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DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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Optimization of stepped plates in the case of smooth yield surfaces

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Author's contribution: The author of this dissertation is responsible for the majority of the research in all phases of preparation of images of papers 1–6. The solution procedures were developed in co-operation with the supervisor; the numerical simulations, including parallel computing implementations of the numerical methods, were implemented by the author of the dissertation; the statement of the problem belongs to the supervisor.

Chapter 1
Introduction

1.1 Review of literature

Thin-walled plates and shells are widely used in the aircraft and space industry. Moreover, plates and shells are used in various fields of engineering and industry. This involves the need for the minimization of the cost and weight of plate and shell structures.

Although the investigation of the elastic plastic behaviour of plates and shells got its start quite early (for instance, the early results on the elastic plastic bending of circular and annular plates are due to Hodge [40], Tekinalp [133], [22], Haythornthwaite [34], Cinquini [16], Lamblin *et al.* [71] and others) there exist only a few papers on optimization of elastic plastic beams, plates and shells.

The problems of optimization of plates and shells made of pure elastic or of ideal inelastic materials are studied by many investigators. Optimal designs of elastic plastic beams with piece wise constant thickness are established by Lepik [91]. It is assumed herein that the beams under consideration are subjected to the distributed transverse pressure of high intensity. In the subsequent paper by Lepik [92] optimal positions for additional supports are established for elastic plastic beams. Lellep and Polikarpus [76], [77] studied the elastic plastic response of circular plates to distributed loading in the case of the Tresca material. Lellep, Puman, *et al.* studied stepped [79] and rotationally symmetric shells [80]. Reviews of these papers can be found in books and review papers by Banichuk [3], Atkočiūnas [2], Kirsch [61], Cherkaev [12], Karkauskas, Čyras, Borkowski [57], Bendsoe [6], Lellep and Lepik [75], Krużelecki and Życzkowski [66], Rozvany [125].

Kaliszky and Logo [55], [56] investigated the elastic plastic behavior of disks and developed an optimization method in the case of presence of constraints imposed on displacements and deformations.

Elastic plastic response of circular and annular plates was studied by Lellep and Polikarpus [78], [77] in the case of sandwich plates made of a material which obeys the Tresca's yield condition. An optimal design of axisymmetric plates subjected to the uniformly distributed transverse pressure has been established by Lellep and

Vlassov [81], [82], [83], [84], [85], [86], [87] in the cases of von Mises and Hill's yield criteria.

In the present research an optimization technique is developed for circular plates made of an ideal elastic plastic material obeying von Mises yield condition and the associated flow law. Necessary optimality conditions are derived with the aid of variational methods of the theory of optimal control.

Problems of optimization of annular plates have been studied by many authors in the case of a pure elastic material (see Banichuk, 1991; Dzjuba [160] and others). On the other hand, there exists an exhaustive list of papers devoted to the optimization of perfectly plastic plates (see Lellep [73]; Lepik [89], [90]). However, there exist only a few papers concerning optimization of plates made of elastic plastic materials. Among such papers one should mention the papers by Lellep and Vlassov [81], [82], [83], [84], [85], [86], [87].

In this study the method of optimization is extended to annular plates subjected to the transverse pressure.

Circular and annular plates are of practical interest in mechanical, civil and ocean engineering where these plates are used as bulkheads of separable sections of submersibles. A purely elastic design of a structural element made of a ductile material and loaded by pressures of high intensity may be overly conservative. Thus, it is reasonable to account for the elastic plastic stages of deformation in the direct problem of determination of the stress strain state of plates as well as in the shape optimization of plates.

The foundations of the stress strain analysis of plates and shells made of composite materials are presented in books by Jones [48], Herakovich [36], Reddy [121], Vinson and Sierakowski [146], Tittle [134]. Although the elastic plastic analysis of plates and shells got its start several decades ago (see Hodge [41], Chakrabarty [14], Kaliszky [54], Sawczuk, Sokół-Supel [129]), new approaches have been developed during the last years (Kojić, Bathe, [65]). Gorji and Akileh [30] utilized the concept of the load analogy to obtain non-linear elastic plastic solutions to annular plates undergoing moderately large deflections.

The problem of determination of the stress-strain state of circular and annular plates under the transverse pressure has a long history. The first attempts in this direction have been undertaken by Sokolovski [164] and Naghdi [110]. Assuming that the material corresponds to the von Mises yield condition Naghdi [110] developed the concept of a deformation-type theory of plasticity and calculated the deflections of simply supported plates of uniform thickness subjected to the concentrated load at the center of the plate. Different extensions of the problem and new calculation algorithms are presented by Lackman [67], Popov *et al.* [118].

Eason [22] studied the problem of the elastic plastic bending of a circular plate which is simply supported at its edge and carries a constant load over a central circular area. The solution is compared with that corresponding to the Tresca yield hexagon. The solution for Tresca plates have been obtained by Tekinalp [133], also by Hodge [39].

Later Turvey [136], Turvey and Lim [138], Turvey and Salehi [137] compiled computer codes for the elasto-plastic analysis of circular plates. The algorithms use the constitutive models based on the Ilyshin full-section yield criterion and the von Mises yield criterion.

Makapatra and Dasgupta [104] developed the mixed finite element method for axisymmetric elasto-plastic problems.

Upadrasta *et al.* [139] used a deformation-type theory of plasticity to implement the method of elastic compensation. The main idea of this method consists in the iterative modification of elastic properties of the material used in the finite element codes to simulate plastic yielding. The predictions obtained for simply supported and clamped plates are compared with the earlier results by Ohashi, Murakami [114], Lim, Turvey [138] and others whereas a good agreement between various predictions was observed.

In the present paper the method of optimization of elastic plastic annular plates is developed. The plates are made of anisotropic materials which obey the yield criteria of Hill and Tsai-Wu. It is assumed that the plates have a sandwich cross section whereas the carrying layers are of piece wise constant thickness.

Although the solution of elastic-plastic plate problems got its start several decades ago ([28], [30], [31]; [124]) there still exists the need for new computer-aided techniques for calculation of elastic-plastic plates (see [146]). The same regards the problems of optimization of plate and shell structures which have a theoretical and practical importance as shown in [5]. Some new approaches for optimal design of steel structures were developed by Farkas and Jármai [25], [26] recently.

Minimum weight problems for axisymmetric plates operating in the range of elastic plastic deformations were studied by Lellep and Polikarpus [78] in the case of the material obeying Tresca's yield condition and the associated flow law. The case of Von Mises material was investigated by Lellep and Vlassov [82], [84].

The problem governing equations [161] in a tensor form can also be found in [162] – [163].

Wavelet theory used in current study is presented in [15], [18], [20], [21], [53], [59], [62], [68], [103], [105], [111], [122], [131], [132], [141], [148], [150]. The method of solving differential equations using Haar wavelet method was presented by Lepik [93] – [100], [102]. The solution of optimal control problem solving via Haar wavelets was also studied by Lepik [101].

1.2 Finite element method

The finite element method (FEM) a numerical procedure for solving mathematical, mechanical and physical problems governed by differential equations or an energy theorem is described in numerious monografies and articles [130]. There are two characteristics that distinguish it from other numerical procedures:

- the method utilizes an integral formulation to generate a system of algebraic equations;
- the method uses continuous piecewise smooth functions for approximating the unknown quantity or quantities.

The second feature distinguishes the finite element method from other numerical procedures that utilize an integral formulation.

The implementation of FEM can be described as follows:

- A structure is divided into several elements (pieces of the structure).
- The FEM reconnects elements at "nodes" as if nodes were pins or drops of glue that hold elements together.
- Current process results in a set of simultaneous algebraic equations.

Note here that in case of continuum numerical calculus we have infinite number of degrees of freedom, in case of FEM it is finite, which gives the origin of the method name: *F̲inite E̲lement M̲ethod*. It is also known as a *Finite Element Analysis* (FEM) and this numerical method provides solutions to problems that would otherwise be difficult to obtain.

FEA/FEM has applications in much broader range of areas: fluid and fracture mechanics, electrostatic and electromagnetics [42], heat flow etc. While this range is growing, one thing will remain the same in case of classical FEM: the theory of how the method works.

Before the appearance of FEM it was implicitly assumed that basis functions N_m of the decomposition

$$\varphi \approx \hat{\varphi} = \psi + \sum_{m=1}^{M} a_m N_m \tag{1.1}$$

were determined by one expression over the whole domain (manifold) Ω, and integrals in the approximation equations were calculated over the whole Ω.

An alternative approach is to divide domain into a number of disjoint subdomains called elements Ω^e and construct the piecewise approximation $\hat{\varphi}$ separately onto each subdomain. Then, basis functions of the approximation can also be determined piecewise by applying different expressions for the different subdomains Ω^e, which are composed the whole region Ω, i.e. $\bigcup_E \Omega^e = \Omega$. Then, definite integrals from approximation equations can be obtained by summing the values of weighting functions W_l and $\overline{W}_l$ of each subdomain (or element):

$$\begin{aligned} \int_{\Omega} W_l R_\Omega d\Omega &= \sum_{e=1}^{E} \int_{\Omega^e} W_l R_\Omega d\Omega, \\ \int_{\Gamma} \overline{W}_l R_\Gamma d\Gamma &= \sum_{e=1}^{E} \int_{\Gamma^e} \overline{W}_l R_\Gamma d\Gamma, \end{aligned} \tag{1.2}$$

where $\sum_{e=1}^{E} \Omega^e = \Omega$ and $\sum_{e=1}^{E} \Gamma^e = \Gamma$. Here E is a number of subdomains (elements) and Γ^e are the parts of the subdomain Ω^e boundary which lies onto domain boundary Γ. Hence, Γ^e summing should be done only by elements Ω^e which have a common boundary section.

If subdomains have rather simple form and their approximation functions basises are determined simultaneously, it will be a good idea to use presented approach in case of domains which consists of such subdomains. This idea of classical FEM is well-known as a method for numerical solution of field problems.

Let us consider the fundamental concepts of the implementation of the method. Generally, many engineering and theoretical phenomena can be expressed by "governing equations" and "boundary conditions". Let $L(\phi)+f=0$ be a governing equation written as the ordinary or partial differential equation and $B(\phi)+g=0$ be a boundary condition obtained from the theories of fluid mechanics, elasticity and plasticity or electrostatic and electromagnetics, etc. Usually we have a set of differential equations which cannot be solved by hand. So, it is reasonable to try to apply the FEM to result in the set of simultaneous algebraic equations

$$\underbrace{[K]}_{\text{Property}}\ \underbrace{\{u\}}_{\text{Behavior}} = \underbrace{\{F\}}_{\text{Action}} \tag{1.3}$$

In the different fields of research property matrix K, behavior and action vectors u and F, respectively, called as shown in Table 1.1.

Table 1.1 Notations in different research field

	Property $[K]$	**Behavior** $\{u\}$	**Action** $\{F\}$
Elasticity	stiffness	displacement	force
Thermal	conductivity	temperature	heat source
Fluid	viscosity	velocity	body force
Electrostatic	dialectric permittivity	electric potential	charge

According to Segerlind [130] the use of the finite element method can be subdivided into five following steps.

1. Discretization of the region. It is very difficult to obtain the algebraic equations for the entire domain. That is why one has to discretize the region by dividing the domain into a small, simple elements. Generally, a field quantity is interpolated by a polynomial over an element. Then, adjacent elements share the degree of freedom at connection nodes. This includes locating and numbering the node points, as well as specifying their coordinates' values.
2. Specification of the approximation. The order of the approximation, linear or quadratic, must be specified and the equations must be written in terms of the unknown nodal values. An equation is written for each element.
3. Derivation of the system of equations, i.e. put all the elements together. When using Galerkin's method, the weighting functions for each unknown nodal value are defined and the weighted residual integral is evaluated. This generates one equation for each unknown nodal value. In the potential energy formulation, the potential energy of the system is written in terms of the modal displacement and then it is minimized. This gives one equation for each of the unknown displacements.

4. Solvution of the sytem of equations $\left[K\right]\left\{u\right\} = \left\{F\right\}$. As the result, unknown variables at nodes are defined.
5. Calculation of quantities of interest: $\left\{u\right\} = \left[K\right]^{-1}\left\{F\right\}$. These quantities are usually related to the derivative of the parameter and include the stress components, and heat flow and fluid velocities.

Summarising briefly, it can be noted that classical FEM/FEA approach uses:

- the concept of piecewise polynomial interpolation;
- by connecting elements together, the field quantity becomes interpolated over the entire structure in piecewise fashion;
- a set of simultaneous algebraic equations at nodes.

Professor Oliver de Weck and Dr. Il Young Kim [147] have classified the advantages and disadvantages of classical FEM technique as shown in Table 1.2.

Table 1.2 Advantages and disadvantages of FEM

Advantages	Disadvantages
1. The FEM can readily handle very complex geometry 2. The FEM can handle a wide variety of problems of different nature 3. The FEM can handle complex restraints 4. The FEM can handle complex loading	1. A general closed-form solution, which would permit one to examine system response to changes in various parameters, is not produced. 2. The FEM obtains only "approximate" solutions, "inherent" errors are unavoidable. 3. Mistakes by users can be fatal.

It is impossible to quote nowadays the date or the author of the invention of the FEA/FEM, and since the historical approach to the problem lies outside of the current research, it is reasonable to adduce hereby only a brief history of FEM evaluation based only onto undeniable facts and irrefutable arguments [147], [155].

- Initially, FEA/FEM was more engineering than a scientific tool originated from the need to solve complex elasticity and structural analysis problems in civil and military industry. In the late 1960s and early 1970s, the FEM was still applied to a wide variety of engineering problems.
- Method development initially traced back to the works by A. Hrennikoff (1941) and R. Courant (1943): Hrennikoff discretized the domain by implementing a lattice analogy; Courant utilized the Ritz method of numerical solution to vibration systems. By M. J. Turner *et al.* in 1956 there was established a broader definition of numerical analysis with the focus onto stiffness and deflection of complex structures.
- The new impetus for FEA/FEM development was obtained in the 1960s and 70s due to the results received by J. H. Argyris (University of Stuttgart), R. W. Clough (US Berkeley), O. C. Zienkiewicz (University of Swansea), R. Gallagher (Cornell University), etc.

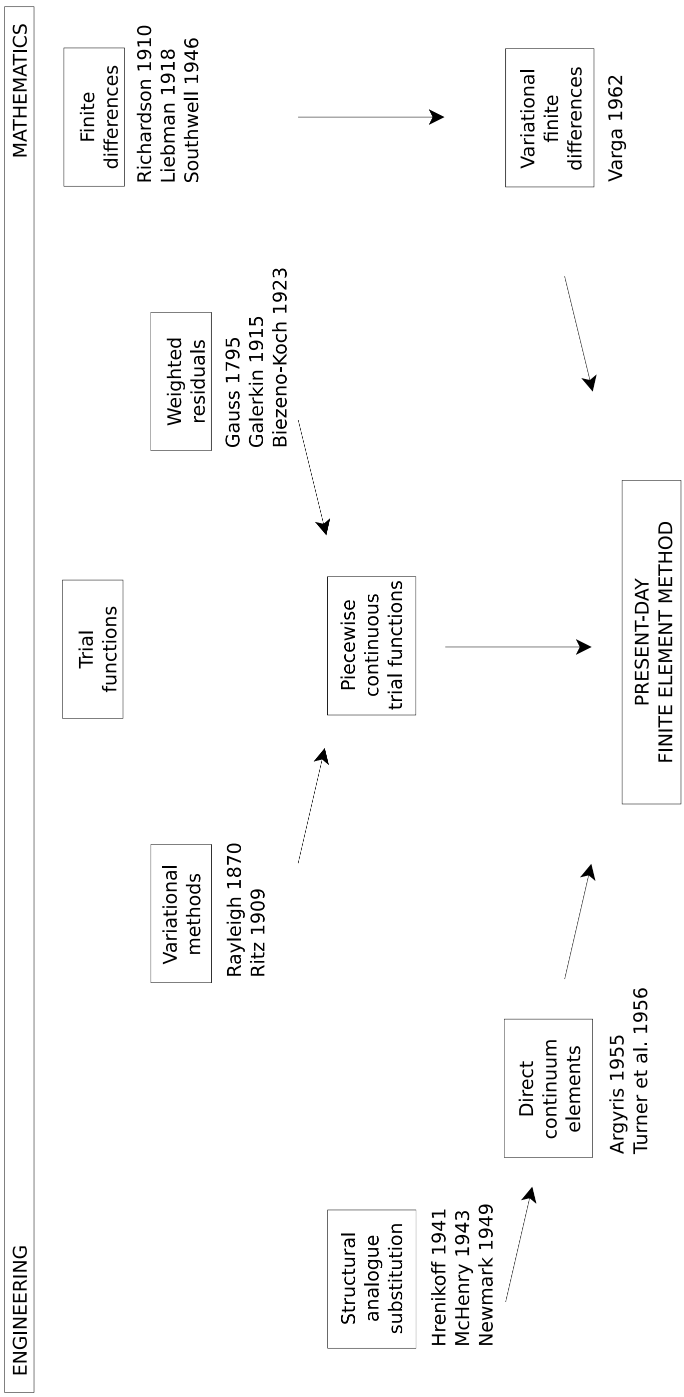

Fig. 1.1 FEM evolution till nowadays

- The term “finite element”was first coined by Clough in 1960. In the early 1960s, engineers used the method for approximate solution of problems in stress analysis, fluid flow, and other areas.
- The first book on the FEA/FEM by O. C. Zienkiewicz and Chung was published in 1967.
- The development of the most known commercial FEM software packages was originated in the 1970s: NASTRAN (NASA, 1971), ABAQUS (1978), ANSYS (1970); ADINA (1974), etc.
- Fig. 1.1 presents the look of O. C. Zienkiewicz (see [155]) onto the process of FEA/FEM evolution which led to the present day concepts of finite element analysis.

During the years 1990 – 2013, the main activities in FEA/FEM research were in FEA/FEM coupling with other numerical technologies to provide methods with possibility to pre-define the result precision and in coupling FEM mostly with boundary element method (BEM), wavelet analysis and with neural networks [33]. Detailed description and implementation of FEM is presented in [60], [69] – [71], [130], [153] – [159].

FEM was used in analysis and optimization of thin-walled elastic plastic structures by Kačianauskas and Čyras [51]. In the paper by Kačianauskas *et al.* [52] parallel discrete element simulation was employed.

Fast development of parallel and high powered computing (HPC) inspires the development of parallel versions of FEM. Due to high importance of the HPC in the optimization part of current research we will discuss the main parallel computing results and their coupling with FEM in the separate subsection.

1.3 Parallel computing with finite element method

The serial finite element method (FEM) discussed in the previous subsection appears to be a strong tool for solving direct problems in engineering and mechanics, yet insufficient for optimization problems, which follows from the fact that CPU clock frequencies are no longer increasing due to the physical restrictions. So, the speed-up of the computation can be obtained only by using multiple cores. Parallel programming is required for utilizing multiple cores and allow application to use more memory applying old models to new length and time scales. Thus, it, in its turn, inspired the new impetus for existing numerical methods coupling with recently appeared parallel computation technologies, which nowadays have led to such well-known parallel programming technologies as MPI [109], OpenMP [115], OmpSs, OpenPALM, CUDA [17], OpenCL, OpenACC [149], PGAS (partitioned global address space) languages (like UPC, CAF, Chapel, X10), etc. For description of mostly used programming languages see [23], [32], [119], [120], [128], [119], [143], etc. For FEM it led to the separate parallel-FEM algorithm creation within the last 15 years of the FEM evaluation. Over the last five years, a strong trend in

FEA/FEM development has been FEM implementation by using the possibilities provided by active HPC (High Performance Computing) development.

Furthermore, the main parallel programming facts will hereby be indicated as well as FEM algorithm main parallelization techniques applying to the current research.

Definition 1.1. *Parallel computation* means executing tasks concurrently, where each task encapsulates a sequential program and local data, and its interface to its environment and the data of those other tasks is remote.

Definition 1.2. *Data dependency* means that the computation of one task requires data from another task in order to proceed.

Solution methods for a linear system can be devided into two main classes: direct methods and iterative methods. Gaussian elimination is the main algorithm for direct methods and in general direct methods are the methods with a fixed number of operations.

Iterative methods, in its turn, can be divided into so-called preconditional Krylov subspace methods (Conjugate Gradient (CG), Conjugate Gradient Squared (CGS), Generalized Minimal Residual (GMRES), Transpose-Free Quasi-Minimal Residual (TFQMR), Biconjugate Gradient Stabilized (BiCGStab), Generalized Conjugate Residual (GCR), etc.), multilevel methods (Geometric Multigrid (GMG) and Algebraic Multigrid (AMG)) and methods based on a combined idea of the last two mentioned method branches. In a general case, iterative method parallelisation depends on solver implementation, for example, some implementation description of ElmerFEM solvers and preconditioning strategies given in [128].

At present, there exist two main classes of parallel FEA/FEM implementation: one is based on domain decomposition methods and the other is based on multigrid methods. Currently, the main representatives of domain decomposition methods with the focus on parallel FEM implementation are ILU (incomplete LU factorization) in parallel (see [37], [44], [45], [58], [117]), additive Schwarz [140], and FETI (finite element tearing and interconnect). In case of multigrid methods we mostly consider algebraic [35], [108], [151] and geometric multigrid methods [43], but algebraic multigrid methods are more popular because in many cases geometric multigrid can not be applied due to the fact that we have no a set of appropriate hierarchical meshes while algebraic multigrid approach uses only matrix $[K]$ to obtain the projectors and the coarse level equations. Due to Non-disclose agreements (NDA) and export restrictions on HPC technologies main algorithms mentioned in current chapter are not quoted. Kacimi *et al.* coples ILU with wavelets for preconditioning [24].

Available numerical releases and representation of the ILU parallel algorithms ([37], [44], [45], [58], [113]), moreover, the same parallelization algorithm can often be presented in different ways depending on system requirements, for example, ILU parallelization algorithm developed by D. Hysom and A. Pothen [46] can be presented either in the form suitable for the message passing computational environments or as the algorithm represented in equivalent serial formulation.

1.3.1 Equivalent sequential version of PILU algorithm

During the last decades the demand for the computational power. Owing to the complexity of the problems, the non-linearity of the geometry or material behaviour, the need to solve larger and more complex problems within reasonable time has reached more and more towards physical limitations of single processor supercomputers. Consequently, in the field of research the concept of parallelization turned out to be of utmost importance. There were two different ways in those days to approach a large scale problems with parallel computers: existing computers mostly under *IX-like OS connected to the distributed network via standard parallelization/clustering software based on MPI (Message Passing Interface) and PVM (Parallel Virtual Machine), which nowadays is transformed into a modern GRID computing idea (multi-core machines can also be added to Grid without restrictions of any kind) from one side, and multi-core/multi-thread based supercomputing in the mid-1990s based on multi-core computing, which is a standard nowadays due to the usage of multi-core/thread CPU-s and GPU-s with implementation from multi-core PC up to supercomputer releases based on high performance computing technologies. Note here, that both mentioned technologies can be combined.

As a rule, currently, parallelization implementations and algorithms strictly depend on available hardware and software, and must not be portable in an easy way, which is one of the reasons why there are a lot of versions of parallel multigrid and algebraic multigrid algorithms.

Consider here in brief parallel AMG algorithm developed by S. Meynen, A. Boersma and P. Wriggers [108]. Assume that the problem is parallelized by using a self defined data decomposition onto a parallel computer with a distributed memory.

The provided algorithms [108] based on different methods have been developed to solve large algebraic systems of equations on parallel computers, e.g., on the conjugate gradient method (see Meyer, Haase and Langer [107]), on Schur-complement decomposition (introduced by Meyer [106]) and the resulting PCG methods can be preconditioned with a hierarchical basis (Yserantant [152]). Additional information on an overlapping domain decomposition [50] and multigrid techniques with hierarchical grid structure can be found, for example, in (Bastian and Wittum [4], Jung and Langer [49]).

1.3.2 Multigrid algorithms

Multigrid methods have been applied to the problems in elasticity by Braess [8], Peisker, Ruse and Stein [116]. Nowadays, it is, in fact, one of the standards for the modern fluid dynamic software. The related techniques are very efficient, however, they need special mesh hierarchy. To circumvent this problem, a concept of algebraic multigrid methods has been developed (see Brandt [9], Ruge [126], Ruge and Stüben [127], Reusken [123], Kočvara [63], Kočvara and Mandel [64]. Sequential versions for problems in solid mechanics coupled with AMG have been developed by Boersma and Wriggers, see [7]. In this context the AMG method is used here as a preconditioner for the CG method. Useful aspects on coupling FEM with other tehnologies described in [27], [29].

One of the first AMG implementation with parallel computation support was done by O. Zienkiewicz in co-operation with R. Taylor. The authors' programming concept regarding the parallel computation principles was that on every processor the same version of a parallel FE program is running independently from the other processors and handles different data due to a domain decomposition $\Omega = \bigcup_{s=0}^{p-1} \Omega_s$ with non-overlapping domains Ω_s defined by elements (due to implementation processor numbering counter always starts from 0). On each of the domains Ω_s standard finite element procedures are employed to discretize the subdomains by isoparametric elements $\Omega = \bigcup_{s=0}^{n_e} \Omega_s$. Since during the period of the development of the disputed approach to the method, the most available so-called massive parallel systems did not allow individual input files for each processor P^S, in the observed algorithm the input file was read by processor P^0 and splitted (spread down) to the other processors by a data transfer. Hence, every processor P^S obtained its own set of input data including the relevant global geometrical data for positioning the mesh plots.

