University of Tartu
Faculty of Science and Technology
Institute of Mathematics and Statistics

Kätlin Kippar

Structural Time Series Models in GDP Analysis

Financial and Actuarial Mathematics

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Supervisor: Toomas Raus

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Abstract

The aim of this thesis is to use structural time series models to estimate the business cycles of Estonia and its five key trading partners, with additional objectives of examining potential economic dependencies between the estimated cycles and evaluating the forecast accuracy of structural models. To address these research questions, structural models with both trigonometric and ARMA cycle formulations were applied to GDP time series data, alongside an alternative cycle estimation method of the Hodrick-Prescott (HP) filter. The resulting cycle estimates were further used to test Granger causality. Additionally, ARIMA models were estimated for comparative purposes in forecasting evaluation. The thesis includes a brief overview of gross domestic product (GDP) and business cycles, a comprehensive overview of the applied methods, a summary of relevant previous research, and the results of the analysis. As a result of this thesis, business cycles were successfully estimated for all the countries considered in the analysis, economic dependencies between Estonia and its trading partners were identifies, and the forecast accuracy of the models was assessed.

CERCS research specialisation: P160 Statistics, operations research, programming, financial and actuarial mathematics.

Key words: Structural time series models, gross domestic product (GDP), business cycle estimation, Hodrick-Prescott filter, Granger causality, forecasting.

SKP analüüs struktuursete aegridade mudelite abil

Magistritöö

Kätlin Kippar

Lühikokkuvõte

Käesoleva magistritöö eesmärk on hinnata Eesti ja tema viie suurima eksportpartneri majandustsükleid kasutades selleks struktuurseid aegridade mudeleid. Lisaks soovitakse hinnatud majandustüklite põhjal kindlaks teha riikidevahelised potentsiaalsed majanduslikud sõltuvused ning hinnata struktuursete mudelite prognoositäpsust. Selleks modelleeritakse SKP aegridu struktuursete mudelite abil, kasutades selleks nii trigonomeetrilist kui ka ARMA tsükli esitust. Sealjuures hinnatakse majandustüklit ka alternatiivse meetodiga, milleks on Hodrick-Prescott (HP) filter. Saadud tsükli hinnangute alusel testitakse Grangeri standardset põhjuslikkust. Mudelite prognoositäpsuse hindamisel kaasatakse võrdluse saavutamiseks ka ARIMA mudelid. Töös tuuakse lühike ülevaade sisemajanduse koguproduktist (SKP) ja majandustsüklitest, ülevaade töös kasutatavast metoodikast, lühiülevaade varasemate uuringute tulemustest ning saadud tulemused. Magistritöö tulemusena hinnati majandustüklid, tuvastati Eesti ja tema eksportpartnerite vahelisi majanduslikke sõltuvusi ning hinnati mudelite prognoositäpsust.

CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.

Märksõnad: Struktuursed aegridade mudelid, sisemajanduse koguprodukt (SKP), majandustsüklite hindamine, Hodrick-Prescott filter, Grangeri põhjuslikkus, prognoosimine.

Contents

Introduction

Gross Domestic Product (GDP) is a key indicator of country's economic performance, used to describe the overall economic activity. Economic activity tends to follow a cyclical pattern and is characterized by periods of growth and decline that vary both in duration and magnitude (Škare and Stjepanović, 2016). Various methods have been developed to estimate and analyse these economic patterns, one of which are structural time series models.

The aim of the thesis is to analyze the GDP time series of Estonia and its five largest export partners with the objective of determining the economic (or business) cycles of each country and exploring potential interdependencies between these cycles. The central focus of this thesis is on the application of structural time series models for determining the economic cycles and forecasting, with the resulting cycle estimates used in assessing interdependencies between Estonia and its key trading partners. In addition to structural models, the Hodrick-Prescott filter for determining the economic cycle and ARIMA models for forecasting are considered.

The thesis is divided into four sections. The first section introduces background information on GDP and economic cycles, along with the description and formulation of one of the cycle estimation methods - the Hodrick-Prescott (HP) filter. The following chapter provides a comprehensive overview of the main methods used in this thesis, starting with a concise overview of ARIMA models, followed by a detailed overview of structural time series models in the univariate case with separate descriptions and formulations for each of the model components. The chapter concludes with a definition of Granger causality and the presentation of the Granger causality test in its F-test formulation. The third chapter provides a brief overview of previous research on the use of structural time series models for estimating business cycles. The final chapter of this thesis presents the estimated business cycles of Estonia and its five biggest export partners, together with the results of the Granger causality analysis and the forecast accuracy evaluation.

1 GDP and Business Cycles

Gross domestic product (GDP) is a measure that provides an overview of the country's economic activity over some specified period of time. Over the years, economists have sought to find the best way to define GDP - an economic model that would give a precise overview of a nation's income. Discussions have emerged on various aspects of what should be accounted for in GDP calculations, including what qualifies as an expense or income and how one can measure the input of services provided. With its rich history and alterations over the years, the current definition of GDP in a very simplified form is the sum of exports minus imports and the money spent in the country itself. (Coyle, 2014)

As GDP is reliant on consumption and production in the country, it is also affected by changes in the economy as a whole. In many instances GDP is used in further defining economic patterns, e.g. recessions, or business cycles. (Abberger and Wolfgang, 2008) A business or economic cycle refers to recurring phases of economic growth, fluctuation, and decline. This concept is based on the idea that such economic fluctuations follow an identifiable and repeated pattern over time. A standard way of identifying the stages of a business involves determining turning points in economic data. Periods of two consecutive quarters of negative GDP growth are usually classified as a recession, while sustained positive growth is referred to as an expansion or recovery phase. (Abberger and Wolfgang, 2008; Škare and Stjepanović, 2016)

In the economic literature, several methods are proposed for identifying the business cycles. One of those methods is the Hodrick-Prescott filter (hereafter, the HP filter) for separating the trend and cycle components of a given series. For a series $y_t = \mu_t + \varphi_t$, where μ_t represents the trend and φ_t represents the cyclical component, the HP filter de-trends the series y_t by solving the following optimization problem

$$\min_{\mu_t} \sum_{t=1}^{T} \left((y_t - \mu_t)^2 + \lambda_{HP} \left((\mu_{t+1} - \mu_t) - (\mu_t - \mu_{t-1}) \right)^2 \right),$$

where the first term penalizes large residuals, and the second penalizes lack of smoothness in the trend. The smoothing parameter λ_{HP} controls the trade-off between these two criteria. This optimization problem is typically solved using computational algorithms, such as the Kalman filter. (Ravn and Uhlig, 2002; Kaiser and Maravall, 1995) It is important to note that, in practice, the series y_t is often logarithmically transformed before the estimation process and that, for quarterly data, a commonly suggested value for the smoothing parameter λ_{HP} is 1600. (Škare and Stjepanović, 2016) It should also be noted that the HP filter has received criticism, and some sources do not consider it a reliable method for estimating business cycles, particularly when applied to non-stationary series, as this may significantly affect the results. Nevertheless, the HP filter remains one of the standard methods used in the estimation of economic cycles. (Ravn and Uhlig, 2002; Škare and Stjepanović, 2016)

In addition to Hodrick-Prescott filter, other approaches such as band-pass filtering and structural time series models are used for identifying the business cycles. The main focus of this thesis is on using the structural time series models for the trend and cycle estimation, for which the theory will be given in the following section.

Knowing the behaviour of GDP, including its rising and lowering patterns, is of great importance in decision making in multiple government and financial sectors. Thus, the ability to use statistical methods to forecast and assess GDP growth is essential. (Jansen, Xiaowen, and De Winter, 2016)

2 Methods

Section 2.1 presents an overview of autoregressive integrated moving average (ARIMA) models, based on the work of P. J. Brockwell and R. A. Davis (2002).

Section 2.2 provides a concise overview of structural time series models and their components, based on the work of A. C. Harvey (1990) and an article by J. T. Jalles (2009).

Section 2.3 introduces the theory behind the Granger causality test and its statistical implementation, based on the work of G. Kirchgässner, J. Wolters, and U. Hassler (2013), complemented by an article by A. Shojaie and E. B. Fox (2022).

2.1 Overview of ARIMA Models

A time series $\{y_t\}$ is a sequence of observations made at time points $t = 1, 2, \ldots, T$, where $t = 1$ denotes the first observation and $t = T$ denotes the most recent observation. Henceforth, the series $\{y_t\}$ is considered to be a realization of a random process $\{Y_t\}$, through which the properties and formulas are defined. It is assumed that the properties of the random process $\{Y_t\}$ extend to its realizations.

The aim of time series analysis is to find a suitable method to model the underlying process of the observed values and to make accurate forecasts for future observations. The general approach for modeling a time series consists of several important steps: examining the series, ensuring stationarity (if required) and choosing the appropriate model. In time series literature, if not specified otherwise, the definition of weak stationarity is used in determining the stationarity of a series. By definition, a process $\{Y_t\}$ is **second order weakly stationary** if

1. $E(Y_t)$ does not depend on time t;
2. $Cov(Y_{t+h}; Y_t)$ does not depend on time t for any integer value of h.

In other words, a (weakly) stationary series is characterized by a series' mean value and variance that stays consistent over time and covariation between observations that does not depend on time. Whether a series is stationary can be determined in multiple ways. By visual evaluation, one can observe stationarity if regardless of the fluctuations of observations, the overall direction of the movements stays the same as that of the series' mean value. Another way is to observe the behaviour of the autocorrelation function (ACF) values - a stationary series is characterized by rapidly decreasing ACF. Another approach is to perform a unit root test[1]. In case of non-stationary series, stationary can be achieved through differencing, that is, working with the process $(Y_t - Y_{t-1})$ instead of Y_t.

One of the methods used in time series analysis is autoregressive integrated moving average models, also known as ARIMA models. ARIMA is a combination of autoregressive (AR) and moving average (MA), where the stationarity of the series is obtained by differencing (I).

The following provides a brief overview of the components of ARIMA models.

Autoregressive Process AR(p).

Let $\{Y_t\}$ be weakly stationary process with zero-mean. $\{Y_t\}$ is said to follow

[1] A unit root test was proposed by Dickey and Fuller to check whether a time series is stationary or needs differencing. It tests for the presence of a unit root in the autoregressive component of the series.

a p-th order causal autoregressive AR(p) process, if for any time moment t,

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + A_t,$$

where $\{A_t\}$ is a white noise process with zero-mean and finite variance σ_A^2 and $\phi_1, ...\phi_p$ are the coefficients of the AR(p) process. The process is causal if $\text{Cov}(A_t, Y_j) = 0$ for all $j < t$ - that is, the current innovation A_t is uncorrelated with the past values of the process.

Moving Average Process MA(q).

A weakly stationary process with zero-mean $\{Y_t\}$ is said to follow a q-th order moving average MA(q) process if for any time moment t

$$Y_t = A_t + \theta_1 A_{t-1} + \cdots + \theta_q A_{t-q},$$

where $\{A_t\}$ is a white noise process with zero-mean and finite variance, and $\theta_1, ..., \theta_q$ are the coefficients of the MA(q) process.

Autoregressive Moving Average Process ARMA(p,q).

A weakly stationary process with zero-mean $\{Y_t\}$ follows a causal ARMA(p,q) process if, for any time moment t,

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + A_t + \theta_1 A_{t-1} + \cdots + \theta_q A_{t-q}, \tag{1}$$

where $\{A_t\}$ is a white noise process with zero mean and finate variance, $\phi_1, ...\phi_p$ are the AR(p) coefficients, and $\theta_1, ..., \theta_q$ are the MA(q) coefficients.

Formula (1) can also be written in the following form

$$\phi(B)Y_t = \theta(B)A_t, \tag{2}$$

where

$$\phi(k) = 1 - \sum_{i=1}^{p} \phi_i k^i, \qquad \theta(k) = 1 + \sum_{i=1}^{q} \theta_i k^i.$$

In equation (2) B represents a backward shift operator, i.e. $B^m Y_t = Y_{t-m}$.

If a process $\{Y_t\}$ has a non-zero mean ν and follows an AR, MA or ARMA process, then the mean-adjusted process $\{Y_t - \nu\}$ follows the respective process in its standard zero-mean form.

