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Contents

List of publications

This dissertation is based on the following four publications.

I. P. Gallagher and T. Koivisto, Symmetry **13**, 2076 (2021), arXiv:2103.05435 [gr-qc]

II. P. Gallagher et al., Phys. Rev. D **105**, 125010 (2022), arXiv:2202.05657 [hep-th]

III. P. Gallagher et al., Phys. Rev. D **109**, L061503 (2024), arXiv:2311.07464 [hep-th]

IV. P. Gallagher, Phys. Rev. D **110**, 085010 (2024), arXiv:2403.02578 [hep-th]

Although not included, the following preprint is topical and referenced.

V. P. Gallagher, *Nontrivial constitutive laws and unified structures in constrained BF theory*, 2025, arXiv:2504.14062 [hep-th]

Author's contribution

In article [1], I developed the Lagrangian and Equations of Motion derivation of the Khronon integration constant dark matter dust in collaboration with Tomi Koivisto, in addition to general discussion and analysis.

Article [2] is in significant part the work of myself, from problem conception and literature review, to the analysis and proposal of new Lagrangians. I thank my collaborators Tomi Koivisto and Luca Marzola for discussing, verifying and further developing these ideas.

I was a coauthor in article [3], participating in the discussions and study of the Khronon gravity covariant phase space and Noether currents, developing the interpretation (w.r.t. covariant frame-dragging, isometries, shift symmetries), contributing in analysis and verification, and exploring future developments, some continued in [4].

Article [4] and the preprint [5] are written solely by myself.

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Chapter 1

Introduction

1.1 On motifs of unification

The fundamental theories of physics can describe a vast landscape of phenomena, having identified the fundamental constituents of Nature, really particles and fields, and the laws that govern their dynamics — the equations of motion, when taken to a pragmatic extreme. This is, of course, a simplistic recourse into what defines a theory of physics, as, for example, it does not detail the symmetry content that defines the field interactions, or the geometry and topology of the backdrop spacetime, or the causal ordering relations that are of similar fundamental importance, not to enter questions of quantum structure. Nevertheless, we are presented with the Standard Model of particle physics, a quantum field theory that describes particle phenomena to extreme accuracy [6], and General Relativity, which models gravitational phenomena to a similarly extreme accuracy [7–11]. But they are separate theories, in several senses: General Relativity is not a quantum field theory of gravity, and the most straightforward quantum geometrodynamic formulations are non-renormalizable [12–14]. Similarly, and of primary relevance here, the symmetry structure of the gauge theories in the Standard Model and that of gravity as described in General Relativity is separate, broken, and does not have an unambiguous unified origin [15]. Put into more definite terms, General Relativity is not immediately situated in a unified gauge theory of Yang-Mills type, and the arena of symmetry transformations is not in fiberwise automorphisms, but in spacetime diffeomorphisms[1]. Quite clearly, there are still many unexplained physical phenomena as well, of which dark matter and dark energy

[1]This insight has been attributed to Bryce DeWitt, per M. Hamilton referring to E. Witten's Newton Lecture [16, 17]. It is not well to speculate on the implications, but in the author's opinion this difference should vanish in the unified phase.

are particular examples, as they elude a clear solution in the common fundamental theories of physics [18, 19], suggesting a necessary extension. This preludes a recent Cartan-geometric theory of gravity [20], which proposes a solution to the question of time and a geometric origin to cold (mimetic) dark matter, through a careful symmetry breaking of a single vector field, the "Khronon", which together with the Lorentz connection generates spacetime geometry. The purpose of this dissertation is to establish how the Khronon Lorentz gauge theory of gravity should be reconciled with particle theory for unification.

More conservatively, the material here presents one line of thought how Yang-Mills theory and electromagnetism, temptingly termed collectively internal gauge theory, should be made compatible with the Khronon theory of gravity. Matter is suggested to correspond to "internal" space, loosely such as that of a particle or an observer, while gravity relates to spacetime (i.e. "external") degrees of freedom. Field theory geometry would formalize it in terms of e.g. fiber bundles[2]. An experienced reader might already predict some conclusions that will be drawn, in that gauge-gravity unification will generally, without additional principles, be rather hopelessly *ad hoc*, and the focus should be on a search for a creative structure that might facilitate any given theory, and then work on specific issues of any given approach. Hopefully this dissertation will provide guidance how this can be specifically done for the Khronon theory of gravity, in e.g. modifying the simplicity constraints, so any interested party can skip over less prospective lines of thought. That is, *a priori* predicating any specific Lagrangian is somewhat hopeless, as the models are plethora. But to give a further example on theory *construction*, Grand-Unified Theories of the Standard Model interactions and the like often require, imply, or, depending on the perspective, predict, processes and phenomena such as proton decay, Lorentz violation, among many other possibilities and a plethora of new particles, all in various scheme-dependent mixtures. Currently this conflicts with observations [6], so is forced to a practically so-far unobservable regime, while the inclusion of gravity pushes to unification schemes beyond Grand Unified Theory. Nevertheless, a definite, concrete continuation would be embedding the Khronon into specific Lagrangian formulations of graviGUT-s, and working out the phenomenology in rigor.

[2] Note that the distinction is not entirely unambiguous. The metric, the common vessel for gravitational degrees of freedom, is a section of a tensor bundle. The soldering form, as used in e.g. tetradic Palatini gravity, "solders" the tangent spaces unto the base manifold. There are plenty of other formulations of gravity [21]. This internal-external distinction is only brought up because in the author's opinion it is a fundamental geometric issue in classical unification of gauge theory and gravity, but this should hardly be taken as the necessary premise for unification.

But the initial premise or motivation for this study was actually rather definite in its suggested direction. A particular example of recent devise in particle physics symmetry structure was the work of Furey [22], which reformulated the Standard Model representation theory in terms of the algebra $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$, a tensor product of the only normed division algebras over real numbers: the real numbers $\mathbb{R}$ themselves, the complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$. The list of normed division algebras is entirely limited by Hurwitz's[3] theorem [23], so if there was a canonical fitting of particles into numbers, or division algebras to be unambiguous, it would also naturally limit the true particle content, in consistency with what is indeed observed. There could be a natural explanation why the observed field content is limited. Even more so, the quaternions are very suggestive of relativistic geometry, as the quaternionic-imaginary units i, j, k seem to correspond to spacelike components, and the remainder quaternionic-real unit 1 relates to the timelike component in 4-dimensional spacetime geometry. Indeed, this has been rigorously considered [24]. Now, apart from the spin connection, given that the Khronon theory only introduces a single Lorentz vector τ^a (or, Lorentz-valued scalar, contingent on the interpretation) to produce the spacetime coframe $e^a \equiv \mathrm{D}\tau^a$, a rather curious hypothesis follows, in that perhaps this scalar could become quaternion-valued instead, and when further extended and embedded in some manner into the remainder of the division algebras, be used to describe the remainder of the interactions: the electromagnetic, the weak, and the strong force. This could be considered a unification of gravity with internal gauge interactions, and if successful, this would be akin to a theory of everything — or at least, of all known interactions.

This did not come to be. Or rather, as will be concluded, there are several ways to possibly make sense of unifying the Khronon theory of gravity with the remainder of the gauge interactions, but they range from difficult to substantiate in their interest, to difficult in actually working out the details of a definite proposal. Moreso, the author did not take the unjustifiably ambitious aim of proposing a theory of everything as the goal, but the aim was rather in geometrizing gauge theories such that there was greater "symmetry" between internal gauge theories and gravity, such that it could facilitate a "unified phase" which would be "broken" to our standard understanding of spacetime, gravity and gauge theory, and hold a "natural" way to introduce an analogue of the Khronon vector into the remaining gauge theories — thus, if the Khronon became valued in division algebras, the

[3] There are many similar theorems, where the list of allowed division algebras is limited, provided some additional assumptions. For example, there is Frobenius's, Wedderbun's, Mazur's theorem etc.

split between gravity and internal gauge theory would be relatively straightforward. Note the quotation marks, as these terms are not used currently in any precise definition, but as basic motivations. Throughout, there will be a rather messy invocation of both physical and mathematical notions, as on one hand, the premise for the Khronon theory was taken to be more intricate than a crude reading would provide, and on the other, a clear and satisfactory proposal was not reached. That is, the intent was to work through the construction of spacetime, from topology and continuity, to causality and geometry, to introducing fermionic and bosonic matter, and this all attempted in possibly more interesting, nonstandard terms. The motivation to construct spacetime came from ideas of "pregeometry" [25] and the presence of a more fundamental clock vector field in the Khronon. The crudest reading would have been just in modified (Lagrangian) simplicity constraints [26–28], but in hindsight would have provided a variety of definite models to work through. As a premonition, let it be suggested that an application of the Khronon simplicity constraints to models of Loop Quantum Gravity, Spin Foams, Causal Dynamical Triangulation, or immediately to various unified models (e.g. Smolin's [29]) would provide a rather definite avenue of research, which, at the time of this writing, has not really been considered in extent. But the aspiration, perhaps naive or hopeless, was to develop a more immediate model of spacetime geometry conformant to the clock field, with internal interactions on equal Lagrangian footing to gravity — and this did not happen in a properly satisfactory way.

Returning to unification in more general terms, the purpose of gauge-gravity unification in specific requires a slightly different motivation than some other topics might need, in theoretical physics in general and even more so than in the disciplines more immediately phenomenological, observational or experimental. The purpose of this procedure is not immediately obvious when taken in any greater detail, even if the desire for greater simplicity and symmetry is. That is, the aspiration is understandable, but the exact phenomenology that is (*required* or *desired*) to be explained is not as direct. There are several issues that bear immediate emphasis.

Firstly, unification in physics has consistently proven to be of qualitatively different character — there is a definite meaning of unification in contemporary field theory and restricted to interactions [15], but such constriction might not prove to be always useful or meaningful. So, the merging of space and time into a single covariant spacetime entity is not a unification of interactions by itself, as it is most directly a change in the background geometry. There is a rather intricate substantiation in e.g. the sense of deformations of Lie algebras[4] [30], but more than that, introducing relativity

[4]This requires further introducing the notion of stability or rigidity of Lie algebras.

underlies the unification of the electric and magnetic forces into the single electromagnetic interaction. This, in turn, is not immediately of similar type to electroweak unification, which follows the symmetry breaking pattern $U(1)_Y \times SU(2)_L \to U(1)_{EM} \times SU(2)_W$, and requires introduction of the Higgs mechanism, thus producing massive W and Z bosons. Essentially, it is a specification of internal geometry[5]. With the Strong Force appended, the Standard Model symmetry group $U(1)_Y \times SU(2)_L \times SU(3)_C$ could be thought of as a complete theory of the electromagnetic, weak and strong force, but clearly the strong force is separate from the remainder of interactions, as it is only appended by a direct product, and even electroweak theory is not a proper unification *per se* of electromagnetism and the weak interaction, as the product is not semi-simple.

This somewhat arbitrary symmetry structure can already be brought up as a possible motivation for Grand Unification, where the basic method is to work out how a larger, unified gauge group is restricted to the Standard Model subgroup, and this is further supported by e.g. the convergence in the renormalization group flow of the Standard Model gauge couplings, alongside a host of other questions (the Hierarchy, Parameter, Fermion problem etc.) [31, 32]. What is more difficult to substantiate, however, is how gravity should be appended to this list, as the arena of interaction is utterly different. A pure symmetry extension argument is difficult to take for the entire truth, and seems contradictory with some standard results, e.g. the Coleman-Mandula theorem [33]. Indeed, this dissertation provides some similar observer-structure arguments why gauge-gravity unification would be of yet again a different type, and aspired to search for a qualitatively different resolution, by facilitating a certain unified topological phase, albeit not entirely successfully.

Let it be emphasized that unification has a definite meaning in terms of particle physics [15]. In particular, Krasnov & Percacci define it in terms of a four-step process, starting from identifying the unified group $G \supset G_A \times G_B$, organizing the particle representations, and then by means of an order parameter restricting to the broken phase $G_A \times G_B$, which is expected to be the stabilizer of the potential's minimum; the fourth and final step being the writing of the action functional proper. Although, strictly speaking, even Electroweak theory does not follow this exactly (the gauge

Then it can be found that the Poincaré algebra is a deformation of the Galilei algebra, but can further be deformed into the de Sitter or anti-de Sitter algebra, which are stable and simple.

[5] Or in less poetic terms, a specification of Lagrangian symmetry content. This is not unreconcilable terminology e.g. per Lie group differential geometry, but the author's implication is that gauge-gravity unification will have to distinguish spacetime geometry from internal space geometry.

group is only semi-simple there), this is an almost unavoidable description of particle-physics unification. For gravity, the situation is *arguably* slightly different due to the existence of (and almost in a contradictory manner) significantly more restrictive results like the Coleman-Mandula theorem [33] — in terms of Poincaré invariant scattering theory, only a trivial direct product of the internal symmetry group and the Poincaré group is permitted. In turn, the gauge-gravity unified phase might be expected to be a more drastic departure, e.g. with a zero soldering form order parameter, which is essentially a transition to a metrically degenerate region of spacetime. Supersymmetric theories would be subject to the Haag–Łopuszański–Sohnius theorem [34], but generally there are many methods to work around such restrictions (e.g. conformal symmetry, graded Lie algebras, spontaneously broken symmetries etc.).

Secondly, it is not even unambiguously clear if gravity requires unification with other forces, in a graviGUT or otherwise. Three of the interactions follow a clear Maxwell/Yang-Mills premise, such that any modification is observationally almost completely forbidden. Gravity is significantly more ambiguous in its many formulations [21, 35]. Nevertheless, General Relativity can be thought of as a particle theory of a spin-2 bosonic force carrier, the graviton [36–38], so it is not unimaginable that it comes from a unified origin as all other forces. Passing to the geometry of field theory, the justification why gravity should necessarily originate from the same common origin as the other interactions is less clear, as the sectors of relevant geometry are simply different; consider the Atiyah exact sequence [39, 40] for a formalization in terms of symmetries.

Thirdly, there are a plethora of ways how to attempt unification of gravity with the other interactions. An excellent review is in [15], and certainly there will be still more attempts to come. What needs to be emphasized is rather the ontological multitude of these unification attempts — the arena is extensively colorful. On one hand, gravity can be reformulated as a pure connection field [41], akin to the other interactions (and the connection in the Khronon theory can be understood to fulfill a similar purpose). Or, the geometry can be changed, as in [42–45], where the Cartan model space is to be rolled over an underlying manifold. In the other end there is the popular case of String Theory [46], which utterly and entirely changes the premise: rather than (quantum fields of) point-particles, the case is for higher-dimensional objects, strings and branes, in higher-dimensional spaces suitably compactified. In contrast, a much more straightforward approach would be to consider BF-type models, so that the auxiliary B-field obtains whatever physical value through some constraining process, contingent on what sector of interactions it is to describe. The separation of gravity and internal gauge theory is very suggestive, and rather minimalistic, as it becomes a very literal split of the BF-form into the Einstein-Hilbert/Cartan or Maxwell/Yang-Mills Lagrangian. Several attempts have already been made before [29, 47–50], and this dissertation considers a little more of the structure of this type. However, more than that, what we observe is a variety of models, each contending to different merits, and providing a different set of problems. It is not straightforward to assign preference to any specific one. Gravity introduces a mixture of several notions, including causality, topology and metric geometry, thus it is only natural that this complexity persists in the variety of unification options possible. The only path forward is to test and evaluate each, weighed by intuition, and hope that the correct solution eventually presents itself. Until then, and until verified by the scholarly community at large, the inverse implies that any particular approach will remain contentious to varying extent, and can only be handled with caution.

As a final remark, on terminology and a guideline for further development, note that a unified theory of gravity and internal gauge theory is often taken as one part of a (or "the") Theory of Everything. For a more extensive discussion what a "Theory of Everything" is even understood to require, refer to e.g. [51, 52]. Gauge-gravity unification in whatever sense would clearly be a theory of all interactions[6]. It should be further consistently unified with a theory of quantum gravity, but would also require some model of the quantum measurement, which is far too often neglected in direct development, and would require at least a guideline (and preferably an actual derivation) for how higher-complexity phenomena such as sentience and consciousness should emerge — otherwise, it would not be possible to rule out the possibility of a greater set of *fundamental* natural principles that should be appended for a true theory of everything. A generic suggestion that macroscopic particle interactions would necessarily unlock the origin of e.g. consciousness is difficult to be taken as a definitive solution. A Theory of Physical Everything can be more restrained, and limit its attention to physical materia, particles and fields and emergent phenomena a few layers above, such as up to thermodynamics, the span of classical and nonrelativistic quantum mechanics, cosmology etc. But what is a very unique and relevant line of thought to consider is that a true Theory of Everything might never even be possible, due to limitations of mathematical and logical structure. In specific, reference is to Gödel's incompleteness theorems [53] and Tarski's undefinability theorem [54], on the limited extent of algorithmically provable truth about arithmetic in the

[6]Interactions that are presently observed, that is. This dissertation will not go into depth on the possibilities for a fifth force, dark sector interactions etc., or how the possibility of such extensions would affect the meaning of a Theory of Everything — of increasingly tangential relevance for physics research.

confines of any system of axioms, and on the concept of truth in first-order logic, respectively. However, it can be argued that the implication is significantly more extensive, as any particular limited Lagrangian theory will be unable to prove its own logic, and indeed, this was the topic of an excellent recent essay [55] and has been argued far earlier as well (e.g. F. Dyson and S. Hawking outside the research setting [56, 57]). That is, there will necessarily be something "outside" the confines of physical interest, which cannot be derived from a Theory of "Everything". Nevertheless, this should not be necessarily understood as a deficiency of any such programme, as a physical theory is not necessarily expected to implicate the logic that is used to define itself, the structure of what is "outside" is unclear, and indeed, application of e.g. Gödel's incompleteness theorem outside of its actual setting can be considered simply invalid. But regardless, let us relent, as the aim of unification here was more limited in its scope, and only acknowledge the argument that there may be an infinite array of more phenomena, including *physical* phenomena, that cannot be limited to the confines of any single Lagrangian theory. So, let us instead observe how one line of thought leads its course to demise.

1.2 A summary of the included articles

The initial aspiration of the author's work was to develop a gauge-gravity unified theory based on the "Khronon" Lorentz gauge theory [20], which would have been fit into the division algebra model of Furey [22]. The Khronon vector field τ^a, or Lorentz-valued scalar if avoiding reference to tangent geometry, would have instead become quaternion-valued, eventually generically division-algebra valued, thus becoming the link between spacetime geometry and algebra. There are several open questions in the Khronon theory of gravity, e.g. the exact mechanism how the spontaneous symmetry breaking should take place, but at the time the most urgent one was of incorporating matter fields in general, and for unification, the case of including Maxwell and Yang-Mills gauge fields in particular.

The published work follows the idea of facilitating a zero-metric topological phase in terms of first order formalism and BF theory. The introduction of an auxiliary field permits a (rather *ad hoc* and formal) split and isolation of covariant and contravariant components. Spelling out the Hodge star requires the inverse metric, but it is possible to play around this in first order formalism and avoid explicit reference, despite as arbitrary as unlinking the dual spaces may seem — and entirely unnecessary in the *metric* phase of the theory, which necessarily should be as close to standard theory as possible on observational grounds. In the author's view, the metric, broken phase should *exactly* reduce to the standard theory of matter fields. Nevertheless, the quadratic Maxwell-Yang-Mills potential and various rewritings of the auxiliary field established a somewhat greater similarity between the Lagrangian gauge-gravity sectors and the Khronon theory in particular, leading to introducing the "Isokhronon" theory — although ultimately not viable. It similarly lends to a rather surprisingly extensive amount of related work, e.g. premetric theory [58], deformations of topological BF theory [59], bimetric theory [60], speculation on the cosmological constant and the Kodama state [61], double copy structure [62], and more. The hypothesis was further that by virtue of some constraining mechanism the B-field would obtain the relevant physical values — so with the internal Maxwell-Yang-Mills theory suitably rewritten, the position of the extended Khronon in both the spacetime hypersurface basis $*(e^a \wedge e^b)$ as well as the internal gauge theory B-field would be more clear, and somehow naturally fall into place. If all were successful, this would have eventually resulted in a gauge-gravity unified theory, which in the broken phase reproduces standard theory of matter and gravity, while the unified phase corresponds to a zero metric ground state and a topological field theory.

The topological field theory questions provide another perspective into premetric theory, as developed by Hehl & Obukhov [58], extended to gravity by Koivisto, Hohmann & Marzola [63], and which served as significant motivation for the work of this dissertation. That is, as the author suggests, the physical premetric axioms can be understood as another expression of introducing the basic principal bundle structure, outfit with the generic BF-theory action, and by virtue of a *constitutive law*, constrained to physical Maxwell-Yang-Mills form. So, as this work has established, the B-field can entirely be understood as the "excitation" field of premetric fare — and more generally, premetric theory can be understood as the constraining of a topological field theory to physical, metric form. First order formalism in gauge theory is rather basic and fundamental and has been discovered and rediscovered throughout the study of field theory, but obtains a more nontrivial interpretation as the parent system in the study of dualities [64–67], where apparently different theories in actuality describe the same physical system, such as vacuum Proca and Kalb-Ramond field theory. As the author's work [4, 5] claimed, this is simply an identity of the parent system, trivially integrating over the auxiliary fields — nontrivial modifications require nontrivial (degenerate) constitutive law structure for the auxiliaries.

The particular aim of the unified theory hypothesis, that the internal gauge theories should somehow admit a suitable formulation paralleling that of gravity, proved unsuccessful — here ultimately traceable to the simple fact that the interactions are independent, at least so in the broken phase. Nevertheless, this work suggests several locations and mechanisms that could instead be investigated further, especially so in adapting simplicity constraints to the Khronon waywiser geometry, and studying all steps of the constraining process. Perhaps future work can deal with the more definite suggestions to an ultimately amicable resolution.

As presented earlier, this dissertation is based on the author's work in four research papers, published in peer-reviewed journals, and one *preprint*, which is only included for the sake of completeness of the author's thought. These papers constitute the primary novel component of the dissertation, and illustrate the approach to gauge-gravity unification that was attempted to be developed and resolved. A brief summary follows.

1. Article [1] achieved two main results: it provided a Lagrangian and equations of motion derivation of the cold dark matter dust of the Khronon theory which was earlier derived only in Hamiltonian formulation [20], and it proposed a unimodular *theory* of ΛCDM cosmology, emergent from integration constants. The Khronon theory has a curiously nontrivial interplay between the Einstein-prototype coframe and connection equations of motion in how the gravitational degrees of freedom are spread across the τ^a vector field and the self-dual $^{+}\omega_{ab}$ and anti-self-dual $^{-}\omega_{ab}$ connection, while ideal dust (mimetic [68–70], as [20] refers to it) is collected into an integration constant 3-form.

2. Article [2] focused on the issue of introducing matter fields to the Khronon theory of gravity. This was primarily following the premise of the unification approach that was chosen early on: the Khronon field τ^a would be the main object to develop for unified gauge theory geometry, thus would have to be introduced into the standard Maxwell-Yang-Mills Lagrangian, so that the action would be formally as similar as possible to the standard Einstein-Cartan-like Khronon gravitational Lagrangian. This would better facilitate the Lorentz-valued scalar τ^a to become valued in other algebras, and, hopefully, *reveal* the unified dynamics in a straightforward manner.

   This required introducing new principles, as otherwise it would not be possible to prefer any or restrict to specific matter couplings in any manner, and only introduce a list of all possible Lagrangian terms constructed from the dynamical quantities — in hindsight, a more prospective effort. At the same time, compared to metric $g_{\mu\nu}$ geometry, it is more difficult to reject nil values and regions $\tau^a = 0$, which would provide a zero metric (topological) phase. Permitting such regions was argued to resolve a multiple of problems simultaneously: correspond to the unified phase, which would be topological as it loses access to nondegenerate metric values, correspond to a natural ground state of gravity, and seemingly correspond to (a notion of) a "pregeometric" phase, where spacetime metric geometry is yet to be formed. By reformulating the basic gauge theory actions, such that they no longer make explicit reference to Hodge dualization or the inverse metric, and bookkeeping the tensor spaces and their duals *without a priori relation* inside the degenerate region, it was argued that a first order formulation of gauge theory appears as the natural Lagrangian for gauge-gravity unification.

While the Dirac spinorial action did not require any addition, the Maxwell-Yang-Mills theory Lagrangians had a natural "kinetic cycle" first order formulation, depending on the specific auxiliary field introduced — a Lie-algebra valued 0-form G_{ab}, 1-form u^a or 2-form B. Various interpretations for the introduction of these fields and the quadratic potential were attempted (e.g. linear transformations, bimetric theory, premetric excitations, dual field strength auxiliary, cosmological constant), and it was argued that the premise of premetric theory is dynamically resolved in terms of introducing auxiliary fields of this kind. A novel modification was considered — the Isokhronon ϕ^a theory, with a *similar* background effect to the Khronon theory of gravity, viz. an integration constant $\mathrm{D}X_a = 0$. Some different algebraic organizations of the unified Lagrangian were attempted.

3. Article [3] recollected previous work in the gravity and matter Lagrangians, and, in terms of the covariant phase space formalism, studied their symplectic structure and conserved currents and charges. The Noether symmetries of Lorentz transformations and diffeomorphisms were considered, alongside energy-momentum and hypermomentum currents. Some specific issues such as shift symmetries and frame-dependent charges were analyzed; the appearance of dust was linked with the shift symmetry of the Khronon. Altogether, the first order formulations of scalar field theory, Maxwell-Yang-Mills gauge fields, Dirac fermions and Khronon gravity with the cosmological constant were considered, spanning the entirety of the usual Lagrangian content of the Standard Model and the theory of gravity. Several questions were nuanced, such as the symplectic structure of boundary-equivalent formulations of the Khronon theory of gravity — classically, there is no expectation of substantial dynamical difference, but as canonical quantization directly refers to the Poisson, or symplectic structure, the distinction is no longer as obviously irrelevant for the case of e.g. quantum theory and black hole thermodynamics.

4. Article [4] considered two main topics: firstly, a canonical study of the Isokhronon theory was performed, in both the noncovariant Dirac-Bergmann algorithm and the covariant phase space formulation, and secondly, the duality structure of a variety of theories was studied, for the cases of Maxwell, Proca, Kalb-Ramond and Chern-Simons theory, and non-Abelian duality rotations in differential geometric terms. Outside non-Abelian questions (e.g. duality rotations), much of the study considered the electromagnetic $U(1)$ case for simplicity. Yang-Mills generalization would be relatively straightforward.

   Applying the Dirac-Bergmann algorithm to the Isokhronon theory resulted in a constraint algebra equivalent with the usual Maxwell constraint structure, including e.g. the Gauss constraint and only two propagating photon degrees of freedom, despite the covariantly constant $\mathrm{D}X_a = 0$ magnetization and polarization background — and provided a suitably torsionful background geometry. This marks a stark contrast to the Khronon theory, where proper dust degrees of freedom appear in a similar integration constant fashion [20, 71]. The torsion-dependence, also visible when integrating the Lagrangian by parts, decidedly argues against the viability of the Isokhronon theory. Some technical remarks on background fields and Noether's theorems were considered. The symplectic geometry derived from the covariant phase provided a new and immediate argument for the quantum equivalence (viz. the BRST complex, geometric and deformation quantization) of first order gauge theory with the standard, second order form, consistent with previous results [72, 73]. More generally, a symplectomorphic phase space was seen as an argument for canonical quantum equivalence.

   The duality implications of introducing auxiliary fields was found to be of particular pertinence, as they define the parent action (Lagrangian, generating functional, etc.) in duality terms. Then, the path integral formulation provides an especially prompt analysis of the effects of such additions, as in many relevant cases they seemingly allow for a literal integration over the auxiliary field measure, in terms of e.g. Gaussians or Dirac delta functions. It was shown that this remains so in the recently found S-dual of Maxwell-Chern-Simons theory [74], provided the vacuum structure is maintained (e.g. disconnected propagators). Proca-Kalb-Ramond duality was the subject of some recent controversy [75, 76], but it was shown that this remains the case in vacuum Proca and Kalb-Ramond theory, and instead quartic self-interactions simply break the dual description (the quartically self-interacting theories no longer possess a common parent partition function that would integrate to both one and the other). The crucial conclusion is that dualities of this type are simply identities for the defining system (e.g. the path integral) — the integration can be performed directly. Finally, the failure of non-Abelian electric-magnetic dualities and rotations [77] were shown to immediately follow from differential geometry, e.g. the lack of a non-Abelian Poincaré lemma, loss of homogeneous structure for non-Abelian connections $F[B] \neq aF[A]$, or solving linear algebraic systems.

5. The preprint [5] was intended as a conclusion to the approach for gauge-gravity unification as was initially (!) attempted by the author. In particular, three main points were argued.

   First, it was argued that gauge-gravity unification immediately runs into *geometric* issues, before even theorems like Coleman-Mandula necessarily manifest. That is, gravity deals[7] with external symmetries and reductions of the frame bundle. The tangent bundle is canonically induced and soldered onto the base manifold, and is tangent by definition, while internal gauge groups are defined on a separately appended principal bundle. Similarly, observer-signal information arguments, formulated in terms of a "signal triplet", suggest a geometry change in the unified phase. Note that if the solder form is taken as the order parameter, the unified phase can be signified by a zero value [15], i.e. degenerate geometry. However, here the author's arguments were termed "heuristic", as they are contingent on the specific formulation of spacetime and unification. A more standard particle physics view might not care about dimensionality or tangent geometry vs. particle excitations in the Hilbert space.

   Secondly, the author has argued that the constitutive law in premetric theory can be understood as reducing the B-field of BF theory fare to some function of the field strength F. In turn, auxiliary fields can be simply dismissed from the Lagrangian, cf. dualities and parent systems. Therefore, the only option for nontrivial phenomenology would be a degenerate constitute law, with multiple possible minima. This kind of degeneracy is also somewhat reminiscent of spontaneous symmetry breaking, but not of the same Lie group source. The case of Plebanski theory has already been extensively studied [78], and the paper explored options how to facilitate a similar multi-phase phenomenology in electromagnetism.

[7] Rather, can deal, to be objective. There are many formulations of gravity, and it is difficult to exclusively claim only one interpretation.

Finally, the folly of the author's attempts for a clear internal gauge analogy to the Khronon was formulated in terms of a commutative diagram being ill-defined. As Maxwell-Yang-Mills theory describe independent interactions, it is not surprising that there is no canonical carryover from one sector to the other. However, the idea of this approach was nevertheless put forward, as a new concept to consider. The most straightforward way to unify the Khronon with other interactions was considered in terms of constraints, with Smolin's unified model [29] as a specific example.

1.3 Statements

In particular, the dissertation argues for the following summarized statements:

1. The Cartan Khronon theory admits a consistent description in its equations of motion, Lagrangian, Hamiltonian, (Noether) symmetries and matter couplings. The Lorentz symmetry breaking is crucial for the metric phase and the cosmological dust phenomenology.

2. The Khronon waywiser constraint provides a simple inclusion principle into any other field theory model, gauge-gravity unification in particular. The interest and significance of the constraint alone, however, is uncertain and requires case-specific analysis.

3. First order formalism in gauge theory, and BF theory in particular, are a formalization of premetric ideas and prospective grounds for studying the split of internal gauge vs. gravity. The Isokhronon, however, is not consistent, albeit one of the more immediately obvious candidate analogies to the Khronon. Lagrangian analogy is an independent concept.

4. A class of field theory dualities (e.g. Proca-Kalb-Ramond) is path integral identity. Correspondingly, nontrivial phenomenology development for auxiliary fields in this class implicates degenerate constitutive laws.

5. The core issue with gauge-gravity unification is geometric, establishing what the internal gauge-gravity unified phase even is. The author prefers a topological phase, and a requirement for consistent separation of the frame bundle, possibly in higher-dimensional manifolds. However, this issue is also heavily interpretation-dependent — and due to a lack of experimental data, difficult to unanimously solve.

1.4 Structure of the dissertation

This dissertation is divided into two main components. In the first part, a general overview, a cover letter in a sense, of the mathematical and physical material is provided. This material is primarily referential and does not include novel research. Similarly, it is not to be considered exhaustive, as the topics require and deserve a significantly more detailed discussion to preserve their entire nuance and purpose. However, an introduction to primary topics and theories is nevertheless provided.

First, chapter 2 provides an introduction to the relevant differential geometry, in terms of manifolds and fiber bundles. Gauge theory is introduced in terms of connections on principal bundles, and the Cartan geometry motivation for the Khronon waywiser tetrad is provided. A brief reference to matter fields in terms of associated vector bundles and spin geometry is included. Then, chapter 3 provides the main dynamical premise for the study. The topics include the variational bicomplex and covariant phase space, premetric theory and topological field theories. A description of electromagnetism and Yang-Mills theory, including dualities and gauge fixing, and the dynamics of gravity is provided. Finally, chapter 4 discusses the main topics of unification. The Khronon theory of gravity is introduced, alongside a discussion of its currently most pressing issues. Various avenues of unification are briefly introduced, that were touched upon at various points in study, including particle physics theories (Krasnov & Percacci), division algebras (Furey), and Cartan-geometry motivated approaches (Westman & Zlosnik). A short section on the (lack of) pertinent observational signatures is provided. This dissertation is concluded in chapter 5, with a discussion of possible future directions, and a summary of the author's line of thought on the Khronon theory and unification.

The second part consists of an as-is reproduction of the author's publications [1–4], which include the work done by the author during the doctoral programme, and which compose the scientific novelty of the dissertation.

Chapter 2

Spacetime and matter

As the Khronon theory of gravity is, in its core, an argument on the specific geometry of spacetime, and namely the geometry of (co)frames and (co)tetrads, we begin with a brief introduction to manifold geometry, up to Cartan geometry, so to in a later chapter consider self-dual complexified gravity — which is the setting that was used to define the Cartan Khronon theory [20]. Generally, this mathematics lays at the core of models of spacetime and relativity. Likewise, as much of the work was dedicated to attempts at reinterpreting internal gauge theory, a similarly brief exposition on principal bundle geometry is provided. Ultimately, this is standard exposition, written in the modern language of differential forms, and will not contain much in the way of new material, except for the emphasis of the presentation. In physical terms, the chapter considers the differential geometric premise of classical field theory on curved spacetime — complete classical field theory would be obtained by introducing a Lagrangian to select actually physical field configurations, which will be the subject of the following chapter.

