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Derivations on quandle algebras

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KVANDLI ALGEBRA DERIVATSIOONID

Magistritöö

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Lühikokkuvõte

Käesolevas magistritöös käsitletakse kvandlite ja sõlmede vahekorda ja derivatsioone kvandli algebratel. Esimeses kahes peatükis defineeritakse ja vaadeldakse räkke, kvandleid, räki ringe ja algebraid. Kolmas peatükk annab ülevaate varem uuritust kvandli algebra derivatsiooni algebrate alal. Kahes viimases peatükis uuritakse konjugatsiooni klassidega rühma konjugatsiooni ja dihedraalse konjugatsiooni kvandli D_n algebra derivatsiooni algebraid. Tõestasime, et konjugatsiooni klassidega konjugatsiooni kvandel allub ühele sümmeetriale ja kahele summale, ja paaritu n puhul on D_n algebra derivatsiooni algebra triviaalne.

CERCS teaduseriala: P120 Arvuteooria, väljateooria, algebraline geomeetria, algebra, rühmateooria.

Märksõnad: kvandel, räkk, sõlm, kristall, derivatsioon, dihedraalne rühm, sümmeetriline rühm, konjugatsiooni klass.

DERIVATIONS ON QUANDLE ALGEBRAS

Master's thesis

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Abstract

The aim of this thesis is to outline the connection between quandles and knots and to build on the notion of derivation for quandle algebras. The first two chapters define and discuss racks and quandles, and rack rings and algebras. The third chapter is an overview of previously studied derivation algebras of quandle algebras, and the two final chapters investigate the derivation algebras of two types of conjugation quandles: the symmetric conjugation quandles and the dihedral conjugation quandles. We proved, that the derivation algebra of a conjugation quandle algebra with conjugacy classes obeys one symmetry and two sums. Furthermore, the derivation algebra of a dihedral conjugation quandle D_n algebra is trivial when n is odd.

CERCS research specialisation: P120 Number theory, field theory, algebraic geometry, algebra, group theory.

Key Words: quandle, rack, knot, crystal, derivation, dihedral group, symmetric group, conjugacy class.

Contents

Introduction

The notion of self-distributive algebraic structures with a unique left inverse has been introduced separately multiple times. In 1943 Mituhisa Takasaki introduced the kei [14], a structure meant to convey reflection without being associative. At around the same time, John Conway and Gaving Wraith were exchanging letters about wracks: the rules outlined in Gavin Wraith's personal story about knots [15] for wracks are the same as those of Takasaki's kei. Both David Joyce and Sergei Matveev independently published articles on quandles and their relation to knots in 1982, with Joyce coining the current nomenclature for kei: involute quandles [9, 12]. These structures were generalized into what is now known as a rack thanks to Roger Fenn and Colin Rourke who coined the term in 1992 [8].

The study of group rings began in 1854 with Arthur Cayley's article on group theory [6]. Group rings, and the group algebra as specialization, were generalized to involve racks and quandles instead of groups in the late 2010s [5, 7]. In 2021, Elhamdadi et al. further defined derivation on quandle algebras and studied the derivation algebras of all isomorphism classes of 3- and 4-dimensional quandles and dihedral quandles. [7]. On the other hand, the basic structure of a rack has been expanded on by Viktor Abramov and Emanuele Zappala: the binary self-distributivity of a rack has been used to construct ternary self-distributive objects using different Lie algebras. Finally, it was proven that these structures can serve as representations of the infinite (framed) braid group. [2]

The first chapter of this thesis covers the rack and quandle structures, their relation to knots, and some examples of quandles; the involute quandle, trivial quandle, dihedral quandle, and the conjugation quandle are discussed. Crystals as defined by Kauffman [10] are brought up as a generalization of quandles. The structure of a knot as it pertains to knot theory is explored, as well as the representation of a 3-dimensional knot on a 2-dimensional plane. The three basic moves that can be made on a knot to change its shape without breaking the knot, the Reidemeister moves, are related to the three conditions of a quandle.

