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A characterization of denting points in Banach spaces

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Abstract. A Banach space has the Radon–Nikodým Property if and only if every non-empty closed bounded convex subset of it has a denting point. The Radon–Nikodým property also implies the Krein–Milman Property – every closed bounded convex subset has an extreme point. It is a long-standing problem to show whether these two properties are actually different or not. The main aim of this thesis is to study denting points in various Banach spaces and prove a famous characterization of denting points as extreme points which are simultaneously points of continuity.

CERCS research specialisation: P140 Series, Fourier analysis, functional analysis.

Keywords: Radon–Nikodým Property, Krein–Milman Property, denting point, extreme point.

Hammaspunktide kirjeldus Banachi ruumides

Bakalaureusetöö
Karl Oskar Kuuse

Lühikokkuvõte. Banachi ruumil on Radon–Nikodými omadus, kui tema mis tahes mittetühjal kinnisel kumeral ja tõkestatud alamhulgal on olemas hammaspunkt. Radon–Nikodými omadusest järeldub Krein–Milmani omadus – iga kinnine kumer ja tõkestatud hulk omab ekstreemumpunkti. On pikaaegne lahtine küsimus, kas need omadused on üldse teineteisest erinevad. Bakalaureusetöö peamine eesmärk on uurida hammaspunkte erinevates Banachi ruumides ja tõestada kuulus kirjeldus, et hammaspunktid on ekstreemumpunktid, mis on samaaegselt ka pidevuspunktid.

CERCS teaduseriala: P140 Jadad, Fourier analüüs, funktsionaalanalüüs.

Märksõnad: Radon–Nikodými omadus, Krein–Milmani omadus, hammaspunkt, ekstreemumpunkt.

Contents

Introduction

A well-investigated geometric property of Banach spaces is the *Radon–Nikodým Property* (RNP for short) because it emphasizes the interplay between the topological, geometrical, and measure theoretical structures of a Banach space. The RNP is known to have plentiful equivalent characterizations (see [DU77, pages 217-218]). The roots of a purely geometric characterization date back to H. B. Maynard (see [May74]) who first proved that a Banach space has the RNP if and only if every non-empty bounded subset of it is σ-dentable (see [May74, Definition 2.2]). Soon after that, W. J. Davis and R. R. Phelps improved the result (see [DP74]) and obtained that a Banach space has the RNP if and only if every non-empty bounded subset of it is dentable (see [DP74, Definitions]). It was also later shown that a Banach space has the RNP if and only if every non-empty closed bounded convex subset of it has a denting point (see Definition 2.4).

Closely related to the RNP is the *Point of Continuity Property* (PCP for short). A Banach space has the PCP if and only if every non-empty closed bounded subset of it has a point of continuity (see Definition 3.1). It is known that the RNP implies the PCP, however, the converse is not true because J. Bourgain and H. P. Rosenthal constructed a Banach space with the PCP, but failing the RNP (see [BR80]).

The RNP also implies the *Krein–Milman Property* (KMP for short) – every closed bounded convex subset has an extreme point (see Definition 1.18). This implication follows from the fact that every denting point is also an extreme point (see Lemma 2.10). Despite the seemingly stronger nature of the RNP over the KMP, it is a long-standing open problem to show whether these two properties are actually different or not.

The main aim of this thesis is to give a comprehensive overview of denting points in various Banach spaces and describe their relations to extreme points, points of continuity, and σ-denting points.

The content of the thesis is split into four chapters. In Chapter 1, we introduce the necessary background from general topology and functional analysis, including topics such as nets, weak and weak* topologies and slices. We also define extreme points and present examples in different classical Banach spaces.

In Chapter 2, we define denting and σ-denting points and show that if A is a bounded subset of some Banach space, then the relationships in Figure 1 hold, where $\mathrm{PC}(A)$, $\mathrm{dent}(A)$, σ-$\mathrm{dent}(A)$, and $\mathrm{ext}(A)$, respectively, denote the sets of all points of continuity, denting points, σ-denting points, and extreme points of A.

$$\begin{tikzcd} \mathrm{dent}(A) \arrow[r, phantom, "\overset{\text{Lemma 2.5}}{\underset{\text{Example 2.6}}{\subsetneq}}"] \arrow[d, phantom, "\text{Remark 3.5}\ \not\supset\ \cap\ \text{Lemma 3.2}"] & \sigma\text{-}\mathrm{dent}(A) \arrow[r, phantom, "\overset{\text{Lemma 2.7}}{\underset{\text{Example 2.8}}{\subsetneq}}"] & \mathrm{ext}(A) \\ \mathrm{PC}(A) & & \end{tikzcd}$$

Figure 1: Relationships between $\mathrm{dent}(A)$, σ-$\mathrm{dent}(A)$, $\mathrm{ext}(A)$, and $\mathrm{PC}(A)$

We additionally present the denting point characterization through slices and describe the set of denting points of the closed unit ball for various classical Banach spaces, which are summarized in Table 1. Finally, we show that if A is a closed bounded convex subset, then σ-$\mathrm{dent}(A) = \mathrm{ext}(A)$.

Banach space X	Extremal structure of B_X	References
$c_0, L_{[0,1]}$	$\mathrm{ext}(B_X) = \mathrm{PC}(B_X) = \varnothing$	Remark 2.11 Remark 2.18
c, ℓ_∞	$\mathrm{PC}(B_X) = \varnothing$, $\mathrm{ext}(B_X) =$ $\{(x_n) \in B_X \colon \lvert x_n\rvert = 1 \text{ for every } n \in \mathbb{N}\}$	Remark 2.16 Remark 2.18
ℓ_1	$\mathrm{PC}(B_X) = S_X$, $\mathrm{dent}(B_X) = \mathrm{ext}(B_X) =$ $\{\lambda e_n \in B_X \colon \lambda \in S_{\mathbb{K}}, n \in \mathbb{N}\}$	Remark 2.19 Example 3.4
$\ell_p, p \in (1, \infty)$	$\mathrm{PC}(B_X) = \mathrm{dent}(B_X) = \mathrm{ext}(B_X) = S_X$	Lemma 2.20 Lemma 3.3
$\dim(X) < \infty$	$\mathrm{PC}(B_X) = B_X$, $\mathrm{dent}(B_X) = \mathrm{ext}(B_X)$	Lemma 1.10 Corollary 3.10

Table 1: Extremal structure of the closed unit ball in various Banach spaces

In Chapter 3, we present the proof of a classical theorem by B.-L. Lin, P.-K. Lin, and S. L. Troyanski, which is formulated as follows.

Theorem ([LLT88, Theorem])**.** Let A be a closed bounded convex subset of a Banach space X. Then $\mathrm{dent}(A) = \mathrm{ext}(A) \cap \mathrm{PC}(A)$.

As a corollary of the previous theorem, we also get that $\mathrm{ext}(B_X) = \mathrm{dent}(B_X)$, whenever $\dim(X) < \infty$.

In Chapter 4, we describe the extreme points and points of continuity of the closed unit ball in the p-direct sum space $X \oplus_p Y$ of two Banach spaces. Furthermore, we use the theorem from the previous chapter to also describe its denting points, like summarized in Table 2.

Direct sum space Z	Denting point description	Reference
$X \oplus_\infty Y$	$(x, y) \in \mathrm{dent}(B_Z) \Leftrightarrow$ (a) $x \in \mathrm{dent}(B_X)$; (b) $y \in \mathrm{dent}(B_Y)$	Theorem 4.4
$X \oplus_p Y$, $p \in [1, \infty)$	$(x, y) \in \mathrm{dent}(B_Z) \Leftrightarrow$ (a) $(\|x\|, \|y\|) \in \mathrm{ext}(B_{\ell_p^2})$; (b) $\frac{x}{\|x\|} \in \mathrm{dent}(B_X)$, if $x \neq 0$; (c) $\frac{y}{\|y\|} \in \mathrm{dent}(B_Y)$, if $y \neq 0$	Theorem 4.11

Table 2: Denting points of the closed unit ball in the direct sum spaces

Throughout this thesis, we will be working with normed spaces X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and we will denote the closed unit ball of X as B_X and the unit sphere as S_X. We will denote the diameter of $A \subset X$ as $\mathrm{diam}(A)$ and denote $B_r(x) \coloneqq \{y \in X \colon \|x - y\| \leq r\}$ for some $x \in X$ and $r > 0$.

1 Definitions and preliminaries

In this section, we will introduce essential definitions and results required in the later chapters of this thesis.

1.1 Nets

Let us recall some basic definitions and facts about nets in topological spaces.

Definition 1.1 (see [FHH$^+$11, Definition 17.1])**.** A non-empty set I with a binary relation $\preceq$ on I satisfying for all $\alpha, \beta, \gamma \in I$:

(1) $\alpha \preceq \alpha$;

(2) if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$;

(3) for every α and β in I, there exists γ in I such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$,

is called a *directed set*.

Definition 1.2 (see [FHH$^+$11, Definition 17.2])**.** A *net* in a non-empty set X is a mapping f from a directed set I into X. We often write $x_\alpha \coloneqq f(\alpha)$ and denote the net f as $(x_\alpha)_{\alpha \in I}$ or just (x_α).

Let J be an directed set and $g\colon J \to I$ a mapping with the property that for any given $\alpha_0 \in I$, there exists $\beta_0 \in J$ such that $\alpha_0 \preceq g(\beta)$ whenever $\beta_0 \preceq \beta$. Then the net $(x_{g(\beta)})_{\beta \in J}$ is called a *subnet* of net $(x_\alpha)_{\alpha \in I}$. If there is no confusion, then we often denote the subnet just as (x_β).

Definition 1.3 (see [FHH$^+$11, Definition 17.4])**.** We say that a net $(x_\alpha)_{\alpha \in I}$ in a topological space (X, τ) *converges* to some point $x \in X$ if for every neighbourhood U of x, there exists $\alpha_0 \in I$ such that $x_\alpha \in U$ whenever $\alpha \succeq \alpha_0$. We denote this as $x_\alpha \xrightarrow{\tau} x$ or $\tau\text{-}\lim_\alpha x_\alpha = x$, but often omit the reference to τ if it is clear from the context.

It is well-known that if X is a Hausdorff topological space, then the limit of a convergent net is unique (see [Meg98, Proposition 2.1.17]).

Using nets and the definition of closure points through neighborhoods, it is straightforward to prove the following useful proposition by ordering the neighbourhoods of a point by the reverse inclusion.

Proposition 1.4 (see [Meg98, Proposition 2.1.18]). *Let (X, τ) be a topological space and A some subset of X. Then $x \in X$ belongs to the closure of A if and only if there exists some net (x_α) of A elements such that $x_\alpha \to x$.*

We can also define limit inferior and limit superior for nets of real numbers in a similar way to sequences.

Definition 1.5 (see [Meg98, Definition 2.5.20]). Let (r_α) be a net in $\mathbb{R}$. Then

$$\liminf_{\alpha} r_\alpha \coloneqq \lim_{\alpha} \inf_{\alpha \preceq \beta} \{r_\beta\}$$

and

$$\limsup_{\alpha} r_\alpha \coloneqq \lim_{\alpha} \sup_{\alpha \preceq \beta} \{r_\beta\}$$

are respectively called the *limit inferior* and *limit superior* of (r_α).

The following useful properties regarding the limit inferior and superior of nets hold.

Proposition 1.6 (see [Meg98, Exercise 2.55]). *Let (r_α) and (s_α) be some nets in $\mathbb{R}$ over the same index set. Then the following properties hold.*

(*a*) $\liminf_\alpha r_\alpha \leq \limsup_\alpha r_\alpha$.

(*b*) $\liminf_\alpha r_\alpha = \limsup_\alpha r_\alpha$ *if and only if* $\lim_\alpha r_\alpha$ *exists. In this case* $\liminf_\alpha r_\alpha = \limsup_\alpha r_\alpha = \lim_\alpha r_\alpha$.

(*c*) *If* (r_α) *and* (s_α) *are bounded, then* $\liminf_\alpha r_\alpha + \liminf_\alpha s_\alpha \leq \liminf_\alpha (r_\alpha + s_\alpha)$.

The next lemma is going to be useful to us in Chapter 4.

Lemma 1.7. *Let (r_α) be a bounded net in $\mathbb{R}$ such that $\limsup_\alpha r_\alpha > r$ for some $r \in \mathbb{R}$. Then there exists $\eta > 0$ and a subnet (r_β) of (r_α) such that $r_\beta > r + \eta$ for every β.*

Proof. Let I denote the index set of the net (r_α). For simplicity, let us denote $M_\alpha \coloneqq \sup_{\alpha \preceq \beta} \{r_\beta\}$, which is finite for every $\alpha \in I$ as the net is bounded. Since $\lim_\alpha M_\alpha = \limsup_\alpha r_\alpha > r$, there exists some $\eta > 0$ and $\alpha_0 \in I$ such that $M_\alpha >$

$r+\eta$ for every $\alpha \in I$, $\alpha \succeq \alpha_0$. Let us construct an index set $J \subset I$ such that $(r_\beta)_{\beta\in J}$ is a subnet and $r_\beta > r+\eta$ for every $\beta \in J$. For this, let us choose

$$J \coloneqq \{\alpha \in I \colon \alpha_0 \preceq \alpha, r_\alpha > r+\eta\}.$$

Notice that J is a directed set with the same binary relation $\preceq$ from I. Indeed, since $M_{\alpha_0} > r+\eta$, there exists some $\alpha \in I$, $\alpha \succeq \alpha_0$ such that $r_\alpha > r+\eta$, hence $\alpha \in J$ and J is not empty. The set J also immediately satisfies conditions (1) and (2) of Definition 1.1. Let $\alpha, \beta \in J$ be arbitrary. Then there exists $\gamma \in I$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Since α and β succeed α_0, we also have $\alpha_0 \preceq \gamma$. Therefore, $M_\gamma > r+\eta$, which in turn implies that there exists some $\gamma' \in I$, $\gamma' \succeq \gamma$ such that $r_{\gamma'} > r+\eta$. But now we have $\gamma' \in J$ and it also succeeds both α and β, so the condition (3) of the definition is satisfied and J is indeed a directed set.

