on what happens near this value of r ($r = 1.333$) in the case $\nu = \frac{1}{2}$, Table 5.3 is presented. This table shows the dependence of the convergence rate on the nonuniformity parameter r , when r is increasing by steps of 0.1.

In the proof of Theorem 3.2.4 (see [15]) it is actually shown that

$$\|u - y\|_\infty \leq cN^{-r(2-\nu)} \sum_{l=1}^N l^{r(2-\nu)-m-1}, \quad (5.11)$$

Table 5.2: Method 1 $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$

$\nu = -\frac{1}{4}$	r=1		r=1.2		r=1.4		r=1.6	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	7.9E-4	4.1	6.8E-4	4.6	7.5E-4	4.4	8.6E-4	4.3
32	1.3E-5	3.9	8.9E-6	4.1	9.4E-6	4.2	1.1E-5	4.2
256	2.2E-7	3.9	1.4E-7	4.0	1.4E-7	4.0	1.6E-7	4.0
1024	1.4E-8	3.9	8.5E-9	4.0	8.8E-9	4.0	1.0E-8	4.0
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.4)$	ε'_N	$\varrho'(4.0)$
4	1.3E-2	2.2	9.6E-3	2.6	6.9E-3	3.2	5.5E-3	3.5
32	1.0E-3	2.4	4.4E-4	2.8	1.9E-4	3.4	9.2E-5	4.0
256	7.9E-5	2.4	2.0E-5	2.8	4.9E-6	3.4	1.4E-6	4.0
1024	1.4E-5	2.4	2.5E-6	2.8	4.3E-7	3.4	9.0E-8	4.0
$\nu = 0$	r=1		r=1.333		r=1.667		r=2	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	4.7E-3	3.5	3.7E-3	4.0	3.9E-3	4.1	4.6E-3	4.0
32	1.1E-4	3.6	5.9E-5	4.0	5.9E-5	4.0	6.9E-5	4.0
256	2.2E-6	3.7	9.2E-7	4.0	9.2E-7	4.0	1.1E-6	4.0
1024	1.6E-7	3.7	5.8E-8	4.0	5.7E-8	4.0	6.6E-8	4.0
N	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(2.5)$	ε'_N	$\varrho'(3.2)$	ε'_N	$\varrho'(4.0)$
4	7.3E-2	1.9	4.7E-2	2.4	3.0E-2	3.0	2.6E-2	3.6
32	9.6E-3	2.0	3.1E-3	2.5	9.6E-4	3.2	4.2E-4	4.0
256	1.2E-3	2.0	1.9E-4	2.5	3.0E-5	3.2	6.6E-6	4.0
1024	3.0E-4	2.0	3.0E-5	2.5	3.0E-6	3.2	4.1E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.2		r=1.4		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(3.5)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	3.2E-3	2.6	2.2E-3	3.2	1.4E-3	3.9	3.0E-3	3.8
32	1.5E-4	2.8	6.0E-5	3.2	3.0E-5	3.6	2.9E-5	4.6
256	7.0E-6	2.7	1.7E-6	3.3	5.9E-7	3.7	3.8E-7	4.1
1024	9.1E-7	2.8	1.6E-7	3.3	4.2E-8	3.8	2.3E-8	4.0
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.5)$	ε'_N	$\varrho'(1.6)$	ε'_N	$\varrho'(4.0)$
4	5.0E-2	1.3	4.5E-2	1.4	4.0E-2	1.5	1.9E-2	2.6
32	1.9E-2	1.4	1.4E-2	1.5	9.8E-3	1.6	3.3E-4	4.0
256	6.9E-3	1.4	4.0E-3	1.5	2.3E-3	1.6	5.2E-6	4.0
1024	3.5E-3	1.4	1.7E-3	1.5	8.7E-4	1.6	3.2E-7	4.0
$\nu = \frac{9}{10}$	r=1		r=1.455		r=1.909		r=20	
N	ε_N	$\varrho(2.1)$	ε_N	$\varrho(3.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	2.1E-3	1.7	1.3E-3	2.2	7.1E-4	3.1	5.7E-3	1.1
32	3.1E-4	2.0	5.7E-5	2.9	1.6E-5	3.6	1.4E-4	4.7
256	3.3E-5	2.1	2.4E-6	2.9	3.4E-7	3.6	9.3E-7	5.3
1024	7.1E-6	2.1	2.8E-7	2.9	2.5E-8	3.7	4.3E-8	4.4
N	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(4.0)$
4	2.3E-2	1.5	2.9E-2	0.9	3.3E-2	0.7	5.7E-2	1.0
32	3.3E-2	1.0	3.0E-2	1.1	2.6E-2	1.1	3.0E-3	4.1
256	2.9E-2	1.1	2.3E-2	1.1	1.8E-2	1.1	3.5E-5	4.0
1024	2.5E-2	1.1	1.8E-2	1.1	1.3E-2	1.1	2.2E-6	4.0

Table 5.3: Method 1 $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$ and $\nu = \frac{1}{2}$

N	r=1		r=1.1		r=1.2		r=1.3	
	ε_N	$\varrho(2.8)$	ε_N	$\varrho(3.1)$	ε_N	$\varrho(3.5)$	ε_N	$\varrho(3.9)$
4	3.2E-3	2.6	2.6E-3	2.9	2.2E-3	3.2	1.8E-3	3.5
32	1.5E-4	2.8	9.1E-5	3.1	6.0E-5	3.2	4.1E-5	3.4
256	7.0E-6	2.7	3.4E-6	3.0	1.7E-6	3.3	9.7E-7	3.5
1024	9.1E-7	2.8	3.7E-7	3.0	1.6E-7	3.3	7.7E-8	3.6
N	r=1.4		r=1.5		r=1.6		r=1.7	
	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	1.4E-3	3.9	1.3E-3	3.9	1.2E-3	3.9	1.1E-3	3.9
32	3.0E-5	3.6	2.3E-5	3.8	1.8E-5	3.9	1.6E-5	4.0
256	5.9E-7	3.7	4.0E-7	3.9	3.0E-7	3.9	2.5E-7	4.0
1024	4.2E-8	3.8	2.7E-8	3.9	1.9E-8	4.0	1.6E-8	4.0

which is asymptotically equivalent to (5.9). From the Table 5.4 we see, that the ratios of the right-hand side of (5.11) behave similarly to the observed convergence rate, which explains the slow convergence of the observed rate to the theoretical one.

Next we consider the case $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{6}$, when the corresponding quadrature formula is exact for all polynomials up to order 2. In this case the assumptions of Theorem 3.4.1 hold. This gives us the error estimate

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{3}{2-\nu}, \\ N^{-3}(1 + \log N) & \text{for } r = \frac{3}{2-\nu}, \\ N^{-3} & \text{for } r > \frac{3}{2-\nu}, \end{cases}$$

the estimate for the derivative is again (5.10). The ratios of the error

Table 5.4: The ratios of the right-hand side of (5.11) for $\nu = \frac{1}{2}$

N	r = 1.1	r = 1.2	r = 1.3	r = 1.4	r = 1.5	r = 1.6
4	2.5	2.6	2.8	3.0	3.2	3.4
8	2.6	2.8	3.0	3.2	3.3	3.5
16	2.8	3.0	3.2	3.3	3.5	3.6
32	2.9	3.1	3.3	3.4	3.6	3.7
64	2.9	3.2	3.4	3.5	3.7	3.8
128	3.0	3.2	3.4	3.6	3.7	3.8
256	3.0	3.3	3.5	3.7	3.8	3.9
512	3.1	3.3	3.5	3.7	3.8	3.9
1024	3.1	3.3	3.6	3.7	3.9	3.9