Autoregressive Integrated Moving Average Process ARIMA(p,d,q). In case the series $\{Y_t\}$ is not stationary, differencing the series to obtain stationarity is used. Let the difference operator Δ be defined as

$$\Delta Y_t = Y_t - Y_{t-1} = (1 - B)Y_t,$$

then the jth difference is given by

$$\Delta^j Y_t = \Delta^{j-1} Y_t - \Delta^{j-1} Y_{t-1} = (1 - B)^j Y_t,$$

where $j = 1, 2, ...$ and the order j is chosen based on how many differences are needed to achieve stationarity.

Let d be a non-negative integer. Then the process $\{Y_t\}$ is said to follow an **ARIMA(p,d,q)** process if the differenced process $(1-B)^d Y_t$ follows a causal ARMA(p,q) process.

2.2 Overview of Structural Time Series Models

Structural time series models represent a structured method for the analysis of time series. Unlike other methods, structural models make it possible to recognize and model different underlying processes of a series separately. Such processes are the trend, the cycle, and the seasonal process. The trend represents the overall direction of a series, the cycle represents the cyclical nature of a series and is usually talked about when considering economic data, and the seasonal process represents the underlying behaviours of the series caused by effects associated with specific dates and/or periods of time, such as holidays or seasonal changes.

At its core, a time series model is a regression model in which the explanatory variables are functions of time. Thus, an univariate time series model with trend, cyclic and seasonal component can be formulated as

$$y_t = \mu_t + \varphi_t + \gamma_t + \varepsilon_t, \tag{3}$$

where μ_t represents the trend, φ_t the cycle and γ_t corresponds to the seasonal component. Random disturbance term ε_t is assumed to be normally and independently distributed with zero mean and variance σ^2_ε, that is, $\varepsilon_t \sim NID(0; \sigma^2_\varepsilon)$.

In their essence, structural time series models provide the flexibility to recognize the behavioural effects of different model components by taking into consideration their stochastic nature and change over time. The formulation of a structural time series model depends on the specific characteristics of the time series being studied and may contain different variation of components, where not all components have to be present or time-varying. The following subsections will discuss some different formulations for components in itself and will also provide some different combinations of model components.

2.2.1 Trend

The trend represents the long-term direction of a time series. In the simplest case, a time series model can be formulated using only the trend, denoted by μ_t, and the irregular disturbance component, denoted by ε_t, i.e.,

$$y_t = \mu_t + \varepsilon_t \qquad \varepsilon_t \sim NID(0; \sigma_\varepsilon^2).$$

It is important to note that the trend component may not always be present in a time series or may be present but not depend on time. However, when the trend term is present and depends on time, the following cases apply in formulating it.

One approach to modeling a time-varying trend is to use a level[2] trend that can change on a local scale, known as a **local level model** or **random walk plus noise model**. In a local level model, the level component is updated by adding the value from the previous period and a random disturbance term, and it is formulated as follows

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t \qquad & \varepsilon_t &\sim NID(0; \sigma_\varepsilon^2); \\ \mu_t &= \mu_{t-1} + \eta_t \qquad & \eta_t &\sim NID(0; \sigma_\eta^2), \end{aligned} \tag{4}$$

where ε_t and η_t are uncorrelated random disturbance terms, and where η_t dictates the level's fluctuations. If ε_t is assumed to be zero ($\sigma_\varepsilon^2 = 0$) then equation (4) implies that the series y_t follows a trend component that evolves as a random walk. More generally, the local level model is equivalent to an

[2]Level references to an underlying mean value of the time series that remains constant over time.

ARIMA(0,1,1) process, where the non-stationarity arises from the level component μ_t. To obtain stationarity, the model is differenced with respect to μ_t, which cancels out the fluctuations in the stochastic level and thereby stabilizes the mean of the time series. Let Δ denote the first-difference operator, so that $\Delta\mu_t = \mu_t - \mu_{t-1}$. Taking the first difference of the local level model (4) with respect to μ_t yields the following formulation

$$
\begin{aligned}
&\Delta y_t = \Delta\mu_t + \Delta\varepsilon_t \Rightarrow \\
&\Delta y_t = \Big(\big(\mu_{t-1} + \eta_t\big) - \mu_{t-1}\Big) + \Delta\varepsilon_t \Rightarrow \\
&\Delta y_t = \eta_t + \Delta\varepsilon_t.
\end{aligned}
$$

Another way that describes the change of the trend over time is assuming the trend to be linear and allowing the change on a local scale, i.e., **local linear trend model**. Local linear trend model is in its formulation similar to the local level model with the difference of an additional parameter β_t that represents the slope of a linear model. The formulation of a local linear trend model is

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t & \varepsilon_t &\sim NID(0;\sigma^2_\varepsilon);\\
\mu_t &= \mu_{t-1} + \beta_t + \eta_t & \eta_t &\sim NID(0;\sigma^2_\eta);\\
\beta_t &= \beta_{t-1} + \xi_t & \xi_t &\sim NID(0;\sigma^2_\xi),
\end{aligned}
\tag{5}
$$

where ξ_t and ε_t are uncorrelated irregular processes. When the slope parameter β is constant, i.e., $\sigma^2_\xi = 0$, the model becomes a local level with drift type. If, in addition to ξ_t being zero ($\sigma^2_\xi = 0$), the irregular component ε_t is also assumed to be zero ($\sigma^2_\varepsilon = 0$), then the series y_t follows a trend component that evolves as a random walk with drift. The local linear model (5) can be further modified by introducing a damping factor $\rho \in (0, 1]$ to the slope component

$$\beta_t = \rho\beta_{t-1} + \xi_t,$$

resulting in the **damped local linear trend model**.

In the standard formulation, the local linear trend model (5) is equivalent to an ARIMA(0,2,2) process, where non-stationarity arises from both the level component μ_t and the slope component β_t. To achieve stationarity, the model is first differenced with respect to μ_t

$$\Delta y_t = \Delta\mu_t + \Delta\varepsilon_t \Rightarrow$$

$$\Delta y_t = \beta_t + \eta_t + \Delta\varepsilon_t.$$

As stochasticity of the slope component β_t still introduces non-stationarity, a second difference with respect to β_t is required. The local linear trend model then takes the following form

$$\Delta^2 y_t = \Delta\beta_t + \Delta\eta_t + \Delta^2\varepsilon_t \Rightarrow$$

$$\Delta^2 y_t = \big(\beta_t - \beta_{t-1}\big) + \Delta\eta_t + \Delta^2\varepsilon_t \Rightarrow$$

$$\Delta^2 y_t = \xi_t + \Delta\eta_t + \Delta^2\varepsilon_t.$$

2.2.2 Cycles

The cycle represents a behavioural pattern in a time series that lacks a fixed frequency. Cycles are commonly discussed in the context of business and macroeconomic data. For example, economic cycles, such as recessions and booms, are crucial for understanding time series behaviour but are harder to predict due to their dependence on external economic factors and lack of a fixed pattern.

When the cyclical component is assumed to be deterministic, it is characterized by a perfectly periodic pattern, which can be expressed using a cosine, sine, or a combination of both functions, with parameters for amplitude and phase. A cycle defined as a combination of cosine and sine functions is given by

$$\varphi_t = \alpha \cos(\lambda t) + \beta \sin(\lambda t), \tag{6}$$

where λ, which is measured in radians, defines the frequency, $\sqrt{(\alpha^2 + \beta^2)}$ defines the amplitude and $\tan^{-1}(\frac{\beta}{\alpha})$ defines the phase of the cycle.

The same function (6) can also be defined recursively, by incorporating a rotation matrix, in the following way

$$\begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \varphi_{t-1} \\ \varphi_{t-1}^* \end{pmatrix}, \tag{7}$$

where $\varphi_0 = \alpha$ and $\varphi_0^* = \beta$. This claim is proved by mathematical induction.

Proof. Basis. Let $t = 1$, then

$$\begin{aligned} \varphi_1 &= \cos(\lambda)\alpha + \sin(\lambda)\beta \\ \varphi_1^* &= -\sin(\lambda)\alpha + \cos(\lambda)\beta \end{aligned} \tag{8}$$

which holds as (8) is equal to (6) when $t = 1$.

Hypothesis. Let's assume that for $t = k$

$$\begin{aligned}\varphi_k &= \cos(\lambda k)\alpha + \sin(\lambda k)\beta \\ \varphi_k^* &= -\sin(\lambda k)\alpha + \cos(\lambda k)\beta.\end{aligned}$$

Inductive. Let's show that for $t = k + 1$ the equivalence also holds. From equation (7) it follows that

$$\begin{aligned}\varphi_{k+1} &= \cos(\lambda)\varphi_k + \sin(\lambda)\varphi_k^* \\ \varphi_{k+1}^* &= -\sin(\lambda)\varphi_k + \cos(\lambda)\varphi_k^*.\end{aligned}$$

When φ_k and φ_k^* are substituted into the expression for φ_{k+1}

$$\begin{aligned}\varphi_{k+1} = \cos(\lambda) \cdot \Big(\cos(\lambda k)\alpha + \sin(\lambda k)\beta\Big) + \sin(\lambda) \cdot \Big(-\sin(\lambda k)\alpha + \cos(\lambda k)\beta\Big) = \\ = \frac{1}{2}\Big(\cos\big(\lambda(1-k)\big) + \cos\big(\lambda(k+1)\big)\Big) \cdot \alpha + \frac{1}{2}\Big(-\sin\big(\lambda(1-k)\big) + \sin\big(\lambda(k+1)\big)\Big) \cdot \beta + \\ + \frac{1}{2}\Big(\cos\big(\lambda(1-k)\big) - \cos\big(\lambda(k+1)\big)\Big) \cdot \alpha + \frac{1}{2}\Big(\sin(\lambda(1-k)) + \sin(\lambda(k+1))\Big) \cdot \beta = \\ = \cos\big((k+1)\lambda\big)\alpha + \sin\big((k+1)\lambda\big)\beta.\end{aligned}$$

Thus, by mathematical induction, the formula holds for all k. □

Since it has been established that equation (7) extends equation (6), the stochasticity of the cycle component can be introduced by adding a random disturbance term to equation (7), as shown below

$$\begin{pmatrix}\varphi_t \\ \varphi_t^*\end{pmatrix} = \bar{\rho}\begin{pmatrix}\cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}\varphi_{t-1} \\ \varphi_{t-1}^*\end{pmatrix} + \begin{pmatrix}\kappa_t \\ \kappa_t^*\end{pmatrix}, \tag{9}$$

where κ_t and κ_t^* are uncorrelated random processes with common variance σ_κ^2, and the damping factor $\bar{\rho} \in (0, 1]$ increases the flexibility of the cyclical component.

When the cyclical component φ_t is assumed to be non-deterministic, it can also be expressed through an ARMA(p,q) process. In that case it is formulated as follows

$$\varphi_t = \sum_{i=1}^{p} \phi_i \varphi_{t-i} + \alpha_t + \sum_{i=1}^{q} \theta_i \alpha_{t-i} + \zeta_t, \qquad (10)$$

where process φ_t is assumed to be stationary process with zero-mean, $\phi_1, ..., \phi_t$ denote the autoregressive process coefficients, $\theta_1, ..., \theta_t$ denote moving average process coefficients, α_t is a white noise process with zero-mean and variance σ_α^2 and $\zeta_t \sim NID(0; \sigma_\zeta^2)$ is the irregular term of the cycle component. (Koopman and Ooms, 2012)

Having established the theoretical foundation and formulation of the cycle component, various combinations of potential structural models can now be defined. **The cyclical model** is a structural model that combines a cyclical component and a deterministic trend component and is formulated as

$$y_t = \mu + \varphi_t + \varepsilon_t \qquad \varepsilon_t \sim NID(0; \sigma_\varepsilon^2).$$

The trend cycle model is a combination of cyclical term and a time-varying trend component, formulated as

$$\begin{aligned} y_t &= \mu_t + \varphi_t + \varepsilon_t && \varepsilon_t \sim NID(0; \sigma_\varepsilon^2); \\ \mu_t &= \mu_{t-1} + \eta_t && \eta_t \sim NID(0; \sigma_\eta^2). \\ \beta_t &= \beta_{t-1} + \xi_t && \xi_t \sim NID(0; \sigma_\xi^2). \end{aligned}$$

More distinguishable example is **the cyclical trend model** which defines the cycle component inside the trend component. Cyclical trend model can be formulated as follows

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t & \varepsilon_t &\sim NID(0;\sigma^2_\varepsilon);\\
\mu_t &= \mu_{t-1} + \varphi_{t-1} + \beta_{t-1} + \eta_t & \eta_t &\sim NID(0;\sigma^2_\eta);\\
\beta_t &= \beta_{t-1} + \xi_t & \xi_t &\sim NID(0;\sigma^2_\xi).
\end{aligned}
$$

2.2.3 Seasonality

Seasonality represents a pattern in the behaviour of a time series that continues to reoccur after a certain period of time, with fixed frequency[3].