It remains to show that $(r_\beta)_{\beta\in J}$ is a subnet of $(r_\alpha)_{\alpha\in I}$. Let $g\colon J \to I$ from Definition 1.2 be the identity map and let $\alpha \in I$ be arbitrary. Then there exists some $\beta \in I$ such that $\alpha \preceq \beta$ and $\alpha_0 \preceq \beta$. Therefore, we once again have $M_\beta > r+\eta$, which implies the existence of some $\gamma \in I$, $\gamma \succeq \beta$ such that $r_\gamma > r+\eta$, i.e. $\gamma \in J$ and we also have $\alpha \preceq \gamma'$ for every $\gamma' \succeq \gamma$. In conclusion, $(r_\beta)_{\beta\in J}$ is indeed a subnet of $(r_\alpha)_{\alpha\in I}$ with the desired property $r_\beta > r+\eta$ for all $\beta \in J$. $\square$

1.2 Weak and weak* topologies

Throughout this subsection, let X denote a normed space unless stated otherwise. The following definitions and facts can be found, for example, in [Meg98, Sections 2.4 – 2.6]

Definition 1.8. The *weak topology* on X is the topology whose neighborhood basis at $x_0 \in X$ consists of all sets of the form

$$\{x \in X \colon |x_i^*(x) - x_i^*(x_0)| < \varepsilon,\ i = 1, \ldots, n\}, \tag{1.1}$$

where $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$.

The *weak* topology* on X^* is the topology whose neighborhood basis at $x_0^* \in X^*$

consists of all sets of the form

$$\{x^* \in X^* \colon |x^*(x_i) - x_0^*(x_i)| < \varepsilon,\ i = 1, \ldots, n\}, \tag{1.2}$$

where $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$.

The sets of the form (1.1) and (1.2) are open themselves in their respective topologies and they are sometimes referred to as the basic neighbourhood sets. Additionally, the weak topology of X is in fact the coarsest topology on X that makes every element of X^* continuous and the weak* topology of X^* is the coarsest topology on X^* that makes every element of X continuous. Both of these topologies are also known to be Hausdorff and it is useful to keep in mind that if (x_α) is a net in X and $x \in X$, then $x_\alpha \xrightarrow{w} x$, i.e. (x_α) converges to x in the weak topology, if and only if $f(x_\alpha) \to f(x)$ in $\mathbb{K}$ for every $f \in X^*$ (see [Meg98, Propostition 2.4.4]).

If a subset of X has some property that holds in the weak topology of X, we say that this property holds *weakly*. For example, we say that $A \subset X$ is *weakly open*, if A is open in the weak topology of X. Analogously, if a subset of X^* has some property that holds in the weak* topology of X^*, then we say that this property holds *weak** or *weakly**. Additionally, if $A \subset X$, then we say that some subset $B \subset X$ is *relatively weakly open* in A, if there exists some weakly open set $U \subset X$ such that $B = U \cap A$. If the set A is clear from context, we just say that B is relatively weakly open. Once again, we use a similar notion for a relatively weakly* open set.

The following is another well-known fact regarding the weak topology.

Proposition 1.9 (see [FHH$^+$11, Proposition 3.89])**.** *Let X be an infinite-dimensional normed space. Then every non-empty weakly open set is unbounded.*

The previous result implies that in an infinite-dimensional normed space X, every non-empty relatively weakly open subset of B_X intersects the unit sphere. As shown in the next lemma, this does not hold in finite-dimensional spaces, because in this case the weak and norm topologies coincide.

Lemma 1.10. *If X is a finite-dimensional normed space, then its weak topology and norm topology coincide.*

Proof. It is clear that the norm topology is finer than the weak topology. Let us also show that the norm topology is contained in the weak topology. For this, it suffices to fix some $x_0 \in X$ and show that there exist sets of arbitrarily small diameter from the weak neighborhood basis at x_0. Let us fix $\varepsilon > 0$ and let $N := \dim(X)$. Let $(e_i)_{i=1}^N$ be a normalized basis for X, i.e. every element $x \in X$ can be uniquely expressed as $\sum_{i=1}^N w_i e_i$, where $w_i \in \mathbb{K}$, and the norm of all the basis vectors e_i is 1. Let us define functionals $f_i \colon X \to \mathbb{K}, x \mapsto w_i$, which are clearly in X^*, and construct the weak neighbourhood basis set

$$U := \left\{ x \in X \colon |f_i(x) - f_i(x_0)| < \frac{\varepsilon}{N},\ 1 \leq i \leq N \right\}.$$

If $x \in U$ and $x_0 = \sum_{i=1}^N w_i^0 e_i$, then

$$\begin{aligned} \|x - x_0\| &= \left\| \sum_{i=1}^N (w_i - w_i^0) e_i \right\| \leq \sum_{i=1}^N |w_i - w_i^0| \leq N \cdot \max_{1 \leq i \leq N} |w_i - w_i^0| \\ &= N \cdot \max_{1 \leq i \leq N} |f_i(x) - f_i(x_0)| \leq N \cdot \frac{\varepsilon}{N} = \varepsilon. \end{aligned}$$

Therefore, $\operatorname{diam}(U) \leq 2\varepsilon$, which completes the proof. □

The next classical theorem and its corollary tell us that every bounded and weakly* closed set in X^* is actually weakly* compact.

Theorem 1.11 (Banach–Alaoglu Theorem; see [Meg98, Theorem 2.6.18])**.** *The closed unit ball B_{X^*} is weakly* compact.*

Corollary 1.12 (see [Meg98, Corollary 2.6.19])**.** *The weak* closure of any bounded subset of X^* is weakly* compact.*

We will also use the fact that the norm of X is weakly lower semicontinuous.

Theorem 1.13 (see [Meg98, Theorem 2.5.21])**.** *If a net (x_α) in X converges to an element $x \in X$ in the weak topology, then*

$$\|x\| \leq \liminf_{\alpha} \|x_\alpha\|.$$

An analogous result holds for weak* convergence.

Theorem 1.14 (see [Meg98, Theorem 2.6.14])**.** *If a net (x_α^*) in X^* converges to an element $x^* \in X^*$ in the weak* topology, then*

$$\|x^*\| \le \liminf_{\alpha} \|x_\alpha^*\|.$$

1.3 Slices and extreme points

Throughout this subsection, let X denote a normed space unless stated otherwise.

Definition 1.15. Let A be a non-empty bounded subset of X. A *slice* of A is a set of the form

$$S(A, x^*, \alpha) = \left\{ x \in A\colon\ \operatorname{Re} x^*(x) > \sup_{y \in A} \operatorname{Re} x^*(y) - \alpha \right\},$$

where $x^* \in X^*$ and $\alpha > 0$.

It is also clear from the previous definition that a slice is always non-empty. Furthermore, let us note the following.

Remark 1.16. If A is a non-empty bounded subset of X, then every slice of A is a relatively weakly open set in A. Indeed, let $S := S(A, f, \alpha)$, $f \in X^*$, $\alpha > 0$, be some slice of A and let $x \in S$ be arbitrary. We can find some $\varepsilon > 0$ such that $\operatorname{Re} f(x) > \sup_{y \in A} \operatorname{Re} f(y) - \alpha + \varepsilon$. Now choose the set

$$V_x := \{y \in A\colon |f(x) - f(y)| < \varepsilon\}.$$

Note that V_x is a relatively weakly open subset of A. Let $z \in V_x$. Then $|\operatorname{Re} f(x) - \operatorname{Re} f(z)| \le |f(x) - f(z)| < \varepsilon$, which implies

$$\operatorname{Re} f(z) > \operatorname{Re} f(x) - \varepsilon > \sup_{y \in A} \operatorname{Re} f(y) - \alpha.$$

This means that $z \in S$, i.e. $V_x \subset S$. Therefore, as $S = \bigcup_{x \in S} V_x$, then S is a relatively weakly open subset of A.

The following theorem is one of the central results of Banach space theory.

Theorem 1.17 (Hahn–Banach separation theorem; see [FHH+11, Theorem 2.12])**.** *Let A be a closed convex set in X. If $x_0 \notin A$, then there is $f \in X^*$ such that*

$$\operatorname{Re} f(x_0) > \sup_{x \in A} \operatorname{Re} f(x).$$

The next concept will play an important role throughout this thesis.

Definition 1.18. Let $A \subset X$. Element $x \in A$ is said to be an *extreme point* of A if whenever $x = \lambda y + (1-\lambda)z$, for some $y, z \in A$ and $\lambda \in (0,1)$, then $x = y = z$.

By $\operatorname{ext}(A)$, we denote the set of all extreme points of A. As extreme points are mainly studied in convex subsets, it is worth noting that if A is convex, then $x \in \operatorname{ext}(A)$ if and only if $x = \frac{y+z}{2}$, for some $y, z \in A$, implies $x = y = z$ (see [Chu73, Lemma 2.3]).

It is also worth noting that $x \in \operatorname{ext}(B_X)$ implies $\|x\| = 1$ (see [Chu73, Lemma 2.5]) and it is also not difficult to show that extreme points of the closed unit ball in $\mathbb{R}$ and $\mathbb{C}$ are exactly all of the elements with modulus 1 (see [Chu73, Lemma 2.6]).

In the following, we will describe the set of extreme points of the closed unit ball for some classical Banach spaces. This knowledge will be useful to us in the next chapter when studying denting points. As the spaces that we will be working with are all well-known, we will not be recalling their definition here explicitly, but one can find them, for example, in [Meg98, Section 1.2]. Note that in sequence spaces, by e_n, $n \in \mathbb{N}$, we denote the sequence with 1 at the n-th coordinate and 0 elsewhere.

Example 1.19 (see [Chu73, Theorem 4.1])**.** In the space c_0, we have $\operatorname{ext}(B_{c_0}) = \varnothing$. Indeed, let $x = (x_n) \in B_{c_0}$. As x converges to 0, it has some coordinate $n_0 \in \mathbb{N}$ such that $|x_{n_0}| < \frac{1}{2}$. Now we can choose two distinct B_{c_0} elements $y \coloneqq x + \frac{1}{4}e_{n_0}$ and $z \coloneqq x - \frac{1}{4}e_{n_0}$ such that $x = \frac{y+z}{2}$. Therefore, x is not an extreme point.

Example 1.20. In the space c, we have

$$\operatorname{ext}(B_c) = \left\{(x_n) \in B_c \colon |x_n| = 1 \text{ for every } n \in \mathbb{N}\right\}.$$

Indeed, if $x = (x_n) \in B_c$ has some coordinate n_0 such that $|x_{n_0}| < 1$, then $x_{n_0} = \frac{\lambda+\mu}{2}$ for some $\lambda, \mu \in B_{\mathbb{K}}$, $\lambda \neq \mu$, because x_{n_0} is not an extreme point of $B_{\mathbb{K}}$. Therefore, we can construct two distinct B_c elements $y = (y_n)$ and $z = (z_n)$ such that $y_{n_0} = \lambda$, $z_{n_0} = \mu$ and $y_n = z_n = x_n$ elsewhere. Now we have $x = \frac{y+z}{2}$, so x is not an extreme point.

On the other hand, if $x = (x_n) \in B_c$ is an element such that $|x_n| = 1$ for every $n \in \mathbb{N}$ and $x = \frac{y+z}{2}$, for some $y = (y_n), z = (z_n) \in B_c$, then $x_n = \frac{y_n+z_n}{2}$, for every $n \in \mathbb{N}$. As x_n is an extreme point of $B_{\mathbb{K}}$, we have $y_n = z_n$ for every $n \in \mathbb{N}$, which implies $y = z$. Therefore, $x \in \operatorname{ext}(B_c)$.

Example 1.21 (see [Chu73, Theorem 4.2])**.** In the space ℓ_1, we have

$$\operatorname{ext}(B_{\ell_1}) = \{\lambda e_n \colon \lambda \in \mathbb{K},\, |\lambda| = 1,\, n \in \mathbb{N}\}.$$

In order to see this, take $x = (x_n) \in S_{\ell_1}$. If x is not of the form λe_k for some $\lambda \in S_{\mathbb{K}}$, $k \in \mathbb{N}$, then there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = |x_{n_0}|e^{i\theta}$, where $|x_{n_0}| \in (0,1)$ and $\theta \in \mathbb{R}$. Choose $y = (y_n)$ to be a sequence with $y_{n_0} = e^{i\theta}$ and zeroes elsewhere. Also, choose $z := \frac{1}{1-|x_{n_0}|}(x - x_{n_0}e_{n_0})$. As $\|y\| = 1$ and

$$\|z\| = \frac{1}{1-|x_{n_0}|}\left(\sum_{n=1}^{\infty} |x_n| - |x_{n_0}|\right) = \frac{1}{1-|x_{n_0}|}(1-|x_{n_0}|) = 1,$$

then $y, z \in B_{\ell_1}$. Notice that $x = |x_{n_0}|y + (1-|x_{n_0}|)z$. We also have $y \neq z$, because $\|x\| = 1$ and $|x_{n_0}| \in (0,1)$ implies that x has at least two non-zero coordinates, while y has only one non-zero coordinate. Therefore, x is not an extreme point. This means that an extreme point of B_{ℓ_1} must have the form λe_k.