It increasingly appears that the basic and most direct application of pseudo-Riemannian geometry and fiber bundle structure is slowly nearing exhaustion, in the sense that to develop a truly new theory either ever-more-sophisticated structure has to be overlaid onto an underlying geometry, or the underlying geometry has to be changed to ever-more-intricate structure (or more likely, both). Let us only mention the amplituhedron [79], noncommutative geometry [80, 81], and orbifolds [82–84] as an arbitrary selection of nontrivial extensions — indeed, even teleparallel relativity embedded into metric-affine geometry [85, 86] is of a generalized class compared to the basic Levi-Civita structure. It is readily apparent that a thorough physical *and* mathematical understanding of differential geometry is inescapable — even gauge fixing in the BRST antifield formalism is justified in terms of the geometry of the covariant phase space [87].

At the same time, many topics will not be included at any significant length, if at all, but would deserve substantial discussion for a more precise understanding of the geometry of physics. For specific examples, let us note the geometric description of nonrelativistic physics, e.g. in terms of the Galileian group [88–90], or discretization procedures, including Regge calculus [91], or the generic method of attaching structures onto underlying spaces in terms of sheaves in category theory. Not only would this hold theoretical value, but would actually be relevant for continuing work with the Khronon (e.g. discrete Khronon quantum spacetime per adapting to Regge calculus, spin foams, etc.). Hopefully the present mention of these ideas will serve as some motivation and guidance.

The basic references for this chapter are [16, 58, 92, 93], and the mathematics itself is well understood. Beware the expedite tempo of introduced concepts and definitions; a full mathematical treatise is, unfortunately, beyond the scope of this dissertation.

2.1 Geometry: from topology to manifolds

The common model for spacetime springs from $\mathbb{R}^n$-like continuity, which immediately suggests manifold geometry. Defining this structure requires the notion of continuity as provided by general topology, e.g. simplest in its point-set formulation, so that a "local homeomorphism" to $\mathbb{R}^n$ could bring over the apparatus of mathematical analysis in a natural fashion.

Definition 1. *A topology $\tau \subset \mathcal{P}(M)$ on a set M is the collection of subsets such that*

1. $\emptyset \in \tau$, $M \in \tau$,
2. $i \in \mathcal{I} : O_i \in \tau \Rightarrow \bigcup_{i\in\mathcal{I}} O_i \in \tau$,
3. $i \in \mathcal{N}_n = \{1, 2, \ldots, n\} : O_i \in \tau \Rightarrow \bigcap_{i\in\mathcal{N}_n} O_i \in \tau$.

The pair (M, τ) is called a topological space. Sets $O_i \in \tau$ are called open sets.

Very often, the practitioner is imprecise and abuses terminology, e.g. referring to just M as a topological space; this covering overview will similarly be rather relaxed in the terminology, as the aim is only to summarize notions and ideas, but not necessarily their rigor. Many definitions follow, e.g. closed sets A are those whose complement $M \setminus A$ is open, a basis of the topology is a collection of sets such that each open set is the union of some sub-family of the basis, and so on. A trivial topology is such that $\tau = \{\emptyset, M\}$, while a discrete topology is such that $\tau = \mathcal{P}(M)$. General topology can similarly be constructed in different formulations, e.g. instead in terms of neighborhoods or closed sets, and the definitions can easily be adjusted, e.g. here it is sufficient to only require the intersection of *two* sets to be contained in the topological space, as finite intersections follow immediately.

Arguably the most important notion that can be derived from this basic, abstract structure is the notion of continuity.

Definition 2. *A mapping $f : M \to N$ between two topological spaces (M, τ_M) and (N, τ_N) is called continuous if for any open set $O \in \tau_N$ the preimage is also open, $f^{-1}(O) \in \tau_M$.*

A continuous bijection, whose inverse is also continuous, is called a homeomorphism.

It is a common exercise to show that the epsilon-delta definition of continuity in $\mathbb{R}^n$ is fully consistent with this definition, in the standard open-ball topology of $\mathbb{R}^n$. Recall that open balls around p and of radius r are those sets such that

$$O_{p,r} = \{x \in \mathbb{R}^n | \, ||x - p|| < r\}. \tag{2.1}$$

Homeomorphisms define an equivalence class of topological spaces. This allows to define another important notion, that of topological properties. Topological properties are simply any structures that are shared (invariant) between homeomorphic topological spaces. Any listing would be quite varied, but we will only name some categories of more immediate relevance to further mathematics and physics.

1. Separability — a question how *topologically* distinguishable separate points are. The topology binds together points into open sets, but without further structure, there is not generally any possibility of topologically distinguishing these points themselves, i.e. the points are topologically degenerate. Most commonly, a Hausdorff (T_2) space is assumed — every two distinct points have disjoint open neighborhoods. It is not difficult to conjure scenarios with physically significantly more degenerate structure. Gauge theory, for one, works with technically distinct but gauge-equivalent structures, and e.g. a zero metric region would introduce metrically degenerate regions. Commonly, however, these are not considered as degeneracies of the topology, but of some other structure.

2. Compactness — the corresponding adjective evokes the intent. Generally, properties of compactness generalize closed and bounded subsets of Euclidean space, and give a notion on "size". For spacetime, paracompactness is beneficial, as it is related to partition of unity (relevant for e.g. integration) in relation to Hausdorff-separability. A topological space is paracompact if every open cover has an open locally finite refinement. Compactness is important in questions of measure (e.g. probability), but physically, non-compact gauge groups, such as the Lorentz group, will quickly run into quantum issues, e.g. with unitarity.

3. Connectedness — a question how many separate components a topological space might have. In another way, the question on holes and empty spaces between "material" parts. A space is connected if it is not the union of disjoint non-empty open sets. This is physically reasonable, as the separate components are difficult to continuously causally link. Often, a more restrictive requirement of path-connectedness is assumed, where any two points possess a path between them.

4. Countability — a requirement on whether some part of the topological structure can be reduced to a countable cardinality, rather than being of the more troublesome uncountably infinite variety. Manifolds are commonly assumed second-countable, i.e. they possess a countable basis for the topology. This ensures that manifolds can be embedded into some finite-dimensional Euclidean space, and is a useful technical assumption — similarly for manifolds, paracompactness follows from second-countability.

These items only include some common, often technical, assumptions on spacetime. There are many more properties, and varieties of properties, that are studied. For example, cardinality is, in principle, a topological property as well, but physics does not often encounter such fundamental set-theoretic quantities. Metrizability is very relevant, as a metric geometry has proven to be physically more relevant. Most importantly, we are making a fundamental *assumption* on spacetime — that it is Hausdorff-separable, paracompact, path-connected, second-countable, and, soon-to-be-defined, locally Euclidean. The validity of such assumption can only be verified to the extent that the predictions derived from it are consistent with what is observed.

A physicist reader might rightfully be confused why a physics dissertation begins with such an abstract (and incomplete!) introduction to the mathematical apparatus. So far, even the still abstract, but much more computationally useful structure of a manifold was not defined. The point the author wants to drive is of the toolset that is actually implicitly being operated. Quite on the contrary, a theoretical physicist should be very mindful of the model that is being operated to describe Nature, and of every cog in the machine that can be calibrated, adjusted and replaced to develop new *models*, with the aspiration of eventually reaching something even more foundational — and the careful awareness of everything that can be broken along the way. So let us provide a fundamental justification for physics: the choice of topology defines the model how signals are transmitted through spacetime, considered as a topological space. Signal transmission is *assumed* continuous, in the most general sense. In essence, there is a fundamental assumption of relativity present. Manifold geometry, outside of boundary questions, assumes a locally open ball structure[1]. There is no causality assumed. Rather, this requires a light cone structure and the introduction of the pseudo-Riemannian metric — or perhaps instead causal orderings [94]. Even more, there are also more immediately causally motivated topologies available, such as Zeeman-Göbel and Alexandrov [95–98]. Without introducing the metric, no temporal notions are introduced as well, so in a purely topological regime, there is a loss of causality. In turn, this illustrates what a possible unified phase might appear like — at least in one variant, as the ideas hardly hold a unique preference.

The preceding notions prove sufficient to introduce manifold structure, one of the grandest arenas for differential geometry. Manifolds correspond to geometries which locally appear like $\mathbb{R}^n$, but can have nontrivial global topology. Locally Euclidean topology is formalized in an obvious way through charts and atlases.

Definition 3. *A chart is a homeomorphism $\varphi : U \to \mathbb{R}^n$ from an open set U in a topological space M to Euclidean space. A C^k-atlas is a collection of charts (to $\mathbb{R}^n$) that cover the whole of M, such that all transition functions $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$, $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are of class C^k.*

An atlas can be extended to include all compatible charts, producing a maximal atlas, so the geometric object is unambiguously defined, independently from any particular single atlas representation.

[1]The author argued [5] that this can be interpreted in terms how signal transmission is modelled, in a sequence of steps (choice of topology, choice of causality, etc.) for a spacetime-creation analogy with the premetric theory of electromagnetism [58]. However, interpretations are not calculable, and in practice it is simpler to just postulate a geometry and a Lagrangian field theory on top.

Definition 4. *A manifold M is a topological space*[2] *equipped with a maximal atlas: it is a locally Euclidean*[3] *topological space.*

Charts permit local differential calculus on $\mathbb{R}^n$, while the atlas structure permits nontrivial global topology. Depending on the class of the atlas, we are presented with topological, differential, smooth etc. manifolds. Transition functions can be further characterized, e.g. by the notion of orientation (viz. the sign of the Jacobian determinant, existence of the volume form, etc.) we can define an orientable manifold. Furthermore, chart-compatibility allows to understand mappings $f : M \to N$ between manifolds, in particular in the sense of their (local) differentiability. The most important comparison operation is provided by the notion of diffeomorphism: on smooth manifolds, a smooth bijective mapping with a smooth inverse (or, correspondingly with relaxed differentiability). This introduces the notion of isomorphism into the category of manifolds.

Altogether, a beautiful variety of geometry is derived from this basic construction. For our purposes, we primarily request differential calculus, and then a method to append yet more structure: the manifold structure, as wonderful as it might be, is intrinsically still quite barren and stems from just a topological requirement. For this reason, let us quickly introduce also the notion of fiber bundles before anything else: for a fiber F, the concept is just to permit local trivializations $F \times U$, so the fibers are locally trivially attached to the base manifold, literally so.

Definition 5. *A manifold E is called a fiber bundle, $F \to E \xrightarrow{\pi} M$, with general fiber F, base manifold M and a surjective map π, if for every $p \in M$ there is an open neighborhood $U \subset M$, such that it can be locally trivialized: there is a diffeomorphism $\phi_U : E_U \to U \times F$ such that $pr_1 \circ \phi_U = \pi$, where pr_1 is the projection unto U.*

Depending on the specific structure of the fiber F, we are provided with e.g. line bundles, vector bundles, or, as in the next section, principal bundles. Regardless, it is natural to introduce (smooth) *sections* $s \in \Gamma(E) : M \to E$ of the fiber bundle, the space of all sections being $\Gamma(E)$. The implication of a geometric setting for classical field theory is clear. However, for curiosity, note that this introduces an ambiguity into

[2] The exact topological requirements vary: e.g. Hausdorff-separability and paracompactness is common, for technical reasons.

[3] This will not be discussed at length, but in principle, a *manifold with boundary* is to be considered a separate object, and requires a slightly adapted definition, in permitting some charts to include the edge — to be mappings to the half-space $\{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n \geqslant 0\}$. Further e.g. manifold with corners.

the background spacetime premise: every global smooth section is diffeomorphic to the base manifold. What constitutes spacetime, if this freedom persists? Is it even possible to uniquely determine what is the base manifold (*spacetime*), or does it even hold any meaning? In a sense, all diffeomorphic manifolds can be considered the "same", so it is not necessarily sensible to distinguish a *specific* manifold as *the* spacetime. Partly this will be resolved by introducing a connection, but partly this re-enforces an operational view unto physics, where the capacity for calculation takes the forefront. This becomes more a philosophical question, that bears little difference to calculable physical implication.

Returning to the basic manifold construction, some objects follow naturally. Real (smooth) functions $C^\infty(M)$ are simply mappings $f : M \to \mathbb{R}$, and they define an algebra over the real numbers. Vector fields $v \in TM$ can be (algebraically) defined as derivations of this algebra of functions, collected into the set $\mathrm{Vect}(M)$ of all vector fields on M — and with it, introducing vector and tensor calculus to the manifold setting. There is a multitude of equivalent definitions of tangent vectors, e.g. as equivalence classes of curves $\gamma : (-1, 1) \to M$ through a point p, which can then be extended to tangent vector fields, and used to define the tangent bundle TM. Due to the importance of tangent (frame) geometry, let us provide this in definition.

Definition 6. *A tangent vector at $p \in M$ is an equivalence class $\dot\gamma = [\gamma]$ of curves γ_i through p, $\gamma_i(0) = p$, such that $\gamma_1 \sim \gamma_2$ if for arbitrary local coordinates $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 x(\gamma_1(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 x(\gamma_2(t))$.*

The tangent space T_pM of a manifold M at $p \in M$ is the set of all such tangent vectors at p.

Then, it can be shown that T_pM truly does have the structure of a linear space — refer in particular to [92] for a clear recourse on the many possible definitions of a tangent vector and a tangent space.

Definition 7. *The tangent bundle is the disjoint union*

$$TM = \dot{\bigcup}_{p \int M} T_pM$$

with the canonical projection $\pi : TM \to M$ and has the structure of a smooth real vector bundle.

Tangent geometry is ubiquitous to manifold geometry and automatically introduced — this bears special emphasis. Applications range from modelling spacetime symmetries (Killing vectors) to variational calculus on the variational bicomplex and beyond — some are explored further on in this dissertation. To hint at future sections, unified classical geometry seems largely an issue of separating directions tangent to spacetime from those directed to internal geometry, while gravitationally relevant is the notion of a frame (tetrad in particular), really just bases of tangent spaces.

Covector fields, that is differential 1-forms, become mappings $\omega \in T^*M : TM \to \mathbb{R}$, and further introduce the bundle of exterior k-forms $\Omega^k(M) = \bigwedge^k T^*M$, which can be summed into the entire exterior algebra. The operations $\bigoplus$, $\bigotimes$, $\bigwedge$ hold a natural definition compatible with basic tensor analysis, simply continued fiberwise. Evaluation of a vector on a differential form is mediated by the interior product $\lrcorner$, an antiderivation on the algebra of exterior forms, while the exterior derivative d increases the rank of a differential form. Finally, these structures and objects can be introduced from one manifold to another via $f : M \to N$, by the notion of pushforward f_* and pullback f^*.

Consider all of these structures naturally and immediately present. The cynic, however, will refer to the basis expansion of a differential form,

$$\omega = \frac{1}{k!}\omega_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}, \tag{2.2}$$

and dissuade from any insurmountable advantage of differential calculus on differential forms compared to standard tensor index formalism — this is true, and must be so in principle. But convenience in many senses cannot be dismissed, e.g. consider the definition of de Rham cohomology as given by the complex

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \xrightarrow{\mathrm{d}} \Omega^4(M) \xrightarrow{\mathrm{d}} \dots, \tag{2.3}$$

or in another vein Stokes' theorem condensed to

$$\int_M \mathrm{d}\omega = \oint_{\partial M} \omega. \tag{2.4}$$

Note this requires introducing integral calculus on manifolds through e.g. cochain complexes, which will not be considered at length, despite the relevance to e.g. variational calculus. The statement of the Poincaré lemma is similarly at leisurely capacity: on a contractible smooth manifold, a closed form is also exact,

$$\mathrm{d}\alpha = 0 \Rightarrow \alpha = \mathrm{d}\beta.$$

So, the language of differential forms and adjacent structure will be used because it is convenient in many senses, and in a sense more immediately derived from the geometry, but it cannot replace the computational efficiency as provided by tensor index formalism.

It is not useful to speculate on whether four-dimensional space(time) is sacred in any particular fashion: some results are rigorously known, for example General Relativity in lower dimension is simply a topological Chern-Simons theory[4] [102, 103]. If discussing Lie groups, there are certain exceptional isomorphisms, such as $Spin(4) \cong SU(2) \times SU(2)$ or $Spin(3) \cong SU(2) \cong Sp(1)$, which can be argued to have special physical relevance, by virtue of chirality. On the other hand, quantum field theory introduces deviations from integer dimensions as part of dimensional regularization without extensive qualms on the meaning (cf. [104] on comparison with Wilsonian Effective Field Theory), and fractal dimensions also emerge in quantum gravity [105]. Alternatively, from the Whitney embedding theorem we can infer that higher-dimensional spaces are not by themselves anything inexcusable, as any nontrivial manifold that physics might utilize can be seen as a hypersurface in some higher-dimensional background space.

Definition 8. *Let $f : N \to M$ be a smooth map between manifolds, $\dim N \leqslant \dim M$. If the pushforward f_* is everywhere injective, f is called an immersion. If both f and f_* are injective, f is called an embedding. A submersion requires the pushforward to be surjective (lesser the dimensionality requirements).*

In particular, such notions of inclusion are useful to distinguish possible cusps, corners and edges.

Theorem 1 (Whitney embedding theorem)**.** *Any smooth m-dimensional manifold can be embedded into $\mathbb{R}^{2m}$.*

Note that the embedding theorem is not necessarily a guideline to a better description of geometry. Here it is referred to because it helps position manifold geometry in relation to Euclidean geometry. More importantly, at a point, a hypothesis was considered how spacetime might become more than just a base manifold to attach structures to, so the dynamics of embedded hypersurfaces (cf. branes) is not entirely unthinkable — however, this was not developed to any greater extent.

Nevertheless, continuing this line of thought leads to ideas of e.g. holography [106–108], or the (anti-)de Sitter group, as seen for example in MacDowell-Mansouri [109] or Stelle-West gravity [110]. For the latter in particular note that the Poincaré algebra can be seen as a deformation of the Galilei algebra, and permits a further deformation to (anti-)de Sitter

[4] As was kindly emphasized by a referee, the relation to Chern-Simons theory is *not* exact due to the non-degeneracy of the coframe [99–101]. Nevertheless, unified field theories might want to relax the requirement of non-degeneracy.

as well — so there is plenty of motivation to deform spacetime geometry in this direction.

Physically, we would generally assume that we are provided with a topological space M that is, as mentioned before, Hausdorff-separable, paracompact, path-connected, second-countable, locally Euclidean, and furthermore e.g. oriented. More curiously, note that despite the introduction of calculus, a manifold is otherwise actually a rather barren object: essentially, its definition is a topological requirement, which just permits significant computational carry-over. There is no surprise that still more structure is required for any physical geometry.

For the last motif of this section, consider the question: if we are to truly build a new fundamental theory of spacetime, how should all of this structure, and more, appear? The Khronon proposes a step forward in the *metric* structure, but the reader should also acknowledge that the implication of any such aspiration is possibly significantly more grand than at first glance might appear. That is, should the goal be to find some mechanisms to produce also the entirety of geometry, from topology to differential calculus? We are at a complicated choice of strategy — the most successful theories of fundamental physics, General Relativity and quantum field theory (in the representation of the Standard Model, and close extensions) have an exhaustive definition and formalism, but their extension appears anything but easy or obvious. Yet, without extending them *somehow*, it is not entirely clear if a unification of gravity and the rest of the interactions is meaningful or even possible.

2.2 Principal bundles and gauge theory

To a large part, this dissertation works with gauge theory. However, a general definition of gauge theory itself will not be attempted, because, apart from somewhat being semantic meandering, partly the importance of different aspects are not unanimously agreed upon[5], and partly, the full precision of the definition is of relatively little physical relevance, compared to calculation of observables from specific Lagrangians. An interested reader can refer to e.g. Trautman [111, 112], and shortly, internal and external symmetry transformations will also be introduced. Generalizing these notions even further could lead to a less than useful extreme, arguing for gauge theory to refer to just the case of multiple representatives of some equivalence class related by some "gauge transformations". In practice, the most

[5]From personal anecdote, the exact gauge-theory relevance of, say, first class constraints versus internal and external symmetries has been of slight controversy.

relevant structure to introduce is the geometry of principal bundles, which describes the geometry of electromagnetism and Yang-Mills theory.

The equivalence of differing representatives is an important characteristic of gauge theory, but this span is characterized by the action of a particular Lie group G, so the definition becomes calculable. Equipped with this notion, it is straightforward to introduce the required structures, beginning with the notion of a Lie group itself — loosely put, requiring the group operation to be smooth.

Definition 9. *A Lie group G is a group which is also a smooth manifold, such that its group product and inversion are smooth mappings.*

Purely on principle there is no obstruction, apart from the loss of useful structure, to at least trying to generalize to weaker algebraic structures, e.g. monoids or semigroups, but Lie groups are of practical utility — in relation to rotations, literally so with e.g. the special orthogonal groups $SO(n)$. Groups in general find their utility when understood as collections of possible transformations, and Lie groups represent continuous such transformations. Thus, we are importantly provided with the general linear group $GL(n, \mathbb{K})$, its various (closed) subgroups (the classical Lie groups, SL, $(S)O$, $(S)U$, Sp), the exceptional compact Lie groups G_2, F_4, E_6, E_7, E_8, and the spin groups $Spin$. Further note the classification of compact Lie groups, which prove an isomorphism with (products of) the preceding groups. Hamilton [16] gives a particularly clean interpretation in terms of the Peter-Weyl theorem, as compact Lie groups can be seen as embedded Lie subgroups of *some* $SO(n)$, i.e. literally as rotations on $\mathbb{R}^n$.

The *nonlinear* effect of a group on some space (set) is provided by its group action, while the linear effect is studied in terms of representation theory. The mathematical model is straightforward, as it follows homomorphisms, i.e. structure preservation, to the group of all transformations of the space in question. The purpose is subtle in its utility: we can characterize objects by their behaviour under various transformations. It is quite serendipitous that the Standard Model Lagrangian then only requests the simplest of multiplet representations of $U(1)_Y \times SU(2)_W \times SU(3)_C$, rather than some horrendous structures — or perhaps a linear approximation in rotations is a foregone conclusion at low energies.

Definition 10. *A representation of a Lie group G on a vector space V is a homomorphism to the space of linear isomorphisms, $\rho : G \to GL(V)$.*

Definition 11. *A left action of a group G on a set M is the mapping*

$$\begin{aligned} \Phi : G \times M &\to M \\ (g, p) &\mapsto \Phi(g, p) = g \cdot p, \end{aligned}$$

which satisfies

1. $(g \cdot h) \cdot p = g \cdot (h \cdot p)$, $\forall p \in M, g, h \in G$,
2. $e \cdot p = p$, $\forall p \in M$.

The choice between left and right actions is mostly arbitrary convention.

Linear representation theory is vast and of immense utility [113, 114]. Nonlinear theory, with generic group actions as an example, is more difficult to handle in general, but has natural structures when characterizing group actions, e.g. left translations, orbits, stabilizers, and possible characteristics of the actions themselves, e.g. free, transitive, faithful, principal (so the action is free, and the map $(p, g) \mapsto (p, p \cdot g)$ is closed — the name is consistent with the geometry, as this action can also be used to define a principal bundle). In particular the space of orbits O_p allows to define a quotient space with respect to the action,

$$M/G = \{O_p \subset M | p \in M\}. \tag{2.5}$$

This is relevant for constructing associated bundles, and generally in gauge theory as well.

Definition 12. *Consider the fiber bundle $G \to P \xrightarrow{\pi} M$, with general fiber G and a smooth right action $P \times G \to P$. P is a principal G-bundle if*

1. *the action preserves the fibers, and is simply transitive on them,*
2. *there exists a principal bundle atlas for P. That is, there is a fiber bundle atlas of G-equivariant charts $\phi_i : P_{U_i} \to U_i \times G$ such that*

$$\phi_i(p \cdot g) = \phi_i(p) \cdot g, \forall p \in P_{U_i}, g \in G.$$

The principal bundle is the fundamental object of study of classical gauge theory. Many structures are naturally introduced, e.g. morphisms of principal bundles require an equivariant smooth mapping, "preserving" the principal bundle geometry. Note "quantum" principal bundles (in the sense of noncommutative geometry) as a further development [115], or how Hamiltonian analysis rather focuses on the constraint algebra [87]. Regardless, principal bundle theory is *the* basic modern differential geometry of gauge theory. A particular purpose of this covering introduction is also to emphasize how structures can be introduced or already be naturally present — regardless whether a field theorist chooses to engage with the differential geometry of principal bundles, from necessity or convenience, this structures is implicitly present still.

Some examples about the variety of principal bundles are in order. Gauge interactions are described by principal bundles of the corresponding symmetry group: for example, the electromagnetic $U(1)$, weak $SU(2)$ and strong $SU(3)$, or the Standard Model symmetry group $U(1)_Y \times SU(2)_L \times SU(3)_C$ (requiring the important notable addition of symmetry breaking). However, note that the frame bundle $Fr_{GL}(M)$, constructed pointwise from bases of T_pM, is similarly a $GL(n)$ principal bundle, $GL(n) \to Fr_{GL}(M) \xrightarrow{\pi} M$. The frame bundle is naturally induced, the gauge interaction principal bundles have to be appended *ad hoc*. Generally speaking, it is possible to imagine the idea of attaching the option or possibility (i.e. the capacity to do so) of a transformation to each spacetime point — rather quickly, the conclusion would be to simply attach the entire Lie group itself, ultimately leading to the notion of a principal bundle.

Principal bundles can be "compared" as well by virtue of the notion of morphisms. Let P and P' be correspondingly G and G' principal bundles over the same M, and let $f : G \to G'$ be a Lie group homomorphism. Then a bundle morphism $H : P \to P'$ is an f-equivariant smooth bundle map; if f is also an embedding, then H is called a G-reduction of P', and $\mathrm{Im}(H)$ is a principal G-subbundle of P'. What is curious is how relativity appears as an SO-reduction of the frame bundle [116, 117] — nevertheless, the standard way to see it is that a Riemannian metric (and an orientation) defines an SO-structure on M. The author had toyed with whether the notion of principal bundle reduction was primary, so a chosen reduction of the frame bundle would have introduced the corresponding notion of relativity to spacetime. That is, whether the metric defines the reduction, or the reduction would induce the remainder of relativity structure. More than that, the reduction of specifically the *frame* bundle, which is tangent to spacetime and of relevance to gravity, appears special compared to appending arbitrary principal bundle structures — the question appears how should this be handled in a *unified* geometry. However, this premise did not result in a particularly calculable conclusion, which would rather operate with Lagrangian field theory. Nevertheless, the author cannot help but imagine this question of decomposition of the frame bundle to be at the core of any potential unified classical geometry. If to speculate, a unified geometry might have the frame bundle reduced to a unified group, and the dimensionality might be a question of representation theory.

Introducing a SO-structure is akin to moving to orthogonal frames — sections of the frame bundle are tetrads. The specific choice of $SO(1,3)$ on a four-dimensional manifold is physically motivated. Comparatively, a pseudo-Riemannian metric of signature (s,t) on a manifold M is a non-degenerate fiberwise symmetric section of the tensor bundle

$$g \in \Gamma(T^*M \otimes T^*M). \tag{2.6}$$

At each point it induces a non-degenerate symmetric bilinear form of signature (s,t) on the tangent vectors

$$g_p : T_pM \times T_pM \to \mathbb{R}. \tag{2.7}$$

This allows to define orientation and time-orientation, depending on the specific reduction of the frame bundle, and introduces a canonical correspondence between the tangent and cotangent bundle in terms of the musical isomorphisms $\sharp$ and $\flat$.

In particular, the metric introduces the notion of a Hodge duality between differential k and $n-k$ forms, where n is the dimension of the manifold M. This is a curious and important notion, and due to the explicit appearance of the (inverse) metric, the author's [2] took it as a particular object to try work around. Nevertheless, there is no inherent reason for disdain, of course, as the Hodge star is naturally present as soon as a metric is introduced. Especially important is the introduction of an inner product onto differential forms, which the entirety of field theory Lagrangians use[6]. So, we have the Hodge star

$$* : \Omega^k(M) \to \Omega^{n-k}(M), \tag{2.8}$$

with

$$\alpha \wedge *\beta = \alpha^{\mu_1\cdots\mu_k}\beta_{\mu_1\ldots\mu_k}\mathrm{Vol}_g,\ \ \alpha,\beta \in \Omega^k(M). \tag{2.9}$$

Note the g-volume form Vol_g and a very curious mixing with Lorentzian duality on the hypersurface basis, $*(e^a \wedge e^b)$ vs. $\epsilon^{ab}{}_{cd}(e^c \wedge e^d)$ vs. $\star(e^a \wedge e^b)$. Moreso, the latter is a matter of self-dual gravity and the complexified Lorentz group, but moving over to Hodge duality (rather than the internal duality operator $\star$) is one way how to e.g. try generalize the Khronon to other dimensions. In practical terms, the Hodge dual can be calculated as

$$*\alpha = \frac{\sqrt{\det g}}{k!(n-k)!}\alpha_{\mu_1\ldots\mu_k}g^{\mu_1\nu_1}\ldots g^{\mu_k\nu_k}\epsilon_{\nu_1\ldots\nu_k}\mathrm{d}x^{\nu_{k+1}} \wedge \ldots \mathrm{d}x^{\nu_n}, \tag{2.10}$$

[6]This question is more subtle than at first glance. On one hand, due to the presence of the metric, the Hodge dual also introduces some relation to causality to a field theory action, so it is a natural candidate to utilize for including time evolution. In more Effective Field Theory terms, any and all Lagrangian respecting some symmetry could be written down and studied. Some Lagrangian terms turn out to be topological, or otherwise unremarkable. What should the extent of the theory be?!

using the particularly nonlinear inverse metric[7], while the Hodge inner product

$$\langle \alpha, \beta \rangle_g = \int \alpha \wedge *\beta \tag{2.11}$$

has a significantly more prosaic component expression in $\alpha^{\mu_1 \cdots \mu_k} \beta_{\mu_1 \ldots \mu_k}$. In this sense, exhausting all index contractions is a somewhat simpler operation to execute than to exhaust all relevant combinations of $\lrcorner, \wedge, *, \ldots$

Returning to understanding the geometry, the fibers of a principal bundle are isomorphic to the structure group almost by definition, so the basic geometry is thus quite straightforward — the principal bundle geometry simply attaches Lie groups to each point on the base manifold. Although globally there is a possibility for a "twist", the following theorem is quite severe.

Theorem 2. *If M is a contractible manifold and G is a Lie group, then every principal G-bundle over M is trivial.*

The apparatus that has been introduced so far is sophisticated, but its most striking relevance is almost entirely in global geometry, as locally, and even for many physically relevant global spacetimes, this structure is necessarily a trivial bundle. When considering e.g. quantum field theory, which is one of the primary practical areas for application of Yang-Mills-type gauge theory, the common assumption is that of trivially contractible Minkowski space background and observable states being only asymptotic.

Definition 13. *A gauge transformation is a principal bundle automorphism.*

Note a crucial point: gauge theory symmetry transformations, i.e. gauge transformations, are automorphisms of the principal bundle, while gravity works with the base manifold itself, and its symmetry group, i.e. diffeomorphisms of spacetime. The gauge group of the principal bundle (and corresponding physical theory!) is the generally infinite-dimensional automorphism group $\mathcal{G}(P) = \mathrm{Aut}(P)$. For General Relativity, the analogous gauge group is $\mathrm{Diff}(M)$. This is, essentially, an orthogonal difference between the symmetries in electromagnetism and Yang-Mills theory, and gravity of General Relativity type. Hamilton [16] refers to DeWitt through Witten [17] for this insight. Correspondingly, these diffeomorphisms are

[7] In four dimensions this could be calculated as

$$g^{\mu\nu} = \frac{4\epsilon^{\alpha_1\alpha_2\alpha_3\mu}\epsilon^{\beta_1\beta_2\beta_3\nu} g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} g_{\alpha_3\beta_3}}{\epsilon^{\rho_1\rho_2\rho_3\rho_4}\epsilon^{\sigma_1\sigma_2\sigma_3\sigma_4} g_{\rho_1\sigma_1} g_{\rho_2\sigma_2} g_{\rho_3\sigma_3} g_{\rho_4\sigma_4}}.$$

respectively internal and external symmetries, and can be very concisely formalized in terms of the Atiyah exact sequence [39, 40]

$$0 \to P \times_G \mathfrak{g} \to TP/G \to TM \to 0, \tag{2.12}$$

This "orthogonality" is also the reason why the author [5] had termed electromagnetism and Yang-Mills theory collectively as internal gauge theory, while gravity could be recognized as external gauge theory — although this is not standard terminology, and only serves a semantic reinforcement on the distinction.

We are at a point where the question of comparing objects at different points becomes relevant, that is the issue of parallel transport; similarly, we are still bereft of some crucial dynamical components and structural relations — gauge bosons and interacting fields. Uncharged, unassociated fields can exist without as significant additional structure in separate fiber bundles. For this purpose, we are required to introduce correspondingly connections on principal bundles and associated vector bundles. The usual justification and explanation stems from the simple fact that two tangent spaces T_pM and T_qM at differing points $p \neq q$ cannot have an *a priori* relation, so there is no general method to actually compare vectors (tensors, differential forms, etc.) at different locations. The required structure is the connection, which further allows to take covariant derivatives, and write a covariant theory of particle and field dynamics. In relativity, we would generally already have introduced the metric, for the sake of causality. Then, restricted to tangent spaces on the base manifold, the particularly natural choice is the unique metric-compatible and torsion-free Levi-Civita connection.

However, the principal bundle perspective allows us to introduce, simultaneously, connections as gauge bosons of any symmetry group and principal bundle of interest, *and* the connection relevant for relativity, when restricting interest to a particular principal bundle — the frame bundle. Perhaps the physical relevance of this lengthy exposition now also appears more justified: as the aim was to (attempt to) *unify* the gravitational interaction with other internal gauge interactions, we should trace the locations where there is a distinction between the interactions. The method to overcome this necessarily extends somewhat beyond the basic geometry, but the issue is very core to the geometry of spacetime and matter.