As with other structural time series components, the seasonality term can be defined as either deterministic or time-varying. However, unlike trend and cyclical components, a deterministic seasonality requires that the seasonal effects sum up to zero.

To satisfy this requirement, the seasonal effect for the final period of the seasonal cycle is determined as the negative sum of all preceding seasonal effects. Let p denote the frequency of the seasonal cycle, dividing it into s periods (e.g., for quarterly seasonality, $p = 4$, and the seasonal cycle consists of $s = 4$ periods, where each value of s corresponds to a different quarter of the year). If the final season of the cycle is represented as s_{last}, the deterministic

[3] E.g., there is a rise in the sales of toy stores at the end of the year (around Christmas time). This is a pattern that can be expected to repeat itself at yearly intervals, creating a seasonal effect with a period or frequency of one year.

seasonal component is expressed as follows:

$$\gamma_t = \begin{cases} \gamma_t, & \text{if } t \text{ belongs to season } s \;/\{s_{last}\}; \\ -\sum\limits_{i=1}^{s-1} \gamma_i, & \text{if } t \text{ belongs to season } s_{last}. \end{cases}$$

There exist multiple approaches to allow the seasonal component to vary over time. One way is to let the seasonal component sum up to a random disturbance term instead of zero, that is $\sum\limits_{i=0}^{s-1} \gamma_{t-i} = \omega_t$, where $\omega_t \sim NID(0; \sigma_\omega^2)$. Alternatively, the seasonal component can be defined as the sum of the previous periods' seasonal effect and a random disturbance term, effectively allowing it to evolve as a random walk. This can be formulated as

$$\gamma_{t,i} = \gamma_{t,i-1} + \omega_{t,i}.$$

Similarly to the cycle term, the stochasticity of the seasonal component can be obtained by incorporating a trigonometric function. Due to the symmetric nature of the cosine and sine functions, the fixed value of the seasonal term is defined as $\gamma_t = \sum\limits_{i=1}^{\lfloor s/2 \rfloor} \gamma_{t,i}$, where the values corresponding to periods $i = 1, ..., \lfloor s/2 \rfloor$ are calculated through the trigonometric function which is formulated in the following way

$$\begin{pmatrix} \gamma_{t,i} \\ \gamma_{t,i}^* \end{pmatrix} = \begin{pmatrix} \cos \lambda_i & \sin \lambda_i \\ -\sin \lambda_i & \cos \lambda_i \end{pmatrix} \begin{pmatrix} \gamma_{t-1,i} \\ \gamma_{t-1,i}^* \end{pmatrix} + \begin{pmatrix} \omega_{t,i} \\ \omega_{t,i}^* \end{pmatrix},$$

where $t = 1, ..., T$ and λ_i is the frequency of the pattern, measured in radians. It should also be emphasized that a seasonal component can be defined as the sum of multiple seasonal terms that describe the different variations of patterns. For example, a pattern associated with the beginning of the month and the pattern associated with the end of that same month. This approach allows to describe patterns where a period in one year corresponds to a different period in the following year. Such instances arise most often in the case of moving holidays, e.g. Easter, Thanksgiving and Chinese New Year. The formulations and more in-depth explanation of these methods can be found in (Jalles, 2009; Harvey, 1990).

2.2.4 Lagged Values and Exogenous Variables

In some cases, incorporating additional information to the series, by including lagged values and/or exogenous variables, can improve the performance of the model. In such cases, this additional information can improve the model's ability to capture the dynamics of the series and enhance the predictive power of the model. Adding lagged values and exogenous variables transforms the structural model equation (3) into the following form

$$y_t = \mu_t + \varphi_t + \gamma_t + \sum_{\tau=1}^{r} \psi_\tau y_{t-\tau} + \sum_{j=1}^{k} \sum_{\tau=0}^{s_k} \delta_{j\tau} x_{j,t-\tau} + \varepsilon_t,$$

where $x_{j,t}$ are exogenous variables and ψ_τ and $\delta_{j\tau}$ are unknown coefficients. More in-depth explanation can be found in (Jalles, 2009; Harvey, 1990).

2.2.5 State Space Representation of Structural Models

Structural time series models can be represented using the state space framework. A state space models consist of two equations: a measurement equation that expresses the observed variables through an l-dimensional state vector, and a transition equation that describes how the state vector evolves over time. For example, the local linear trend model (5) has the following state space form

$$y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_t \\ \beta_{t+1} \end{pmatrix} + \varepsilon_t \tag{11}$$

$$\begin{pmatrix} \mu_t \\ \beta_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \beta_t \end{pmatrix} + \begin{pmatrix} \eta_t \\ \xi_{t+1} \end{pmatrix}, \tag{12}$$

where $(\mu_t \quad \beta_t)'$ is the 2-dimensional state vector. The first equation (11) is the measurement equation that defines y_t as a function of the current state, and the second equation (12) is the transition equation that describes the evolution of the state vector over time.

Once the model is expressed in state space form, various algorithms can be applied for estimation and forecasting, with one of the most widely used algorithms being the Kalman filter. For estimation, both maximum likelihood and generalized least squares methods can be employed.

2.3 Granger Causality Test

Granger causality test is used to determine whether the values of one series help predict the values of the other.

Let f and g be two time series, let $\mathrm{OLP}(f_{t+1}|f^*)$ denote the optimal linear predictor for the series f that takes into account all the current and past values of the series f, i.e. $f^* = \{f_t, f_{t-1}, ..., f_{t-l}, ...\}$, and likewise let $g^* = \{g_t, g_{t-1}, ..., g_{t-l}, ...\}$. By definition, the series g is **Granger causal** to the series f if and only if

$$\sigma^2\Big(f_{t+1} - \mathrm{OLP}(f_{t+1}|f^* \cup g^*)\Big) < \sigma^2\Big(f_{t+1} - \mathrm{OLP}(f_{t+1}|f^*)\Big).$$

In other words, series g is Granger causal to series f if incorporating the current and past values of g the prediction for f leads to a smaller prediction error in terms of error variance, than when using only the past values of f. This definition of Granger causality is fundamentally a concept of predictability rather than causation; however, Granger argued that taking into account the time-ordered nature of the values, such predictability can be interpreted as one having an effect and a cause on the other.

To determine whether one time series causes the other, the Granger causality test was introduced. It was determined that the relationship can be evaluated using various statistical methods, such as the F-test or alternative tests like the Wald test. In this thesis, the formulation of Granger causality is presented using the F-test.

When testing if g Granger causes f, the full model is compared to the restricted model. In this case the full model is defined as follows

$$f_t = a_0 + \sum_{k=1}^{k_f} a_k^{ff} f_{t-k} + \sum_{k=1}^{k_g} a_k^{fg} g_{t-k} + u^{ft}, \tag{13}$$

where a_{ff}^k represent the regression coefficients of lagged values of f, a_{fg}^k represent the regression coefficients of the lagged values of g, a_0 is the intercept term and u_{ft} denotes the error of the model. The corresponding restricted model is obtained by setting coefficients of the lagged values of g in formula (13) to zero, i.e. $a_{fg}^k = 0$ for all $k = 1, ..., k_g$.

It is also possible to test whether f Granger causes g, in that case the full model would be defined as

$$g_t = a_0' + \sum_{k=1}^{k_g} a_k^{gg} g_{t-k} + \sum_{k=1}^{k_f} a_k^{gf} f_{t-k} + u^{gt}, \tag{14}$$

where a_{gg}^{k} represent the regression coefficients of lagged values of g, a_{gf}^{k} represent the regression coefficients of the lagged values of f, a_0' is the intercept term and u_{gt} denotes the error of the model. The corresponding restricted model is obtained by setting coefficients of the lagged values of f in formula (14) to zero, i.e. $a_k^{gf} = 0$ for all $k = 1, ..., k_f$.

In total, there are four possible outcomes: 1. g Granger-causes f; 2. there is no Granger causality from g to f; 3. f Granger-causes g; 4. there is no Granger causality from f to g.

In the following, the null and alternative hypotheses as well, as the F-test formulation, are presented for the case where g is tested for Granger causality on f, corresponding to the full model formula (13). In testing whether f Granger causes g, the formulas are defined analogously.

The hypotheses are defined as follows

$$H_0 : a_1^{fg} = a_2^{fg} = \cdots = a_k^{fg} = 0,$$

$$H_1 : \exists j \in 1, ..., k_g : a_j^{fg} \neq 0.$$

Under the null hypothesis, the lagged values of g do not improve the prediction of f, that is, g does not Granger cause f. Rejecting the null hypothesis indicates Granger causality from g to f.

Let RSS_{res} denote the residual sum of squares of the restricted model, and RSS_{full} the residual sum of squares of the full model. The F-statistic is calculated as

$$\text{F-statistic} = \frac{\big(RSS_{res} - RSS_{full}\big)/((k_f + k_g) - k_f)}{RSS_{full}/\big(T - (k_f + k_g)\big)},$$

where T denotes the number of observations. Under the null hypothesis, the F-statistic follows the F-distribution with $\big(k_g, T - (k_f + k_g)\big)$ degrees of freedom.

An important consideration when testing for Granger causality is the choice of lag order. Including more lags allows for better capturing of causal effects, but reduces the power of the test. As the test results depend on the number of lags, it is crucial to select the lag order appropriately. It is suggested that the lag order can be determined using information criteria such as the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC).

3 Previous research

Structural time series models have been widely applied in previous research to analyze the behaviour of GDP and its underlying components. This section provides an overview of key findings from studies that have employed univariate structural models.

The study by Y. S. Nakstad (2006) analyzes seasonally-adjusted, logarithmically transformed real GDP time series of United States, Euro Area, United Kingdom, Sweden, and Japan, covering the period from the first quarter of 1960 to the second quarter of 2005. The study employs univariate trend-cycle models, where the trend component is specified as a fixed level with stochastic slope - corresponding to the local linear trend model (Formula (5)) with the level irregular term η_t is set to zero. The cyclical component is modeled using a trigonometric specification, and estimation is conducted using STAMP software.

Nakstad estimates the structural models both with and without added intervention terms and them to forecast ten quarters (2.5 years) ahead. Interventions are defined as significant unusual movements in the GDP time series, such as instant shocks (e.g. nationwide strike) or structural brakes (e.g. major tax policy changes). These innovations are identified with the help of STAMP software and are incorporated in the models via dummy variables. The analysis shows that including intervention terms affects the flexibility of the trend component. Forecast accuracy is evaluated against OECD report forecasts, and the results indicate that allowing too much stochasticity in the trend leads to poor forecast performance, despite a close fit to the observed series. In contrast, limiting the stochastic variation of the trend improves forecast accuracy but results in a weaker in-sample fit.

The study by M. A. Cuevas (Cuevas, 2002) analyzes the relationship between Venezuela's real GDP and real oil prices, with the aim of identifying associations between the two. The time series are estimated as separate structural models, with an exogenous variable representing the 1974 oil price shock added to the oil price model. Both models incorporate a stochastic trend and a trigonometric cycle component. The analysis uses data from 1970 to 2000 and is conducted using STAMP software. The results indicate successful model estimation where the estimated cycles exhibited opposite-phases with similar frequencies, suggesting a strong association between the real GDP and real oil prices.