The second chapter defines first the rack ring, a precursor to the quandle ring defined in [7], and then the rack algebra, which is more general than the quandle ring.

Derivations on rack algebras are discussed in chapter 3. After defining derivations on rack algebras and discussing their representations as $n \times n$ matrices, where n is the dimension of the rack, the derivation matrix of a 3-dimensional quandle is computed using proposition 3.1, which was proven in [7]. The theorem on dihedral quandle derivation algebras proved by Elhamdadi et al. in [7] is included without its proof as an example of what the derivation algebra of a set of quandles (in this case n-dimensional dihedral quandles) can look like. This is followed by a proposition outlining the form of the derivation algebra of an n-dimensional trivial quandle.

The fourth chapter pertains to n-dimensional symmetric groups and permutations in general. The structure of the conjugation quandle on a symmetric group is discussed, specifically the presence of conjugacy classes. The conjugation quandles of cyclic groups, which are isomorphic to some sub-groups of symmetric groups, are shown to be isomorphic to trivial quandles. The chapter ends with a theorem on the derivation algebras of conjugation quandles on groups with conjugacy classes.

The fifth, and final, chapter deals with dihedral conjugation quandles, which are not isomorphic to dihedral quandles. The operations of the dihedral conjugation quandle are computed based on the group's intrinsic multiplication operation. This thesis ends with a theorem proving, that the derivation algebra of an n-dimensional dihedral conjugation quandle in case of an odd n is trivial.

1 Racks and quandles

Racks and quandles have been introduced independently many times, from both knot-theoretic and algebraic standpoints. Certain types of quandles are established in the first section of this chapter, including involute quandles (formerly known as kei), trivial quandles, dihedral quandles, and conjugation quandles. Section 2 presents the relation between knots and quandles, beginning with an introduction to knots, their diagrams, the concept of knot equivalence, and knot invariants. This chapter ends with a generalization of the quandle; crystals were defined by Kauffman as an alternative algebraic structure to represent knots [10].

Definition 1.1. A *rack* is a set R supplied with an operation $\triangleright : R \times R \to R$ such that for all $a, b, c \in R$

R1 $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$ (self distributivity),

R2 there is a unique $d \in R$ such that $d \triangleright a = b$.

A *quandle* Q is a rack with a third additional condition:

Q3 for all $a \in Q$, $a \triangleright a = a$.

Various notations have been used for the operation $\triangleright$ on a rack or quandle: $a * b$, $a \circ b$, a^b [5, 12, 8]. Occasionally the symbol $\triangleleft$ is used as the operation of the quandle, and some even define the rack with two operations $(\triangleright, \triangleleft)$, but in this thesis only $\triangleright$ will be used as an operation: $\triangleleft$ is only used symbolically to represent the unique element d in R2 as $a \triangleleft b$.

1.1 Examples of quandles

Definition 1.2. An *involute quandle*, or a *kei*, is a quandle on which the unique right inverse $a \triangleleft b$ in $(a \triangleleft b) \triangleright a = b$ is equal to $a \triangleright b$.

Alternatively, the second quandle axiom R2 is replaced by

$$(a \triangleright b) \triangleright b = a \ , \quad \forall a, b \in Q \ .$$

Definition 1.3. A quandle is *trivial* if for all $a, b \in Q$, $a \triangleright b = a$.

As a result, the left inverse $b \lhd a$ (such that $(b \lhd a) \rhd b = a$) of a trivial quandle is equal to a.

Definition 1.4. The *dihedral quandle* Z_n is the quandle on the set $\mathbb{Z}_n$ where the operation $\rhd$ is defined as

$$a \rhd b = 2b - a \pmod{n} \ . \tag{1}$$

Using the previous notation for the left inverse we find that $a \lhd b = 2a - b \pmod{n}$, since

$$(a \lhd b) \rhd a = b \implies 2a - a \lhd b = b \ .$$

Furthermore, for any $a, b \in Z_n$

$$(a \rhd b) \rhd b = 2b - (a \rhd b) = 2b - (2b - a) = a \ ,$$

which makes the dihedral quandle Z_n involutary.