The different domains were connected via a data exchange between the processors at distinct stages within the algorithm. Remind here that only the nodes on the boundaries Γ_s of each subdomain Ω_s were affected by communication which has to be performed mainly during the solution phase of the algebraic set of equations.

S. Meynen *et al.* [108] there was discussed in detail the parallel AMG coupling with theoretical results for thin shells undergoing non-linear elastoplastic deformations according to von Mises yield condition provided by P. Wriggers *et al.*

According to S. Meynen *et al.* [108] the parallelization of the algebraic multigrid method is performed in two steps, namely: in the executed only once setup phase, coarse-grid stiffness matrices, transfer operators and coarse-grid matrices are computed on all levels and then, in the second phase, the system of equations is solved iteratively.

Determination of coarse grid-points starts with a split of the nodes on the coupling boundaries. to different processors. This step includes also data exchange. After finishing the task, the remaining interior nodes should be subdivided into the coarse and fine grid-points independently from nodes on the other processors, and owing to this, without any communication. Then, the parts belonging to coupling boundaries should be computed, after this the interior parts can be calculated in parallel in each subdomain. Note here, that owing to these preliminary operations, a totally parallel construction of the coarse grid matrix is possible and within this process the fine-grid matrix is split into parts on the subdomains.

To simplify the notation rewrite system (1.3) in the form

$$Kv = f \tag{1.4}$$

Then, during the iterative solution of the finite element equations a parallel smoothing operator called $S_p(u, f)$ is needed, for example, the parallel Gauss–Seidel algorithm or an incomplete Cholesky decomposition can be implemented here. Presented on the algebraic multigrid algorithm (pAMG) [108] solve a set of linear equations (1.4) starting with pAMG$(1, v, f, \nu)$ and taking the original stiffness matrix K as K_1 and the force vector f as the right-hand side. Within the parallel algebraic multigrid method all operators will be assigned with a rising level index l, and let level one denote the finest and l_{max} – the coarsest levels, respectively. Let ν be a cycle parameter, then, by extending a two-grid method to a multigrid method, the solution of the coarse-grid problem is replaced on each level by ν sweeps of the multigrid scheme given above ($\nu = 1$ leads to a so-called V-cycle and $\nu = 2$ results in a W-cycle). We use the parallel algebraic multigrid method (pAMG) as a preconditioner for another iterative solver, the conjugate gradient method, it means we will also state the CG algorithm. Note here, that in each CG-iteration, the pAMG is applied to the precondition of the system of equations [108].

In the current thesis a new analytical numerical technique of optimization of circular and annular plates is developed. It is assumed that the material of plates is an ideal elastic plastic material obeying a non-linear yield condition and the associated flow law. Obtained results are compared with existing solutions of other authors. Calculations have been implemented by the FEM and the method based on Haar wavelets.

Chapter 2
Optimization of elastic plastic circular plates made of von Mises material

2.1 Introduction

Thin-walled plates and shells are widely used in the aircraft and space industry. Moreover, plates and shells are used in various fields of engineering and industry. This involves the need for the minimization of the cost and weight of plate and shell structures.

The problems of optimization of plates and shells made of pure elastic or of ideal inelastic materials are studied by many investigators. Reviews of these papers can be found in books and review papers by Banichuk [3], Atkočiūnas [2], Kirsch [61], Cherkaev [12], Karkauskas, Čyras, Barkowski [57], Bendsoe [6], Lellep and Lepik [75], Krużelecki and Życzkowski [66], Rozvany [125].

Elastic plastic response of circular and annular plates was studied by Lellep and Polikarpus [78], [76], [77] in the case of sandwich plates made of a material which obeys the Tresca's yield condition. Optimal designs of axisymmetric plates subjected to the uniformly distributed transverse pressure have been established by Lellep and Vlassov [81], [81], [82], [83], [84], [85], [86], [87] in the cases of von Mises and Hill's yield criteria.

In the present chapter an optimization technique is developed for circular plates made of an ideal elastic plastic material obeying von Mises yield condition and the associated flow law. Necessary optimality conditions are derived with the aid of variational methods of the theory of optimal control.

2.2 Problem formulation

Let us consider the elastic plastic bending of a circular plate of radius R. It is assumed that the plate is subjected to the axisymmetric transverse pressure of intensity $p = p(r)$, where r is the current radius (Fig. 2.1). In what follows we focus the attention on the axisymmetric response of the plate assuming that the hypotheses of Kirchhoff hold good in the regions of elastic and plastic deformations.

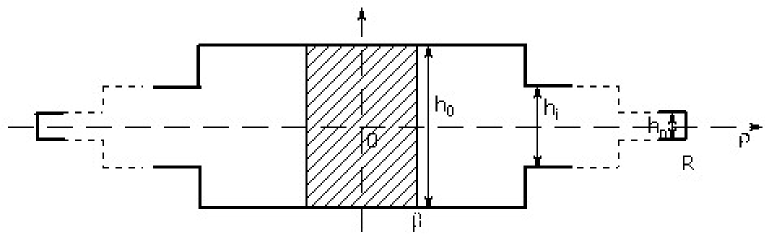

Fig. 2.1 Circular elastic-plastic plate

Furthermore, the plates with a sandwich cross section will be considered. A sandwich plate is a structure which consists of two carrying layers of thickness h and of a layer of a core material between the rims. Let the thickness of carrying layers be piece wise constant, e.g.

$$h = h_j \tag{2.1}$$

for $r \in (a_j, a_{j+1})$, where $j = 0, \ldots, n$ and $a_0 = 0$, $a_{a+1} = R$. At the same time the layer of the core material is of constant thickness H. According to this concept the thickness of the rim is much smaller than the quantity H.

The quantities h_j $(j = 0, \ldots, n)$ and a_i $(i = 1, \ldots, n)$ are preliminarily unknown constant parameters. The aim of the paper is to determine the design parameters and the stress-strain state of the plate so that a given cost function attains its minimum value whereas pertinent boundary conditions and additional constraints are met.

As regards the formulation of an optimization problem, one can find from the literature a lot of different particular problems (Rozvany, [125]; Banichuk, [3]).

However, in the present paper the attention will be confined to the problems of minimum weight under constrained deflections. Also, we will study the problem of minimization of the central deflection under constrained material consumption. The total mass of the plate of piece constant thickness can be presented as

$$\bar{V} = 2\rho_2 \pi \sum_{j=0}^{n} h_j (a_{j+1}^2 - a_j^2) + \pi \rho_1 H R^2, \tag{2.2}$$

where ρ_2 and ρ_1 stand for the densities of carrying layers and the core material, respectively.

As we are interested in the reducing of the cost of carrying layers instead of (2.2), the cost of a rim will be used in the form

$$V = \sum_{j=0}^{n} h_j (a_{j+1}^2 - a_j^2) \tag{2.3}$$

When minimizing the cost criterion (2.3) the deflections should be constrained. Thus, it is reasonable to demand that

$$W(r) \leqslant W_0 \tag{2.4}$$

for $r \in [0, R]$ where W_0 is a given constant and W stands for the transverse displacement. Evidently, the transverse deflection is maximal at the center of the plate. Thus, the constraint (2.4) can be replaced by

$$W(0) = W_0. \tag{2.5}$$

Evidently, (2.5) can be treated as the boundary condition for the system of basic equations.

2.3 Governing equations

Let M_1, M_2 be bending moments in the radial and circumferential direction, respectively, and Q be the shear force applied at the edge of an element in the transverse direction.

These generalized stresses are related to each other by equilibrium equations. In the frame works of the linear theory of plates equilibrium conditions of a plate element can be presented as (Reddy [121]; Vinson [144], [145]; Chakrabarty [13]; Ventsel and Krauthammer [142])

$$\begin{gathered}\frac{d}{dr}(rM_1) - M_2 - rQ = 0,\\ \frac{d}{dr}(rQ) + Pr = 0.\end{gathered} \tag{2.6}$$

Principal curvatures of the middle surface of the plate consistent with equilibrium equations (2.6) have the form

$$\begin{aligned}\kappa_1 &= -\frac{d^2W}{dr^2},\\ \kappa_2 &= -\frac{1}{r}\frac{d^2W}{dr^2}.\end{aligned} \tag{2.7}$$

It is worthwhile to mention that equations (2.6) and (2.7) are independent of the constitutive behaviour of the material. These hold good equally in elastic and plastic regions of a plate.

It appears that the plate will be subdivided into elastic and plastic regions. In an elastic region of the plate the principal stresses σ_1, σ_2 and strain components

$$\varepsilon_1 = z\kappa_1, \quad \varepsilon_2 = z\kappa_2$$

satisfy the Hooke's law for plane stress state.

Here z stands for the axis of coordinate transverse to the middle surface of the plate.

Integrating of these relations leads to the generalized Hooke's law. The latter can be presented for $r \in (a_j, a_{j+1})$ where $j = 0, \ldots, n$ as

$$M_1 = D_j(\kappa_1 + \nu\kappa_2),$$
$$M_2 = D_j(\kappa_2 + \nu\kappa_1). \tag{2.8}$$

In (2.8) and henceforth ν is the Poisson modulus and

$$D_j = \frac{EH^2 h_j}{2(1-\nu^2)}, \tag{2.9}$$

where E stands for the Young modulus.

In plastic regions of the plate relations (2.8) are not valid as the stress state lies on a yield surface. In the present paper it is assumed that the material of the plate obeys von Mises yield condition written in generalized stresses as

$$M_1^2 - M_1 M_2 + M_2^2 - M_{0j}^2 \leqslant 0 \tag{2.10}$$

for $r \in (a_j; a_{j+1});\ j = 0, \ldots, n$. In (2.10) M_{0j} stands for the yield moment

$$M_{0j} = \sigma_0 H h_j, \tag{2.11}$$

σ_0 being the yield stress of the material.

Note that the non-strict inequality (2.10) undergoes into an equality if the stress state corresponds to a plastic region. However, in an elastic state the inequality (2.10) is satisfied as a strict inequality.

It is known from the theory of plasticity that in a plastic region the associated flow law, or the gradientality law holds good (see Chakrabarty [13]; Kaliszky [54]; Sawczuk [129]).

According to the gradientality law in a plastic region the relations

$$\kappa_1 = \lambda(2M_1 - M_2),$$
$$\kappa_2 = \lambda(2M_2 - M_1), \tag{2.12}$$

hold good whereas λ is a non-negative scalar multiplier.

Eliminating quantity λ from (2.12) gives

$$\frac{\kappa_1}{\kappa_2} = \frac{2M_1 - M_2}{2M_2 - M_1}, \tag{2.13}$$

whereas

$$\lambda = \frac{\kappa_1}{2M_1 - M_2}. \tag{2.14}$$

Substituting the curvatures κ_1, κ_2 from (2.7) to (2.13) leads to the equation

$$\frac{d^2W}{dr^2} = \frac{dW}{rdr}\frac{2M_1 - M_2}{2M_2 - M_1}. \tag{2.15}$$

Equations (2.6) and (2.15) together with the equation $\Phi_j = 0$ present the set of governing equations for a plastic region. Here Φ_j stands for the left hand side of the inequality (2.10).

It appears to be reasonable to introduce an auxiliary variable

$$Z = \frac{dW}{dr}. \tag{2.16}$$

This enables to present the governing equations in the form of the first order equations

$$\begin{aligned} \frac{dW}{dr} &= Z, \\ \frac{dZ}{dr} &= \frac{Z}{r}\frac{2M_1 - M_2}{2M_2 - M_1}, \\ \frac{dM_1}{dr} &= \frac{M_2}{r} - \frac{M_1}{r} + Q. \end{aligned} \tag{2.17}$$

Constraints (2.10) will be transformed into the form

$$M_1^2 - M_1M_2 + M_2^2 - M_{0j}^2 + \Theta_j^2 = 0 \tag{2.18}$$

for $r \in (a_j, a_{j+1})$, $j = 0, \ldots, n$. The quantities Θ_j in (2.18) are certain unknown functions of current radius to be determined later. Note that the quantity Q in (2.17) is to be handled as a given function. Indeed, it follows from the second equation in the system (2.6) that

$$Q = -\frac{1}{r}\int_0^r P(r)\, r\, dr \tag{2.19}$$

where P is the intensity of the distributed lateral loading. Since $P(r)$ is assumed to be a given function it infers from (2.19) that Q is given, as well.

For an elastic region it follows from (2.7), (2.8) and (2.16) that

$$M_2 = \frac{z}{r}D_j(\nu^2 - 1) + \nu M_1 \tag{2.20}$$

and

$$\frac{dZ}{dr} = -\frac{M_1}{D_j} - \frac{\nu}{r}Z \tag{2.21}$$

for $r \in (a_j, a_{j+1})$, $j = 0, \ldots, n$.

Substituting (2.20) in (2.6) and taking (2.16), (2.21) into account leads to the system of equations

$$\begin{aligned}
&\frac{dW}{dr} = Z, \\
&\frac{dZ}{dr} = -\frac{M_1}{D_j} - \frac{\nu}{r} Z, \\
&\frac{M_1}{dr} = (\nu^2 - 1) D_j \frac{Z}{r^2} - \frac{1-\nu}{r} M_1 + Q
\end{aligned} \tag{2.22}$$

which holds good in each elastic region for $r \in (a_j, a_{j+1})$, $j = 0, \ldots, n$.

As it was mentioned above, the solution of the problem subdivides the plate into elastic and plastic regions, respectively. Let us denote these subregions of the plate by S_e and S_p, respectively. In what follows we will treat S_e and S_p as one-dimensional subregions consisting of an interval or of the sum of intervals.

It is worthwhile to mention that the sandwich plate is studied herein. Carrying layers of the plate are assumed to be relatively thin and thus, the stress state can be either pure elastic or pure plastic; no elastic plastic state occurs. This means that the sum of sets

$$S_e \cup S_p = [0, R].$$

Summarizing the results, one can say that the stress strain state of the plate is prescribed by (2.22) for $r \in S_e$ and by (2.17) for $r \in S_p$ whereas the equation (2.18) holds good for each $r \in [0, R]$.

In the case of a simply supported plate at the boundary

$$M_1(R) = 0, \quad W(R) = 0 \tag{2.23}$$

whereas at the center

$$Z(0) = 0, \quad M_1(0) = M_2(0). \tag{2.24}$$

2.4 Necessary optimality conditions

The problem posed above will be considered as a particular problem of optimal control. It consists in the minimization of the cost function (2.3) accounting for the state equations (2.17) for $r \in S_p$ and (2.20), (2.22) for $r \in S_e$ and (2.18) for each value of r. Of course, appropriate boundary requirements have to be satisfied, as well.

The variables W, Z, M_1 in (2.17), (2.18), (2.22) will be treated as state variables and M_2, Θ_j as the controls whereas r is the independent variable and h_j $(j = 0, \ldots, n)$ and a_i $(i = 1, \ldots, n)$ are constant parameters. The problem set up herein belongs to the class of problems with discontinuous state equations (Bryson [10]; Hull [47]). In fact, it is evident that right hand sides of equations (2.17) and (2.22) differ from each other. However, the state variables W, Z, M_1 are continuous everywhere, in particular at the boundary points between elastic and plastic regions.

It is resonable to introduce the following notations. Let K_e be the set of such natural numbers that for $j \in K_e$ intervals (a_j, a_{j+1}) are elastic, e.g. $S_{ej} = (a_j, a_{j+1}) \in S_e$

for $j \in K_e$. Similarly, we expect that $S_{pj} = (a_j, a_{j+1}) \in S_p$ if $j \in K_p$. For the sake of simplicity it is assumed that the interval $S_{ep} = (a_k, a_{k+1})$ is the unique interval where exist both, the elastic and plastic regions, respectively. Let (r_{k_0}, k_{k_1}) be a plastic region where $r_{k_0} \in (a_k, a_{k+1})$ and $r_{k_1} \in (a_k, a_{k+1})$ and (a_k, r_{k_0}), (r_{k_1}, a_{k+1}) – elastic regions.

Evidently, in particular cases can be $r_{k_0} = a_k$ or $r_{k_1} = a_{k+1}$. Let

$$S_{ek} = S_{ek_0} \cup S_{ek_1}, \qquad S_{pk} = (r_{k_0}, r_{k_1})$$

where $S_{ek_0} = (a_k, r_{k_0})$, $S_{ek_1} = (r_{k_1}, a_{k+1})$ and let K_{p_1}, K_{e_1} be such that

$$S_e = \bigcup_{j \in K_{e_1}} S_{ej}, \qquad S_p = \bigcup_{j \in K_{p_1}} S_{pj}.$$

In other words, $K_{e_1} = K_e \cup \{K\}$ and $K_{p_1} = K_p \cup \{K\}$. Here $\{K\}$ denotes a set which contains the number k.

In order to derive necessary conditions of optimality one has to introduce an extended functional (Bryson [10]; Hull [47]; Lellep [74])

$$\begin{aligned} J_* = V + \sum_{j \in K_{e1}} \int_{S_{ej}} \Bigg\{ & \psi_1 \left(\frac{dW}{dr} - Z \right) + \psi_2 \left(\frac{dW}{dr} + \frac{M_1}{D_j} + \frac{\nu}{r} Z \right) + \\ + \psi_3 \left(\frac{dM_1}{dr} + (1-\nu^2) D_j + \frac{1-\nu}{r} M_1 - Q \right) & + \varphi_{0j} \left(M_2 + D_j (1-\nu)^2 \frac{Z}{r} - \nu M_1 \right) \Bigg\} dr + \\ + \sum_{j \in K_{p1}} \int_{S_{pj}} \Bigg\{ & \psi_1 \left(\frac{dW}{dr} - Z \right) + \psi_2 \left(\frac{dZ}{dr} - \frac{Z}{r} \frac{2M_1 - M_2}{2M_2 - M_1} \right) + \\ & + \psi_3 \left(\frac{dM_1}{dr} - \frac{M_2}{r} + \frac{M_1}{r} - Q \right) \Bigg\} dr + \\ & + \sum_{j=0}^{n} \int_{a_j}^{a_{j+1}} \varphi_j \left(M_1^2 - M_1 M_2 + M_2^2 - M_{0j}^2 + \Theta_j^2 \right) dr \end{aligned} \tag{2.25}$$

In (2.25) ψ_1, ψ_2, ψ_3 stand for adjoint (conjugate) variables, φ_{0j}, $\varphi_j (j = 0, \ldots, n)$ are unknown Langrange' multipliers and the material volume V is given by (2.3). Multipliers φ_{0j} and $\varphi_j (j = 0, \ldots, n)$ are introduced in order to account for the constraints (2.20) and (2.18), respectively. Evidently, in elastic regions constraints (2.18) are inactive.

Calculating the variation of (2.25) and equalizing it to zero leads to the equation

$$
\begin{aligned}
\delta J_* = \sum_{j=0}^{n} \Bigg\{ & \Delta h_j (a_{j+1}^2 - a_j^2) + 2h_j (\Delta a_{j+1} \cdot a_{j+1} - \Delta a_j \cdot a_j) + \\
& + \varphi_j \Big(2M_1 \delta M_1 - M_1 \delta M_2 - M_2 \delta M_1 + 2M_2 \delta M_2 + 2\Theta_j \delta \Theta_j - 2M_{\Theta j} \delta M_{0j} \Big) \Bigg\} + \\
& + \sum_{j \in K_{e_1}} \Bigg\{ \int\limits_{S_{ej}} \Bigg\{ \psi_1 \delta \frac{dW}{dr} - \psi_1 \delta Z + \psi_2 \delta \frac{dZ}{dr} + \psi_2 \frac{\delta M_1}{D_j} - \frac{\psi_2 M_1}{D_j^2} \Delta D_j + \psi_2 \frac{\nu}{r} \delta Z + \\
& + \psi_3 \delta \frac{dM_1}{dr} + \frac{\psi_3 (1-\nu^2)}{r^2} \big(D_j \delta Z + Z \Delta D_j \big) + \frac{1-\nu}{r} \delta M_1 \psi_3 + \\
& + \varphi_{0j} \Big(\delta M_2 + \frac{1-\nu^2}{r} \big(D_j \delta Z + Z \Delta D_j \big) - \nu \delta M_1 \Big) \Bigg\} dr + F_{ej} \cdot \Delta S_{ej} \Bigg\} + \\
& + \sum_{j \in K_{p_1}} \Bigg\{ \int\limits_{S_{pj}} \Bigg\{ \psi_1 \delta \frac{dW}{dr} - \psi_1 \delta Z + \psi_2 \delta \frac{dZ}{dr} - \frac{\psi_2}{r} \delta Z \frac{2M_1 - M_2}{2M_2 - M_1} - \\
& - \frac{\psi_2}{r} Z \frac{(2\delta M_1 - \delta M_2)(2M_2 - M_1)}{(2M_2 - M_1)^2} + \frac{\psi_2}{r} Z \frac{(2\delta M_2 - \delta M_1)(2M_1 - M_2)}{(2M_2 - M_1)^2} + \\
& + \psi_3 \delta \frac{dM_1}{dr} - \psi_3 \frac{\delta M_2}{r} + \psi_3 \frac{\delta M_1}{r} \Bigg\} dr + F_{pj} \Delta S_{pj} \Bigg\} = 0
\end{aligned}
\tag{2.26}
$$

In (2.26) F_{ej} stands for the integrand in (2.25) corresponding to the integral over an elastic region S_{ej} and ΔS_{ej} is the variation of the boundary of the elastic region. Similarly, F_{pj} is the integrand in (2.25) for a plastic region S_{pj}.

However, it immediately infers from (2.25), (2.17) and (2.22), that $F_{pj} = 0$, $F_{ej} = 0$ for $j \in K_{p_1}$ and $i \in K_{e_1}$, respectively.

Integrating the terms

$$
\psi_1 \delta \frac{dW}{dr}, \quad \psi_2 \delta \frac{dZ}{dr}, \quad \psi_3 \delta \frac{dM_1}{dr}
$$

in (2.26) by parts due to arbitrariness of variations of state variables δW, δZ, δM_1 one can state that for $r \in S_{ei}$; $i \in K_{e_1}$

$$
\begin{aligned}
\frac{d\psi_1}{dr} &= 0, \\
\frac{d\psi_2}{dr} &= -\psi_1 + \frac{\nu}{r} \psi_2 + (1-\nu)^2 D_i \frac{\psi_3}{r^2} + \frac{1-\nu^2}{r} D_i \varphi_{0i}, \\
\frac{d\psi_3}{dr} &= \varphi_i (2M_1 - M_2) + \frac{\psi_2}{D_i} + (1-\nu) \frac{\psi_3}{r} + \nu \varphi_{0i}
\end{aligned}
\tag{2.27}
$$

and for $r \in S_{pj}$, $j \in K_{p_1}$

$$\frac{d\psi_1}{dr} = 0,$$
$$\frac{d\psi_2}{dr} = -\psi_1 - \frac{\psi_2}{r}\frac{2M_1 - M_2}{2M_2 - M_1}, \tag{2.28}$$
$$\frac{d\psi_3}{dr} = \varphi_j(2M_1 - M_2) - \frac{\psi_2}{r}\frac{3M_2 Z}{(2M_2 - M_1)^2} + \frac{\psi_3}{r}.$$

The variations of controls δM_2, $\delta\Theta_j$ $(j = 0, \ldots, n)$ are independent, as well, in (2.26). Therefore, the coefficients before these variations must vanish. Thus, for $j = 0, \ldots, n$

$$\varphi_j \Theta_j = 0 \tag{2.29}$$

for $r \in S_{ej}, j \in K_{e_1}$

$$\varphi_j(2M_2 - M_1) + \varphi_{0j} = 0, \tag{2.30}$$

and for $r \in S_{pj}, j \in K_{p_1}$

$$\varphi_j(2M_2 - M_1) + \frac{\psi_2}{r} Z \frac{3M_1}{(2M_2 - M_1)^2} - \frac{\psi_3}{r} = 0. \tag{2.31}$$

Note finally that (2.27), (2.30) hold good in the intervals (a_i, a_{i+1}) for $i \in K_e$ and also for $r \in (a_k, r_{k_0})$ and (r_{k_1}, a_{k+1}). Similarly, (2.28), (2.31) are satisfied for $r \in (a_j, a_{j+1})$ for $j \in K_p$ and for $r \in (r_{k_0}, r_{k_1})$.