The study by G. Bulligan *et al.* (2019) examines the relation between Italy's real GDP and real credit to the private sector from the first quarter of 1970 to the third quarter of 2016. A univariate trend-cycle model is estimated for both time series, with the trend component specified as a local linear trend and the cyclical component modeled using the trigonometric formulation. The software used for the estimation is not explicitly specified; however, the MATLAB software is mentioned in the acknowledgments, suggesting that it was likely used for the analysis. The resulting cycle estimates align with findings from previous literature: real GDP displays relatively short cycles of approximately 14 quarters, while credit to the private sector exhibits longer cycles of around 30 and 70 quarters.

In general, the application of structural time series models to macroeconomic data, such as GDP, is a widely studied topic with a broad range of available resources. Examples of other macroeconomic data that have been analyzed include regional income per capita in the United States (Carvalhoa and Harvey, 2005), and the unemployment rate in Barbados (Mamingi, Williams, and Browne, 2014).

4 Empirical Study

4.1 Exploring the Data and Tools

Estonia is a small country that is economically dependent on its export partners and their economic state. Thus, in addition to analyzing Estonia's economy, it is important to consider the possible economic relations between the countries that influence Estonian economy the most. This thesis focuses on analyzing the GDP time series of Estonia and its 5 biggest export partners for the year 2024 - Finland, Latvia, Lithuania, Sweden, and Germany - which are also presented in the following table.

Table 1: Estonia's biggest export destinations for the year 2024. (Statistics Estonia, 2025a)

Country	Total export in €	Per cent of all export
Finland	2 742 631 341 (2.74B)	15.8%
Latvia	1 937 018 392 (1.94B)	11.1%
Sweden	1 553 452 115 (1.55B)	8.93%
Lithuania	1 404 121 256 (1.4B)	8.07%
Germany	1 268 563 108 (1.27B)	7.29%

Estonia's total export for the year 2024 was €166 billion, with Finland as one of its key trading partners, contributing €2.74 billion (Statistics Estonia, 2025a).

The analysis was conducted using seasonally adjusted real gross domestic product (GDP) time series data from the Federal Reserve Bank of St. Louis's FRED database (FRED, 2025). As the main focus of this thesis is to estimate the cyclical and trend components of GDP, using data where seasonal effects have been already removed improves the accuracy of these estimates. The analysis covers quarterly data for the period from January 1, 1995, to October 1, 2024.

For structural model analysis, the STAMP software is often used; however, this analysis was conducted using the R software, due to its accessibility and widespread availability. By using R, this study introduces a degree of variability to the existing literature, showcasing the flexibility and potential of open-source software in structural time series analysis, offering an alternative to the commonly used STAMP. This study primarily relies on the autostsm, autoarima, lmtest, and mFilter packages. Both autoarima and autostsm packages enable automatic time series modeling using ARIMA and structural time series methods, respectively. The autostsm package implements a maximum likelihood-based approach to time series modeling, using the Kalman filter for estimation and forecasting. The package provides the *stsm_estimate* function, which automatically detects a suitable model and returns the model together with the estimated coefficients. The *stsm_forecast* function then uses the estimated structural model to generate component estimates for the entire time series, along with future predictions. Packages lmtest and mFilter contain the functions for the Granger causality test and Hodrick-Prescott filter.

In the modeling process of structural time series models, the trend, drift, and cyclical components are specified as time-varying, and the models are estimated for trend-cycle decomposition. This choice of preset conditions comes from the understanding that business cycles cannot be captured by perfect periodic functions, nor can the trend component be adequately represented by a strictly linear function, as economic activity is described by alternating periods of growth and decline, which may vary in both duration and intensity.

The following subsections present the results of the analysis, beginning with the structural time series models and the estimation of the cyclical and trend components, followed by the results of the Granger causality test, and concluding with a comparison of the forecast accuracy between structural and ARIMA models.

4.2 Business Cycle Estimation Using Different Methods

To identify the overall behaviour and activity of the economy, annual GDP growth can be measured. It reflects how much the economic activity has changed compared to the same quarter of the previous year, and is calculated using the following formula

$$\text{annual GDP growth (\%)} = 100 \cdot \frac{(\text{GDP}_t - \text{GDP}_{t-4})}{\text{GDP}_{t-4}}.$$

Figures Figure 1 and Figure 2 present a comparison of annual GDP growth across the Baltic countries and between Estonia and the larger economies considered in this analysis, that is, Finland, Germany, and Sweden. One of the more noticeable shifts in GDP growth, seen in the figures, is the 2010 recession. For the Baltic countries, it resulted in a nearly 20% decline in economic activity compared to the previous year, while the larger countries experienced a 10% decline. This illustrates how recessions tend to have a greater impact on smaller economies like the Baltics than on the larger ones such as Germany and Sweden. However, this pattern does not hold for the 2020 COVID-19 crisis, which led to a similar decline in economic activity across all countries. The figures also reveal that economic activity in the larger countries fluctuates less over time, indicating greater stability. This observation lends support to the idea that smaller countries are more economically dependent on the economy of larger countries. Thus, analyzing the underlying patterns of economic data and the potential interrelationships between the countries becomes increasingly important.

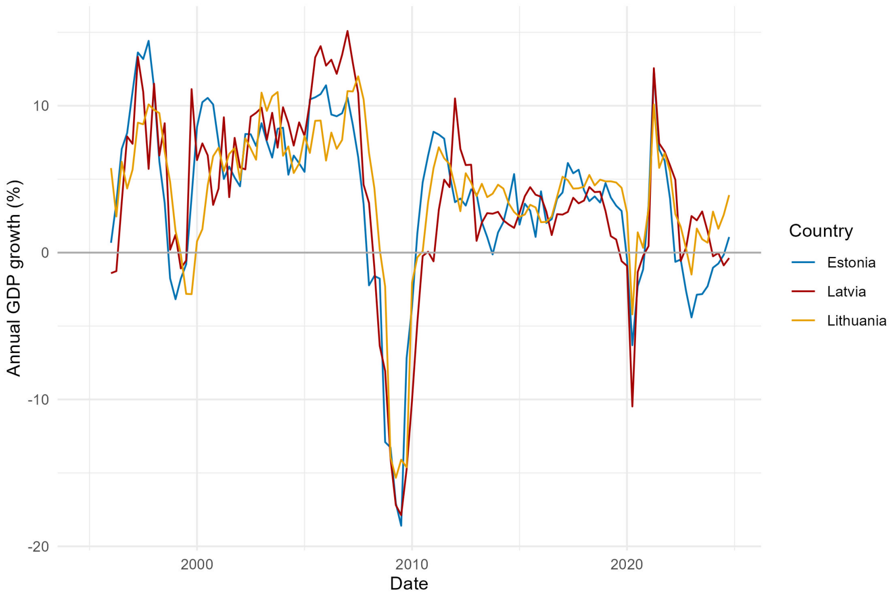

Figure 1: Comparison of the annual GDP growth of the Baltic countries.

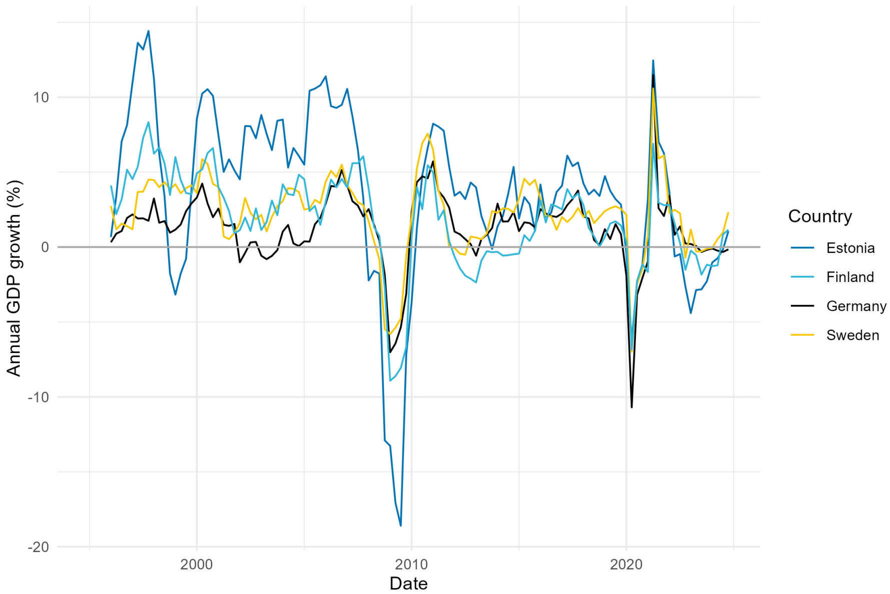

Figure 2: Comparison of the annual GDP growth of Estonia, Finland, Germany, and Sweden.

Another important observation that can be made from Figure 1 and Figure 2 is that overall economic growth from 1995 to 2024 has been highly volatile and generally increasing. This behaviour suggests that the usage of multiplicative models, often suggested in the economic literature, may not be the correct approach. This conclusion is further supported by a comparison of the forecast accuracy of both multiplicative and additive models, with the latter demonstrating better performance in predicting future values. Therefore, the main results presented in this thesis are based on the additive models, with the results from the multiplicative models included in the appendices for reference.

This thesis models GDP data using structural time series models with two alternative formulations for the cyclical component: the trigonometric specification (see Formula (9)) and the ARMA specification (see Formula (10)). Figures 3 and 4 present the estimated business cycles from the structural models assuming a trigonometric cycle formulation. The corresponding trend estimates are shown in Figure 5.

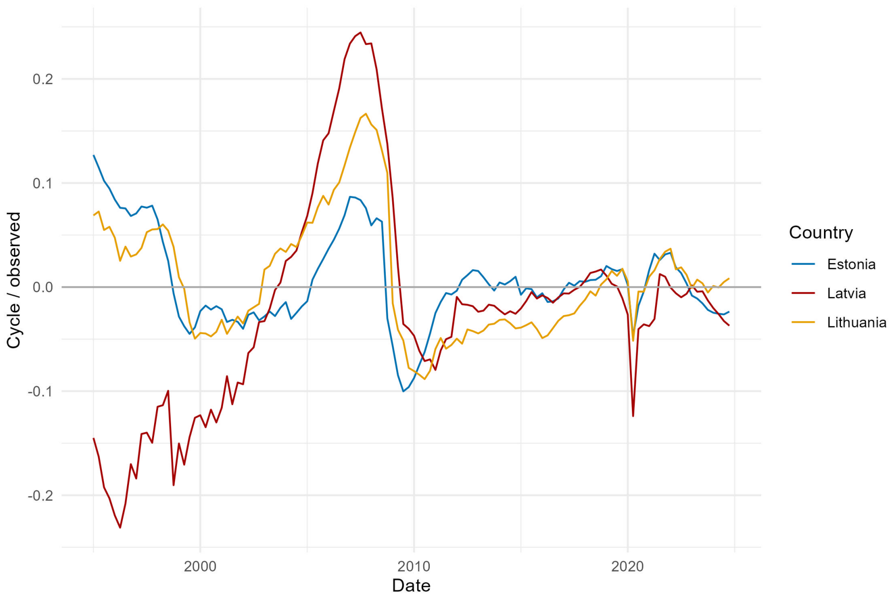

Figure 3: Structural model business cycle estimates using the trigonometric cycle formulation for the Baltic countries.