Definition 1.5. The *conjugation quandle* on a group $(G, \cdot)$ is the quandle $(G, \rhd)$ with the quandle operation defined as

$$a \rhd b = b^{-1} \cdot a \cdot b \ ,$$

where $a, b \in G$.

Let us check if the quandle axioms R1, R2, and Q3 hold for a conjugation quandle. For brevity, all $\cdot$ will be omitted. First, due to the associativity and existence of the inverse in the group G, self distributivity (R1) holds for any $a, b, c \in G$:

$$\begin{aligned}(a \rhd c) \rhd (b \rhd c) &= (c^{-1}bc)^{-1}(c^{-1}ac)(c^{-1}bc) \\ &= (c^{-1}b^{-1}c)(c^{-1}ac)(c^{-1}bc) \\ &= c^{-1}(b^{-1}ab)c = (a \rhd b) \rhd c \ .\end{aligned}$$

The unique left inverse (R2) is defined by $a \lhd b = aba^{-1}$, since

$$(a \lhd b) \rhd a = a^{-1}(aba^{-1})a = b \ .$$

Finally,

$$a \rhd a = a^{-1}aa = a \ ,$$

so Q3 holds. A conjugation quandle on a commutative group is a trivial quandle:

$$a \triangleright b = b^{-1}ab = b^{-1}ba = a \ .$$

1.2 Relation to knots

A knot as it appears in reality is a fastening created by looping and tying one or more strings together. In mathematics, a knot refers to the looping or tying of a single string, with its two ends glued together. Formally, a knot is defined as follows:

Definition 1.6. A *knot* K is a continuous function $K : [0,1] \to \mathbb{R}^3$ which is injective in $(0,1)$ and $K(0) = K(1)$.

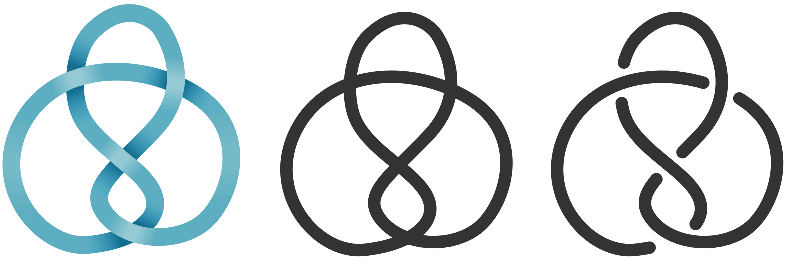

Figure 1: The shaded figure-eight knot (left), its shadow (middle), and its diagram which uses segment deletion to represent undercrossings (right): the last is the most common presentation of a knot in knot theory.

In other words, a knot is the embedding of a circle in 3-dimensional Euclidian space. Embedding the knot in 3-dimensional Euclidian space preserves its real-life behaviour: a knot that cannot be untied in real life embedded in 4-dimensional Euclidian space may be untied. This definition of a knot also supplies it with an orientation, since one can follow the knot from its start $K(0)$ to its end $K(1)$. Historically the shape of a knot has been conveyed using shading: the knot is laid out at a certain angle, then the general form (the shadow) is drawn and at each crossing the string that is on top of another part of the string remains the same, while on each side of it the undercrossing string is made darker to signify its distance from the viewer compared to the string on top of it (figure 1). In knot theory it is customary to instead delete these parts that would otherwise be shaded, as seen in figure 1.

Two knots are said to be equivalent if it is possible to deform one knot into the other without cutting the string. The formal definition employs a family of functions.