On the contrary, if $x = \lambda e_k$, where $\lambda \in S_{\mathbb{K}}$, $k \in \mathbb{N}$, and $x = \frac{y+z}{2}$ for some $y, z \in B_{\ell_1}$, then $\lambda = x_k = \frac{y_k+z_k}{2}$. As λ is an extreme point of $B_{\mathbb{K}}$, we have $y_k = z_k = \lambda$, which implies $y = z$, since other coordinates of y and z must therefore be 0. In conclusion, $x \in \operatorname{ext}(B_{\ell_1})$.

Example 1.22. It is shown in [Chu73, Corollary 4.9] that $\operatorname{ext}(B_{L_1[0,1]}) = \varnothing$. Additionally, by [Chu73, Theorem 4.3], we have

$$\operatorname{ext}(B_{\ell_\infty}) = \{(x_n) \in B_{\ell_\infty} \colon |x_n| = 1 \text{ for every } n \in \mathbb{N}\},$$

and by [Chu73, Theorem 4.27], it holds that

$$\mathrm{ext}(B_{C[0,1]}) = \left\{x \in B_{C[0,1]} \colon |x(t)| = 1 \text{ for every } t \in [0,1]\right\}.$$

Also, in Lemma 2.20, we will show that $\mathrm{ext}(B_{\ell_p}) = S_{\ell_p}$ for $p \in (1, \infty)$.

In Chapter 3, we will also make use of the weak* version of Choquet's lemma. Note that by *weak* slices* of some X^* subset, we mean slices whose defining functional comes from the predual X.

Lemma 1.23 (Choquet's lemma; see [Lan12, Lemma 2.12])**.** *Let A be a weak* compact convex set in X^* and x^* an extreme point of A. Then weak* slices of A containing x^* form a neighborhood basis at x^* in the relative weak* topology of A.*

2 Denting and σ-denting points

Let X denote a Banach space. Let us introduce the main definitions of this thesis.

Definition 2.1. The *convex hull* of $A \subset X$ is the set

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \colon n \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in A \right\}.$$

The *closed convex hull* of A is defined as the closure of the convex hull and denoted as $\overline{\operatorname{conv}}(A)$.

Note that $\operatorname{conv}(A)$ and $\overline{\operatorname{conv}}(A)$ are respectively the smallest convex set and the smallest closed convex set containing A. Additionally, elements of the form $\sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \in [0,1]$ and $\sum_{i=1}^{n} \lambda_i = 1$, are referred to as convex combinations of the elements $x_1, \dots, x_n$.

Definition 2.2 (see [DP74, Definitions])**.** The *σ-convex hull* of $A \subset X$ is the set

$$\sigma(A) = \left\{ \sum_{i=1}^{\infty} \lambda_i x_i \colon \lambda_i \geq 0, \sum_{i=1}^{\infty} \lambda_i = 1, x_i \in A, \text{ the respective series converges} \right\}.$$

If A is bounded in the previous definition, then all of the infinite convex sums in $\sigma(A)$ exist.

Remark 2.3. If $A \subset X$, then the following inclusions hold:

$$A \subset \operatorname{conv}(A) \subset \sigma(A) \subset \overline{\operatorname{conv}}(A).$$

Indeed, the first two inclusions are obvious. If $x \in \sigma(A)$, then $x = \sum_{i=1}^{\infty} \lambda_i x_i$ for some elements like specified in the definition of σ-convex hull. Therefore, we can define elements $y_n \coloneqq \sum_{i=1}^{n} \lambda_i x_i + (\sum_{i=n+1}^{\infty} \lambda_i) x_1 \in \operatorname{conv}(A)$ and since $y_n \xrightarrow[n\to\infty]{} x$, we have $x \in \overline{\operatorname{conv}}(A)$.

Definition 2.4 (see [DP74, Definitions])**.** Let $A \subset X$ be a bounded set. We say that $x \in A$ is a *denting* (respectively, *σ-denting*) *point* of A, if for all $\varepsilon > 0$, $x \notin \overline{\operatorname{conv}}\left(A \setminus B_\varepsilon(x)\right)$ (respectively, $x \notin \sigma\left(A \setminus B_\varepsilon(x)\right)$).

With $\operatorname{dent}(A)$ and σ-$\operatorname{dent}(A)$, we respectively denote the set of all denting points and σ-denting points of A. Note that one could also study denting and σ-denting points in unbounded subsets, but they are not of interest to us in this thesis.

The following useful lemma follows directly from the definitions.

Lemma 2.5. *Let $A \subset X$ be a bounded set. If $x \in \operatorname{dent}(A)$, then $x \in \sigma$-$\operatorname{dent}(A)$.*

Proof. Let $x \in \operatorname{dent}(A)$. Then for every $\varepsilon > 0$, we have $x \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$. Since $\sigma(A \setminus B_\varepsilon(x)) \subset \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$ by Remark 2.3, we also have $x \notin \sigma(A \setminus B_\varepsilon(x))$. Hence $x \in \sigma$-$\operatorname{dent}(A)$. $\square$

The next example demonstrates that the reverse implication of the previous lemma does not generally hold, i.e. every σ-denting point is not a denting point.

Example 2.6. Let $X = c$ or $X = \ell_\infty$ and denote

$$\mathbb{1} := (1, 1, 1, \dots) \in B_X.$$

Let us show that $\mathbb{1}$ is a σ-denting point of B_X, but not a denting point.

First, let us show that it is a σ-denting point. Suppose that $\mathbb{1}$ is not a σ-denting point. This means that there exists some $\varepsilon > 0$ such that $\mathbb{1} \in \sigma(B_X \setminus B_\varepsilon(\mathbb{1}))$, i.e. $\mathbb{1}$ can be expressed as some infinite convex sum $\sum_{n=1}^{\infty} \lambda_n x_n$, where for every $n \in \mathbb{N}$, we have $x_n = (\alpha_k^n) \in B_X \setminus B_\varepsilon(\mathbb{1})$ and $\lambda_n > 0$, since we can present some positive λ_n as an infinite sum, if required. This means that

$$\|\mathbb{1} - x_1\| = \sup_k |1 - \alpha_k^1| > \varepsilon,$$

so there exists $k_0 \in \mathbb{N}$ such that $|1 - \alpha_{k_0}^1| > \varepsilon$. This also implies that $1 - \operatorname{Re} \alpha_{k_0}^1 > \delta > 0$, as otherwise we would have $\operatorname{Re} \alpha_{k_0}^1 = 1$ and $\operatorname{Im} \alpha_{k_0}^1 > \varepsilon$, which contradicts the fact that $x_1 \in B_X$. As the equality $\mathbb{1} = \sum_{n=1}^{\infty} \lambda_n x_n$ implies coordinate-wise equalities, we have

$$\begin{aligned}1 = \operatorname{Re} \sum_{n=1}^{\infty} \lambda_n \alpha_{k_0}^n &= \sum_{n=1}^{\infty} \lambda_n \operatorname{Re} \alpha_{k_0}^n = \lambda_1 \operatorname{Re} \alpha_{k_0}^1 + \sum_{n=2}^{\infty} \lambda_n \operatorname{Re} \alpha_{k_0}^n \\ &< \lambda_1 - \lambda_1 \delta + \sum_{n=2}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n - \lambda_1 \delta = 1 - \lambda_1 \delta < 1,\end{aligned}$$

which is a contradiction. Therefore, $\mathbb{1}$ is a σ-denting point of B_X.

In order to show that $\mathbb{1}$ is not a denting point, we want to show that there exists some $\varepsilon > 0$ such that for every $\delta > 0$, we can find a $x \in \operatorname{conv}(B_X \setminus B_\varepsilon(\mathbb{1}))$ such that $\|\mathbb{1} - x\| < \delta$, i.e. every neighbourhood of $\mathbb{1}$ contains an element of $\operatorname{conv}(B_X \setminus B_\varepsilon(\mathbb{1}))$. For this, let us choose $\varepsilon = 1$ and fix an arbitrary $\delta > 0$. Also, fix $m \in \mathbb{N}$ so that $\frac{2}{m} < \delta$. Now choose elements y_i, $i \in \{1, 2, \ldots, m\}$, such that the i-th coordinate of y_i is -1 and the rest are all equal to 1. Then for every y_i, we have $\|\mathbb{1} - y_i\| = 2 > 1$ and $\|y_i\| = 1$, i.e. $y_i \in B_X \setminus B_1(\mathbb{1})$. Now for the element

$$x \coloneqq \frac{1}{m} \sum_{i=1}^{m} y_i \in \operatorname{conv}(B_X \setminus B_1(\mathbb{1})),$$

we have

$$\|\mathbb{1} - x\| = 1 - \frac{1}{m}(m-2) = \frac{2}{m} < \delta.$$

In conclusion, $\mathbb{1} \in \overline{\operatorname{conv}}(B_X \setminus B_1(\mathbb{1}))$ and $\mathbb{1}$ is not a denting point of B_X.

We can prove another simple yet useful lemma connecting denting points and extreme points.

Lemma 2.7. *Let $A \subset X$ be a bounded set. If $x \in \sigma\text{-}\operatorname{dent}(A)$, then $x \in \operatorname{ext}(A)$.*

Proof. Suppose that x is not an extreme point of A. Then there exist $y, z \in A$ and $\lambda \in (0,1)$ such that $y \neq z$ and $x = \lambda y + (1-\lambda)z$. This also means that $y \neq x$ and $z \neq x$. Therefore, there exists $\varepsilon > 0$ such that $\|x - y\| > \varepsilon$ and $\|x - z\| > \varepsilon$, which means $y, z \notin B_\varepsilon(x)$. But now we have $x = \lambda y + (1-\lambda)z \in \sigma(A \setminus B_\varepsilon(x))$, so x is not a σ-denting point of A. $\square$

The previous result also does not hold in reverse though, as demonstrated by the following examples.

Example 2.8. Let $X \coloneqq \ell_1$ and

$$K \coloneqq \{(x_k) \in B_{\ell_1} \colon x_k \neq 0 \text{ for finitely many } k \in \mathbb{N}\}.$$

It is easy to see that K is convex. Also, denote $e \coloneqq \sum_{k=1}^{\infty} 2^{-k} e_k$, where elements e_k are the sequences with 1 at the k-th position and 0 elsewhere. Let us look at the set

$$A \coloneqq \operatorname{conv}(K \cup \{e\}) = \{\lambda x + (1-\lambda)e \colon x \in K, \lambda \in [0,1]\},$$

where the previous equality holds by the fact that K is convex. Notice that $e \in \operatorname{ext}(A)$. Indeed, let $e = \frac{y+z}{2}$ for some $y, z \in A$, where $y = \lambda_1 x_1 + (1-\lambda_1)e$, $z = \lambda_2 x_2 + (1-\lambda_2)e$, for some $x_1, x_2 \in K$, $\lambda_1, \lambda_2 \in [0,1]$. Additionally, let $f_n \colon \ell_1 \to \mathbb{K}$ denote the linear functionals, which map the sequences $x = (x_k) \in \ell_1$ to their n-th coordinate, i.e. $f_n(x) = x_n$, for any $n \in \mathbb{N}$. As x_1 and x_2 are finitely supported, there exists some $N \in \mathbb{N}$ such that $f_N(x_1) = f_N(x_2) = 0$. Then we also have $f_N(e) = f_N\left(\frac{y+z}{2}\right) = \frac{(1-\lambda_1)f_N(e)+(1-\lambda_2)f_N(e)}{2}$ and since $f_N(e) \neq 0$, we get $\lambda_1 = \lambda_2 = 0$. Therefore, we have $y = z = e$, so e is indeed an extreme point. On the other hand, for every $n \in \mathbb{N}$, we have

$$\|e - e_n\| = \left\|\left(2^{-1}, \dots, 2^{-(n-1)}, 2^{-n} - 1, 2^{-(n+1)}, \dots\right)\right\| = \sum_{\substack{k=1 \\ k \neq n}}^{\infty} 2^{-k} + 1 - 2^{-n} > \frac{1}{2},$$

and $\sum_{k=1}^{\infty} 2^{-k} = 1$, i.e. $e \in \sigma(A \setminus B_{\frac{1}{2}}(e))$, as the elements e_n belong to A. This means that $e \notin \sigma\text{-dent}(A)$. In conclusion, we have constructed a bounded convex set, which contains an extreme point that is not a σ-denting point.

Example 2.9. Consider the set L in B_{ℓ_1} given by

$$L \coloneqq \{e_n \colon n \in \mathbb{N}\} \cup \{e\},$$

where the vector e is defined as in Example 2.8. Then L is a non-convex closed and bounded set. Similarly to the above, one can check that e is an extreme point of L, which is not σ-denting.

To sum up, we have shown that for a bounded $A \subset X$, the following relationships hold

$$\operatorname{dent}(A) \underset{\neq}{\overset{\text{Lemma 2.5}}{\subset}} \sigma\text{-}\operatorname{dent}(A) \underset{\neq}{\overset{\text{Lemma 2.7}}{\subset}} \operatorname{ext}(A),$$
$$\text{Example 2.6} \qquad\qquad \text{Example 2.8}$$

which also implies that every denting point is an extreme point. Let us formulate the fact $\operatorname{dent}(A) \subset \operatorname{ext}(A)$ as a separate lemma for reference.