Substituting (2.27) – (2.31) into (2.26) the equation (2.26) can be written as

$$\sum_{j=0}^{n} \Delta h_j (a_{j+1}^2 - a_j^2) - 2\varphi_j M_{0j} \Delta M_{0j} + 2\sum_{j=1}^{n} a_j (h_{j-1} - h_j)\Delta a_j +$$
$$+ \sum_{i \in K_{e1}} \int_{S_{ei}} \left\{ -\frac{\psi_2 M_1}{D_i^2} + \frac{\psi_3}{r^2} Z(1-\nu^2) + \frac{\varphi_{0i}}{r} Z(1-\nu^2) \right\} \Delta D_i dr +$$
$$+ \sum_{i \in K_e} Y\Big|_{a_i}^{a_{i+1}} + \sum_{j \in K_p} Y\Big|_{a_j}^{a_{j+1}} + Y\Big|_{a_k'}^{r_{k_0}} + Y\Big|_{r_{k_0}}^{r_{k_1}} + Y\Big|_{r_{k_1}'}^{a_{k+1}} = 0 \tag{2.32}$$

where the notation

$$Y = \psi_1 \delta W + \psi_2 \delta Z + \psi_3 \delta M_1 \tag{2.33}$$

is used. Moreover, it can be easily rechecked that

$$\sum_{j=0}^{n} h_j (\Delta a_{j+1} \cdot a_{j+1} - \Delta a_j \cdot a_j) = \sum_{j=1}^{n} a_j (h_{j-1} - h_j)\Delta a_j \tag{2.34}$$

As the parameters a_j $(j = 0, \ldots, n)$ are unknown parameters the current problem belongs to the class of problems with moving boundaries. In this case the variations $\delta W(a_j)$, $\delta Z(a_j)$, $\delta M_1(a_j)$ are not independent. Arbitrary can be considered the total variations

$$\Delta W(a) = \delta W(a\pm) + \left.\frac{dW}{dr}\right|_{a\pm 0} \cdot \Delta a$$
$$\Delta Z(a) = \delta Z(a\pm) + \left.\frac{dZ}{dr}\right|_{a\pm 0} \cdot \Delta a \tag{2.35}$$
$$\Delta M_1(a) = \delta M_1(a\pm) + \left.\frac{dM_1}{dr}\right|_{a\pm 0} \cdot \Delta a$$

Evidently, the variations ΔM_{0j} and ΔD_j are not independent, as

$$\Delta M_{0j} = \sigma_0 H \Delta h_j; \qquad \Delta D_j = \frac{EH^2 \Delta h_j}{2(1-\nu^2)} \tag{2.36}$$

Making use of (2.36) one easily obtains from (2.32) that due to arbitrariness of Δh_j

$$a_{j+1}^2 - a_j^2 - 2\sigma_0 H \cdot M_{0j}\varphi_j + \delta_{ej}\frac{EH^2}{2(1-\nu^2)}\int\limits_{S_{ej}} \left\{ -\frac{\psi_2}{D_j^2}M_1 + \right.$$
$$\left. + \frac{\psi_3}{r^2}Z(1-\nu^2) + \frac{\varphi_{0j}}{r}Z(1-\nu^2)\right\} dr = 0 \tag{2.37}$$

for $j = 0, \ldots, n$ where

$$\delta_{ej} = \begin{cases} 1, & \text{if } j \in K_{e_1} \\ 0, & \text{if } j \bar{\in} K_{e_1} \end{cases} \tag{2.38}$$

Note that if $j = k$ in (2.37) then the domain of integration consists of two intervals (a_k, r_{k_0}) and (r_{k_1}, a_{k+1}), respectively.

Substituting the variations of state variables according to (2.35) into (2.32), (2.33) and accounting for the equations (2.37) yields

$$\sum_{j=1}^{n} \left\{ 2a_j(h_{j-1} - h_j)\Delta a_j - [\psi_1(a_j)\delta W(a_j) + \psi_2(a_j)\delta Z(a_j) + \psi_3(a_j)\delta M_1(a_j)] \right\} +$$
$$+ \psi_1(R)\delta W(R) + \psi_2(R)\delta Z(R) + \psi_3(R)\delta M_1(R) - \psi_1(0)\delta W(0) - \psi_2(0)\delta Z(0) -$$
$$- \psi_3(0)\delta M_1(0) - \left[\psi_1(r_{k_0})\delta W(r_{k_0}) + \psi_2(r_{k_0})\delta Z(r_{k_0}) + \psi_3(r_{k_0})\delta M_1(r_{k_0})\right] -$$
$$- \left[\psi_1(r_{k_1})\delta W(r_{k_1}) + \psi_2(r_{k_1})\delta Z(r_{k_1}) + \psi_3(r_{k_1})\delta M_1(r_{k_1})\right] = 0 \tag{2.39}$$

where the square brackets denote the jump of corresponding variable, e.g.

$$[\psi(a_j)y(a_j)] = \psi(a_j + 0)y(a_j + 0) - \psi(a_j - 0)y(a_j - 0) \tag{2.40}$$

Variations $\delta W(a_j)$, $\delta Z(a_j)$, $\delta M_1(a_j)$, also the variations of state variables at $r = r_{k_0}$ and $r = r_{k_1}$ can not be considered as arbitrary independent variations because the coordinates a_j, r_{k_0}, r_{k_1} themselves are subjected to the variation (see Hull [47]; Ahmed [1]). However, the variations $\delta W(0)$, $\delta Z(0)$, $\delta M_1(0)$ and $\delta W(R)$, $\delta Z(R)$,

$\delta M_1(R)$ are independent. Accounting for the boundary conditions (2.5), (2.23), (2.24) it follows from (2.39) that

$$\psi_3(0) = 0, \qquad \psi_2(R) = 0. \tag{2.41}$$

Let us introduce the total variations at $r = r_k$

$$\begin{aligned}
\Delta W(r_k) &= \delta W(r_k \pm 0) + \left.\frac{dW}{dr}\right|_{r_k \pm 0} \cdot \Delta r_k \\
\Delta Z(r_k) &= \delta Z(r_k \pm 0) + \left.\frac{dZ}{dr}\right|_{r_k \pm 0} \cdot \Delta r_k \\
\Delta M_1(r_k) &= \delta M_1(r_k \pm 0) + \left.\frac{dM_1}{dr}\right|_{r_k \pm 0} \cdot \Delta r_k
\end{aligned} \tag{2.42}$$

In similar way one can define the total variations of state variables at points r_{k_0} and r_{k_1}. Substituting the total variations according to (2.42) in (2.39) and taking into account the transversality conditions (2.41) with boundary conditions (2.5), (2.23), (2.24) one obtains

$$\begin{aligned}
\sum_{j=1}^{n} \Bigg\{ & 2a_j(h_{j-1} - h_j)\Delta a_j - \psi_1(a_j + 0)\left(\Delta W(a_j) - \frac{dW(a_j + 0)}{dr}\Delta a_j\right) - \\
& - \psi_2(a_j + 0)\left(\Delta Z(a_j) - \frac{dZ(a_j + 0)}{dr}\Delta a_j\right) - \\
& - \psi_3(a_j + 0)\left(\Delta M_1(a_j) - \frac{dM_1(a_j + 0)}{dr}\Delta a_j\right) + \\
& + \psi_1(a_j - 0)\left(\Delta W(a_j) - \frac{dW(a_j - 0)}{dr}\Delta a_j\right) + \\
& + \psi_2(a_j - 0)\left(\Delta Z(a_j) - \frac{dZ(a_j - 0)}{dr}\Delta a_j\right) + \\
& + \psi_3(a_j - 0)\left(\Delta M_1(a_j) - \frac{dM_1(a_j - 0)}{dr}\Delta a_j\right) - \\
& - \psi_1(r_{k_0} + 0)\left(\Delta W(r_{k_0}) - \frac{dW(r_{k_0} + 0)}{dr}\Delta r_{k_0}\right) - \\
& - \psi_2(r_{k_0} + 0)\left(\Delta Z(r_{k_0}) - \frac{dZ(r_{k_0} + 0)}{dr}\Delta r_{k_0}\right) - \\
& - \psi_3(r_{k_0} + 0)\left(\Delta M_1(r_{k_0}) - \frac{dM_1(r_{k_0} + 0)}{dr}\Delta r_{k_0}\right) + \\
& + \psi_1(r_{k_0} - 0)\left(\Delta W(r_{k_0}) - \frac{dW(r_{k_0} - 0)}{dr}\Delta r_{k_0}\right) + \\
& + \psi_2(r_{k_0} - 0)\left(\Delta Z(r_{k_0}) - \frac{dZ(r_{k_0} - 0)}{dr}\Delta r_{k_0}\right) +
\end{aligned}$$

$$
\begin{aligned}
&+\psi_3(r_{k_0}-0)\left(\Delta M_1(r_{k_0})-\frac{dM_1(r_{k_0}-0)}{dr}\Delta r_{k_0}\right)-\\
&-\psi_1(r_{k_1}+0)\left(\Delta W(r_{k_1})-\frac{dW(r_{k_1}+0)}{dr}\Delta r_{k_1}\right)-\\
&-\psi_2(r_{k_1}+0)\left(\Delta Z(r_{k_1})-\frac{dZ(r_{k_1}+0)}{dr}\Delta r_{k_1}\right)-\\
&-\psi_3(r_{k_1}+0)\left(\Delta M_1(r_{k_1})-\frac{dM_1(r_{k_1}+0)}{dr}\Delta r_{k_1}\right)+\\
&+\psi_1(r_{k_1}-0)\left(\Delta W(r_{k_1})-\frac{dW(r_{k_1}-0)}{dr}\Delta r_{k_1}\right)+\\
&+\psi_2(r_{k_1}-0)\left(\Delta Z(r_{k_1})-\frac{dZ(r_{k_1}-0)}{dr}\Delta r_{k_1}\right)+\\
&+\psi_3(r_{k_1}-0)\left(\Delta M_1(r_{k_1})-\frac{dM_1(r_{k_1}-0)}{dr}\Delta r_{k_1}\right)\Bigg\}=0 \quad (2.43)
\end{aligned}
$$

Since the total variations $\Delta W(a_j)$, $\Delta Z(a_j)$, $\Delta M_1(a_j)$ are independent, it immediately infers from (2.43)

$$
\begin{aligned}
\psi_1(a_j+0)-\psi_1(a_j-0)&=0\,,\\
\psi_2(a_j+0)-\psi_2(a_j-0)&=0\,,\\
\psi_3(a_j+0)-\psi_3(a_j-0)&=0
\end{aligned}
\tag{2.44}
$$

for each $j=1,\dots,n$. This means that $[\psi_i(a_j)]=0$ for $i=1,2,3$. In other words, adjoint variables are continuous at $r=a_j$ $(j=1,\dots,n)$.

Due to the arbitrariness of total variations of state variables at $r=r_{k_0}$ and $r=r_{k_1}$ it follows from (2.43) that

$$
\begin{aligned}
[\psi_i(r_{k_0})]&=0\,,\\
[\psi_i(r_{k_1})]&=0
\end{aligned}
\tag{2.45}
$$

for each $i=1,2,3$.

Arbitrary are the increments $\Delta a_j (j=1,\dots,n)$, Δr_{k_0} and Δr_{k_1} in (2.43). Due to the arbitrariness on Δa_j one has

$$
\begin{aligned}
&2a_j(h_{j-1}-h_j)+\psi_1(a_j)\left[\frac{dW(a_j)}{dr}\right]+\\
&\qquad+\psi_2(a_j)\left[\frac{dZ(a_j)}{dr}\right]+\psi_3(a_j)\left[\frac{dM_1(a_j)}{dr}\right]=0 \quad (2.46)
\end{aligned}
$$

for each $j=1,\dots,n$. Note that when deriving the last equation the continuity of adjoint variables (2.44) has been taken into account.

Similarly, it follows from (2.43) that

$$\psi_1(r_{k_0})\left[\frac{dW(r_{k_0})}{dr}\right] + \psi_2(r_{k_0})\left[\frac{dZ(r_{k_0})}{dr}\right] + \psi_3(r_{k_0})\left[\frac{dM_1(r_{k_0})}{dr}\right] = 0 \quad (2.47)$$

and

$$\psi_1(r_{k_1})\left[\frac{dW(r_{k_1})}{dr}\right] + \psi_2(r_{k_1})\left[\frac{dZ(r_{k_1})}{dr}\right] + \psi_3(r_{k_1})\left[\frac{dM_1(r_{k_1})}{dr}\right] = 0. \quad (2.48)$$

2.5 Numerical results

The results of calculations are presented in Fig. 2.2 – 2.15 and Tables 2.2 – 2.5. In Tables 2.2 – 2.5 the values of non-dimensional quantities p, α_i, γ_i and w_0 are exposed. These quantities are defined by 2.49.

The load deflection relations for elastic plastic von Mises plates are presented in Fig. 2.2 and Fig. 2.3. Fig. 2.2 corresponds to the plate of constant thickness whereas Fig. 2.3 is associated with the one-stepped plate. Different curves in Fig. 2.2 correspond to plates with thicknesses $h_1 = 0.80$, $h_2 = 0.85$, $h_3 = 0.90$, $h_4 = 0.95$, $h_5 = 1$. However, curves labelled with 1, 2, 3, 4, 5 in Fig. 2.3 correspond to steps located at $a_1 = 0.18R$, $a_1 = 0.20R$, $a_1 = 0.33R$, $a_1 = 0.43R$ and $a_1 = 0.71R$, respectively (here $h_1 = 0.4h_0$). It can be seen from Fig. 2.2 and Fig. 2.3 that with increasing the mass of the plate the deflection w_0 decreases at each loading level.

Table 2.1 Numbers of curves

p	$2,8000$	$3,000$	$3,2000$	$3,4000$	$3,6000$
#	5	4	3	2	1

Table 2.2 Plastic region interval for $\gamma_1 = 0,2$; $\alpha = 0,4$

p	$1,6000$	$1,6300$	$1,8000$	$1,9000$	$2,0000$	$2,1000$
η	–	$0,0701$	$0,2431$	$0,3726$	$0,5318$	$0,6702$

Table 2.3 Plastic region interval for $\gamma_1 = 0,4$; $\alpha = 0,4$

p	$1,9821$	$2,1000$	$2,2000$	$2,3000$	$2,4000$	$2,5000$
η	$0,5793$	$0,6014$	$0,6207$	$0,6552$	$0,6893$	$0,7604$

Table 2.4 Optimal parameters for the deflection $w_0 = 0.2500$, $n = 2$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.3781	0.4093	0.6002	0.7069	0.8103	0.8873
α_1	0.2478	0.4012	0.4092	0.4791	0.5368	0.5990
γ_2	0.6102	0.7150	0.7292	0.7415	0.8561	0.8997
α_2	0.3007	0.4816	0.5602	0.5217	0.7204	0.8021
e	12.47%	17.61%	17.9%	11.20%	09.17%	12.76%

Table 2.5 Optimal parameters for the deflection $w_0 = 0.2500$, $n = 4$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.4374	0.4861	0.5019	0.5343	0.6276	0.6733
α_1	0.1711	0.1891	0.2351	0.2671	0.2714	0.2904
γ_2	0.4837	0.5302	0.6072	0.7039	0.7161	0.7996
α_2	0.3349	0.4014	0.5039	0.4261	0.3997	0.3817
γ_3	0.6483	0.7318	0.7514	0.8624	0.8220	0.8356
α_3	0.4602	0.4692	0.6080	0.5357	0.5886	0.6112
γ_4	0.6092	0.5477	0.5513	0.6489	0.7952	0.8075
α_4	0.8794	0.7213	0.8495	0.9251	0.9075	0.9014
e	19.14%	12.91%	14.03%	17.40%	10.12%	12.84%

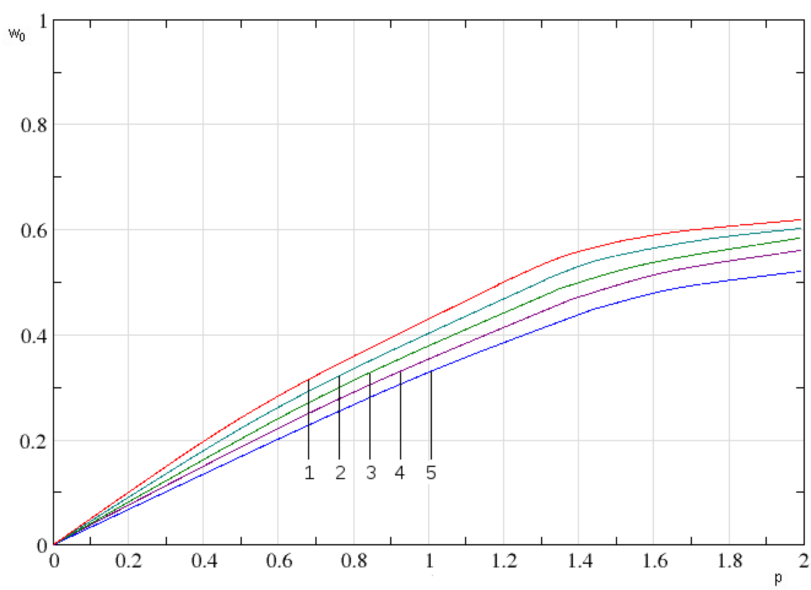

Fig. 2.2 Load-deflection relations for plates of constant thickness

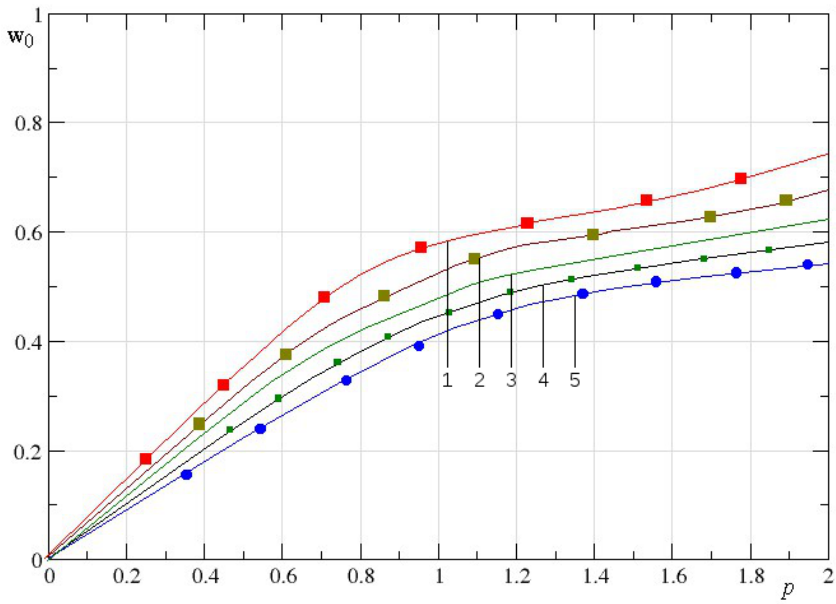

Fig. 2.3 Deflections for $\alpha_1 = 0.18$, $\alpha_1 = 0.20$, $\alpha_1 = 0.33$, $\alpha_1 = 0.43$, $\alpha_1 = 0.71$.

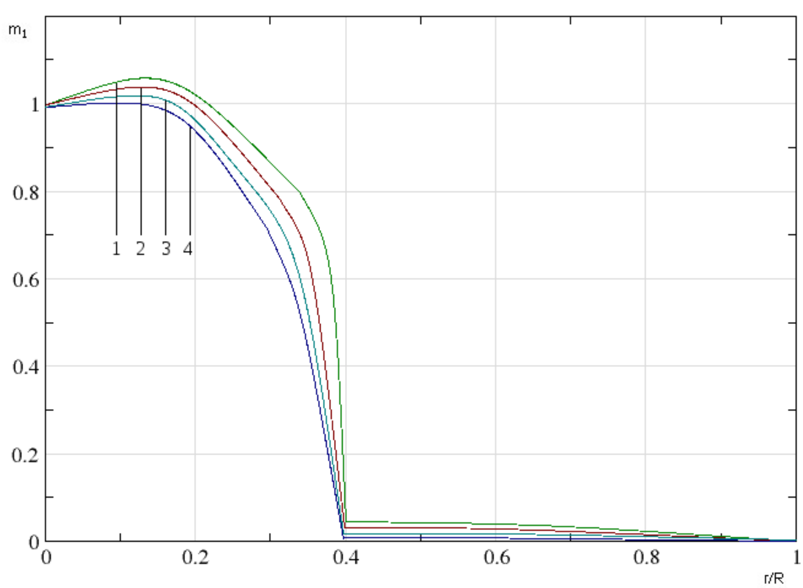

Fig. 2.4 Bending moment $m_1(\rho)$ for $\alpha = 0.4$, $\gamma_1 = 0.2$.

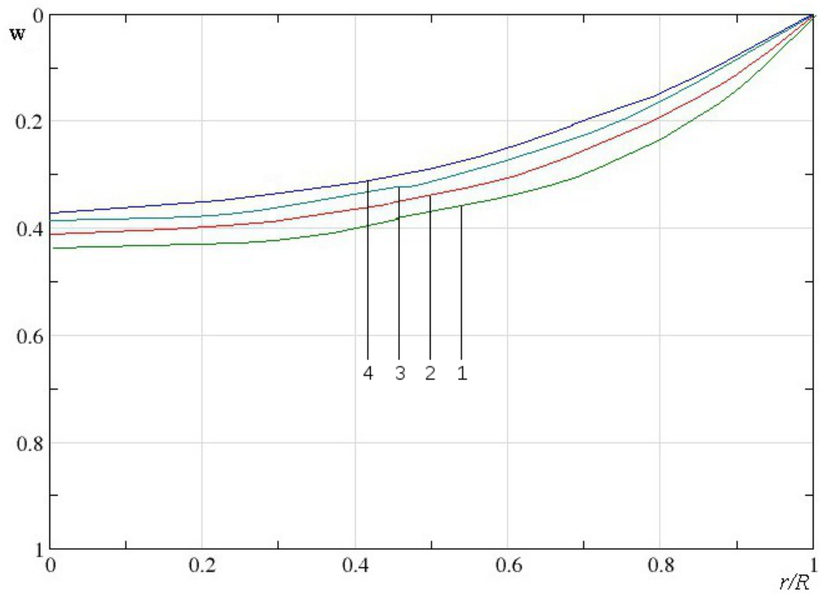

Fig. 2.5 Deflections of the plate ($\alpha = 0.4$, $\gamma_1 = 0.2$)

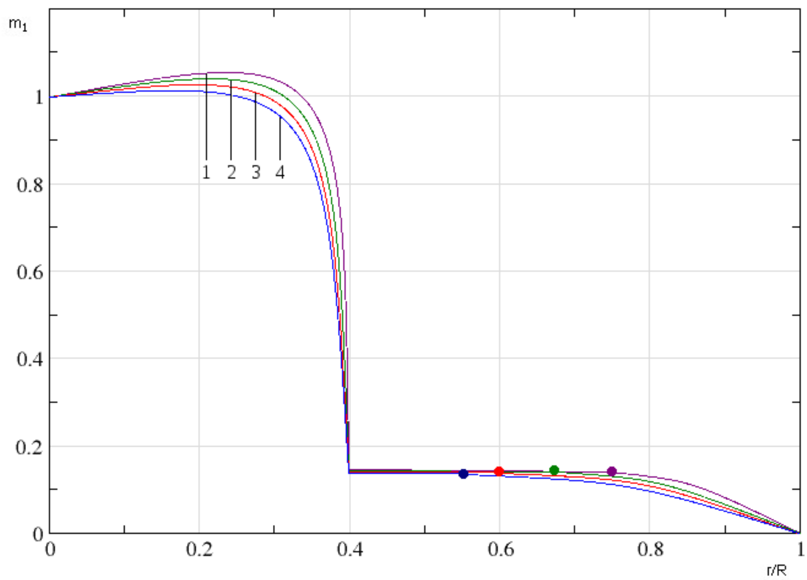

Fig. 2.6 Radial moment $m_1(\rho)$ for $\alpha = 0.4$, $\gamma_1 = 0.4$.

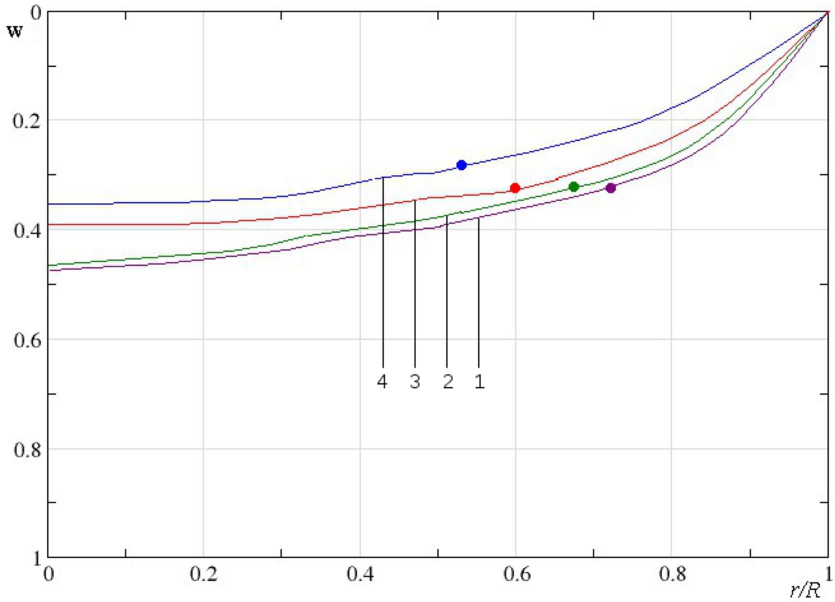

Fig. 2.7 Transverse deflections for $\alpha = 0.4$, $\gamma_1 = 0.4$.