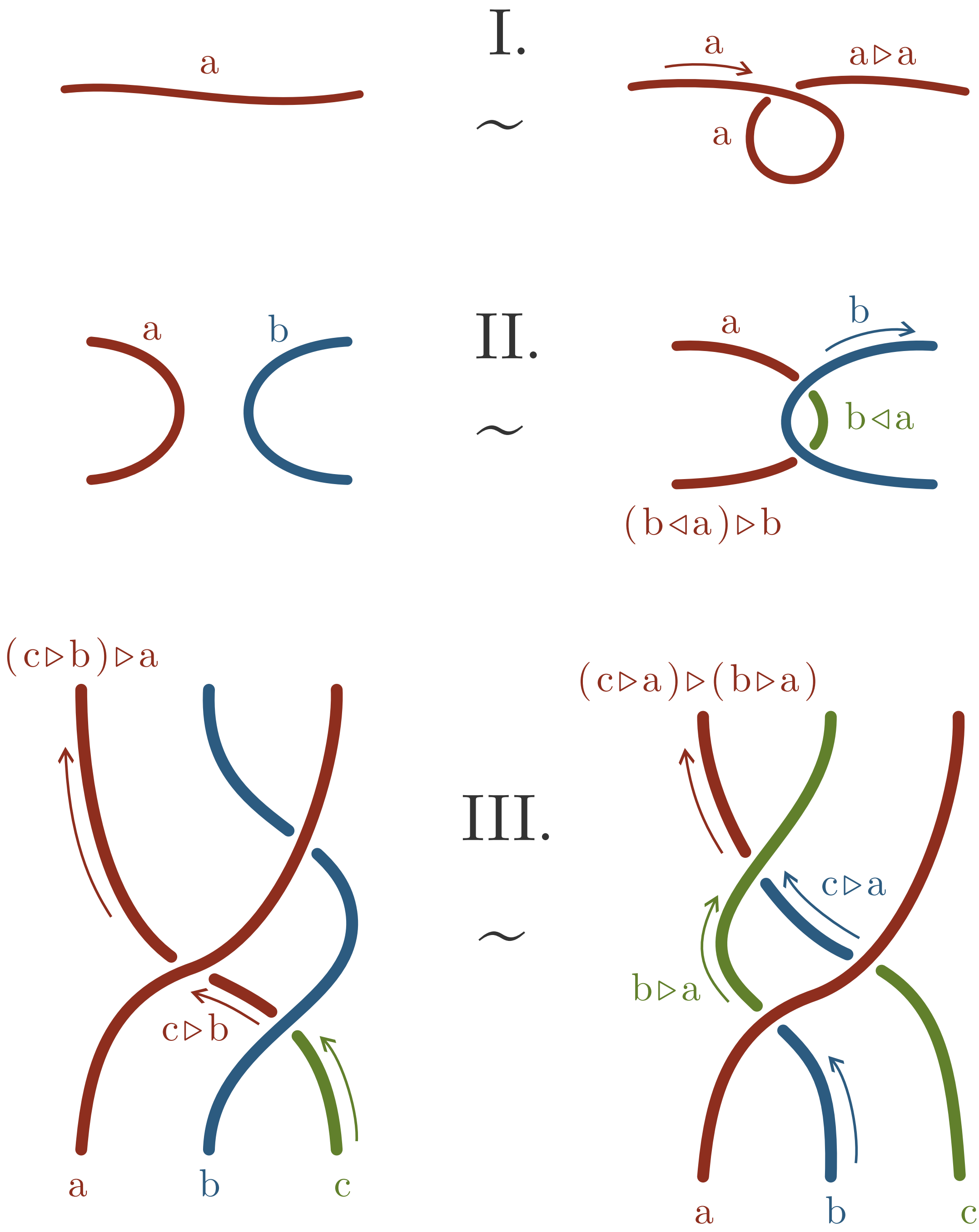

Figure 2: The three Reidemeister moves with quandle notation as well as tricolorability. As tricolorability holds (see Definition 1.8) for the three Reidemeister moves, tricolorability is a knot invariant.