Lemma 2.10. *Let $A \subset X$ be a bounded set. If $x \in \operatorname{dent}(A)$, then $x \in \operatorname{ext}(A)$.*

Remark 2.11. As we know from Examples 1.19 and 1.22 that $\operatorname{ext}(B_{c_0}) = \varnothing$ and $\operatorname{ext}(B_{L_1[0,1]}) = \varnothing$, then B_{c_0} and $B_{L_1[0,1]}$ are examples of closed unit balls with no denting or σ-denting points.

At the end of the paper [DP74], W. J. Davis and R. R. Phelps asked the following question: *In spaces with the RNP does every closed bounded convex set have a denting point (σ-denting point, extreme point)?* Nowadays, we know that the answer to this question is positive because the RNP is characterized in the terms that every closed bounded convex set has a denting point (see [DU77, pages 217-218]); hence, also a σ-denting point and an extreme point. However, this question and the examples of σ-denting points which are not denting points in the literature (see [May74], [DP74], [DU77, Example V.3.5]) all seemed to have overlooked the fact that σ-denting points and extreme points actually coincide in closed bounded convex subsets of a Banach space. At least, it seems that R. D. Bourgin knew it (see [Bou83, page 18]), but he did not bother to explicitly state it nor include a proof for it.

Proposition 2.12. *Let A be a closed bounded convex subset of X. Then* $\operatorname{ext}(A) = \sigma\text{-}\operatorname{dent}(A)$.

Proof. By Lemma 2.7, we know that $\sigma\text{-}\operatorname{dent}(A) \subset \operatorname{ext}(A)$. Let us show that every extreme point of A is a σ-denting point. Assume that $x \notin \sigma\text{-}\operatorname{dent}(A)$. Then there exists some $\varepsilon > 0$ such that $x = \sum_{n=1}^{\infty} \lambda_n x_n$, $\lambda_n \in [0,1]$, $\sum_{n=1}^{\infty} \lambda_n = 1$, and $x_n \in A \setminus B_\varepsilon(x)$ for every $n \in \mathbb{N}$. We can additionally assume, without loss of generality, that $\lambda_1 \neq 0$. We also know that $\lambda_1 \neq 1$, since otherwise we would have $x = x_1$, which contradicts the fact that $x_1 \in A \setminus B_\varepsilon(x)$. Now we can write

$x = \lambda_1 x_1 + (1-\lambda_1)\sum_{n=2}^{\infty} \frac{\lambda_n}{1-\lambda_1} x_n$. Denote $y := \sum_{n=2}^{\infty} \frac{\lambda_n}{1-\lambda_1} x_n$ and notice that y is a convex series of $A \setminus B_\varepsilon(x)$ elements. Therefore, using Remark 2.3, we have

$$y \in \sigma(A \setminus B_\varepsilon(x)) \subset \sigma(A) \subset \overline{\mathrm{conv}}(A).$$

As A is both closed and convex, we have $\overline{\mathrm{conv}}(A) = A$, hence $y \in A$. In conclusion, we have that $x = \lambda_1 x_1 + (1-\lambda_1)y$, i.e. x lies between two distinct points x_1 and y of set A, so x is not an extreme point of A. Note that $x_1 \neq y$ in the previous, because otherwise we would once again have $x = x_1$. □

Remark 2.13. The proof of Proposition 2.12 heavily relies on closedness and convexity of the set. Example 2.8 and 2.9 show that both are necessary for the validity of Proposition 2.12.

As the closed unit ball of a Banach space is convex and bounded, we get the following direct corollary.

Corollary 2.14. *Let X be a Banach space. Then* $\mathrm{ext}(B_X) = \sigma\text{-}\mathrm{dent}(B_X)$.

So, although σ-dentability was an important tool for establishing a purely geometric characterization of the RNP, the notion of σ-denting points is not of interest to us when it comes to the study of the extremal structure of the unit balls in Banach spaces. For this reason, in the rest of the thesis, we will focus our attention on denting points.

The next lemma is an useful tool for finding denting points among other things.

Lemma 2.15. *Let $\Phi\colon X \to X$ be an onto linear isometry and let A be some bounded convex subset of X. Then $x \in A$ is an extreme or σ-denting or denting point of A if and only if $\Phi(x) \in B := \Phi(A)$ is respectively an extreme or σ-denting or denting point of B.*

Proof. Note that since Φ is a linear isometry, it is injective and continuous. Therefore, it has an inverse Φ^{-1}, which is also a linear isometry from X to X. Also, $\Phi^{-1}(B) = A$ and it is easy to check that B is bounded and convex. This means that it suffices to show that if x has one of the properties in A, then so does $\Phi(x)$ in B.

Let $x \in A$ be an extreme point. If $\Phi(x) = \frac{y+z}{2}$ for some $y, z \in B$, then we have

$$x = \Phi^{-1}(\Phi(x)) = \Phi^{-1}\left(\frac{y+z}{2}\right) = \frac{\Phi^{-1}(y) + \Phi^{-1}(z)}{2}.$$

As x is an extreme point, we get $\Phi^{-1}(y) = \Phi^{-1}(z)$, which implies $y = z$. In conclusion, $\Phi(x)$ is an extreme point of B.

Assume that $\Phi(x)$ is not a σ-denting point, i.e. there exists $\varepsilon > 0$ such that $\Phi(x)$ can be expressed as a convex sum $\sum_{n=1}^{\infty} \lambda_n x_n$, where $x_n \in B \setminus B_\varepsilon(\Phi(x))$ for every $n \in \mathbb{N}$. By the continuity of Φ^{-1}, we have

$$x = \Phi^{-1}\left(\sum_{n=1}^{\infty} \lambda_n x_n\right) = \sum_{n=1}^{\infty} \lambda_n \Phi^{-1}(x_n)$$

and, for every $n \in \mathbb{N}$, we also have $\|x - \Phi^{-1}(x_n)\| = \|\Phi(x) - x_n\| > \varepsilon$, i.e. $\Phi^{-1}(x_n) \in A \setminus B_\varepsilon(x)$. This means that x is not a σ-denting point of A.

If we assume that $\Phi(x)$ is not a denting point, then there exsits $\varepsilon > 0$ such that $\Phi(x) \in \overline{\operatorname{conv}}(B \setminus B_\varepsilon(\Phi(x)))$. This means that there exist some elements $x_n \in \operatorname{conv}(B \setminus B_\varepsilon(\Phi(x)))$, for every $n \in \mathbb{N}$, such that $x_n \xrightarrow[n\to\infty]{} \Phi(x)$. Every x_n can be represented as some finite convex sum $\sum_{k=1}^{N_n} \lambda_k^n y_k^n$, $N_n \in \mathbb{N}$, where $y_k^n \in B \setminus B_\varepsilon(\Phi(x))$ for every $1 \leq k \leq N_n$. Since we have $\|x - \Phi^{-1}(y_k^n)\| = \|\Phi(x) - y_k^n\| > \varepsilon$ for every y_k^n and

$$\Phi^{-1}(x_n) = \sum_{k=1}^{N_n} \lambda_k^n \Phi^{-1}(y_k^n),$$

we also have $\Phi^{-1}(x_n) \in \operatorname{conv}(A \setminus B_\varepsilon(x))$ for every $n \in \mathbb{N}$. Continuity of Φ^{-1} gives us that $\Phi^{-1}(x_n) \xrightarrow[n\to\infty]{} x$, i.e. $x \in \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$. Hence x is not a denting point of A. $\square$

Now we are able to show that the closed unit balls of spaces c and ℓ_∞ contain no denting points even though they have extreme points. For this, we will be using the previous lemma for the special case, where our set A is B_X, which means that its image is also B_X.

Lemma 2.16. *Let $X \coloneqq c$ or $X \coloneqq \ell_\infty$. The closed unit ball B_X contains no denting points.*

Proof. We know that $\operatorname{dent}(B_X) \subset \operatorname{ext}(B_X)$ by Lemma 2.10. Therefore, it suffices to show that no extreme point of B_X is a denting point. Let $x = (x_n) \in \operatorname{ext}(B_X)$ be arbitrary. By Examples 1.20 and 1.22, this means that $|x_n| = 1$ for every $n \in \mathbb{N}$, i.e. there exist $\theta_n \in \mathbb{R}$ such that $x_n = e^{i\theta_n}$. We can now define a mapping $\Phi\colon X \to X$ given by $\Phi(y) = (e^{-i\theta_n} y_n)$ for some $y = (y_n) \in X$. Note that if $X = \ell_\infty$, then Φ is clearly well-defined. If $X = c$, then the convergence of x implies the convergence of $(e^{-i\theta_n})$, which in turn guarantees the convergence of $\Phi(y)$ for all $y \in X$, so Φ is again well-defined. It is also clear that Φ is a linear isometry. Additionally, if $(y_n) \in X$, then its origin is the element $(e^{i\theta_n} y_n) \in X$, i.e. Φ is onto. Now we have that $\Phi(x) = \mathbb{1}$ and we have shown in Example 2.6 that $\mathbb{1}$ is not a denting point of B_X. Therefore, by Lemma 2.15, x is not a denting point of B_X, which completes the proof. $\square$

Denting points can also be described using slices, as we show in the following proof that is adapted from [Bou83, Proposition 2.3.2].

Proposition 2.17. *Let $A \subset X$ be some bounded set. Then $x \in A$ is a denting point of A if and only if A has slices of arbitrarily small diameter containing x.*

Proof. Suppose first that $x \in \operatorname{dent}(A)$. This means that for an arbitrary $\varepsilon > 0$, $x \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$. By the Hahn–Banach separation theorem (see Theorem 1.17), there exists $f \in X^*$ and $\delta \in \mathbb{R}$ such that

$$\operatorname{Re} f(x) > \delta > \sup_{y \in \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))} \operatorname{Re} f(y).$$

Let us choose

$$\alpha \coloneqq \sup_{y \in A} \operatorname{Re} f(y) - \delta \geq \operatorname{Re} f(x) - \delta > 0$$

and the slice $S \coloneqq S(A, f, \alpha)$. Now we have that $x \in S$, since

$$\operatorname{Re} f(x) > \delta = \sup_{y \in A} \operatorname{Re} f(y) - \alpha.$$

We also have $S \subset B_\varepsilon(x) \cap A$. Indeed, if $z \in S$, then $z \in A$ and

$$\operatorname{Re} f(z) > \sup_{y \in A} \operatorname{Re} f(y) - \alpha = \delta > \sup_{y \in \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))} \operatorname{Re} f(y),$$

which ensures that $z \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$ and in turn implies $z \in B_\varepsilon(x)$. This means that $\operatorname{diam}(S) \leq 2\varepsilon$ and this quantity can be made arbitrarily small.

Now assume that A has slices of arbitrarily small diameter containing x. For $\varepsilon > 0$, we want to show $x \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$. Choose a slice $S := S(A, f, \alpha)$, $f \in X^*$, $\alpha > 0$, such that $x \in S$ and $\operatorname{diam}(S) < \varepsilon$. Denote $\delta := \sup_{y \in A} \operatorname{Re} f(y) - \alpha$ and remark that

$$\overline{\operatorname{conv}}(A \setminus B_\varepsilon(x)) \subset \overline{\operatorname{conv}}(A \setminus S) \subset \operatorname{Re} f^{-1}[(-\infty, \delta]].$$

The first inclusion holds because $S \subset B_\varepsilon(x)$. For the second inclusion, assume that $z \in \overline{\operatorname{conv}}(A \setminus S)$. This means that there exists some sequence (z_n) of $\operatorname{conv}(A \setminus S)$ elements such that $z_n \xrightarrow[n \to \infty]{} z$. As every z_n can be expressed as some convex sum $\sum_{i=1}^{N_n} \lambda_i^n w_i^n$, $N_n \in \mathbb{N}$, $w_i^n \in A \setminus S$, we have

$$\operatorname{Re} f(z_n) = \sum_{i=1}^{N_n} \lambda_i^n \operatorname{Re} f(w_i^n) \leq \sum_{i=1}^{N_n} \lambda_i^n \delta = \delta.$$

Note that in the previous, the inequality holds thanks to the fact that the elements w_i^n are not in the slice S. This means that for all $n \in \mathbb{N}$, we have $\operatorname{Re} f(z_n) \leq \delta$. Now, since $\operatorname{Re} f(z_n) \xrightarrow[n \to \infty]{} \operatorname{Re} f(z)$, we also have $\operatorname{Re} f(z) \leq \delta$, i.e. $z \in \operatorname{Re} f^{-1}[(-\infty, \delta]]$ and the inclusion holds. Finally, note that $\operatorname{Re} f(x) > \delta$, so $x \notin \operatorname{Re} f^{-1}[(-\infty, \delta]]$, which also means $x \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(x))$. $\square$

Remark 2.18. In light of the previous result, it is interesting to note that every relatively weakly open subset of B_c, and in particular every slice of B_c, has a diameter of 2.

Indeed, let W be some relatively weakly open subset of B_c. Let us first show that $e_n \xrightarrow{w} 0$, where $e_n \in c$ still denotes the sequence with 1 at the n-th coordinate and 0 elsewhere. By the properties of the weak topology, it is equivalent to show that $f(e_n) \to 0$ for every $f \in c^*$.

It is known that $c^* \cong \ell_1$, where the isometric isomorphism is defined as $T\colon \ell_1 \to c^*$, $(Tx)(y) = x_0 \lim_{n\to\infty} y_n + \sum_{n=1}^{\infty} x_n y_n$, where $x = (x_n)_{n=0}^{\infty} \in \ell_1 = \ell_1(\mathbb{N} \cup \{0\})$ and $y = (y_n)_{n=1}^{\infty} \in c$ (see [FHH+11, Exercise 2.31]).