Table 5.5: Method 1 $\eta_1 = \frac{1}{4}, \eta_2 = \frac{5}{6}$

$\nu = -\frac{1}{4}$	r=1		r=1.189		r=1.378		r=1.6	
N	ε_N	$\varrho(4.8)$	ε_N	$\varrho(6.4)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	7.8E-4	4.5	4.4E-4	6.0	3.4E-4	7.1	4.2E-4	7.5
32	7.7E-6	4.7	1.8E-6	6.3	7.3E-7	7.8	8.4E-7	8.0
256	7.2E-8	4.8	6.8E-9	6.4	1.5E-9	7.8	1.6E-9	8.0
1024	3.2E-9	4.8	1.7E-10	6.4	2.5E-11	7.9	2.6E-11	8.0
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.3)$	ε'_N	$\varrho'(4.0)$
4	1.4E-2	2.2	1.1E-2	2.6	7.7E-3	3.1	6.1E-3	3.5
32	1.1E-3	2.4	5.0E-4	2.8	2.2E-4	3.3	1.0E-4	4.0
256	8.5E-5	2.4	2.3E-5	2.8	6.2E-6	3.3	1.6E-6	4.0
1024	1.5E-5	2.4	2.9E-6	2.8	5.7E-7	3.3	1.0E-7	4.0
$\nu = 0$	r=1		r=1.275		r=1.550		r=2	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(5.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	4.4E-3	3.8	2.1E-3	5.6	1.7E-3	7.3	2.7E-3	6.6
32	7.2E-5	4.0	1.1E-5	5.8	3.7E-6	7.7	5.8E-6	7.9
256	1.1E-6	4.0	5.4E-8	5.9	8.0E-9	7.8	1.1E-8	8.0
1024	7.1E-8	4.0	1.6E-9	5.9	1.3E-10	7.8	1.8E-10	8.0
N	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.9)$	ε'_N	$\varrho'(4.0)$
4	7.8E-2	1.9	5.4E-2	2.3	3.8E-2	2.8	2.8E-2	3.7
32	1.0E-2	2.0	4.0E-3	2.4	1.5E-3	2.9	4.6E-4	4.0
256	1.3E-3	2.0	2.8E-4	2.4	6.2E-5	2.9	7.2E-6	4.0
1024	3.2E-4	2.0	4.8E-5	2.4	7.2E-6	2.9	4.5E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.533		r=2.067		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(4.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	3.3E-3	2.6	1.1E-3	4.5	7.7E-4	6.0	2.6E-3	4.5
32	1.6E-4	2.8	9.9E-6	4.9	2.0E-6	7.5	6.7E-6	7.8
256	7.0E-6	2.8	8.4E-8	4.9	4.6E-9	7.6	1.3E-8	8.0
1024	8.8E-7	2.8	3.4E-9	4.9	7.9E-11	7.7	2.1E-10	8.1
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.7)$	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(4.0)$
4	5.3E-2	1.3	3.9E-2	1.5	2.7E-2	1.9	2.0E-2	2.5
32	2.0E-2	1.4	8.2E-3	1.7	3.2E-3	2.0	3.6E-4	4.0
256	7.3E-3	1.4	1.7E-3	1.7	3.8E-4	2.0	5.6E-6	4.0
1024	3.6E-3	1.4	5.7E-4	1.7	9.0E-5	2.0	3.5E-7	4.0
$\nu = \frac{9}{10}$	r=1		r=1.909		r=2.818		r=20	
N	ε_N	$\varrho(2.1)$	ε_N	$\varrho(4.3)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	2.2E-3	1.7	7.3E-4	3.1	4.5E-4	4.7	5.9E-3	1.1
32	3.2E-4	2.0	1.2E-5	4.1	1.7E-6	7.1	1.6E-4	5.7
256	3.3E-5	2.1	1.5E-7	4.3	4.1E-9	7.5	3.9E-7	7.9
1024	7.3E-6	2.1	8.1E-9	4.3	7.2E-11	7.6	6.0E-9	8.0
N	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.2)$	ε'_N	$\varrho'(4.0)$
4	2.5E-2	1.2	3.4E-2	0.7	3.4E-2	0.9	4.6E-2	1.0
32	3.5E-2	1.0	2.7E-2	1.1	2.0E-2	1.2	2.4E-3	3.9
256	3.0E-2	1.1	1.8E-2	1.1	1.1E-2	1.2	3.7E-5	4.0
1024	2.6E-2	1.1	1.4E-2	1.1	7.4E-3	1.2	2.3E-6	4.0

Table 5.6: Method 1 $\eta_1 = \frac{3-\sqrt{3}}{6}, \eta_2 = \frac{3+\sqrt{3}}{6}$

$\nu = -\frac{1}{4}$	r=1		r=1.189		r=1.378		r=1.6	
N	ε_N	$\varrho(4.8)$	ε_N	$\varrho(6.4)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	5.9E-4	4.5	3.3E-4	6.0	2.4E-4	7.5	3.4E-4	7.1
32	5.8E-6	4.7	1.3E-6	6.4	4.7E-7	8.0	7.0E-7	7.9
256	5.5E-8	4.8	5.2E-9	6.4	9.1E-10	8.0	1.4E-9	8.0
1024	2.4E-9	4.8	1.3E-10	6.4	1.4E-11	8.0	2.1E-11	8.0
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.3)$	ε'_N	$\varrho'(4.0)$
4	1.1E-2	2.2	7.8E-3	2.6	5.7E-3	3.1	4.7E-3	3.9
32	8.4E-4	2.4	3.7E-4	2.8	1.6E-4	3.3	7.3E-5	4.0
256	6.3E-5	2.4	1.7E-5	2.8	4.6E-6	3.3	1.1E-6	4.0
1024	1.1E-5	2.4	2.2E-6	2.8	4.2E-7	3.3	7.2E-8	4.0
$\nu = 0$	r=1		r=1.275		r=1.550		r=2	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(5.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	3.4E-3	3.8	1.6E-3	5.6	1.1E-3	7.7	1.9E-3	6.9
32	5.6E-5	4.0	8.4E-6	5.8	2.2E-6	8.0	3.9E-6	8.0
256	8.8E-7	4.0	4.2E-8	5.9	4.2E-9	8.1	7.6E-9	8.0
1024	5.5E-8	4.0	1.2E-9	5.9	6.4E-11	8.1	1.2E-10	8.0
N	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.9)$	ε'_N	$\varrho'(4.0)$
4	5.9E-2	1.9	4.1E-2	2.3	2.8E-2	2.8	2.0E-2	3.7
32	7.8E-3	2.0	3.0E-3	2.4	1.2E-3	2.9	3.4E-4	4.0
256	9.8E-4	2.0	2.1E-4	2.4	4.6E-5	2.9	5.2E-6	4.0
1024	2.4E-4	2.0	3.6E-5	2.4	5.4E-6	2.9	3.3E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.533		r=2.067		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(4.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	2.7E-3	2.6	9.4E-4	4.6	5.7E-4	6.2	1.9E-3	4.8
32	1.3E-4	2.8	8.2E-6	4.9	1.2E-6	8.1	5.6E-6	7.7
256	5.8E-6	2.8	6.8E-8	4.9	2.1E-9	8.2	1.1E-8	8.0
1024	7.2E-7	2.8	2.8E-9	4.9	3.1E-11	8.2	1.8E-10	8.0
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.7)$	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(4.0)$
4	4.1E-2	1.2	3.0E-2	1.5	2.1E-2	1.9	1.5E-2	3.0
32	1.6E-2	1.4	6.4E-3	1.7	2.5E-3	2.0	2.6E-4	4.0
256	5.7E-3	1.4	1.3E-3	1.7	3.0E-4	2.0	4.1E-6	4.0
1024	2.9E-3	1.4	4.5E-4	1.7	7.1E-5	2.0	2.6E-7	4.0
$\nu = \frac{9}{10}$	r=1		r=1.909		r=2.818		r=20	
N	ε_N	$\varrho(2.1)$	ε_N	$\varrho(4.3)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	1.9E-3	1.8	6.3E-4	3.2	3.6E-4	6.0	6.2E-3	1.1
32	2.7E-4	2.0	8.9E-6	4.3	9.8E-7	7.8	1.6E-4	5.6
256	2.9E-5	2.1	1.1E-7	4.3	1.8E-9	8.2	3.7E-7	7.7
1024	6.3E-6	2.1	6.2E-9	4.3	2.6E-11	8.3	5.9E-9	7.9
N	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.2)$	ε'_N	$\varrho'(4.0)$
4	2.5E-2	1.4	2.7E-2	0.9	2.7E-2	0.9	5.9E-2	1.0
32	2.8E-2	1.0	2.2E-2	1.1	1.6E-2	1.2	2.8E-3	4.2
256	2.4E-2	1.1	1.5E-2	1.1	8.9E-3	1.2	2.8E-5	4.3
1024	2.1E-2	1.1	1.1E-2	1.1	6.0E-3	1.2	1.7E-6	4.0

estimates are

$$x_r = \begin{cases} 2^{r(2-\nu)} & \text{for } 1 \leq r < \frac{3}{2-\nu}, \\ 8 & \text{for } r > \frac{3}{2-\nu}. \end{cases}$$

Corresponding numerical results are presented in Table 5.5. We can see that the convergence rate is much better in this case (compared with Table 5.2) and agrees well with the estimate of Theorem 3.4.1.