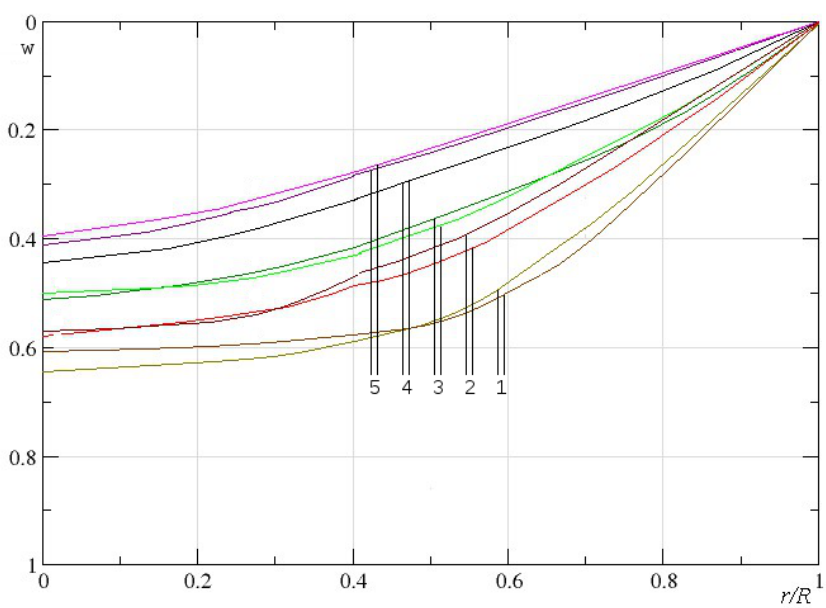

Fig. 2.8 Deflections of the plate (comparison with ABAQUS for $\alpha = 0.4$, $\gamma_1 = 0.35$)

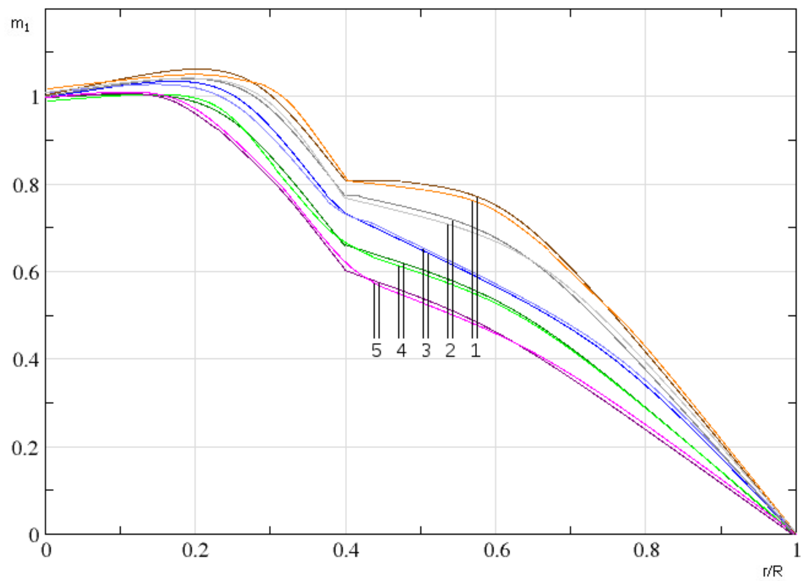

Fig. 2.9 Radial bending moment (comparison with wavelet method for $\alpha = 0.4$, $\gamma_1 = 0.9$)

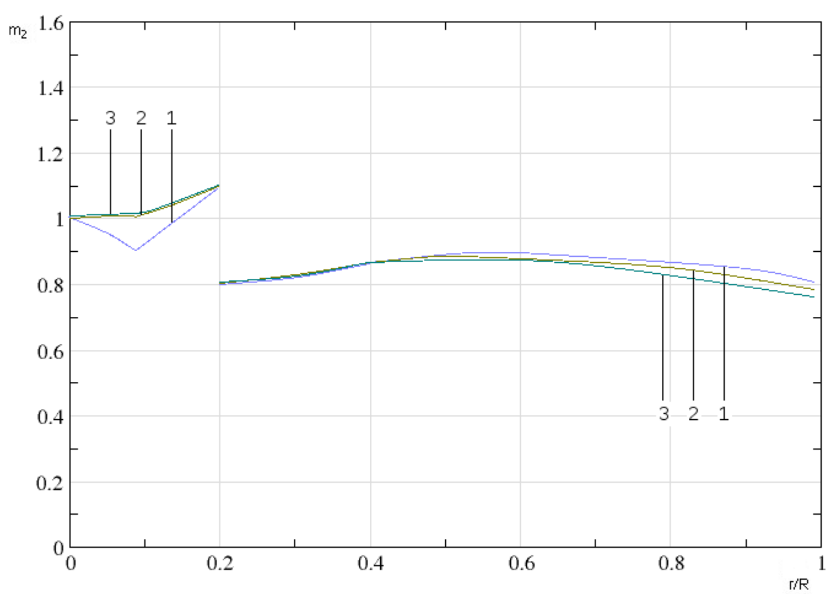

Fig. 2.10 Hoop moment for $a = 0.2R$

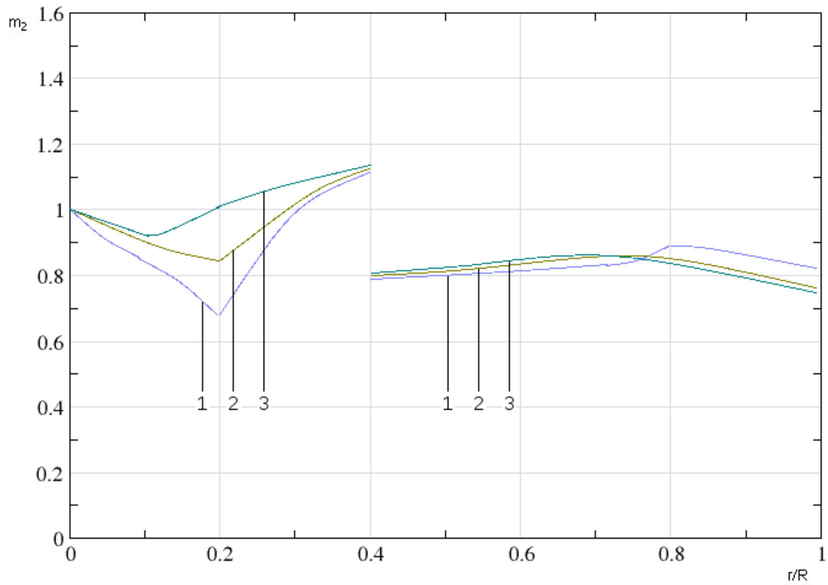

Fig. 2.11 Hoop moment for $a = 0.4R$

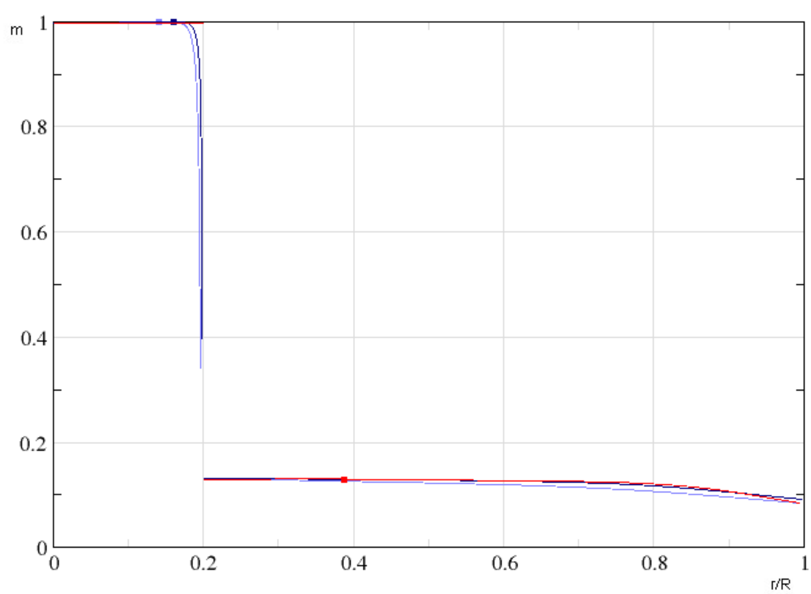

Fig. 2.12 The quantity m for $a = 0.2R$

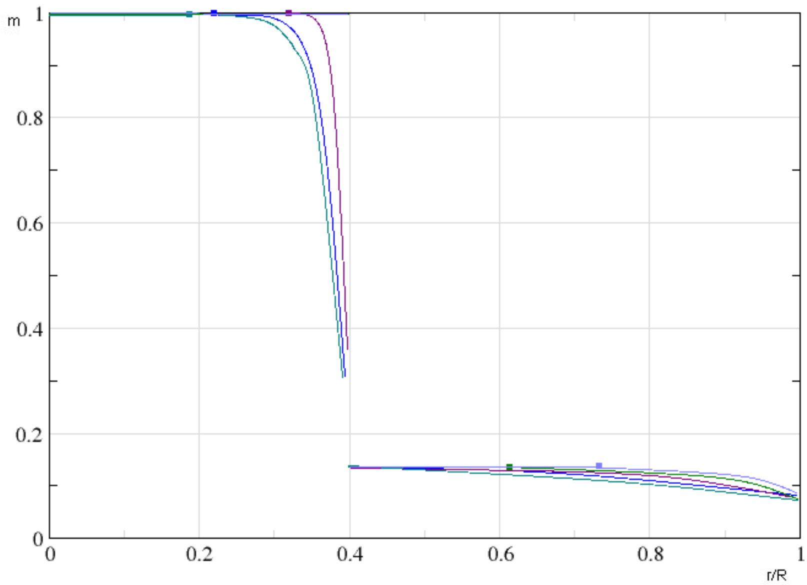

Fig. 2.13 The quantity m for $a = 0.4R$

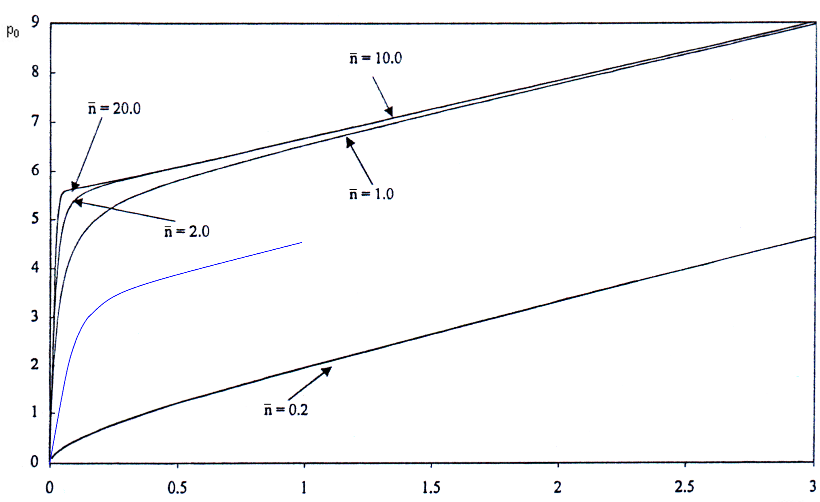

Fig. 2.14 Load deflection relations (comparison of results)

The curves labelled with 1, 2, 3, 4, 5 in Fig. 2.4 – 2.13 correspond to different values of the load intensity. Corresponding load intensities are accommodated in the Table 2.1. In Table 2.2 and 2.3 η stands for the length of the single plastic region located at the center of the plate. Thus $r_{k_0} = 0$, $r_{k_1} = \eta$ and $k = 0$ at the center of the plate.

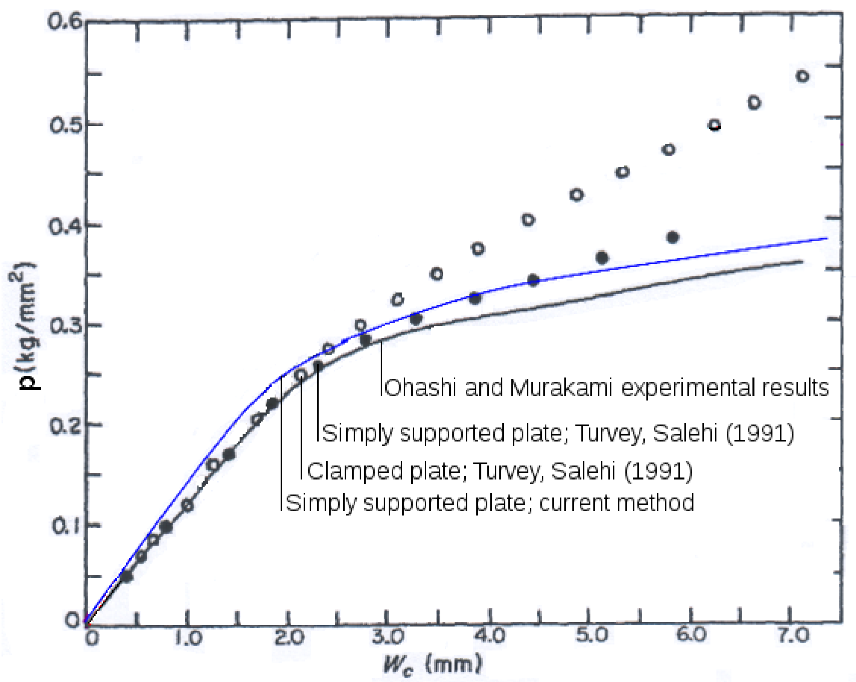

Fig. 2.15 Load deflection relations (comparison of results)

For the solution of the direct problem which consists in the determination of the stress strain state of the plate for given distribution of the material the Finite Element Codes FEMLAB and ABAQUS were used.

In what follows the results regarding to plates with two different thicknesses h_0 and h_1 will be discussed. Corresponding plate will be called one-stepped plate. The load-deflection relations for one-stepped plates are presented in Fig. 2.2 – 2.3. Here w_0 stands for the transverse deflection at the centre of the plate and p is the non-dimensional load intensity whereas

$$p = P \cdot \frac{M_{00}}{R^2}, \quad \alpha_i = \frac{a_i}{R}, \quad \gamma_i = \frac{h_i}{h_0}, \quad w_0 = \frac{W_0}{H}, \quad m_{1,2} = \frac{M_{1,2}}{M_{00}} \tag{2.49}$$

Here M_{00} stands for the yield moment for a section with thickness h_0. Optimal values of design parameters are accommodated in Tables 2.4 and 2.5 for two- and four-stepped plates, respectively. It can be seen that in contrast to the elastic case the optimal values of design parameters depend on the load intensity. In the last rows of Tables 2.4; 2.5 the values of the quantity $e = (1 - V/V_*) \cdot 100\%$ are accommodated. Here V stands for the optimal (minimal) value of the plate volume and V_* is the volume of the reference plate of constant thickness. It can be seen from Table 2.4 that in the case of two-stepped plates and the deflection $w_0 = 0.25$ one can save more than 17% of the material.

The distributions of deflections of simply supported plates are plotted in Fig. 2.5; Fig. 2.7; Fig. 2.8. Here $a = 0.4R$; Fig. 2.5 corresponds to the case when $h_1 = 0.2h_0$ and Fig. 2.7, Fig. 2.8 are associated with the $h_1 = 0.4h_0$. The distributions of radial bending moments corresponding to these cases are portrayed in Fig. 2.4, Fig. 2.6 and Fig. 2.9, respectively.

It can be seen from Fig. 2.5; Fig. 2.7 and Fig. 2.8 that the higher the transverse pressure is the larger is the deflection at each point of the plate, as might be expected.

The asterisks in Fig. 2.6; Fig. 2.7 indicate the length of the plastic region in the plate. The matter that the maximum of the radial bending moment is not achieved at the center of the plate means that a sort of unloading takes place during the elastic plastic stage of deformation of circular plates.

In Fig. 2.9 the bending moments obtained by the use of the finite element method are compared with those obtained by the wavelet method. One can see that the results favourably agree with each other.

Transverse deflections calculated by the code ABAQUS and by the currect method are compared in Fig. 2.8. One can see that the results are quite close to each other.

Distributions of the hoop moment $m_2 = M_\Theta/M_0$ are presented in Fig. 2.10 and Fig. 2.11 for plates with $a = 0.2R$ and $a = 0.4R$, respectively. It reveals from Fig. 2.10, Fig. 2.11 that the hoop moment has discontinuities at $r = a$, as might be expected.

The quantity

$$m = m_1^2 - m_1 m_2 + m_2^2 \tag{2.50}$$

is portrayed in Fig. 2.12, Fig. 2.13 for plates with $a = 0.2R$ and $a = 0.4R$, respectively. Here the ratio $h_1/h_0 = 0.4$.

Note that the Mises yield condition can be presented as

$$m - \gamma_j^2 \leqslant 0 \tag{2.51}$$

for $r \in (a_j; a_{j+1})$; $j = 0, \ldots, n$. According to the assumption about the stress profile in plastic regions (2.51) takes the form $m = \gamma_j^2$. However, in elastic regions the inequality (2.51) must be converted into a strict inequality. Calculations carried out and Fig. 2.12, Fig. 2.13 demonstrate that (2.51) is satisfied at each point of the plate.

The load deflections relations obtained by the current method are compared with the results of Updastra, Peddieson, Buchanan [139] and Turvey [136], [137], [138] in Fig. 2.14 and Fig. 2.15, respectively.

The method of Upadrasta *et al.* [139] exploits the idea of elastic compensation to iteratively modify the elastic properties of the material to simulate the plastic yielding in finite element codes. In Fig. 2.14 the load deflection relations calculated by Upadastra *et al.* are presented for different values of the material parameter $\bar{n}$. The curve labelled with 1 in Fig. 2.14 corresponds to the method used in current study. The results are more or less coinciding if $\bar{n} \approx 0.74$.

The load-deflection relations calculated by the current method are compared with those obtained by Turvey, Salehi [137] also by Ohashi and Murakami [114] in Fig. 2.15. It can be seen from Fig. 2.15 that the results of the current study are in good correlation with the results of Turvey and with the data of the experiments conducted by Ohashi and Murakami [114].

2.6 Concluding remarks

A method for numerical investigation of axisymmetric plates subjected to the distributed transverse pressure loading was presented. The material of plates studied herein is assumed to be an ideal elastic plastic material obeying the non-linear yield condition of von Mises and the associated flow law. The strain hardening as well as geometrical non-linearity are neglected in the present investigation.

Making use of the variational methods of the theory of optimal control necessary optimality conditions are derived for plates of piece wise constant thickness. For calculations a computer code in the Linux environment is created. Calculations carried out showed that the obtained results are in good correlation with those obtained by ABAQUS when solving the direct problem of determination of the stress strain state of the plate.

Chapter 3
Optimization of annular plates made of a von Mises material

3.1 Introduction

Problems of optimization of annular plates have been studied by many authors in the case of a pure elastic material (see Banichuk [3]; Dzjuba [160] and others). On the other hand, there exists an exhaustive list of papers devoted to the optimization of perfectly plastic plates (see Lellep [73]; Lepik [89]). However, there exist only a few papers concerning to the optimization of plates made of elastic plastic materials. Among such papers one should mention the papers by Lellep and Vlassov [82] – [87], Kaliszky, Logo [55].

In this section a method of optimization is developed for annular plates subjected to the transverse pressure.

3.2 Problem formulation

Let us study the response of the annular plates with radii a and R to the quasistatic transverse pressure of intensity $P = P(r)$, where r is the current radius. It is assumed that the plates have an ideal sandwich cross section with two rims of thickness $h = h_j$ for $j = 0,\dots,n$ and $x \in (a_j,\ a_{j+1})$ (Fig. 3.1). These rims are separated by a layer of the core material with thickness H. The latter is considered as a constant over the plate. It is reasonable to take $a_0 = a$ and $a_{n+1} = R$.

Under these assumptions one can present the volume of the plate (actually we calculate the volume of a layer) as

$$V = \sum_{j=0}^{n} h_j \left(a_{j+1}^2 - a_j^2 \right). \tag{3.1}$$

The quantities a_j and h_j are treated as unknown parameters which will be defined so that the cost function attains the minimal value, provided the governing equations and additional constrains with boundary conditions are satisfied.

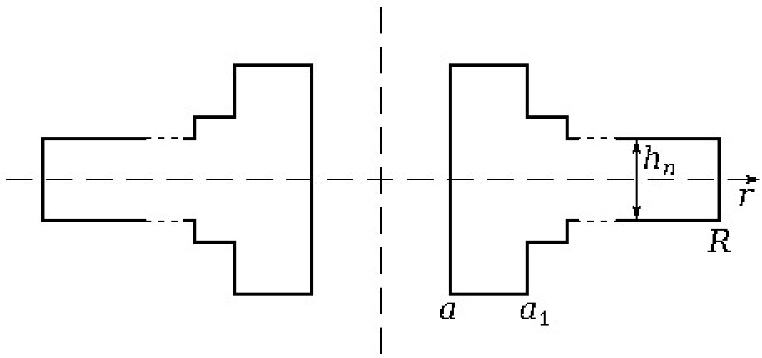

Fig. 3.1 Stepped annular plate

In the literature one can find various types of cost functions due to the large variety of optimization problems. In the present paper we shall treat two kinds of problems: 1) the minimum weight problem for constrained deflections and 2) minimization of the deflections at the free edge of the plate. In the first case the cost function can be presented by (3.1), in the second case $J = W(a)$, where W stands for the transverse deflection.

Assume that the plate under consideration is simply supported at the outer edge and it is absolutely free at the inner edge. Thus, the boundary conditions at the inner edge are

$$M_1(a) = 0, \qquad Q(a) = 0, \tag{3.2}$$

where M_1 is the radial bending moment and Q stands for the radial shear force. Note that the bending moment in the circumferential direction will be denoted by M_2. It is well-known (see Reddy, [121]) that

$$Q = \frac{1}{r}\left[\frac{d}{dr}\big(rM_1\big) - M_2\right]. \tag{3.3}$$

At the simply supported outer edge the boundary conditions have the form

$$M_1(R) = 0, \qquad W(R) = 0. \tag{3.4}$$

Note that in the case of the minimum weight problem $W(a) = W_0$, where W_0 is a given constant.

3.3 Basic equations and assumptions

We are investigating the elastic plastic response of axisymmetric plates to the axisymmetric loading $P(r)$. The plates have a sandwich-type cross section. It is widely recognized that stresses across thin carrying layers in a sandwich plate are constant. Thus, in a cross section of a sandwich plate the stress state can be either a pure elastic or a pure plastic state. Therefore, the plate must be subdivided into elastic and plastic regions. Let these regions be denoted by S_e and S_p, respectively. Evidently, the sum of these regions $S_e \cup S_p$ coincides with the interval $[a,\ R]$.

It is well-known that the stress state in a pure elastic axisymmetric plate satisfies the equations (see Reddy, [121]; Vinson, [144]; Lellep and Vlassov, [82]),

$$
\begin{aligned}
&\frac{dW}{dr} = Z,\\
&\frac{dZ}{dr} = -\frac{M_1}{D_j} - \frac{\nu}{r}Z,\\
&\frac{dM_1}{dr} = -\left(1-\nu^2\right)D_j\frac{Z}{r^2} - \frac{1-\nu}{r}M_1 + Q,\\
&\frac{dQ}{dr} = -\frac{Q}{r} - p
\end{aligned}
\tag{3.5}
$$

for $r \in (a_j,\, a_{j+1})$; $j = 0, \ldots n$. Here Z is an auxiliary variable, whereas

$$
D_j = \frac{EH^2 h_j}{2(1-\nu^2)}. \tag{3.6}
$$

In (3.5), (3.6) E and ν stand for the Young and Poisson modules, respectively, whereas D_j is the bending stiffness coefficient of the segment of the plate with thickness h_j.

Note that the system (3.5) is identical to the system (2.22) with (2.19) used in the previous section.

It is assumed that the material of the plate obeys von Mises yield condition presented as

$$
M_1^2 - M_1 M_2 + M_2^2 - M_{0j}^2 \leqslant 0 \tag{3.7}
$$

for $r \in (a_j, a_{j+1})$; $j = 0, \ldots n$. Here M_{0j} stands for the limit moment, e.g.

$$
M_{0j} = \sigma_0 H h_j, \tag{3.8}
$$

σ_0 being the yield stress of the material.

In an elastic region the inequality (3.7) is satisfied as a strict inequality. However, in a plastic region (3.7) is transformed into an equality. Introducing a new variable Θ_j the inequality (3.7) can be presented as an equality

$$
M_1^2 - M_1 M_2 + M_2^2 - M_{0j}^2 + \Theta_j^2 = 0 \tag{3.9}
$$

The latter holds good in an arbitrary region of the plate.

The associated flow law and equilibrium equations furnish the governing equations for a plastic region as (Lellep and Vlassov, [82]),

$$
\begin{aligned}
\frac{dW}{dr} &= Z, \\
\frac{dZ}{dr} &= \frac{z}{r}\frac{2M_1 - M_2}{2M_2 - M_1}, \\
\frac{dM_1}{dr} &= \frac{M_2}{r} - \frac{M_1}{r} + Q, \\
\frac{dQ}{dr} &= -\frac{Q}{r} - p.
\end{aligned}
\tag{3.10}
$$

It is worth while to emphasize that (3.10) is satisfied for $r \in S_p$ whereas (3.5) holds good for $r \in S_e$. In the elastic regions one has an additional requirement

$$
M_2 - \frac{Z}{r} D_j(\nu^2 - 1) - \nu M_1 = 0, \tag{3.11}
$$

the consequence from Hooke's law, geometrical relations and equilibrium conditions.

The posed problem will be treated as a particular problem of the theory of optimal control, the variables W, Z, M_1, Q being the state variables and M_2, Θ_j – controls.

Let us denote the state vector by $\mathbf{X} = (X_1, X_2, X_3, X_4)$. Thus,

$$
X_1 = W; \quad X_2 = Z; \quad X_3 = M_1; \quad X_4 = Q \tag{3.12}
$$

and the left hand sides of equations (3.5) and (3.10) can be denoted by X_i', $(i = 1, \ldots, 4)$. Denoting the right hand sides of equations (3.5) and (3.10) by f_{ei} and f_{pi}, respectively, one can write

$$
X_i' = f_{ei}(X_1, X_2, X_3, X_4, D_j) \tag{3.13}
$$

for $r \in S_{ej}$ and

$$
X_i' = f_{pi}(X_1, X_2, X_3, X_4, M_2) \tag{3.14}
$$

for $r \in S_{p_i}$; $(i = 1, \ldots, 4)$.