Therefore, for every $f \in c^*$, there exists some $x = (x_n)_{n=0}^{\infty} \in \ell_1$ such that $f = Tx$. In particular, $f(e_n) = x_n$ for every $n \in \mathbb{N}$. As $x \in \ell_1$ implies $x_n \to 0$, we also have $f(e_n) \to 0$. Hence $e_n \xrightarrow{w} 0$ as desired.

Since we know that every weakly open set in an infinite-dimensional normed space is unbounded (see Proposition 1.9), then there must also exist some $x_0 = (x_n^0) \in W \cap S_c$. Define elements $x_n \coloneqq x_0 + (1 - x_n^0)e_n \in B_c$ and $y_n \coloneqq x_0 - (1 + x_n^0)e_n \in B_c$. Since $x_n \xrightarrow{w} x_0$ and $y_n \xrightarrow{w} x_0$, there exists some $N \in \mathbb{N}$ such that $x_N, y_N \in W$. But then we also have $\|x_N - y_N\| = \|2e_N\| = 2$ and in particular $\operatorname{diam}(W) = 2$.

In some sense, this fact tells us that B_c is very 'far' from having denting points, even if it has extreme points. Note that the same thing can be shown analogously for c_0 and it is also known to be true in ℓ_∞, $L_1[0,1]$, and $C[0,1]$.

Using the description of denting points through slices, we can show that extreme points and denting points coincide in B_{ℓ_1}.

Lemma 2.19. *In ℓ_1, we have* $\operatorname{dent}(B_{\ell_1}) = \operatorname{ext}(B_{\ell_1}) = \{\lambda e_n \colon \lambda \in S_{\mathbb{K}}, n \in \mathbb{N}\}$.

Proof. We already know that $\operatorname{dent}(B_{\ell_1}) \subset \operatorname{ext}(B_{\ell_1})$. Let us show that the reverse inclusion holds. For this, recall that by Example 1.21, $\operatorname{ext}(B_{\ell_1})$ is made up of sequences of the form λe_k, $\lambda \in \mathbb{K}$, $|\lambda| = 1$, $k \in \mathbb{N}$.

Let us first show that the sequence e_k is a denting point of B_{ℓ_1}. Let $k \in \mathbb{N}$ be fixed and choose the functional $f \in X^*$ given by $f(x) = x_k$, i.e. f maps $x \in \ell_1$ to its k-th coordinate. Now look at the slice

$$S \coloneqq S(B_{\ell_1}, f, \alpha) = \{x \in B_{\ell_1} \colon \operatorname{Re} x_k > 1 - \alpha\},$$

where $\alpha \in (0,1)$. Since $f(e_k) = 1$, we have $e_k \in S$. For an arbitrary $x \in S$, we have

$$\begin{aligned}\|x-e_k\| &= \sum_{\substack{n=1\\n\neq k}}^{\infty}|x_n| + |x_k-1| \leq \sum_{\substack{n=1\\n\neq k}}^{\infty}|x_n| + |\operatorname{Re} x_k - 1| + |\operatorname{Im} x_k| \\ &\leq \alpha+\alpha+\sqrt{\alpha(2-\alpha)} = 2\alpha + \sqrt{\alpha(2-\alpha)}.\end{aligned}$$

Note that in the previous, we use the fact that $x \in S$ implies $\operatorname{Re} x_k > 1-\alpha$. In total, this gives us that $\operatorname{diam}(S) \leq 2(2\alpha + \sqrt{\alpha(2-\alpha)})$, and by choosing α sufficiently small, we can make this quantity smaller than any $\varepsilon > 0$. We have shown that B_{ℓ_1} has slices of arbitrarily small diameter containing e_k, therefore, by Proposition 2.17, e_k is a denting point of B_{ℓ_1}.

Finally, in order to finish the proof, let $x \in \operatorname{ext}(B_{\ell_1})$ be arbitrary. Then $x = \lambda e_k$ for some $\lambda \in \mathbb{K}$, $|\lambda| = 1$ and $k \in \mathbb{N}$. Define the map $\Phi\colon \ell_1 \to \ell_1, x \mapsto \lambda^{-1}x$, which is an onto linear isometry. As $\Phi(x) = e_k$ and e_k is a denting point, then x is also a denting point by Lemma 2.15. We indeed have $\operatorname{ext}(B_{\ell_1}) \subset \operatorname{dent}(B_{\ell_1})$. □

Our next aim is to show that denting points and extreme points coincide in ℓ_p spaces for every $p \in (1,\infty)$. For this, we will use the well-known fact that if $p, q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q} = 1$, then $\ell_p^* \cong \ell_q$. Furthermore, the respective isometric isomorphism can be defined as

$$T\colon \ell_q \to \ell_p^*,\ (Tx)(y) = \sum_{n=1}^{\infty} x_n y_n, \tag{2.1}$$

where $x = (x_n) \in \ell_q$ and $y = (y_n) \in \ell_p$ (see [Meg98, Theorem C.12]).

We will also make use of Clarkson's inequalities, that are proven in [Cla36, Theorem 2]. They state that if $x, y \in \ell_p$, then for $p \geq 2$, we have

$$\left\|\frac{x-y}{2}\right\|^p + \left\|\frac{x+y}{2}\right\|^p \leq \frac{1}{2}\left(\|x\|^p + \|y\|^p\right) \tag{2.2}$$

and for $1 < p < 2$, we have

$$\left\|\frac{x-y}{2}\right\|^q + \left\|\frac{x+y}{2}\right\|^q \leq \left(\frac{1}{2}\left(\|x\|^p + \|y\|^p\right)\right)^{\frac{q}{p}}, \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ once again.

Lemma 2.20. *In the space ℓ_p, $p \in (1, \infty)$, we have* $\operatorname{dent}(B_{\ell_p}) = \operatorname{ext}(B_{\ell_p}) = S_{\ell_p}$.

Proof. To prove the statement, it suffices to show that any $x \in S_{\ell_p}$ is a denting point of B_{ℓ_p}, as then we have $S_{\ell_p} \subset \operatorname{dent}(B_{\ell_p}) \subset \operatorname{ext}(B_{\ell_p}) \subset S_{\ell_p}$.

Let us first prove the case where $x = (x_n) \in S_{\ell_p}$ is such an element, that $x_n \in \mathbb{R}_{\geq 0}$ for every $n \in \mathbb{N}$, i.e. x_n are non-negative real numbers. For this, let $q = \frac{p}{p-1}$ and notice that the sequence $x' = (x_n^{p-1})$ belongs to ℓ_q, since

$$\|x'\|_q^q = \sum_{n=1}^{\infty} |x_n^{p-1}|^q = \sum_{n=1}^{\infty} x_n^{(p-1)q} = \sum_{n=1}^{\infty} x_n^p = 1.$$

Therefore, using the isometric isomorphism (2.1), we can pick the functional $f \coloneqq Tx' \in \ell_p^*$, which is given by $f(y) = \sum_{n=1}^{\infty} x_n^{p-1} y_n$, $y = (y_n) \in \ell_p$. We also know that $\|f\| = \|x'\| = 1$ and $f(x) = 1$. Let us look at the slice

$$S \coloneqq S(B_{\ell_p}, f, \alpha) = \left\{ y \in B_{\ell_p} \colon \operatorname{Re} f(y) > 1 - \alpha \right\},$$

where $\alpha \in (0, 1)$. It is clear that $x \in S$. Now let $y \in S$ be arbitrary, hence $\operatorname{Re} f(y) > 1 - \alpha$. Note that then $\frac{x+y}{2} \in S$, because

$$\operatorname{Re} f\left(\frac{x+y}{2}\right) = \frac{1}{2} \operatorname{Re} f(x) + \frac{1}{2} \operatorname{Re} f(y) > \frac{1}{2}(1-\alpha) + \frac{1}{2}(1-\alpha) = 1 - \alpha.$$

This in turn gives us

$$1 - \alpha < \operatorname{Re} f\left(\frac{x+y}{2}\right) \leq \left| f\left(\frac{x+y}{2}\right) \right| \leq \|f\| \left\| \frac{x+y}{2} \right\| = \left\| \frac{x+y}{2} \right\|.$$

Now if $p \geq 2$, using Clarkson's inequality (2.2), we get

$$\left\| \frac{x-y}{2} \right\|^p \leq \frac{1}{2} \left(\|x\|^p + \|y\|^p \right) - \left\| \frac{x+y}{2} \right\|^p < 1 - (1-\alpha)^p.$$

In total, we have $\|x - y\| < 2(1 - (1-\alpha)^p)^{\frac{1}{p}}$. For the case $1 < p < 2$, using the inequality (2.3), we have

$$\left\|\frac{x-y}{2}\right\|^q \leq \left(\frac{1}{2}\left(\|x\|^p + \|y\|^p\right)\right)^{\frac{q}{p}} - \left\|\frac{x+y}{2}\right\|^q < 1 - (1-\alpha)^q,$$

hence $\|x-y\| < 2(1-(1-\alpha)^q)^{\frac{1}{q}}$. In both cases, it is clear that as $\alpha \to 0$, the upper estimate for $\|x-y\|$ approaches 0. Therefore, for any $\varepsilon > 0$, we can always choose $\alpha > 0$ sufficiently small such that the diameter of $S(B_{\ell_p}, f, \alpha)$ is less than ε. This means that x is a denting point of B_{ℓ_p}.

Finally, if $x = (x_n) \in S_{\ell_p}$ is arbitrary, then $x' = (|x_n|) \in S_{\ell_p}$ is a denting point of the closed unit ball by the previously shown case. For every $n \in \mathbb{N}$, we can fix some $\theta_n \in \mathbb{R}$ such that $x_n = |x_n|e^{i\theta_n}$. Using this, we can define an onto linear isometry Φ from ℓ_p to ℓ_p, given by $\Phi(y) = (e^{-i\theta_n} y_n)$, $y = (y_n) \in \ell_p$. As currently $\Phi(x) = (|x_n|) = x'$ is a denting point, then $x \in \mathrm{dent}(B_{\ell_p})$ by Lemma 2.15. $\square$

3 Points of continuity and the main theorem

Throughout this section, let X denote a Banach space. The main goal of this chapter is to prove a classical result, which was first proven in [LLT88] and states that if A is a closed bounded convex subset of X, then the denting points of A are exactly the elements, which are simultaneously extreme points and points of continuity. For the proof, we will be following the general idea provided by H. Rosenthal in [Ros88, pages 29-30]. First, let us define points of continuity and go over some basic properties.

Definition 3.1. Let A be a bounded subset of X. We say that $x \in A$ is a *point of continuity* (PC) of A, if the identity mapping $(A, \text{weak}) \to (A, \text{norm})$ is continuous at x.

Let A be a bounded subset of X^*. We say that $x^* \in A$ is a *weak* point of continuity* (w^*-PC) of A, if the identity mapping $(A, \text{weak*}) \to (A, \text{norm})$ is continuous at x^*.

By $\text{PC}(A)$, we denote the set of all points of continuity of A. Using the topological definition of a mapping's continuity at some point, we can also describe points of continuity as follows.

Lemma 3.2. *Let A be a bounded subset of X. Then $x \in \text{PC}(A)$ if and only if, for every $\varepsilon > 0$, there exists a relatively weakly open subset W of A such that $x \in W$ and $\text{diam}(W) < \varepsilon$.*

We can formulate the definition of w^*-PC analogously. This fact and the Lemma 2.17 make it easy to see that if A is a non-empty bounded subset of X, then $\text{dent}(A) \subset \text{PC}(A)$, because slices are relatively weakly open.

We also know by Lemma 1.10, that $\text{PC}(B_X) = B_X$ whenever $\dim(X) < \infty$. This is not the case in infinite-dimensional spaces, as demonstrated by the next useful fact.

Lemma 3.3. *Let X be an infinite-dimensional Banach space. Then $\text{PC}(B_X) \subset S_X$.*

Proof. Let $x \in B_X$ and $y \in S_X$. By the reverse triangle inequality, we have $\|x - y\| \geq \|y\| - \|x\| = 1 - \|x\|$. We also know from Proposition 1.9 that every relatively weakly open subset of B_X contains a point from the unit sphere, since X is infinite dimensional. Combining these two facts, we get that if $\|x\| < 1$, then the diameter of any relatively weakly open subset of B_X containing x is at least $1 - \|x\| > 0$, i.e. x is not a PC of B_X by Lemma 3.2. $\square$

For an example, let us describe the points of continuity of B_{ℓ_1}.