Finally, in order to show that it is not possible to get any further speedup of the convergence of u to y we present the numerical results in the case of Gaussian parameters $(\eta_1 = \frac{3-\sqrt{3}}{6}, \eta_2 = \frac{3+\sqrt{3}}{6})$, that are exact for polynomials of order $2m - 1 = 3$. As we can see from Table 5.6, we do not get any further improvement in the convergence rate, although the actual errors are slightly smaller due to a smaller error coefficient.

5.2.6. Numerical results for Method 2

Numerical results in case of Method 2 are presented in Tables 5.7 - 5.10. The Table 5.7 shows the errors in the supremum norm of the approximate solution and its derivative in case of collocation parameters that are nodes of quadrature formula that is exact for polynomials of order 2. In this case the theoretical error estimates from Theorem 3.5.2 are

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{3}{2-\nu}, \\ N^{-3}(1 + \log N) & \text{for } r = \frac{3}{2-\nu} = 1, \\ N^{-3} & \text{for } r > \frac{3}{2-\nu} \text{ or } r = \frac{3}{2-\nu} > 1, \end{cases}$$

$$\|u' - y'\|_\infty \leq c N^{-(1-\nu)} \quad \text{for } r = 1$$

and

$$\|u' - y'\|_{\varepsilon, \infty} \leq c_\varepsilon \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r \leq \frac{2}{1-\nu}, \\ N^{-2} & \text{for } r > \frac{2}{1-\nu}. \end{cases}$$

As we can see, the agreement with theoretical estimates is very good for all values of r and ν and we may conclude, that in case of the test equation, the error estimates of Theorem 3.5.2 correspond to the leading term of the errors $\|u - y\|_\infty$ and $\|u' - y'\|_\infty$, which are dominating even for small values of N . Moreover, it seems that the estimate for the derivative of the error,

$$\|u' - y'\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r \leq \frac{m}{1-\nu}, \\ N^{-m} & \text{for } r > \frac{m}{1-\nu} \end{cases}$$

may be valid.

Table 5.7: Method 2 $\eta_1 = \frac{1}{6}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5}{6}$

$\nu = -\frac{1}{4}$	r=1		r=1.189		r=1.378		r=1.6	
N	ε_N	$\varrho(4.8)$	ε_N	$\varrho(6.4)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	2.0E-4	4.4	1.1E-4	5.9	9.8E-5	8.1	1.4E-4	7.4
32	2.0E-6	4.7	4.6E-7	6.3	1.8E-7	8.1	2.9E-7	8.0
256	1.9E-8	4.8	1.8E-9	6.4	3.6E-10	8.0	5.6E-10	8.0
1024	8.3E-10	4.8	4.4E-11	6.4	5.6E-12	8.0	8.7E-12	8.0
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.3)$	ε'_N	$\varrho'(4.0)$
4	1.7E-2	2.3	1.2E-2	2.8	8.8E-3	3.3	7.2E-3	3.9
32	1.3E-3	2.4	5.6E-4	2.8	2.5E-4	3.3	1.1E-4	4.0
256	9.4E-5	2.4	2.6E-5	2.8	6.9E-6	3.3	1.8E-6	4.0
1024	1.7E-5	2.4	3.3E-6	2.8	6.3E-7	3.3	1.1E-7	4.0
$\nu = 0$	r=1		r=1.275		r=1.550		r=2	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(5.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	1.1E-3	3.8	5.1E-4	5.5	3.8E-4	7.5	7.1E-4	6.3
32	1.8E-5	4.0	2.7E-6	5.8	7.4E-7	8.0	1.6E-6	7.9
256	2.8E-7	4.0	1.3E-8	5.9	1.4E-9	8.0	3.1E-9	8.0
1024	1.8E-8	4.0	3.9E-10	5.9	2.3E-11	8.0	4.9E-11	8.0
N	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.9)$	ε'_N	$\varrho'(4.0)$
4	9.1E-2	2.0	6.2E-2	2.4	4.3E-2	2.9	3.3E-2	3.9
32	1.1E-2	2.0	4.4E-3	2.4	1.7E-3	2.9	5.2E-4	4.0
256	1.4E-3	2.0	3.1E-4	2.4	6.8E-5	2.9	8.1E-6	4.0
1024	3.6E-4	2.0	5.3E-5	2.4	7.9E-6	2.9	5.1E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.533		r=2.067		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(4.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	6.8E-4	2.5	2.4E-4	4.4	1.8E-4	7.3	8.4E-4	3.8
32	3.3E-5	2.8	2.1E-6	4.9	3.3E-7	8.1	2.3E-6	7.8
256	1.5E-6	2.8	1.8E-8	4.9	6.4E-10	8.0	4.7E-9	8.0
1024	1.9E-7	2.8	7.3E-10	4.9	1.0E-11	8.0	7.3E-11	8.0
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.7)$	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(4.0)$
4	6.3E-2	1.4	4.4E-2	1.7	3.0E-2	2.0	2.5E-2	3.3
32	2.2E-2	1.4	8.9E-3	1.7	3.5E-3	2.0	4.0E-4	4.0
256	7.9E-3	1.4	1.8E-3	1.7	4.1E-4	2.0	6.2E-6	4.0
1024	4.0E-3	1.4	6.2E-4	1.7	9.8E-5	2.0	3.9E-7	4.0
$\nu = \frac{9}{10}$	r=1		r=1.909		r=2.818		r=20	
N	ε_N	$\varrho(2.1)$	ε_N	$\varrho(4.3)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	4.2E-4	2.3	1.2E-4	5.2	1.7E-4	7.9	3.5E-3	1.0
32	5.3E-5	2.0	1.8E-6	4.3	2.4E-7	8.6	8.8E-5	6.6
256	5.8E-6	2.1	2.3E-8	4.3	4.3E-10	8.1	1.6E-7	8.2
1024	1.3E-6	2.1	1.2E-9	4.3	6.8E-12	7.9	2.4E-9	8.1
N	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.1)$	ε'_N	$\varrho'(1.2)$	ε'_N	$\varrho'(4.0)$
4	4.5E-2	0.9	4.2E-2	1.1	3.8E-2	1.2	3.6E-2	1.0
32	4.0E-2	1.1	2.9E-2	1.1	2.1E-2	1.2	2.7E-3	3.9
256	3.2E-2	1.1	2.0E-2	1.1	1.2E-2	1.2	4.2E-5	4.0
1024	2.8E-2	1.1	1.5E-2	1.1	8.0E-3	1.2	2.6E-6	4.0

We have also presented Table 5.8, showing the numerical results in the case, where the Gaussian collocation parameters are used. We can see that the agreement with the theoretical estimates is again very good, and there is no further improvement in the convergence rate.

Numerical results about superconvergence at the collocation points are presented in Tables 5.9-5.10. The errors at the collocation points are denoted by

$$\delta_N = \{\max |u(t_{jk}) - y(t_{jk})| : k = 1, \dots, m + 1; j = 1, \dots, N\}.$$

Table 5.9 corresponds to the case, where the collocation parameters are the knodes of quadrature formula that is exact for polynomials of order 3. According to Theorem 3.6.1 the theoretical error estimate in this case is

$$\delta_N \leq c \begin{cases} N^{-r(3-\nu)} & \text{for } 1 \leq r < \frac{4}{3-\nu}, \\ N^{-4}(1 + \log N) & \text{for } r = \frac{4}{3-\nu}, \\ N^{-4} & \text{for } r > \frac{4}{3-\nu}. \end{cases} \quad (5.12)$$

As we can see, the numerical results are in good accordance with the estimate (5.12), if r is not very close to the value $\frac{m+2}{3-\nu}$ ($\frac{m+2}{3-\nu} = 1.23$ if $\nu = -\frac{1}{4}$, $\frac{m+2}{3-\nu} = 1.33$ if $\nu = 0$, $\frac{m+2}{3-\nu} = 1.6$ if $\nu = \frac{1}{2}$ and $\frac{m+2}{3-\nu} = 1.9$ if $\nu = \frac{9}{10}$). Similarly to the Table 5.2, if r is close to the critical value, after which the maximal convergence rate is achieved, the observed convergence rate is smaller than the predicted one, but approaches slowly to the predicted theoretical value.