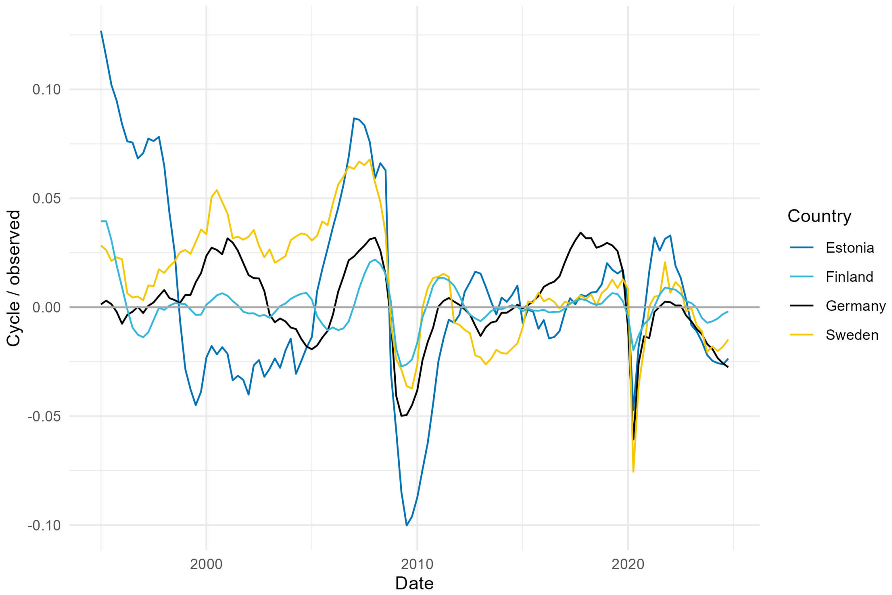

Figure 4: Structural model business cycle estimates using the trigonometric cycle formulation for Estonia, Finland, Germany, and Sweden.

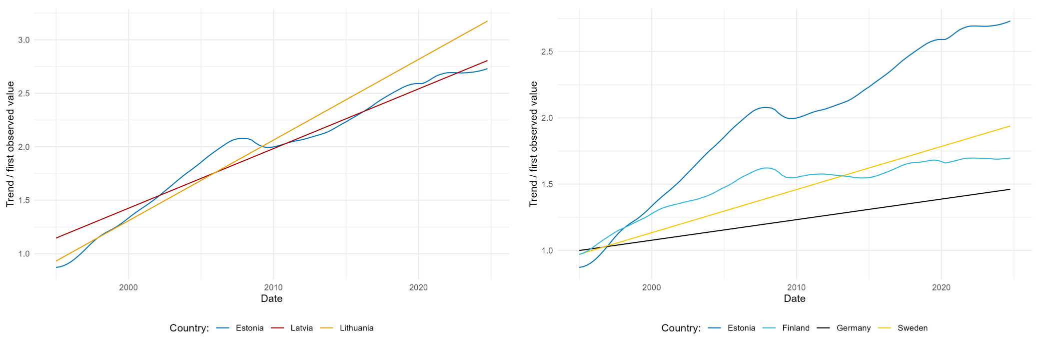

Figure 5: Structural model trend estimates using the trigonometric cycle formulation.

Figures 6 and 7 present the estimated business cycles from the structural models assuming an ARMA cycle formulation. The corresponding trend estimates are shown in Figure 8.

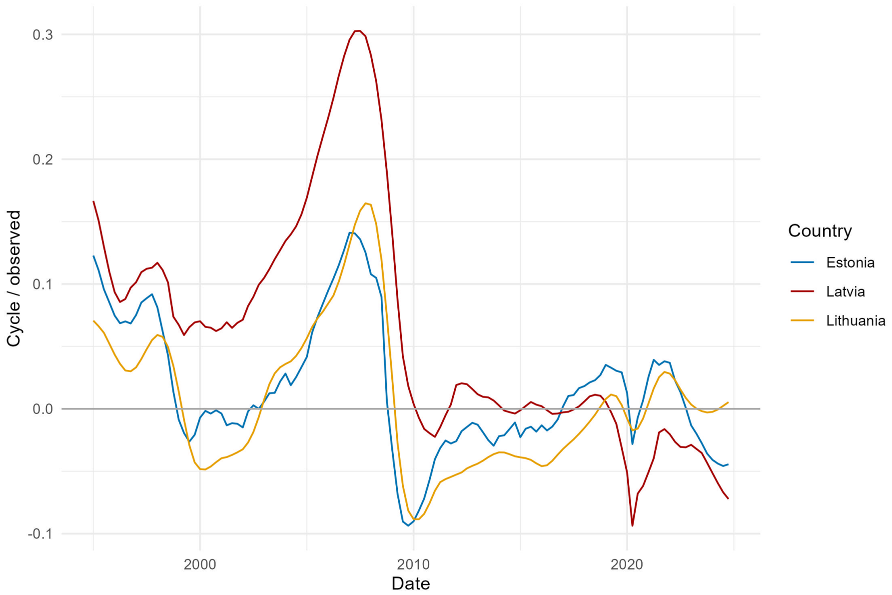

Figure 6: Structural model business cycle estimates using the ARMA cycle formulation for the Baltic countries.

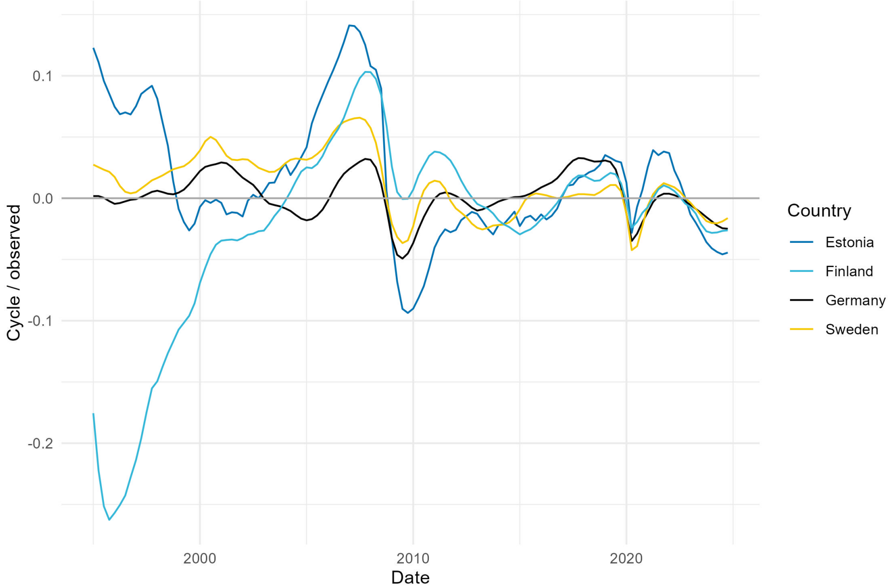

Figure 7: Structural model business cycle estimates using the ARMA cycle formulation for Estonia, Finland, Germany, and Sweden.

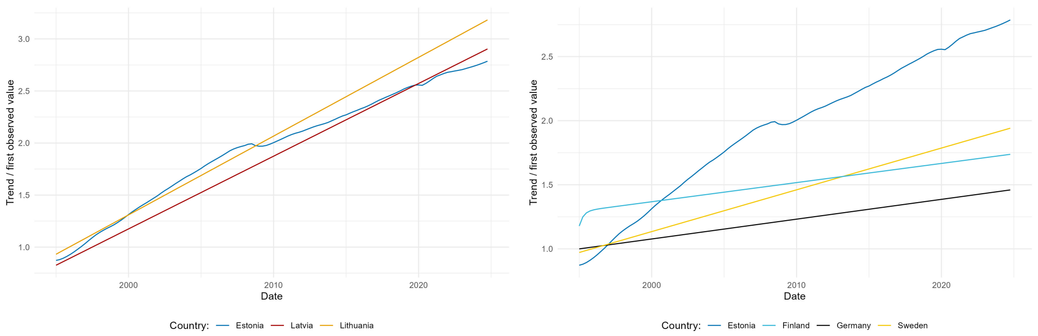

Figure 8: Structural model trend estimates using the ARMA cycle formulation.

To improve the interpretability and comparability of the cycle components across countries, the values shown in the figures are divided by the actual (observed) values. This way, the figures represent the proportion of GDP attributed to the cyclical component. Similarly, the trend estimates are normalized by dividing them by the first observed value of the corresponding time series, allowing for clearer visual comparison across countries with different absolute GDP levels.

When comparing the cyclical component estimates from the trigonometric and ARMA formulations (Figures 3, 4, and Figures 6, 7), the results are largely consistent — both formulations capture similar peaks and troughs, including the recessions of 2010 and 2020, along with the recovery periods before and after. These patterns align closely with the observed annual GDP growth shown in Figures 1 and 2. The trigonometric formulation tends to reflect more short-term fluctuations, whereas the ARMA-based estimates appear smoother. Some discrepancies emerge in the early part of the sample period (1995 – 2002). For instance, Latvia's cycle estimate starts at a notably lower lever for the trigonometric case than in the ARMA case, while the opposite is observed for Finland. These inconsistencies in the initial period may stem from the automatically determined starting values used during the model estimation process. One more notable thing in the figures of trigonometric and ARMA cycle estimates is the estimated start of Estonia's business cycle, which is relatively high compared to the starting values of the larger economies' cycles. However, this is an expected result, since Estonia experienced rapid growth in economic activity starting in 1995, triggered by the collapse of the Soviet Union. As a transition economy, Estonia experienced an average annual growth rate of 5.8% from 1995 to 2002 (Rindzeviciute, 2004), which explains the higher initial value for the estimated cycle component. Similarly, the low starting value for Finland's business cycle in the ARMA cycle formulation can be explained by the deregulation crisis of the early 1990s. This was one of the most significant crises the Finnish economy faced during peacetime, caused by the simultaneous decline of Soviet and Western markets. (Kalela et al., 2001) The lower estimate for the Finnish cycle matches this recovery phase, though it was not captured by the trigonometric cycle formulation.

For all countries, structural models with both trigonometric and ARMA cycle formulations estimated the trend component to follow a damped local linear trend model. This means that all trend estimates are non-deterministic by definition. However, when comparing estimated trends (Figures 5 and 8), many appear visually close to deterministic, such as Germany's trend. This effect arises from the relatively small estimated variances of the irregular and slope components, σ_η and σ_ξ (see Formula (5)), and, in some cases, from a damping factor ρ that is close to zero. Despite these differences in smoothness, all countries show an increasing trend in GDP over time.

The estimated business cycle and trend comparisons for the Baltic countries and for Estonia, Finland, Germany, and Sweden in the multiplicative case are presented in Appendix 1.

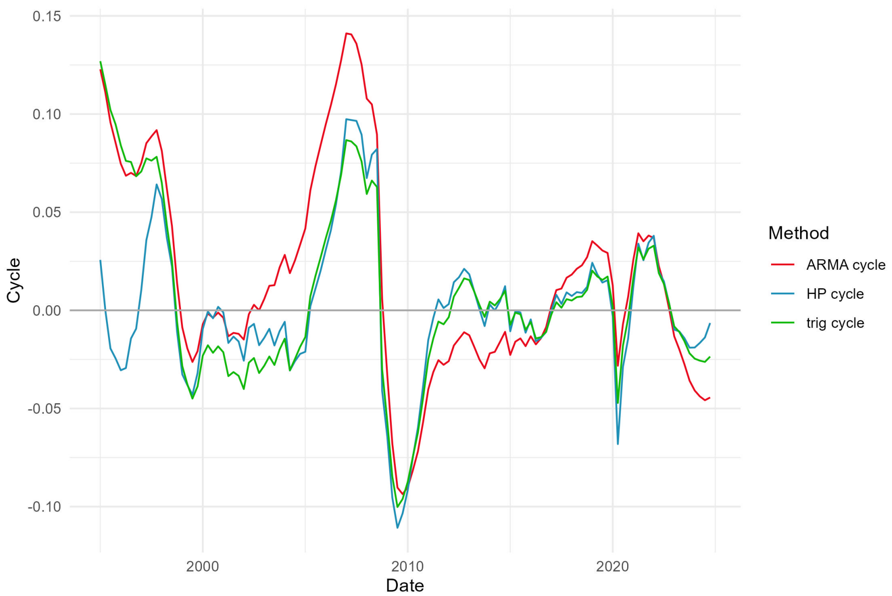

Figure 9: Comparison of different cycle estimation methods for Estonia's GDP time series.

Figure 9 provides a comparison of different cycle estimation methods for Estonian GDP data. Specifically, it shows the structural model cycle estimates using the ARMA formulation (red) and the trigonometric formulation (green), alongside the cycle estimated using the Hodrick–Prescott (HP) filter (blue). To allow visual comparability, the structural model cycles have been scaled by dividing through with the observed GDP values, while the HP filter was applied to log-transformed data. In general, the cycle estimates capture similar periods of growth and decline. In the later periods trigonometric and HP filter estimates closely follow each other, while ARMA-based estimate tends to lead the turning points slightly and places more emphasis on periods of economic growth. Near the end of the estimation period, the three methods show different results for the timing and magnitude of the latest turning point.