Note that in the intrinsic geometry of fiber bundles there is no clear establishment *how* the fibers are actually situated in the bundle. We know there is always a local trivialization, but this is also the topological extent that is admitted, without having introduced any extra structure. Let $P_x = \pi^{-1}(\{x\})$ be a fiber over $x \in M$, and let $p \in P_x$ be a point in the fiber. Then, quite naturally, we can introduce the vertical tangent space $V_p = T_p(P_x)$ at p, and collect these spaces to the vertical tangent bundle V. By definition, the horizontal tangent space H_p is the complement to the vertical space, such that $T_pP = V_p \oplus H_p$.

Definition 14. *An Ehresmann connection on a principal bundle P is a right-invariant distribution H, i.e. a vector subbundle of the tangent bundle.*

Only this notion allows us to unambiguously write

$$TP = V \oplus H, \tag{2.13}$$

otherwise this decomposition is simply not uniquely defined. We are also immediately led to the more practical definition of a connection 1-form.

Definition 15. *A connection 1-form on P is a Lie-algebra valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ with the properties*

1. $r_g^* A = \mathrm{Ad}_{g^{-1}} \circ A$, $\forall g \in G$,
2. $A(\tilde{X}) = X$, $\forall X \in \mathfrak{g}$, *where $\tilde{X}$ is the corresponding fundamental vector field.*

It can be shown that this definition is equivalent to defining the Ehresmann connection.

We have finally reached the definition of the bosonic fields mediating gauge interactions — the connection forms *are* the gauge fields of particle physics fare. However, note a subtle issue, in that A is defined on the principal bundle P, *not* the base manifold. To be complete, we should pull the connection 1-form down using a section $s : M \to P$, to $s^*(A)$, so we are provided with a usual spacetime field.

In terms of differential geometry, the inhomogeneous (and non-Abelian) transformation law

$$A \to gAg^{-1} + g\mathrm{d}g^{-1} \tag{2.14}$$

is easily substantiated, rather than just sprung upon us on Lagrangian consistency grounds, as would generally be explained in physical field theory introductions — it is calculable from principal bundle automorphisms[8]. In

[8]The author cannot help, however, to worry about the circularity of the argumentation. Do we care about principal bundles because they are primary, or because we have found them to be relevant up to our theories in use? The question whether we find connections on principal bundles because of gauge transformations, or in reverse, is slightly easier to solve — these formulations exist simultaneously.

practical terms, the bosons A can simply be seen as Lie-algebra valued 1-forms with a particularly curious transformation law[9].

The connection is a colorful structure, which introduces still other structure — most immediately, the curvature F. In a computationally-minded view, we will define the curvature 2-form through Cartan's structure equations, although there are other (equivalent) options. So, the curvature of a connection is provided by the structure equation

$$F = \mathrm{d}A + A \wedge A. \tag{2.15}$$

We will introduce the exterior covariant derivative soon, but some useful properties can be mentioned preemptively. For instance, the Bianchi identity $\mathrm{D}F = 0$ is easy to check. Returning to frame bundle geometry, the corresponding connection is the affine (or Lorentz, when suitably reduced) connection ω_{ab}, which for the metric-compatible case is antisymmetric, $\omega_{ab} = -\omega_{ab}$, and in Riemann-Cartan geometry (2.15) can be called Cartan's first structure equation. Then, Cartan's second structure equation gives the torsion

$$T^a = \mathrm{D}e^a. \tag{2.16}$$

For post-Riemannian geometry, also note nonmetricity,

$$Q_{ab} = \mathrm{D}g_{ab}. \tag{2.17}$$

Nonmetricity is signified by a symmetric component in the Lorentz connection ω_{ab}, and is crucial in the symmetric teleparallel family of theories [118–120]. We will not consider theories with nonmetricity, but will permit torsion, as in Einstein-Cartan theory [121–124].

The final construction that is required is to introduce representations to matter fields — interactions, essentially. This is formalized in terms of associated vector bundles, and the geometric implication is that there is a sort-of internal symmetry (or structure, geometry) that is probed by the gauge group. The natural candidate construction is to request that the matter fields, as expressed as vectors in some suitable vector space, further transform in a particular linear representation of the gauge group of the principal bundle — this requires some configurations to be naturally considered gauge-equivalent.

Definition 16. *Let P be a principal G-bundle, and V a vector space carrying a G-representation ρ. The vector bundle*

$$E = P \times_\rho V = (P \times V)/G$$

[9] A transformation law which is easy to imagine being generalized with some nonminimal, nonlinear additions. However, this has not proven as vastly relevant for physics — nor developed in this dissertation, despite the curiosity of the possibility.

is called the vector bundle associated to the principal bundle P and the representation ρ.

The simple exterior derivative is not gauge-covariant, so a generalization is required. This is justified on grounds of physical principles, as invariance of the Lagrangian under symmetry transformations can be understood to be a fundamental characteristic of the theory, so a consistent gauge behaviour is required; it is as much as a mathematical condition as well, as it would otherwise lead to inconsistent gauge behaviour. As we are entrenched in exterior calculus, we introduce the exterior covariant derivative — the simplest definition can be found in e.g. Fecko [92] or Hehl & Obukhov [58].

Definition 17. *The exterior covariant derivative on P is the horizontal part*

$$\mathrm{D}\alpha = \mathrm{Hor}\ \mathrm{d}\alpha.$$

We are provided with an explicit formula to calculate exterior covariant derivatives of e.g. horizontal k-forms α of type ρ (i.e. forms with values in the representation space),

$$\mathrm{D}\alpha = \mathrm{d}\alpha + \rho'(A) \dot{\wedge} \alpha, \tag{2.18}$$

where $\dot{\wedge}$ is the exterior product adapted to Lie-algebra valued forms, which we will suppress in the following. The more widely used 0-form notions, i.e. fancily rephrased standard tensor component analysis, can be achieved by means of the interior product, so we would consider components or directional derivatives. Of course, a more intrinsic and equivalent definition from the notion of derivatives along lines can similarly be invoked, cf. [16].

Definition 18. *Let s be a section of an associated vector bundle E and $v \in \mathrm{Vect}(M)$ a vector field on the base manifold M. The covariant derivative is $\nabla : \Gamma(E) \to \Omega^1(M, E)$ such that locally*

$$(\nabla_v s)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Pi_\gamma)^{-1}(s(\gamma(t))).$$

Here Π_γ is the parallel transport along the curve γ, defined using the horizontal lift so that $p \mapsto \gamma_p^*(b)$. Carried over to vector bundles, $[p, v] \mapsto [\Pi_\gamma, v]$.

There are several important auxiliary steps here that are not presented, e.g. showing that the derivative is not dependent on the specific curve γ used — refer to [16] for details.

Definition 19. *For a covariant derivative ∇ on a vector bundle E, the exterior covariant derivative is the mapping*

$$\begin{aligned} \mathrm{D} : \Omega^k(M, E) &\to \Omega^{k+1}(M, E) \\ \omega &\mapsto \mathrm{D}\omega = \mathrm{d}\omega_i \otimes e_i + (-1)^k \omega_i \wedge \nabla e_i, \end{aligned}$$

For algebra, the exterior covariant derivative is a (graded) derivation, i.e. satisfies the graded Leibniz rule, is linear etc. Handling exterior calculus just requires adapting to a different set of rules.

We are now (barely) complete with a (barely!) sufficient amount of the intrinsic geometry required to describe classical field theory on arbitrary manifold backgrounds, provided the reader weighs the comments provided here with a preceding understanding of differential geometry. However, what this introduction provides is only a part of the physical premise — the matter content and the spacetime premise. Dynamics, however, requires a sense of trajectories, and purposeful dynamics requires calculable equations to define these trajectories. Before considering dynamics, we will consider a slight reformulation of connections, and the inclusion of fermions.

2.3 Cartan geometry

The Cartan Khronon theory [20] is Cartan *waywiser* geometry adapted to the dynamics of self-dual General Relativity. Although similar in philosophy, the waywiser constraint $e^a \equiv \mathrm{D}\tau^a$ is *not* strictly part of the standard definition of Cartan geometry. Nevertheless, the inescapable geometric topic at hand is that of Cartan geometry: waywisers and the moving frames, *repère mobile*, the Cartan connection, and model spaces. Then the preceding mathematics is the natural apparatus to introduce Cartan-geometric ideas, but it will not be purely a rewriting of manifold theory in new terms. The Cartan connection is a generalization of the affine connection[10] [125], and rather than a locally linear space, we are provided a locally homogeneous space [126]. Nevertheless, we will still struggle with the computational issue, in that the backdrop geometry matters little in comparison with the observables, which will verify or falsify any physical theory. Throughout this section, the primary references will be the series by Westman & Zlosnik [42–45], but also refer to [127] for more details. In particular, [128] is of fundamental importance for a precise definition for

[10]Differential geometry has many notions of *connection*, sometimes used interchangeably, but generally of a subtle and complicated difference. So, we are presented with linear, affine, principal, Cartan, Ehresmann, Koszul, and so on — all connections. To avoid a lengthy digression on this topic, let the limit be here only their mention, and that calculation-wise we would only care what properties the forms and tensors have.

the notion of the observer space — the author of this dissertation provided a heuristic argument [5] for incompatibility of unified geometry with observer structure, arguing for the information that an "observer" might even have access to, and a rigorous argument would certainly have to operate on the notion of timelike unit vector fields.

Put figuratively, the motive of Cartan waywiser geometry is to derive an understanding of the underlying (manifold) geometry by rolling a model space along it, and tracing the change of a contact vector from the model space to the underlying geometry — akin to measuring distances using a waywiser. Several useful simplifying notions apply: e.g. the rolling is to happen without slipping, and the model space is to be sufficiently well-behaved[11]. We will introduce two constructions: first, the Cartan geometry proper [126–130], as usually found in more mathematical literature, and then Cartan *waywiser* geometry, as presented by Westman & Zlosnik [42–45] (but see [122, 131–135] for earlier work and discussion), so a more intently physical description is provided. In particular, the Cartan waywiser variables are the contact vector and the connection, so it is easy to fit them into standard physical discourse, but introduction of a contact vector is a separate requirement from what is standardly understood as Cartan geometry.

Cartan geometry formalizes the idea that a geometry is locally homogeneous. More specifically, the space is locally the quotient G/H, i.e. a Klein geometry. A homogeneous space (e.g. a homogeneous manifold) possesses a transitive group action on it, i.e. it has a single orbit, any two points can be related by the group action, $y = g \cdot x$.

Definition 20. *A Klein geometry is a pair (G, H) where G is a Lie group and H is a closed Lie subgroup, such that the coset space G/H is connected.*

In principle, the Klein geometry is also a (right) principal H-bundle $G \to G/H$, with the fibers being (left) cosets of H. Let some x on the base space be fixed. Let us refer to [126] to explain the geometric meaning of homogeneous space — distinct points can instead be identified with cosets of the stabilizer H_x (that is, there is a injective correspondence). This leads to the idea to generally define the surroundings of a point as the coset space G/H, i.e. the Klein geometry. The issues is that taken as-is, spacetime globally is not homogeneous — at best it is locally Minkowski, $ISO(3,1)/SO(3,1)$ [128].

[11] As this is not a mathematical treatise, we will not discuss what is "well-behaved" exactly. Moreso, the waywiser constraint can be taken to be quite flexible (say, in unified geometry, even lacking immediate relation to the base manifold coframe). Theoretical physics, (un)fortunately, can be very malleable with the concepts.

Cartan geometry formalizes the notion of locally homogeneous spaces. The crucial object is the Cartan connection, which allows to "roll without slipping" the Klein geometry along the base manifold; in particular, see [127, 136]. Consider the definition provided by [126, 128] and in particular [130], which is quoted.

Definition 21. *A Cartan geometry $(\pi : P \to M, A)$ modeled on the Klein geometry (G, H) is a principal right H-bundle $\pi : P \to M$ with the Cartan* connection *A, a $\mathfrak{g}$-valued 1-form on P such that*

1. $(r_h)^* A = \mathrm{Ad}_{h^{-1}} \circ A$, $\forall h \in H$,
2. $A(\tilde{X}) = X$, $\forall X \in \mathfrak{h}$, *where $\tilde{X}$ is the corresponding fundamental vector field.*
3. $A_p : T_p P \to \mathfrak{g}$ *is a linear isomorphism* $\forall p \in P$.

The only *new* requirement is on the linear isomorphisms, relating the tangent spaces on M with those on the Klein geometry. Principal bundle structures (curvature, torsion, Bianchi identities) apply, suitably adapted, and do not need to be replicated. Wise [126] presents a very clean interpretation on the defining conditions of Cartan geometry: to be brief, this comes down to that elements of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ can be interpreted as infinitesimal transformations (rotations) of the model space and the point of tangency. This notion of "infinitesimal rotation" is also at the crux of the waywiser geometry, where it is introduced through the contact vector.

Before discussing waywisers, let us present what notion of the metric is naturally present, first in the sense of the tangent geometry [127].

Theorem 3. *Let $(\pi : P \to M, A)$ be a Cartan geometry on the Klein geometry (G, H). There is a canonical bundle isomorphism $TM \cong P \times_H \mathfrak{g}/\mathfrak{h}$. For all $p \in P$ with $\pi(p) = x$, there is a canonical linear isomorphism $\varphi_p : T_x M \to \mathfrak{g}/\mathfrak{h}$ such that $\varphi_{ph} = \mathrm{Ad}_{h^{-1}} \varphi_p$.*

This is an expected result, in principle, as it clarifies how exactly "locally homogeneous" is interpreted in the tangent bundle. The bundle $P \times_H \mathfrak{g}/\mathfrak{h}$ is sometimes called the fake tangent bundle [126]. If the Lie algebras have more structure, it is possible to introduce the Cartan connection more directly into the coframes.

Definition 22. *A Cartan geometry is called reductive if the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$, where $\mathfrak{h}$ and $\mathfrak{z}$ are subrepresentations of the adjoint representation of H.*

Then, the Cartan connection $A \in \Omega^1(P, \mathfrak{g})$ decomposes into

$$A = \omega + \varepsilon, \tag{2.19}$$

where $\omega \in \Omega^1(P, \mathfrak{h})$ and $\varepsilon \in \Omega^1(P, \mathfrak{z})$. Let $x \in M$, $v \in T_xM$, $p \in P_x$ and $\omega \in T_pP$, such that $\pi_*(w) = v$. Then, a coframe $e : TM \to P \times_H \mathfrak{g}/\mathfrak{h}$ can be defined by

$$e(v) = p(\varepsilon(w)), \tag{2.20}$$

which is well-defined and relates the fake frame bundle and the proper frame bundle $GL(M)$ [130].

However, what is not introduced is any other detail of the waywiser notation. Put another way, we also lack any objects to generate the quotient G/H, and mathematical Cartan geometry proper takes it as primitively available. So, a description of geometry of a waywiser requires extending the Cartan-geometric framework by more objects. Let us follow the physics-oriented resolution, as developed by Westman & Zlosnik [42–45], but the premise is hardly unique: for example, consider [122, 131–135] as further examples.

The object that is required is a vector τ^a in the fundamental representation of G. Depending on the context, τ^a can be called the contact vector, or the (Cartan) radius vector. So, the full set of waywiser variables is (τ^a, A_{ab}), i.e. the contact vector and the Cartan connection. Consider the infinitesimal change

$$\tau^a + \delta\tau^a = \delta\Omega^a{}_b\tau^b = (\delta^a_b - \delta x^\mu A_\mu{}^a{}_b)\tau^b. \tag{2.21}$$

The Cartan connection is understood to define the notion of "rolling without slipping". Even more, if we assume another contact vector τ'^a as a *repère*, coinciding with the contact vector at $\tau^a(x_1)$ and rotating along with the model space, we can measure the difference moving $x_1 \to x_2$,

$$\begin{aligned}\delta\tau^a &= \tau^a(x_2) - \tau'^a(x_2) = \tau^a(x_2) - (\delta^a_b - \delta x^\mu A_\mu{}^a{}_b)\tau^b(x_1) \\ &\approx \delta x^\mu(\partial_\mu\tau^a + A_\mu{}^a{}_b\tau^b) \\ &= \delta x^\mu \mathrm{D}_\mu\tau^a.\end{aligned} \tag{2.22}$$

The motivation can be seen graphically, consider figure 2.1.

Note the implicit assumption $\tau^2 = \text{const}$, where relaxing this requirement would lead to nonmetricity. However, retaining this consistent measure of length, we are suggested to identify $(\delta\tau)^2$ as the measure of distance on the base manifold, as understood with a waywiser,

$$\mathrm{d}s^2 = \eta_{ab}\delta\tau^a\delta\tau^b. \tag{2.23}$$

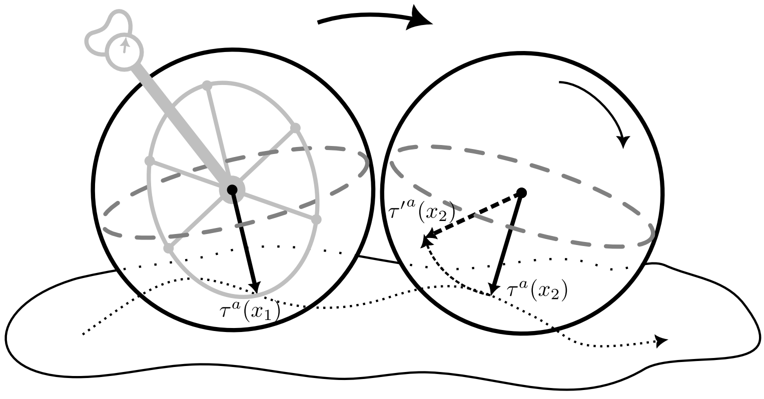


Figure 2.1: The vexing graphic representation how to measure distances using a waywiser, inspired and adapted from [42, 45]. With the motion of a waywiser's state established as $\delta\tau^a \approx \delta x^\mu \mathrm{D}_\mu \tau^a$, it is straightforwardly generalizable. However, this illustration, just as the calculation (2.22) and the identification (2.23), are better taken as *definition*, rather than derivation — there are too many notions implicitly assumed.

Note the introduction of η_{ab} which should correspond to a metric on the model space! Generally, $\eta_{ab} = \mathrm{diag}(+,\ldots,+,-,\ldots,-)$ in the Euclidean or Lorentzian tangent sense. What bears greater emphasis, however, is the requirement on the coframe,

$$e^a \equiv \mathrm{D}\tau^a. \tag{2.24}$$

This is of core importance in the Khronon theory of gravity [20].

The author would call (2.24) the waywiser coframe constraint. It is a core part of any Khronon-like theory of gravity and its importance has to be emphasized. In three dimensions, it has also been called the Bartels frame [137, 138]. In particular, a suitable adaptation can be used to define any Khronon-like theory in any other premise (other dimensions, other groups, different topologies, real or complex, etc.). From a purely functional point of view, the entire geometry discussion can be simply dismissed and only the field theory Lagrangian retained, so it is (2.24) that provides the substantial difference from standard Einstein(-Cartan, etc.) gravity[12]. The author's aspiration [2] was to introduce a similar vector to Maxwell-Yang-Mills theory, formulated in first order form, and through the excitation (in premetric terminology — the B-field, from more common BF-theory fare). In hindsight, a full similarity was perhaps doomed to fail from

[12]The waywiser constraint alone is *not* sufficient to reproduce Khronon phenomenology. Rather, the symmetry group, such as $Spin(4)$ or $SO_{\mathbb{C}}(1,3)$, is as important, and the contact vector has to further be gauge-fixed. Consider [71, 138] for further discussion.

tangent geometry premises alone, but at the time, this seemed as a natural way to develop gauge-gravity matter couplings. Possible more prospectives directions of the Khronon development will be discussed later — let us just hint that there are several, from simply appending the waywiser constraint (that is, to the standard pseudo-Riemannian coframe), to a Standard Model Higgs-like extension (which would change the representation of the contact vector).

We find that standard manifold metric geometry can be derived through a lighter or smaller set of variables (τ^a, ω_{ab}), than in the Palatini formalism $(g_{\mu\nu}, \Gamma^\rho{}_{\mu\nu})$ or (e^a, ω_{ab}), by a count of configuration space dimensions (vector vs. symmetric tensor vs. (co)frame). The value of this, off-shell or on-shell, can ultimately only be understood by understanding the phenomenology already present, and then further development of extended models that would utilize the different field content. The physical interpretation is rather colorful, and will be explored in further sections. An important motivation of this dissertation was in trying to find new geometrizations of τ^a, especially in terms of internal gauge theory — a task still not replete in its possibilities.

2.4 Fermions, spinors

Let us conclude on a topic that was not extensively developed in the articles that constitute this dissertation, but is important both simply intrinsically, due to the existence of fermionic matter, and also theoretically, as there was a line of thought that several of the fields, including the titular Khronon field, might instead by constructed from spinors. At the same time, there is another difficult theoretical obstruction in how the fermionic fields should limit to or exist in the hypothesized topological unified phase — and as spin geometry is based upon the spin group, it is not readily apparent how to handle this phase transition in the fermionic sector.

When studying the representation theory of the Lorentz group or the Poincaré group, it becomes clear that it possesses unique objects which transform in a particularly nontrivial manner under rotations. These are the fabled half-spin representations, and of primarily relevance are the Weyl and Dirac spinors[13]. For reference, consider [140] for a more extensive discussion of the representation theory of the Lorentz group, although this

[13]Rather than higher spin theories, such as Rarita-Schwinger vector-spinors [139], which have not been found in any fundamental capacity. From a representation theory perspective, an extremely limited set of possible *elementary* fields have actually been observed in Nature.

dissertation mainly follows the material by Hamilton [16]. The classical reference for spin geometry is the book by Lawson & Michelsohn [141].

There are several options available for deriving spin(or) representations associated with the special orthogonal $SO(n)$ and pseudo-orthogonal $SO(m,n)$ groups, as well as just the orthogonal groups. The most direct approach would be to consider just the representation theory, which is an approach easiest to motivate in physics terms. E.g. for the Lorentz group, establishing the representations of the Lie algebra then establishes the representations of the group itself, and allows to classify all objects that the Lorentz group can even "sense". Then, a series of isomorphisms becomes of importance:

$$\mathfrak{so}(3,1) \cong \mathfrak{su}_{\mathbb{C}}(2) \oplus \mathfrak{su}_{\mathbb{C}}(2) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}). \tag{2.25}$$

However, Dirac γ-matrices in Clifford algebras have to be introduced separately, and it is not possible to define dynamics without an understanding of the Dirac operator. At the same time, the representations of the Clifford algebra associated with the (pseudo-)orthogonal group likewise contain spinors, and studying the Clifford algebra allows to naturally define the *Pin* and *Spin* groups; further, $Spin(m,n)$ is connected for $max(m,n) \geqslant 2$, it is the universal cover (with some dimensional constraints) and a double cover of the underlying $SO(m,n)$. Note the heavy emphasis on actually properly defining the spinorial objects, i.e. groups and representations: the field theory extension is relatively straightforward through the associated vector bundle and principal bundle construction, although contingent on some requirements on the manifold.

Definition 23. *Let V be a $\mathbb{K}$-vector space and Q a symmetric bilinear form on V. A Clifford algebra of (V,Q) is $(\mathrm{Cl}(V,Q),\gamma)$, where*

1. $\mathrm{Cl}(V,Q)$ *is an associative $\mathbb{K}$-algebra with unit element* 1*,*

2. *the Clifford relation is satisfied, in that $\gamma : V \to \mathrm{Cl}(V,Q)$ is a linear map with*
$$\{\gamma(v),\gamma(w)\} = -2Q(w,Q)\cdot 1, \forall v,w \in V,$$

3. *the Universal Property is satisfied, in that if A is some other associative $\mathbb{K}$-algebra with unit element 1 and $\delta : V \to A$ a $\mathbb{K}$-linear map with*
$$\{\delta(v),\delta(w)\} = -2Q(v,w)\cdot 1, \forall v,w \in W,$$

there there exists a unique algebra homomorphism $\phi : \mathbb{C}\lessdot(V,Q) \to A$*, such that*

$$\begin{array}{ccc} V & \xrightarrow{\ \gamma\ } & \mathrm{Cl}(V,Q) \\ & {\scriptstyle\delta}\searrow & \downarrow{\scriptstyle\phi} \\ & & A \end{array}$$

commutes.

To be complete, also consider the definitions of the *Pin* and *Spin* groups. Let η be the standard bilinear form of $SO(s,t)$ on $\mathbb{R}^{s,t}$. Define S_+, S_- as the sets of vectors $v \in \mathbb{R}^{s,t}$ with positive or negative unit norm, and $S_\pm^{s,t} = S_+^{s,t} \cup S_-^{s,t}$. Then,

$$\mathrm{Pin}(s,t) = \{v_1 v_2 \cdots v_r | v_i \in S_\pm^{s,t}, r \geqslant 0\}, \tag{2.26a}$$

$$\begin{aligned} Spin(s,t) &= \mathrm{Pin}(s,t) \cap \mathrm{Cl}^0(s,t) \\ &= \{v_1 v_2 \cdots v_{2r} | v_i \in S_\pm^{s,t}, r \geqslant 0\}, \end{aligned} \tag{2.26b}$$

$$Spin^+(s,t) = \{v_1 \cdots v_{2p} w_1 \cdots w_{2q} | v_i \in S_+^{s,t}, w_j \in S_-^{s,t}, p,q \geqslant 0\}. \tag{2.26c}$$

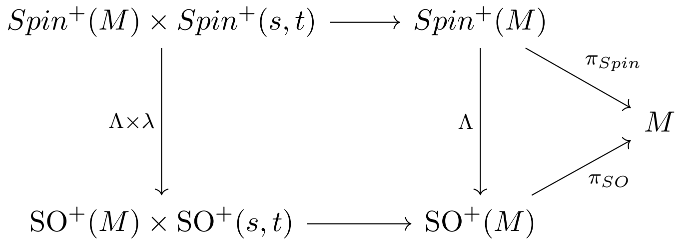
These are the pin group, spin group and orthochronous spin group respectively, with the structure of a Lie group. The Lorentz group is not compact, nor simply connected — there are four distinct connected components, contingent on the behaviour under time reversal T and parity P, with the identity in the proper orthochronous Lorentz group[14] $SO^+(3,1)$. The orthochronous spin group $Spin^+(s,t)$ is a double covering $\lambda : Spin^+(s,t) \to \mathrm{SO}^+(s,t)$ of the orthochronous Lorentz group, and is generally connected if $s \geqslant 2$ or $t \geqslant 2$.

We will not consider the representation theory at greater length — for Clifford algebras, the target is in homomorphisms to the endomorphism algebra of a vector space. For example, the Dirac spinor representations would then consider complex Clifford algebras, as endomorphisms of $\mathbb{C}^N$; the definition of the dimension N is complicated [16]. For the Lorentz group, the irreducible representations are classified as (s,t) with two half-integer numbers. The representation theory of the *Spin* group can be done in Clifford algebra form, and it is similar to that of the corresponding rotation group [141, 142]. Spinorial field theory is reproduced with associated vector bundles to spin group principal bundle geometry.

Quoting [16], the geometry is as follows.

[14] Readers beware of some abuse of terminology in literature, on what exactly is the Lorentz group or the spin group. For example, Hamilton [16] terms $O(1,t) \cong O(t,1)$ as the Lorentz group, and $SO^+(3,1)$ as the (or rather, a!) proper orthochronous Lorentz group, and similarly for pin and spin groups. Physics discourse often refers to just $SO(3,1)$ as the Lorentz group.

Definition 24. *A spin structure on M is a $Spin^+(s,t)$-principal bundle $\pi_{Spin} : Spin^+(M) \to M$ with a double covering $\Lambda : Spin^+(M) \to SO^+(M)$, such that the following diagram commutes,*



Note that $SO^+(M)$ refers to the $SO^+(s,t)$ frame bundle. Spin structures impose topological requirements on the base manifold. Namely and to be precise, $SO^+(M)$ admits a spin structure if and only if the second Stiefel-Whitney class vanishes, $w_2(M)$. In particular, parallelizable manifolds, i.e. those with a trivial tangent bundle, have $w_2(M) = 0$, but this is not an exhaustive list: e.g. all sphere S^n are also spin. And as promised, spinor fields are just sections of the associated complex vector bundle $Spin^+(M) \times_\kappa \Delta$, where $\kappa : Spin^+(s,t) \to GL(\Delta)$ is a spinor representation. In particular, the spin connection[15] defines the spin-covariant derivative of a spinor,

$$\mathrm{D}\psi = \mathrm{d}\psi + \frac{1}{8}[\gamma_a, \gamma_b]\gamma^{ab}\psi, \tag{2.27}$$

where $\gamma_a = \rho \circ \gamma(e_a)$ on a representation ρ of the Clifford algebra basis are the *mathematical* gamma matrices (vs. the physical $\Gamma_a = -i\gamma_a$).

Let us move on to more practical comments. Recently, the Spin group was taken as the basis for reinterpreting the Khronon theory of gravity [138]. To be brief, the phenomenology was found to be similar to the standard Khronon theory [20] in terms of various gravitational phenomenology (exact solutions, e.g. black holes, but also cosmology, perturbations, etc.), but was devoid of necessary complexification and exclusively the self-dual formulation. The discussion during the author's doctoral studies rather concerned a slightly different possibility, in that the titular Cartan Khronon vector field was a spinor bilinear, e.g. $\tau^a \to \bar{\psi}\gamma^a\psi$. In principle, a similar substitution can be understood for MacDowell-Mansouri and Stelle-West theory, where a vector field in (Anti-)de Sitter breaks the symmetry to Lorentz[16]. To a large extent, this remained incomplete, but note that

[15] Sometimes interchangeably called the Lorentz connection, although this is somewhat an abuse of terminology and more precisely speaking the spin connection is on the spin structure.

[16] As suggested by M. Hohmann, work in progress.

$\bar{\psi}\gamma^a\psi$ would *not* be sufficient to reproduce standard vector phenomenology, as the pseudo-vector contribution $\bar{\psi}\gamma^a\gamma^5\psi$ is missing. That is, the chirality element

$$\omega = \lambda e_1 \cdots e_n \in \mathrm{Cl}(s,t) \otimes \mathbb{C}, \tag{2.28}$$

or the usual "fifth" gamma matrix in physics convention

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.29}$$

is crucial for well-defined vector-equivalence. Put another way, the vector norm squared

$$(\bar{\psi}\gamma^a\psi)^2 = \eta_{ab}(\bar{\psi}\gamma^a\psi)(\bar{\psi}\gamma^b\psi) \tag{2.30}$$

is necessarily positive-definite, and cannot produce negative norm vectors. The substitution

$$\tau^a \to \bar{\psi}\gamma^a\psi + \bar{\psi}\gamma^a\gamma^5\psi \tag{2.31}$$

is rather trivial, however, and is unlikely to produce significantly interesting phenomenology, although the appending of a Dirac kinetic Lagrangian $\bar{\psi}(i\not{D} - m)\psi$ would still remain to be clarified. There are some possible approaches, including rather adventurous, e.g. considering a ray or line space of spinors, but a resolution is still pending. In this vein, consider the recent pregeometric work by Wetterich [143–145] for related ideas, or even further extension into twistors [146, 147]. Unfortunately, they remain only as possible inspiration for future work.

On geometric foundations, spin structures require the vanishing of the second Stiefel-Whitney class, which is in particular satisfied for parallelizable manifolds, i.e. a trivial tangent bundle. Sometimes parallelizability is thus seen as a fundamental requirement of spacetime. The author cannot be entirely in agreement with this motivation — by suitably cutting away parts of the non-parallelizable manifold, it is possible to imagine creating a parallelizable manifold, which would then admit a spinor bundle. This suggests that, loosely put in a limiting case, we are rather presented with some discontinuity, which can be thought of as a shock wave or jump in the field configurations. The validity of such solutions can be argued in greater length, e.g. in terms of well-definedness or singularity of momentum, but more likely this becomes a somewhat impractical discussion, as before such phenomena should even be considered, we would likely have to understand the proper definition of quantum fields on quantum geometry (gravity), and a clear notion how to interconnect local phenomena to global geometries, e.g. through a better understanding of the measurement process. In the context of quantum field theory, this issue essentially does not appear, because Minkowski space is trivially parallelizable.

Chapter 3

Dynamics

To wit, note so far we are definitively incomplete in our theory definition, as we lack any dynamic principles. Generally, this is understood to mean differential equations that define the trajectories, or the geometry of the fields in our situation, but we could just as well consider e.g. flows on phase spaces in a canonical or phase space formulation of physics, or a causal sequence of objects in a suitable ordering. Differential equations become just means to establish the trajectories, *calculate* their geometry as understood when embedded to $\mathbb{R}^n$ — perhaps there is something to say on the inevitability of such a description given assumptions up to smoothness. But a dynamical principle could be something as abstract as a rule establishing what section are allowed in the space of all sections. Further assumptions are required to avoid degeneracy, e.g. forbidding multiple sections correspondent to the same boundary conditions, or those correspondent to different ones from intersecting. The most trivial dynamical principle would be that all configurations are allowed, and would be trivially inconsistent with physical reality. So, a *solution space* has to be reduced from the entirety of the *configuration space*, by means of some general rule. The history of physics has determined the Lagrangian formulation to be sufficiently generic and beneficial, and the covariant phase space formalism allows interpreting the solution space as *the* phase space.

3.1 Variational bicomplex. Covariant phase space

Earlier, it was alluded that physical trajectories can be understood as geometric objects in their own right — lines and hypersurfaces, embedded in higher-dimensional (possibly infinitely so) configuration spaces. The list of physically relevant differential equations is certainly severely limited and curious in properties, but their purpose can be seen to define the solutions as geometric objects. In the phase space perspective, the states are rather considered to constitute points in the phase space geometry, and flows correspond to dynamics. It is, of course, well known that the Hamiltonian (canonical, phase space, flow — cf. Poisson brackets) description is generally consistent with the Lagrangian (variational, configuration space, trajectory[1]) viewpoint, the Legendre transformation being a particular interrelation, but the *covariant* phase space geometry [87, 148, 149] provides another, near-immediate link between the Lagrangian and the canonical formalism. In specific, it allows deriving the symplectic structure for a Lagrangian (field) theory, thus justifying the "phase space" denomination. The derivation of the Peierls bracket, cf. the Poisson bracket, from symplectic structure proves useful for e.g. quantization, while the basic operation of the first variation allows immediate operation of Noether's theorems.