We have also presented Table 5.10, which corresponds to the Gaussian collocation parameters. As we can see, the numerical experiments in this case are also in good agreement with the error estimate (5.12) and no further improvement in the convergence rate can be observed.

5.2.7. Comparison of Methods 1 and 2

Theoretical estimates and numerical experiments show that, in terms of uniform convergence, Method 2, with arbitrary collocation parameters, for computing approximate solution, as well as an approximation for the derivative of the solution, is equivalent to the Method 1, if the conditions of the Theorem 3.4.1 are satisfied (see Tables 5.5 and 5.7).

An advantage of Method 2 is, that if we use a special choice of collocation parameters, it is possible to obtain faster convergence (superconvergence) at the collocation points.

Table 5.8: Method 2 $\eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$

$\nu = -\frac{1}{4}$	r=1	r=1.189	r=1.378	r=1.6
N	ε_N $\varrho(4.8)$	ε_N $\varrho(6.4)$	ε_N $\varrho(8.0)$	ε_N $\varrho(8.0)$
4	2.2E-4 4.7	1.2E-4 6.3	1.0E-4 7.7	1.5E-4 7.1
32	2.0E-6 4.7	4.7E-7 6.4	2.1E-7 8.0	3.2E-7 7.9
256	1.9E-8 4.8	1.8E-9 6.4	4.0E-10 8.0	6.3E-10 8.0
1024	8.4E-10 4.8	4.4E-11 6.4	6.3E-12 8.0	9.8E-12 8.0
N	ε'_N $\varrho'(2.4)$	ε'_N $\varrho'(2.8)$	ε'_N $\varrho'(3.3)$	ε'_N $\varrho'(4.0)$
4	1.5E-2 2.3	1.1E-2 2.7	7.8E-3 3.2	6.4E-3 3.9
32	1.1E-3 2.4	5.0E-4 2.8	2.2E-4 3.3	1.0E-4 4.0
256	8.4E-5 2.4	2.3E-5 2.8	6.1E-6 3.3	1.6E-6 4.0
1024	1.5E-5 2.4	2.9E-6 2.8	5.6E-7 3.3	9.9E-8 4.0
$\nu = 0$	r=1	r=1.275	r=1.550	r=2
N	ε_N $\varrho(4.0)$	ε_N $\varrho(5.9)$	ε_N $\varrho(8.0)$	ε_N $\varrho(8.0)$
4	1.1E-3 4.0	5.2E-4 5.8	4.2E-4 7.7	7.9E-4 6.6
32	1.7E-5 4.0	2.6E-6 5.9	8.3E-7 8.0	1.8E-6 7.9
256	2.7E-7 4.0	1.3E-8 5.9	1.6E-9 8.0	3.5E-9 8.0
1024	1.7E-8 4.0	3.8E-10 5.9	2.5E-11 8.0	5.5E-11 8.0
N	ε'_N $\varrho'(2.0)$	ε'_N $\varrho'(2.4)$	ε'_N $\varrho'(2.9)$	ε'_N $\varrho'(4.0)$
4	8.0E-2 2.0	5.5E-2 2.4	3.8E-2 2.9	2.9E-2 3.9
32	1.0E-2 2.0	3.9E-3 2.4	1.5E-3 2.9	4.6E-4 4.0
256	1.3E-3 2.0	2.8E-4 2.4	6.0E-5 2.9	7.2E-6 4.0
1024	3.2E-4 2.0	4.7E-5 2.4	7.0E-6 2.9	4.5E-7 4.0
$\nu = \frac{1}{2}$	r=1	r=1.533	r=2.067	r=4
N	ε_N $\varrho(2.8)$	ε_N $\varrho(4.9)$	ε_N $\varrho(8.0)$	ε_N $\varrho(8.0)$
4	6.8E-4 2.8	2.2E-4 4.9	1.8E-4 7.4	8.7E-4 4.2
32	3.0E-5 2.8	1.9E-6 4.9	3.7E-7 8.0	2.6E-6 7.6
256	1.3E-6 2.8	1.6E-8 4.9	7.2E-10 8.0	5.2E-9 8.0
1024	1.6E-7 2.8	6.4E-10 4.9	1.1E-11 8.0	8.2E-11 8.0
N	ε'_N $\varrho'(1.4)$	ε'_N $\varrho'(1.7)$	ε'_N $\varrho'(2.0)$	ε'_N $\varrho'(4.0)$
4	5.5E-2 1.4	3.8E-2 1.7	2.7E-2 2.0	2.2E-2 3.2
32	2.0E-2 1.4	7.8E-3 1.7	3.1E-3 2.0	3.5E-4 4.0
256	7.0E-3 1.4	1.6E-3 1.7	3.6E-4 2.0	5.5E-6 4.0
1024	3.5E-3 1.4	5.5E-4 1.7	8.7E-5 2.0	3.4E-7 4.0
$\nu = \frac{9}{10}$	r=1	r=1.909	r=2.818	r=20
N	ε_N $\varrho(2.1)$	ε_N $\varrho(4.3)$	ε_N $\varrho(8.0)$	ε_N $\varrho(8.0)$
4	5.2E-4 2.2	1.2E-4 4.7	1.3E-4 8.8	2.9E-3 1.0
32	4.8E-5 2.2	1.5E-6 4.3	2.6E-7 8.2	6.8E-5 6.5
256	4.8E-6 2.1	1.9E-8 4.3	4.9E-10 8.0	1.7E-7 7.9
1024	1.0E-6 2.1	1.0E-9 4.3	7.6E-12 8.0	2.7E-9 8.0
N	ε'_N $\varrho'(1.1)$	ε'_N $\varrho'(1.1)$	ε'_N $\varrho'(1.2)$	ε'_N $\varrho'(4.0)$
4	3.8E-2 0.9	3.7E-2 1.0	3.3E-2 1.2	3.8E-2 1.0
32	3.5E-2 1.1	2.6E-2 1.1	1.9E-2 1.2	2.5E-3 3.8
256	2.8E-2 1.1	1.7E-2 1.1	1.0E-2 1.2	3.7E-5 4.0
1024	2.5E-2 1.1	1.3E-2 1.1	7.0E-3 1.2	2.3E-6 4.0

Table 5.9: Method 2 $\eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{3}{4}$

$\nu = -\frac{1}{4}$	r=1	r=1.189	r=1.378	r=1.6
N	δ_N $\varrho(9.5)$	δ_N $\varrho(14.6)$	δ_N $\varrho(16.0)$	δ_N $\varrho(16.0)$
4	2.3E-5 9.1	1.1E-5 12.6	1.3E-5 14.9	2.1E-5 13.2
32	2.8E-8 9.4	4.1E-9 14.2	3.4E-9 15.8	5.7E-9 15.8
256	3.3E-11 9.5	1.4E-12 14.3	8.5E-13 15.9	1.4E-12 16.0
1024	3.7E-13 9.5	6.9E-15 14.4	3.1E-15 16.8	4.4E-15 19.4
$\nu = 0$	r=1	r=1.275	r=1.550	r=2
N	δ_N $\varrho(8.0)$	δ_N $\varrho(14.2)$	δ_N $\varrho(16.0)$	δ_N $\varrho(16.0)$
4	9.9E-5 6.4	3.4E-5 11.3	2.9E-5 15.3	7.1E-5 12.2
32	2.4E-7 7.7	1.8E-8 13.0	7.0E-9 16.1	1.6E-8 16.5
256	4.9E-10 8.0	7.0E-12 13.7	1.7E-12 16.0	3.8E-12 16.1
1024	7.8E-12 8.0	3.7E-14 13.8	6.7E-15 15.9	1.5E-14 16.0
$\nu = \frac{1}{2}$	r=1	r=1.533	r=2.067	r=4
N	δ_N $\varrho(5.7)$	δ_N $\varrho(14.2)$	δ_N $\varrho(16.0)$	δ_N $\varrho(16.0)$
4	1.2E-4 5.4	3.1E-5 11.8	5.1E-5 13.2	3.7E-4 5.8
32	6.0E-7 6.0	1.5E-8 13.1	1.2E-8 16.0	1.6E-7 15.0
256	2.9E-9 5.9	6.2E-12 13.6	3.1E-12 16.0	4.1E-11 15.9
1024	8.6E-11 5.8	3.3E-14 13.7	1.1E-14 16.5	1.6E-13 16.3
$\nu = \frac{9}{10}$	r=1	r=1.909	r=2.818	r=20
N	δ_N $\varrho(4.3)$	δ_N $\varrho(16.0)$	δ_N $\varrho(16.0)$	δ_N $\varrho(16.0)$
4	3.3E-4 2.7	5.4E-5 6.9	7.0E-5 9.4	1.9E-3 1.0
32	5.6E-6 4.3	3.0E-8 13.1	1.7E-8 16.4	3.3E-5 8.0
256	6.7E-8 4.5	1.2E-11 14.0	4.0E-12 16.0	9.9E-9 16.7
1024	3.3E-9 4.5	5.6E-14 14.4	5.5E-14 4.5	3.9E-11 15.6