For the other countries, HP filter cycle estimates do not align as closely with the ARMA and trigonometric cycle estimates. The corresponding figures for these countries can be found in Appendix 2, where both the additive and multiplicative cases are presented.

Next, the theoretical forms of the structural time series models estimated for the Estonian GDP time series are presented. Both structural models are trend-cycle models with a damped local linear trend formulation, where an additional intercept term is included in the slope equation. This intercept term is estimated using R's autostsm package, which defines the slope equation in this way. Unlike for the trend component, the autostsm package does not define a damping factor for the trigonometric cycle formulation.

In case of the trigonometric cycle formulation, Estonia's structural time series model is expressed as

$$y_t = \mu_t + \varphi_t + \varepsilon_t; \quad \varepsilon_t \sim NID(0; 0.0195^2);$$

$$\mu_t = \mu_{t-1} + \beta_t + \eta_t; \quad \eta_t \sim NID(0; 0.0002^2);$$

$$\beta_t = 0.0062 + 0.791\beta_{t-1} + \xi_t; \quad \xi_t \sim NID(0; 0.0194^2),$$

where the cyclical component φ_t follows a trigonometric specification with $\lambda = 0.2034$:

$$\begin{pmatrix} \varphi_t \\ \varphi_t^* \end{pmatrix} = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \varphi_{t-1} \\ \varphi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \kappa_t \\ \kappa_t^* \end{pmatrix}, \quad \kappa_t \sim NID(0; 0.94^2).$$

For the ARMA cycle formulation, Estonia's structural model takes the following form, with the cyclical component specified as an ARMA(2,2) process

$$y_t = \mu_t + \varphi_t + \varepsilon_t; \quad \varepsilon_t \sim NID(0; 0.0224^2);$$

$$\mu_t = \mu_{t-1} + \beta_t + \eta_t; \quad \eta_t \sim NID(0; (1.0 \cdot 10^{-6})^2);$$

$$\beta_t = 0.0113 + 0.629\beta_{t-1} + \xi_t; \quad \xi_t \sim NID(0; 0.0219^2);$$

$$\varphi_t = 1.7357\varphi_{t-1} - 0.7721\varphi_{t-2} + \alpha_t + 1.238\alpha_{t-1} - 2.2378\alpha_{t-2} + \zeta_t; \quad \zeta_t \sim NID(0; 0.0219^2).$$

It is important to note that any similarity in estimated coefficient values is coincidental or due to rounding. For instance, the equality of σ_ξ and σ_κ arises solely from rounding.

The estimated coefficient values for the structural time series models in both the additive and multiplicative cases for the other countries are presented in Appendix 3.

4.3 Relations Between Business Cycles

This subsection focuses on determining whether estimated business cycles indicate any relations between Estonia and its 5 biggest export partners. Granger causality test allows to determine whether the values of one country's business cycle lead the values of the other, and vice versa. Thus, revealing potential economic dependencies between the countries. For each country, the Granger causality test was conducted on the estimated cycle values, normalized by dividing them by the respective observed GDP values. Table 2 presents the results of the Granger causality test of whether Estonia's business cycle Granger causes the other countries' business cycle values and vice

Table 2: Results of the Granger causality tests assessing whether a given country's business cycle Granger-causes Estonia's (p-val Granger →EE), and whether Estonia's business cycle Granger-causes the given country's (p-val Granger EE →). Significance codes: '***' $p < 0.001$, '**' $p < 0.01$, '*' $p < 0.05$, '.' $p < 0.1$, and blank for $p \geq 0.1$.

Country	Cycle type	Lag order	p-val Granger →EE	p-val Granger EE→
Latvia	Trigonometric	8	0.0898 .	0.0123*
	ARMA	3	0.0113*	0.2526
	HP	4	0.0049**	0.0006***
Lithuania	Trigonometric	4	0.6909	< 0.0001***
	ARMA	5	< 0.0001***	0.0242*
	HP	4	0.4266	< 0.0001***
Sweden	Trigonometric	7	0.9095	0.4008
	ARMA	6	0.1282	0.2884
	HP	8	0.8432	0.3258
Finland	Trigonometric	4	0.0159*	0.2536
	ARMA	4	0.0055**	0.7583
	HP	4	0.3019	0.0014**
Germany	Trigonometric	4	0.0217*	0.6665
	ARMA	4	< 0.0001***	0.4852
	HP	4	0.1550	0.0192*

versa for all cycle estimation methods. A small p-value indicates the presence of a Granger-causal relationship, and the lag order is determined using the Akaike Information Criteria (AIC).

The test results show that Latvia's business cycle Granger-causes Estonia's business cycle in both ARMA and HP case while Estonia appears to Granger-cause Latvia in the trigonometric and HP cases. This suggests a potential two-way economic dependency between Estonia and Latvia. A similar result is observed with Lithuania. Interestingly, the Granger causality test does not detect any dependencies between Swedish and Estonian economies. An important discovery; however, are the dependencies between Estonia and Finland, and Estonia and Germany. Both trigonometric and ARMA cases, the business cycle values of Finland and Germany lead Estonia's business cycle values. This result is important as it shows that economic activity that happens in Germany and Finland will most likely cause a similar change in the Estonia's economic activity. For example, if there happens to be a period of decline in Finland's economic activity, it will most likely cause a period of decline in Estonia's economic activity as well. However, when using the Hodrick-Prescott (HP) filter cycle estimates, the Granger causality test indicates the opposite direction of dependence: from Estonia to both Finland and Germany. This result in likely misleading, as it is improbable that a small economy like Estonia's would exert a causal influence on significantly larger economies such as Germany's. This interpretation is supported by the fact that the HP filter is not considered to be the most reliable method for isolating the cyclical components.

One factor that could have influenced the Granger test results was the initial estimate period from 1995 to 2002, which exhibited highly volatile and inconsistent behaviour before stabilizing in later years. To account for this, the Granger test was repeated using only the estimated values from 2002 to 2024. As a result, three previously detected dependencies disappeared, while two new cases of Granger causality emerged. The new detected dependencies include Lithuania Granger-causing Estonia in the case of HP filter, and Sweden Granger-causing Estonia in the case of ARMA cycle formulation. Meanwhile, three dependencies identified earlier that were no longer detected: Estonia Granger-causing Latvia (trigonometric), Finland Granger-causing Estonia (trigonometric), and Germany Granger-causing Estonia (trigonometric). This suggests that the Granger causality results for the trigonometric cycle formulation are highly sensitive to the values of the initial estimate period. This can be seen, for example, in Figure 4, where from 2002 onward, fluctuations in the business cycles tend to occur simultaneously across countries. In contrast, Figure 7 shows that in the ARMA cycle formulation, the fluctuations of the other countries' cycles occur slightly before Estonia's cycle.

The results of the Granger test for the multiplicative model case are provided in Appendix 4.

4.4 Forecasting

This subsection focuses on evaluating the forecast accuracy of structural time series models, considering both the trigonometric and ARMA cycle formulations. Their predictive performance is assessed in comparison to that of ARIMA models.

A prediction horizon of one and a half years (six quarters) is used to compare forecast performance. The choice is motivated by the fact that in Estonia the upcoming year's national budget is addressed during the summer period; therefore, being able to accurately forecast the economic activity till the end of the following year is important. Forecast accuracy is evaluated in two ways: one approach considers only the sixth-step prediction, while the other accounts for all six forecast steps. The evaluation is based on a recursive control-test set framework, also known as the rolling forecast approach. The initial training set includes quarterly GDP values from January 1, 1995, to October 1, 2020. Based on this data, both structural and ARIMA models are estimated using automatic model selection and then used to produce forecasts six periods ahead. The process is repeated 11 times, each time extending the training set by one additional quarter. The final training set ends on March 1, 2023. The accuracy of the forecasts is evaluated using two common measures: root mean square error (RMSE) and mean absolute percentage error (MAPE). In case when only the sixth prediction is evaluated RMSE and MAPE are calculates as follows

$$\mathrm{RMSE}_6 = \sqrt{\frac{1}{11}\sum_{j=1}^{11}(y_{j+6} - \hat{y}_{j+6})^2}\ ; \quad \mathrm{MAPE}_6 = \frac{1}{11}\sum_{j=1}^{11}\frac{|y_{j+6} - \hat{y}_{j+6}|}{y_{j+6}}.$$

Table 3 presents the RMSE and MAPE results for the case where only the sixth prediction is evaluated. The results show that the ARIMA model generally provides the most accurate predictions 1.5 years ahead. The exceptions are Sweden and Finland, where the most accurate forecasts are obtained using the structural model with the ARMA cycle formulation. When comparing models with ARMA and trigonometric cycle formulations, the forecast accuracy is relatively similar in most cases, with the largest difference being 0.69 percentage points in Estonia's time series. However, the ARMA formulation demonstrates slightly better overall predictive performance, producing lower forecast errors in 5 of the 6 countries. When considering results by country, Germany's GDP forecasts for the sixth period were the most accurate overall, with the ARIMA model achieving a mean absolute percentage error of 1.5%, corresponding to a root mean square error of 12 467.91 units. However, the overall prediction errors are quite large, with mean absolute prediction

Table 3: Evaluation of forecast accuracy for the 6th value prediction measured by RMSE and MAPE.

	RMSE_6			**MAPE_6**		
Country	**Trig**	**ARMA**	**ARIMA**	**Trig**	**ARMA**	**ARIMA**
Estonia	324.64	361.34	303.22	0.0562	0.0631	0.0523
Latvia	309.91	321.42	274.41	0.0420	0.0404	0.0346
Lithuania	171.25	158.07	234.57	0.0132	0.0120	0.0177
Sweden	22 336.63	21 574.77	23 632.44	0.0180	0.0171	0.0180
Finland	1 319.30	1 280.85	1 561.15	0.0242	0.0217	0.0282
Germany	19 349.19	15 857.74	12 467.91	0.0239	0.0185	0.0151

errors ranging from 1.2% to 6.3%. A major contributor to these errors is likely the occurrence of two significant non-economic crises during the forecast period: the COVID-19 pandemic and the onset of the Russia–Ukraine war. The economic effects of the COVID-19 pandemic began in the second quarter of 2020, immediately after the forecast evaluation period started. The economic impact of the Russia–Ukraine war emerged around the second quarter of 2022. These events appear to have had the greatest impact on Estonia, Latvia, and Finland, where prediction errors are notably higher. As shown in Figure 10, all models substantially overestimated actual GDP values, leading to large forecast deviations. Similar overestimation patterns can be observed for other countries as well (see Appendix 5, Figure 15). For additional context, Table 4 presents the annual GDP growth forecasts from Estonia's Ministry of Finance for 2021–2024, which were published in the summer of 2021. A comparison with the actual outcomes shows that these official forecasts also significantly overestimated growth, with the divergence from actual values becoming especially evident starting in 2022.

Table 4: GDP forecasts by the Estonian Ministry of Finance made in summer 2021 (Estonian Ministry of Finance, 2021), compared to the actual GDP growth (Statistics Estonia, 2025b).