Definition 1.7. The knots K_1 and K_2 are *equivalent* if there is a continuous family of homeomorphisms $\{h_u : \mathbb{R}^3 \to \mathbb{R}^3 \text{ for } 0 \leq u \leq 1\}$ such that $h_0(K_1(t)) = K_1(t)$ and $h_1(K_1(t)) = K_2(t)$.

Knots which cannot be made simpler (through the removal or smoothing of arcs) through deformation are called prime knots. The simplest knot in knot theory is the *unknot*, also known as the *trivial* knot: one single arc with no crossings. Generally the mirror images of prime knots are not included when listing them, but these mirror images may not be equivalent to their originals.

Example 1.1. The trefoil knot cannot be deformed into its mirror image (figure 4, but the figure-eight knot (figure 1) is equivalent to its own mirror image.

Aside from taking the mirror image of a knot, the orientation of a knot can be changed as well. Using the notation of a knot outlined in Definition 1.6, its copy with reversed orientation is defined as $K'(t) = K(1-t)$.

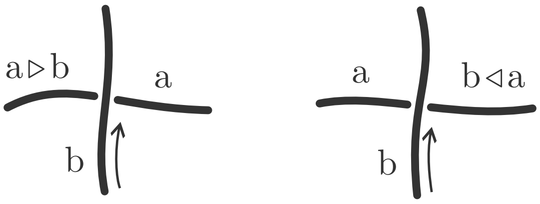

Figure 3: Two ways of representing intersections using quandles.

In order to formalize proofs concerning knot equivalences, the ways one could deform knots were classified into three moves called the Reidemeister moves (figure 2). While effective in proving knot equivalences, applying these to solving and simplifying more complex knots can take time, so algebraic and numeric representations were developed for knots, resulting in *knot invariants*. Different knot invariants can differentiate knots to different extents: the trefoil knot is not equal to its mirror image, however the Alexander and Conway polynomials, two well-known polynomial knot invariants, cannot distinguish them. One of the weaker knot invariants is tricolorability, which involves the coloring of every arc in a knot using only 3 colours.

Definition 1.8. A knot is *tricolorable* if each arc in a knot diagram is one of three colors, all three colors have been used, and at each crossing the three involved arcs must all be either different colors or the same color.

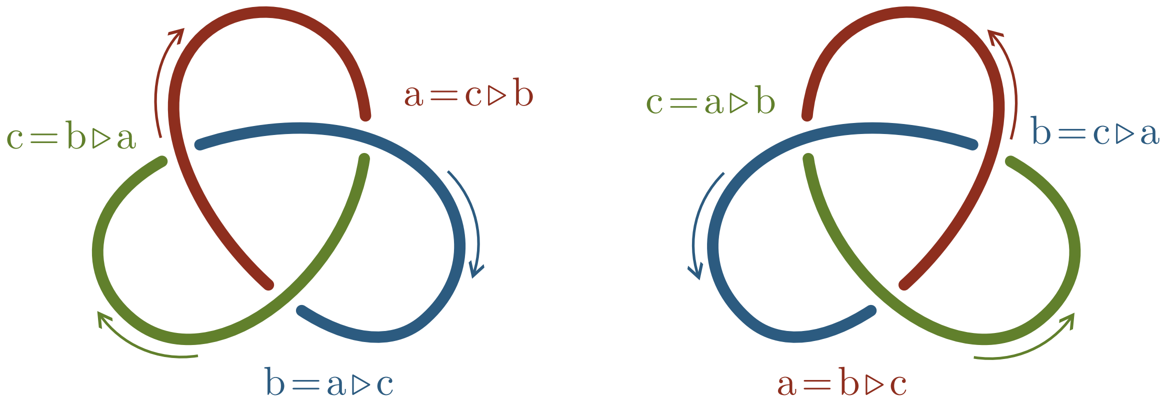

Figure 4: The (left-handed) trefoil knot and its mirror image. Its Alexander-Briggs notation is 3_1.