Here S_{ej} stands for the interval $(a_j,\, a_{j+1})$ under the condition that it belongs wholly to the elastic region.

Optimality conditions are defined with the help of the theory of optimal control [1], [3], [47], [75].

Let K_e be the set of integers defined as follows. Namely, for $j \in K_e$ the region $S_{ej} = (a_j,\, a_{j+1})$ is an elastic region. Similarly, let K_p include such values of indexes $j \in K_p$ for which the region $S_{pj} = (a_j,\, a_{j+1})$ is a fully plastic region.

For the sake of simplicity it is assumed that there exists only a unique interval $(a_k,\, a_{k+1})$ including the both, elastic and plastic regions. Let plastic deformations take place for $r \in (r_{k_0},\, r_{k_1})$. It is reasonable to assume that in the region of constant thickness $h = h_k$ plastic deformations take place for $r \in (r_{k_0},\, r_{k_1})$ whereas intervals $(a_k,\, r_{k_0})$ and $(r_{k_1},\, a_{k+1})$ remain elastic. In particular cases it can be $a_k = r_{k_0}$ or $a_{k+1} = r_{k_1}$.

Let us denote the set of integers which contains the number k and all elements of K_e by K_{e1} and let

$$S_{ek} = S_{ek_0} \cup S_{ek_1} \tag{3.15}$$

where S_{ek_0}; S_{ek_1} are the subintervals of (a_k, a_{k+1}) associated with elastic deformations.

Similarly, it is expected that K_{p1} is the set of integers which contains besides elements K_p the integer k. Thus, one can write

$$S_e = \bigcup_{j \in K_{e1}} S_{ej}, \qquad S_p = \bigcup_{j \in K_{p1}} S_{pj}. \tag{3.16}$$

3.4 Derivation of optimality conditions

The problem posed above will be treated as a particular problem of optimal control. This problem involves control functions M_2, Q_j, $(j = 0, \ldots, n)$ as well as concentrated parameters a_j $(j = 1, \ldots, n)$, h_i $(i = 0, \ldots, n)$ whereas the right-hand sides of state equations are discontinuous when passing the boundary between S_e and S_p.

In order to deduce the conditions of optimality of the cost criterion (3.1) subjected to constraints (3.5) and (3.11) in elastic regions and state equations (3.10) in plastic regions and equalities (3.9) in both regions one can employ the method of an augmented functional (Bryson, [10]; Banichuk, [3]; Hull, [47]; Lellep and Vlassov, [82], [83]). In the present case the augmented functional can be presented as (here the notations (3.12) – (3.14)) are used)

$$J_* = V + \sum_{j \in K_{e1}} \int_{S_{ej}} \left\{ \sum_{i=1}^{4} \psi_i \left(X_i' - f_{ei} \right) + X_j \left(M_2 + D_j \left(1 - \nu^2 \right) \frac{X_2}{r} - \nu X_3 \right) \right\} dr +$$
$$+ \sum_{j \in K_{p1}} \int_{S_{pj}} \left\{ \sum_{i=1}^{4} \psi_i \left(X_i' - f_{pi} \right) \right\} dr + \sum_{j=0}^{n} \int_{a_j}^{a_{j+1}} \varphi_j \left(X_3^2 - X_3 M_2 + M_2^2 - M_{0j}^2 + \Theta_j^2 \right) dr \tag{3.17}$$

Here the functions X_j $(j \in K_{e1})$ and φ_j $(j = 0, \ldots, n)$ stand for Lagrange multipliers and ψ_1, ψ_2, ψ_3, ψ_4 adjoint (conjugate) variables.

Calculating the total variation of the functional (3.17) one has to take into account that the quantities a_j $(j = 1, \ldots, n)$ and h_i $(i = 0, \ldots, n)$ are preliminarily unknown. Unfixed are also boundary points between S_{ej} and S_{pj}. Therefore the variations of state variables at $r = a_j$ must be calculated by the following sample (Ahmed, [1]; Hull, [47]; Lellep, [73])

$$\Delta Y(a) = \delta Y(a \pm 0) + Y'(a \pm 0) \cdot \Delta a \tag{3.18}$$

where δY is the weak variation and ΔY the total variation of Y at $r = a$.

The total variation of (3.17) can be presented as

$$\Delta J_* = 2\sum_{j=1}^{n} a_j\big(h_{j-1}-h_j\big)\Delta a_j + \sum_{j=0}^{n}\Big\{\big(a_{j+1}^2-a_j^2\big)\Delta h_j +$$
$$+\int_{a_j}^{a_{j+1}} \varphi_j\big(2X_3\delta X_3 - X_3\delta M_2 - M_2\delta X_3 + 2M_2\delta M_2 - 2M_{0_j}\delta M_{0_j} + 2\Theta_j\delta\Theta_j\big)dr\Big\}+$$
$$+\sum_{j\in K_{e1}}\int_{S_{ej}}\Big\{\sum_{i=1}^{4}\big(\psi_i\delta X_i' - \psi_i\delta f_{ei}\big) + X_j\big(\delta M_2 + \frac{1-\nu^2}{r}\big(X_2\delta D_j + D_j\delta X_2\big) - \nu\delta X_3\big)\Big\}dr+$$
$$+\sum_{j\in K_{p1}}\int_{S_{pj}}\sum_{i=1}^{4}\big(\psi_i\delta X_i' - \psi_i\delta f_{pi}\big)dr \quad (3.19)$$

Integrating by parts one easily obtains

$$\int_{a_j}^{a_{j+1}} \psi_i\delta X_i' dr = \int_{a_j}^{a_{j+1}} \psi_i'\delta X_i dr +$$
$$+\psi_i(a_{j+1}-0)\delta X_i(a_{j+1}-0) - \psi_i(a_j+0)\delta X_i(a_j+0) \quad (3.20)$$

Making use of (3.5), (3.12) one can find

$$\begin{aligned}
\delta f_{e1} &= \delta X_2,\\
\delta f_{e2} &= -\frac{1}{D_j}\delta X_3 + \frac{X_3}{D_j^2}\delta D_j - \frac{\nu}{r}\delta X_2,\\
\delta f_{e3} &= \frac{\nu^2-1}{r^2}\big(X_2\delta D_j + D_j\delta X_2\big) - \frac{1-\nu}{r}\delta X_3 + \delta X_4,\\
\delta f_{e4} &= -\frac{1}{r}\delta X_4
\end{aligned} \quad (3.21)$$

for $r\in S_{ej}$, $j\in K_{e1}$.

Similarly, it infers from (3.10), (3.12) that

$$\begin{aligned}
\delta f_{p1} &= \delta X_2,\\
\delta f_{p2} &= \frac{\delta X_2}{r}\frac{2X_3-M_2}{2M_2-X_3}+\\
&+\frac{X_2}{r\big(2M_2-X_3\big)^2}\big\{\big(2\delta X_3-\delta M_2\big)\big(2M_2-X_3\big)-\\
&-\big(2\delta M_2-\delta X_3\big)\big(2X_3-M_2\big)\big\},\\
\delta f_{p3} &= \frac{1}{r}\big(\delta M_2-\delta X_3\big)+\delta X_4,\\
\delta f_{p4} &= -\frac{1}{r}\delta X_4
\end{aligned} \quad (3.22)$$

for $r \in S_{pj}$, $j \in K_{p1}$.

Making use of (3.6), (3.8) one can recheck that for $j \in K_{e1}$

$$\delta D_j = \frac{EH^2}{2\left(1-\nu^2\right)}\Delta h_j \tag{3.23}$$

and for $j \in K_{p1}$

$$\delta M_{0_j} = \sigma_0 H \Delta h_j \tag{3.24}$$

Substituting (3.20) – (3.24) in (3.19) and starting from the equation $\Delta J_* = 0$ one can deduce the requirements which have to be met by the optimal solution. Due to the independence of variations of state variables $\delta X_i \, (i = 1,2,3,4)$ one obtains the adjoint equations for $r \in S_{ej}$, $j \in K_{e1}$

$$\begin{aligned}
\psi_1' &= 0,\\
\psi_2' &= -\psi_1 + \frac{\nu}{r}\psi_2 + (1-\nu^2)\left(D_j\frac{\psi_3}{r^2} + \frac{D_j}{r}X_j\right),\\
\psi_3' &= \frac{\psi_2}{D_j} + \frac{1}{r}(1-\nu)\psi_3 + \varphi_j(2X_3 - M_2) - \nu X_j,\\
\psi_4' &= -\frac{\psi_4}{r}
\end{aligned} \tag{3.25}$$

and for $r \in S_{pj}$, $j \in K_{p1}$

$$\begin{aligned}
\psi_1' &= 0,\\
\psi_2' &= -\psi_1 - \frac{\psi_2}{r}\frac{2X_3 - M_2}{2M_2 - X_3},\\
\psi_3' &= -\frac{\psi_2}{r}\frac{3X_2M_2}{(2M_2 - X_3)^2} + \frac{\psi_3}{r} + \varphi_j(2X_3 - M_2),\\
\psi_4' &= -\frac{\psi_4}{r}
\end{aligned} \tag{3.26}$$

The variations of control functions $\delta\Theta_j$ are independent in (3.19). Thus,

$$\varphi_j \Theta_j = 0 \tag{3.27}$$

for $j = 0, \dots n$.

Making use of (3.19), (3.23), (3.24) and taking into account that the increments Δh_j are arbitrary one can write for each $j = 0, \dots, n$

$$\begin{aligned}
&a_{j+1}^2 - a_j^2 - 2\sigma_0 H M_{0_j}(a_{j+1} - a_j)\varphi_j + \\
&\quad + \frac{EH^2\delta_{ej}}{2(1-\nu^2)}\int\limits_{S_{ej}}\left\{-\frac{\psi_2}{D_j^2}X_3 + \frac{\psi_3}{r^2}X_2(1-\nu^2) + \frac{X_j}{r}(1-\nu^2)X_2\right\}dr = 0
\end{aligned} \tag{3.28}$$

In (3.28) the following notation is used:

$$\delta_{ej} = \begin{cases} 1, & \text{if } j \in K_{e1} \\ 0, & \text{if } j \bar{\in} K_{e1} \end{cases} \tag{3.29}$$

It is worthwhile to mention that if $j = k$ in (3.28) then the integration domain S_{ek} consists of the intervals (a_k, r_{k_0}) and (r_{k_1}, a_{k+1}), as it was stipulated in the previous section of the paper.

Evidently, the variation δM_2 is also an independent one in (3.19). Therefore, one has

$$\varphi_j(2M_2 - X_3) + X_j = 0 \tag{3.30}$$

for $r \in S_{ej}$, $j \in K_{e1}$ and

$$\varphi_j(2M_2 - X_3) + \frac{\psi_2}{r} X_2 \frac{3X_3}{(2M_2 - X_3)^2} - \frac{\psi_3}{r} = 0 \tag{3.31}$$

for $r \in S_{pj}$, $j \in K_{p1}$.

Substituting (3.25) – (3.31) in the equation $\Delta J_* = 0$ and integrating the terms $\psi_i \delta X_i'$, $(i = 1, \ldots, 4)$ results in

$$2\sum_{j=1}^{n} a_j\big(h_{j-1} - h_j\big)\Delta a_j + \sum_{j \in K_e} Y\Big|_{a_{j+0}}^{a_{j+1}-0} + \\ + \sum_{j \in K_p} Y\Big|_{a_{j+0}}^{a_{j+1}-0} + Y\Big|_{a_k+}^{r_{k_0}-} + Y\Big|_{r_{k_0}+0}^{r_{k_1}-0} + Y\Big|_{r_{k_1}+0}^{a_{k+1}-0} = 0 \tag{3.32}$$

where

$$Y = \sum_{i=1}^{4} \psi_i \delta X_i. \tag{3.33}$$

The variations of state variables δX_i, $(i = 1, \ldots, 4)$ can be considered as independent quantities at $r = a$ and $r = R$. Accounting for the boundary conditions (3.2) and (3.4) one can present the transversality conditions as

$$\psi_2(a) = 0 \tag{3.34}$$

at $r = a$ and

$$\psi_2(R) = o, \qquad \psi_4(R) = 0 \tag{3.35}$$

at the outer edge of the plate.

At the intermediate points $r = a_j\,(j = 1, \ldots, n)$ and at $r = r_{k_0}$, $r = r_{k_1}$ the independent quantities are the total variations of state variables

$$\Delta X_i(a_j) = \delta X_i(a_j \pm 0) + X_i'(a_j \pm 0) \cdot \Delta a_j \tag{3.36}$$

and

$$\begin{aligned}\Delta X_i(r_{k_0}) &= \delta X_i(r_{k_0} \pm 0) + X_i'(r_{k_0} \pm 0)\Delta r_{k_0},\\ \Delta X_i(r_{k_1}) &= \delta X_i(r_{k_1} \pm 0) + X_i'(r_{k_1} \pm 0)\Delta r_{k_1},\end{aligned} \tag{3.37}$$

where $i = 1, 2, 3, 4$ and $j = 1, \ldots, n$.

It is worthwhile to remind that the sum of sets K_e and K_p e.g.

$$K = K_e \cup K_p$$

includes the integers $0, 1, \ldots, n$. Therefore, inserting (3.34), (3.35) in (3.32) on can present the equation (3.32) as

$$\begin{aligned}&\sum_{j=1}^{n}\left\{2a_j\big(h_{j-1} - h_j\big)\Delta a_j - \sum_{i=1}^{4}\Big[\psi_i(a_j)\delta X_i(a_j)\Big]\right\} -\\ &\qquad - \sum_{i=1}^{4}\left\{\Big[\psi_i(r_{k_0}\delta X_i(r_{k_0}))\Big] + \Big[\psi_i(r_{k_1})\delta X_i(r_{k_1})\Big]\right\} = 0\end{aligned} \tag{3.38}$$

Here the square brackets denote the jumps of corresponding variables, e.g.

$$\big[\psi(a)\delta X(a)\big] = \psi(a+0)\delta X(a+0) - \psi(a-0)\delta X(a-0) \tag{3.39}$$

Substituting the total variations ΔX_i defined according to (3.36), (3.37) in (3.38) and bearing in mind that ΔX_i are independent variations, one can easily conclude that

$$\psi_i(a_j + 0) = \psi_i(a_j - 0) \tag{3.40}$$

for $i = 1, \ldots, 4$; $j = 1, \ldots, n$. It infers from (3.32) also that

$$\begin{aligned}\psi_i(r_{k_0}) &= \psi_i(r_{k_0} - 0)\\ \psi_i(r_{k_1}) &= \psi_i(r_{k_1} + 0)\end{aligned} \tag{3.41}$$

where $i = 1, 2, 3, 4$.

Because of the continuity of adjoint variables required by (3.40), (3.41) the equation (3.38) takes the form

$$\begin{aligned}&\sum_{j=1}^{n}\left\{2a_j\big(h_{j-1} - h_j\big)\Delta a_j + \sum_{i=1}^{4}\psi_i(a_j)\Big[X_i'(a_j)\Big]\Delta a_j\right\} +\\ &\qquad + \sum_{i=1}^{4}\left\{\psi_i(r_{k_0})\Big[X_i'(r_{k_0})\Big]\Delta r_{k_0} + \psi_i(r_{k_1})\Big[X_i'(r_{k_1})\Big]\Delta r_{k_1}\right\} = 0\end{aligned} \tag{3.42}$$

Due to the arbitrariness of increments Δa_j it follows from (3.42)

$$\sum_{i=1}^{4}\psi_i(a_j)\Big[X_i'(a_j)\Big] + 2a_j(h_{j-1} - h_j) = 0 \tag{3.43}$$

for each $j = 1, \dots, n$.

In the similar way one obtains

$$\sum_{i=1}^{4} \psi_i(r_{k_j}) \left[X_i'(r_{k_j}) \right] = 0 \tag{3.44}$$

for $j = 0, 1$.

3.5 Numerical results and discussion

In order to solve the problem up to the end one has to integrate the systems of equations (3.5) and (3.10) separately in elastic and plastic regions. The integration is implemented numerically by making use of different computer codes. Before the integration one has to define the hoop moment M_2 from (3.9) and substitute it to the set (3.10) in each plastic region. After that one can determine the adjoint variables according to (3.25) and (3.26) for elastic and plastic regions, respectively. The design parameters will be determined from (3.28) and (3.43).

The results of calculations are presented for one-stepped plates in Fig. 3.2 – 3.11 and Tables 3.1 – 3.2.

The results obtained by FEM are compared with those calculated by the wavelet method.

In Fig. 3.2 the load-deflections curve are presented for annular plates loaded by the uniformly distributed transverse pressure. Fig. 3.2 – 3.8 correspond to the plate simply supported at the outer edge and absolutely free at the inner edge. In calculations the inner radius of the plate $a = 0.2R$. The load intensities corresponding to curves 1 – 5 are following: $p_1 = 1{,}98$, $p_2 = 2{,}81$, $p_3 = 3{,}02$, $p_4 = 2{,}31$, $p_5 = 3{,}21$.

Distributions of deflections w and radial bending moments for stepped annular plates with thicknesses h_0 and h_1 are depicted in Fig. 3.4 – 3.9. Here $a = 0.2R$ and the notation

$$\alpha_0 = \frac{a}{R}, \quad \alpha = \frac{a_1}{R}, \quad \gamma = \frac{h_1}{h_0} \tag{3.45}$$

is used. In Fig. 3.2 – Fig. 3.6 different curves correspond to different values of the transverse load intensity. Corresponding values of the load are presented above.

In Fig. 3.5 and Fig. 3.8 the deflection and radial moment are depicted for various values of the parameter α_1 and in Fig. 3.6 and Fig. 3.9 for different values of the ratio γ. Here different curves are associated with $\gamma = 0.4$; $\gamma = 0.5$; $\gamma = 0.6$; $\gamma = 0.7$ and $\gamma = 0.8$, respectively. It can be seen from Fig. 3.3 and Fig. 3.5 that when the load intensity increases then transverse deflections monotonically increase, as might be expected. Fig. 3.7 demonstrates the matter that similar relationship holds good between the moment and the loading, as well.

It can be seen from Fig. 3.7 – 3.9 that the radial bending moment is a continuous but not continuously differentiable function of the current radius r. The jumps of slopes of the derivative m_1' are admissible because at these points the thickness has