We will not present details of the Lagrangian formulation of dynamics in deep sophistication, although this is possible in terms of e.g. jet bundles [150, 151], among other options. Rather, let us re-emphasize that the Lagrangian formulation of dynamics is based on a (linear) comparison of trajectories, weighted by the action, so that the physical trajectories are at a variational minimum. For simplicity, we will restrict to field theory, although a conclusion drawn from this dissertation is that more generalized dynamics should also be considered and developed — for example, dynamical generation of the underlying topology, or nontrivial base spaces (e.g. graphs, discrete geometries). The method to achieve this does not appear natural in traditional Lagrangian field theory, and only suggestions can be given in this direction, e.g. gluing topological spaces together, using nontraditional products and sums (of e.g. the partition function, or using the step function, etc.), considering dynamics in other spaces (for example, flows in the space of all topologies), or introducing some other conditional logic into the equations. This, however, is highly speculative, and far too distant from ordinary physics to be confidently evaluated in any depth.

We are presented with some dynamical quantities, in our case a collection of fields[2] ϕ. These classical fields are just sections of some fiber bundle, and the set of sections $\Gamma(E)$ can be considered to be the configuration space.

[1]Not to imply that the Hamiltonian formulation does not define trajectories, simply that the action principle compares trajectories directly. Similarly, it is not to imply that Lagrangians and Hamiltonians are somehow in contention over configuration vs. phase space.

[2]Let it be remarked that it is entirely reasonable to also consider *all* fields simply as *one* field with a particular internal structure, e.g. the vector spaces might decompose into a direct sum $\bigoplus_{i=1}^{n} \phi_i \sim \phi_1 + \ldots + \phi_n$. This viewpoint is useful for a unified treatment of matter fields, as it quickly distinguishes trivial from nontrivial structure.

We are led to the Lagrangian n-form $L(\phi)$, and the action

$$S[\phi] = \int L(\phi), \tag{3.1}$$

a functional of the dynamical objects. To compare different field configurations, we need some sense of ordering — in terms of smoothness and continuity, and because the action should be calculable for any field configuration ϕ, we can consider "small" deviations $\delta\phi$. This can be made rigorous, here most relevant by virtue of the variational bicomplex [152, 153]. So, fields are understood to rather be on the infinite-dimensional jet bundle $J^\infty(E)$ — very loosely, jets correspond to order-by-order Taylor series expansion and partial derivative equivalence classes of sections, so we are provided the derivative components as local coordinates. Bundle constructions apply: we can introduce vector fields, differential forms, project along the fibers of $\pi : J^\infty(E) \to M$, *et cetera.* We find that on $J^\infty(E)$ differential forms bi-grade into

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)), \tag{3.2}$$

as does the *total* exterior derivative $\mathbf{d} : \Omega^p(J^\infty(E)) \to \Omega^{p+1}(J^\infty(E))$ into a horizontal component $\mathrm{d} : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E))$ along the base manifold, and a vertical component $\delta : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s+1}(J^\infty(E))$ along field configurations,

$$\mathbf{d} = \mathrm{d} + \delta. \tag{3.3}$$

Definition 25. *The variational bicomplex is the bi-graded algebra $\Omega^{*,*}(J^\infty(E))$, with the exterior derivatives* d *and* δ*.*

Now, δ provides a formalization to the notion of variation. Note, that formally nilpotence of all the exterior derivatives implies anticommutation

$$\mathbf{d}^2 = 0 \Rightarrow \mathrm{d}\delta = -\delta\mathrm{d}, \tag{3.4}$$

but this can often be waived by convention.

The reason why such attention is given to a more rigorous variational calculus is that the variation defines the pre-symplectic form on the space of solutions, thus giving rise to the notion of a *covariant* phase space [87, 148, 149]. At the same time, as the variations are generic, they can be restricted to symmetry variations, and give rise to Noether's theorems [154–156]. Both questions were studied in [3, 4]. Recall that standard Hamiltonian analysis is built upon the $3+1$ spatial-temporal split: we introduce a unit timelike vector n, corresponding to a "flow of time". Further, this defines a foliation into spacelike hypersurfaces $M = \mathbb{R} \times \Sigma$, which can be understood to represent space at "instants of time"[3], and is indeed the basis for the

[3]Which clearly is a thoroughly *non*-relativistic notion.

ADM formalism of canonical analysis of gravity [157]. This foliation is parametrized by a single monotonously increasing time variable σ, such that $n \lrcorner \mathrm{d}\sigma = \mathcal{L}_n \sigma = +1$. Note the Lie derivative and Cartan's homotopy formula (or "Cartan's magic formula")

$$\mathcal{L}_n \alpha = \mathrm{d} n \lrcorner \alpha + n \lrcorner \mathrm{d}\alpha. \tag{3.5}$$

In more practical terms, we can put the preceding mathematical apparatus to use, and split any arbitrary differential form α in terms of a longitudinal (temporal) component $\alpha_\perp$ and transverse (spatial) component $\underline{\alpha}$, such that

$$\alpha = \mathrm{d}\sigma \wedge \alpha_\perp + \underline{\alpha}, \tag{3.6}$$

where

$$\alpha_\perp = n \lrcorner \alpha, \tag{3.7a}$$

$$\underline{\alpha} = (\mathbb{1} - {}^\perp)\alpha = n \lrcorner (\mathrm{d}\sigma \wedge \alpha), \ \ n \lrcorner \underline{\alpha} = 0. \tag{3.7b}$$

The standard understanding of the Legendre transform follows. The top-rank Lagrangian splits into

$$S[\alpha] = \int \mathrm{d}\sigma \wedge L_\perp, \tag{3.8}$$

and as usual, variation defines canonical momenta π_α so that

$$\delta_{\underline{\dot{\alpha}}} L_\perp = \delta \underline{\dot{\alpha}} \wedge \pi_\alpha, \tag{3.9}$$

with the Legendre transform to the canonical Hamiltonian

$$H_c = \sum_\alpha \underline{\dot{\alpha}} \wedge \pi_\alpha - L_\perp. \tag{3.10}$$

The Poisson brackets of functionals read

$$\{F, G\} = \int \sum_\alpha \left(\frac{\delta F}{\delta \underline{\alpha}} \wedge \frac{\delta G}{\delta \pi_\alpha} - \frac{\delta G}{\delta \underline{\alpha}} \wedge \frac{\delta F}{\delta \pi_\alpha} \right). \tag{3.11}$$

Note there are a multitude of bracket structures for various formulations of dynamics. For instance, consider the Peierls bracket, which generally coincides with the Poisson bracket on the covariant phase space, see [148, 158–160] for further details. The variety is somewhat subtle. Poisson brackets would most immediately define a Poisson geometry, but a symplectic geometry is often sufficient[4], in addition to multisymplectic geometry etc. Nevertheless, it is valid to understand phase geometry as symplectic geometry.

[4] As an example, consider that every symplectic manifold defines a Poisson structure in an obvious manner. The converse is not generally true. Poisson geometry correspondence would rather consider foliations (symplectic leaves, viz. the Weinstein splitting theorem) [161].

Definition 26. *A symplectic structure on a manifold M is the choice of a particular closed non-degenerate differential 2-form ω.*

A manifold with a symplectic structure is called a symplectic manifold.

The equivalence (isomorphism) of symplectic manifolds follows as standard in mathematics, requiring compatibility of the additional structure.

Definition 27. *A symplectomorphism $f : M \to N$ between symplectic manifolds M and N with symplectic forms ω and ω' is a diffeomorphism that preserves the symplectic structure, $f^*\omega' = \omega$.*

Several procedures are thoroughly more Hamiltonian-based than Lagrangian, including canonical quantization, but also the Dirac-Bergmann algorithm, and indeed the entirety of Dirac constraint analysis [162] into primary, secondary, first-class and second-class constraints. The constraint structure of the Isokhronon theory [2] was studied in [4], and both the initial Khronon paper [20] as well as the chiral extension [71] understood the phenomenology in terms of Hamiltonian analysis, so a brief discussion is warranted. The aim of constraint analysis is to establish the constraint algebra of a theory, so to identify the basic degrees of freedom and both the evolution and constraint equations they follow. This can clarify the system in detail, e.g. whether it is consistent (e.g. field-dependence of the constraints), over or under constrained, etc. We obtain the following sequence of steps for the basic algorithm; see also [87, 163, 164].

1. Derive the initial canonical description in terms of the canonical variables (fields α and momenta π_α) and the primary Hamiltonian H_p. The canonical Hamiltonian H_c is the Legendre-transformed Lagrangian $L_\perp$, which appends the primary constraints C as Lagrange multipliers to form a primary Hamiltonian.

2. Study the time evolution $\{H, C\} = 0$ of constraints, i.e. the consistency conditions. That is, beginning with $H = H_p$, consistent time evolution might introduce additional constraints, which are not satisfied by previous constraints or evolution equations. Then, additional (secondary, tertiary, etc.) constraints have to be introduced and appended to H, until the process terminates to a total Hamiltonian H_T or the system proves to be inconsistent.

3. Consider the constraint algebra, studying Poisson brackets $\{C_i, C_j\}$. Constraints whose brackets vanish with all others are first class constraints and signify gauge symmetries, while any nonvanishing bracket implies a second class constraint. Altogether, this provides an understanding of the constraint surface, including e.g. dynamical degrees

of freedom — constraints are subtracted from the initial count of variables, and first class constraints are counted twice.

The Isokhronon (as to be introduced later) is not physically viable, per torsion-dependence in the constraints among other issues, but the degree of freedom count is curious: we only obtain two degrees of freedom, as expected for Maxwell electromagnetism, despite the presence of an electromagnetic background freedom $\mathrm{D}X_a = 0$ — an integration constant freedom *somewhat* similar to the Khronon [20], but dynamically insufficiently admissible.

Let us return to the variation of the Lagrangian, and introduce the covariant phase space in the presentation of [149]. To again emphasize, there are several formulations of covariant canonical geometry, including multisymplectic geometry and beyond [165]. Let us consider a base manifold with boundary. Without significant delay, we can now write the action

$$S[\phi^a, \mathrm{d}\phi^a] = \int_M L + \oint_{\partial M} \ell, \tag{3.12}$$

so its variation is

$$\delta S = \int \delta(\phi^a \wedge E_a + \mathrm{d}\Theta) + \oint_{\partial M} \delta\ell. \tag{3.13}$$

Define the *pre-symplectic 2-form* as an integral over a Cauchy surface

$$\tilde{\Omega} = \int_\Sigma \delta(\Theta - \mathrm{d}C), \tag{3.14}$$

and it is a *closed* form on the pre-phase space $\tilde{P}$, which consists of solutions to the equations of motion. It can be shown that this integral is independent of the Cauchy surface. The phase space proper $\mathcal{P}$ is obtained by factoring out (gauge) symmetries, which are expressed as a group action, generated by zero modes of $\tilde{\Omega}$. The symplectic form proper Ω is induced by mapping to orbits $f : \tilde{\mathcal{P}} \to \mathcal{P}$. Let $X, Y \in T_q\tilde{\mathcal{P}}$ and $\tilde{X}, \tilde{Y}$ be their pushforwards through f_*. It is possible to show that

$$\Omega(X, Y) = \tilde{\Omega}(\tilde{X}, \tilde{Y}) \tag{3.15}$$

is well-defined, i.e. independent of the representatives $q, \tilde{X}, \tilde{Y}$. Thus, we are provided with a closed, non-degenerate 2-form on the *manifold* $\mathcal{P}$, which is constituted of solutions to the equations of motion, and it deserves an interpretation as the covariant[5] phase space.

As we are already intricately involved in Lagrangian theory, let us also introduce the Noether machinery. It was already discussed that groups induce transformations of the basic fields (sections of the associated vector bundles), and thus also on the Lagrangian. The group transformations where the variation of Lagrangian is at most an exact form, so the action is varied at most by a boundary term, are of fundamental importance, as they represent symmetries of the trajectories — and, by Noether's theorems, induce conserved quantities. Deriving Noether's theorems is straightforward from the general variation (3.13), and was considered in [3, 4]. In particular, the Lagrangian variation can be expanded as

$$\delta L[\phi^a, \mathrm{d}\phi^a] = \delta\phi^a \wedge \frac{\partial L}{\partial \phi^a} + \mathrm{d}\delta\phi^a \wedge \frac{\partial L}{\partial \mathrm{d}\phi^a}, \tag{3.16}$$

which rearranges to the standard symmetry statement

$$\begin{aligned}\delta S = &\int_M \left[\delta\phi^a \wedge \left(\frac{\partial L}{\partial \phi^a} - (-1)^p \mathrm{d}\frac{\partial L}{\partial \mathrm{d}\phi^a}\right) + \mathrm{d}\left(\delta\phi^a \wedge \frac{\partial L}{\partial \mathrm{d}\phi^a}\right)\right] + \oint_{\partial M} \delta\ell \\ \equiv &\int \mathrm{d}K.\end{aligned} \tag{3.17}$$

The transformations are arbitrary, so $\delta\phi^a$ can be chosen in whichever way is preferred: whether vanishing on the boundary or not, constrained to be on-shell or not. We will neglect the pure boundary Lagrangian ℓ.

1. Noether's first theorem, by which on-shell, to each global symmetry of the Lagrangian, the theory admits a conserved current

   $$J = \Theta - K, \ \ \mathrm{d}J = 0. \tag{3.18}$$

   The corresponding conserved charge is just the integral

   $$Q = \int J. \tag{3.19}$$

[5] An important remark, as also mentioned in the author's [4]: the covariant phase space is only as covariantly consistent as the solutions to the equations of motions are. It is known that interacting Rarita-Schwinger theory has issues with superluminal propagation [139, 166–168], and this issue would persist in the "covariant" phase space as well. At the same time, although the 3+1 formalism is not *explicitly* covariant, there is no actual inherent issue with relativity, as all objects still *collectively* transform in a covariant manner.

2. Noether's second theorem, that the parameters of the local symmetry transformation are required to satisfy a certain constraint; e.g. when parametrized as in [4] (however, not in the most general manner)

$$\delta\phi^i(\lambda_\alpha, \mathrm{d}\lambda_\alpha) = \sum_\alpha \left(\delta\lambda_\alpha \wedge a^i_\alpha + \mathrm{d}\delta\lambda_\alpha \wedge b^i_\alpha\right) \tag{3.20}$$

then

$$a^i_\alpha \wedge E_i = (-1)^k \mathrm{d}(b^i_\alpha \wedge E_i). \tag{3.21}$$

3. Boundary theorem, which implies that the variation's boundary contribution must vanish separately, leading to another collection of constraints [154, 169]; per the preceding parametrization

$$\sum_\alpha \mathrm{d}\left(\delta\lambda_\alpha \wedge b^i_\alpha \wedge \left(\frac{\partial L}{\partial\phi^i} - (-1)^p \mathrm{d}\frac{\partial L}{\partial \mathrm{d}\phi^i}\right)\right) = \mathrm{d}(\Theta - K). \tag{3.22}$$

This is purposefully left incomplete — Brading & Brown [169] vs. Utiyama [170, 171] use different forms of the three equations. This generally contingent on the parametrization, but the point of emphasis is in the independence of various variations.

Note that background fields may break the implications of Noether's theorems, even if the background fields also take part in the symmetry transformation [169, 172].

Noether's theorems have long been the subject of extended mathematical and philosophical discussion. From the author's view, it can only be added that they are an important structural implication of the variational system: they allow to characterize the symmetries by deriving their complementary conserved quantities. Here, "structure" is intentionally left imprecise, as the applications are too varied to count, from geometric characterization of trajectories to simplifying dynamics (classical *and* quantum) by invariant quantities, to providing observables for measurement.

3.2 Premetric theory. Topological field theory

Replete with formalism, we can now specialize into specific actions for the dynamic objects of interest — in particular, the internal gauge fields, i.e. separate principal bundle connections A, and gravitation embedded into tangent bundle geometry, e.g. through the coframe e^a and Lorentz connection ω_{ab}. But in a manner of delight, it is now easy to briefly just collect a listing of all the basic top-rank gauge-invariant differential forms [5, 173], i.e. Lagrangians, constructed from the fields and basic geometric operations like $\wedge$, $*$ and D, and find that they all have relevance[6]. However, near-all of dynamics is concentrated into a select few, namely the Einstein-Hilbert (or Einstein-Cartan) and Maxwell-Yang-Mills Lagrangians. Nevertheless, a more immediately physically interpretable introduction of notions is also merited, and premetric theory [58] precisely corresponds to some of the ideas developed in this dissertation, as premetric theory aims to develop theory as far as possible without introducing any metric structure. For the case of electromagnetism, Hehl & Obukhov list six physical axioms[7].

1. Conservation of electric charge, introducing a *twisted*[8] charge-current density 3-form, such that over any 3-volume
$$\oint J = 0.$$
2. Lorentz force density in experimentally observed form
$$f_a = (\partial_a \lrcorner F) \wedge J,$$
for the field strength 2-form F (that is, curvature).
3. Conservation of magnetic flux, requiring that for any closed 2-surface F satisfies
$$\oint F = 0.$$
4. Local energy-momentum formula postulated to be
$$T_a = \frac{1}{2}(F \wedge (\partial_i \lrcorner B) - B \wedge (\partial_i \lrcorner F)).$$
Here B is the *excitation* $n-2$ form, which appears from the requirement that current be conserved, by the Poincaré lemma
$$\mathrm{d}J = 0 \Rightarrow J = \mathrm{d}B.$$
It also trivially appears from the action
$$S = \int B \wedge F,$$
which will be discussed shortly.

[6]Providing justification for "polynomial simplicity" [43] to be promoted to almost a principle, and validating Lovelock's theorem [174].

[7]That is, there is significant implicit mathematical structure nevertheless assumed. It might be more accurate to call this list *principles* rather than axioms.

[8]Twisted differential forms introduce a further sign of the Jacobian when changing coordinates compared to standard forms.

5. Local and linear constitutive law, reducing the count of free variables and relating the excitation and field strength as

$$B = \lambda_0 * F.$$

   To comment, the constitutive law of a gauge theory generally refers to the relation between the excitation B and field strength F.

6. Split of internal and external currents

$$J = J^{\mathrm{int}} + J^{\mathrm{ext}},$$

   and their conservation separately,

$$\mathrm{d}J^{\mathrm{int}} = 0,\ \mathrm{d}J^{\mathrm{ext}} = 0.$$

To the author's knowledge, this axiomatics has not been continued to non-Abelian theory in any greater extent, although much of the ideas can be read off if a suitable covariant generalization of the operations is assumed — namely, the exterior covariant derivative D rather than the exterior derivative d. At the same time, for example, because the exterior covariant derivative does *not* obey a Poincaré lemma $\mathrm{D}\alpha = 0 \not\Rightarrow \alpha = \mathrm{D}\beta$, the collection of conservation laws does not imply a simple realization of the field strength. On the other hand, gravitational premetric theory has seen more development [63, 175, 176].

More critically, note that the theory of connections on principal bundles produces an immediate mapping between these ideas and the precise mathematical structures: even the avenues for generalization are just as apparent (e.g. generalizations in the constitutive law $B = \kappa(F)$ in the prototype Yang-Mills Lagrangian $B \wedge F$), even if they are slightly de-emphasized. For the context of the dissertation, articles [2, 5] understood the concept of premetric theory as a physical restriction of topological BF theory, with the primary distinction of physical versus mathematical introduction of notions — for example, the constitutive law can be understood as just a constraint term on the excitation B-field. Perhaps somewhat provocatively, it might be reasonable to understand any derivation of physical theory from topological field theory as a realization of the constitutive law of premetric fare.

On the other hand, this leads to a line of thought, that *any* physical theory descends from a series of simpler, less restricted or more general Lagrangians, through essentially a constraining process, until finally reaching its physical form [2, 5]. In this sense, *topological* field theories [177] appear as a natural candidate for the initial theories, and indeed this has been both a curious and a mathematically intricate perspective, as it has been found how both Yang-Mills theory and gravity can appear as a deformation of BF-theory [59, 178]; the cycle of papers by Addazi et al. [179–181] is also noteworthy (in particular, e.g. the Ricci flow to BF theory). For the author of this dissertation, the interest in topological field theory arose from a possibly greater similarity between internal gauge theory and gravity, where the distinction between (Maxwell-)Yang-Mills theory and gravity can be worked out possibly dynamically (hopefully, as an appendix to Lagrangian field theory) *and* geometrically (so the process of generating the split of principal bundle geometry to the tangent bundle and internal bundles could be observed in detail). The hypothesis was that in a truly gauge-gravity unified phase, there would be no distinction between internal gauge interactions and gravity, so there possibly also would not be a standard pseudo-Riemannian 4-manifold geometry. Unfortunately, this ideal was not realized — if to repeat the attempt, the mechanisms for the split of some principal bundle to the frame bundle and an internal principal bundle would be studied in greater detail. Two options seem reasonable: either to consider an abstract frame bundle in higher dimensions with a unified group, so the broken phase splits off the spacetime frame bundle and an internal gauge group, or to see how some part of the unified group splits off and reduces the structure group of the frame bundle of spacetime.

Definition 28. *A (quantum) field theory is topological, if according to some notion, it does not depend on any (pseudo-)Riemannian metric. In particular, the vacuum expectation value of observables should be metric-independent,*

$$\frac{\delta}{\delta g^{\mu\nu}}\langle O_1 \cdots O_k \rangle = 0. \tag{3.23}$$

Depending on the exact mechanism, topological quantum field theories (TQFT-s) are of two primary types [182]:

1. *Schwarz-type TQFT-s, where the action functional $S[\phi_i]$ is independent of the metric.*

2. *Witten-type TQFT-s (or, cohomological field theories), which arise from a topological twist in a supersymmetric theory.*

Schwarz-type TQFT-s are more direct, as the defining system itself is independent from the metric, but Witten-type TQFT-s are more subtle in this sense, as the action is permitted to hold a metric-dependence, but under certain conditions, prove to be nevertheless metric-independent. In particular, four conditions, including introducing an exact symmetry transformation δ with requirements on observables and the stress-energy tensor, would then be considered — however, here we will not consider Witten-type TQFT-s in any extent.

The purpose and relevance of T(Q)FT-s is rather colorful. On purely mathematical terms, a topological quantum field theory allows to compute topological invariants (partly by definition, but also consider Atiyah's classic papers [183, 184]), but is is known that e.g. 3-dimensional quantum General Relativity is a Chern-Simons theory[9] [102, 103], among other serendipitous relations. Topological quantum field theories can be exactly solvable, so they also provide a testing ground for general (axiomatic and otherwise) quantum field theory. Recall that the Chern-Simons action is defined on a 3-manifold so that

$$S_{\mathrm{CS}} = \int \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right). \tag{3.24}$$

However, we are rather interested in what is known as BF theory, by the name of the two fields in the action

$$S_{\mathrm{BF}} = \int \mathrm{Tr}(B \wedge F). \tag{3.25}$$

BF theory is well-understood, and exactly solvable [177, 185]. More generally, any invariant nondegenerate bilinear form over $\mathfrak{g}$ could be considered. The author has argued that topological BF theory is one further formalization of premetric ideas [58]. Indeed, the equations of motion

$$\mathrm{D}B = (J), \tag{3.26a}$$
$$F = 0 \tag{3.26b}$$

already include the prototype inhomogeneous Maxwell-Yang-Mills equation in premetric conserved current form, suitably generalized to non-Abelian theory (and with the matter current (J) only mentioned for utility). The field strength constraint $F = 0$ is very severe, but ultimately only indicates that a constitutive law $B = \kappa(F)$ is missing. This was termed the "kinetic cycle" when executed in Lagrangian theory [2][10], and furthermore, we will see that BF theory greatly symmetrizes internal gauge theory and gravity dynamics, while the distinction comes in the constitutive law, and the geometric sector under study [5].

[9] Albeit, again, not exactly so [99–101], as mentioned earlier.

[10] But is also referred to as just a quadratic potential, the first order formalism of Maxwell of Yang-Mills gauge theory, or the parent system formalism.

3.3 Electromagnetism and Yang-Mills theory. Duality

We are finally complete with context to move on to the definition of physical dynamics of interest — Yang-Mills theory and electromagnetism, that any consistent physical internal gauge theory aims to reproduce, at least as a limiting case. The theory of principal bundles provided the setting, structure and field content of gauge theory. The *dynamical* content is fixed with the action

$$S_{\text{gauge}}[A, e^a] = \int \frac{1}{2g^2} \operatorname{Tr}_G(F \wedge *F). \tag{3.27}$$

The basic dynamic variable is a connection A on a principal G-bundle, where G is generally assumed to be a compact Lie group, as otherwise this would lead to a variety of problems in e.g. the quantum theory [186–189]. Electromagnetism uses the Abelian group $U(1)$, and Yang-Mills is based upon non-Abelian groups like $SU(2)$ (Weak interaction) or $SU(3)$ (Strong interaction). The coupling constant g determines the strength and character of the interaction, while the trace Tr_G ensures that the action is invariant under G-transformations, i.e. G is a symmetry of the action — which by Noether's theorems leads to conserved currents. Despite many attempts for generalization, e.g. in terms of non-linear electrodynamics [190], no deviation from quantum field theory based upon this Lagrangian has been observed. Writing out all possible field theory Lagrangians only in $\wedge$ and $*$, no low-order nontopological generalization can even exist.

Generally, Lagrangians of this type compose the topic of study of (microscopic, quantum and classical) electrodynamics, optics, electroweak theory and chromodynamics, particularly so when further equipped with matter. Let us focus on some dynamical implications with a more specialized focus, handling Yang-Mills theory and electromagnetism as simultaneously as possible — this is not difficult, given the covariant continuation $\mathrm{d} \to \mathrm{D}$, but will become incurably different in the sense of e.g. dualities. Note there is only one nontrivial equation of motion,

$$\frac{1}{g^2} \mathrm{D} * F = (J), \tag{3.28a}$$

as the Bianchi identity

$$\mathrm{D}F = 0 \tag{3.28b}$$

is trivially satisfied by construction. We have included possible matter couplings J by hand, but in principle, they would appear from matter Lagrangians, should some matter fields ϕ^i be charged and thus require the gauge-covariant derivative D rather than d — this is the *minimal* coupling procedure, but it is hardly the only option[11].

Generally solving (3.28a) is difficult, and even more so when J is constituted of dynamical fields itself, leading to a coupled set of differential equations. In principle, the exterior covariant derivative possesses a formal inversion in a star-shaped neighborhood in terms of the linear homotopy operator [191, 192], but in effect it comes down to numerical integration. For Abelian theories, $\mathrm{D} \to \mathrm{d}$, so J is exact, thus closed $\mathrm{d}J = 0$, and provides the conservation of current automatically — which in premetric theory was introduced as a separate axiom (note the order of the sequence). Non-Abelian theory introduces self-interactions,

$$\frac{1}{g^2}(\mathrm{d} * F + [A, *F]) = J, \tag{3.29}$$

so either a covariant sense of "conservation" through D has to adopted, or the self-interactions have to be included into an effective current J_{eff} for more standard Stokes theorem volume-to-boundary flux integrals. The gauge-covariance in such a premetric interpretation would be somewhat open, as it would require a reinterpretation of the transformation laws in the self-interaction contribution.

It is, however, possible to study simpler-behaved classes of solutions. Consider the vacuum equations

$$\mathrm{D} * F = 0, \tag{3.30a}$$

$$\mathrm{D}F = 0. \tag{3.30b}$$

There is a sense of *duality* $*F \leftrightarrow F$, in that, taken as 2-forms, both $*F$ and F seem to be completely interchangeable. This is true even more so for electromagnetism, as then the Poincaré lemma implies that the dual field strength is also an exact form, $*F = \mathrm{d}b$, i.e. the field strength of another gauge field b. Indeed, this is the premise of (vacuum!) electric-magnetic duality, which was found already early in the 20th century, but proves to be significantly more general. Refer to [64, 65, 193] for an excellent overview, and to the author's [4] for some differential geometry on this topic — here, we will only present the premise. However, already beware significant difference in non-Abelian theory, because D does *not* define a de Rham cohomology in the way d does.

To be fair, there are several notions of duality present — including Hodge duality on 4-dimensional spacetime. Then, each 2-form α splits

[11] The variety, unfortunately, appears too large to recount. The appearance of higher gauge potential powers is common, in any sense of products and contractions. Idly wondering leads to consider whether the gauge transformation $A \to gAg^{-1} + g\mathrm{d}g^{-1}$ could be generalized as well, but this was not pursued.

into a self-dual ${}^{+}\alpha$ and anti-self-dual component ${}^{-}\alpha$, $\Omega^2(M) = {}^{+}\Omega^2(M) \oplus {}^{-}\Omega^2(M)$, such that

$$\alpha = {}^{+}\alpha + {}^{-}\alpha, \;\; *{}^{\pm}\alpha = \pm\alpha \tag{3.31}$$

with the projector in the Riemannian case

$${}^{\pm}P_{*} = \frac{1}{2}(\mathbb{1} \pm *). \tag{3.32}$$

The same holds for the field strength 2-form F in four dimensions, with (anti-)self-dual field strengths ${}^{\pm}F$ generated by (anti-)self-dual connections ${}^{\pm}A$. These are the instanton solutions of Yang-Mills theory. Refer to Donaldson's early work [194–196], but as an interesting continuation, this general idea of instanton solutions was recently applied to teleparallel gravity, constructing "self-excited gravitational instantons" [197]. As minor addendum, remark the symmetrized set of inhomogeneous equations, constructed from (3.28),

$$\frac{1}{2g^2}\mathrm{D}(*F - F) = \frac{1}{2}J, \tag{3.33a}$$

$$\frac{1}{2g^2}\mathrm{D}(*F + F) = \frac{1}{2}J, \tag{3.33b}$$

essentially written in terms of (anti-)self-dual projectors. With the Bianchi identity $\mathrm{D}F = 0$, these are trivially equivalent to the basic equations. However, forgoing this relation, they allow the initial set (3.28) to be derived in full. At an early stage, this was aspired as a premise to introduce Maxwell-Yang-Mills gauge theory to the Khronon model, but was not developed in full. The hope was that introducing ${}^{\pm}F$ as independent 2-forms with an implicit duality property would allow to avoid explicit reference to the metric. However, the question remained how these 2-forms should be related to the curvature of gravity, and ultimately a direction of first order formalism was (unfortunately) taken.

However, we are interested in the form of duality, where apparently different theories, in the field content or dynamics, in fact describe the same physical system [64, 65]. Apart from Maxwell electric-magnetic duality, another example is Proca-Kalb-Ramond duality, which recently had controversy in the quartic Proca-Kalb-Ramond limits [75]. As the author claimed [4], this type of duality is really partition function identity, which is how we will also introduce it here[12]. Consider a theory defined by the

[12] With a rather adventurous handling of the path integral, and without any rigorous definition of it. A careful and rigorous review is mandated, but here the topic of duality came rather unexpectedly, in relation to first order formalism. A primary concern is understanding what is actually physically meaningful in theory development, and the first order formalism in gauge theory, although with a different configuration space, proves physically identical to standard form; auxiliary fields as they are.

collection of fields $\phi = \{\phi_1, \ldots, \phi_n\}$, with the partition function

$$Z[\phi] = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_n \exp(iS[\phi]). \tag{3.34}$$

The path integral can be understood to include a *literal* integration, so we could, in theory, perform a path integral over one of the field variables ϕ_k, and obtain exactly the same generating functional as we began with:

$$\begin{aligned} Z[\phi] &= \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_{k-1}\mathcal{D}\phi_{k+1} \ldots \mathcal{D}\phi_n \int \mathcal{D}\phi_k \exp(iS[\phi]) \\ &\equiv \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_{k-1}\mathcal{D}\phi_{k+1} \ldots \mathcal{D}\phi_n \; H[\phi \setminus \phi_k], \end{aligned} \tag{3.35}$$

where H is some new functional of the field variables ϕ, without the particular ϕ_k. This, of course, greatly oversimplifies the procedure: it is entirely possible that integration over field variables does *not* yield any simple result, but rather e.g. a propagator, or some other vacuum functional, or some other sense of multi-valued "function"[13]. However, for many relevant cases this procedure can be done exactly, for example involving Gaussians, (generalized) Dirac deltas, or a set of constraints, disjoint propagators, functional determinants, etc. That is, we might get a simple numerical coefficient to a path integral with a different Lagrangian

$$Z[\phi] \to Z[\phi \setminus \phi_k] = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_{k-1}\mathcal{D}\phi_{k+1} \ldots \mathcal{D}\phi_n N_k \exp(iS[\phi \setminus \phi_k]), \tag{3.36}$$

or a set of constraints

$$Z[\phi] \to Z[\phi\setminus\phi_k] = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_{k-1}\mathcal{D}\phi_{k+1} \ldots \mathcal{D}\phi_n C(\phi\setminus\phi_k) \exp(iS[\phi\setminus\phi_k]). \tag{3.37}$$

Or, it will not simplify in any significant manner, and retain a functional dependence on ϕ_k — these examples are hardly exhaustive. However, the particular case of two different field theories ϕ_1 and ϕ_2, such that

$$\begin{gathered} \int \mathcal{D}\phi_1 \exp(iS_1[\phi_1]) = \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp(iS_{\text{parent}}[\phi_1, \phi_2]) = \int \mathcal{D}\phi_2 \exp(iS_2[\phi_2]) \\ Z_1[\phi_1] = Z_{\text{parent}}[\phi_1, \phi_2] = Z_2[\phi_2] \end{gathered} \tag{3.38}$$

is relevant, as it directly implies that the theories S_1 and S_2 define the *same* physical system.

[13] Although somewhat an oxymoron, this is standard terminology. The works of Kleinert often consider multi-valued functions, and recently, these objects appear in the study of amplitudes.