Considering that in the case of Method 2 with $u \in S_m^{(-1)}(\Pi_N)$, the complexity of implementation and computation time are comparable to those of Method 1 with $v \in S_m^{(-1)}(\Pi_N)$ ($u \in S_{m+1}^{(0)}(\Pi_N)$), Method 1 seems to be preferable to Method 2, if the assumptions of Theorem 3.4.1 hold.

5.3. Numerical solution of Fredholm integro-differential equations

In this section we first write out matrix form of the collocation conditions (4.17) and then verify the theoretical results of Chapter 4 by numerical experiments in the case of a test equation.

Table 5.10: Method 2 $\eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$

$\nu = -\frac{1}{4}$	r=1		r=1.189		r=1.378		r=1.6	
N	δ_N	$\varrho(9.5)$	δ_N	$\varrho(14.6)$	δ_N	$\varrho(16.0)$	δ_N	$\varrho(16.0)$
4	7.3E-6	9.2	3.1E-6	14.2	4.7E-6	14.3	7.8E-6	12.6
32	8.7E-9	9.5	1.0E-9	14.5	1.2E-9	15.8	2.2E-9	15.7
256	1.0E-11	9.5	3.4E-13	14.6	3.1E-13	15.8	5.6E-13	15.9
1024	1.1E-13	9.5	4.1E-15	5.7	4.8E-15	4.7	5.9E-15	6.5
$\nu = 0$	r=1		r=1.275		r=1.550		r=2	
N	δ_N	$\varrho(8.0)$	δ_N	$\varrho(14.2)$	δ_N	$\varrho(16.0)$	δ_N	$\varrho(16.0)$
4	3.6E-5	7.1	1.2E-5	12.6	1.9E-5	12.4	4.3E-5	9.8
32	7.9E-8	7.8	4.6E-9	14.0	5.6E-9	15.6	1.5E-8	15.2
256	1.6E-10	8.0	1.6E-12	14.2	1.4E-12	16.0	3.9E-12	15.9
1024	2.5E-12	8.0	8.1E-15	14.2	5.5E-15	15.8	1.5E-14	16.0
$\nu = \frac{1}{2}$	r=1		r=1.533		r=2.067		r=4	
N	δ_N	$\varrho(5.7)$	δ_N	$\varrho(14.2)$	δ_N	$\varrho(16.0)$	δ_N	$\varrho(16.0)$
4	3.8E-5	5.8	6.9E-6	14.0	1.6E-5	12.9	1.2E-4	5.9
32	1.9E-7	5.8	2.0E-9	14.8	3.7E-9	16.4	5.1E-8	15.4
256	9.7E-10	5.7	6.7E-13	14.4	8.7E-13	16.1	1.2E-11	16.2
1024	3.0E-11	5.7	3.3E-15	14.3	5.3E-15	10.8	5.1E-14	15.0
$\nu = \frac{9}{10}$	r=1		r=1.909		r=2.818		r=20	
N	δ_N	$\varrho(4.3)$	δ_N	$\varrho(16.0)$	δ_N	$\varrho(16.0)$	δ_N	$\varrho(16.0)$
4	1.5E-4	3.5	2.4E-5	15.7	9.1E-5	10.0	2.5E-3	1.0
32	2.1E-6	4.4	4.5E-9	17.0	2.1E-8	16.3	4.0E-5	8.8
256	2.3E-8	4.5	9.9E-13	15.8	4.6E-12	16.6	1.2E-8	15.6
1024	1.1E-9	4.5	4.8E-14	2.2	6.4E-14	5.1	4.6E-11	16.6

5.3.1. Collocation method for solving Fredholm integro-differential equations

At first we transform the right-hand side of the equation (4.1)

$$y'(t) = p(t)y(t) + q(t) + \int_0^b K(t,s)y(s)ds, \quad t \in [0, b]$$

to a more convenient form. Using the notation

$$z = y'$$

we have

$$y(t) = y(0) + \int_0^t z(s)ds.$$

Therefore

$$\begin{aligned}
& p(t)y(t) + q(t) + \int_0^b K(t, s)y(s)ds \\
&= p(t) \left[y(0) + \int_0^t z(s)ds \right] + q(t) + \int_0^b K(t, s) \left[y(0) + \int_0^t z(x)dx \right] ds \\
&= y(0)g(t) + p(t) \int_0^t z(s)ds + q(t) + \int_0^b z(s)K_1(t, s)ds,
\end{aligned}$$

where

$$g(t) = p(t) + \int_0^b K(t, s)ds$$

and

$$K_1(t, s) = \int_s^b K(t, x)dx.$$

Denoting

$$K_2(t, s) = \begin{cases} p(t) + K_1(t, s) & \text{if } s < t, \\ K_1(t, s) & \text{if } s > t \end{cases}$$

and using (see (4.4))

$$y(0) = \frac{\gamma}{\alpha + \beta} - \frac{\beta}{\alpha + \beta} \int_0^b z(s)ds$$

we rewrite the right-hand side of (4.1) in the form

$$\begin{aligned}
& p(t)y(t) + q(t) + \int_0^b K(t, s)y(s)ds \\
&= \left[\frac{\gamma}{\alpha + \beta} - \frac{\beta}{\alpha + \beta} \int_0^b z(s)ds \right] g(t) + q(t) + \int_0^b K_2(t, s)z(s)ds \\
&= f_1(t) + \int_0^b K_3(t, s)z(s)ds,
\end{aligned}$$

where

$$f_1(t) = q(t) + \frac{\gamma}{\alpha + \beta} g(t) = q(t) + \frac{\gamma}{\alpha + \beta} \left[p(t) + \int_0^b K(t, s) ds \right] \quad (5.13)$$

and

$$\begin{aligned} K_3(t, s) &= K_2(t, s) - \frac{\beta}{\alpha + \beta} g(t) \\ &= \int_0^t K(t, x) dx - \frac{\beta}{\alpha + \beta} \int_0^b K(t, x) dx + p(t) \begin{cases} \frac{\alpha}{\alpha + \beta} & \text{if } s < t, \\ -\frac{\beta}{\alpha + \beta} & \text{if } s > t. \end{cases} \end{aligned} \quad (5.14)$$

Therefore, using the notation $v_n = v|_{(t_{n-1}, t_n)}$ we can write the collocation conditions (4.17) in the form

$$\begin{aligned} v_n(t_{nj}) &= f_1(t_{nj}) + \int_0^b v(s) K_3(t_{nj}, s) ds \\ &= f_1(t_{nj}) + \sum_{k=1}^{2N} \int_{t_{k-1}}^{t_k} v(s) K_3(t_{nj}, s) ds, \quad j = 1, \dots, m, \quad n = 1, \dots, 2N, \end{aligned}$$

where f_1 and K_3 are given by (5.13) and (5.14), respectively. Using the change of variables

$$s = t_{k-1} + xh_k$$

we get

$$\begin{aligned} v_n(t_{nj}) &= f_1(t_{nj}) + \sum_{k=1}^{2N} h_k \int_0^1 v(t_{k-1} + xh_k) K_3(t_{nj}, t_{k-1} + xh_k) dx, \\ & \quad j = 1, \dots, m, \quad n = 1, \dots, 2N. \end{aligned} \quad (5.15)$$