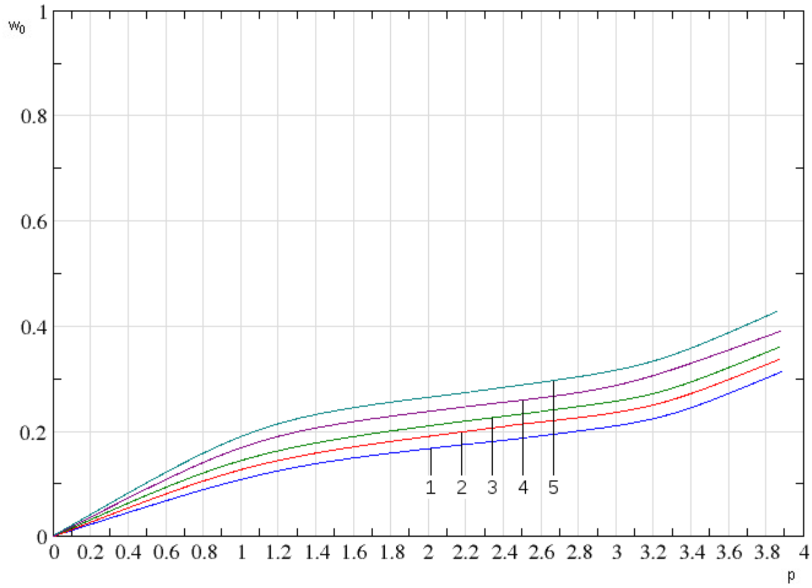

Fig. 3.2 Load-deflection relations for plates of constant thickness

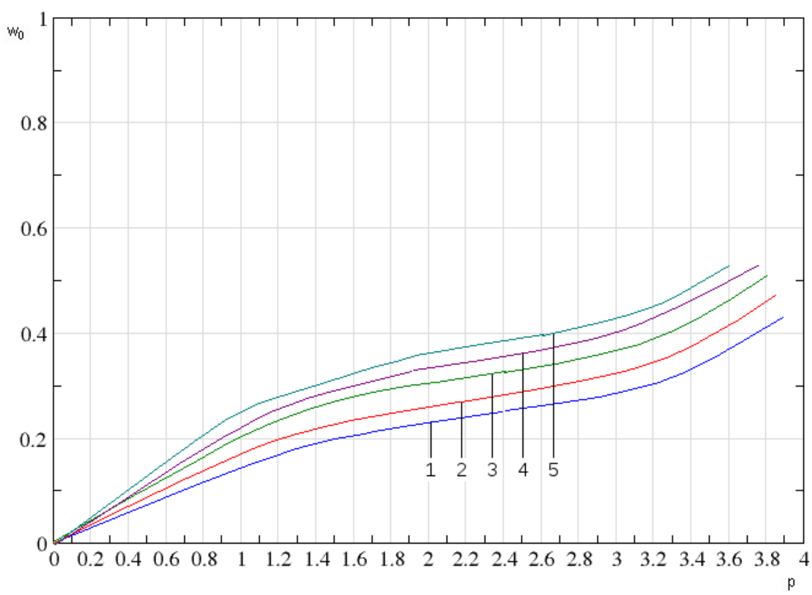

Fig. 3.3 Load-deflection relations of stepped plates

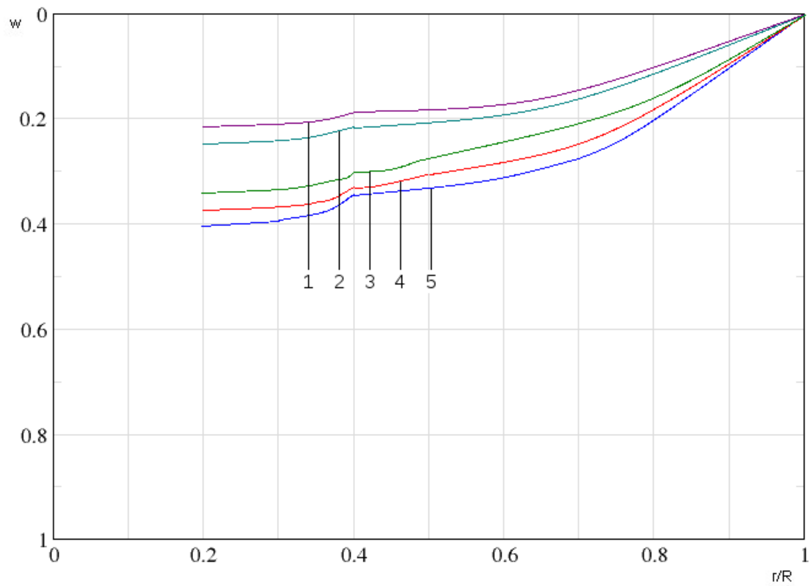

Fig. 3.4 Transverse deflections of the plate w

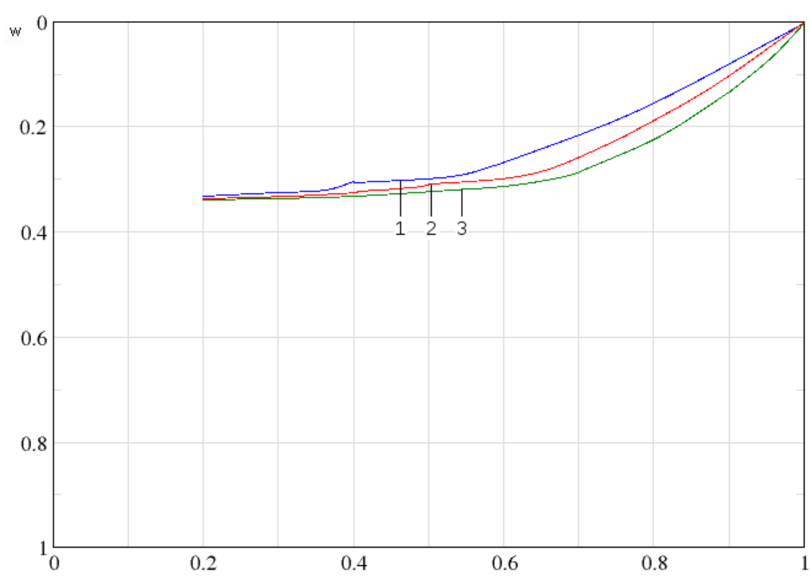

Fig. 3.5 Deflections of the plate for different step locations

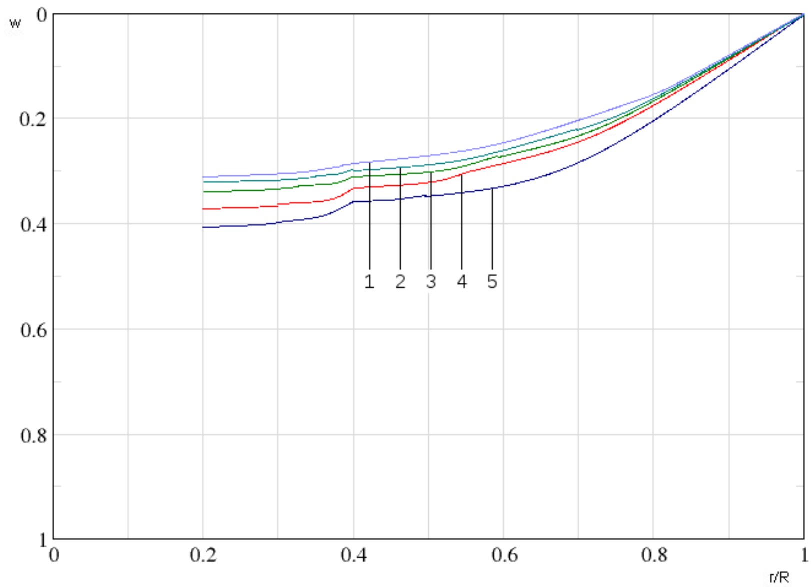

Fig. 3.6 Deflections of the plate w for different ticknesses

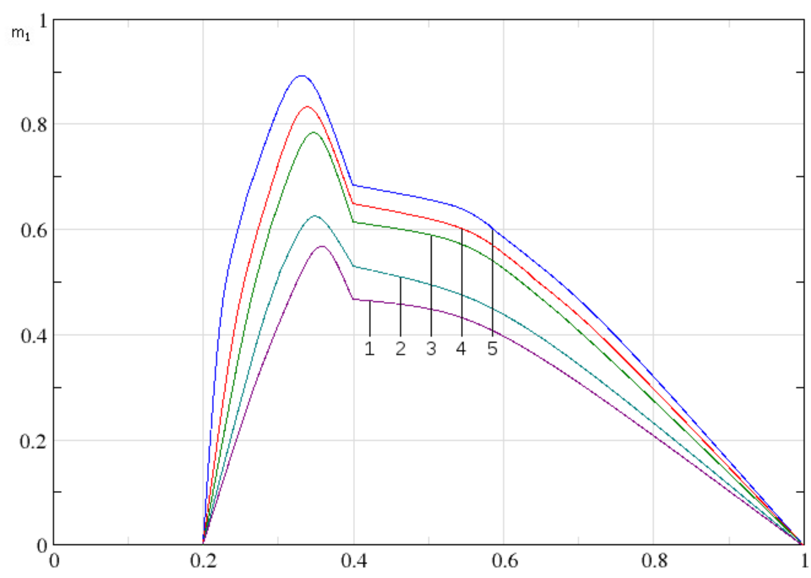

Fig. 3.7 Radial moment m_1

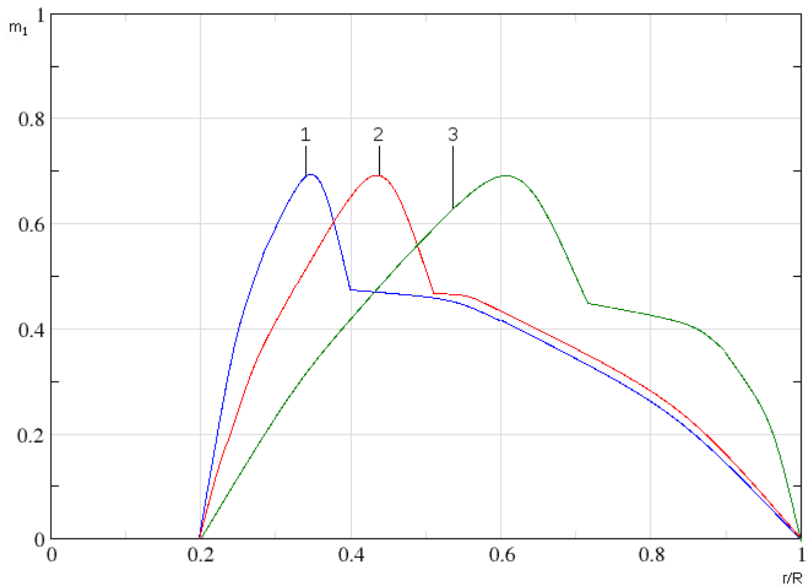

Fig. 3.8 Bending moment m_1 for various step locations

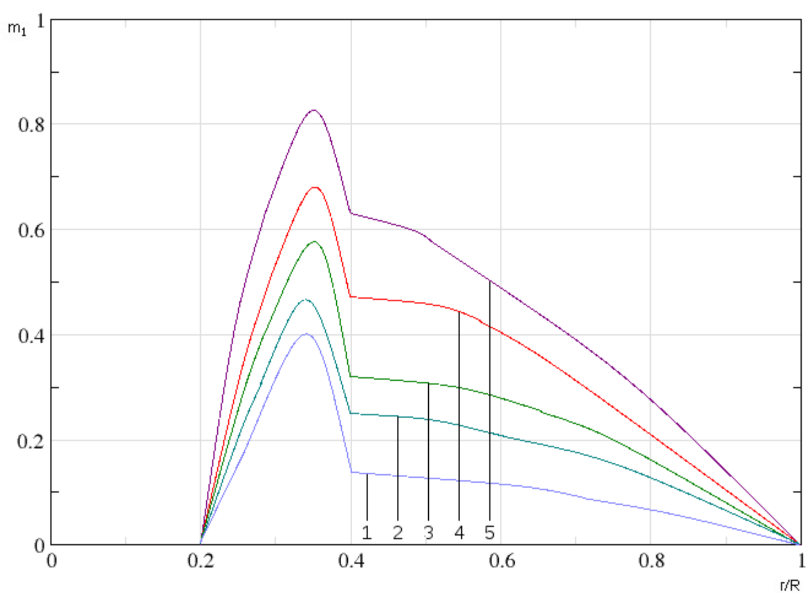

Fig. 3.9 Bending moment for different ratios of thicknesses

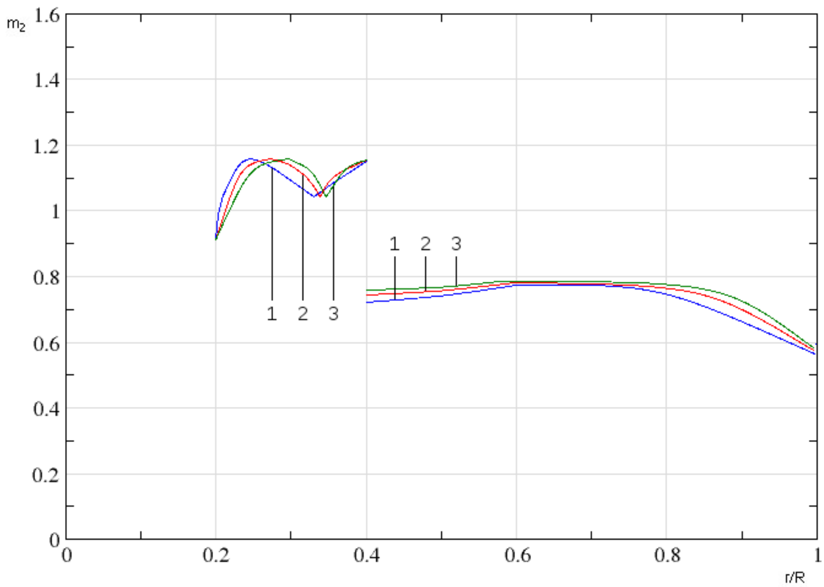

Fig. 3.10 Hoop moment

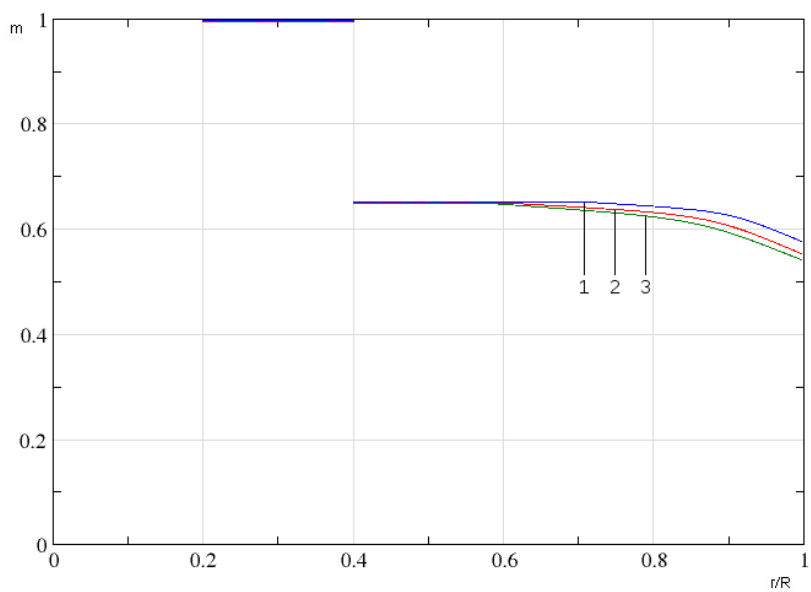

Fig. 3.11 The quantity *m*

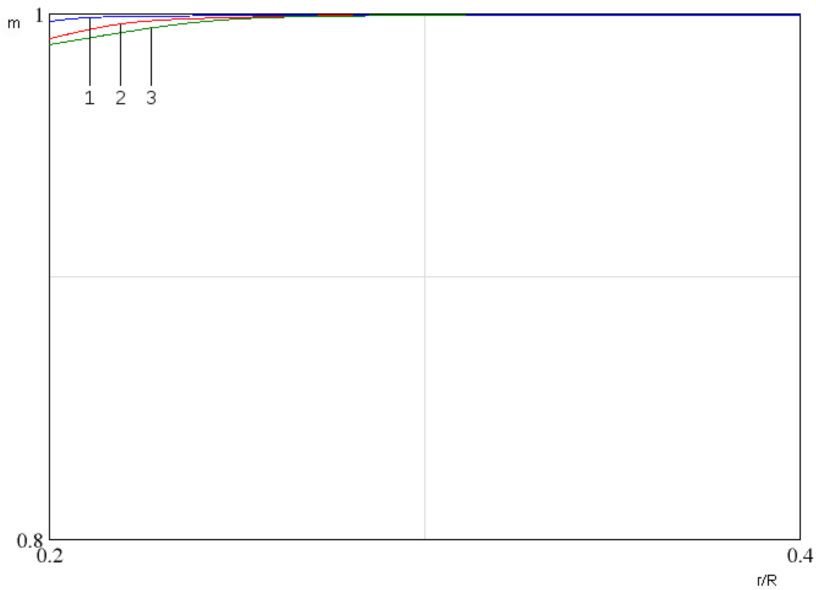

Fig. 3.12 Value m for the range $r/R \in [0.2, 0.4]$

jumps. On the other hand, it can be easily rechecked that the shear force defined by (3.3) preserves its continuity.

It is quite surprising that the radial bending moment m_1 is rather sensitive with respect to the step location (Fig. 3.8) whereas the changes of the step coordinate a_1 cause only slight changes in the distribution of deflections (Fig. 3.5). Similarly, looking at Fig. 3.9 and Fig. 3.6 one can see that the bending moment m_1 is strongly sensitive with respect to the ratio of thicknesses (Fig. 3.9) but the deflection is less sensitive with respect to this ratio.

In order to check whether the obtained solution is statically admissible one has to check if the stresses in elastic regions remain inside the yield surface in the stress space. This means that inequalities (3.7) must be fulfilled as strict inequalities in elastic regions. For this purpose let us introduce a function

$$m = m_1^2 - m_1 m_2 + m_2^2 \tag{3.46}$$

where

$$m_1 = \frac{M_1}{\sigma_0 H h_*}, \quad m_2 = \frac{M_2}{\sigma_0 H h_*}, \tag{3.47}$$

where h_* is the thickness of layers of a reference plate of constant thickness.

Comparing (3.7) and (3.46) one can state that (3.7) is satisfied if $m \leqslant \gamma_j^2$ where $\gamma_j = h_j/h_*$. In calculations one has taken $h_* = h_0$.

Results of calculation showed that the solutions established above are statically admissible.

Table 3.1 Optimal parameters for $w_0 = 0.25$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.8241	0.8002	0.7801	0.7201	0.6144	0.5891
η_1	0.2917	0.3201	0.3271	0.3406	0.4211	0.4309
η_2	0.3289	0.4238	0.3055	0.6320	0.7298	0.8044
α	0.3071	0.3612	0.4072	0.5106	0.6091	0.6675
e	12,74%	14.32%	17.92%	19.21%	18.01%	17.31%

Table 3.2 Optimal parameters for $w_0 = 0.30$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.7140	0.7013	0.7301	0.7814	0.6215	0.5913
η_1	0.4139	0.4031	0.3783	0.4126	0.3817	0.4427
η_2	0.5721	0.5791	0.5604	0.7893	0.8314	0.8204
α	0.4715	0.5003	0.4173	0.6120	0.5169	0.6689
e	14,32%	09.81%	15.39%	17.96%	16.92%	18.02%

The distribution of the hoop moment m_2 and the quantity m are depicted in Fig. 3.10 – 3.12. It can be seen from Fig. 3.10 – 3.12 that the moments m_2 and m are discontinuous, as might be expected. One can see from Fig. 3.11 that the inequality $m < \gamma^2$ (here $\gamma = 0.8$) is satisfied in the elastic regions. However, $m = \gamma^2$ in the plastic region as it was assumed above. The variable m is almost constant in the internal region of the plate (Fig. 3.11). However, studying the situation more exactly one can see that the quantity m is slightly different from the unity in the internal region for $r \leqslant 0.4R$ (Fig. 3.12).

The values of the design parameters α and γ are accommodated in Tables 3.1, 3.2 for plates with a unique step of the thickness for different values of the load intensity. The data presented in Tables 3.1, 3.2 correspond to the plate with the internal radius $a_0 = 0.2R$.

The parameters η_1 and η_2 in Tables 3.1, 3.2 indicate the internal and external radii of the plastic zone. It means that plastic deformations take place for $r \in (\eta_1 R, \eta_2 R)$ whereas the remained parts of the plate are elastic.

It can be seen from Tables 3.1, 3.2 that the plastic zone spreads when transverse pressure increases. It is interesting to note that the coordinates η_1 and η_2 do not increase monotonically with the load intensity.

In order to assess the effectivity of the design the coefficient

$$e = \frac{V}{V_*}$$

is introduced. Here V is the optimal value of the volume defined by (3.1) and V_* stands for the sheet volume of the reference plate of constant thickness h_*.

The values of the coefficient of effectivity are presented in the last rows of Tables 3.1, 3.2. It can be seen from Tables 3.1, 3.2 that the material saving depends essentially on the loading level and also on the admissible deflection at the free edge of the plate. If, for instance, $w_0 = 0.3$ and $p = 2.8$ one can save more than 14% of the material when using the design of one-stepped plates.

3.6 Conclusions

An analytical-numerical study of annular plates operating in the range of elastic plastic deformations was undertaken. The material of plates was assumed to be an ideal elastic plastic material obeying the Mises yield condition. The author succeeded in the analytical derivation of optimality conditions for this highly non-linear problem. The obtained systems of equations were solved by existing computer codes.

The results of calculations have shown that in the case of a minimum weight problem it is possible to establish designs with remarkably smaller material consumption than that of a plate with constant thickness. The material saving depends on the displacement level. For instance, in the case if $a = 0.2R$, $w = 0.25$ and $p = 3.4$ one can save 19,21% of material when using the design with a unique step of the thickness. However, if $p = 2.8$ then the eventual saving is 12,74%. The plotted results have revealed the matter that radial bending moment is continuous but the hoop moment has jumps at the cross sections associated with the steps of the thickness.

Chapter 4
Optimization of anisotropic plates

4.1 Introduction

Circular and annular plates are of practical interest in mechanical, civil and ocean engineering where these plates are used as bulkheads of separable sections of submersibles. A purely elastic design of a structural element made of a ductile material and loaded by pressures of high intensity may be overly conservative. Thus, it is reasonable to account for the elastic plastic stages of deformation in direct problems of determination of the stress strain state of plates as well as in the shape optimization of plates.

In the present section a method of optimization of elastic plastic annular plates is developed. The plates are made of anisotropic materials which obey the yield criteria of Hill and Tsai-Wu. It is assumed that the plates have sandwich cross section whereas the carrying layers are of piece wise constant thickness.

4.2 The cost criterion and main assumptions

The aim of the present investigation is to study the behaviour of elastic plastic circular plates made of composite materials and to establish the optimal design for plates of piece wise constant thickness (Fig. 4.1). It is recognized that the behaviour of structures made of unidirectionally reinforced composites can be prescribed as the response of anisotropic structures. Further, it is assumed that the material of the plate is an anisotropic quasi-homogeneous material obeying the Hill's or Tsai-Wu yield condition. It is assumed that the thickness (Fig. 4.1)

$$h = \begin{cases} h_0, & r \in (a_0, a), \\ h_1, & r \in (a, R), \end{cases} \tag{4.1}$$

where R is the radius of the plate and r – current radius. The quantities h_0, h_1, a are treated as design variables. Note that in the case of sandwich plates the quantity h

is considered as the thickness of carrying layers whereas the thickness of the core material H is assumed to be constant.

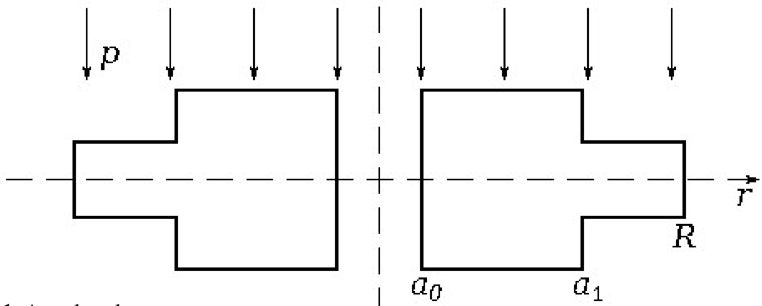

Fig. 4.1 Annular plate

Although in the literature one can find a lot of different formulations of the problems of optimization one of the most important problem of this kind is the minimum weight problem. An exhaustive list of papers can be found in books by Banichuk [3], Kirsch [61], Logo [88], Rozvany [124], Gajewski, Życkowski [28] and others. Discrete material optimization of vibrating plates made of laminated and composite materials was undertaken by several authors. In the paper by Niu, Olhoff, Lund, Cheng [112] the material optimization of plates is accomplished in order to obtain the minimum sound radiation.

In the present study, however, we are looking for minimum weight designs of composite plates. In the case of a single step of the thickness the cost criterion can be presented as

$$J = h_0\left(a^2 - a_0^2\right) + h_1\left(R^2 - a^2\right). \tag{4.2}$$

When minimizing the cost function (4.2) one has to take into account the governing equations of axisymmetric plates. Moreover, the deflection of the plate must be constrained. This constraint can be transformed into the boundary condition

$$W(a_0) = W_0 \tag{4.3}$$

where W is the transverse deflection and W_0 – a given number.

It is assumed that the material of the plate is a unidirectionally reinforced composite which can be treated as an anisotropic quasi homogeneous material. During the bending of the plate caused by the transverse pressure of intensity $P(r)$ the elastic and elastic plastic stages of deformation will take place. We assume that in the plastic stage the material obeys the Tsai-Wu yield criterion (see Daniel, Ishai, [19]).

4.3 Yield criteria for anisotropic materials

It is well-known that the failure and yielding of homogeneous isotropic bodies can be prescribed by the theories of maximal strains or maximal values of tangential stresses. The yield conditions corresponding to these theories are called Mises and Tresca yield conditions. As regards the failure criteria and yield conditions for composite materials compared to those for homogeneous and isotropic bodies it is not so clear what the actual and unique criterion for a particular problem is. One can find a series different criteria suggested by different authors. Probably the most familar to the engineering and scientific community are the criteria developed by Hill [38], Lance and Robinson [72], Barlat at al [5], Tsai and Wu [135]. A review of existing criteria at this time was presented by Burk [11].

In the present study our analysis is resorting to the criteria of Hill and Tsai and Wu. The criterion of Hill is one of the earliest failure and yield criteria for anisotropic materials. It presents a generalization of the isotropic yield behaviour of ductile metals for large strains. Long ago, it was recognized by practicioners and scientists that in a rolling process the metal grains tend to align and thus a self induced anisotropy occurs.

Guided by these considerations, Hill has formulated, an interactive yield criterion which can be presented for a three-dimensional stress state as (Vinson and Sierakowski, [146])

$$\begin{aligned} A_1\left(\sigma_{11}-\sigma_{22}\right)^2+A_2\left(\sigma_{22}-\sigma_{33}\right)^2+A_3\left(\sigma_{33}-\sigma_{11}\right)^2+ \\ +2A_{12}\sigma_{12}^2+2A_{13}\sigma_{13}^2+2A_{23}\sigma_{23}^2=1 \end{aligned} \tag{4.4}$$

where σ_{ij} are the stress components and $A_1, A_2, A_3, A_{12}, A_{13}, A_{23}$ stand for numbers which can be defined on the basis of appropriate experimental data. In the case of unidirectionally reinforced composites the criterion (4.4) simplifies taking the form

$$\begin{aligned} A_1\left[\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{33}-\sigma_{11}\right)^2\right]+A_2\left(\sigma_{22}-\sigma_{33}\right)^2+ \\ +2A_{12}\left(\sigma_{12}^2+\sigma_{13}^2\right)+2A_{23}\sigma_{23}^2=1. \end{aligned} \tag{4.5}$$

In many practically important situations the stress state can be treated as a plane stress state with $\sigma_{33}=0$, $\sigma_{13}=0$, $\sigma_{23}=0$. For the plane state the Hill's criterion takes the form (Fig. 4.2)

$$A_1\left(\sigma_{11}-\sigma_{22}\right)^2+A_2\sigma_{22}^2+A_3\sigma_{11}^2+2A_{12}\sigma_{12}^2=1. \tag{4.6}$$

Further simplifications of (4.6) can be introduced for transversely isotropic materials. In this case (4.6) can be rewritten as

$$\left(\frac{\sigma_{11}}{X}\right)^2+\left(\frac{\sigma_{22}}{Y}\right)^2-\left(\frac{\sigma_{11}\sigma_{22}}{XX}\right)^2+\left(\frac{\sigma_{12}}{S}\right)^2=1. \tag{4.7}$$

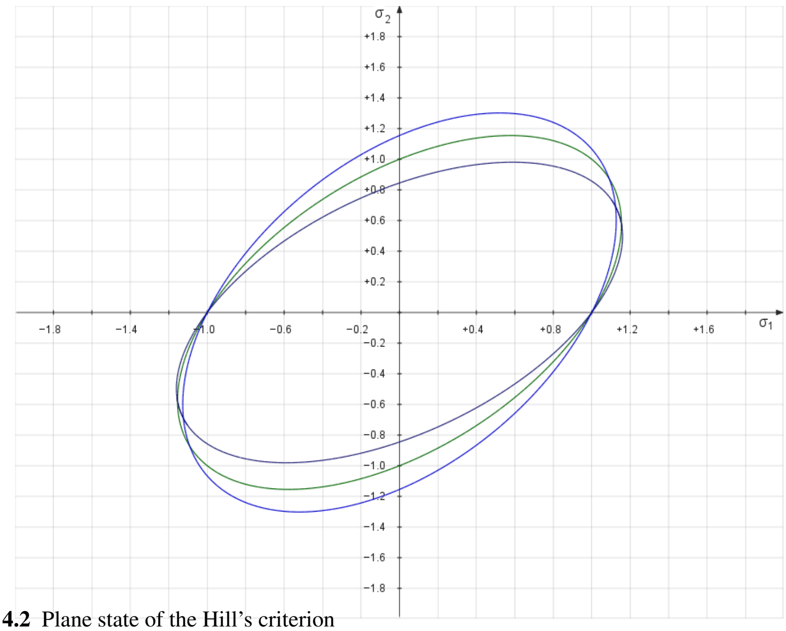

Fig. 4.2 Plane state of the Hill's criterion

In (4.7) the quantities X and Y can be interpreted as yield stresses in directions of coordinate axes and S is the limit value of the tangential stress component.

Somewhat later Tsai and Wu [135] developed a tensor polynomial theory with the failure surface (Daniel, Ishai, [19])

$$\sum_{i=1}^{6}\left(B_i\sigma_{ii}+\sum_{j=1}^{6}B_{ij}\sigma_{ii}\sigma_{jj}\right)=1 \tag{4.8}$$

where B_i and B_{ij} stand for certain numerical coefficients. For the plane state stress the criterion (4.8) can be expressed as

$$\begin{aligned} B_1\sigma_{11}+B_2\sigma_{22}+B_6\tau_6+B_{11}\sigma_{11}^2+B_{22}\sigma_{22}^2+B_{66}\tau_6^2+ \\ +2B_{12}\sigma_{11}\sigma_{22}+2B_{16}\sigma_{11}\tau_6+2B_{26}\sigma_{22}\tau_6=1. \end{aligned} \tag{4.9}$$

where

$$\tau_6=\sigma_{66}. \tag{4.10}$$

4.4 Governing equations for anisotropic plates

It was assumed above that the composite material of the plate is an elastic plastic material. Thus, if the pressure loading is high, then in the plate the regions of elastic and plastic deformations occur. In the elastic regions the Hooke's law holds good and in plastic regions the stress profile reaches on the yield surface. However, in the regions of both type the equilibrium conditions must be satisfied. The equilibrium equations of a plate element are presented by (2.6), provided the stress-strain state remains axisymmetric. As above, M_1, M_2 denote the bending moments in the radial and hoop directions, respectively, whereas Q is the shear force and P – the intensity of the distributed transverse loading.