The crucial implication is that this has only been a change in representation, and not in any physical definition. The different representations can be called dual theories, and the initial generating functional the parent system. Duality in this sense is simply parent system identity, and was one of the main topics in the author's articles [4, 5]. As a particular example, consider what is called the first order formulation of gauge theory,

$$S_{\text{1st YM}} = \int \mathrm{Tr}\left(F[A] \wedge B - \frac{g^2}{2} B \wedge *B\right), \tag{3.39}$$

which introduces a single auxiliary 2-form field B — interpreted as the dual field strength, or the excitation, or the B-field of BF theory fare. This is a rather basic theory description, without an easy-to-find definite origin — it has been discovered, rediscovered, introduced and studied at various points [198–204]. Indeed, the author had called the introduction and elimination of the auxiliary field the "kinetic cycle" [2], and the basic procedure can be related to the Ostrogradski procedure of lowering differential order [205, 206]. It is also an example of a parent system, because in the path integral, we can integrate over[14] either A or B. For simplicity, let us restrict to $U(1)$, so we obtain respectively a Dirac delta and a Gaussian,

$$\begin{aligned} Z_{\text{EM, parent}}[A,B] &= \int \mathcal{D}A\mathcal{D}B \exp\left[i\int\left(\mathrm{d}A \wedge B - \frac{g^2}{2} B \wedge *B\right)\right] \\ \sim Z_{\text{electric}}[A] &= \int \mathcal{D}A \exp\left[i\int -\frac{1}{2g^2}\mathrm{d}A \wedge *\mathrm{d}A\right] \\ \sim Z_{\text{magnetic}}[B] &= \int \mathcal{D}B\delta(\mathrm{d}B) \exp\left[i\int\left(-\frac{g^2}{2} B \wedge *B\right)\right] \\ &\sim \int \mathcal{D}b \exp\left[i\int\left(-\frac{g^2}{2}\mathrm{d}b \wedge *\mathrm{d}b\right)\right], \end{aligned} \tag{3.40}$$

and the A-generated Dirac delta had constrained B to be a closed form, thus by the Poincaré lemma also exact, allowing to introduce a "magnetic" potential b.

We obtain the vacuum electric-magnetic duality in Maxwell theory. Rather than literal integration, it is also possible to take a field transformation perspective, where one of the fields is shifted to its minimum (constitutive law) — both approaches are well-explained in [74]. For Maxwell theory up to quadratic terms such a field shift decouples the fields — essentially, this is going through the standard procedure of calculating a

[14] Beware that the gauge theory path integral without gauge fixing is not well-behaved — we will discuss this in the following section. The curious property, however, is that this technicality can be neglected to illustrate the point, as e.g. the BRST antifield contribution separates [73].

Gaussian. Nevertheless, for the parent system, we consider B rather as a perturbation around the minimum,

$$B \to -\frac{1}{g^2} * F + h, \tag{3.41}$$

and find the action to decouple

$$S_{\text{1st YM}} \to \int \mathrm{Tr}\Big(-\frac{1}{2g^2} F \wedge *F - \frac{g^2}{2} h \wedge *h \Big). \tag{3.42}$$

The path integral measure is invariant under such shifts, and the Gaussian can be evaluated separately. The field shift is also the argument Armoni [74] proposed for deriving the dual of the Maxwell-Chern-Simons theory on a 3-manifold:

$$Z_{\text{MCS}} = \int \mathcal{D}A \exp\Big[i \int -\frac{1}{2g^2} \mathrm{d}A \wedge *\mathrm{d}A + \frac{k}{4\pi} A \wedge \mathrm{d}A \Big], \tag{3.43}$$

with the parent system

$$Z_{\text{MCS parent}} = \int \mathcal{D}A\mathcal{D}b \exp\Big[i \int -\frac{g^2}{2} b \wedge *b + b \wedge \mathrm{d}A + \frac{k}{4\pi} A \wedge \mathrm{d}A \Big] \tag{3.44}$$

shifted to the dual

$$Z_{\text{MCS dual}} = \int \mathcal{D}a\mathcal{D}b \exp\Big[i \int -\frac{g^2}{2} b \wedge *b - \frac{\pi}{k} b \wedge \mathrm{d}b + \frac{k}{4\pi} a \wedge \mathrm{d}a \Big]. \tag{3.45}$$

Paper [4] argued that this is *still* effectively identical to a literal integration, provided the vacuum functional determinant is retained,

$$\begin{aligned} \int \mathcal{D}A \exp\Big[i \int A \wedge \mathrm{d}b + \frac{k}{4\pi} A \wedge \mathrm{d}A \Big] \\ \sim \Big[-\frac{i}{2} \int \mathrm{d}b \wedge \Big(\frac{k}{2\pi} \mathrm{d} \Big)^{-1} \mathrm{d}b \Big] \frac{1}{\sqrt{\det((k/2\pi)\mathrm{d})}} \\ \sim \Big[-i \int \frac{\pi}{k} \mathrm{d}b \wedge b \Big] \int \mathcal{D}a \exp\Big[i \int \frac{k}{4\pi} a \wedge \mathrm{d}a \Big]. \end{aligned} \tag{3.46}$$

Field transformations do raise the question whether dual theories can be continuously transformed (rotated) between themselves. For electromagnetism, this is true, but for non-Abelian theory, this is not possible [77] — the differential geometry is straightforward [4] and can be traced to the fact that there is no covariant generalization of the Poincaré lemma, $\mathrm{D}G = 0 \not\Rightarrow G = F[B]$, nor $G = \mathrm{D}B$, and that curvatures generally do not form a homogeneous space, $F[B] \neq aF[A]$. As electromagnetism is an Abelian theory with the symmetry group $U(1)$, it loses the self-interaction contributions, and standard cohomology theory applies.

Finally, invariance and identity of the parent system is crucial for deriving dual theories. Any additions, e.g. quartic self-interactions, change the underlying theory, and no duality continuation can be assumed. So, Proca and Kalb-Ramond theory are dual, but *quartic* Proca

$$Z_{\mathrm{Proca}^4} = \int \mathcal{D}A \exp\left[i \int -\frac{1}{2g^2} \mathrm{d}A \wedge *\mathrm{d}A + \frac{m^2}{2} A \wedge *A + \lambda * (*(A \wedge *A))^2 \right] \tag{3.47}$$

is *not* dual to quartic Kalb-Ramond

$$Z_{\mathrm{KB}^4} = \int \mathcal{D}B \exp\left[i \int \frac{1}{2m^2} \mathrm{d}B \wedge *\mathrm{d}B - \frac{g^2}{2} B \wedge *B + \lambda * (*(B \wedge *B))^2 \right], \tag{3.48}$$

as they do not integrate to one-another from a common parent system — therefore, the massless limits can be expected to also be different, consistently so, and without any implication to the original Proca and Kalb-Ramond duality [4, 75, 76].

In principle we are not required to introduce any quantum theory, resp. the path integral, and can also consider purely classical duality. However, because the configuration spaces are clearly unequal, this requires studying on-shell behaviour — or a field-shift decoupling, as discussed earlier. The path integral simply allows to operate a single procedure, i.e integration, to derive the same results. An important conclusion, regardless, is that a nontrivial theory extension should either modify the field algebra or geometry in a nontrivial manner (e.g. more unique representations of whatever algebraic structure, non-manifold topologies in the background, new Lagrangian coupling terms, irreducible phase solution space geometries) — or nontrivial potentials, with degenerate minima [5]. The Plebanski theory of gravity [78, 207] is a particular example where the Lagrange multiplier allows for multiple solutions, i.e. the minimum can be understood to be degenerate. The physical choice of the minimum is akin to a spontaneous process, although it would be misleading to claim it to be spontaneous symmetry breaking [208] *per se*, as spontaneous symmetry breaking refers to actual symmetry breaking of Lie group symmetries, between the Lagrangian and the vacuum solutions (states). Manufacturing auxiliary field minima, which in addition to the physical constitutive law $B \sim *F$ contain other false vacuums, is rather a case of weakening the equations of motion[15]. Furthermore, due to the inevitable backreaction to the inhomoge-

[15] The author attempted an interpretation in terms of a weakened sense of "rotation" [5], but this has not been fully successful. A closer study of the function spaces would be merited if a clear interpretation were to be found.

neous Yang-Mills equation of a pure auxiliary field potential [5], candidates like quartic B-potentials

$$V(B) = \lambda * (*(B \wedge *B))^2 \tag{3.49}$$

are not physically viable regardless — which is unfortunate, because it dismisses the aspiration of producing nontrivial quantum phenomenology from possible tunneling between the different phases. It does, however, leave an avenue of continuation by considering weaker Lagrange multiplier constraints, akin to Plebanski theory. However, this remains to be fully resolved.

Note the presence of the Hodge star $*$ in the Maxwell-Yang-Mills Lagrangian, and indeed the action is written as the Hodge inner product. This is the local and linear constitutive law in premetric terms, and the constraint $B \sim *F$ that has to be introduced in BF theory — physically, it establishes a sense of causality into the gauge theory[16]. The Hodge star is also a nonlinear and singular object if the metric became degenerate. As was discussed, it requires the inverse metric in its expression — and if a zero-metric phase was desired, the question became how to bypass it [2]. The proposal was to consider first order gauge theory, as defined earlier, and define the dual field strength B as some expression of a fundamentally contravariant tensor, i.e. naturally situated in the dual space, without any *a priori* relation to the base tensor space. Then, only in a metric phase would a canonical correspondence have been introduced. In definite terms, the rewritings under study were

$$S_u = \int \mathrm{Tr}\left(\frac{g^2}{2}\epsilon_{abcd} u^a \wedge e^b \wedge u^c \wedge e^d + \eta_{ab} u^a \wedge e^b \wedge F\right), \tag{3.50a}$$

$$S_G = \int \mathrm{Tr}\left(\frac{g^2}{24} G^{ab} G^{cd} \eta_{ac}\eta_{bd}\epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l + G^{ab}\eta_{ac}\eta_{bd} e^c \wedge e^d \wedge F\right). \tag{3.50b}$$

The symbols η_{ab} and ϵ_{abcd} are only Lorentz-group invariants, and would be well-defined regardless of the specific behaviour of any specific pseudo-Riemannian metric. That is, the model space in Cartan geometry terms would have been assumed to be well-defined, independent of whatever happens on the base manifold. Furthermore, in a metric phase, this reduces to standard Hodge duality, and the standard first order formalism (3.39), with the added benefit that the action does not become singular if $g = 0$ (and

[16] In fact, note that *all* physically relevant kinetic terms are of the Hodge inner product form — it is a matter of how exactly metric causality relates to field dynamics, but deserves further elaboration and development.

only becomes zero) — at no point does the inverse metric appear in the action, and the contravariant indices can be assumed to be formal, without the requisite interpretation of a canonical mapping to the dual space[17].

The author is quite dissatisfied with this reinterpretation, but at the time, it appeared as the simplest way forward to introduce a potentially zero-metric phase, while also retaining equivalence with standard electromagnetism and Yang-Mills theory. When deviations from General Relativity can still be considered to be permitted, any significant deviations from quantum electrodynamics, electroweak theory, or chromodynamics, would be devastating to a theory's viability — and here, this does not happen, as it reproduces the standard theories both classically *and* quantum-mechanically [4, 13, 202, 203]. Furthermore, the added benefit of greater similarity to the theory of gravity is apparent — internal gauge theory can be thought to be a "rotation" of the surface element [2, 5]

$$e^a \wedge e^b \to u^a \wedge e^b, \tag{3.51}$$

the connection assembled to

$$\tilde{\omega}_{ab} = \omega_{ab} + \eta_{ab} A, \tag{3.52}$$

so the cosmological constant term rotating to a potential, and the trace eliminating unnecessary cross-terms. Then collectively, consider e.g.

$$\begin{aligned} S = \frac{1}{2} \int \mathrm{Tr} \Big[& (i\kappa^{-1} e^a \wedge e^b + u^a \wedge e^b) \\ & \wedge \Big({}^{+}R_{ab} + \eta_{ab} F + \epsilon_{abcd} \Big(2i\Lambda e^c \wedge e^d + \frac{1}{2} u^c \wedge e^d \Big) \Big) \Big], \end{aligned} \tag{3.53}$$

which, as soon to be defined, includes gravity as well as other gauge interactions — but still not in a truly unified setting, as the gauge groups are only trivially separated. Rather, unification would have required a mechanism for the solder form, or the hypersurface basis element, to separate

[17] The crux of the argument is a difference in interpretation — the Lorentz structure is assumed to persist throughout, as a sort-of model space with the Minkowski metric and Levi-Civita symbol as background objects, so the Lorentz indices would be sensible, even though not immediately related to the tangent space. The standard pseudo-Riemannian metric only appears implicitly, through the coframe e^a, so purely formally, the action tends to zero if $e^a \to 0$, rather than a ∞-singularity because of the inverse metric $g^{\mu\nu}$. Such juggling of structures is hardly appealing, and the author did not consider this formalism to be the ultimate solution — e.g. an argument can instead be made for ϵ^{abcd} as a separate object, with F_{ab} untouched. If to reattempt now, this direction would rather be dismissed, and the topological phase would be glued through some other (potentially non-Lagrangian) method.

into SO and internal gauge components. The interpretations, and possible directions of continuation, were multiple: should this construction be interpreted as some "surface excitation" (in the sense that other gauge theory B-fields are understood as deviations $e \wedge e \to e \wedge e + B$), or a "rotation" (so, the process $e \wedge e \to u \wedge e$ as primary), or a "linear transformation" (working with the components $G^I_{ab} T^I$)? To a large extent, these are trivial reinterpretations, at least mathematically so, owing due to the component expression of (Lie-algebra valued) differential forms, but they were suggestive of rather different mathematical approaches to study.

Furthermore, a particular choice of the Lie-algebra valued u^a one-form led to a decidedly novel theory, the so-called "Isokhronon" with the action

$$S_\phi = \int \mathrm{Tr}\left(\frac{1}{2}\epsilon_{abcd}\mathrm{D}\phi^a \wedge e^b \wedge \mathrm{D}\phi^c \wedge e^d + \eta_{ab}\mathrm{D}\phi^a \wedge e^b \wedge F\right). \tag{3.54}$$

Here, the phenomenology is *not* trivially the same of electromagnetism and Yang-Mills, but rather the equations of motion introduce a magnetized-polarized background to the inhomogeneous Yang-Mills equation,

$$\epsilon_{abcd}e^b\mathrm{D}\phi^c \wedge e^d + \eta_{ab}e^b \wedge F = X_a,\ \mathrm{D}X_a = 0, \tag{3.55a}$$

$$\mathrm{D}\phi_a \wedge e^a = *F + \frac{1}{2}(*X_a) \wedge e^a. \tag{3.55b}$$

This somewhat mimics the behaviour of the Khronon theory of gravity, which is also the reason why the Isokhronon was one of the main topics of study [2, 4]. The structure of this theory becomes a bit more clear when shifted to a perturbation around a solution[18] Φ^a of the equations of motion,

$$\phi^a \to \Phi^a + f^a \tag{3.56}$$

so

$$Z_\phi \to \int \mathcal{D}A\mathcal{D}f^a \exp\left[i\int \frac{1}{2} * F \wedge F + \frac{1}{4}(*X_a) \wedge e^a \wedge F \right.$$
$$\left. + \epsilon_{abcd}\mathrm{D}f^a \wedge e^b \wedge \mathrm{D}f^c \wedge e^d\right]. \tag{3.57}$$

Note how the decoupled vector $f^a \sim \partial^a \lrcorner f$ compares to a usual massless vector v action as

$$\int \mathrm{D}\partial_a \lrcorner f \wedge \mathrm{D}\partial_b \lrcorner f \wedge *(e^a \wedge e^b) \text{ vs. } \int \partial_a \lrcorner \mathrm{d}v \wedge \partial_b \lrcorner \mathrm{d}v \wedge *(e^a \wedge e^b) \sim \int \mathrm{d}v \wedge *\mathrm{d}v. \tag{3.58}$$

[18] Note that this requires a choice of the integration constant $X_a,\ \mathrm{D}X_a = 0$ as well!

In hindsight, by a simple integration by parts, it is clear that it introduces a torsion-dependence into the kinetic term[19], $\phi_a T^a \wedge F[A]$, so the mediation of the gauge interaction becomes contingent on the presence of torsion, and is not to be considered a viable extension of electromagnetism and Yang-Mills theory [4, 137]. But even this could be, to some extent, ameliorated in the Khronon theory of gravity, which in principle permitted torsionful Minkowski solutions. More immediately, the phenomenology was novel — and rather tricky. For one, classification of the effects of the integration constant $X_a, \mathrm{D}X_a = 0$ is not immediately obvious, and mimics the Khronon dark matter integration constant. At the same time, torsion permitting, the electromagnetic Isokhronon only propagates two degrees of freedom [4], despite this background, which is entirely unlike the Khronon with the clear additional presence of ideal dust in vacuum. It would appear that the Khronon symmetry breaking is indispensable for the dust generation, which is in turn obtained entirely from the connection sector. The Isokhronon does not possess such greater configuration freedom, such that it could be removed by further symmetry breaking.

On a more fundamental side, there is a question why such actions with inherent backgrounds are not forbidden — the Isokhronon can be generalized to other types of backgrounds, and there was a recent discussion of "shadow charges" in both electromagnetism and gravity [209, 210]. The author suggested that "background independence" (from background fields) could be a fundamental principle to further limit the space of possible Lagrangians, although it is similarly possible that inherent background fields are simply in conflict with the second law of thermodynamics [4]. Nevertheless, the thermodynamic questions are yet to be studied. It seems a reasonable hypothesis that a well of immutable interaction potential might ruin thermodynamics, but formalizing an argument on the (to be selected!) space of theories is yet to be done. There is still standard study possible for the Isokhronon, e.g. canonical quantization or renormalization, but the interest should be balanced with what insight can be gathered from such models — rather, the implications seem to be better applied to further theory construction, than hanging with the Isokhronon model.

3.4 Gauge fixing, BRST

The core of gauge theory is defining objects with internal gauge-redundant description. The physical quantities are gauge-invariant, so we should work with the equivalence classes themselves, rather than the representatives of

[19] As does the *Khronon* theory [20]. However, the Khronon can respite in shadow of the Einstein-Cartan premise.

equivalence classes. Failure to do so has rather severe consequences, to the point that the dynamics is not well-defined. For example, if the gauge redundancy is not removed, then the gauge theory path integral is simply divergent (overcounted). Classically, the initial value problem cannot be uniquely solved, instead there is a residual gauge freedom. This is classically also rather straightforward to resolve — in terms of the preceding differential geometry, a local or global gauge is simply a selection of a local or global section s in the principal bundle P; in terms of field theory, a variety of explicit constraints have been utilized, mainly based on computational convenience. E.g. consider the Lorenz gauge $\partial_\mu A^\mu = 0$ (which is actually an incomplete fixing, as it still permits a gauge transformation $A_\mu + \partial_\mu \varphi$, $\partial_\mu \partial^\mu \varphi = 0$), but the listing is greatly varied.

There are a few ways to include gauge fixing into the theory, from as simple as the addition of Lagrange multipliers (and breaking gauge invariance of the Lagrangian) to the introduction of ghost fields[20]. However, the preceding differential geometry of the covariant phase naturally directs toward the heavily algebro-geometric BRST (Becchi-Rouet-Stora-Tyutin) antifield formalism, see [213] for an overview. The original cycle of papers is spread out. BRST is also intrinsically linked with the Batalin-Vilkovisky (BV) formalism, but their exact distinction is sometimes less than clear; depending on the source, BV might be seen as a generalization of BRST (to more relaxed constraint algebras) or as working on a different sector of the issue (antifields, symmetries, homology and homotopy, etc.). Nevertheless, the basic aim is to introduce a set of "antifields" and ghosts with a collection of superalgebraic axioms and weightings, replacing the initial gauge symmetry with a new fermionic rigid symmetry. This is motivated by renormalization and anomaly cancellation.

A traditional reference is the book by Henneaux & Teitelboim [87], which will also be the primary reference of this section. A complete description, unfortunately, extends well beyond the scope of this dissertation, as it would require treating perturbative (co)homology and supersymmetric algebras. In the context of this dissertation neither is it actually required — there are just a few points to emphasize. That is, gauge fixing can be achieved through the addition of ghost fields, by additive Lagrangian terms and relevant for partition function manipulation. The symplectic structure of the covariant phase space is the natural structure to quantize (using traditional methods), and allows to quickly derive hypotheses on the quantum behaviour of more nontrivial theories (but which has to be actually *verified*). As further development, the Khronon gauge fixing can

[20] And then, the introduction of Nakanishi-Lautrup fields [211, 212] is also reminiscent of a "first order formalism" for gauge theory.

be inspired from BRST and similar methods — but requires care w.r.t. the shift symmetry. However, on more physical and philosophical grounds, the author is not completely convinced that unification and quantum gravity can proceed from traditional methods, or would require new principles and settings — however, refraining from suggesting any speculation in this direction.

For example, the main theorem to refer to is as follows, verbatim quoted.

Theorem 4 (Theorem 8.3 in Henneaux & Teitelboim [87])**.** *(a) If* $\mathcal{H}_k(\delta) = 0$ *for all* $k \neq 0$ *there exists a differential* s *of total ghost number one that combines* d *with* δ*,*

$$\begin{aligned} s &= \delta + \mathrm{d} + \overset{(1)}{s} + \overset{(2)}{s} + \dots, \\ r(\overset{(k)}{s}) &= k,\ \mathrm{gh}(\overset{(k)}{s}) = 1, \\ s^2 &= 0. \end{aligned}$$

(b) Any differential s *that combines* d *and* δ *as preceded, fulfills*

$$H^k(s) = H^k(\mathrm{d}),$$

where the cohomology of d *is computed in* $H_0(\delta)$*, i.e., is defined by the equations*

$$\mathrm{d}x = \delta y,\ x \sim x + \mathrm{d}z + \delta z',$$

with

$$r(x) = 0 = r(z),\ r(y) = 1 = r(z').$$

Applied to the covariant phase space $\mathcal{P} = \tilde{\mathcal{P}}/G$, this theorem allows to construct the BRST symmetry operator s, which has observables in the cohomology at ghost number zero, $H^0(s)$. There is a great wealth of definition to unpack, and the elements are nearly impossible to present even in a very cavalier pacing; or, would be so brief to be meaningless. We could say that the oft-quoted formula

$$s = \delta + \mathrm{d} + \text{“more”} \tag{3.59}$$

includes the Koszul-Tate differential δ (transverse to gauge orbits, restricting from configuration space to the stationary surface of solutions), the longitudinal derivative d (along the gauge orbits), and the “more” component are the $\overset{(k)}{s}$, derivations of resolution degree k. However, this would require introducing homological resolutions, then differentials modulo δ, and then differential structure of a supercommutative (i.e. $\mathbb{Z}_2$-graded[21])

[21] Supersymmetry has proven to be useful for physics theory, but mathematically, other gradings are also possible — consider e.g. “hypersymmetric” $\mathbb{Z}_3$-gradings [214].

algebra, to then return to cohomology — and this for what was ultimately used as argument in little more than a few paragraphs of the author's [4].

Handling gauge fixing is a crucial component of modern gauge theory, so any discussion of gauge theory would be incomplete without this topic. But the implication of interest is somewhat different. For one, the physical (anti)ghost content, including Lagrangian terms, can be calculated in a direct manner by solving what is known as the BRST master equation, so in practical terms we could be content in defining the additional field and Lagrangian content, leaning on relevant theorems explaining the impact on physics — in particular, see [73] for application to first order Yang-Mills theory. For two, the arena of operation is the *covariant phase space*, the symplectic geometry that is constituted by the trajectories, both in the classical and quantum theory. We are led to believe that if two symplectic geometries are the same or equivalent, i.e. symplectomorphic, we are provided the *same* phase space, and as we will shortly see, this suggests quantum equivalence, and was the author's argument why quantum equivalence of first and second-order Yang-Mills theory is trivial [4].

Let us begin with the "one". In addition to basic fields ϕ^i, we introduce (contingent on the symmetries) ghosts C^α and ghosts-of-ghosts C^A, and the corresponding antifields ϕ^*_i, ϕ^*_α and ϕ^*_A, all assigned a suitable grading ϵ and ghost numbers pure gh and antigh, total ghost number being gh = pure gh − antigh. Then,

$$\begin{aligned} \text{gh} \quad & \phi^*_A, \phi^*_\alpha, \phi^*_i, \phi^i, C^\alpha, C^A = -3, -2, -1, 0, 1, 2, \\ \text{pure gh} \quad & \phi^*_A, \phi^*_\alpha, \phi^*_i, \phi^i, C^\alpha, C^A = 0, 0, 0, 0, 1, 2, \\ \text{antigh} \quad & \phi^*_A, \phi^*_\alpha, \phi^*_i, \phi^i, C^\alpha, C^A = 3, 2, 1, 0, 0, 0. \end{aligned} \tag{3.60}$$

Gauge transformations in the fields can be parametrized as

$$\delta\phi^i = R_\alpha{}^i \varepsilon^\alpha, \tag{3.61}$$

under which the action is invariant, $S \to S$. However, there may be further reducibility identities

$$Z_A{}^\alpha R_\alpha{}^i = C_A{}^{ij} \frac{\delta S_0}{\delta \phi^j}, \; C_A{}^{ij} = -C_A{}^{ji}. \tag{3.62}$$

Note the indices! The additional field content establishes its own transformation rules as well, e.g. $\delta\phi^*_\alpha = R_\alpha{}^i \phi^*_i$, $\delta\phi^*_A = -Z_A{}^\alpha \phi^*_\alpha - \frac{1}{2} C_A{}^{ij} \phi^*_i \phi^*_j$ etc.

To bring this into BRST terms, an antibracket (,) is introduced, similar to a Poisson bracket,

$$(F,G) = \frac{\delta^R F}{\delta\phi^i}\frac{\delta^L G}{\delta\phi_i^*} - \frac{\delta^R F}{\delta\phi_i^*}\frac{\delta^L G}{\delta\phi^i} + \frac{\delta^R F}{\delta C^\alpha}\frac{\delta^L G}{\delta\phi_\alpha^*} - \frac{\delta^R F}{\delta\phi_\alpha^*}\frac{\delta^L G}{\delta C^\alpha} + \frac{\delta^R F}{\delta C^A}\frac{\delta^L G}{\delta\phi_A^*} - \frac{\delta^R F}{\delta\phi_A^*}\frac{\delta^L G}{\delta C^A}. \quad (3.63)$$

Field-antifield pairs are naturally conjugate, e.g. $(\phi^i, \phi_j^*) = \delta_j^i$, but more importantly, the antibracket works to generate the BRST symmetry through the BRST-generator S in the sense that

$$sF = (F, S). \quad (3.64)$$

This permits to construct the new action that incorporates ghosts and antifields. Physical requirements are

$$\epsilon(S) = 0,\ \mathrm{gh}(S) = 0, \quad (3.65)$$

while nilpotency $s^2 = 0$ requires

$$(S, S) = 0, \quad (3.66)$$

which is the BRST "master equation", to solve for S. Expanded in antighost number,

$$S = \sum_{n\geqslant 0} \overset{(n)}{S}, \quad (3.67)$$

there are further boundary conditions, e.g. in particular $\overset{(n)}{S} = S_0$, being the original gauge-invariant action. Henneaux & Teitelboim [87] provide several examples how to solve the master equation order-by-order, and [73] applies BRST to first-order Yang-Mills theory for a proof of equivalence to standard form. Up to order two,

$$\begin{aligned} S &= \overset{(0)}{S} + \overset{(1)}{S} + \overset{(2)}{S} + \dots \\ &= S_0 + \phi_i^* R_\alpha{}^i C^\alpha + \phi_\alpha^* \Big(Z_A{}^\alpha C^A + \frac{1}{2} C_{\beta\gamma}{}^\alpha C^\beta C^\gamma \Big) \\ &\quad + \phi_i^* \phi_j^* \Big(-\frac{1}{4} M_{\alpha\beta}{}^{ij} C^\alpha C^\beta + \frac{1}{2} C_A{}^{ij} C^A \Big) + \dots \end{aligned} \quad (3.68)$$

So, the pragmatic extreme is quite straightforward in the algorithm, and we can be content in handling gauge issues in gauge theories.

Concluding very briefly with the "two", however, we would like to consider what even is "quantization". Quantum field theory is an extensively studied and *understood* subject [215, 216], without referring to mathematical treatments, and can be characterized as a theory of operator-valued distributions. At the same time, it is not entirely surprising that General Relativity and the pseudo-Riemannian geometry of spacetime does not admit a simple quantum description in terms of Fock states (essentially, asymptotic plane waves) on a flat Minkowski background (essentially, a single tangent space). What to even quantize? A field theory of the metric? Or should the issue of (quantum) measurements and observers be taken to the forefront? Perhaps the setting should be wholly different, strings or loops or simplices or otherwise? Certainly the author has his opinion — relationalist, observer-focused and geometric — but to avoid unnecessary speculation, refer to a recent and exhaustive handbook on the matter [217] instead. Rather, the curiosity is that the symplectic structure of the covariant phase space admits a well-motivated approach to quantum theory, following Dirac's prescriptions.

So, we are led to geometric [218] and deformation quantization [219]. These are non-perturbative quantizations of a theory. That is, we are to find a mapping $f \mapsto \hat{f}$ from classical observables f to quantum operators $\hat{f}$, such that

1. the mapping is linear,
2. gives $\hat{f}$ as the multiplication operator when $f = \text{const}$,
3. satisfies the Poisson bracket-to commutator quantization rule

$$\{f_1, f_2\} = f_3 \Rightarrow [\hat{f}_1, \hat{f}_2] = -i\hbar \hat{f}_3.$$

By the Gronewold-van Hove theorem [220, 221], these conditions alone are not satisfactory, but nevertheless, they provide a natural direction to follow. Then, geometric quantization aims to explicitly construct the Hilbert space geometry, and finds that the Kostant-Souriau prequantum operator depends on the symplectic potential θ. Deformation quantization rather works with a Poisson manifold (note that a symplectic manifold is a Poisson one as well), and deforms the associative algebra of classical observables to a non-commutative associative algebra. *Both* approaches are directly based on the geometry of the phase space.

The author's interest here was that the covariant phase space can be probed for quantum effects as well — or lack of them. In particular, we can argue that symplectomorphic covariant phase spaces should yield canonically the same quantum theory, e.g. as in the case of first order Yang-Mills theory [4]. In turn, the Isokhronon would provide a quantum-inequivalent gauge theory, but the details become less clear because of the integration constant background. A proper canonical quantization should follow — in particular, how is the Hilbert space built (if it even can be, consistently!) in relation to the new integration constants, background fields. Are transitions between different backgrounds possible, or do they not intersect? This was attempted as a simpler premise for quantization of the Khronon proper, and also to illustrate problems with the "shadow charges" [209, 210], but it might as well have increased the complexity instead. The quantum theory of the Khronon can prove to be quite nontrivial, compare [222], and in the author's opinion, would be a prospective direction to consider.

3.5 Gravity. Relativity, real and complexified

Let us conclude with the theory of gravity. Then, all of General Relativity can be derived from the simple construction that spacetime is provided as a pseudo-Riemannian manifold M with metric g, such that g resides in the stationary points of the (vacuum) Einstein-Hilbert action functional

$$S_{\mathrm{GR}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R \sim \frac{1}{2\kappa}\int *(e^a \wedge e^b) \wedge R_{ab}[\omega_{ab}]. \tag{3.69}$$

Matter Lagrangians can be appended, so variation provides the energy-momentum tensor — or, the energy-momentum 3-form Σ_a (commonly rather T_a, but the letter is already used for torsion). Relevant to our apparent reality is the 4-dimensional case with the Lorentzian metric $(-,+,+,+)$ or $(+,-,-,-)$. In a sense, physical theory ends once a Lagrangian is written down, as it gives a sufficiently total description of the model, and proceeding analysis only tries to make sense of the trajectories (and it is unfair to be dismissive of this — generally, this is extremely sophisticated). There is little room for additional flexibility of theory creation once a Lagrangian formulation is provided[22], apart from handling initial conditions and proposing ever-new models instead. Having fixed a manifold background, however, the Einstein-Hilbert Lagrangian is one of the few lowest-order Lagrangian contributions that can even be included; Lovelock's theorem provides similar justification [174].

This dynamics has been the subject matter of gravity for over a hundred years, with many excellent treatises on the subject [224–226]. Our interest is rather in the formulation. This dissertation cares less how relativity

[22] A *consistent* Lagrangian description, that is. For example, the Einstein-Hilbert action must be appended with the Gibbons-Hawking-York boundary term if the manifold has a boundary [223].

is justified, in the equivalence principles or otherwise, or how solutions are derived analytically and numerically. Rather, the central issue is the *premise*, which is not any less colorful than the *physics* that gravitation provides.

Immediately, note that we are not required to consider the metric g as the only basic dynamical variable, but can just as well consider e.g. the (tetradic) Palatini formalism, where rather the coframe e^a and the Lorentz connection ω_{ab} are the basic dynamical variables. Then, it is simple to relax to Einstein-Cartan formalism, where the Lorentz connection is allowed to in principle possess torsion. The equivalence in vacuum is relatively trivial, as the system of differential equations

$$\epsilon_{abcd} e^b \wedge R^{cd} = 0, \tag{3.70a}$$

$$\mathrm{D} * (e^a \wedge e^b) = 0 \tag{3.70b}$$

trivially implies that the connection is still torsion-free, thus Levi-Civita, and the connection equation of motion produces little new. Uniquely,

$$\omega^{ab} = \frac{1}{2}(\partial^b \lrcorner \mathrm{d}e^a - \partial^a \lrcorner \mathrm{d}e^b + \partial^a \lrcorner (\partial^b \lrcorner \mathrm{d}e_i) e^i). \tag{3.71}$$

Note the variation of the Hodge star,

$$\delta * \alpha = *\delta\alpha - *(\delta e^a \wedge \partial_a \lrcorner \alpha) + \delta e^a \wedge \partial_a \lrcorner * \alpha. \tag{3.72}$$

The observational or physical relevance of hypermomentum to source the connection is still a debated subject [227–231]. In metric-affine couplings, fermionic currents provide a natural hypermomentum source, while General Relativity does not require its presence.

The promotion of the connection to an independent dynamical variable can be somewhat debatable on its merit. Arguments can vary, from the inclusion of fermionic fields to greater similarity to Yang-Mills gauge theory — the author also argued for the latter [5]. Nevertheless, the proper relation between what is commonly known as the Palatini formalism and metric General Relativity is somewhat subtle, the question if and where it can actually introduce anything substantially different. To say that the Palatini formalism is equivalent to metric General Relativity (at least in the vacuum) would be somewhat underselling the subtlety — although the Hamiltonian formulation of Palatini theory coincides with metric theory [232, 233] and the background-effective equivalence was discussed in [234], there would be further discussion on projective transformations [235, 236], or if the theory were to be modified in any manner[23] [237], or on the bound-

[23]Non-minimal matter couplings and higher curvature terms are only one possible questions then. There is an endless wealth of discussion in the definition and distinction for Palatini formalism in *any* theory of gravity.

ary conditions [238]. Refer to [239–241] for even further discussion on the distinction of gravity from Maxwell-Yang-Mills theory.