Choose a basis $\{\varphi_1, \dots, \varphi_m\}$ in the space of polynomials of order $m - 1$

$$\varphi_\ell \in \pi_{m-1}, \quad \ell = 1, \dots, m.$$

Then

$$v_n(t_{n-1} + \tau h_n) = \sum_{\ell=1}^m Y_\ell^{(n)} \varphi_\ell(\tau), \quad \tau \in (0, 1),$$

where $Y_1^{(n)}, \dots, Y_m^{(n)}$ are unknown coefficients, that have to be determined. Thus, we can write the conditions (5.15) as follows:

$$\sum_{\ell=1}^m Y_\ell^{(n)} \varphi_\ell(\eta_j) = f_1(t_{nj}) + \sum_{k=1}^{2N} h_k \sum_{\ell=1}^m Y_\ell^{(k)} \int_0^1 K_3(t_{nj}, t_{k-1} + xh_k) \varphi_\ell(x) dx,$$

$$j = 1, \dots, m, \quad n = 1, \dots, 2N.$$

We have obtained a system of linear algebraic equations for finding quantities $Y_1^{(n)}, \dots, Y_m^{(n)}$ ($n = 1, \dots, 2N$):

$$AY = F,$$

where

$$Y = (Y_{1,1}, \dots, Y_{1,m}, Y_{2,1}, \dots, Y_{2,m}, \dots, Y_{2N,1}, \dots, Y_{2N,m})^T,$$

$$F = (f_1(t_{1,1}), \dots, f_1(t_{1,m}), f_1(t_{2,1}), \dots, f_1(t_{2,m}), \dots, f_1(t_{2N,1}), \dots, f_1(t_{2N,m}))^T$$

and

$$A =$$

$$\begin{pmatrix} a_{1,1,1,1} & \dots & a_{1,1,1,m} & a_{1,1,2,1} & \dots & a_{1,1,2,m} & \dots & a_{1,1,2N,1} & \dots & a_{1,1,2N,m} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{1,m,1,1} & \dots & a_{1,m,1,m} & a_{1,m,2,1} & \dots & a_{1,m,2,m} & \dots & a_{1,m,2N,1} & \dots & a_{1,m,2N,m} \\ a_{2,1,1,1} & \dots & a_{2,1,1,m} & a_{2,1,2,1} & \dots & a_{2,1,2,m} & \dots & a_{2,1,2N,1} & \dots & a_{2,1,2N,m} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{2,m,1,1} & \dots & a_{2,m,1,m} & a_{2,m,2,1} & \dots & a_{2,m,2,m} & \dots & a_{2,m,2N,1} & \dots & a_{2,m,2N,m} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{2N,1,1,1} & \dots & a_{2N,1,1,m} & a_{2N,1,2,1} & \dots & a_{2N,1,2,m} & \dots & a_{2N,1,2N,1} & \dots & a_{2N,1,2N,m} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{2N,m,1,1} & \dots & a_{2N,m,1,m} & a_{2N,m,2,1} & \dots & a_{2N,m,2,m} & \dots & a_{2N,m,2N,1} & \dots & a_{2N,m,2N,m} \end{pmatrix}$$

with

$$a_{n,j,k,\ell} = \begin{cases} \varphi_\ell(\eta_j) - h_n \int_0^1 K_3(t_{nj}, t_{n-1} + xh_n) \varphi_\ell(x) dx & \text{if } n = k, \\ -h_k \int_0^1 K_3(t_{nj}, t_{k-1} + xh_k) \varphi_\ell(x) dx & \text{if } n \neq k, \end{cases}$$

$$j, \ell = 1, \dots, m; \quad n, k = 1, \dots, 2N.$$

5.3.2. Test problem of Fredholm type

Consider the following boundary-value problem:

$$y'(t) = y(t) + q(t) + \int_0^1 |t-s|^{-\nu} y(s) ds, \quad t \in [0, 1], \quad y(0) + y(1) = 2. \quad (5.16)$$

The forcing function q is selected so that $y(t) = t^{2-\nu} + (1-t)^{2-\nu}$ is the exact solution. Actually, this is a problem of type (4.1) with

$$\begin{aligned} p(t) &= 1, \\ q(t) &= (2-\nu)(t^{1-\nu} - (1-t)^{1-\nu}) - (t^{2-\nu} + (1-t)^{2-\nu}) \\ &\quad - (t^{3-2\nu} + (1-t)^{3-2\nu}) \int_0^1 x^{-\nu}(1-x)^{2-\nu} dx \\ &\quad - t^{1-\nu} \int_0^1 x^{-\nu}(1-t-x)^{2-\nu} dx \\ &\quad - (1-t)^{1-\nu} \int_0^1 x^{-\nu}(t+(1-t)x)^{2-\nu} dx, \\ K(t, s) &= |t-s|^{-\nu}, \end{aligned}$$

$\alpha = \beta = 1$, $\gamma = 2$ and $b = 1$. Moreover, it is easy to check that $p, q \in C^{m,\nu}[0, b]$ and $K \in W^{m,\nu}(\Delta)$ with $\nu \in (-\infty, 1)$ and arbitrary $m \in \mathbf{N}$.

5.3.3. Numerical results

Problem (5.16) is solved numerically by the collocation method $\{(4.16), (4.17)\}$, in the case $m = 2$ for $\nu = -\frac{1}{4}$ and $\nu = \frac{1}{2}$. Not surprisingly, the results obtained for the Fredholm equation are very similar to those of Method 1 for Volterra equation.

Parameters $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$ correspond to a quadrature formula which is exact for polynomials up to order 1, therefore only Theorem 4.4.3 applies. The theoretical error estimates are

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{2}{2-\nu}, \\ N^{-2}(1 + \log N) & \text{for } r = \frac{2}{2-\nu}, \\ N^{-2} & \text{for } r > \frac{2}{2-\nu} \end{cases} \quad (5.17)$$

and

$$\|u' - y'\|_\infty \leq c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \leq r < \frac{2}{1-\nu}, \\ N^{-2}(1 + \log N) & \text{for } r = \frac{2}{1-\nu} = 1, \\ N^{-2} & \text{for } r > \frac{2}{1-\nu} \text{ or } r = \frac{2}{1-\nu} > 1. \end{cases} \quad (5.18)$$

Table 5.11: Case $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$

$\nu = -\frac{1}{4}$	r=1		r=1.2		r=1.4		r=1.6	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	2.6E-4	3.9	1.8E-4	4.4	1.5E-4	4.5	1.5E-4	4.8
8	6.5E-5	3.9	4.2E-5	4.2	3.6E-5	4.3	3.5E-5	4.2
16	1.7E-5	3.8	1.0E-5	4.1	8.6E-6	4.1	8.3E-6	4.2
32	4.6E-6	3.8	2.6E-6	4.0	2.1E-6	4.1	2.0E-6	4.1
64	1.2E-6	3.8	6.4E-7	4.0	5.2E-7	4.0	5.0E-7	4.0
128	3.1E-7	3.8	1.6E-7	4.0	1.3E-7	4.0	1.3E-7	4.0
256	8.1E-8	3.9	4.0E-8	4.0	3.2E-8	4.0	3.1E-8	4.0
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.4)$	ε'_N	$\varrho'(4.0)$
4	5.6E-3	2.3	4.0E-3	2.7	29E-3	3.2	2.1E-3	3.7
8	2.4E-3	2.3	1.5E-3	2.8	87E-4	3.3	5.8E-4	3.6
16	1.0E-3	2.3	5.2E-4	2.8	26E-4	3.3	1.5E-4	3.9
32	4.4E-4	2.4	1.9E-4	2.8	78E-5	3.3	3.8E-5	3.9
64	1.9E-4	2.4	6.6E-5	2.8	23E-5	3.4	9.7E-6	4.0
128	7.8E-5	2.4	2.3E-5	2.8	69E-6	3.4	2.4E-6	4.0
256	3.3E-5	2.4	8.3E-6	2.8	21E-6	3.4	6.1E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.2		r=1.4		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(3.5)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(4.0)$
4	2.3E-3	2.5	1.6E-3	3.0	1.2E-3	3.6	1.4E-3	5.2
8	8.8E-4	2.7	5.1E-4	3.2	3.2E-4	3.7	2.6E-4	5.3
16	3.2E-4	2.7	1.6E-4	3.2	8.9E-5	3.6	5.6E-5	4.7
32	1.2E-4	2.8	4.8E-5	3.3	2.5E-5	3.6	1.3E-5	4.4
64	4.2E-5	2.8	1.5E-5	3.2	6.7E-6	3.6	3.1E-6	4.2
128	1.5E-5	2.8	4.6E-6	3.3	1.8E-6	3.7	7.6E-7	4.1
256	5.4E-6	2.8	1.4E-6	3.3	4.9E-7	3.7	1.9E-7	4.0
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.5)$	ε'_N	$\varrho'(1.6)$	ε'_N	$\varrho'(4.0)$
4	4.0E-2	1.4	3.4E-2	1.5	3.0E-2	1.6	1.4E-2	3.7
8	2.8E-2	1.4	2.3E-2	1.5	1.8E-2	1.6	3.6E-3	4.0
16	2.0E-2	1.4	1.5E-2	1.5	1.1E-2	1.6	9.0E-4	4.0
32	1.4E-2	1.4	9.8E-3	1.5	6.9E-3	1.6	2.2E-4	4.0
64	9.8E-3	1.4	6.5E-3	1.5	4.3E-3	1.6	5.6E-5	4.0
128	6.9E-3	1.4	4.3E-3	1.5	2.6E-3	1.6	1.4E-5	4.0
256	4.9E-3	1.4	2.8E-3	1.5	1.6E-3	1.6	3.5E-6	4.0