Example 3.4. In the space ℓ_1, every element of S_{ℓ_1} is a point of continuity of B_{ℓ_1}. Indeed, let $x = \sum_{i=1}^{\infty} x_i e_i$ be some element of S_{ℓ_1} and fix an arbitrary $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\|\sum_{i=1}^{n} x_i e_i\| > 1 - \varepsilon$ and hence $\left\|\sum_{i=n+1}^{\infty} x_i e_i\right\| < \varepsilon$. Let us consider the relatively weakly open subset of B_{ℓ_1}

$$W := \left\{ y = \sum_{i=1}^{\infty} y_i e_i \in B_{\ell_1} \colon |x_i - y_i| < \frac{\varepsilon}{n}, 1 \leq i \leq n \right\}.$$

Clearly, $x \in W$. Now for every $y \in W$, we have

$$\left\| \sum_{i=1}^{n} (x_i - y_i) e_i \right\| = \sum_{i=1}^{n} |x_i - y_i| < n \cdot \frac{\varepsilon}{n} = \varepsilon,$$

therefore

$$\left\| \sum_{i=1}^{n} y_i e_i \right\| \geq \left\| \sum_{i=1}^{n} x_i e_i \right\| - \left\| \sum_{i=1}^{n} (x_i - y_i) e_i \right\| > 1 - 2\varepsilon.$$

This means that $\left\|\sum_{i=n+1}^{\infty} y_i e_i\right\| < 2\varepsilon$, hence we have

$$\|x - y\| \leq \left\| \sum_{i=1}^{n} (x_i - y_i) e_i \right\| + \left\| \sum_{i=n+1}^{\infty} x_i e_i \right\| + \left\| \sum_{i=n+1}^{\infty} y_i e_i \right\| < \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon.$$

In conclusion, $\operatorname{diam}(W) \leq 8\varepsilon$ and as this quantity can be made arbitrarily small, x is indeed a PC of B_{ℓ_1}. By Lemma 3.3, we can further say that $\operatorname{PC}(B_{\ell_1}) = S_{\ell_1}$. Note that we can show analogously that $\operatorname{PC}(B_{\ell_p}) = S_{\ell_p}$ for all $p \in (1, \infty)$.

Remark 3.5. The fact $\operatorname{PC}(B_{\ell_1}) = S_{\ell_1}$ from the previous example also shows us that extreme point and points of continuity do not generally coincide, since we have shown in Example 1.21 that $\operatorname{ext}(B_{\ell_1}) \neq S_{\ell_1}$.

Now we are ready to present the proof of the main theorem of the thesis, which we are going to split into three lemmata. We are also going to be working extensively with the bidual X^{**} of X. It is well-known that there exists a canonical embedding j_X from X into X^{**}, that is defined for $x \in X$ as $j_X(x)\colon f \mapsto f(x)$, for all $f \in X^*$. It can also be shown that j_X is a linear isometry (see [FHH+11, Definition 3.95]). Additionally, if $A \subset X^*$, then by $\overline{A}^{w^*}$ we will denote the weak* closure of A.

The equivalence of (i) and (iii) in the following Lemma 3.6 is given as a fact without proof in [Ros88, page 30].

Lemma 3.6. *Let A be a closed bounded convex subset of X. Let $\tilde{x} \in \overline{j_X(A)}^{w^*}$ in X^{**}. The following statements are equivalent:*

(i) *$\tilde{x}$ is a w^*-PC of $\overline{j_X(A)}^{w^*}$;*

(ii) *For every net $(\tilde{x}_\alpha)$ of $\overline{j_X(A)}^{w^*}$ elements, $\tilde{x}_\alpha \xrightarrow{w^*} \tilde{x}$ if and only if $\tilde{x}_\alpha \to \tilde{x}$ in norm;*

(iii) *There exists $x \in A$ such that $j_X(x) = \tilde{x}$ and $x \in \mathrm{PC}(A)$.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume that $\tilde{x}$ is a w^*-PC of $\overline{j_X(A)}^{w^*}$. Then, for every $\varepsilon > 0$, there exists some relatively weakly* open neighborhood W of $\overline{j_X(A)}^{w^*}$ containing $\tilde{x}$, which has a diameter less than ε. Therefore, if $\tilde{x}_\alpha \xrightarrow{w^*} \tilde{x}$, for some net $(\tilde{x}_\alpha)$ of $\overline{j_X(A)}^{w^*}$, there must exist some α_0 such that if $\alpha \succeq \alpha_0$, then we have $\tilde{x}_\alpha \in W$. Since $\tilde{x} \in W$, we thus have $\|\tilde{x}_\alpha - \tilde{x}\| < \varepsilon$ for every $\alpha \succeq \alpha_0$. Hence $\tilde{x}_\alpha \to \tilde{x}$ in norm. The reverse implication holds generally, since the norm topology is finer than the weak* topology.

(ii) $\Rightarrow$ *(iii)*. Let us first show that $\tilde{x} \in j_X(A)$. Since $\tilde{x} \in \overline{j_X(A)}^{w^*}$, there exists some net $(\tilde{x}_\alpha)$ of $j_X(A)$ elements such that $\tilde{x}_\alpha \xrightarrow{w^*} \tilde{x}$. By *(ii)*, $\tilde{x}_\alpha \to \tilde{x}$ in norm. Notice that $j_X(X)$ is a complete subspace of X^{**}, which itself is complete. Therefore, $j_X(X)$ is norm closed in X^{**}. Using this knowledge and the fact that A is a closed subset of X, we also get that $j_X(A)$ is norm closed in X^{**}. This means that $\tilde{x}_\alpha \to \tilde{x}$ implies $\tilde{x} \in j_X(A)$ by Proposition 1.4.

Now let $x \in A$ be the inverse image of $\tilde{x}$ and assume that x is not a PC of A. Then there exists some $\varepsilon > 0$ such that every relatively weakly open neighborhood of x of the form

$$U := \{y \in A \colon |f_i(y) - f_i(x)| < \delta,\ 1 \leq i \leq n\},$$

for some $n \in \mathbb{N}$, $f_i \in X^*$, $\delta > 0$, has diameter equal to or greater than ε. Let $\mathcal{I}$ denote the set containing all of these relatively weakly open neighbourhoods. This means that for every $U \in \mathcal{I}$, we can choose some element $x_U \in U$ such that $\|x_U - x\| \geq \frac{\varepsilon}{4}$ and we can also look at the corresponding set

$$\tilde{U} := \left\{\tilde{y} \in \overline{j_X(A)}^{w^*} : |f_i(\tilde{y}) - f_i(\tilde{x})| < \delta,\ 1 \leq i \leq n\right\}$$

in X^{**}, which is relatively weakly* open. If $\tilde{\mathcal{I}}$ denotes the set of these relatively weakly* open sets, we can make $\tilde{\mathcal{I}}$ a directed set by defining $\tilde{U} \preceq \tilde{V}$ if and only if $\tilde{U} \supset \tilde{V}$, for some $\tilde{U}, \tilde{V} \in \tilde{\mathcal{I}}$. Therefore, we can construct a net $(\tilde{x}_{\tilde{U}})$, where $\tilde{x}_{\tilde{U}} := j_X(x_U) \in \tilde{U}$. Notice that $\tilde{x}_{\tilde{U}} \xrightarrow{w^*} \tilde{x}$, because $\tilde{\mathcal{I}}$ contains all of the basic neighbourhoods of $\tilde{x}$ in the relative weak* topology of $\overline{j_X(A)}^{w^*}$, but the net does not converge in norm, because $\|\tilde{x}_{\tilde{U}} - \tilde{x}\| \geq \frac{\varepsilon}{4}$, for every possible $\tilde{x}_{\tilde{U}}$. In conclusion, x must be a PC of A if we assume (ii).

$(iii) \Rightarrow (i)$. Let $\varepsilon > 0$ be arbitrary. If the inverse image x of $\tilde{x}$ is a PC of A, then there exists some relatively weakly open subset

$$U := \{y \in A \colon |f_i(y) - f_i(x)| < \delta,\ 1 \leq i \leq n\},$$

where $n \in \mathbb{N}$, $f_i \in X^*$, $\delta > 0$, such that $\operatorname{diam}(U) < \varepsilon$. Then

$$\tilde{U} := \left\{\tilde{y} \in \overline{j_X(A)}^{w^*} : |f_i(\tilde{y}) - f_i(\tilde{x})| < \delta,\ 1 \leq i \leq n\right\}$$

is a relatively weakly* open set in $\overline{j_X(A)}^{w^*}$. If $\tilde{y}, \tilde{z} \in \tilde{U}$, then there exist nets $(\tilde{y}_\alpha)$, $(\tilde{z}_\alpha)$ of $j_X(A)$ elements over the same index set such that $\tilde{y}_\alpha \xrightarrow{w^*} \tilde{y}$ and $\tilde{z}_\alpha \xrightarrow{w^*} \tilde{z}$, hence $\tilde{y}_\alpha - \tilde{z}_\alpha \xrightarrow{w^*} \tilde{y} - \tilde{z}$. We may additionally assume that elements of $(\tilde{y}_\alpha)$ and $(\tilde{z}_\alpha)$ are in $j_X(U)$, since $\tilde{U}$ is a neighbourhood of $\tilde{y}$ and $\tilde{z}$, so there exists some α_0 such that for all $\alpha \succeq \alpha_0$, $\tilde{y}_\alpha$ and $\tilde{z}_\alpha$ are in $\tilde{U}$ and therefore in $j_X(U)$. Additonally, we have $\operatorname{diam}(j_X(U)) < \varepsilon$ since $\operatorname{diam}(U) < \varepsilon$. Now, by the lower semi-continuity of the norm, given in Theorem 1.14, we have

$$\|\tilde{y}-\tilde{z}\| \leq \liminf_{\alpha} \|\tilde{y}_\alpha - \tilde{z}_\alpha\| \leq \operatorname{diam}(j_X(U)) < \varepsilon.$$

Therefore, $\operatorname{diam}(\tilde{U}) \leq \varepsilon$ as $\tilde{y}$ and $\tilde{z}$ were arbitrary. This means that $\tilde{x}$ is a w^*-PC of $\overline{j_X(A)}^{w^*}$, as it belongs to relatively weakly* open subsets of $\overline{j_X(A)}^{w^*}$ of arbitrarily small diameter. □

Lemma 3.7 (see [Ros88, Lemma 3.2])**.** *Let A be a closed bounded convex subset of X. Let $\tilde{x}$ be a w^*-*PC *of $\overline{j_X(A)}^{w^*}$ in X^{**}. If $\tilde{x} = \lambda\tilde{y} + (1-\lambda)\tilde{z}$ for some $\tilde{y}, \tilde{z} \in \overline{j_X(A)}^{w^*}$ and $\lambda \in (0,1)$, then $\tilde{y}$ and $\tilde{z}$ are also w^*-*PC *of $\overline{j_X(A)}^{w^*}$.*

Proof. Let $\tilde{x} = \lambda\tilde{y} + (1-\lambda)\tilde{z}$ for some $\tilde{y}, \tilde{z} \in \overline{j_X(A)}^{w^*}$ and $\lambda \in (0,1)$. Let $(\tilde{y}_\alpha)$ be some net of $\overline{j_X(A)}^{w^*}$ elements such that $\tilde{y}_\alpha \xrightarrow{w^*} \tilde{y}$. Then $\lambda\tilde{y}_\alpha + (1-\lambda)\tilde{z} \xrightarrow{w^*} \tilde{x}$. Note that the previous convex combinations are in $\overline{j_X(A)}^{w^*}$, because it is a convex set. Indeed, if we have some $\tilde{a}, \tilde{b} \in \overline{j_X(A)}^{w^*}$ and $\lambda \in (0,1)$, then there exist nets $(\tilde{a}_\beta)$, $(\tilde{b}_\beta)$ of $j_X(A)$ elements, that converge weakly* to $\tilde{a}$ and $\tilde{b}$ respectively. Since it is clear that $j_X(A)$ is convex as A is convex, elements $\lambda\tilde{a}_\beta + (1-\lambda)\tilde{b}_\beta$ also belong to $j_X(A)$ and converge weakly* to $\lambda\tilde{a} + (1-\lambda)\tilde{b}$, which therefore belongs to $\overline{j_X(A)}^{w^*}$ by Proposition 1.4.

Now using the implication $(i) \Rightarrow (ii)$ from Lemma 3.6, as x is w^*-PC of $\overline{j_X(A)}^{w^*}$, we have $\lambda\tilde{y}_\alpha + (1-\lambda)\tilde{z} \to \tilde{x}$ in norm. This in turn implies

$$\|\tilde{y} - \tilde{y}_\alpha\| = \frac{1}{\lambda}\|\tilde{x} - (\lambda\tilde{y}_\alpha + (1-\lambda)\tilde{z})\| \to 0,$$

i.e. $\tilde{y}_\alpha \to \tilde{y}$ in norm. Therefore, as the net $(\tilde{y}_\alpha)$ was arbitrary, $\tilde{y}$ satisfies statement (ii) from Lemma 3.6, which implies that $\tilde{y}$ is a w^*-PC of $\overline{j_X(A)}^{w^*}$. For $\tilde{z}$, the proof is analogous. □

This next lemma is the first part of [Ros88, Proposition 3.3].

Lemma 3.8. *Let A be a closed bounded convex subset of X. If x is both a* PC *of A and an extreme point of A, then $\tilde{x} := j_X(x)$ is an extreme point of $\overline{j_X(A)}^{w^*}$ in X^{**}.*

Proof. Assume that x is a PC of A, but $\tilde{x}$ is not an extreme point of $\overline{j_X(A)}^{w^*}$. Then there exist some distinct $\tilde{y}, \tilde{z} \in \overline{j_X(A)}^{w^*}$ and $\lambda \in (0,1)$, such that $\tilde{x} =$

$\lambda\tilde{y} + (1-\lambda)\tilde{z}$. By Lemma 3.6, $\tilde{x}$ is w^*-PC of $\overline{j_X(A)}^{w^*}$. Therefore, by Lemma 3.7, $\tilde{y}$ and $\tilde{z}$ are w^*-PC of $\overline{j_X(A)}^{w^*}$ and hence, by Lemma 3.6, there exist $y, z \in A$ which are respectively the inverse images of $\tilde{y}$ and $\tilde{z}$. As j_X is injective, we have $x = \lambda y + (1-\lambda)z$, so x is not an extreme point of A. In conclusion, if x is also assumed to be an extreme point of A, then $\tilde{x}$ must be an extreme point of $\overline{j_X(A)}^{w^*}$. □

Now we have all of the tools to prove the main theorem of the thesis.