Let us denote by S_e and S_p the regions of elastic and plastic deformations, respectively. Employing the concept of a sandwich plate the whole plate is subdivided into elastic and plastic regions. However, in the case of solid plates the regions of elastic and elastic plastic deformations must be studied separately.

It was shown in the previous works (see Lellep, Vlassov, [84]) that the governing equations have the form (3.5) in elastic regions for $r \in (a_j, a_{j+1})$; $j = 0; 1$. At the same time

$$M_2 = \nu M_1 + \frac{1}{r} Z\left(\nu^2 - 1\right) D_j \tag{4.11}$$

and

$$D_j = \frac{EH^2 h_j}{2(1-\nu^2)}. \tag{4.12}$$

In (4.11), (4.12) Z stands for an auxiliary variable; ν is the Poisson modulus and E stands for the Young modulus. It is worthwhile to mention that for $r \in (a_j, a_{j+1})$ $h = h_j$ and $a_2 = R$, $a_1 = a$.

Note that the differential equations (3.5) can be integrated without paying any attention to the equation (4.11). The latter serves for determination of the hoop moment M_2. The boundary conditions for (3.5) are

$$M_1(a_0) = 0; \quad Q(a_0) = 0; \quad M_1(R) = 0 \tag{4.13}$$

if the plate is simply supported at the outer edge and free at the inner edge.

In the plastic regions for $r \in S_p$ the stress state of the plate corresponds to a point lying on the yield surface. It is assumed herein that the yield surface can be presented as

$$\Phi_j(M_1, M_2, h_j) = 0 \tag{4.14}$$

for $r \in (a_j, a_{j+1})$ for $j = 0$ and $j = 1$.

According to the associated flow law the vector of strain rates must be directed towards the outward normal to the yield surface (see Chakrabarty, [14]; Kaliszky, [54]; Jones, [48]). Thus, one has

$$\begin{aligned} \dot{K}_1 &= \lambda_j \frac{\partial \Phi_j}{\partial M_1}, \\ \dot{K}_2 &= \lambda_j \frac{\partial \Phi_j}{\partial M_2} \end{aligned} \tag{4.15}$$

for $r \in (a_j, a_{j+1})$ and $r \in S_p$. In (4.15) λ_j stands for a non-negative scalar multiplier and $\dot{K}_1$, $\dot{K}_2$ denote the strain rate components. In the present paper a deformation-type theory of plasticity is used. Thus, one can replace the strain rates by strain components whereas

$$K_1 = -\frac{d^2W}{dr^2}, \qquad K_2 = -\frac{1}{r}\frac{dW}{dr}. \tag{4.16}$$

Omitting the derivatives with respect to time or a time-like parameter and eliminating λ_j in (4.15) leads to the relations

$$\frac{d^2W}{dr^2} = \frac{Z}{r}\cdot\frac{\dfrac{\partial\Phi_j}{\partial M_1}}{\dfrac{\partial\Phi_j}{\partial M_2}} \tag{4.17}$$

where Z is defined earlier by (3.5).

The governing equations for elastic plastic regions are presented by (2.6), (4.13) – (4.17) Equation (4.15) – (4.17) together with the equilibrium equations result in

$$\begin{aligned}
&\frac{dW}{dr} = Z,\\
&\frac{dZ}{dr} = \frac{Z}{r}\frac{\partial\Phi_j}{\partial M_1}\left(\frac{\partial\Phi_j}{\partial M_2}\right)^{-1},\\
&\frac{dM_1}{dr} = \frac{M_2 - M_1}{r} + Q,\\
&\frac{dQ}{dr} = -\frac{Q}{r} - p
\end{aligned} \tag{4.18}$$

for $r \in S_p$. Note that (4.18) presents the system of governing equations for the plastic region S_p. The system (4.18) is to be integrated together with the yield condition $\Phi_j = 0$ in each plastic region for $r \in S_{pj}$. Here S_{pj} stands for the interval (a_j, a_{j+1}) where the stress state corresponds to a plastic state.

It is well-known in the theory of plasticity that the yield surface is a closed convex surface whereas in elastic regions the inequalities $\Phi_j < 0$ hold good. Introducing new variables Θ_j one can present the inequalities $\Phi_j \leqslant 0$ as equalities

$$\Phi_j + \Theta_j^2 = 0 \tag{4.19}$$

which hold good for each $r \in [a_0, R]$.

4.5 Optimality conditions

The problem posed above is considered as a particular problem of optimization with constraints. In order to get the conditions of optimality let us introduce the extended functional

$$
\begin{aligned}
J_* = J + \int\limits_{S_e} \Big\{ & \psi_1\left(\frac{dW}{dr} - Z\right) + \psi_2\left(\frac{dZ}{dr} + \frac{M_1}{D_j} + \frac{\nu}{r}Z\right) + \\
& + \psi_3\left(\frac{dM_1}{dr} + (1-\nu^2)D_j\frac{Z}{r^2} + \frac{1}{r}(1-\nu)M_1 - Q\right) + \\
& + \psi_4\left(\frac{dQ}{dr} + \frac{Q}{r} + P\right) + \nu_j\left(M_2 - \nu M_1 - \frac{1}{r}Z(\nu^2-1)D_j\right)\Big\}dr + \\
& + \int\limits_{S_p} \Big\{ \psi_1\left(\frac{dW}{dr} - Z\right) + \psi_2\left(\frac{dZ}{dr} - \frac{Z}{r}\frac{\partial \Phi_j}{\partial M_1}\left(\frac{\partial \Phi_j}{\partial M_2}\right)^{-1}\right) + \\
& + \psi_3\left(\frac{dM_1}{dr} - \frac{1}{r}(M_2 - M_1) - Q\right) + \psi_4\left(\frac{dQ}{dr} + \frac{Q}{r} + P\right)\Big\}dr + \\
& + \int\limits_{a_0}^{R} \varphi_0(\Phi_0 + \Theta_0^2)dr + \int\limits_{a_0}^{R} \varphi_1(\Phi_1 + \Theta_1^2)dr.
\end{aligned}
\tag{4.20}
$$

In (4.20) ψ_1 – ψ_4 stand for adjoint (conjugate) variables and ν_j, φ_0, φ_1 are unknown Lagrange multipliers. Here S_e and S_p are the elastic and plastic region, respectively. It is reasonable to assume that the plastic region is located away from the edges of the plate. Thus $S_p = (\eta_0, \eta_1)$ and $S_e = [a_0, \eta_0] \cup [\eta_1, R]$.

Calculating the total variation of (4.20) and equalizing ΔJ_* to zero leads to the set of optimality conditions. Resorting to the technique used in the previous sections and also previous papers by the authors ([82], [84]), one obtains the adjoint equations in the elastic region for $r \in S_{ej}$

$$
\begin{aligned}
\frac{d\psi_1}{dr} &= 0, \\
\frac{d\psi_2}{dr} &= -\psi_1 + \frac{\nu}{r}\psi_2 + (1-\nu^2)D_j\frac{\psi_3}{r^2} - \frac{\nu_j}{r}(\nu^2-1)D_j, \\
\frac{d\psi_3}{dr} &= \frac{\psi_2}{D_j} + (1-\nu)\frac{\psi_3}{r} - \nu\nu_j + \varphi_j\frac{\partial \Phi_j}{\partial M_1}, \\
\frac{d\psi_4}{dr} &= -\frac{\psi_4}{r} - \psi_3.
\end{aligned}
\tag{4.21}
$$

In (4.21) one has to take $j = 0$, if $r \in [a_0, a)$ and $j = 1$, if $r \in (a, R]$ as the elastic deformations occur in two separate regions.

In plastic region for $r \in S_p$ one has

$$
\begin{aligned}
&\frac{d\psi_1}{dr} = 0,\\
&\frac{d\psi_2}{dr} = -\psi_1 - \frac{\psi_2}{r}\frac{\partial \Phi_j}{\partial M_1}\left(\frac{\partial \Phi_j}{\partial M_2}\right)^{-1},\\
&\frac{d\psi_3}{dr} = -\frac{\psi_2}{r}Z\frac{\partial}{\partial M_2}\left[\frac{\partial \Phi_j}{\partial M_1}\left(\frac{\partial \Phi_j}{\partial M_2}\right)^{-1}\right] + \frac{\psi_3}{r} + \varphi_j\frac{\partial \Phi_j}{\partial M_1},\\
&\frac{d\psi_4}{dr} = -\psi_3 + \frac{\psi_4}{r}.
\end{aligned}
\tag{4.22}
$$

Due to the arbitrariness of variations $\delta\Theta_j$

$$\varphi_j\Theta_j = 0$$

for $j = 0$ and $j = 1$. Similarly one can stipulate $\nu_j = 0$ for $r \in S_e$. However, for $r \in S_p$ one has

$$-\frac{\psi_3}{r} - \frac{\psi_2}{r}Z\frac{\partial}{\partial M_2}\left[\frac{\partial \Phi_j}{\partial M_1}\left(\frac{\partial \Phi_j}{\partial M_2}\right)^{-1}\right] + \varphi_j\frac{\partial \Phi_j}{\partial M_2} = 0. \tag{4.23}$$

Since the variations of thicknesses Δh_0, Δh_1 are arbitrary the equation $\Delta J_* = 0$ results in

$$
\begin{aligned}
&a^2 - a_0^2 + \int_{a_0}^{\eta_0}\left\{-M_1\cdot\frac{\psi_2}{D_0^2} + \frac{\psi_3}{r^2}(1-\nu^2)Z - \frac{\nu_0}{r}(\nu^2-1)Z\right\}\frac{\partial D_0}{\partial h_0}dr +\\
&\qquad + \int_{\eta_0}^{a}\left\{-\frac{\psi_2}{r}Z\frac{\partial}{\partial h_0}\left[\frac{\partial \Phi_0}{\partial M_1}\left(\frac{\partial \Phi_0}{\partial M_2}\right)^{-1}\right] + \varphi_0\frac{\partial \Phi_0}{\partial h_0}\right\}dr = 0
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
&R^2 - a^2 + \int_{a}^{\eta_1}\left\{-\frac{\psi_2}{r}Z\left[\frac{\partial}{\partial h_1}\frac{\partial \Phi_1}{\partial M_1}\left(\frac{\partial \Phi_1}{\partial M_2}\right)^{-1}\right] + \varphi_1\frac{\partial \Phi_1}{\partial h_1}\right\}dr +\\
&\qquad + \int_{\eta_1}^{R}\left\{-M_1\frac{\psi_2}{D_1^2} + \frac{\psi_3}{r^2}(1-\nu^2)Z - \frac{\nu_1}{r}(\nu^2-1)Z\right\}\frac{\partial D_1}{\partial h_1}dr = 0
\end{aligned}
\tag{4.25}
$$

Note that the rest of requirements which yield from the equation $\Delta J_* = 0$ are similar to those obtained in the previous sections and in the papers by Lellep, Vlassov [82], [84].

4.6 Final results and discussion

The direct problem of determination of the stress strain state of the plate is solved numericallly with the finite element method. For solution of optimization problems the finite elements are combined with the method of Haar wavelets.

Table 4.1 Optimal parameters for $w_0 = 0.25$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.8492	0.8342	0.8014	0.7892	0.7513	0.7239
η_0	0.3977	0.4116	0.4385	0.4609	0.4863	0.5204
η_1	0.4591	0.4817	0.5247	0.6102	0.6319	0.6404
α	0.4069	0.4215	0.4879	0.5267	0.5862	0.6031
e	9.27%	8.14%	10.17%	12.41%	14.23%	13.07%

Table 4.2 Optimal parameters for $w_0 = 0.30$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.8502	0.8402	0.8211	0.8109	0.7899	0.7876
η_0	0.4094	0.4117	0.4291	0.5231	0.5148	0.5334
η_1	0.4582	0.4826	0.5290	0.6152	0.6411	0.6482
α	0.3819	0.4017	0.4310	0.4719	0.5481	0.6217
e	10.78%	12.03%	14.91%	10.52%	06.89%	11.72%

The results of calculations are presented in Fig. 4.3 – 4.10 and Tables 4.1 – 4.2 for plates with two different thicknesses. The results corresponding to $n = 200$ are depicted in Fig. 4.11 – 4.14 [86]. It is reasonable to use the following notation

$$\alpha_0 = \frac{a_0}{R}, \quad \alpha = \frac{a_1}{R}, \quad \rho = \frac{r}{R},$$
$$\gamma = \frac{h_1}{h_0}, \quad m_{1,2} = \frac{M_{1,2}}{M_0}, \quad w_0 = \frac{W_0}{H}$$

where M_0 is the yield moment of the matrix material.

The load deflection relations of plates with constant thicknesses are presented in Fig. 4.3. Here $a_0 = 0.2R$ and different curves correspond to different values of the plate thickness.

The sensitivity of transverse deflections with respect to the intensity of the pressure loading and to the step is portrayed in Fig. 4.4 and Fig. 4.5, respectively. Different curves in Fig. 4.4 correspond to the load intensities $p_1 = 2.8$; $p_2 = 2.9$; $p_3 = 3.0$; $p_4 = 3.1$ and $p_5 = 3.2$. Fig. 4.5 corresponds to the load intensity $p = 3.09$. Different curves in Fig. 4.5 correspond to the step location at $\alpha = 0.4$ and to the ratios of thicknesses $\gamma = 0.60$; $\gamma = 0.65$; $\gamma = 0.70$; $\gamma = 0.75$; $\gamma = 0.80$ whereas the upper

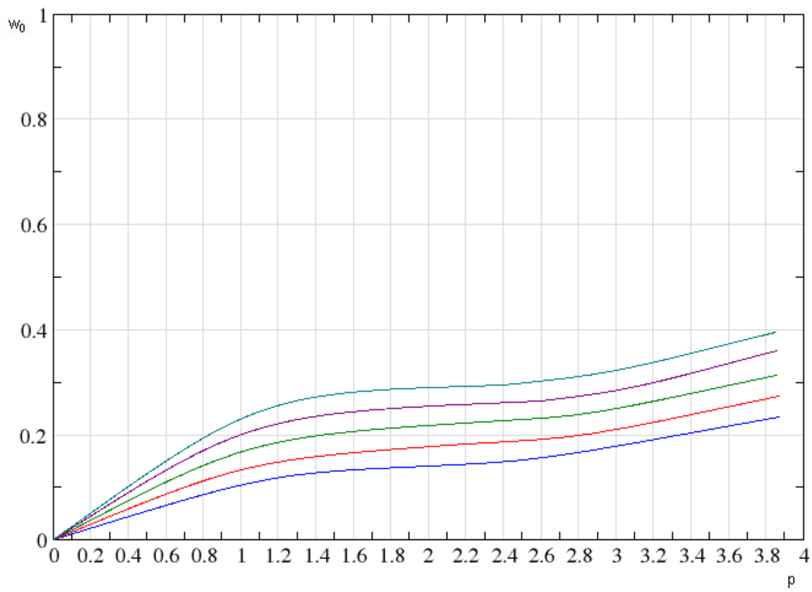

Fig. 4.3 Load-deflection relations

curve is obtained for $h_1 = 0.8h_0$. It can be seen from Fig. 4.5 that the deflection varies monotonically with the thicknesses ratio, as might be expected.

The distributions of the radial bending moment m_1 are presented in Fig. 4.6; 4.7 for different values of the transverse pressure (Fig. 4.6) and for different positions of the step. In both cases $\alpha_0 = 0.2$. Different curves in Fig. 4.6 are associated with the same values of the load as in Fig. 4.4. The curves of Fig. 4.6 are calculated for the case $\alpha = 0.4$ and $\gamma = 0.8$. However, Fig. 4.7 is obtained for the same values of the ratio of thicknesses as Fig. 4.5. Here the step is located at $a_1 = 0.4R$. Calculations carried out showed that the radial bending moment has slope discontinuities at the radius where the thickness has discontinuities. It is somewhat surprising that the moment distribution is strongly unsymmetrical with respect to the centre of the interval (a_0, R).

Distributions of the hoop moment are portrayed in Fig. 4.8 and Fig. 4.9. It can be seen from Fig. 4.9 that the hoop moment m_2 has discontinuities at the cross-section associated with the step location and that in the elastic regions the stress components do not exceed the limit value.

The values of the coefficient of efficiency $e = V/V_*$ are accommodated for different values of the load intensity in Tables 4.1, 4.2. Tables 4.1, 4.2 correspond to the annular plate with the internal radius $a_0 = 0.2R$. Parameters α and γ present the optimal values of design parameters and η_0, η_1 stand for the internal and external radii of the plastic region. Tables 4.1, 4.2 reveal the matter that the step can be located either in the elastic or inside the plastic region. It can be seen from Tables 4.1, 4.2

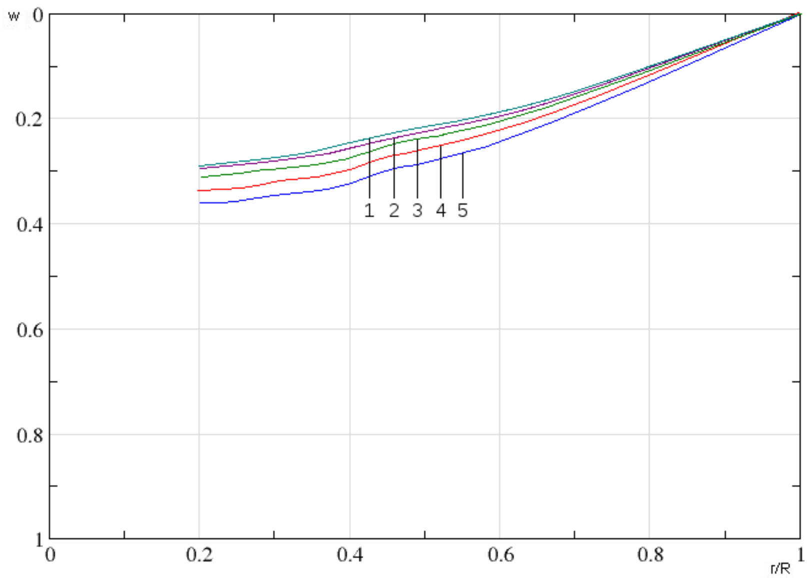

Fig. 4.4 Deflection of the plate for different loadings

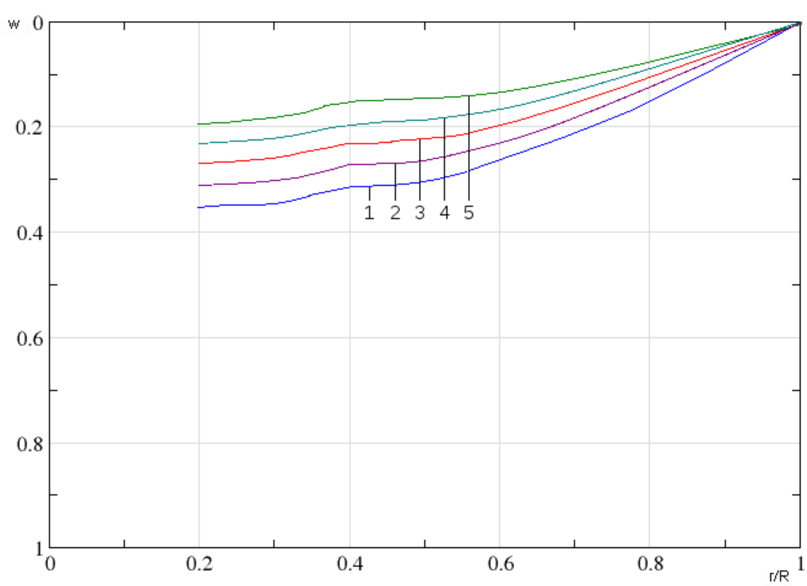

Fig. 4.5 Deflection of the plates with different thicknesses

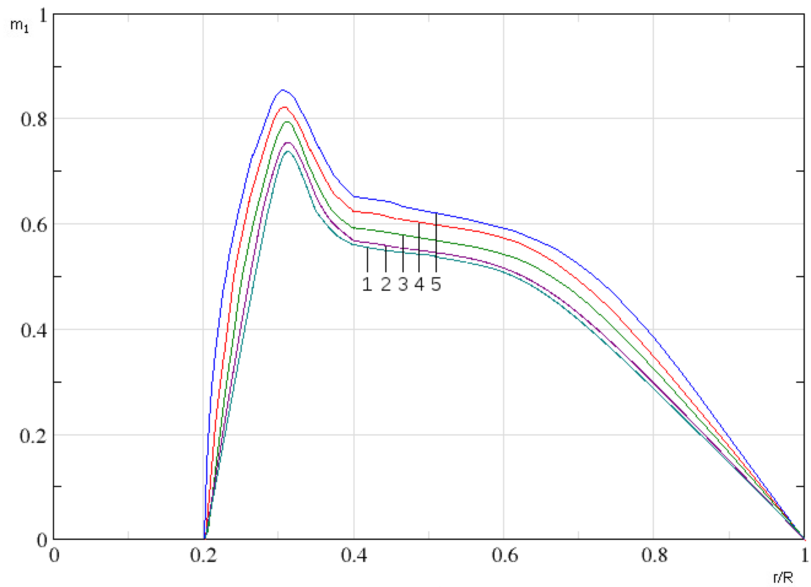

Fig. 4.6 Distributions of the radial bending moment

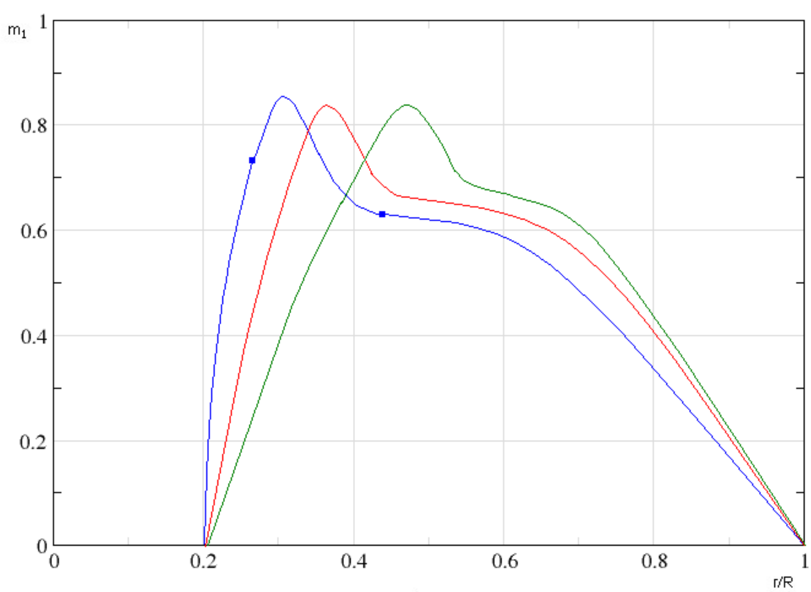

Fig. 4.7 Bending moment m_1 in the cases of different ratios of thicknesses

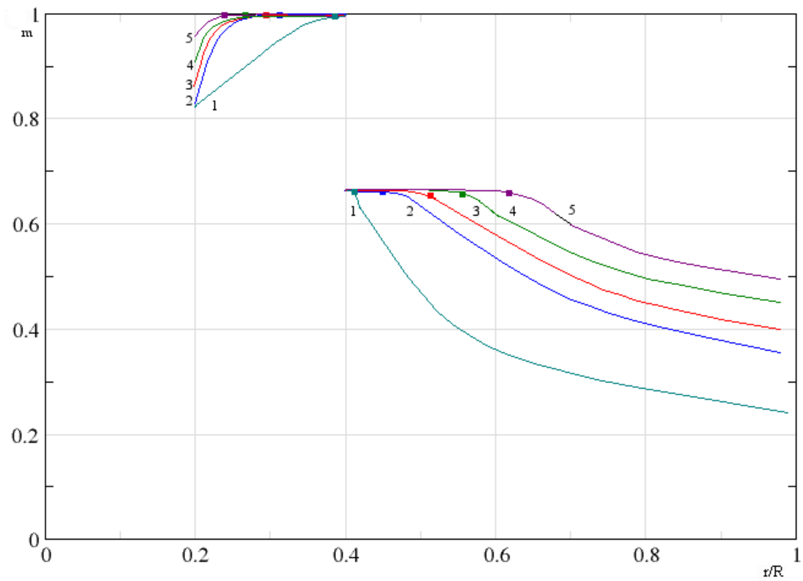

Fig. 4.8 The quantity *m*

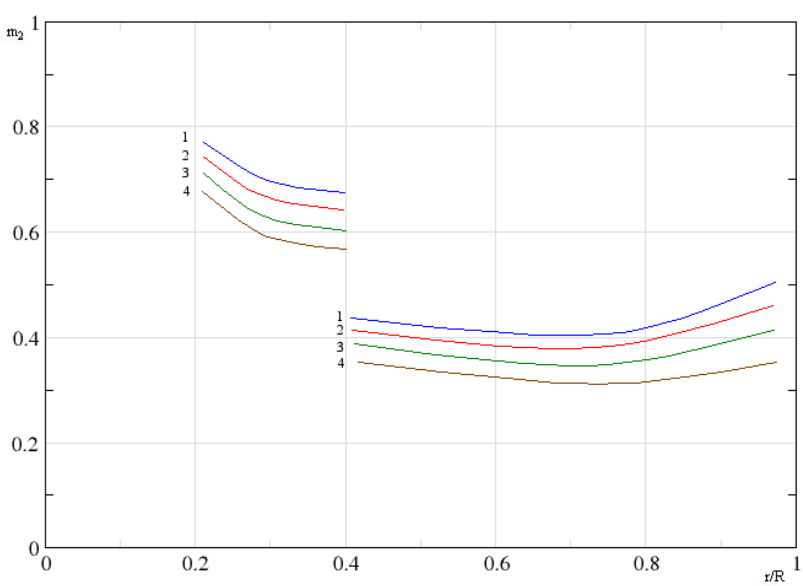

Fig. 4.9 Hoop moment

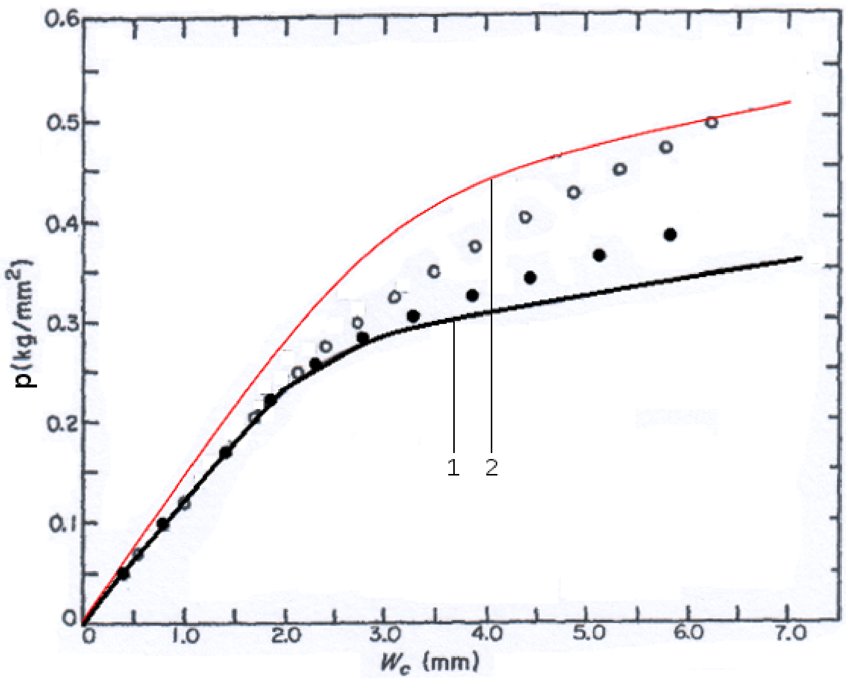

Fig. 4.10 Load deflection relations (comparison with experiment)

that the radii η_0 and η_1 move towards the outer edge of the plate when the pressure intensity increases.