Its invariance under the three Reidemeister moves is seen in figure 2 and the trefoil in figure 4 is tricolored. The weakness of this knot invariant lies in its possible values: a knot is either tricolorable or non-tricolorable. This definition has been generalized to *n-colorability*, as seen in [11].

Definition 1.9. A knot is called *n-colorable* if a solution exists for the set of equations in the form of $2b = a + c \pmod{n}$, where $a, b, c \in \mathbb{Z}_n$ and b is the overcrossing arc and a, c are the two undercrossing arcs, such that at least 2 elements of $\mathbb{Z}_n$ are used.

Finally, a quandle with constraints defined by the structure of a knot is a knot invariant. Each arc is represented by an element of the quandle, and each intersection forms a constraint in the form of $a \triangleright b = c$, where a, b and c are elements of the quandle. The way in which the placement of the quandle elements is determined in these equations is shown in figure 3.

When applying this notation to the three Reidemeister moves as seen in figure 2 one can see how the axioms R1, R2, and Q3 are reflected in the three basic deformations of a knot.

Taking the n-colorability of a knot, if we rewrite the equation representing a crossing as $a = 2b - c$ we find that this knot invariant is a specialization of the quandle knot invariant, where the quandle is the dihedral quandle.

Example 1.2. Take for example the trefoil knot (figure 4). The trefoil knot is the simplest non-trivial knot and it is tricolorable. The quandle Q of the trefoil knot is defined as

$$Q = \{a, b, c : \; a = c \triangleright b, \, b = a \triangleright c, \, c = b \triangleright a\} \; .$$

The quandle of the trefoil knot's mirror image is $Q* = \{a, b, c : \; a = b \rhd c, \; b = c \rhd a, \; c = a \rhd b\}$, which is isomorphic to the quandle of the trefoil knot ($a \to b, b \to a, c \to c$): quandles as knot invariants cannot distinguish between mirror images [9].

Example 1.3. A (q, p)-torus knot, where p, q are integers such that their greatest common factor is 1, is a knot that can be embedded in $\mathbb{R}^3$ with the parametrization

$$[0, 1] \ni t \longmapsto (r \cos 2\pi qt, r \sin 2\pi qt, -\sin 2\pi pt) \in \mathbb{R}^3 \; ,$$

where $r = 2 + \cos 2\pi pt$. This type of knot is called a torus knot, because it lies on the surface of the torus defined by the equation $(r - 2)^2 + z^2 = 1$.

The trefoil knot in example 1.2 is a torus knot with the parameters $q = 2$ and $p = 3$. Figure 5 shows the form of a $(2, p')$-torus knot, where $p' = 11$ is used as the illustrative example. A $(2, p')$-torus knot has the Alexander-Briggs notation of p'_1.

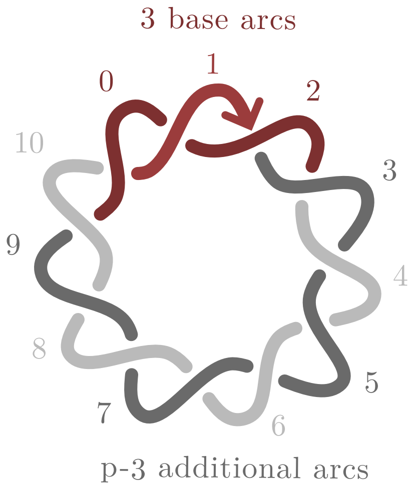

Figure 5: Example of a $(2, p)$-torus knot (p_1), illustrated with the knot $(2, 11)$-torus knot (11_1). The colours are not representative of n-colorability.