So, why should we care? The author proceeded from the *hypothesis* that there is a substantial split from a "unified" phase in the B-field of BF theory fare, something akin to symmetry breaking that would completely distinguish gravity from the remainder of gauge interactions. However, this was, ultimately, nothing more than an unrealized hypothesis. The premise is ultimately only as important as far as its implications go, unless the study of the premise is taken as its own merit. We might as well reasonably so consider gravity as a field theory of spin-2 gravitons [36, 37, 242]. Regardless, the action (3.69) is the target, either exactly or as a limit, for any physical theory of gravity, and this can be achieved in a great manner of ways [21].

Before exploring some further formulations, and primarily so in the context of unification, let us briefly introduce complexified gravity. A short introduction can be found in [243], which is also the main reference for this exposé, but also refer to [233, 244]. Complexification here refers to complexification of *bundles* over a *real* (implicitly 4-dimensional!) manifold M[24]. In particular, e.g. the tensor bundle

$$T_{\mathbb{C}} = \bigoplus_{r,s} T^r_s(M) \otimes \mathbb{C}. \tag{3.73}$$

We are provided two notions of dualization: the external Hodge $*$-dual, and the internal $\mathfrak{so}_{\mathbb{C}}(3,1)$ dual $\star$ defined by

$$\star\sigma_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}\sigma_{cd}. \tag{3.74}$$

Note that for the coframe basis the internal and external dualizations are coincident, $\star(e^a \wedge e^b) \sim *(e^a \wedge e^b)$. The complexified Lie algebra splits into self-dual and anti-self-dual components $\mathfrak{so}^{\pm}_{\mathbb{C}}(3,1)$ — in particular, this applies to the now complex-valued Lorentz connection $\omega_{ab} = {}^+\omega_{ab} + {}^-\omega_{ab}$, and therefore to the curvature $R_{ab} = {}^+R_{ab} + {}^-R_{ab}$, as well. The Einstein-Hilbert action decomposes,

$$S_{\text{GR}} = \frac{i}{2\kappa}\int e^a \wedge e^b \wedge {}^+R_{ab}[{}^+\omega] - \frac{i}{2\kappa}\int e^a \wedge e^b \wedge {}^-R_{ab}[{}^-\omega]. \tag{3.75}$$

However, there is no necessity for both contributions here — it is possible to show (and spelled out in [243]) that we can restrict to only one term, respectively self-dual or anti-self-dual gravity, and still obtain the same equations of motion for the coframe field as the full complexified gravity action.

[24] As opposed to the theory of complex manifolds, locally homeomorphic to $\mathbb{C}^n$.

As a final remark, but not considered in greater length in this dissertation, the Hamiltonian structure can be studied through the $3+1$ split, as defined before and which is the premise for the ADM formalism [157]. In the real case this quickly leads to various issues: the Hamiltonian vanishes, second class constraints, vanishing momenta; for canonical quantum gravity this becomes near-unmanageable with the inner product problem, the problem of time, finding solutions to the Wheeler-DeWitt equation *et cetera* [217, 245]. Giulini [243] traces it to the nondynamical nature of ω_{ab}, i.e. their vanishing momenta — and self-dual gravity allows to rather consider the longitudinal component ${}^{+}\omega_{\perp ab} \sim {}^{+}\omega_{0I}$,

$$\frac{1}{2}\epsilon_{IAB}{}^{+}\omega^{AB} = i^{+}\omega_{0I} \tag{3.76}$$

for time derivatives of ${}^{+}\omega_{ab}$. The spacetime structure group $SO_{\mathbb{C}}(3,1)$ is reduced to $SO_{\mathbb{C}}(3)$, working on the 3-dimensional hypersurfaces. Either through a careful constraint analysis, or with the prescient knowledge of the required results [243], a set of new variables can be defined: the $SO_{\mathbb{C}}(3)$ connection[25] $\underline{A} = 2\underline{\omega}$ on the hypersurfaces Σ, alongside the lapse N, shift $\underline{N}$ and densitized tetrad $\underline{E}$. Reality constraints then remove the intermediary complexification. Apart from just a curious expression of gravitational variables, these quantities laid the basis for quantization in the loop representation [246, 247].

The extent to which the dissertation cares about complexified gravity is twofold: the complexified Lorentz group possesses a self-dual vs. anti-self-dual split, and the Khronon Hamiltonian proves to be self-dual Ashtekar gravity alongside ideal dust. The former is crucial for theory definition, the latter characterizes the phenomenology of the theory.

Consider that this incredibly short introduction is already *sufficient* to soon introduce the Khronon theory of gravity — the basic philosophy is straightforward, as will shortly be clear. We need not consider how to solve the equations for *any* physical situation. Specialized treatises for various aspects of the the Khronon phenomenology can be written[26], and there are plenty of points that *may* prove to be nontrivial, such as the question of black hole entropy, chiral cosmology, quantization, and more. But for theory *construction*, we are complete. Rather, the issue is that if a theory's phenomenology can be sufficiently classified, it becomes difficult

[25] An off-hand comment by Giulini is quite striking in the context of Yang-Mills duality rotations: connections form an affine space, but not a linear one, and cannot generally by added and multiplied with numbers. The author's [4] traced a similar issue as the reason why non-Abelian duality rotations fail, and this comment summarizes the problem.

[26] And the author would like to kindly direct toward the work of fellow student L. Zheng for questions on cosmology in the Khronon theory.

to have conviction in any certainty for nontrivial implications, or to not simply rederive previously already known phenomenology just in a new shade of theory. This is a question of risk balancing, and already in the initial paper [20] was the Khronon theory implicated to follow an Einstein-Aether phenomenology alongside a cosmological dust, per a Hamiltonian comparison. The question of interest and purpose, triviality *versus* nontriviality, complexity against meaningfulness, was a serious concern in choosing the direction of theory development. So, a somewhat high risk approach was settled to, partly by necessity, of testing rather speculative ideas and proposing new principles for theory building — a more modest approach in adapting simplicity constraints is also entirely possible, and any follow-up can certainly exhaust the standard methods.

Chapter 4

Unification

Although the aim of proposing a unique unified theory of the Khronon theory of gravity was not fully achieved, we would be incomplete without reference to the motivation of this study. So, we will finish with a brief exposition on unification, as it is commonly understood in particle physics, and what motivations laid in the basis of the author's work. Note, however, that although a *unique* unified theory was not proposed, the *methods* to approach Khronon unification were proposed several — from modified simplicity constraints to different vector representations. Follow-up work can execute upon these ideas in a straightforward manner.

4.1 The Khronon theory of gravity

Before anything else, we are *finally* at the moment to introduce the theory of gravity that was the origin topic of this study: the Cartan Khronon Lorentz gauge theory of gravity [20]. So, we present the action

$$S_\tau = \int \frac{i}{2\kappa} \mathrm{D}\tau^a \wedge \mathrm{D}\tau^b \wedge {}^+R_{ab}. \tag{4.1}$$

Note the *self-dual* curvature ${}^+R_{ab}$ and the exterior covariant derivative w.r.t. the *full* connection $\mathrm{D}\tau^a = \mathrm{d}\tau^a + \omega^a{}_b\tau^b$. We consider a 4-dimensional real manifold M, and complexified variables upon it, with the symmetry group $SO_{\mathbb{C}}(1,3)$. The basic dynamic variables are the Khronon Lorentz-valued scalar τ^a, and the Lorentz connection $\omega_{ab} = -\omega_{ba}$; note that we permit torsion, but not non-metricity (resp. the symmetric component $\omega_{(ab)}$ vanishes). The corresponding vacuum equations of motion are

$$\mathrm{D}(\mathrm{D}\tau^b \wedge {}^+R_{ab}) = 0, \tag{4.2a}$$

$${}^+\mathrm{D}^+(\mathrm{D}\tau^{[a} \wedge \mathrm{D}\tau^{b]}) = 2\tau^{[a+}R^{b]}{}_i \wedge \mathrm{D}\tau^i, \tag{4.2b}$$

respectively the Khronon and the connection equations of motion. In particular, the Khronon produces the prototype Einstein field equations.

The Lorentz connection ω_{ab} retains its standard meaning as a principal connection on the frame bundle, but we introduce τ^a as a "Lorentz-valued scalar". This is one of several interpretations — we might as well have considered τ^a as a Lorentz vector (or more precisely, components of a vector, represented in an orthonormal basis); it is not unfair to instead desire to introduce a proper manifold vector field $\tau = \tau^a \ni_a$, or a 1-form $\perp = \tau_a e^b$, but the specifics of such an interpretation were not considered to greater extent. In a basic rewriting it should be trivially equivalent, of course, but as the intermixing of spacetime geometry and internal space is quite fundamental to this theory, the implication of introducing a proper tangent-space object, and then forcing upon it a specific value, is slightly more subtle than might appear at first glance — the Khronon vector field is assumed to obtain a timelike[1] expectation value $\tau^a\tau_a < 0$ as a gauge choice, thus spontaneously breaking the Lorentz symmetry of the theory, and, through a particular coframe choice $e^a \equiv \mathrm{D}\tau^a$, introduce the pseudo-Riemannian geometry. Note that the symmetry breaking is crucial to obtain a metric phase of the theory, and indispensable for seeing the Khronon theory as a proper gauge theory — the dynamical degrees of freedom are entirely in the connection field[2].

To request that the action (4.1) has an interpretation as a theory of gravity, in a more restricted fashion, it should have a geometric interpretation analogous to that of General Relativity, i.e. a relation to the pseudo-Riemannian metric structure. There are a handful of options available how to make this exact, with a varying degree of sophistication. Mostly this is contingent on interpreting the geometry (in dynamic variables of the connection ω_{ab}, coframe e^a and Khronon scalar-Lorentz vector τ^a; more abstractly the tangent geometry), its generation (introduction of the physical coframe, and the spontaneous symmetry breaking), and the specific form of the Lagrangian (gauge group, boundary-equivalence, comparison with standard Einstein-Hilbert(-Cartan) theory). Most suggestively, the action (4.1) defines an Einstein-Cartan-like gravity, with the identification

$$e^a \equiv \mathrm{D}\tau^a, \tag{4.3}$$

i.e. the coframe is forced into the form of an exterior covariant derivative

[1]In $(-,+,+,+)$ signature. Convention choices like this are not entirely innocuous, e.g. in de Sitter vs. anti de Sitter Cartan gravity models later on.

[2]The relevance of first class constraints in General Relativity cannot be ignored when discussing gravity as a gauge theory, but clearly the basic variables are different here, resp. a symmetry breaking field and purely a connection. The importance of this difference should be seen as a core point of study for the Khronon theory.

of the Khronon field — generally, and due to the importance of this choice to the theory, this has been called the Cartan frame in 4 dimensions, and Bartels frame in 3 dimensions [137, 138]. The gauge choice forced upon τ^a partly removes it from the dynamical premise. In principle it could be any symmetry breaking to timelike $\tau^2 < 0$, but in practice, the most common choice is the "time gauge" $\tau^a = \tau\delta^a_0$ (or the synchronous symmetry breaking [137]; cf. also the time gauge $\underline{\mathrm{D}}\tau^0 = 0$ in [20]), where τ is here just a real-valued function; other gauges could be Lorentz-rotated or normalized accordingly. Understanding that this system of differential equations truly provides General Relativity accompanied by ideal dust dark matter as the integration constant M_a was one of the topics of paper [1] — it is a matter of algebra, and careful intermixing of the equations. In simple terms, equation (4.2a) can be integrated to

$$\mathrm{D}\tau^b \wedge {}^+R_{ab} = M_a,\ \mathrm{D}M_a = 0, \tag{4.4}$$

and through clever interplay with the connection equation (4.2b), it can be shown that M_a is the energy-momentum of ideal dust.

However, this phenomenology was already presented in the *initial* paper [20], albeit in the Hamiltonian formulation

$$H_\tau = {}^+\omega_\perp^{I0}\epsilon_{IJK}{}^+\underline{\mathrm{D}}(e^J \wedge e^K) + N(P\sqrt{1+\partial^I\tau\partial_I\tau} + \epsilon_{IJK}e^I \wedge {}^+\underline{R}^{JK}) \\ + N^I(P\partial_I\tau + 2\epsilon_{IJK}e^J \wedge {}^+\underline{R}^{0K}), \tag{4.5}$$

and found to reduce to the self-dual formulation of gravity in Ashtekar's variables [246], in the presence of ideal dust [248, 249] (which is also seen as "mimetic dark matter" [68–70]). Deriving the Hamiltonian is a matter of Legendre transformation and fixing definitions, e.g. $e^I \equiv \underline{\mathrm{D}}\tau^I$ — in turn, the action is Legendre-transformed back to

$$S_\tau = \int \mathrm{d}t \wedge (\epsilon_{IJK}e^I \wedge e^J \wedge \partial_t{}^+\underline{\omega}^{0K} + P\partial_t\tau - H_\tau). \tag{4.6}$$

A count of the variables and constraints leads to 3 complex propagating degrees of freedom, reducible to 3 real degrees via reality conditions — gravity, and dust; note rotationless dust defines a preferred frame [248], so $\tau = \sqrt{-\tau^2}$ can also be seen as a measure of (cosmological) time. As paper [20] summarizes, the phenomenology of (4.1) contains solutions of

$$S_{\tau\ \mathrm{Aether}}[g,\tau,\rho] = \frac{1}{2}\int \sqrt{-g}\mathrm{d}^4x\Big(R - \rho(\partial^\mu\tau\partial_\mu\tau + 1)\Big), \tag{4.7}$$

where ρ is a Lagrange multiplier, and the action is in the broken phase. Action (4.7) is an Einstein-Aether theory, for phenomenology see [250–253].

The author has already alluded that the proposition of a Lagrangian is the end of a set of physical ideas, as now every implication (consistency and phenomenology) can be derived, tested, verified and studied — and the initial paper of Khronon gravity already classified the classical system. Trying to work nontrivially around the initial classification is a significant issue in theory development. There have been extended studies into this formulation since then, rederiving the phenomenology in Lagrangian formulation [1], discussions of matter couplings [2–4], and studies of solutions and cosmology [137, 138], but any derivation from the *initial* formulation will struggle to extend beyond what was already classified. Verifying this assertion to be true is a subtle point and well-merited, but it is difficult to be sure in any certainty if or where to expect a nontrivial difference. As such, an extension must necessarily change part of the premise. This was the case in the adaptation to $Spin(4)$ gauge theory [138], and must be the case in any unified theory to be proposed. Note that although [138] considered $Spin(4)$ theory, the analysis of cosmology (e.g. Friedmann equations, perturbations, FLRW solutions), solutions (black holes, gravitational waves) and matter couplings would be expected to persist by analogy to the $SO_{\mathbb{C}}(1,3)$ theory as well.

The author would rather emphasize some questions that, as of writing this dissertation, are not entirely yet clarified.

1. Although arguably the classical dynamics is sufficiently understood, the spontaneous symmetry breaking of τ^a to whatever vector v^a is *not* actually entirely clear, and is rather mostly done by hand. Introducing this as a part of the theory is most simply done by fixing a Lagrange multiplier[3]

$$S_{\tau\to\delta} = \int \lambda_a(\tau^a - t\delta^a_0), \tag{4.8}$$

   akin to the naivest by-hand gauge fixing in electromagnetism (and thus here explicitly breaking Lorentz invariance). A quadratic constraint like $\eta_{ab}(\tau^a - t\delta^a_0)(\tau^b - t\delta^b_0) = 0$ or $\eta_{ab}\tau^a\tau^b - t^2 = 0$ can be explored as well, the latter in particular being more analogous to the Stelle-West Lagrange multiplier (cf. $\lambda(\tau^2 - C)$). Any of these options would introduce a manifest preferred direction, although they perhaps would not be as up-to-taste a spontaneous process as those moderated by a Higgs-like potential (cf. simply integrating over the Lagrange multiplier in the partition function).

[3] Note that here we use t instead of τ. This is an open question — it might be plausible to instead consider the actual norm $t \to \sqrt{\eta_{ab}\tau^a\tau^b}$, or t might be seen as an independent field (or perhaps literally a constant over all of spacetime, likely losing interpretation as time), or some other combination of gauge fixing.

Nevertheless, this development remains a somewhat open problem — including in phenomenology. The Lorentz-*breaking* that the time gauge introduces, viz. a preferred frame, is not studied in greater detail. It is known that Lorentz-violation is constrained to minuscule width, or conversely, Lorentz-covariance is verified to extreme precision [252–256]. In this sense, the Khronon theory deserves clarification on the range, scale and effect of the spontaneous symmetry breaking it involves.

Reality conditions are another component that can be made more explicit, but they are not as pressing compared to the symmetry breaking. In principle, appending reality conditions is understood [233], and the initial paper [20] provides several options how to proceed to a real $SO(3,1)$ theory, and to some part the complexified Khronon has been supplanted by the $Spin(4)$ theory [138] which is real as-is. But for the sake of completeness, reality conditions should, of course, be considered a part of the complexified theory.

2. While classical phenomenology can be considered to be well understood, the *quantum* phenomenology is not explored in significant depth. Namely, standard approaches are yet to be studied, including canonical quantization, path integrals, and e.g. comparison with Loop Quantum Gravity, bar more adventurous directions like Regge calculus or String Theory infrared limits. Renormalization is a particularly poignant question to study, either in Euclidean $SO(4)$ or in $Spin(4)$ theory. The author has argued that the integration constant seems to behave like a classical background in the *Iso*khronon theory [4], but Khronon gravity has proper dust degrees of freedom, the quantization of which requires some consideration. Similarly, it was argued [3] that such a vector-model might help in Loop Quantum Gravity, but an actual verification of this is yet open. Consider [222] as example that the inclusion of dust might improve quantum behaviour — reworking this with the Khronon would be very curious.

3. The Khronon is inherently Cartan (waywiser!) geometric [42–45], but the generation of manifold differential geometry is, in the author's view, not entirely clarified — if the Khronon was understood as *pre*geometric. It is possible to take a crude and in principle true view that the Khronon theory posits either the coframe or the hypersurface basis on a preexisting 4-dimensional manifold to obtain a

specific form,

$$e^a = \mathrm{D}\tau^a,$$
$$*(e^a \wedge e^b) = \star(\mathrm{D}\tau^a \wedge \mathrm{D}\tau^b),$$

but there seems to be a some subtlety left to clarify, how exactly the Lorentzian metric is imposed onto the base manifold's tangent structure. In the sense of field theory, this is an entirely secondary point, as the question comes down to the field content and the dynamics of it, which the action completely explains, but geometrically, it is not entirely clear how the Lorentz group as an internal symmetry group imposes a reduction of the frame bundle's structure group, without forcing this by hand in the explicit symmetry breaking $\tau^a \to \tau\delta^a_0$ and geometry identification $e^a \equiv \mathrm{D}\tau^a$.

The author's motive was that this question should be in the basis of gauge-gravity unification — that is, gravity is necessarily tangent geometry of the base manifold, and *not* just an appended structure on top. Unification, in turn, would require internal gauge theory bundles to be separated from the frame bundle geometry[4]. The Cartan Khronon has a clear "pregeometric" phase, where the symmetry breaking and the identifications are not yet done, so the issue of the relation to the underlying manifold appears immediately.

Note that this might be easy to bypass and ignore when accepting that the structures (tangent and frame bundle, possible metrics, all possible field theories) are always present and only gain importance based on identification and interrelation (i.e., what fields are utilized in the couplings, fermionic and otherwise — the actual Lagrangian content), or if we are just studying the geometry that is generated by the Khronon coframe.

4. The motivation for considering first order actions appeared from considering matter couplings in the Khronon theory, i.e. the Dirac spinor and Yang-Mills actions. If the Khronon vector were allowed to vanish, in a unified or "pregeometric" phase (i.e. before the symmetry breaking and the generation of the pseudo-Riemannian metric) the Hodge star component expression would become singular. There are

[4]Taken to an even greater extreme, the dimensionality of the base manifold also becomes an open question. It is quite tempting to consider a gauge-gravity unified phase such that unified group is the structure group of some different manifold, but clearly the question is entirely open — from the definition of the geometry itself to the dynamics that would mediate its behaviour.

several ways to resolve this, but the author does not consider them fully satisfactory yet.

The articles [2–4] considered the first order approach, where purely formally the inverse metric was removed. However, this is not necessarily a physically relevant issue to the metric phase, as clearly Nature persists in a metric phase where signals of spacetime topological phases are close to nonexistent[5] — so, alternatively, the standard matter couplings and actions, Hodge duality included, could be maintained as-is. Clearly, in the symmetry-broken metric phase the metric exists, and standard Maxwell-Yang-Mills and Dirac actions are targeted for the sake of consistency. Rather, the issue is in resolving the phase transition — [5] considered constraints and potentials for BF theory, but the exact dynamical (i.e. put into Lagrangian framework[6]) mechanism remains an open question.

That is, although the matter couplings in the metric phase are known, the transition to and from the unified phase is *not* known. Similarly, a proper gauge-gravity unified Khronon model has not been actually provided — a crude approach, as discussed in the author's [5], would be to append Khronon simplicity constraints $B \to e^a \wedge e^b \sim \mathrm{D}\tau^a \wedge \mathrm{D}\tau^b$ to any other unified model, such as Smolin's Plebanski model [29], but this is not entirely aligned with the Khronon geometric ideas, which would rather prefer Westman & Zlosnik's Cartan geometric proposals [42–45]. In essence, although the simplest method for deriving various Khronon-unified models is of course visible through the simplicity constraints, the geometry of the unified phase, the phase transition, and the matter couplings there are still an open question.

5. The phenomenological and theoretical viability and the predictions of the Khronon, as is the case with any theory, must continuously be verified against any set of new data. To reiterate, the Cartan Khronon allows (predicts, interpretation-contingent) ideal dust degrees of freedom as a dark matter candidate *in the vacuum*[7], purely

[5]There are *some* areas where there is room for speculation for the existence of a topological phase, say singularities like the Big Bang or black holes, some forms of dark matter, spacetime microstructure or a quantum gravity zero metric ground state, but without actual mathematical models behind them this remains speculation.

[6]Whether this means additional Lagrangian terms or some other dynamical expression, e.g. a "dynamical" topology of spacetime (essentially, topology becomes contingent on field content, in some unspecified manner), remains to be seen.

[7]The issue is more complicated when matter is included, i.e. hypermomentum and spin currents. Generally, there might be nontrivial couplings between the "dust"(-like) integration constant and matter or the dust might not be ideal (even in a singly chiral model), and this analysis is pending investigation.

as a geometric effect (through an integration constant). The dust, in turn, defines a preferred frame, and the gauge choice introduces spontaneous Lorentz symmetry breaking.

That is, there is no requirement for dark matter microstructure — an observation of dark matter particles would certainly cast a shadow of doubt on the validity of this theory. At the same time, strictly ruling it out entirely would require a verification of exactly nil (ideal) dust; the requirement for effective or informal disproof is of course lesser, as the commonly-referred Occam's razor dictates. Currently, no dark matter particles have been observed; however, there has been evidence of dark matter self-interactions, per various dragging [257–260]. Any mixture of different dark matter contributions is yet indeterminate.

But if not observed, the question would remain why this seemingly consistent construction is even theoretically permitted. To be accurate, physics does not have to eliminate theories that are not observed, but it would still be desirable to introduce additional principles that reject more of the unobserved possibilities — even if this is more a philosophical than a physical point.

4.2 Further formulations of gravity

It is purposeful to briefly consider a few other Lagrangian formulations of gravity as well, as they are relevant in both the ideas they introduce as well practically useful for the task of unification. Consider [21] for a monograph treatment; here, let us only mention the theories of MacDowell-Mansouri and the Plebanski formulations of gravity. The former introduces the idea of breaking into General Relativity from a larger structure group (de Sitter), while the latter introduces General Relativity by constraining a topological field theory (BF theory).

For the case of MacDowell-Mansouri and Stelle-West, consider also [126]. The direction is to consider a larger symmetry group while remaining on a four-dimensional manifold background, specifically (anti-)de Sitter $SO(4,1)$ or $SO(3,2)$ instead of the Lorentz group $SO(3,1)$; this will determine the sign of the cosmological constant (correspondingly $\Lambda > 0$ and $\Lambda < 0$). The action of MacDowell-Mansouri [109] is of Yang-Mills form, with only the connection as a dynamic variable:

$$S_{\mathrm{MM}}[\omega] = \int R^{AB} \wedge R^{CD} Q_{ABCD}. \tag{4.9}$$

Here we use MacDowell and Mansouri's original notation, introducing *constants* Q_{ABCD}: the crucial step is the projection to $\mathfrak{so}(3,1)$, which breaks the symmetry to $SO(3,1)$ and is done by hand. Consider [126, 261] for further details. There are several explicit expressions for MacDowell-Mansouri gravity in circulation, including utilizing the Stelle-West vector τ^A to explicitly write

$$S_{\text{MM}}[\omega] \sim \frac{1}{2\kappa\Lambda} \int R^{AB} \wedge *R^{CD} \tau^E \epsilon_{ABCDE}, \tag{4.10}$$

or e.g. by manipulating the definition of the trace — in principle, they are equivalent (implementation subtlety non-withstanding), but emphasize different parts of the algebra and geometry. Regardless, the further development of Stelle and West [110] was to break the symmetry spontaneously by means of a Lagrange multiplier term (potential), loosely put

$$S_{\text{SW}} = S_{\text{MM}} + \int \lambda(\tau^A \tau_A - l^2) \tag{4.11}$$

Straightforward analysis shows that this reduces to General Relativity in the broken phase, and further development is yet possible: consider e.g. [262] for spinor condensate and [263, 264] for BF formulations. The *success* of the extent to which the Macdowell-Mansouri and Stelle-West formulations actually mimic Yang-Mills theory is a separate matter: at least in form, the Lagrangians certainly are much more analogous to Yang-Mills theory.

We have already introduced BF theory as a particularly generic template for gauge theory dynamics and as the mathematical premise for introducing premetric theory. An exact formulation of gravity was provided by Plebanski [207], by virtue of introducing a more nontrivial Lagrange multiplier constraint structure rather than the simplistic $B_{ab} = R_{ab}$; for some modern treatments, see e.g. [265, 266]. Consider the $SO(4)$ Plebanski action[8]

$$S_{\text{Plebanski}}[B, \omega, \lambda] = \frac{1}{2\kappa} \int B^{ab} \wedge R_{ab} - \frac{1}{2} \lambda_{abcd} B^{ab} \wedge B^{cd}, \tag{4.12}$$

for a B-field, $SO(4)$-connection ω_{ab} and Lagrange multiplier λ_{abcd}. Note the symmetries of the Lagrange multiplier,

$$\lambda_{abcd} = \lambda_{cdab} = -\lambda_{bacd}, \tag{4.13}$$

[8] This version is as presented in [29, 267]. There are multiple versions of the "Plebanski action" in circulation, differing mildly on the exact group, position and expression of invariants, presence of the cosmological constant etc. — cf. also [78, 207, 265, 266] etc. The crucial notion is that the basic variables are an auxiliary B-field, the connection, and a Lagrange multiplier, which forces the B-field to obtain a coframe expression, see [266].

and

$$\epsilon^{abcd}\phi_{abcd} = 0. \tag{4.14}$$

The Lagrange multiplier equations

$$B^{ab} \wedge B^{cd} = \frac{1}{4!}\epsilon_{ijkl}B^{ij} \wedge B^{kl}\epsilon^{abcd} \tag{4.15}$$

can be shown [78] to constrain the B-field to a branch of the hypersurface basis $+e^a \wedge e^b$, $-e^a \wedge e^b$, $+\frac{1}{2}\epsilon^{ab}{}_{cd}e^c \wedge e^d$ or $-\frac{1}{2}\epsilon^{ab}{}_{cd}e^c \wedge e^d$. The equivalence to General Relativity becomes rather apparent; the cosmological constant can be appended directly. The constraints applied to the B-field are the simplicity constraints, and crucial in restricting to General Relativity [78, 268, 269]. Smolin further proposed a unified model [29],

$$S_{\text{Pl.G}} = \frac{1}{G}\int B^A \wedge R_A - \frac{1}{2}\phi_{AB}B^A \wedge B^B + \frac{g}{2}\phi^{AB}\phi_{AB}B^C \wedge B_C. \tag{4.16}$$

The action was generalized to a larger group $G \supset SO(4)$, and it can be shown that Yang-Mills theory and gravity are obtained in a perturbative expansion. To add to the discussion of dualities from earlier, this weaker or degenerate constraint structure is one of the few options to obtain any extended nontrivial phenomenology with auxiliary fields.

We are presented with a varied array of different possible variational principles to obtain the same gravity dynamics — Plebanski constraints allow for other branches as well, but it retains General Relativity as a particular case. What we are *not* presented with is any relation to internal gauge theory. Part of the author's aspirations for a unified theory of the Khronon and other gauge interactions was that there would be a preferred internal gauge theory Lagrangian that would suggest the exact unification scheme. In large part this is not feasible due to the simple independence of different interactions, at least in the broken phase. Indeed, there is no *canonical* correspondence between the internal gauge and gravity excitations, which can be formalized in that the commutative diagram

$$\begin{array}{ccc} & \underset{B}{\Omega^{n-2}(\mathfrak{g})} & \\ \swarrow^{C_{\text{int}}} & & \searrow^{C_{\text{ext}}} \\ \underset{B_{\text{int}}=*F}{\Omega^{n-2}(\mathfrak{g}_{\text{int}})} & \underset{\phi_{\text{e-i}}}{\overset{\phi_{\text{i-e}}}{\rightleftarrows}} & \underset{B_{\text{ext}}=*(e^a\wedge e^b)}{\Omega^{n-2}(\mathfrak{g}_{\text{ext}})} \end{array} \tag{4.17}$$

is generally ill-defined [5]. In turn, this presents some structure that does not appear to have been introduced before: the constitutive mappings C_{int}, C_{ext}, $\phi_{\text{i-e}}$ and $\phi_{\text{e-i}}$ can, in principle, be considered as new concepts, fixing the particular interrelation between the excitations of various different interactions; their beneficence outside of explaining why the author's particular approach to Lagrangian development was unsuccessful remains to be established, of course. In turn, the hope for a canonically preferred internal gauge Lagrangian corresponding to gravity is not entirely unwarranted, provided the unification scheme is fixed — this would be fixing the constitutive mappings. This can be summarized in that unification is hopelessly *ad hoc*, but it should rather be taken as a guideline that new principles are required, and the space for Lagrangian engineering is still open[9].

4.3 Unification in particle physics and beyond

"Unification" has a rather strict and limited meaning in particle physics, even though the dissertation itself pursued more lax notions; as will soon be seen, for gauge-gravity unification this is not entirely a downside. In gauge theory terms, we are provided two Lie groups G_A and G_B describing two types of gauge interactions, that are to be described by a single unified field. Percacci & Krasnov [15, 270] describe particle physics unification as a four-step process.

1. The gauge groups G_A and G_B must be found as *commuting* subgroups of a unifying gauge group G. Often, G is also assumed to be e.g. simple and compact with a single coupling constant, or at most a direct product of such groups, with an additional discrete symmetry [16].

   Note that $G_A \times G_B$ corresponds to the broken phase, as the different gauge groups are trivially added together. Much of the focus in this dissertation was *only* on the broken phase, and what could then be derived from it to the unified phase — perhaps naive from the start, but the author would argue that there is a qualitative difference in the gauge theories of gravity and Maxwell-Yang-Mills [5]. The geometric setting is different (frame bundle vs. separate principal bundle), the symmetries are different (manifold diffeomorphisms vs. bundle automorphisms), and the fields are different (metric objects possibly in addition to the connection — or, e.g. the Khronon scalar). It is worthwhile to understand how these qualitative differences come to be[10].

[9] That is, optimism and pessimism.

[10] The issue here is that this will be heavily model-dependent and much of the meaning is only assigned in relation to other structures. Say, the connection may be on the frame bundle, but its dynamical relevance is because it is coupled to every field and its properties are defined by the structure group.

2. All known particles must then be first put into irreducible representations of the unifying gauge group G, which is then decomposed into a direct sum in terms of $G_A \times G_B$ — generally, a restricted representation ρ_H of an irreducible representation ρ to an embedded Lie subgroup $H \subset G$ might instead be reducible. The decomposition must result in correct quantum numbers; generally, this also results in new (yet unobserved!) particles. This decomposition is known as the branching rule.

3. An order parameter (field) must then be identified — in essence, a Higgs-like field. This can be a scalar with a linear representation of G and an orbit diffeomorphic to coset $G/(G_A \times G_B)$, but not necessarily. The soldering form (or, tetrad) might also serve as an order parameter.

4. A G-invariant action functional S is provided. Paraphrasing [15], the requirement (a) is fundamental: there must be a dynamical mechanism for the appearance of phenomena of type A and B. In less nebulous terms, this is often the introduction of a G-invariant potential for the order parameter, such that the potential's minimum is an orbit with stabilizer $G_A \times G_B$ — the shape of the potential contingent on various parameters then selects between G (the unified phase) and $G/(G_A \times G_B)$ (the broken phase).

   Requirement (b) is the physically always implicit necessity that the model be conformant to reality — in particular, any new (and often inevitable) particles be unobservable in collider experiments so far. A large mass is rather standard for this.

Rather quickly the critical reader notes that neither the Standard Model nor even the Electroweak theory actually follow this pattern exactly. The symmetry group of the Standard Model is the trivial direct product

$$G_{\mathrm{SM}} = U(1)_Y \times SU(2)_L \times SU(3)_C, \tag{4.18}$$

which includes the Electroweak sector

$$G_{\mathrm{EW}} = U(1)_Y \times SU(2)_L. \tag{4.19}$$

The Standard Model appends the chromodynamics group $SU(3)_C$ trivially. Electroweak theory at least satisfies this list of requirements with the Higgs field as the order parameter, except that $G_{\mathrm{EW}} = U(1)_Y \times SU(2)_L$ is not semi-simple. Nevertheless, this is a clear and unambiguous list of requirements for any theory that claims to unify some fundamental forces.