	2021	2022	2023	2024
Actual annual GDP growth	7.2%	0.1%	–3.0%	–0.3%
Ministry of Finance prediction	9.5%	4.0%	2.6%	2.9%

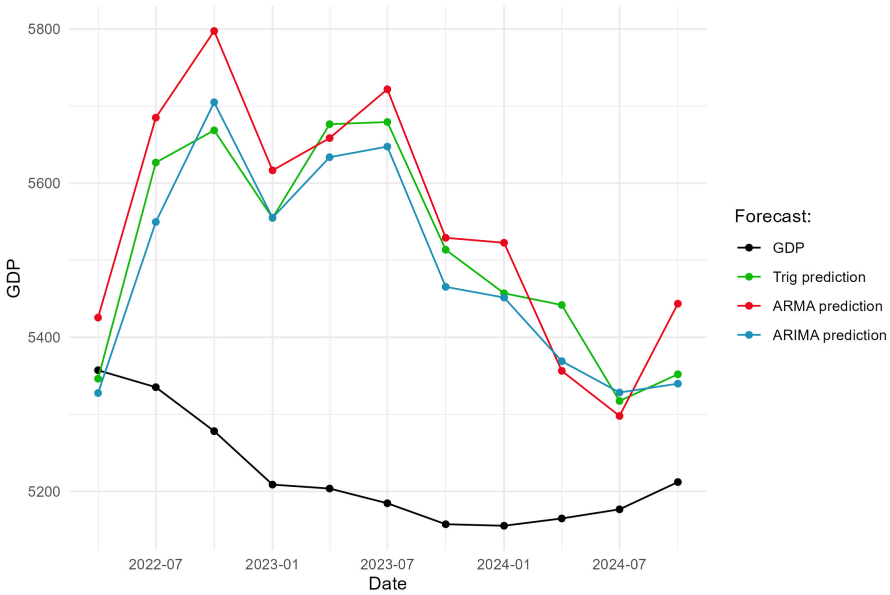

Figure 10: Actual GDP values for Estonia compared to sixth-step forecasts from structural and ARIMA models (2022–2024).

In case when all the six prediction steps are evaluated RMSE and MAPE are calculated as follows

$$\text{RMSE} = \sqrt{\frac{1}{6}\frac{1}{11}\sum_{j=1}^{11}\sum_{k=1}^{6}(y_{j+k} - \hat{y}_{j+k})^2} \; ; \quad \text{MAPE} = \frac{1}{6}\frac{1}{11}\sum_{j=1}^{11}\sum_{k=1}^{6}\frac{|y_{j+k} - \hat{y}_{j+k}|}{y_{j+k}}.$$

Table 5 presents the RMSE and MAPE results for the case where all six forecast steps are evaluated. This approach captures both short- and long-term prediction accuracy. Based on the results, ARIMA models produced the most accurate forecasts in three cases, followed by the ARMA formulation in two cases, and the trigonometric formulation in one case. When focusing on the structural models only, the forecast accuracy between the ARMA and trigonometric cycle formulations remains very close across all countries, with the largest difference being 0.35 percentage points in Germany's time series. The ARMA formulation slightly outperforms the trigonometric one, producing lower forecast errors in 4 out of the 6 countries. Germany's GDP forecasts again achieved the highest accuracy, with the ARIMA model producing a mean absolute percentage error of 0.99%, corresponding to a root mean square error (RMSE) of approximately 9006.43 units. Across all countries, the general mean absolute prediction error ranges from 0.9% to 3.6%.

Table 5: Evaluation of forecast accuracy across all six prediction steps measured by RMSE and MAPE.

Country	**RMSE**			**MAPE**		
	Trig	**ARMA**	**ARIMA**	**Trig**	**ARMA**	**ARIMA**
Estonia	216.60	234.87	202.06	0.0339	0.0363	0.0316
Latvia	221.93	229.51	226.13	0.0287	0.0262	0.0277
Lithuania	182.01	161.24	214.33	0.0130	0.0118	0.0158
Sweden	17 547.13	16 884.45	18 489.32	0.0133	0.0128	0.0137
Finland	934.47	956.37	1 122.36	0.0154	0.0158	0.0186
Germany	13 069.33	10 649.61	9 006.43	0.0147	0.0112	0.0099

The RMSE and MAPE results for the multiplicative model case together with the figures comparing the actual and 6th value predictions are presented in Appendix 5.

Conclusions

This thesis examined the GDP time series data of Estonia and its five biggest trading partners - Latvia, Lithuania, Finland, Sweden, and Germany. To estimate the business cycles of these countries, structural models with both trigonometric and ARMA cycle formulations were applied, alongside an alternative cycle estimation method of the Hordick-Prescott (HP) filter. The estimated cycles showed promising results, successfully capturing key turning points in economic activity. When comparing the structural model cycle formulations, the trigonometric cycle estimates tended to capture more small-scale fluctuations, while the ARMA-based cycle estimates appear smoother.

To explore potential economic dependencies between Estonia's and its key trading partners' business cycles, the Granger causality test was performed. The test results indicated possible two-way economic dependencies between Estonia and the other Baltic countries. In addition, economic dependencies from Finland and Germany to Estonia were detected. No such relationship was detected between Estonia and Sweden.

To evaluate forecast accuracy of the structural models, a rolling forecast approach was used with a prediction horizon of six quarters. For comparison, ARIMA models were also assessed using the same procedure. Forecast accuracy was evaluated in two ways: by examining the error of the sixth prediction alone, and by averaging error across all six prediction steps. The results were inconclusive, as the structural and ARIMA models performed similarly overall. However, when comparing the structural models, the ARMA formulation generally outperformed the trigonometric formulation in terms of forecast accuracy.

Given the scope and depth of this thesis, the analysis was limited to univariate structural time series models. For further research, extending the framework to multivariate time series models with a common cycle specification could offer further insights.

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Appendix 1. Cycle and Trend Estimates

Figure 11: Trigonometric cycle and trends estimates for all countries using multiplicative models.

Note: Latvia's cycle estimate appears unrealistic in the multiplicative trigonometric case. With the exception of Latvia, key turning points are generally well captured.

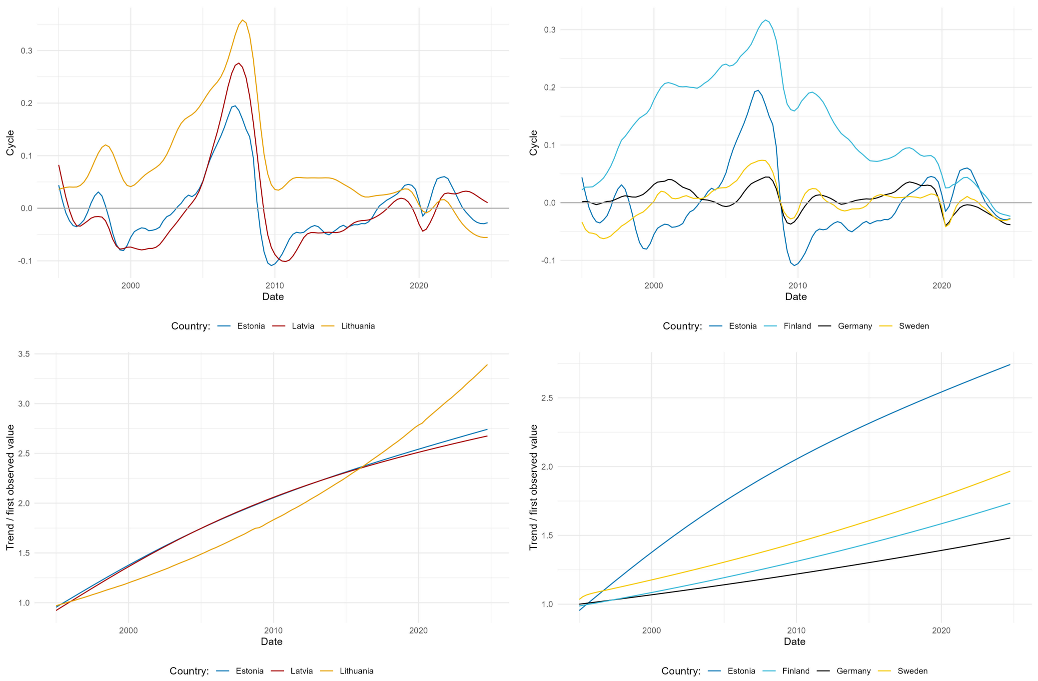

Figure 12: ARMA-based cycle and trends estimates for all countries using multiplicative models.

Note: Finland's ARMA-based cycle estimate appears implausible. In the multiplicative model, ARMA-based cycle estimates also appear smoother than those from the trigonometric formulation.

Appendix 2. Comparison of Cycle Extraction Methods

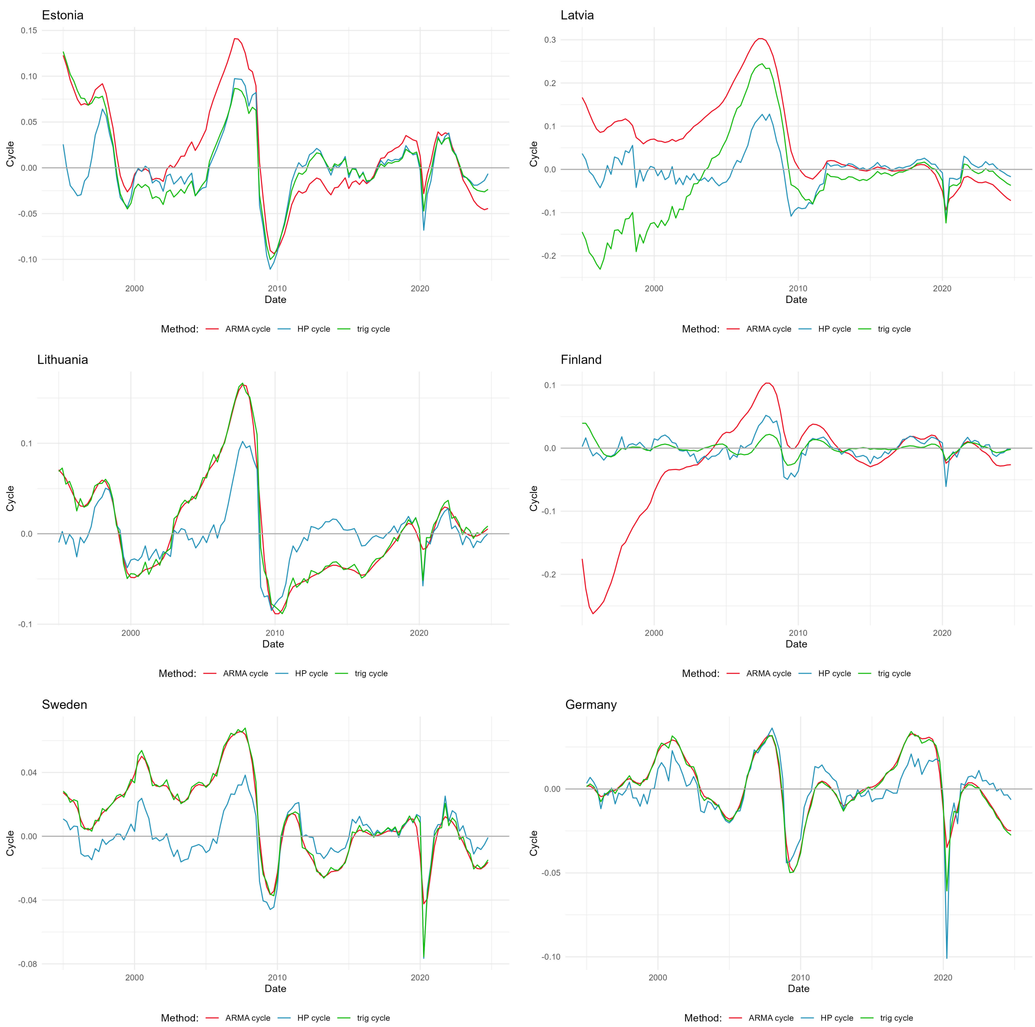

Figure 13: Comparison of different cycle estimation methods for other countries using additive models.

Note: Overall, the three methods behave similarly, though HP filter estimates tend to differ the most.

Figure 14: Comparison of different cycle estimation methods for other countries using multiplicative models.

Note: The key turning points are detected by all methods, but the multiplicative models yield unrealistic results in two cases.

Appendix 3. Estimated Coefficients of Structural Models

Table 6: Estimated model coefficients for Estonia, Latvia, and Lithuania (additive models). Note: b is the slope intercept estimated by R.