Numerical results for this case are presented in Table 5.11.

As we can see from Table 5.11, the observed errors of u' behave exactly according to the right-hand side of the estimate (5.18) starting from $N = 64$, that demonstrates the sharpness of the error estimate.

The observed errors of u are also in good agreement with theoretical estimates of Theorem 4.4.3 except in the cases when r is close to the value $\frac{2}{2-\nu}$, after which the maximal theoretical convergence rate is achieved. If r is close to the critical value then the observed convergence rate is smaller than the one predicted by the error estimate (5.17) but approaches slowly to the theoretical value.

The results of the numerical experiments in the case $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{6}$, when the corresponding quadrature formula is exact for all polynomials up to order 2, are presented in Table 5.12. We can see that the convergence rate of the approximate solution is better in this case (compared with Table 5.11) and agrees well with the estimate

$$\|u - y\|_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leq r < \frac{3}{2-\nu}, \\ N^{-3}(1 + \log N) & \text{for } r = \frac{3}{2-\nu}, \\ N^{-3} & \text{for } r > \frac{3}{2-\nu} \end{cases}$$

of Theorem 4.5.1.

If we use Gaussian parameters $\left(\eta_1 = \frac{3-\sqrt{3}}{6}, \eta_2 = \frac{3+\sqrt{3}}{6}\right)$, that are exact for polynomials of order $2m - 1 = 3$, we do not get any further improvement in the convergence rate. Corresponding numerical experiments are presented in Table 5.13

Table 5.12: Case $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{6}$

$\nu = -\frac{1}{4}$	r=1		r=1.189		r=1.378		r=1.6	
N	ε_N	$\varrho(4.8)$	ε_N	$\varrho(6.4)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	1.3E-4	4.5	7.8E-5	5.9	5.2E-5	7.3	4.6E-5	7.1
8	2.8E-5	4.6	1.3E-5	6.1	6.8E-6	7.6	5.4E-6	8.5
16	5.9E-6	4.7	2.1E-6	6.2	8.8E-7	7.7	6.6E-7	8.1
32	1.2E-6	4.7	3.3E-7	6.3	1.1E-7	7.7	8.2E-8	8.1
64	2.6E-7	4.7	5.2E-8	6.3	1.5E-8	7.8	1.0E-8	8.1
128	5.5E-8	4.7	8.3E-9	6.3	1.9E-9	7.8	1.2E-9	8.2
256	1.2E-8	4.7	1.3E-9	6.2	2.6E-10	7.2	1.5E-10	8.2
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.3)$	ε'_N	$\varrho'(4.0)$
4	6.0E-3	2.2	4.4E-3	2.7	3.2E-3	3.1	2.3E-3	3.6
8	2.6E-3	2.3	1.6E-3	2.7	9.9E-4	3.2	6.4E-4	3.6
16	1.1E-3	2.3	5.8E-4	2.8	3.0E-4	3.3	1.7E-4	3.8
32	4.7E-4	2.4	2.1E-4	2.8	9.3E-5	3.3	4.2E-5	3.9
64	2.0E-4	2.4	7.5E-5	2.8	2.8E-5	3.3	1.1E-5	4.0
128	8.5E-5	2.4	2.7E-5	2.8	8.6E-6	3.3	2.7E-6	4.0
256	3.6E-5	2.4	9.6E-6	2.8	2.6E-6	3.3	6.7E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.533		r=2.067		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(4.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	1.5E-3	2.8	5.7E-4	4.6	3.2E-4	6.3	6.6E-4	8.2
8	5.4E-4	2.8	1.2E-4	4.8	4.6E-5	7.0	7.5E-5	8.8
16	1.9E-4	2.8	2.4E-5	4.9	6.1E-6	7.5	9.6E-6	7.8
32	6.7E-5	2.8	4.9E-6	4.9	8.0E-7	7.6	1.2E-6	8.2
64	2.4E-5	2.8	9.9E-7	4.9	1.0E-7	7.7	1.5E-7	7.9
128	8.4E-6	2.8	2.0E-7	4.9	1.4E-8	7.7	1.9E-8	7.9
256	3.0E-6	2.8	4.1E-8	4.9	1.8E-9	7.6	2.4E-9	7.9
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.7)$	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(4.0)$
4	4.2E-2	1.4	2.9E-2	1.7	2.0E-2	2.1	1.6E-2	3.6
8	2.9E-2	1.4	1.7E-2	1.7	9.6E-3	2.0	4.0E-3	4.0
16	2.1E-2	1.4	9.8E-3	1.7	4.7E-3	2.0	1.0E-3	4.0
32	1.5E-2	1.4	5.8E-3	1.7	2.3E-3	2.0	2.5E-4	4.0
64	1.0E-2	1.4	3.4E-3	1.7	1.1E-3	2.0	6.3E-5	4.0
128	7.3E-3	1.4	2.0E-3	1.7	5.5E-4	2.0	1.6E-5	4.0
256	5.1E-3	1.4	1.2E-3	1.7	2.7E-4	2.0	3.9E-6	4.0