However, physics has been rife with various “unifications” that do not really follow this particle physics logic. Perhaps too simplistic nowadays, but the unification of space and time into a spacetime pseudo-Riemannian 4-manifold does not provide any natural selector for separating space and time again[11]; in fact, $3+1$ decomposition has been criticized on the basis that timelike congruences (viz. observers) might simply lack an associated spacelike foliation [275]. In a similar vein, unification of electricity and magnetism into the electromagnetic force is also inherently a relativistic consequence — there is no clear selector for an “electric” and “magnetic” interaction, nor would it be Lorentz-invariant. The Aharonov-Bohm effect suggests that the 4-potential should be considered the fundamental field of electromagnetism.

But for the purpose of particle physics, despite involving two coupling constants, Electroweak $U(1)_Y \times SU(2)_L$ theory is the closest “standard”[12] particle theory there is to a unification example, as it demonstrates the expected procedure — note the notion of symmetry breaking is here fundamental to unification; we will follow Hamilton [16], but any quantum field theory exposition will discuss the premise just as well. The principle is straightforward and can be understood classically: the Lagrangian possesses a greater set of symmetries, that solutions to the equations of motion (and in particular, quantum states) might not, i.e. the symmetry is broken to a smaller Lie group, which is the stabilizer of a particular vacuum vector.

That is, we are provided a gauge theory with connection A on a principal G-bundle P, a Higgs field $\Phi \in \Gamma(P \times_\rho W)$, and a potential V. In the *vacuum*, we have a flat connection A_0, $F(A_0) = 0$ and a covariantly constant Higgs field Φ_0, $\mathrm{D}\Phi = 0$, at the minimum of V. The space of minima $w_0 \in W$ of V constitute the vacuum manifold — in particular, the action of the symmetry on w_0 naturally induces a smaller symmetry group, the stabilizer $H \subset G$, which is the *unbroken* subgroup. If H is a proper subgroup, the symmetry is spontaneously broken — the generators of the Lie algebra $\mathfrak{g}$ can be divided into unbroken generators $\mathfrak{h}$, and the remainder $\mathfrak{h}^\perp$. Expanding the Higgs-Yang-Mills Lagrangian in terms of perturbations around a minimum, $\Phi = w_0 + \varphi$, introduces the Nambu-Goldstone bosons and Higgs bosons — the former can be removed in unitary gauge, the latter cannot, and are new physical scalar fields. The Higgs kinetic term

[11] Note a rather beautiful geometric theory of Euclidean geometry and Galilean relativity [271–274]. The author has speculated whether relativistic unification, so to say, could be reconciled with particle physics through this formalism, but nothing concrete is yet to be stated.

[12] That is, with widely accepted theoretical, observational and experimental consistency. Full (Grand or otherwise) unifications so far lack unambiguous supporting evidence, despite theoretical merit.

$(\mathrm{D}\Phi)^\dagger \wedge *\mathrm{D}\Phi$ after this expansion ultimately induces mass terms to the broken gauge bosons. The author's attempt [5] to generalize this notion to constitutive laws was based on a particular characteristic of the Higgs mechanism — the degenerate vacuum structure, so a solution of any one vacuum vector is chosen "spontaneously". This mimics the degenerate solution structure of e.g. Plebanski theory, but in the case of pure potentials, inevitably introduces classical backreaction. A more proper nomenclature would be just in degenerate equations of motion, but the purpose was quite straightforward: by virtue of how auxiliary fields are just integrated out of the theory, one of the few ways they might introduce nontrivial phenomena is through this degeneracy (which would be even more curious in quantum theory — hopeless renormalization issues notwithstanding).

In the case of Electroweak theory, $W = \mathbb{C}^2$ and $G = U(1)_Y \times SU(2)_L$. The potential is the standard quadratic-quartic (sombrero) potential

$$V(w) = -\mu\,||w||^2 + \lambda\,||w||^4\,, \tag{4.20}$$

which has a $U(1)$ ring of solutions in the minimum. This generates masses for the W and Z bosons, while the photon γ of the unbroken $U(1)_Q$ remains massless. In principle, this is also the procedure for Grand Unification, albeit with a different group structure. It is a small exercise to list all possible Grand Unification groups up to some rank [16] (proposition 9.5.2), i.e. the maximal dimension of an embedded torus subgroup. The only permitted GUT groups up to rank 6 are the following:

Rank 4: $SU(3)^2$, $SU(5)$,

Rank 5: $SU(6)$ and $Spin(10)$ (or, $SO(10)$ theory),

Rank 6: $SU(3)^3$, $SU(4)^2$, $SU(7)$ and E_6.

All have been studied to various extent, but the author has no strong preference for any one unification group in particular — lacking any observational implications, only theoretical arguments of varying strength remain. Until more direct evidence is actually obtained, the ground for any particular unification group is simply too unsolid to be strongly beholden to[13].

What we also find, rather, is that the Khronon is largely agnostic to any particularities on what happens in the internal gauge sector, and so is the

[13] A methodological remark cannot be overlooked: if an observation was successful, one (possibly for a period of time, some) or none of the proposals might be validated. In the latter case, physics theory is quite flexible to introduce grand new principles and ideas which will explain the observation. In the former case, the popular hindsight will be adamant in correctness. In the run-up, however, objective personal conviction is a very *subtle* question.

Isokhronon, or any derivative of them — perhaps a trivial implication, as gravity is an independent interaction, and symmetry structure is separate to Lagrangian theory. But worse than that, this remains the case for gauge-gravity unifications. This is an issue that plagued the dissertation very early on — if the Khronon can be simply adapted to any particular unification scheme, it is not clear where the nontrivial interest might be. This led to the author's unsuccessful gamble on new principles that might produce a nontrivial unification (viz. toying with the Isokhronon as a first step). The contrary would have been to work out how to *exactly* fit the Khronon into some previous scheme, which would have been steady progress, with less guarantee of breakthrough, but also less probability of failure[14].

There are graviGUT schemes which include gravity into a GUT unification, but consider e.g. the graviweak theory of Nesti & Percacci [276], which was a topic of interest at points due to $U(1)$ not collecting in a straightforward algebraic manner [2], thus prompting toward unification of (Khronon) gravity with non-Abelian theories only; the author also suggested that the electromagnetic interaction could be assembled from the left-over anti-self-dual connection in self-dual gravity, $A = z^{ab-}\omega_{ab}$ and $F[A] = \mathrm{d}(z^{ab-}\omega_{ab})$, but this was not developed to greater extent.

For graviweak theory, the order parameter is a generalized [277] soldering form $e^{\bar{a}a}$, with A_{ab} as the graviweak gauge field (conjugate $\bar{A}_{\bar{a}\bar{b}}$) of the graviweak symmetry group $SO_{\mathbb{C}}(3,1) \cong SO_{\mathbb{C}}(4) \cong SL_{\mathbb{C}}(2)_+ \times SL_{\mathbb{C}}(2)_-$. Here $a, b, \ldots$ are in the vector representation of the *real* $SO_{\mathbb{R}}(3,1)$ (cf. $\bar{a}, \bar{b}, \ldots$ for the imaginary part; together the graviweak group); meanwhile $m, n, \ldots$ and $u, v, \ldots$ can be used for $SL_{\mathbb{C}}(2)_+$ and $SL_{\mathbb{C}}(2)_-$ respectively (Lorentz and isoLorentz). Introducing the generalized torsion 2-form

$$T^{\bar{a}a} = \mathrm{d}e^{\bar{a}a} + \bar{A}^{\bar{a}}{}_{\bar{b}} \wedge e^{\bar{b}a} + A^{a}{}_{b} \wedge e^{\bar{a}b} \tag{4.21}$$

and curvature

$$R^{\bar{a}a\bar{b}b} = R^{ab}\delta^{\bar{a}\bar{b}} + \bar{R}^{\bar{a}\bar{b}}\delta^{ab} \tag{4.22}$$

gives the actions

$$S_{R1} = \frac{1}{2\kappa_1}\int R^{\bar{a}a\bar{b}b} \wedge e^{\bar{c}c} \wedge e^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}, \tag{4.23a}$$

$$S_T = a_1 \int \left(t^{\bar{a}a\bar{b}b}_{\bar{e}e} T^{\bar{e}e} + (t^2) e^{\bar{a}a} \wedge e^{\bar{b}b} \right) \wedge e^{\bar{c}c} \wedge e^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}, \tag{4.23b}$$

$$S_{R2} = \frac{1}{g_2^2}\int \left(r^{\bar{a}a\bar{b}b}_{\bar{e}e\bar{f}f} R^{\bar{e}e\bar{f}f} + (r^2) e^{\bar{a}a} \wedge e^{\bar{b}b} \right) \wedge e^{\bar{c}c} \wedge e^{\bar{d}d}\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}, \tag{4.23c}$$

[14] Ultimately, it will be up to the reader to decide which path would have been better. As additional challenge, the amount of hindsight should be minimized.

where $t^{\bar{a}a\bar{b}b}_{\bar{e}e}$ and $r^{\bar{a}a\bar{b}b}_{\bar{e}e\bar{f}f}$ are auxiliary fields, and $\epsilon_{(\bar{a}a)(\bar{b}b)(\bar{c}c)(\bar{d}d)}$ is an $SO_{\mathbb{C}}(4)$ invariant. Study follows: there are the topological $\langle e \rangle = 0$ and Minkowski space $\langle e^{m4}{}_{\mu} \rangle = M\delta^m_\mu$ vacuum expectation values, and a (massive) isospin triplet appears in addition to the massless graviton. The broken phase has global Lorentz and local weak isospin invariance.

For the the case of the Khronon, there are two issues.

1. How to introduce $e^{\bar{a}a} \sim \mathrm{D}\tau^a$, if it is possible?

2. How to introduce the self-dual connection ${}^{+}\omega_{ab}$ into the Einstein-Cartan action, rather than the full $SO_{\mathbb{C}}(3,1) \cong SL_{\mathbb{C}}(2)_+ \times SL_{\mathbb{C}}(2)_-$?

The latter can be forced, to see if *just* the self-dual connection would lead to a consistent theory; if not, then perhaps this condition can be dropped, at the expense of likely losing dark matter phenomenology. The former can be forced as well: the Lorentz-scalar τ^a can be extended to $\tau^{\bar{a}a}$ for $e^{\bar{a}a} \to \mathrm{D}\tau^{\bar{a}a}$ — in principle, this automatically leads to a *new* theory, the consistency, and interest, of which is to be studied in a new publication. What we gather from this short discourse, however, is that the actuality of unified Khronon theories are entirely dependent on simplicity or coframe constraint consistency and compatibility with a $Spin(4)$ or $SO^+_{\mathbb{C}}(1,3)$ curvature.

Let us reemphasize that the relation of gravity to the underlying manifold is much stronger than that of other (internal) gauge theories. Gravity works with the frame bundle $Fr(M)$ and through it with the tangent bundle TM, which is canonically induced by manifold structure on M; the coframe $e : TM \to \mathbb{R}$ introduces the notion of tangency to inherent geometry [40, 278], and is a crucial addition[15] to the Lorentz connection ω on TM. External symmetries are diffeomorphisms of the base manifold, rather than principal bundle automorphisms, cf. the Atiyah exact sequence earlier. Restricted to the standard geometric interpretation of General Relativity, it is not entirely surprising that a standard unification of gravity and other gauge theories does not succeed. This is most famously formalized in the theorem of Coleman & Mandula [33], which we will quote; there is also e.g. O'Raifeartaigh's theorem [279] preceding it, among others.

Theorem 5 (The Coleman-Mandula theorem)**.** *Let G be a connected symmetry group of the S-matrix. The following five conditions are assumed to hold.*

[15]However, not unavoidable — there exist reformulations of gravity with only the connection as the basic variable [41]. In this sense, philosophical deliberation has not gotten in the way of theory development, even though a General Relativity limit would still require an eventual interpretation in terms of e and ω on M.

1. *(Lorentz invariance) G contains a subgroup locally isomorphic to the Poincaré group.*

2. *(Particle-finiteness) All particle types correspond to positive-energy representations of the Poincaré group. For any finite mass M, there are only a finite number of particle types with mass less than M.*

3. *(Weak elastic analyticity) Elastic scattering amplitudes are analytic functions of* [Mandelstam variables] *center-of-mass energy s, and invariant momentum transfer t, in some neighborhood of the physical region, except at normal thresholds.*

4. *(Occurrence of scattering) Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|p, p'\rangle$ be the two-particle state made from these. Then* [the nontrivial part of the S-matrix ($S = \mathbb{1} - i(2\pi)^4\delta(P_\mu - {P_\mu}')T$)]
$$T|p, p'\rangle \neq 0,$$
except perhaps for certain isolated values of s. (At almost all energies, any two plane waves scatter.)

5. *(An ugly technical assumption* [sic!]*) The generators of G, considered as integral operators in momentum space, have distributions for their kernels. There is a neighborhood of the identity in G such that every element of G in this neighborhood lies on some one-parameter group $g(t)$. Further, if x and y are any two states in* [the set of single-particle states whose momentum-space wave functions are test functions] *$\mathcal{D}$, then*
$$\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}(x, g(t)y) = (x, Ay)$$
exists at $t = 0$, and defines a continuous function of x and y, linear in y and antilinear in x.

Then, G is locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

The most direct or naive gauge-gravity unification schemes are thus ruled out, but with sufficiently clever handling, the Coleman-Mandula theorem is hardly a significant limitation. Supersymmetry bypasses these assumptions (and instead runs into the Haag-Lopuszanski-Sohnius theorem [34]). More generically, the (vacuum expectation value of the) soldering form e simply vanishes in the unified phase, thus the Poincaré invariance assumption is violated, while in the broken phase the symmetry groups are in a trivial product, as required [15]. The author has argued that the Coleman-Mandula theorem simply expresses a geometry phase transition, and an analogy is heuristically apparent even classically, when trying to construct spacetime from possible tangent space notions [5]. Then, we would like to explicitly construct spacetime as a field of observers. Rather, the proposal was to introduce a signal triplet

$$o = (\tau, v, \rho) \tag{4.24}$$

of proper time τ, observer state v in a representation ρ of some total unified group G. If ρ does not decompose into a sum $\rho_{\text{ext}} \oplus \rho_{\text{int}}$, there is no clear possibility of discerning what should be the tangent space of the base manifold and what is internal space, at least based on the signals that could be transmitted — but this requires a reinterpretation of spacetime construction, of course; namely, *conflating* (!) the notion of observer state and bundle tangent spaces, which might hardly be desirable. Furthermore, for a proper theory, this observer-signal notion would require proper dynamics, which is difficult to do without any notions external to the observer. It is likely significantly simpler and more productive to take a global perspective from the beginning, considering some other geometry than the manifold or overlaying some other algebraic structure onto the base topology — but this is yet to be explored.

4.4 Division algebras

Abstract algebra is rife with various algebraic structures. We will only focus on *one* particular structure, that has piqued physical curiosity apart from just mathematical utility, and briefly discuss part of the motivation for this direction.

Definition 29. *An algebra A over a field $\mathbb{K}$ is a (finite-dimensional) $\mathbb{K}$-vector space with a bilinear map $\cdot$, $A \times A \to A$ and a unit element $1 \in A$, such that $\forall a \in A : 1 \cdot a = a = a \cdot 1$. For conciseness, also $a \cdot b = ab$.*

A is a division algebra if $ab = 0 \Rightarrow a = 0 \vee b = 0$.

An algebra is normed if the vector space's norm $|| \cdot ||$ satisfies $||ab|| = ||a|| \cdot ||b||$.

The interpretation is, as usual in mathematics, up to the reader — in a sense, this e.g. generalizes the notion of a "(complex) number", but this is only one option among many. Adding some more natural structure, the space of possible division algebras quickly restricts in an extraordinary manner. Note two exemplary theorems in this direction.

Theorem 6 (Hurwitz's theorem on normed division algebras)**.** *The only normed real division algebras are (isomorphic to) the real numbers* $\mathbb{R}$*, complex numbers* $\mathbb{C}$*, quaternions* $\mathbb{H}$*, and octonions* $\mathbb{O}$ *(correspondingly of dimensions 1,2,4 and 8).*

Theorem 7 (Frobenius's theorem on associative division algebras)**.** *The only associative real finite dimensional division algebras are (isomorphic to)* $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Consider [280] for more details — there is also e.g. Wedderburn's, Mazur's, Gelfand's etc. theorems.

From a theoretical physics point of view, there is plenty of structures in Nature that appear very limited[16]. Spacetime seems four-dimensional[17], and the particle content of leptons, hadrons, mesons etc. is very much finite, and of seemingly rather arbitrary composition, insofar that the Standard Model symmetry group and the three generations of quarks and leptons is not subject to some further principles. Under general field theory logic, it is also quite natural to consider fields subject to various algebraic requirements, as it is essentially a probe to the interactions and particle content or their internal structure — Lie group representations have been the most productive, as they have a clear interpretation and at the same time are varied and adaptable enough to describe most symmetry content[18], but e.g. the quantum description of fermions requires Grassmann algebras, i.e. anticommuting numbers. It is hardly a leap of faith to consider what other algebraic structure could be applied, particularly so as it provides a very clear analysis in terms of Lagrangian dynamics.

A closer look into quaternions raises more intrigue. Quaternions are a 4-dimensional real vector space, spanned by basis vectors $1, i, j, k$. The "1" spans the real part of the quaternion, while i, j, k are the (generalized, in

[16]A rather philosophical issue has to be raised: the division algebra sequence is only relevant if Nature somehow is immediately related to this limited variety. Beyond octonions, there can be e.g. sedenions $\mathbb{S}$ (losing the multiplicative norm and not being alternative), generally the Cayley-Dickson construction, or hypercomplex numbers etc. This is a difficult direction to take for building a theory, due to the great variety of properties to try match to Nature, and just as large a variety of algebraic structures to work with.

[17]At least on a macroscopic scale, and codified into the Standard Model and General Relativity. It is hardly a significant formal obstruction to add (compactified, immeasurable) extra dimensions, cf. Kaluza-Klein theory, String Theory etc. The relevance is still unclear, however. Similarly, the increase in complexity is not innocuous — these kinds of modification introduce a subtle variety of a new kind of theory-building freedom.

[18]A continuous infinitesimal transformation already suggests Lie algebras. Associativity, invertibility, and existence of a unit, thus defining a group, are very generic requirements, and manifold-like continuity is commonplace in physical thought.

a sense) imaginary units, which satisfy

$$i^2 = j^2 = k^2 = ijk = -1, \qquad \begin{array}{c|cccc} \downarrow \cdot \rightarrow & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} \tag{4.25}$$

The table is read $a \cdot b = c$, where a is from the first column, b from the first row; quaternions are not commutative, but still associative — octonions lose even that, but remain e.g. alternative ($x(xy) = (xx)y$ and $(yx)x = y(xx)$). Expanded in the standard basis, a quaternion over the real numbers is then an element of the algebra

$$a = a_0 1 + a_1 i + a_2 j + a_3 k \sim \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} a_0 + a_1 i' & a_2 + a_3 i' \\ -a_2 + a_3 i' & a_0 - a_1 i' \end{pmatrix}, \tag{4.26}$$

where a_i are real numbers, w and z are complex numbers with a separate imaginary unit i', unrelated to the quaternion algebra. Quaternions over *complex* numbers $\mathbb{H}(\mathbb{C}) = \mathbb{C} \otimes \mathbb{H}$ are built in a similar manner, except that the coefficients become complex-valued. Standard constructions follow, e.g. conjugates, norms, real (or scalar) part $\Re$ (i.e. component $a_0 1$) and imaginary (or vector) part $\Im$ (i.e. component $a_1 i + a_2 j + a_3 k$). In particular for $\mathbb{C} \otimes \mathbb{H}$, there are three natural conjugates to introduce: the complex conjugate * of the $\mathbb{C}$ component ($i' \to -i'$), the quaternion conjugate $\tilde{\cdot}$ of the $\mathbb{H}$ component ($i, j, k \to -i, -j, -k$), and their composition † (sometimes referred to as the Hermitian conjugate [22]). Quaternionic matrices require slightly more careful handling for their determinant and inverse [16].

Relativity can be built using complex quaternions [24]. Then, 4-vectors become simply

$$x = (i't, x^1, x^2, x^3) = i't1 + x^1 i + x^2 j + x^3 k, \tag{4.27}$$

with the Minkowski-metric magnitude

$$|x|^2 = xx^* = -t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{4.28}$$

Analysis follows. We will not reproduce the explicit constructions; Girard develops the cases of Special Relativity and classical electromagnetism and up to General Relativity in detail — there should not be any major surprises, as the rewriting (4.27) (attributed already to Minkowski!) allows rewriting any Lorentzian 4-vector rather in terms of a single quaternion (or, "minquat" — Minkowski quaternion), so the tangent space is well-established, and multivector calculus is defined in terms of $\mathbb{H} \otimes \mathbb{H}$. What instead to emphasize is that this is currently little more than a change in representations, as the unit quaternions $Sp(1)$ are isomorphic to $Spin(3)$ and $SU(2)$,

$$Sp(1) \cong SU(2) \cong Spin(3), \tag{4.29}$$

and are the universal cover of the 3-dimensional rotation group $SO(3)$. We have gained in a new representation, and a curious relation to 4 dimensions, separate in time 1 and space i, j, k, but not necessarily in fundamentally new information — the isomorphisms (4.29) explicitly force the quaternion perspective to contest with the Lie algebra one anyway.

Nevertheless, algebraic creativity has persisted, and uncovered yet more relations between physics and division algebras. Here, the reference is to one of the original motivating works for this dissertation — the expression of Standard Model symmetries entirely in division algebra structure, per Furey [22]. We will only relay the basic structure. Furey considers the "Dixon algebra" $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ [281–284]. Very loosely put, causality can be understood to appear as a sequence of the algebra's operations, so particles are to be represented by ideals (as they "persist under propagation").

Definition 30. *A left ideal of an algebra A is a linear subspace $U \subset A$ that is closed under multiplication on the left by any $a \in A$, i.e. $\forall a \in A, u \in U$: $a \cdot u \in U$.*

A right ideal is defined analogously; the notion of ideal carries over to other structures as well, e.g. rings and Lie algebras.

Considering the entire subspace as the particle itself certainly integrates out whatever internal freedom the quantity might have, but will have to contend with standard interpretations, e.g. the standard Fock space number states in quantum field theory. Furey's dissertation [22] works out the details how to derive Standard Model representations in terms of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Generalized ideals in $\mathbb{C} \otimes \mathbb{H}$ are introduced to derive the basic dynamical quantities: the subspace U is to be closed under a new mapping (generalized multiplication) $m : A \times A \to A$, so the three actions respectively produce ($a \in A$ and $u \in U$)

$$\text{Complex-invariant: } m_c(a, u) = auP + a^*uP^* \Rightarrow \text{spinors}, \tag{4.30a}$$

$$\text{Hermitian-invariant: } m_h(a, u) = aua^\dagger \Rightarrow \text{4-vectors}, \tag{4.30b}$$

$$\text{Quaternionic-invariant: } m_q(a, u) = au\tilde{a} \Rightarrow \text{Scalars, field strength tensor}, \tag{4.30c}$$

where P is a particular projector to $\mathbb{C} \times \mathbb{H}$. *One* generation of quarks is introduced via the octonionic chain algebra $\mathbb{C} \otimes \overleftarrow{\mathbb{O}} \cong Cl(6)$, spanned by left-products as maps

$$\overleftarrow{o_k \dots o_1}(a) = o_1 \dots o_k \cdot a,\ \ k = 1, 2, \dots \tag{4.31}$$

Quarks and leptons are fit into algebra through minimal left ideals of $\mathbb{C}\otimes\overleftarrow{\mathbb{O}}$ (i.e. the only left ideals that it contains is $\{0\}$ and itself). Ladder operators can be constructed. An option for *three* generations is found by a different partition of $Cl(6)$. Overall analysis of various Clifford algebra combinations, ideals and maximally totally isotropic subspaces, in comparison to the Standard Model representations and interactions, further e.g. suggests chirality of $SU(2)_L$ and grand-unification contributions in $Cl(10)$. Furey has since developed the division algebra line of thought significantly further [285–292] — in various algebraic methods. Our concern is slightly different, however.

Division algebras provide quite a striking new way to organize Standard Model particles. The hypothesis considered at the beginning of the author's work was that the Khronon was to become "quaternion-valued", and when further extended to other division algebras, it would drive other internal gauge interactions. The exact meaning of this was to be determined — most obviously, the Khronon could rather become a scalar field valued in whatever other algebraic structure, which would be in part constrained to a Lorentz vector representation, and in another part to internal gauge objects (which could be the connection or a form excitation B, most likely). This was one of the main motivations for considering the Isokhronon action, which was also motivated from the geometrization of the excitation (i.e. attempting to provide a coframe-like interpretation to the B-field), rather than just trying to fit the Yang-Mills connections in with the gravitational field(s), the Lorentz connection and the Khronon vector.

There were several issues that obstructed from considering the Khronon from a division algebra perspective, and instead pushed toward continuing Lagrangian and spacetime geometry questions. For one, a rather open question at first was how Yang-Mills theory should even be introduced. This has now been resolved quite multifold, at least in principle. Geometrizing the excitation B-field is one option, another would be to include just the standard Yang-Mills actions in the metric phase. However, the transition from the gauge-gravity unified phase to the metric phase remains unclear — and this is not an algebraic issue of unification, but rather a dynamical issue in the Lagrangian or a geometric issue in the manifold and principal bundles. Furthermore, division algebras would necessarily contest with any other perspective on unification. That is, it is not entirely clear what benefit a $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ perspective would have provided compared to, say, $Spin(10)$ or $SU(5)$. If there was a canonical mapping between division algebras and particle representations, it would benefit in providing an explanation to the limited selection of particles that are observed, but it is not clear that this is the case. For the Khronon, an argument for division algebras would be a canonical appearance of a Lorentz vector and complexified connection in the division algebra structure. It is not, in principle, difficult to introduce various different structure into the Khronon action — [138] considers the Khronon action subject to a different symmetry group, $Spin(4)$ vs. $SO^+_{\mathbb{C}}(1,3)$. Similarly, it is not difficult to rather consider a quaternion[19] $\tau = \tau_0 + \tau_1 i + \tau_2 j + \tau_3 k$ rather than a Lorentz-valued scalar τ^a, but it is not clear if any new phenomenology is actually achieved.

4.5 Unification in waywiser gravity

Although the author did not pursue this direction, it has to be emphasized that there is a particular Cartan-geometric unification proposal already published in literature, the cycle of work by Westman & Zlosnik [42–45]. Adapting this would be the most natural and prospective unification scheme to attempt for the Cartan Khronon theory [20] — and indeed, it should be one of the foremost options to consider when pursuing the Khronon unification further. Despite having been acquainted to this early on, the reason why this dissertation did not develop this scheme was because of the obviousness of this option, a fear of triviality to the point that it was not entirely clear if anything novel could have actually been said, and because there was an ambition to develop new (pregeometric) principles for unification. However, for continuation work, a definite resolution should be developed in working out the details how to put the Khronon (either [20] or the $Spin(4)$ development [138]) into the unified Cartan geometry framework; this procedure should be significantly more straightforward than what was attempted here.

Westman & Zlosnik consider the general Cartan-gravity action

$$S_{\text{Cartan}} = \int a_{ABCD} F^{AB} \wedge F^{CD} + b_{ABCD} \mathrm{D}\tau^A \wedge \mathrm{D}\tau^B \wedge F^{CD} + c_{ABCD} \mathrm{D}\tau^A \wedge \mathrm{D}\tau^B \wedge \mathrm{D}\tau^C \wedge \mathrm{D}\tau^D. \quad (4.32)$$

The theory is on a *four*-dimensional spacetime with either the de Sitter $SO(1,4)$ or anti-de Sitter $SO(2,3)$ symmetry group — the capital letters $A, B, C, \ldots = 0, 1, 2, 3, 4$ refer to (anti-)de Sitter indices, ϵ_{ABCDE} is the

[19] Note the immediate generalization to a quaternion, rather than a minquat! So desired, it could also be generalized to be octonion-valued, if the connection, curvature and trace is adapted so that the Lagrangian is still ultimately a scalar.

respective totally antisymmetric symbol and η_{AB} the corresponding orthonormal metric. As done in waywiser geometry, the dynamical variables are $\{\tau^A, A^{AB}\}$, where τ^A is the contact vector and A^{AB} the (anti-)de Sitter connection. Note that D is here the derivative w.r.t. the full connection, and F is its field strength (curvature). The contact vector then defines the symbols

$$a_{ABCD} = a_1 \epsilon_{ABCDE} \tau^E + a_2 \tau_A \tau_C \eta_{BD} + a_3 \eta_{AC} \eta_{BD}, \tag{4.33a}$$

$$b_{ABCD} = b_1 \epsilon_{ABCDE} \tau^E + b_2 \tau_A \tau_C \eta_{BD} + b_3 \eta_{AC} \eta_{BD}, \tag{4.33b}$$

$$c_{ABCD} = c_1 \epsilon_{ABCDE} \tau^E. \tag{4.33c}$$

The scalars a_i, b_i, c_1 are allowed to depend on $\tau^2 = \tau_A \tau^A$. The tetradic Palatini action is recovered under the gauge choice $\tau^A = \ell \delta^A_4$: the remaining components are identified with the spacetime tangent structure — the Lorentz connection in particular $\omega_{ab} = A_{ab}$, and the coframe $e^a = \mathrm{D}_\omega \tau^a$, with the derivative w.r.t. ω only, i.e. it is the usual $SO(1,3)$ exterior covariant derivative. Here $a, b, c, \ldots = 0, 1, 2, 3$ as before. It can be shown that

$$F^{ab}[A] = R^{ab}[\omega] \pm \frac{1}{\ell^2} e^a \wedge e^b, \tag{4.34a}$$

$$F^{a4}[A] = \mp \frac{1}{\ell^2} \mathrm{D}_\omega e^a \mp \frac{1}{\ell} T^a. \tag{4.34b}$$

For convenience, we will identify $\mathrm{D}_\omega = \mathrm{D}$. The interested reader can refer to the cycle [42–45] for analysis on solutions (cosmological solutions, signature change etc., equations of motion analysis), geometry (frames, connections, hypersurfaces, etc.), and other phenomenology (e.g. quintessence, dark energy), and symmetry breaking (dynamical and otherwise).

In particular, a unified theory electromagnetism and gravity was proposed in [43]: a suitable generalization to a higher-dimensional group rolling on underlying spacetime, so that $SO(3,3)$ or $SO(1,5) \to SO(1,3) \times U(1)$ through two contact vectors $V^{\mathcal{A}}$ and $W^{\mathcal{A}}$. Beginning with the action

$$S_{SO(1,5)/SO(3,3)} = \xi \int \epsilon_{\mathcal{ABCDEF}} V^{\mathcal{E}} W^{\mathcal{F}} e^{\mathcal{A}} \wedge e^{\mathcal{B}} \wedge F^{\mathcal{CD}} + \chi \int V_{[\mathcal{A}} W_{\mathcal{B}]} V_{[\mathcal{C}} W_{\mathcal{D}]} F^{\mathcal{AB}} \wedge F^{\mathcal{CD}}, \tag{4.35}$$

under the gauge fixings $V^{\mathcal{A}} = \ell \delta^{\mathcal{A}}_4$ and $W^{\mathcal{A}} = \mu \delta^{\mathcal{A}}_5$ with

$$A^{\mathcal{AB}} = \begin{pmatrix} \omega^{ab} & \frac{e^a}{\ell} & \frac{B^a}{\mu} \\ -\frac{e^b}{\ell} & 0 & \frac{B}{\mu\ell} \\ -\frac{B^b}{\mu} & -\frac{B}{\mu\ell} & 0 \end{pmatrix}, \tag{4.36}$$

the action decomposes to

$$\begin{aligned} S_{SO(1,5)/SO(3,3)} \rightarrow & \int \xi\mu\ell\epsilon_{abcd}e^a \wedge e^b \wedge F^{cd} + \chi(\mu\ell)^2 F_{45} \wedge F^{45} \\ = \quad & \mu\ell\xi \int \epsilon_{abcd}e^a \wedge e^b \wedge \left(R^{cd} \pm \frac{1}{\ell^2} e^c \wedge e^d\right) \\ & + \int \left(\pm \frac{\ell}{\mu}\xi\epsilon_{abcd}e^a \wedge e^b \wedge B^c \wedge B^d - 2\chi \mathrm{d}B \wedge e_a \wedge B^a \right) \\ & + \chi \int (\mathrm{d}B \wedge \mathrm{d}B + e_a \wedge B^a \wedge e_b \wedge B^b). \end{aligned} \tag{4.37}$$

It is shown that this action reproduces the standard Maxwell equations — note that [43] considers a waywiser geometry extension of the internal gauge fields B^a (or generally some B^A), so that they carry an extra index. The standard gauge field is B, identified with $V_a B^a \sim V_A B^A$. In particular, the Yang-Mills-Cartan action is

$$S_{\mathrm{YMC}} = \frac{1}{2g^2} \int \mathrm{Tr} \left(\frac{\mu}{2} \epsilon_{ABCDE} \tau^E e^C \wedge e^D \wedge B^A \wedge B^B + e_A \wedge B^A \wedge F[B] \right). \tag{4.38}$$

We can add that this is essentially first order formalism for Yang-Mills theory [2], working with the auxiliary field B^A in some other representation, rather than the Lie algebra adjoint (cf. B^A vs. u^a). Clearly, thought had converged in this matter coupling, even though the motivation was slightly different, viz. zero-metric phases vs. rolling on the base manifold.

But the unification extension to non-Abelian theory was not yet resolved: there is a suggestion in [45] that the soldering form should be constructed from an enlarged gauge structure, ${e^a}_\mu \rightarrow {e^{\mathcal{A}}}_\mu$, which should then somehow break down to standard gravity and gauge theory, but this procedure is yet not entirely certain. Indeed, the author argued for a similar case [2] with the Lie-algebra valued $u^a = u^{aI} T^I$ as an analogy of the coframe field, but this does not resolve the topological issue of the assumed spacetime 4-manifold, which seems to prefer $SO(1,3)$ rather than any other gauge group[20].

The similarity of ideas with the Khronon action is rather plain. Both the (anti-)de Sitter Cartan gravity theory as well as the Khronon have as a fundamental geometric requirement that the coframe be restricted to a particular form $\mathrm{D}\tau^a$. Apart from various topological terms which can be similarly appended to the Khronon theory, the distinction is what symmetry group is introduced first: the Khronon works on $SO_{\mathbb{C}}(1,3)$ and $SO^+_{\mathbb{C}}(1,3)$,

[20] An extreme example would be trying to reduce the frame bundle structure group to Abelian $U(1)$, which would be an entirely foreign sense of relativity.