Theorem 3.9 (see [LLT88, Theorem])**.** *Let A be a closed bounded convex subset of X. Then $x \in A$ is a denting point of A if and only if x is both a* PC *of A and an extreme point of A.*

Proof. If $x \in A$ is a denting point, then we already know from Lemma 2.10 that it is also an extreme point. It is also a PC of A, since there exist slices of A that contain x and have arbitrarily small diameter.

Let x be a PC and an extreme point of A. By Lemma 3.8, $\tilde{x} := j_X(x)$ is an extreme point of $\overline{j_X(A)}^{w^*}$. By Corollary 1.12, $\overline{j_X(A)}^{w^*}$ is a weak* compact convex subset of X^{**}. Hence, Choquet's lemma (see Lemma 1.23) implies that weak* slices of $\overline{j_X(A)}^{w^*}$ containing $\tilde{x}$ form a neighborhood basis at $\tilde{x}$ in the relative weak* topology of $\overline{j_X(A)}^{w^*}$. As $\tilde{x}$ is a w^*-PC of $\overline{j_X(A)}^{w^*}$ by Lemma 3.6, there exists some $\tilde{U} \subset \overline{j_X(A)}^{w^*}$ containing $\tilde{x}$ that is a relatively weakly* open subset and satisfies $\operatorname{diam}(\tilde{U}) < \varepsilon$ for an arbitrary $\varepsilon > 0$. Since weak* slices form a neighborhood basis at $\tilde{x}$, there exists some weak* slice $\tilde{S} := S(\overline{j_X(A)}^{w^*}, f, \alpha)$, $f \in X^*$, $\alpha > 0$, such that $\tilde{S} \subset \tilde{U}$ and $\tilde{x} \in \tilde{S}$. But now $S := S(A, f, \alpha)$ is a slice of A that contains x and has a diameter of less than ε, because $j_X(S) \subset \tilde{S}$. Therefore, x is a denting point of A by Proposition 2.17. □

By combining the previous theorem and Lemma 1.10, which tells us that $\mathrm{PC}(A) = A$ in a finite-dimensional space, we get the next immediate corollary.

Corollary 3.10. *Let X be a finite-dimensional Banach space and let A be a closed bounded convex subset of X. Then* $\operatorname{dent}(A) = \operatorname{ext}(A)$.

The previous result obviously also implies that in a finite-dimensional Banach space, we always have $\operatorname{dent}(B_X) = \operatorname{ext}(B_X)$.

4 Extremal structure of direct sums of Banach spaces

The goal of this chapter is to describe denting points in direct sum spaces. First some basic definitions and facts, which one can find, for example, in [Meg98, Section 1.8].

Definition 4.1. Let X and Y be some normed spaces. The *p-direct sum* of X and Y, $1 \leq p \leq \infty$, is the normed space whose underlying vector space is the vector space sum $X \times Y$ of X and Y, and whose norm is given by

$$\|(x,y)\|_p = \begin{cases} (\|x\|^p + \|y\|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max\{\|x\|, \|y\|\}, & \text{if } p = \infty, \end{cases}$$

for every $(x,y) \in X \times Y$. We denote the p-direct sum space as $X \oplus_p Y$.

It is also well-known that the normed space $X \oplus_p Y$ is a Banach space if and only if X and Y are Banach spaces (see [Meg98, Theorem 1.8.6]). Additionally, the dual space of $X \oplus_p Y$ is isometrically isomorphic to $X^* \oplus_q Y^*$, where $q = 1$ if $p = \infty$, $q = \infty$ if $p = 1$, and $q = \frac{p}{p-1}$ if $p \in (1, \infty)$ (see [Meg98, Theorem 1.10.13]). Furthermore, if $h \in (X \oplus_p Y)^*$ is identified by $(f,g) \in X^* \oplus_q Y^*$, then $h(x,y) = f(x) + g(y)$, for all $(x,y) \in X \oplus_p Y$.

Let us first explore the denting points of $X \oplus_p Y$ when $p = \infty$. We are going to do this by first describing the extreme points and points of continuity.

Lemma 4.2. *Let X and Y be Banach spaces. Then $(x,y) \in \operatorname{ext}(B_{X\oplus_\infty Y})$ if and only if $x \in \operatorname{ext}(B_X)$ and $y \in \operatorname{ext}(B_Y)$.*

Proof. Let $(x,y) \in \operatorname{ext}(B_{X\oplus_\infty Y})$. If $x = \frac{1}{2}(x_1 + x_2)$ for some $x_1, x_2 \in B_X$, then $(x,y) = \frac{1}{2}((x_1,y) + (x_2,y))$, which implies $(x_1,y) = (x_2,y)$, i.e. $x_1 = x_2$. This means that x is an extreme point of B_X. Note that in the previous, the infinity norm guarantees that elements (x_1,y) and (x_2,y) are in the closed unit ball of $X \oplus_\infty Y$. The case of y can be proven analogously.

Let $x \in \operatorname{ext}(B_X)$ and $y \in \operatorname{ext}(B_Y)$. Assume that $(x,y) = \frac{1}{2}((x_1,y_1) + (x_2,y_2))$ for some $(x_1,y_1),(x_2,y_2) \in B_{X\oplus_\infty Y}$. This means that $x = \frac{1}{2}(x_1 + x_2)$ and $y =$

$\frac{1}{2}(y_1 + y_2)$, but since both are extreme points, we have $x_1 = x_2$ and $y_1 = y_2$, so $(x_1, y_1) = (x_2, y_2)$. Hence (x, y) is indeed an extreme point. $\square$

Lemma 4.3. *Let X and Y be Banach spaces. Then $(x, y) \in \mathrm{PC}(B_{X \oplus_\infty Y})$ if and only if $x \in \mathrm{PC}(B_X)$ and $y \in \mathrm{PC}(B_Y)$.*

Proof. Let us denote $Z := X \oplus_\infty Y$ and let $\varepsilon > 0$ be arbitrary.

Let $(x_0, y_0) \in \mathrm{PC}(B_Z)$. Then there exists a relatively weakly open set

$$W := \{(x, y) \in B_Z \colon |h_i(x, y) - h_i(x_0, y_0)| < \delta, 1 \leq i \leq n\},$$

$n \in \mathbb{N}, h_i = (f_i, g_i) \in Z^* = X^* \oplus_1 Y^*, \delta > 0$, such that $\mathrm{diam}(W) < \varepsilon$. Now we can construct a relatively weakly open set in B_X as

$$U := \{x \in B_X \colon |f_i(x) - f_i(x_0)| < \delta, 1 \leq i \leq n\}.$$

Notice that $x_0 \in U$ and if $x \in U$, then $(x, y_0) \in W$. Hence, we have $\|x - x_0\| = \|(x, y_0) - (x_0, y_0)\| < \varepsilon$. Therefore, $\mathrm{diam}(U) \leq 2\varepsilon$, which implies that $x_0 \in \mathrm{PC}(B_X)$. The case $y_0 \in \mathrm{PC}(B_Y)$ can be proven analogously.

Let $x_0 \in \mathrm{PC}(B_X)$ and $y_0 \in \mathrm{PC}(B_Y)$. Then there exist relatively weakly open sets of the form

$$U := \{x \in B_X \colon |f_i(x) - f_i(x_0)| < \delta_1, 1 \leq i \leq n\},$$

$n \in \mathbb{N}, f_i \in X^*, \delta_1 > 0$, and

$$V := \{y \in B_Y \colon |g_i(y) - g_i(y_0)| < \delta_2, 1 \leq i \leq m\},$$

$m \in \mathbb{N}, g_i \in Y^*, \delta_2 > 0$, such that diameters of U and V are less than ε. But now we can construct a relatively weakly open set of B_Z as

$$W := \{(x, y) \in B_Z \colon |h_i(x, y) - h_i(x_0, y_0)| < \min\{\delta_1, \delta_2\}, 1 \leq i \leq n + m\},$$

where $h_i = (f_i, 0) \in Z^*$, for $1 \leq i \leq n$, and $h_i = (0, g_{i-n}) \in Z^*$, for $n + 1 \leq i \leq n + m$. If $(x, y) \in W$, then also $x \in U$ and $y \in V$. Hence $\|(x, y) - (x_0, y_0)\| = \max\{\|x - x_0\|, \|y - y_0\|\} < \varepsilon$, which implies $\mathrm{diam}(W) \leq 2\varepsilon$. In conclusion,

$(x_0, y_0) \in \mathrm{PC}(B_Z)$. $\square$

Using Theorem 3.9 and Lemmata 4.2 and 4.3, we can conclude an analogous result for denting points.

Theorem 4.4. *Let X and Y be Banach spaces. Then $(x, y) \in \mathrm{dent}(B_{X \oplus_\infty Y})$ if and only if $x \in \mathrm{dent}(B_X)$ and $y \in \mathrm{dent}(B_Y)$.*

Now, let $1 \leq p < \infty$. Let us begin again by describing the extreme points and points of continuity of $B_{X \oplus_p Y}$.

Lemma 4.5 (see [DS08, Propositions 2 and 3])**.** *Let X and Y be Banach spaces. If $(x, y) \in \mathrm{ext}(B_{X \oplus_p Y})$, then*

(a) $(\|x\|, \|y\|) \in \mathrm{ext}(B_{\ell_p^2})$;

(b) $\frac{x}{\|x\|} \in \mathrm{ext}(B_X)$, *whenever* $x \neq 0$;

(c) $\frac{y}{\|y\|} \in \mathrm{ext}(B_Y)$, *whenever* $y \neq 0$.

Proof. Let (x, y) be an extreme point of $B_{X \oplus_p Y}$. This also implies that $\|(x, y)\| = 1$ and therefore, if $x = 0$ or $y = 0$, then $(\|x\|, \|y\|)$ is either $(1, 0)$ or $(0, 1)$, which are extreme points of $B_{\ell_p^2}$. Now let both x and y be non-zero. If we assume that $(\|x\|, \|y\|) = \frac{1}{2}((r_1, s_1) + (r_2, s_2))$ for some $(r_1, s_1), (r_2, s_2) \in B_{\ell_p^2}$, then we also have $(x, y) = \frac{1}{2}\left(\left(r_1 \frac{x}{\|x\|}, s_1 \frac{y}{\|y\|}\right) + \left(r_2 \frac{x}{\|x\|}, s_2 \frac{y}{\|y\|}\right)\right)$. Since the elements on the right-hand side of the previous equality are in $B_{X \oplus_p Y}$, we have $\left(r_1 \frac{x}{\|x\|}, s_1 \frac{y}{\|y\|}\right) = \left(r_2 \frac{x}{\|x\|}, s_2 \frac{y}{\|y\|}\right)$, which implies $r_1 = r_2$ and $s_1 = s_2$. Hence $(\|x\|, \|y\|)$ is an extreme point of $B_{\ell_p^2}$.

Assume that $x \neq 0$. If $\frac{x}{\|x\|} = \frac{1}{2}(x_1 + x_2)$ for some $x_1, x_2 \in B_X$, then $(x, y) = \frac{1}{2}((\|x\| x_1, y) + (\|x\| x_2, y))$, which implies that $x_1 = x_2$. Therefore, $\frac{x}{\|x\|}$ is indeed an extreme point of B_X. The case for $y \neq 0$ is analogous. $\square$

Lemma 4.6 (see [DS08, Proposition 4])**.** *Let X and Y be Banach spaces. If $x \in B_X$ and $y \in B_Y$ are elements satisfying the conditions*

(a) $(\|x\|, \|y\|) \in \mathrm{ext}(B_{\ell_p^2})$;

(b) $\frac{x}{\|x\|} \in \mathrm{ext}(B_X)$, *whenever* $x \neq 0$;

(c) $\frac{y}{\|y\|} \in \operatorname{ext}(B_Y)$, *whenever* $y \neq 0$,

then $(x,y) \in \operatorname{ext}(B_{X\oplus_p Y})$.