Table 5.13: Case $\eta_1 = \frac{3-\sqrt{3}}{6}, \eta_2 = \frac{3+\sqrt{3}}{6}$

$\nu = -\frac{1}{4}$	r=1		r=1.189		r=1.378		r=1.6	
N	ε_N	$\varrho(4.8)$	ε_N	$\varrho(6.4)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	1.4E-4	4.4	8.1E-5	6.0	5.1E-5	7.4	4.7E-5	7.4
8	3.1E-5	4.6	1.3E-5	6.2	6.8E-6	7.5	5.5E-6	8.4
16	6.6E-6	4.7	2.1E-6	6.3	8.3E-7	8.2	6.5E-7	8.5
32	1.4E-6	4.7	3.3E-7	6.3	1.0E-7	8.2	7.7E-8	8.4
64	3.0E-7	4.7	5.1E-8	6.4	1.2E-8	8.2	9.3E-9	8.3
128	6.2E-8	4.7	8.1E-9	6.4	1.5E-9	8.1	1.1E-9	8.1
256	1.3E-8	4.7	1.3E-9	6.3	2.1E-10	7.4	1.5E-10	7.5
N	ε'_N	$\varrho'(2.4)$	ε'_N	$\varrho'(2.8)$	ε'_N	$\varrho'(3.3)$	ε'_N	$\varrho'(4.0)$
4	4.5E-3	2.3	3.3E-3	2.7	2.4E-3	3.2	1.7E-3	3.8
8	1.9E-3	2.3	1.2E-3	2.7	7.4E-4	3.2	4.6E-4	3.6
16	8.3E-4	2.3	4.3E-4	2.8	2.3E-4	3.3	1.2E-4	3.9
32	3.5E-4	2.4	1.6E-4	2.8	6.9E-5	3.3	3.0E-5	3.9
64	1.5E-4	2.4	5.6E-5	2.8	2.1E-5	3.3	7.6E-6	4.0
128	6.3E-5	2.4	2.0E-5	2.8	6.3E-6	3.3	1.9E-6	4.0
256	2.6E-5	2.4	7.1E-6	2.8	1.9E-6	3.3	4.8E-7	4.0
$\nu = \frac{1}{2}$	r=1		r=1.533		r=2.067		r=4	
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(4.9)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	1.5E-3	2.7	5.4E-4	4.7	3.1E-4	5.8	7.5E-4	7.0
8	5.5E-4	2.8	1.1E-4	4.9	4.0E-5	7.8	8.2E-5	9.2
16	2.0E-4	2.8	2.2E-5	4.9	4.9E-6	8.1	9.3E-6	8.8
32	7.0E-5	2.8	4.6E-6	4.9	5.9E-7	8.2	1.1E-6	8.5
64	2.5E-5	2.8	9.3E-7	4.9	7.2E-8	8.3	1.3E-7	8.3
128	8.8E-6	2.8	1.9E-7	4.9	8.6E-9	8.3	1.6E-8	8.2
256	3.1E-6	2.8	3.8E-8	4.9	1.1E-9	8.1	2.0E-9	8.1
N	ε'_N	$\varrho'(1.4)$	ε'_N	$\varrho'(1.7)$	ε'_N	$\varrho'(2.0)$	ε'_N	$\varrho'(4.0)$
4	3.3E-2	1.5	2.3E-2	1.7	1.5E-2	2.1	1.2E-2	3.8
8	2.3E-2	1.4	1.3E-2	1.7	7.5E-3	2.1	3.0E-3	4.0
16	1.6E-2	1.4	7.7E-3	1.7	3.7E-3	2.0	7.4E-4	4.0
32	1.1E-2	1.4	4.5E-3	1.7	1.8E-3	2.0	1.9E-4	4.0
64	8.1E-3	1.4	2.7E-3	1.7	8.8E-4	2.0	4.7E-5	4.0
128	5.7E-3	1.4	1.6E-3	1.7	4.3E-4	2.0	1.2E-5	4.0
256	4.0E-3	1.4	9.2E-4	1.7	2.1E-4	2.0	2.9E-6	4.0

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Kokkuvõte

Tükiti polünoomiaalsed kollokatsioonimeetodid nõrgalt singulaarsete integro-diferentsiaalvõrrandite lahendamiseks

Käesoleva töö põhiliseks uurimisobjektiks on tükiti polünoomiaalse kollokatsioonimeetodi lahendusalgoritmid nõrgalt singulaarse Volterra integro-diferentsiaalvõrrandi algväärtusülesande

$$\begin{aligned} y'(t) &= p(t)y(t) + q(t) + \int_0^t K(t,s)y(s)ds, & 0 \leq t \leq b, \\ y(0) &= y_0, & y_0 \in \mathbb{R} = (-\infty, \infty) \end{aligned} \quad (1)$$

ja nõrgalt singulaarse Fredholmi integro-diferentsiaalvõrrandi rajaülesande

$$\begin{aligned} y'(t) &= p(t)y(t) + q(t) + \int_0^b K(t,s)y(s)ds, & 0 \leq t \leq b, \\ \alpha y(0) + \beta y(b) &= \gamma, & \alpha, \beta, \gamma \in \mathbb{R}, \alpha + \beta \neq 0 \end{aligned} \quad (2)$$

ligikaudseks lahendamiseks. Ülesande (1) korral eeldatakse, et tuum $K(t, s)$ on m korda pidevalt diferentseeruv, kui $0 \leq t \leq b$, $0 \leq s < t$, kusjuures leidub reaalarv $\nu < 1$ nii, et kõigi tingimust $i + j \leq m$ rahuldavate mittenegatiivsete täisarvude i ja j korral kehtib võrratus

$$\left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^j K(t, s) \right| \leq c \begin{cases} 1, & \text{kui } \nu + i < 0, \\ 1 + |\log |t - s||, & \text{kui } \nu + i = 0, \\ |t - s|^{-\nu - i}, & \text{kui } \nu + i > 0, \end{cases} \quad (3)$$

kus $c = c(K)$ on mingi positiivne konstant ja $0 \leq t \leq b$, $0 \leq s < t$. Ülesande (2) korral eeldatakse, et K on m korda pidevalt diferentseeruv piirkonnas $\{(t, s) \in \mathbb{R}^2 : 0 \leq t \leq b, 0 \leq s \leq b, t \neq s\}$ ning rahuldab selles piirkonnas mingi fikseeritud reaalarvu $\nu < 1$ korral võrratust (3) kõigi mittenegatiivsete täisarvude i ja j puhul, mille korral $i + j \leq m$. Funktsioonide p ja q sileduse kohta tehakse teatud eeldused, mis on rahuldatud kõigi lõigus $[0, b]$ pidevate ja m korda pidevalt diferentseeruvate funktsioonide korral ning võimaldavad käsitleda ka selliseid funktsioone, mille tuletised alates

mingist järgust võivad olla tõkestamata punkti 0 läheduses ülesande (1) korral ning punktide 0 ja b läheduses ülesande (2) korral.

Ülesande (1) numbriliseks lahendamiseks on töös vaadeldud kahte meetodit, mis tuginevad ülesande (1) ümberformuleerimisele teatavaks teist liiki Volterra integraalvõrrandiks ning viimase lahendamisele kollokatsioonimeetodiga tükiti polünoomiaalsete koordinaatfunktsioonide korral.

Mõlema meetodi korral on tõestatud meetodi koondumine ning tuletatud koonduvuskiiruse hinnang, lähtudes ainult eeldusest, et vastavate tükiti polünoomiaalsete funktsioonide defineerimiseks vajamineva võrgu maksimaalse osalõigu pikkus läheneb nullile. Mõlema meetodi koonduvuskiirust on uuritud ka kvaasiühtlase võrgu kasutamise korral ning spetsiaalse ebaühtlase võrgu kasutamise korral, milles võrgu sõlmed paiknevad tihedamalt integreerimislõigu $[0, b]$ alguspunkti 0 läheduses, kus ülesande (1) lahendi tuletised võivad tõkestamatult kasvada. Saadud tulemustest järeldub muuhulgas, kuidas tihendada kasutatavat võrku punkti 0 ümbruses nii, et saavutada meetodi kõrgeim võimalik koonduvusjärk $\mathcal{O}(N^{-m})$, kui kasutatakse $m - 1$ järku tükiti polünoomiaalset aproksimatsiooni. Samuti on mõlema meetodi korral saadud uusi nn superkoonduvuse tulemusi spetsiaalsete ebaühtlaste võrkude korral tingimusel, et kollokatsioonipunktide valik rahuldab teatud täiendavaid eeldusi.

Ülesande (2) puhul on tõestatud lahendi siledust kirjeldav tulemus ning analüüsitud eespool mainitud meetoditega analoogilise meetodi koonduvust spetsiaalse erikujulise võrgu korral. On tuletatud nii koonduvuskiiruse hinnang suvaliste kollokatsioonipunktide määramisel kasutatavate parameetrite väärtuste korral kui ka superkoonduvuse tulemus juhul, kui mainitud parameetrid rahuldavad teatud lisaeldust.

Töö viimases osas on kontrollitud teoreetiliste koonduvushinnangute täpsust ulatuslike numbriliste eksperimentide läbiviimise teel. Arvutustulemustest järeldub, et töös saadud veahinnangud on järgu poolest mitteparandatavad.

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List of Publications

1. R. Kangro, I. Parts, *Superconvergence in the maximum norm of a class of piecewise polynomial collocation methods for solving linear weakly singular Volterra integro-differential equations*, J. Integral Equations Appl., **15**, No. 4, 2003, 403–427. MR2058811 (2005f:45013), Zbl pre02114500
2. I. Parts, *Optimality of theoretical error estimates for spline collocation methods for linear weakly singular Volterra integro-differential equations*, (accepted for publication in Proc. Estonian Acad. Sci. Phys. Math.).
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