In Fig. 4.10 the load deflection relations calculated in the present paper are compared with those obtained by Ohashi and Murakami [114] and with experimental data. Fig. 4.10 shows that the curves are comparatively close to each other in the range of small deflections. In Fig. 4.10 curve 1 presents the experimental data by Ohashi and Murakami and curve 2 corresponds to the results obtained by the current method. The circles indicate theoretical predictions by Ohashi and Murakami for simply supported and clamped plates made of Mises material.

Table 4.3 Optimal parameters for the deflection $w_0 = 0.2500$, $n = 2$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.2411	0.3173	0.5001	0.5306	0.6705	0.7922
α_1	0.2612	0.4703	0.4158	0.5031	0.6079	0.7143
γ_2	0.5794	0.6234	0.7102	0.7302	0.7215	0.8761
α_2	0.4123	0.4712	0.5023	0.5719	0.7524	0.8653
e	15.27%	13.40%	19.13%	12.17%	15.10%	16.32%

Table 4.4 Optimal parameters for the deflection $w_0 = 0.2500$, $n = 4$

p	2.8000	3.0000	3.2000	3.4000	3.6000	3.8000
γ_1	0.2739	0.4253	0.4115	0.5215	0.6033	0.6524
α_1	0.1907	0.2074	0.2553	0.2713	0.2931	0.3107
γ_2	0.4427	0.4731	0.5387	0.7014	0.6592	0.7342
α_2	0.3402	0.4417	0.5319	0.4536	0.4867	0.5128
γ_3	0.6257	0.6881	0.8672	0.6613	0.4293	0.5702
α_3	0.5158	0.5209	0.6142	0.5894	0.6087	0.6449
γ_4	0.5642	0.5100	0.4782	0.5933	0.6997	0.7029
α_4	0.7894	0.8597	0.9130	0.9305	0.9324	0.9412
e	21.10%	14.42%	17.16%	13.04%	08.06%	11.92%

Similar to the section 2.5 of the current research optimal values of design parameters for circular plates are accommodated in Table 4.3 and Table 4.4 for two- and four-stepped plates obeying Hill's yield condition and associated flow law, respectively. It can be seen from Table 4.3 that in the case of two-stepped plates, loading $p = 2.8$ and the deflection $w_0 = 0.25$ one can save more than 15% of the material.

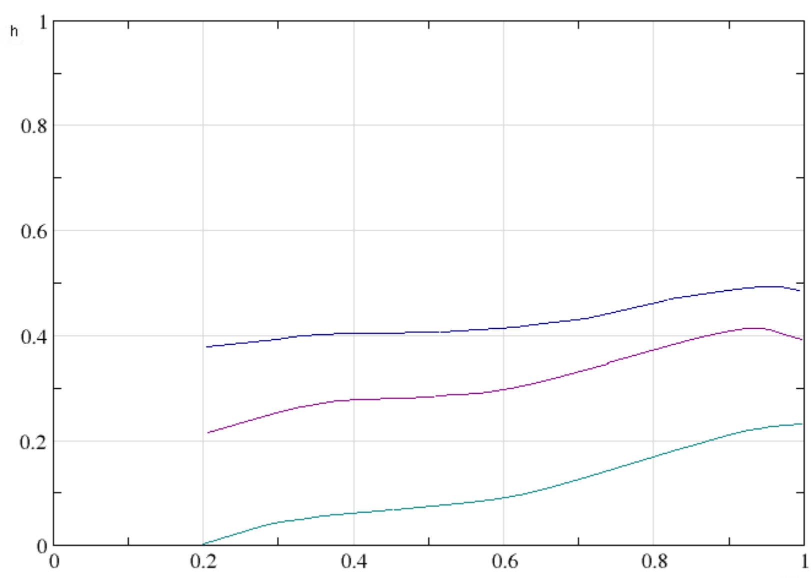

Fig. 4.11 Thickness distribution

For circular and annular plates obeying a non-linear yield condition and the associated gradientality law with a big number of steps the optimization problem was solved numerically making use of finite elements and the method based on wavelets.

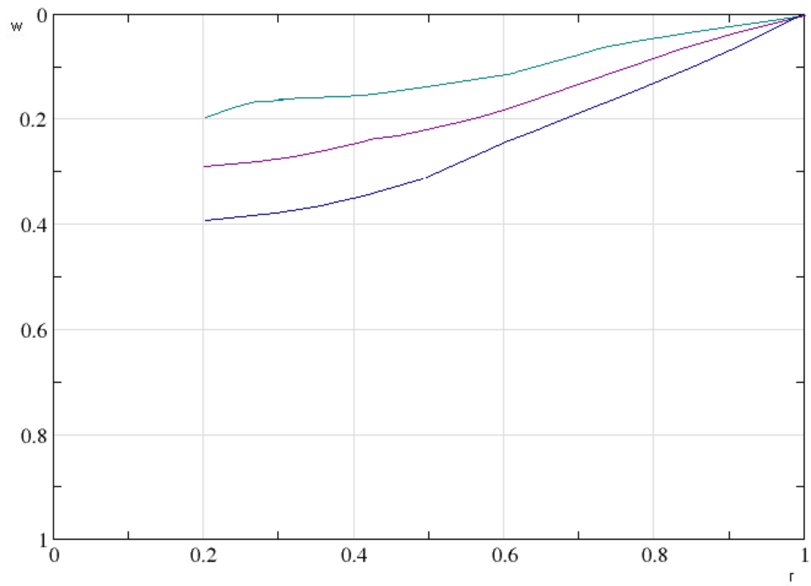

Fig. 4.12 Deflections of the plate

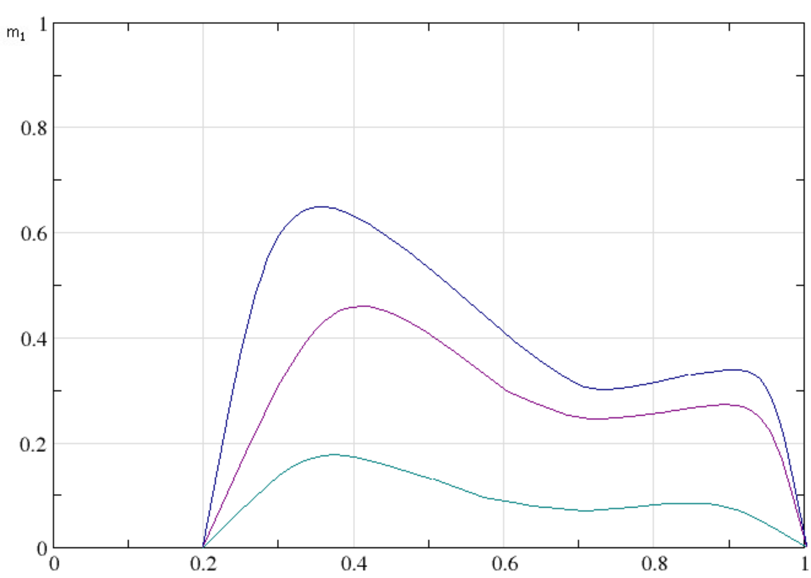

Fig. 4.13 Radial moment

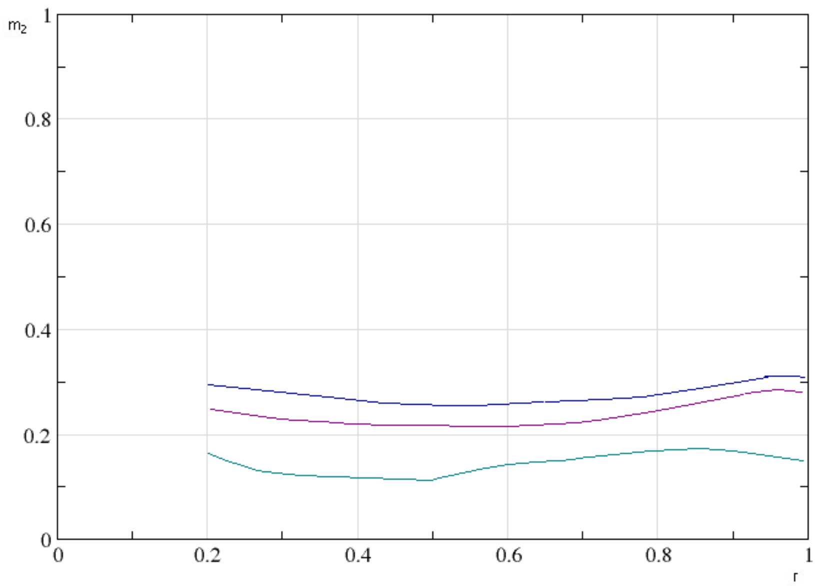

Fig. 4.14 Hoop moment

The results of calculations are presented in Fig. 4.11 – Fig. 4.14 in the case when the number of steps $n = 200$ and the material of plates obeys the Tsai-Wu non-linear yield condition. Fig. 4.11 corresponds to the annular plate with the internal radius $a = 0.2R$. In Fig. 4.11 the distributions of the optimal thickness are presented for $p = 3.4$, $p = 3.6$ and $p = 3.8$. The transverse deflections corresponding to these values of the pressure loading are depicted in Fig. 4.12. In Fig. 4.13, 4.14 the principal moments are portrayed for different values of the transverse pressure. Fig. 4.13 corresponds to the radial moment and Fig. 4.14 to the hoop moment. It can be seen from Fig. 4.14 that the bending moment in the hoop direction is almost constant. This matter was observed in the rigid-plastic analysis of circular and annular plates, as well (see Sawczuk and Sokół-Supel [129]). However, the radial bending moment changes quite rapidly in the neighbourhood of edges of the plate. Maximal values of the bending moment increase together with the pressure (Fig. 4.13) as might be expected. However, when increasing infinitely the number of steps the optimal solution may not exist. The matter that the optimal solution for plate problems does not exist in the classes of continuous and piece wise continuous functions was observed earlier in the case of rigid-plastic materials (see Lellep [73]). This effect is known as the “solid plate paradox”.

4.7 Concluding remarks

The methods of analysis and optimization of plates with piece wise constant thicknesses developed earlier for homogeneous isotropic materials are extended to plates made of anisotropic materials. The yielding of materials is assumed to take place according to the criterion Tsai-Wu and the associated gradientality law.

The traditional bending theory is used, non-linear effects are neglected in the current study.

Optimal designs of annular plates of piece wise constant thickness are developed. The effectivity of designs is assessed numerically. The calculations are implemented with the help of the finite element method and with the method based on Haar wavelets. The comparison of results obtained by different methods shows that the results are quite close to each other.

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Summary

Optimization of stepped plates in the case of smooth yield surfaces

The current work is devoted to the theory of analysis and optimization of stepped circular and annular plates subject to smooth yield surfaces. Chapter 1 provides the brief historical review of the problem and of the finite element method. The basic ideas of parallel computation, also of the multigrid method are presented herein, as well.

In Chapter 2 a method for numerical investigation of axisymmetric plates subjected to the distributed transverse pressure loading was presented. The material of plates studied herein is assumed to be an ideal elastic plastic material obeying the non-linear yield condition of von Mises and the associated flow law. The strain hardening as well as geometrical non-linearity are neglected in the present investigation. Calculations carried out showed that the obtained results are in good correlation with those obtained by ABAQUS when solving the direct problem of determination of the stress strain state of the plate.

In Chapter 3 an analytical-numerical study of annular plates operating in the range of elastic plastic deformations was undertaken. The material of plates was assumed to be an ideal elastic plastic material obeying the Mises yield condition. The author succeeded in the analytical derivation of optimality conditions for this highly non-linear problem. The obtained systems of equations were solved by existing computer codes.

In Chapter 4 the methods of analysis and optimization of plates with piece wise constant thicknesses developed earlier for homogeneous isotropic materials are extended to plates made of anisotropic materials. The plastic yielding of the material is assumed to take place according to the criterion Tsai-Wu and the associated gradientality law. The traditional bending theory is used, non-linear effects are neglected in the current study.

Kokkuvõte

Astmeliste plaatide optimiseerimine siledate voolavuspindade korral

Käesolevas väitekirjas vaadeldakse Misese, Hilli ning Tsai-Wu materjalist valmistatud elastsete plastsete astmeliste plaatide optimiseerimisega seotud küsimusi.

Antud dissertatsioon põhineb autori seitsmel teaduslikul publikatsioonil, millest kuus on avaldatud viimase kolme aasta jooksul.

Käesolev dissertatsioon koosneb neljast peatükist, kirjanduse loetelust ning autori elulookirjeldusest. Esimene peatükk on sisuliselt ülevaade numbriliste meetodite rakendamisest konstruktsioonielementide optimiseerimisel. Selles peatükis antakse ülevaade plaatide ja koorikute optimiseerimisele pühendatud töödest, samuti kirjeldatakse lõplike elementide meetodi ja paralleelarvutuse ajaloolist arengut. Käesoleva uurimise raames on kasutatud lõplike elementide meetodit ning Haari lainikute meetodit harilike ja osatuletistega diferentsiaalvõrrandite lahendamiseks ning on rakendatud kõrgproduktiivse ja paralleelarvutuse põhimõtteid.

Teises peatükis vaadeldakse sandwich-tüüpi sümmeetrilise elastse-plastse ümarplaadi painet ühtlaselt jaotatud koormuse mõjul ning otsitakse miinimumkaaluga projekti ette antud maksimumläbipainde korral. Eeldatakse, et plaadi materjal vastab Misese voolavustingimusele. Optimaalse lahendi leidmiseks on kasutatud lõplike elementide meetodit.

Kolmandas peatükis uuritakse eelmises peatükis püstitatud probleeme sümmeetriliste elastsete-plastsete astmeliste rõngasplaatide puhul. Optimaalse lahendi leidmiseks on kasutatud lõplike elementide meetodit ning Haari lainikute meetodit, viimast kasutatakse ka harilike diferentsiaalvõrrandite lahendamiseks.

Neljandas peatükis on uuritud anisotroopsete rõngasplaatide painet ning on leitud miinimumkaaluga projektid Hilli ja Tsai-Wu voolavustingimuste puhul. Arvutamisel on kasutatud Haari lainikute meetodit.

Väitekirjas on välja töötatud paralleelarvutuse metoodika, mis annab võimaluse numbriliselt lahendada elastsete-plastsete plaatide optimiseerimisprobleeme. Saadud lahendeid on võrreldud Ohashi ja Murakami, Turvey ning Upadrasta tulemustega. Töös saadud tulemused on heas kooskõlas teiste autorite töödega. Uurimistöö käigus ilmnes, et optimiseerimisülesannete puhul on mõistlikum kasutada lainikute meetodit, mille paralleeliseerimine hoiab rohkem kokku arvuti ressurssi.

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I consider it, hereby, necessary to express my gratitude to my supervisor professor Jaan Lellep for his time, consideration, support and advice. Special thanks to all my collegues for cooperation. In fact, it would not have been possible to present this dissertation without the results of the previous research works done by the member of the Estonian Academy of Science Professor Emeritus Ülo Lepik. Also, I appriciate the help given to me by PRACE and Estonian Mathematics and Statistics Doctoral School. The partial support from the Estonian Science Foundation Grant No. 9110 "Optimization of structural elements" and from the target financed project SF0180081508 "Models of applied mathematics and mechanics" is acknowledged. The last but not least, I am thankful to my family for motivation and patience.

Curriculum vitae

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Education

1993 – 1996	Tallinn High School #6, Gold medal
1996 – 2000	University of Tartu, Faculty of Mathematics, Baccalaureus Scientiarum in Mathematics (BSc)
2000 – 2003	University of Tartu, Faculty of Mathematics and Computer Science, Magister scientiarum in Mathematics (MSc)
2003 –	University of Tartu, Faculty of Mathematics and Computer Science, doctoral studies in mathematics

Advanced trainings in 2012 – 2013

1. FEM Workshop at CSC – IT Center for Science Ltd, Espoo, Finland
2. Fortran 2003/2008 at CSC – IT Center for Science Ltd, Espoo, Finland
3. Introduction to Parallel Programming with MPI & OpenMP at CSC – IT Center for Science Ltd, Espoo, Finland
4. Advanced Parallel Programming at CSC – IT Center for Science Ltd, Espoo, Finland
5. OpenACC Programming for Parallel Accelerated Supercomputers – an alternative to CUDA from Cray perspective at HLRS, Stuttgart, Germany
6. Introduction to Unified Parallel C (UPC) and Co-array Fortran (CAF) at HLRS, Stuttgart, Germany
7. 12th VI-HPC Tuning Workshop at JSC – Jülich Supercomputing Centre at Forschungszentrum Jülich, Jülich, Germany

Languages: Estonian, English, Russian

Employment

2004 – 2006	Administrative authority of the Ministry of Economic Affairs and Communications, chief IT specialist in the Estonian Motor Vehicle Registration Centre
2006 – 2008	ITEX Consulting Llc, Research & Development division
2008 – 2009	Administrative authority of the Ministry of Defence, higher state public servant

Fields of scientific interest

Finite element method, wavelet analysis, optimization theory, scientific computing, including parallel application development for high performance computing.

Research activities

The optimization of stepped plates in the case of smooth yield surfaces with FEM and wavelet analysis applied for solving ODE/PDE coupled with resources provided by HPC was the main direction of the scientific activities. The obtained results were presented at the following conferences:

1. OAS 2011, International Conference on Optimization and Analysis of Structures: August 25–27, 2011, Tartu, Estonia.
2. 13th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering (MACMESE '11): November 3–5, 2011, Catania, Sicily, Italy.
3. EngOpt 2012, 3rd International Conference on Engineering Optimization: July 1–5, 2012, Rio de Janeiro, Brazil.
4. 8th WSEAS International Conference on Applied and Theoretical Mechanics (MECHANICS '12): December 29 – 31, Montreux, Switzerland.
5. DFE 2013, Design Fabrication and Economy of Metal Structures 2013: April 24 – 26, 2013, Miskolc, Hungary.
6. OAS 2013, The 2nd International Conference Optimization and Analysis of Structures: August 25–27, 2013, Tartu, Estonia.

Full list of the publications

1. J. Lellep and B. Vlassov, Elastic-plastic bending of plates, *In: Proceedings of the 16th Nordic Seminar Computational Mechanics*, Trondheim, Norway; 16-18 October, 2003. Editors: K. M. Mathisen; T. Kvamsdal; K. M. Okstad. Trondheim: Norwegian University of Science and Technology, 2003, pp. 83–86.
2. J. Lellep and B. Vlassov, Optimization of elastic plastic annular and circular plates, *In: OAS 2011, International Conference on Optimization and Analysis of Structures: August 25-27, 2011, Tartu, Estonia. Abstracts.* Editors: J. Lellep, E. Puman, University of Tartu Press, 2011.
3. J. Lellep and B. Vlassov, Optimization of elastic plastic circular plates made of Von Mises material. *Proceedings of the WSEAS International Conferences, Mathematical Methods and Techniques in Engineering & Environmental Science*, 13th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering (MACMESE '11), (Catania, Sicily, Italy November 3–5, 2011). Editors M. Demiralp, Z. Bojkovic and A. Repanovici, ISBN: 978-1-61804-047-3, WSEAS Press, 2011, pp. 233–238.

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6. J. Lellep and B. Vlassov, Optimization of Elastic Plastic Annular Plates Made of Composite Materials, *In: Advances in Circuits, Systems, Automation and Mechanics. Proceedings of the* 8^{th} *WSEAS International Conference on Applied and Theoretical Mechanics (MECHANICS '12), (Montreux)*. Editors: D. Biolek, K. Volk, Kok Mun Ng, WSEAS Press, pp. 154–159, ISBN: 978-1-61804-146-3.
7. J. Lellep and B. Vlassov, Optimization of Stepped Elastic Plastic Plates, *In: Advanced Materials Research*. Publishing Editor: T. Wohlbier; editors: Jiang Feng and Yijin Wu, Durnten-Zurich, Switzerland, Trans Tech Publications Ltd., pp. 209–214, ISSN: 1022-6680, ISBN: 978-3-03785-729-8.
8. J. Lellep and B. Vlassov, Optimization of Elastic Plastic Plates Made of Homogeneous and Composite Materials, *In: Design, Fabrication and Economy of Metal Structures*. Editors: K. Jármai, J. Farkas, Springer-Verlag, 2013, pp. 43–48, ISBN (Print): 978-3-642-36690-1, ISBN (Online): 978-3-642-36691-8.
9. J. Lellep and B. Vlassov, Elastic plastic elliptical plates, *In: Proceedings of the 2nd International Conference Optimization and Analysis of Structures*. Editors: J. Lellep, E. Puman, University of Tartu Press, 2013, pp. 94–100.

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2. Fortran 2003/2008 at CSC – IT Center for Science Ltd, Espoo, Soome
3. Introduction to Parallel Programming with MPI & OpenMP at CSC – IT Center for Science Ltd, Espoo, Soome
4. Advanced Parallel Programming at CSC – IT Center for Science Ltd, Espoo, Soome
5. OpenACC Programming for Parallel Accelerated Supercomputers – an alternative to CUDA from Cray perspective at HLRS, Stuttgart, Saksamaa
6. Introduction to Unified Parallel C (UPC) and Co-array Fortran (CAF) at HLRS, Stuttgart, Saksamaa
7. 12th VI-HPC Tuning Workshop at JSC – Jülich Supercomputing Centre at Forschungszentrum Jülich, Jülich, Saksamaa

Keelteoskus: eesti, inglise, vene

Teenistuskäik

2004 – 2006	Majandus ja kommunikatsiooniministeeriumi haldusala, Eesti Riiklik Autoregistrikeskus, IT peasptsialist
2006 – 2008	ITEX Consulting OÜ, arendustalitus
2008 – 2009	Kaitseministeeriumi haldusala, kõrgem riigiametnik

Teaduslikud huvid

Lõplike elementide meetod, lainikute analüüs, optimiseerimisteooria ning kõrgproduktiivse paralleelarvutuse toega teadustarkvara loomine superarvutitele.

Teaduslik tegevus

Optimiseerimisteooria ning lainikute analüüs. Plaatide- ja koorikute analüüs ja optimiseerimine kasutades lõplike elementide meetodit ning Haari lainikute meetodit harilike ja osatuletistega diferentsiaalvõrran-dite lahendamiseks ning rakendades kõrgproduktiivse ja paralleelarvutuse põhi-mõtteid. Tulemustega on esinetud kuuel teaduskonverentsil:

1. OAS 2011, International Conference on Optimization and Analysis of Structures: August 25–27, 2011, Tartu, Estonia.
2. 13th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering (MACMESE '11): November 3–5, 2011, Catania, Sicily, Italy.
3. EngOpt 2012, 3rd International Conference on Engineering Optimization: July 1–5, 2012, Rio de Janeiro, Brazil.
4. 8$^{\text{th}}$ WSEAS International Conference on Applied and Theoretical Mechanics (MECHANICS '12): December 29 – 31, Montreux, Switzerland.
5. DFE 2013, Design Fabrication and Economy of Metal Structures 2013: April 24 – 26, 2013, Miskolc, Hungary.
6. OAS 2013, The 2nd International Conference Optimization and Analysis of Structures: August 25–27, 2013, Tartu, Estonia.

Publikatsioonid

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