Proof. Let $x \in B_X$ and $y \in B_Y$ satisfy the conditions of the statement. Let us assume $(x,y) = \frac{1}{2}((x_1,y_1)+(x_2,y_2))$, where $(x_1,y_1),(x_2,y_2) \in B_{X\oplus_p Y}$. Notice that

$$\begin{aligned}1 = \|(x,y)\| &= \frac{1}{2}\|(x_1+x_2, y_1+y_2)\| = \frac{1}{2}\left(\|x_1+x_2\|^p + \|y_1+y_2\|^p\right)^{\frac{1}{p}} \\ &\leq \frac{1}{2}\left((\|x_1\|+\|x_2\|)^p + (\|y_1\|+\|y_2\|)^p\right)^{\frac{1}{p}} = \frac{1}{2}\|(\|x_1\|,\|y_1\|)+(\|x_2\|,\|y_2\|)\|_p \\ &\leq \frac{1}{2}(\|(\|x_1\|,\|y_1\|)\|_p + \|(\|x_2\|,\|y_2\|)\|_p) = \frac{1}{2}(1+1) = 1,\end{aligned}$$

from which we get

$$\frac{1}{2}\left((\|x_1\|+\|x_2\|)^p + (\|y_1\|+\|y_2\|)^p\right)^{\frac{1}{p}} = 1.$$

Now denote $r := 2\|x\| - \frac{1}{2}(\|x_1\|+\|x_2\|)$ and $s := 2\|y\| - \frac{1}{2}(\|y_1\|+\|y_2\|)$. We have $|r| \leq \frac{1}{2}(\|x_1\|+\|x_2\|)$, since if $r<0$, then $|r| = -r \leq \frac{1}{2}(\|x_1\|+\|x_2\|)$, and if $r \geq 0$, then $|r| = r \leq (\|x_1\|+\|x_2\|) - \frac{1}{2}(\|x_1\|+\|x_2\|) = \frac{1}{2}(\|x_1\|+\|x_2\|)$. Analogously, we have $|s| \leq \frac{1}{2}(\|y_1\|+\|y_2\|)$. This ensures that

$$\|(r,s)\|_p = (|r|^p + |s|^p)^{\frac{1}{p}} \leq \frac{1}{2}\left((\|x_1\|+\|x_2\|)^p + (\|y_1\|+\|y_2\|)^p\right)^{\frac{1}{p}} = 1,$$

i.e. $(r,s) \in B_{\ell_p^2}$. Now

$$(\|x\|,\|y\|) = \frac{1}{2}\left((r,s) + \frac{1}{2}(\|x_1\|+\|x_2\|, \|y_1\|+\|y_2\|)\right)$$

and since $(\|x\|,\|y\|)$ is an extreme point of $B_{\ell_p^2}$, we have $r = \frac{1}{2}(\|x_1\|+\|x_2\|)$, hence $\|x\| = \frac{1}{2}(\|x_1\|+\|x_2\|)$. Analogously, $\|y\| = \frac{1}{2}(\|y_1\|+\|y_2\|)$. This in turn gives us

$$(\|x\|,\|y\|) = \frac{1}{2}((\|x_1\|,\|y_1\|)+(\|x_2\|,\|y_2\|))$$

and using the extreme point definition, we have $\|x\| = \|x_1\| = \|x_2\|$ and $\|y\| = \|y_1\| = \|y_2\|$. If $x = 0$, then $x_1 = x_2 = 0$. If $x \neq 0$, then $\frac{x}{\|x\|} = \frac{1}{2}\left(\frac{x_1}{\|x\|} + \frac{x_2}{\|x\|}\right)$ implies $x_1 = x_2$, since $\frac{x}{\|x\|}$ is an extreme point of B_X and $\frac{x_1}{\|x\|}, \frac{x_2}{\|x\|} \in B_X$. By the same argument, we get $y_1 = y_2$. This gives us that $(x_1, y_1) = (x_2, y_2)$. Therefore, (x, y) is indeed an extreme point of $B_{X \oplus_p Y}$. $\square$

Combining Lemmata 4.5 and 4.6, we get the following description for extreme points.

Theorem 4.7. *Let X and Y be Banach spaces. Then $(x, y) \in \operatorname{ext}(B_{X \oplus_p Y})$ if and only if $x \in B_X$ and $y \in B_Y$ satisfy conditions*

(a) $(\|x\|, \|y\|) \in \operatorname{ext}(B_{\ell_p^2})$;

(b) $\frac{x}{\|x\|} \in \operatorname{ext}(B_X)$, *whenever* $x \neq 0$;

(c) $\frac{y}{\|y\|} \in \operatorname{ext}(B_Y)$, *whenever* $y \neq 0$.

When it comes to the points of continuity of $B_{X \oplus_p Y}$, we already know that if $X \oplus_p Y$ is a finite-dimensional space, which clearly holds if and only if X and Y are finite-dimensional spaces, then $\mathrm{PC}(B_{X \oplus_p Y}) = B_{X \oplus_p Y}$ as a consequence of Lemma 1.10. Therefore, it remains to describe points of continuity, when $X \oplus_p Y$ is an infinite-dimensional space. For this, let us first prove two lemmas.

Lemma 4.8. *Let X and Y be Banach spaces. If $(x, y) \in \mathrm{PC}(B_{X \oplus_p Y})$, then*

(a) $\frac{x}{\|x\|} \in \mathrm{PC}(B_X)$, *whenever* $x \neq 0$;

(b) $\frac{y}{\|y\|} \in \mathrm{PC}(B_Y)$, *whenever* $y \neq 0$.

Proof. Let $x \neq 0$ and let (x_α) be some net of B_X such that $x_\alpha \xrightarrow{w} \frac{x}{\|x\|}$. Then we also have $\|x\| x_\alpha \xrightarrow{w} x$, hence $((\|x\| x_\alpha, y))$, which is a net in $B_{X \oplus_p Y}$, converges weakly to (x, y). As (x, y) is a point of continuity, then also $(\|x\| x_\alpha, y) \to (x, y)$ in norm, which implies that $x_\alpha \to \frac{x}{\|x\|}$ in norm. Therefore, $\frac{x}{\|x\|} \in \mathrm{PC}(B_X)$. The case for $y \neq 0$ is analogous. $\square$

Lemma 4.9. *Let X and Y be Banach spaces. If $x \in B_X$ and $y \in B_Y$ are elements satisfying the conditions*

(a) $\|(x, y)\| = 1$;

(b) $\frac{x}{\|x\|} \in \mathrm{PC}(B_X)$, *whenever* $x \neq 0$;

(*c*) $\frac{y}{\|y\|} \in \mathrm{PC}(B_Y)$, *whenever* $y \neq 0$,

then $(x, y) \in \mathrm{PC}(B_{X \oplus_p Y})$.

Proof. Let (x_α, y_α) be some net in $B_{X\oplus_p Y}$ such that $(x_\alpha, y_\alpha) \xrightarrow{w} (x, y)$. In order to prove the statement, it suffices to show that $(x_\alpha, y_\alpha) \to (x, y)$ in norm. Note that the weak convergence of (x_α, y_α) implies that $x_\alpha \xrightarrow{w} x$ in X. Indeed, since the operator $T\colon X \oplus_p Y \to X$, $(x, y) \mapsto x$ is linear and norm-to-norm continuous, it is also weak-to-weak continuous (see [Meg98, Theorem 2.5.11]), hence $x_\alpha = T(x_\alpha, y_\alpha) \xrightarrow{w} T(x, y) = x$. We have $y_\alpha \xrightarrow{w} y$ in Y by the same argument.

By Theorem 1.13, we have $\|x\| \leq \liminf_\alpha \|x_\alpha\|$ and $\|y\| \leq \liminf_\alpha \|y_\alpha\|$. Let us assume that $\limsup_\alpha \|x_\alpha\| > \|x\|$. Then, by Lemma 1.7, we can extract a subnet (x_β) of (x_α) such that for some $\eta > 0$, we have $\|x_\beta\| \geq \|x\| + \eta$ for every β. So by the property (*c*) in Proposition 1.6, we have

$$\begin{aligned}\liminf_\beta \|(x_\beta, y_\beta)\|^p &= \liminf_\beta \left(\|x_\beta\|^p + \|y_\beta\|^p\right) \geq \liminf_\beta \|x_\beta\|^p + \liminf_\beta \|y_\beta\|^p \\ &\geq (\|x\| + \eta)^p + \|y\|^p > \|x\|^p + \|y\|^p = 1,\end{aligned}$$

but this is impossible, since $\|(x_\beta, y_\beta)\| \leq 1$ for every β. Therefore, we must have $\limsup_\alpha \|x_\alpha\| \leq \|x\|$. Using the properties (*a*) and (*b*) from Proposition 1.6, we get that $\liminf_\alpha \|x_\alpha\| = \limsup_\alpha \|x_\alpha\|$ and hence $\lim_\alpha \|x_\alpha\| = \|x\|$. We can show analogously that $\lim_\alpha \|y_\alpha\| = \|y\|$.

Notice that if $x = 0$, then $\lim_\alpha \|x_\alpha\| = 0$, i.e. $x_\alpha \to x$ in norm. Let us assume that $x \neq 0$. Then $x_\alpha \xrightarrow{w} x$ and $\|x_\alpha\| \to \|x\|$ imply that $\frac{x_\alpha}{\|x_\alpha\|} \xrightarrow{w} \frac{x}{\|x\|}$. Note that in the previous, the fact that x is non-zero allows us to assume that there exists some α_0 such that $\alpha \succeq \alpha_0$ implies $x_\alpha \neq 0$. Now by the assumption that $\frac{x}{\|x\|} \in \mathrm{PC}(B_X)$, we have $\frac{x_\alpha}{\|x_\alpha\|} \to \frac{x}{\|x\|}$ in norm. Therefore, we also have $x_\alpha \to x$ in norm, since

$$\|x_\alpha - x\| = \|x\| \cdot \left\| \frac{x_\alpha}{\|x\|} - \frac{x}{\|x\|} \right\| \leq \|x\| \cdot \left(\left\| \frac{x_\alpha}{\|x\|} - \frac{x_\alpha}{\|x_\alpha\|} \right\| + \left\| \frac{x_\alpha}{\|x_\alpha\|} - \frac{x}{\|x\|} \right\| \right).$$

Using the same argumentation, we get that $y_\alpha \to y$ in norm. But now we also have $(x_\alpha, y_\alpha) \to (x, y)$ in norm, as

$$\|(x_\alpha, y_\alpha) - (x, y)\|^p = \|x_\alpha - x\|^p + \|y_\alpha - y\|^p.$$

This means that we indeed have $(x, y) \in \mathrm{PC}(B_{X \oplus_p Y})$. $\square$

Combining Lemmata 4.8 and 4.9 and using the fact that every PC of the closed unit ball in an infinite-dimensional Banach space has norm 1 as shown in Lemma 3.3, we get the following description for points of continuity.

Theorem 4.10. *Let X and Y be Banach spaces such that $X \oplus_p Y$ is an infinite-dimensional Banach space. Then $(x, y) \in \mathrm{PC}(B_{X \oplus_p Y})$ if and only if $x \in B_X$ and $y \in B_Y$ satisfy conditions*

(*a*) $\|(x, y)\| = 1$;

(*b*) $\frac{x}{\|x\|} \in \mathrm{PC}(B_X)$, *whenever* $x \neq 0$;

(*c*) $\frac{y}{\|y\|} \in \mathrm{PC}(B_Y)$, *whenever* $y \neq 0$.

Finally, we can use the proven results to describe the denting points of $B_{X \oplus_p Y}$.

Theorem 4.11. *Let X and Y be Banach spaces. Then $(x, y) \in \mathrm{dent}(B_{X \oplus_p Y})$ if and only if $x \in B_X$ and $y \in B_Y$ satisfy the following conditions*

(*a*) $(\|x\|, \|y\|) \in \mathrm{ext}(B_{\ell_p^2})$;

(*b*) $\frac{x}{\|x\|} \in \mathrm{dent}(B_X)$, *whenever* $x \neq 0$;

(*c*) $\frac{y}{\|y\|} \in \mathrm{dent}(B_Y)$, *whenever* $y \neq 0$.

Proof. If $(x, y) \in \mathrm{dent}(B_{X \oplus_p Y})$, then $(x, y) \in \mathrm{ext}(B_{X \oplus_p Y})$ and $(x, y) \in \mathrm{PC}(B_{X \oplus_p Y})$. By Theorem 4.7, we have $(\|x\|, \|y\|) \in \mathrm{ext}(B_{\ell_p^2})$. If $x \neq 0$, then $\frac{x}{\|x\|}$ is and extreme point of B_X by Theorem 4.7 and a point of continuity of B_X by Lemma 4.8. Therefore, by Theorem 3.9, $\frac{x}{\|x\|}$ is a denting point of B_X. The case for $y \neq 0$ is analogous.

Let x and y satisfy the given condition. As we know that every denting point is both an extreme point and a PC, then by Theorem 4.7 and Lemma 4.9, we have $(x, y) \in \mathrm{ext}(B_{X \oplus_p Y})$ and $(x, y) \in \mathrm{PC}(B_{X \oplus_p Y})$, respectively. Here we also used the fact that the condition (a) implies $\|(x, y)\| = 1$. By Theorem 3.9, we have $(x, y) \in \mathrm{dent}(B_{X \oplus_p Y})$. $\square$

Summary

The main aim of this thesis was to give a comprehensive overview of denting points in various Banach spaces and describe their relations to extreme points, points of continuity, and σ-denting points.

In Chapter 1, we recalled various necessary definitions and results from general topology and functional analysis such as nets, weak and weak* topologies, slices and extreme points.

In Chapter 2, we showed that if A is a bounded subset, then $\mathrm{dent}(A) \subset \sigma\text{-}\mathrm{dent}(A) \subset \mathrm{ext}(A)$, and demonstrated with examples that these inclusions may be strict. Furthermore, we showed that if A is additionally assumed to be closed and convex, then $\sigma\text{-}\mathrm{dent}(A) = \mathrm{ext}(A)$. We also showed that denting points of A can be defined as elements contained in slices of arbitrarily small diameter and used this fact to describe the denting points of the closed unit ball of ℓ_p.

In Chapter 3, we showed that for a bounded set A, we have $\mathrm{dent}(A) \subset \mathrm{PC}(A)$, which may be a strict inclusion. We also proved a classical theorem that states if A is a closed bounded convex subset, then $\mathrm{dent}(A) = \mathrm{ext}(A) \cap \mathrm{PC}(A)$. This allowed us to conclude that if the Banach space X is finite-dimensional, then we always have $\mathrm{dent}(B_X) = \mathrm{ext}(B_X)$.

In Chapter 4, we studied the points of continuity and extreme points of the closed unit ball in the direct sum space $X \oplus_p Y$ of two Banach spaces, using the theorem from Chapter 3 to also describe the denting points. We concluded that if $p = \infty$, then $(x, y) \in B_{X \oplus_\infty Y}$ if and only if $x \in \mathrm{dent}(B_X)$ and $y \in \mathrm{dent}(B_Y)$, while the case for $1 \leq p < \infty$ requires more specific conditions.

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