		Estonia		**Latvia**		**Lithuania**	
	Coef.	**Trig**	**ARMA**	**Trig**	**ARMA**	**Trig**	**ARMA**
Main	σ_ε	0.01952	0.02242	0.02339	0.04553	0.01064	0.03426
	σ_η	0.00016	$1.0 \cdot 10^{-6}$	0.00002	0.00002	$9.3 \cdot 10^{-7}$	0.00001
Trend	ρ	0.79098	0.62899	0.00860	0.02289	0.00005	0.00008
	b	0.00620	0.01129	0.02555	0.03152	0.02928	0.02937
	σ_ξ	0.01936	0.02191	0.00004	0.00004	$1.7 \cdot 10^{-6}$	0.00001
	λ	0.20342	...	0.11651	...	0.14252	....
	σ_κ	0.05500	...	0.07803	...	0.05408	....
	AR1	...	1.7357	...	1.36192	...	2.30035
Cycle	AR2	...	-0.77207	...	-0.02646	...	-1.82521
	AR3	...	...	...	-0.36104	...	0.50180
	MA1	...	1.23803	...	...	...	...
	MA2	...	-2.23784	...	...	...	...
	σ_ζ	...	0.02191	...	0.05040	...	0.01507

Table 7: Estimated model coefficients for Sweden, Finland, and Germany (additive models).

		Sweden		**Finland**		**Germany**	
	Coef.	**Trig**	**ARMA**	**Trig**	**ARMA**	**Trig**	**ARMA**
Main	σ_ε	0.01517	0.04137	0.05405	0.05623	0.06713	0.08595
	σ_η	0.00001	$3.9 \cdot 10^{-6}$	$1.2 \cdot 10^{-7}$	0.00006	$4.8 \cdot 10^{-6}$	$4.8 \cdot 10^{-6}$
Trend	ρ	0.00026	0.00004	0.71472	0.41255	0.00006	0.00012
	b	0.03042	0.0305	0.00886	0.01106	0.02889	0.02879
	σ_ξ	$1.5 \cdot 10^{-6}$	0.00001	0.03696	0.00005	$4.8 \cdot 10^{-6}$	0.00001
	λ	0.03207	...	0.41700	...	0.23879	....
	σ_κ	0.88574	...	0.03697	...	0.08800	....
	AR1	...	1.37572	...	1.56585	...	1.52264
Cycle	AR2	...	-0.46344	...	-0.59593	...	-0.63979
	MA1	...	-0.50108	...	0.03145	...	...
	MA2	...	-0.49879	...	...	...	...
	σ_ζ	...	0.03122	...	0.05374	...	0.05182

Table 8: Estimated model coefficients for Estonia, Latvia, and Lithuania (multiplicative models). Note: b is the slope intercept estimated by R.

	Coef.	Estonia Trig	Estonia ARMA	Latvia Trig	Latvia ARMA	Lithuania Trig	Lithuania ARMA
Main	σ_ε	0.02546	0.02584	0.04094	0.04540	0.00006	0.02874
	σ_η	$8.2{\cdot}10^{-9}$	0.00002	$3.6{\cdot}10^{-8}$	0.00002	0.00002	0.00003
Trend	ρ	0.83270	0.97310	0.8044	0.97451	0.12638	0.00004
	b	0.00510	0.00027	0.00627	0.00016	0.02378	0.03138
	σ_ξ	0.02546	$4.6{\cdot}10^{-7}$	0.02915	0.00001	0.00001	0.01183
	λ	0.48909	...	0.02297	...	0.09430	....
	σ_κ	0.02546	...	0.02915	...	0.05194	....
	AR1	...	1.67943	...	1.80108	...	2.49338
Cycle	AR2	...	-0.73346	...	-0.83466	...	-2.15479
	AR3	...	...	...	...	...	0.65765
	MA1	...	...	...	-0.57498	...	...
	MA2	...	...	...	-0.42222	...	...
	σ_ζ	...	0.03628	...	0.01930	...	0.01185

Table 9: Estimated model coefficients for Sweden, Finland, and Germany (multiplicative models).

	Coef.	Sweden Trig	Sweden ARMA	Finland Trig	Finland ARMA	Germany Trig	Germany ARMA
Main	σ_ε	0.00301	0.03387	0.04236	0.05069	0.05243	0.07842
	σ_η	0.00001	0.00001	0.00014	0.00002	0.00001	0.00001
Trend	ρ	0.00008	0.43879	0.95652	0.00017	0.00004	0.00005
	b	0.02754	0.01552	0.00058	0.03264	0.03019	0.03005
	σ_ξ	0.00002	0.00001	0.00824	0.00006	0.00001	0.00002
	λ	0.00002	...	0.33833	...	0.15441	...
	σ_κ	0.06456	...	0.05241	...	0.10007	...
	AR1	...	1.47282	...	1.66088	...	1.46491
Cycle	AR2	...	-0.53246	...	-0.66436	....	-0.55549
	MA1	...	-0.46388	...	-0.93349	...	...
	MA2	...	0.13013	...	...	...	...
	σ_ζ	...	0.03535	...	0.03470	....	0.05834

Appendix 4. Multiplicative Model: Granger Causality Test

Table 10: Multiplicative model results of the Granger causality tests assessing whether a given country's business cycle Granger-causes Estonia's (p-val Granger →EE), and whether Estonia's business cycle Granger-causes the given country's (p-val Granger EE →). Significance codes: '***' $p < 0.001$, '**' $p < 0.01$, '*' $p < 0.05$, '.' $p < 0.1$, and blank for $p \geq 0.1$.

Country	Cycle type	Lag order	p-val Granger →EE	p-val Granger EE→
Latvia	Trigonometric	10	0.3098	< 0.0001***
	ARMA	4	< 0.0001***	0.7609
	HP	4	0.0049**	0.0006***
Lithuania	Trigonometric	6	0.0676 .	< 0.0001***
	ARMA	6	< 0.0001***	< 0.0001***
	HP	4	0.4266	< 0.0001***
Sweden	Trigonometric	9	0.0553 .	< 0.0001***
	ARMA	3	0.8784	0.1945
	HP	8	0.8432	0.3258
Finland	Trigonometric	4	0.0506 .	0.0101*
	ARMA	4	0.3206	0.0076**
	HP	4	0.3019	0.0014**
Germany	Trigonometric	8	0.2336	< 0.0001***
	ARMA	4	0.0150*	0.3381
	HP	4	0.1550	0.0192*

Note: Several unreasonable dependencies are detected with multiplicative models, such as Estonia Granger-causing Sweden and Finland. These likely stem from unrealistic cycle estimates.

Appendix 5. Forecast Accuracy

Figure 15: Forecasted vs actual GDP for all countries using additive structural models (sixth-step predictions).

Note: The six-step-ahead forecasts (1.5 years) tend to overestimate actual GDP values during the prediction period.

Table 11: Evaluation of forecast accuracy for the 6th value prediction measured by RMSE and MAPE using multiplicative models.

	$\mathbf{RMSE_6}$			$\mathbf{MAPE_6}$		
Country	**Trig**	**ARMA**	**ARIMA**	**Trig**	**ARMA**	**ARIMA**
Estonia	397.13	380.60	398.38	0.0667	0.0604	0.0714
Latvia	319.70	380.21	272.68	0.0407	0.0450	0.0342
Lithuania	503.40	565.17	437.59	0.0386	0.0474	0.0328
Sweden	33 431.64	35 672.70	30 563.61	0.0278	0.0298	0.0241
Finland	1 288.13	1 209.66	1 211.79	0.0221	0.0194	0.0221
Germany	32 733.59	27 482.03	14 125.54	0.0419	0.0338	0.0173

Note: ARIMA models are applied to log-transformed data. Structural and ARIMA models perform similarly overall. Trigonometric and ARMA cycle models also yield close results. MAPE ranges from 1.7% to 7.1%, compared to 1.2%–6.3% in the additive case.

Table 12: Evaluation of forecast accuracy across all six prediction steps measured by RMSE and MAPE using multiplicative models.

	RMSE			**MAPE**		
Country	**Trig**	**ARMA**	**ARIMA**	**Trig**	**ARMA**	**ARIMA**
Estonia	259.21	236.98	255.25	0.0390	0.0323	0.0396
Latvia	228.40	259.91	201.93	0.0278	0.0280	0.0244
Lithuania	349.41	359.35	301.79	0.0248	0.0248	0.0207
Sweden	23 919.05	24 189.84	22 032.27	0.0182	0.0184	0.0162
Finland	901.53	871.83	923.09	0.0147	0.0141	0.0157
Germany	21 984.52	18 476.61	9 803.24	0.0254	0.0206	0.0108

Note: Forecast accuracy remains similar across methods. ARIMA slightly outperforms structural models (4 vs 2), and ARMA and trigonometric cycle models perform equally well (3 vs 3). MAPE ranges from 1.1% to 4.0%, compared to 1.0%–3.6% in the additive case.

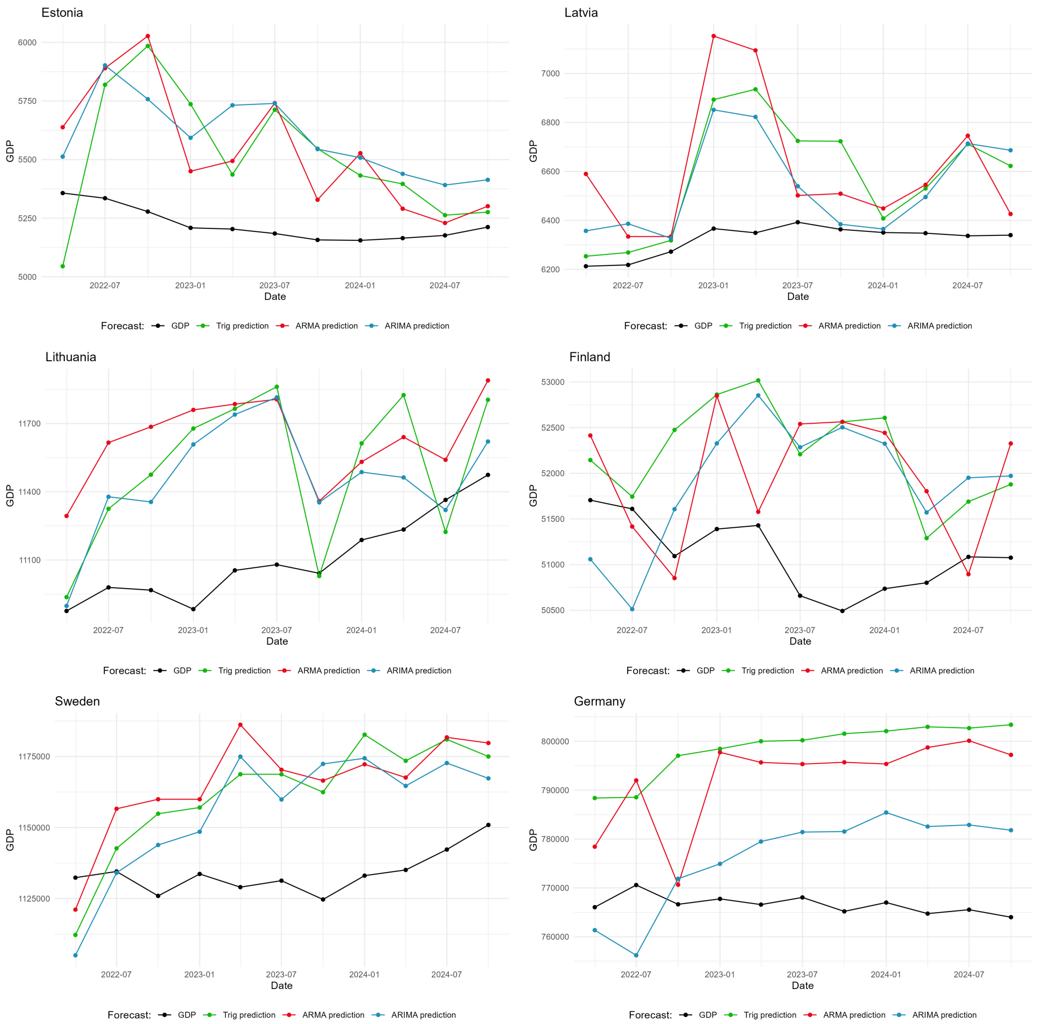

Figure 16: Forecasted vs actual GDP for all countries using multiplicative structural models (sixth-step predictions).

Note: Forecasts based on the multiplicative models also overestimate actual GDP values over the six-step (1.5-year) horizon.

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