Westman & Zlosnik study restrictions of $SO(1,4)$ and $SO(2,3)$ to $SO(1,3)$; and the most recent extension of the Khronon was to $Spin(4)$. It raises the question on the direction of development, however — should the focus be on the symmetry groups, or the representation of the contact vector, or the symmetry breaking potentials or constraints in the Lagrangian, or the matter couplings, or the bundle geometry? Or perhaps something entirely different?

4.6 Observational signatures

To put it bluntly, there is no observational or experimental verification of any specific unification scheme. It is very tempting to conclude the chapter with this single sentence, but nevertheless, we will refer to some recent results. There are some constraints on various processes some unified theories might exhibit (say, proton decay or Lorentz violation) and some indirect suggestions on unified phenomena (e.g. the seeming convergence of coupling constants at high energy). In a similar vein, ΛCDM currently persists, despite some recent data on Hubble tension, dark matter self-interactions, and numerical simulations that are critical against what a pure-ΛCDM phenomenology might suggest. This will not be a complete review: Beyond the Standard Model (BSM) physics is a field of active research, and both specialist and layman up-to-date documentation can be easily found online; neither are concrete observational constraints entirely relevant for pure theory development of Khronon gravity. Experiment and observation does, however, force a theorist to be very critical of the equations.

Currently, the Khronon does not have a definite preference for any specific gauge-gravity unification or Grand-Unified model[21], but any proposal in this direction would be riddled with usual Grand-Unification contention. All Grand-Unified models will differ from the Standard Model simply by virtue of unification of the fundamental forces (e.g. the convergence of coupling constants), but the exact character of this difference is model-dependent [293–297]. Often, new particles are introduced — none have been observed that would implicate any particular grand unification; by virtue of construction, the mass of these new particles can and has to be made so large as to be unobservable at currently achievable energy scales. Proton decay is another possible observational signature, through model-dependent interaction channels — it has not been observed[22], and several

[21] Apart from two loose suggestions, so that either $SO^+_{\mathbb{C}}(1,3)$ or $Spin(4)$ would be included in the full unified group.

[22] Singular candidate events notwithstanding. The lower bound on the mean proton lifetime has been estimated to be more than 10^{32}–10^{34} years (dependent on specific

models have been ruled out [298, 299]. The Review of Particle Physics [6] provides an exhaustive reference on the status of current particle physics experiment and observation. The summary can be taken as that no particular unified model has been implicated so far. Generally speaking, GUT-scale phenomena are currently inaccessible in contemporary colliders, as of the time of writing (and for a good while into the future as well).

The by-hand choice of the gauge of the Khronon field can be seen as spontaneous Lorentz symmetry breaking, introducing a preferred frame. Einstein-Aether theories have similar phenomenology. To date, the breaking of Lorentz symmetry has not been observed [252–256]. However, this is far too simplistic a statement. Establishing *how* a theory *exactly* might break Lorentz symmetry is not straightforward, and generally comparison with observation requires theory-dependent analysis. At the same time, strictly speaking, the parameter range of various models has not been exhausted either[23], so, at the time of writing, neither have Einstein-Aether theories been ruled out. The author would characterize the matter as "to be resolved further on".

Khronon gravity can be seen to make a prediction in the ideal dust it allows for in the vacuum, corresponding to Cold Dark Matter. The inclusion of matter or both gravitational chiralities would complicate the phenomenology considerably, so the dust might no longer be ideal. On the other hand, Cold Dark Matter is part of the Standard Model of Cosmology, but there has been a growing variety of problems and concerns that suggest a necessary extension. Recently, Hubble tension, that is the difference of the Hubble parameter derived from early and late universe measurements, has come to the forefront [300–304]. Numerical simulations have run into certain discrepancies, such as the core-cusp problem (a difference in dark matter density profiles in low-mass galaxies from observation) or the dwarf galaxy problem (observations indicating them in a lesser quantity than simulations), among others [305–310]. The deviations from ΛCDM cosmology are extensively studied [302] (and unfortunately, beyond the scope of this dissertation). The unimodular extension of Khronon gravity [1] produces the cosmological constant Λ as an integration constant — recent data [311] has suggested Λ might be varying over time. It is not currently clear how the Khronon theory would mold to all of these problems simultaneously (or if it even needs to!). In spite of issues that might be levied against it, it is

measurement, confidence interval, etc.), depending on the specific experiment.

[23] As an example, [253] constrains the deviation of the speed of the Einstein-Aether spin-2 graviton from the speed of light (in natural units) to approx. 10^{-15}–10^{-16}. There are other observables also under consideration, e.g. certain rotational invariants, modifications to the photon dispersion relation, etc. This is a heavily phenomenological topic that is unfortunately beyond the scope of this dissertation.

not possible to *rule out* Khronon gravity based on ΛCDM tests either — even if Dark Matter did not possess a *component* of (ideal) dust, at least classically the Khronon gravity zero-dust solution seemingly persists.

Are we even at need for unification, gauge-gravity or Grand Unification or otherwise? Quantum Gravity is a pressing theoretical issue, as the Standard Model is a quantum field theory, but General Relativity is not, and it is not yet unanimously resolved how they should coexist simultaneously and consistently, under a single framework. Even if quantum gravity signatures have not been observed, the theoretical issue persists, as some form of gravity and quantum field theory *must* be accessible simultaneously. *Unification*, however, argues for a simplification of interactions — but without neither an observational basis or strict theoretical necessity. We are left with indirect implications and theoretical desires. Krasnov & Percacci [15] provide four arguments:

1. Convergence of coupling constants under renormalization is a standard argument for unification. A dimensionless coupling constant can be assigned to gravity as well, which converges with the other interactions near the Planck scale, perhaps suggesting unification with gravity.

2. The Kaluza-Klein construction includes the Higgs field, gauge fields, and the metric together into a single unified higher-dimensional metric. The possibility and coincidence of this is very curious.

3. Standard Model fermions can be put into a single irreducible spinor representation of a graviGUT model, complexified to $SO_{\mathbb{C}}(14)$ in particular. Again, the coincidence is striking.

4. That the General Relativity connection is not a propagating degree of freedom is suggestive of a Higgs-like mechanism in gravity, where the difference of an independent connection and the Levi-Civita connection obtains a large mass.

The persuasiveness of these arguments is to be decided by the reader. For the author, the arbitrariness of $SO(1,3)$ *versus* $U(1)_Y \times SU(2)_L \times SU(3)_C$ on the frame bundle against an internal principal bundle is already sufficient to consider if there exists some mechanism to reduce or actuate this split. A manifold seems to introduce a $G_{\text{ext}} \times G_{\text{int}}$ split, but finding a geometric mechanism that removes this distinction is alluring[24] — just as its relevance yet remains physically to be determined.

[24] That is, a mechanism distinct from just making the tetrad the order parameter or the like. To emphasize, this comment is alluding to the author's own intuition, which might not be true.

Chapter 5

Conclusion

5.1 Further directions

There remains a wealth of work to be done — including actually finding a satisfactory solution to the very initial problem of incorporating the Khronon theory into a division algebra unified framework. Chapter 4.1 already includes a list of pressing issues for the Khronon theory of gravity, so for convenience, let us only restate a summary.

1. The theory and phenomenology of the Khronon spontaneous symmetry breaking $\tau^a \to \tau\delta_0^a$ should be better studied. The reality conditions, depending on the theory, can be made more explicit.

2. Quantum phenomenology should be extensively studied, including the (Euclidean and otherwise) path integral, canonical quantization, simplicity constraints in spin foams and Loop Quantum Gravity.

3. The differential geometry generated by the Khronon should be made mathematically more precise, particularly so between the tangent space geometry and Lorentzian internal space — and see whether a more nontrivial interpretation than simplicity constraints is possible.

4. The transition to the gauge-Khronon unified phase should be developed, in both the dynamics (Higgs mechanism, waywiser constraints) and the geometry. In particular, explicit adaptations to various unified models should be studied.

5. The phenomenology should be continuously verified, e.g. the validity of the (ideal *and* non-ideal in extended theory) dust degrees of freedom and quantum Hilbert space, and in relation to recent observational issues (Hubble tension, dark matter self-interactions etc.).

These are the problems most immediately pressing for the Khronon theory *per se* [20], but the model has since been extended to $Spin(4)$ theory [138], and a prospective path is to continue to introduce ever-more modifications. In chapter 4.3 it was suggested that the Khronon could be put into graviweak [276] form by reinterpreting the indices $\tau_a \to \tau^{\bar{a}a}$ of $SO_{\mathbb{C}}(1,3) \cong SO_{\mathbb{C}}(4)$ — it is straightforward to write essentially the waywiser tetrad $\mathrm{D}\tau^a$ to whatever dimensionality and structure group, and then see if interesting phenomenology might follow[1]. The contact vector τ^a is a 0-form object, and standard analysis would be to test any and all couplings to other fields in any and all contexts, and see the phenomenology that might follow. However, theory development in this manner is rather *ad hoc*, and it is difficult to promise whether anything physically relevant will appear.

The basic case of duality is well-understood now (for a while now, truthfully) — it is an identity transformation in the parent path integral. Similarly, the relation of BF theory, premetric theory, first order formalism, and parent systems is now discussed more explicitly and collected together. Premetric theory development would further require a more explicit discussion of non-Abelian theory. A nontrivial continuation of the premise of dualities would instead consider when the auxiliary field is not constrained uniquely, which will provide both interesting quantum phenomenology as well as a rather difficult problem for interpreting this situation. That is, if

$$\int \mathcal{D}\lambda \exp\left[i \int \lambda C(A,B)\right] \sim \delta(C(A,B)) \tag{5.1}$$

is possible and has multiple zeros, then the integration cannot be performed uniquely. Instead, all minima would be included — for the Plebanski theory path integral see [78]. However, [78] did not consider this as a duality between Plebanski theory and the new quantum systems. This is a question which seems to be yet unresolved. The author studied pure B-potentials in [5], but the general case of such (degenerate and otherwise) dual structures (including mapping between e.g. observables) can be considered as a continuation topic this dissertation has motivated.

The Khronon matter couplings should now be much more clear [2–4], appended with the suggestions in this dissertation. What can be considered further is the study of Yang-Mills analogies with gravity [2, 4] — that

[1] As a specific example, the Khronon could be generalized to other dimensions. Four dimensions is special with e.g. the exceptional isomorphism $Spin(4) \cong SU(2) \times SU(2)$, but as a testable premise, it is possible to consider e.g. compactification to three dimensions, or the requirement for self-duality and including only the waywiser constraint — or, considering just arbitrary (products of) groups in arbitrary dimensions.

is, the first order formalism u-field can be used to define a (bi)metric-like structure in $g^{\mathrm{YM}}_{\mu\nu} = \mathrm{Tr}_G(u^a{}_\mu u^b{}_\nu \eta_{ab})$, and the geometry can be studied further. Similarly, a full Yang-Mills gravitational analogue

$$S_{u\text{-GR}} \sim \int \mathrm{Tr}_G \left(\frac{1}{2g^2} u \wedge u \wedge F + \Lambda_G u \wedge u \wedge u \wedge u \right) \tag{5.2}$$

can be curious to consider in various structure groups G and various dimensionalities (and when it is well-defined).

Background fields still provide some theoretical interest. Quantum Electrodynamics in plane wave backgrounds is well-developed [312], and generally the background field method is well-understood [313]. However, the case of inherent backgrounds holds some confusion. For the Dirac-Bergmann algorithm, the Isokhronon provided standard electromagnetic degrees of freedom even in the case of the $X_a, \mathrm{D}X_a = 0$ background (provided a consistent torsion) [3]. The Khronon introduces an additional dust degree of freedom to the graviton [1, 20, 71], despite a similar appearance of the integration constant. When exactly is a degree of freedom introduced? Is the anti-self-dual contribution essential, or is it possible to manufacture other gauge theory (e.g. electromagnetic) constitutive laws which consistently introduce more degrees of freedom? Is the Khronon gauge fixing crucial? On a more fundamental note, the consistency of these inherent backgrounds should be verified. Renormalizability is an oft-quoted principle, but the thermodynamic validity could be checked as well. Namely, whether an inherent background is consistent with the second law of thermodynamics, or whether there is an infinite pool of energy provided in the background. In this sense, the Khronon should be safe, as it introduces a proper dust degree of freedom, but the mixture of quantities remains curious. Finally, the exact Hilbert space structure should be clarified, in general and e.g. to the presence of background quantum transitions. This is also a minor question for the Shadow Charges proposal [209, 210].

A variety of different interpretations were attempted in comparison of Yang-Mills theory and gravity; they all should be developed in more sophisticated terms. For BF theory, ultimately, a constraining process must restrict the B-field to its physical values — as pure B-potentials are not really viable due to the backreaction of the inhomogeneous equation [5], this leaves a Lagrange multiplier constraint structure, similar to Plebanski theory [207]. Explicit degenerate constitutive laws should be manufactured and studied. In comparison with gravity, the relevance of the hypersurface basis $e^a \wedge e^b$ and $*(e^a \wedge e^b)$ appears crucial [2, 5]. When the gauge field is taken as an "excitation" or perturbation [2],

$$k_{ij}{}^{ab} e^i \wedge e^j = \left(\frac{i}{\kappa} \delta^a_i \delta^b_j + \frac{1}{4} B_{ij} \eta^{ab} + \frac{1}{4} \eta_{ij} B^{ab} \right) e^i \wedge e^j, \tag{5.3}$$

collected into the action

$$S_k = \int \mathrm{Tr}\left((e^a \wedge e^b) {k_{ab}}^{ij} \wedge ({}^{+}R_{ij} + \eta_{ij} F + {k_{ij}}^{uv} (e^c \wedge e^d) \epsilon_{cduv}) \right), \quad (5.4)$$

the suggestion of deforming hypersurfaces in "internal gauge directions" should be properly formalized. The relevance and meaning of the object mimicking a connection

$$\tilde{\omega}_{ab} = {}^{+}\omega_{ab} + \eta_{ab} A \quad (5.5)$$

should be clarified. Alternatively, the Yang-Mills excitation can be understood to be "rotated" relative to the hypersurface basis [2, 5] — in particular, the applicability of (generalized) symmetry breaking is something to understand in greater detail. The issue why the Isokhronon direction proved unsuccessful was very concisely formulated in terms of the ill-defined constitutive diagram; it would be curious to see if there is any greater utility possible to introduce there.

Spinorial theory still promises a rich avenue for theory development, as various standard theories are rewritten in terms of spinor quantities, bilinears — although the Dirac equation squares to the standard wave equation, Fermi-Dirac statistics open a greater possibility for modified phenomenology. The most obvious modification is $\tau^a \to \bar{\psi}(\gamma^a + \gamma^5 \gamma^a)\psi$ for the Khronon, or really for any other vector, but this is somewhat trivial. Alternatively, twistors [146] can be a further option, but the geometry becomes much more complicated.

This returns to the issue of the difference in geometry sectors between Maxwell-Yang-Mills theory and General Relativity — the hypothesis was that a unified phase should lose this distinction. An embedded hypersurface approach (or really, branes in String Theory) might prove more prospective for this. Alternatively, the base manifold dimensionality questions might be attempted to solve in a manner similar to junction conditions, see e.g. [314]. There seems to be a significant geometry transition in the unified phase [5], already suggestive from the Coleman-Mandula theorem. A wealth of ideas have been already considered before [25].

A primary hypothesis was that there should be a zero-metric topological phase, where all interactions will obtain the same Lagrangian form. However, the geometry and dynamics of the phase transition is yet to be handled satisfactorily (and explicitly!). This includes actually deriving solutions which include a zero-metric region, and verifying the limits of all relevant quantities when tending to this region. A particular candidate choice is in studying black hole solutions, which include an unavoidable curvature singularity, even for the Khronon [137]. Nevertheless, it may be more productive to just handle the topological region by gluing boundaries to each other. The *physicality* of these regions is a separate issue, and would be verified by studying solutions.

To finish with a (very) distant aspiration — more generally than the quantization of the specific Khronon model, the question of quantum geometry prevails. More than a fancy name for quantum gravity, the author would rather expect several physical phenomena to define a new mathematical structure, which can be interpreted in terms of a quantized metric. The signal triple was an attempt to consider what information an observer might even admit — and a linear representation theory is a natural proposal. If to speculate, perhaps quantum information theory could provide further inspiration — generally, relativistic quantum information theory is a work in progress [315–319], so even apart from the author's speculation, it is a worthwhile direction to understand.

In this vein, with the extreme success of quantum field theory, it is tempting to e.g. interpret its tangent-like Minkowski-space formalism happening at a literal spacetime event, i.e. a single point. Perhaps the Fock space could model the quantum tangent space instead, the quantum measurement would introduce another axis of time, independent as it is from the unitary time evolution, and a global quantum geometry could be built point-wise. However, this proposal heavily requires a definite mathematical structure to operate on. To wit, there is a structure that *must* be naturally present in quantum field theory even as it is, because purely implicitly there is a model of quantum measurement viz. time evolution, (metric) geometry of events, and observer-tangent space relation present, but its mathematics should be explicitly written out[2]. The competition is fierce [217], and the suggestions here remain speculative.

5.2 Summary

This dissertation initially set out to provide a gauge-gravity unified model for the Khronon theory of gravity [20]. The aspiration was to introduce the Khronon into the division algebra scheme of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ [22], changing from a Lorentz-valued scalar to a division algebra valued object, but as work progressed [1–5], focus ever shifted toward unification-geometric, Lagrangian and duality questions. The author will provide a seemingly con-

[2] As a definite example, the quantum measurement defines a probabilistic distributions of outcomes to which the system can collapse — this is a probabilistic evolution, and can be seen as a new axis of time, or a set of them, and should be formalized. Then, specific choices on these extra axes define a specific model of quantum measurement. The danger is that the freedom of choosing the measured observable rather introduces an *infinity* of axes, but introducing unitary evolution to quantum measurement sequences by themselves is nothing foreign to what is already present in the theory.

tradictory answer to the initial problem: at the culmination of this work, unification of the gravitational interaction and internal gauge fields has not yet been achieved, but the method to work the Khronon into any unification proposal has been extensively developed and a significant part of it is now well-understood. Although real and complex numbers, quaternions and octonions did not ultimately take part in the development of unification, the geometry and dynamics, including proposals for new models in the simplicity constraints or extending the Khronon, should be much more straightforward to develop. Only a choice of constraints, waywiser and curvature, or changing the representation of the Khronon scalar is required. Geometric questions remain, in how the base manifold and tangent bundle geometry should be reconciled with internal principal bundle geometry, resp. the geometry of gravity as in General Relativity *versus* the geometry of Maxwell-Yang-Mills theory, but they are also significantly more subtle and can promise more mathematical interest — perhaps in another treatment. To be very clear, adapting the Khronon waywiser coframe will allow to develop any number of preceding gauge-gravity unified models. Even in this dissertation new suggestions were provided for e.g. quaternions and graviweak theory, but studying the phase transition to the unified phase remains to be exhausted.

The position of auxiliary fields, first order systems, and parent system symmetries and transformations in the role of dualities, as in e.g. Proca-Kalb-Ramond [64, 65], but also in terms of premetric theory [58], should be abundantly clear now. Dualities are simply parent system identities, especially obvious as path integral identity operations, e.g. in Gaussian and Dirac delta integration. Premetric theory can be seen as a physical description of principal bundle geometry and constraining topological field theory. Developing nontrivial phenomenology for the auxiliary fields would require a degenerate constraint structure, but this direction would rather work with Lagrange multipliers than with (generalized) potentials, as a development of Plebanski-like ideas [78, 207] to both gravity and internal gauge theory. Dualities continue to prove useful, interesting and mathematically curious, as can be seen e.g. in the case of Maxwell-Chern-Simons [74], and limit inconsistencies [4, 75, 76]. The classical differential geometry is straightforward, and allows to quickly explain the problems with duality rotations in non-Abelian theory in terms of cohomology, or the Poincaré lemma. It would be interesting to see if this idea of dualities as essentially moving between representatives of an equivalence class could be pushed further, to dualities of other classes. Inter-gauge rotations or perturbations require further study.

The mathematical structure of the background fields generated by integration is something that deserves more attention, but some progress has been made. The Khronon case is largely exhausted, both in Hamiltonian [20] and Lagrangian [1] formalism. The ideal dust energy-momentum is an integration constant, and the extra degree of freedom appears from interplay of self-dual and anti-self-dual connection components, the Einstein Field Equations and the connection equation of motion. The case for the Isokhronon is more confusing [4], as the degree of freedom count does not show additional degrees of freedom from the covariantly constant background — perhaps unsurprisingly so, as there is no other sectors to pull from, so we are left with only a covariantly constant background. Similarly, the Isokhronon is not a proper Higgs field. This is partly a mathematical physics problem, to trace when exactly the constitutive law structure changes the Dirac-Bergmann algorithm degree of freedom count. It becomes a more physical problem, if the quantum case is to be studied — the Hilbert space structure should be clarified, particularly so for background transitions. The quantum Khronon theory would promise more novel phenomenology than considering classical solutions.

The Khronon theory is now better understood. Much of the classification and interpretation was already provided in the initial paper [20], but verification and extension required further work. Now, the Lagrangian formalism has been verified [1], matter couplings have been studied [2, 3], particularly so for internal gauge theory [4], and the use of the Khronon constitutive law in unification has been clarified [5]. This dissertation has considered Khronon matter couplings in the metric and topological phase, Lagrangian analogy between internal gauge theory and gravity, Khronon simplicity constraints for unification, and the mathematics how the integration constant appears in conserved currents and Lagrangian theory. There has been significant work in extending the Khronon. The complexified $SO_{\mathbb{C}}(3,1)$ Khronon has been continued to a $Spin(4)$ gauge theory [138], with extensive analysis of cosmology and exact and perturbative solutions. Solutions, black holes and cosmology, have also been considered for the Khronon theory proper [137]. It can be confidently stated that the understanding of the behavior, i.e. structure and phenomenology, of the Khronon theory has significantly improved. However, it clearly has not yet exhausted its potential — it is the author's hope that this dissertation will serve as both inspiration and reference for future development in this regard.

Two important continuation topics bear special emphasis: providing a definite mechanism for the spontaneous symmetry breaking $\tau^a \to \tau\delta^a_0$, and studying the quantum phenomenology of the Khronon Lorentz gauge theory of gravity (or, the same for the $Spin(4)$ theory). The symmetry breaking so far has been done by hand. However, writing a dynamical mechanism, either a scalar potential or a Lagrange multiplier constraint, will be necessary to complete the definition of the theory, and give a definite meaning to the symmetric phase. Quantum phenomenology, in the author's opinion, is likely the most probable location for interesting Khronon phenomenology, as the *classical* theory was already classified in the initial paper and this has been better understood and verified since — it should be verified that the resulting quantum theory really is Ashtekar's self-dual gravity alongside dust, and this can be done canonically, in path integral, spin foams and Loop Quantum Gravity, or otherwise. Dust has already benefited quantization of gravity [222], it would be noteworthy to see if the Khronon can achieve this intrinsically. If it proves true that the Khronon can assist in solving [1] the "covariance crisis" in Loop Quantum Gravity [320], it would be a significant contribution, but it remains to be verified.

The author considers this dissertation to lay the premise for much more successful approaches to the Khronon theory, having pushed through various unsuccessful lines of thought on internal-external gauge unification. The general approach to unification and the Khronon theory development should now be more than clear, and what remains is to evaluate specific details. However, the problem to weigh is the analysis against its value — what odds are there for something remarkable to appear, and what is truly required at this stage? For the author, the most pressing questions reside outside of what was studied in this dissertation, and outside of what the author has even worked on so far: to develop the quantum measurement in all sense of theory and experiment, and to achieve a satisfactory theory of quantum gravity. A notion of geometry, both local and global, subject to some notion of quantization, will unavoidably appear, and it would be most curious to understand and develop it in precision. For the Khronon, however, the author's collection of thoughts reaches its conclusion. Only one final question remains: *Quo vadis?*

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Kokkuvõte

Sisemise kalibratsiooniteooria analoogiast Cartani Kroononi gravitatsiooniteooriaga

Käesoleva dissertatsiooni algseks sihiks oli leida kalibratsiooni ja gravitatsiooni ühendteooria Kroononi gravitatsiooniteooria jaoks [20]. Esialgne soov oli liita Kroonon jagamisega algebrate $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ struktuuri [22], nii et Kroononi Lorentzi-väärtuseline skalaar asendataks jagamisega algebrate väärtuseliseks objektiks, aga töö edenemise käigus [1–5] liikus fookus ühendamisgeomeetriliste, Lagranžiaani ja duaalsusküsimuste poole. Autor annaks näiliselt vasturääkiva vastuse esialgsele probleemile: gravitatsioonilise vastasmõju ja seesmiste kalibratsiooniväljade ühendamist ei ole veel saavutatud, kuid meetodeid, kuidas paigutada Kroononit mistahes ühendamisettepanekusse, on ulatuslikult arendatud ja suur osa on nüüd hästi mõistetud. Kuigi reaal- ja kompleksarvud, kvaternioonid ja oktonioonid kokkuvõttes ei osalenud ühendamisarendustes, siis geomeetria ja dünaamika, sealhulgas uute mudelite väljapakkumine, kasutades lihtsuskitsendusi või laiendades Kroononit ennast, peaks olema oluliselt otsesem arendussuund Kroononiteooria jaoks. On tarvis üksnes valida sobivad kitsendused, st. kas mõõtratta-raami või kõveruse jaoks, või vahetada Kroononi skalaari esitust. Geomeetrilised küsimused aga endiselt jäävad püsima, kuidas alusmuutkonna ja puutujakihtkonna geomeetria peaks olema ühitatud seesmise printsipaalkihtkonna geomeetriaga, vastavalt üldrelatiivsusteooria *versus* Maxwell-Yang-Millsi teooria geomeetria, kuid need küsimused on ka märgatavalt keerulisemad ja võivad pakkuda rohkem matemaatilist huvi, vahest arendatuna mõnes teises töös ja käsitluses. Et rõhutada: Kroononi lihtsuskitsenduste kohandamine võimaldab arendada edasi palju erinevaid varem välja pakutud kalibratsiooni-gravitatsiooniteooria ühendmudeleid. Dissertatsioonis endaski on mõned uued soovitused välja pakutud, nt. kvaternioonide ja gravinõrga teooria osas, aga faasisiirde uurimine ühendatud faasi ei ole veel ammendatud.

Lisaväljade, esimest järku süsteemide ja peremeessüsteemide sümmeetriate ja teisenduste positsioon duaalsustes, nii nagu Proca-Kalb-Ramondi teoorias [64, 65], aga ka eelmeetrilises teoorias [58], peaks olema nüüd küllalt selge. Duaalsused on kõigest põhisüsteemi identsused, mis on eriti ilmsed teeintegraali identsusteisendustena, nt. Gaussi või Diraci delta integreerimise näol. Eelmeetrilist teooriat võib vaadata kui printsipaalkihtkondade geomeetria ja topoloogilise väljateooria kitsendamise füüsikalist kirjeldust. Mittetriviaalse fenomenoloogia arendamine nõuaks kõdunud kitsenduste struktuuri, aga see suund pigem arendaks Lagranži kordajaid, mitte lisaväljade (üldistatud) potentsiaale, Plebanski-laadsete mõtete jätkuna [78, 207] nii gravitatsiooniteoorias kui ka seesmises kalibratsiooniteoorias. Duaalsused osutuvad jätkuvalt kasulikuks ning füüsikaliselt ja matemaatiliselt huvitavaks, nagu on näha näiteks Maxwell-Chern-Simonsi teooria [74] ja piirväärtusebakõlade [4, 75, 76] näitel. Siin on klassikaline diferentsiaalgeomeetria võrdlemisi selge ja võimaldab kiiresti seletada mitte-Abeli teooria duaalsuspöörete probleeme kohomoloogia või Poincaré lemma kaudu. Oleks huvitav uurida, kas mõtet duaalsustest kui ekvivalentsiklassi esindajate vahetusest saaks arendada mõnda teise duaalsuste klassi. Kalibratsioonivahelised pöörded ja häiritused nõuavad edasisi uuringuid.

Integreerimisest tulenevate taustaväljade matemaatiline struktuur on valdkond, mis väärib rohkem tähelepanu, aga mõningast progressi on juba saavutatud. Kroononi näide on valdavalt ammendatud, nii Hamiltoni [20] kui ka Lagranži [1] formalismis. Ideaalne tolm on tõepoolest integreerimiskonstant ning lisavabadusastmed ilmuvad enesekaasse ja antienesekaasse seostuse komponentide, Einsteini väljavõrrandite ja seostuse liikumisvõrrandite koosmõjus. Isokroononi juht on veidi rohkem segadust tekitav [4], kuna vabadusastmete loend ei too sisse kovariantselt konstantsesse tausta lisavabadusastmeid — vahest see ei ole üllatav, sest ei ole ka mõnd teist sektorit, kust neid vabadusastmeid tõmmata, mistõttu jääbki alles üksnes kovariantselt konstantne taust. See on osaliselt matemaatilise füüsika probleem, et teha kindlaks, millal täpselt konstitutiivse seaduse struktuur muudab Dirac-Bergmanni algoritmi vabadusastmete loendit. See muutub füüsikalisemaks probleemiks, kui uurida kvantjuhtu — Hilberti ruumi struktuuri tuleks selgitada, eriti taustasiirete osas. Kvantkroonon paistab lubavat uut fenomenoloogiat, rohkemgi kui klassikaline juht.

Arusaam Kroononi teooriast on nüüd parenenud. Suur osa klassifikatsioonist ja tõlgendusest oli juba pakutud välja algses artiklis [20], aga kontroll ning edasiarendamine nõudis rohkemat tööd. Nüüd on kontrollitud Lagranži formalismi [1], uuritud mateeriaseostusi [2, 3], eriti seesmise kalibratsiooniteooria jaoks [4], ning on selgitatud Kroononi konstitutiivse seaduse tähtsust ühendamise seisukohast [5]. See dissertatsioon on uurinud Kroononi teooria jaoks mateeriaseostusi meetrilises ja topoloogilises faasis, lagranžiaani analoogiaid seesmise kalibratsiooniteooria ja gravitatsiooniteooria vahel, Kroononi lihtsuskitsendusi ühendamise tarbeks ning matemaatilisi detaile integreerimiskonstandi ilmumisest jäävates voo-ludes ja lagranžiaanis. Kroononit on oluliselt edasi arendatud. Kompleksifitseeritud $SO_{\mathbb{C}}(3,1)$ Kroonon on laiendatud $Spin(4)$ kalibratsiooniteooriaks [138], ulatusliku kosmoloogia ning häirituslike ja analüütiliste lahendite analüüsiga. Lahendeid, mustasid auke ja kosmoloogiat, on uuritud ka Kroononi teooria enda jaoks [137]. Võib öelda, et Kroononi teooria käitumine, st. struktuur ja fenomenoloogia, on tunduvalt parenenud, kuid ta ei ole oma kogu potentsiaali veel ammendanud — autor loodab, et see doktoritöö annab nii inspiratsiooni kui ka selgust edasiseks arendustööks.

Kaks olulist jätkusuunda väärivad eraldi tähelepanu: tuleb leida konkreetne mehhanism spontaanse sümmeetriarikkumise $\tau^a \to \tau\delta^a_0$ jaoks ning tuleb uurida Kroononi kvantfenomenoloogiat (või sama ka $Spin(4)$ teooria jaoks). Seni on sümmeetriat rikutud üksnes käsitsi. Sobiva dünaamilise mehhanismi välja kirjutamine, kas skalaarpotentsiaali või Lagranži kordajate näol, on oluline, et teooria definitsioon lõpetada ja et anda konkreetne tähendus sümmeetrilisele faasile. Autori arvates on kvantfenomenoloogia tõenäoliselt kõige huvitavam koht Kroononi fenomenoloogia uurimiseks, sest *klassikaline* teooria oli juba klassifitseeritud esimeses artiklis ning sellest saati on seda kõigest õpitud paremini tundma ja kontrollitud — tuleks kontrollida, et ka vastav kvantteooria on tõepoolest sama kui Ashtekari enesekaasne gravitatsiooniteooria koos tolmuga, ning seda võib uurida kanooniliselt, teeintegraalidega, spinnivahtudega või silmuskvantgravitatsiooniteoorias või mõnel teisel viisil. Tolm on juba aitanud gravitatsiooni kvantiseerimisele kaasa [222], ning oleks märkimisväärne, kui Kroonon annab sama tulemuse. Kui Kroononi teooria aitab lahendada [1] "kovariantsuskriisi" silmuskvantgravitatsiooniteoorias [320], siis see oleks oluline tulemus. Seda tuleks aga tegelikult kontrollida.

Autori silmis see dissertatsioon annab aluse oluliselt edukamatele lähenemistele Kroononiteooriale, olles seesmise-välimise kalibratsiooniteooria ühendamises mitu ebaõnnestunud mõttesuunda ammendanud. Üldine lähenemine kroononiteooria ühendamiseks on nüüd selge ja järel on üksnes detailide kontrollimine. Probleem on aga kaaluda analüüsi tema võimaliku väärtuse vastu — mis on võimalus, et ilmub midagi märkimisväärset, ja mida oleks selles etapis tõepoolest vaja teha? Autori jaoks on peamised küsimused väljaspool seda, mida sai uuritud selles dissertatsioonis, ja väljaspool seda, mis alal autor on seni töötanud: arendada arusaama kvantmõõtmisest, nii teooria kui ka eksperimendi vallas, ning jõuda sobiva kvantgravitatsiooniteooriani. Paratamatult ilmub mingi geomeetria mõiste, nii lokaalselt kui ka globaalselt ja kohandatuna mingile arusaamale kvantiseerimisest, ja oleks väga huvitav seda mõista ning täpselt arendada. Kroononiteooria jaoks aga on üks kogum mõtteid jõudnud oma kokkuvõtteni. Ainult üks viimane küsimus on jäänud alles: *Quo vadis?*

Publications

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Retsenseerinud ajakirjadele Physical Review D, Physical Review Letters, International Journal of Geometric Methods in Modern Physics, Foundations of Physics.

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