Proposition 1.1. *Let K be a $(2, p)$-torus where p is a prime number and $p \geq 3$. Then all its crossings can be computed on the p-dimensional dihedral quandle $(Q, \rhd_Q)$. Furthermore, the quandle $(Q_K, \rhd_K)$ of the knot K is defined by*

$$Q_K = \left\{ \mathbb{Z}_p : (i + 1) \rhd_K \left(i + \frac{p+1}{2} \right) = (i + 1) \rhd_Q \left(i + \frac{p+1}{2} \right) \right\} . \tag{2}$$

Proof. The intersections of the knot K take the form

$$i = (i+1) \triangleright_K \left(i + \frac{p+1}{2}\right), \quad i \in \{0, ..., p-1\},$$

as can be derived from the diagram of the knot in figure 5. Meanwhile, for $i, p \in Q$

$$\begin{aligned}(i+1) \triangleright_Q \left(i + \frac{p+1}{2}\right) &= 2i + p + 1 - (i+1) \pmod p \\ &= i + p \pmod p = i\ .\end{aligned}$$

□

1.3 Crystals as a generalization

A crystal $C(K)$ of a knot is a generalization of the fundemental group and the quandle [10]. A crystal is supplied with two operations: the right cross ($\rceil$) and the left cross ($\lceil$), but, unlike the representation of knots using quandles, the orientation of both the overcrossing and undercrossing arcs decides the representation of the intersection, as seen in figure 6.

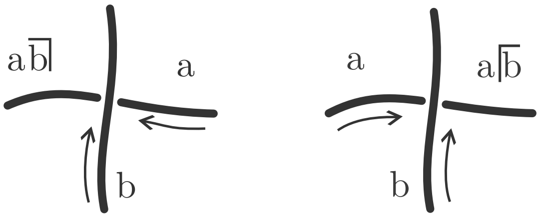

Figure 6: Intersections using the right and left cross operations.

Each arc in a knot diagram is referred to as an operand and each time an arc crosses over another it assumes the operator mode (compare the arc b to its function as the operator $\overline{b}|$ or $|\overline{b}$ in figure 6).

Since the products of intersections, such as $a\,\overline{c}|$, can participate in intersections and form more complex products (such as $a\,\overline{b\,\overline{c}|}|$). These products are referred to as operator products, with 1 representing the identity operator. Operator products are included in the crystal as well, forming a subalgebra called the operator algebra $\pi(K)$.

The crystal $C(K)$ is written as a union of the operator algebra $\pi(K)$ and the primary crystal $C_0(K)$, which is composed of all elements in the form of $a\,\overline{c}|$ and $a|\overline{c}$. As a result, the operator algebra $\pi(K)$ acts on the primary crystal $C_0(K)$ by right multiplication, and the primary crystal maps to the operator algebra when it's the arc that crosses over two others.

The crystal axioms are defined on the operator algebra based on the Reidemeister moves. As opposed to the quandle with a single operation outlined at the start of the chapter, a crystal is supplied with two operations. Due to this, each axiom has more variations. The first axiom C1 is derived from the third Reidemeister move, in which there are four variations for the arc orientations. The second crystal axiom C2 is derived from the second Reidemeister move, in which there are two distinct ways to orient the arcs. For all $a, b, c \in C(K)$,

C1 $\lceil\overline{a\lceil\overline{c}} = \overline{c}\rceil\lceil\overline{a}\,\lceil\overline{c}\;,\;\lceil\overline{a\,\overline{c}\rceil} = \lceil\overline{c}\,\lceil\overline{a}\,\overline{c}\rceil,\;\overline{a\lceil\overline{c}}\rceil = \overline{c}\rceil\,\overline{a}\rceil\lceil\overline{c}\;,\;\overline{a\,\overline{c}\rceil}\rceil = \lceil\overline{c}\,\overline{a}\rceil\,\overline{c}\rceil,$

C2 $\overline{a}\rceil\lceil\overline{a} = 1 = \lceil\overline{a}\,\overline{a}\rceil.$

The crystal axioms C1 and C2 correspond to the quandle axioms R1 and R2. Kauffman claimed, that the axioms C1 and C2 can be used to derive C3 - a crystal axiom analogous to Q3.

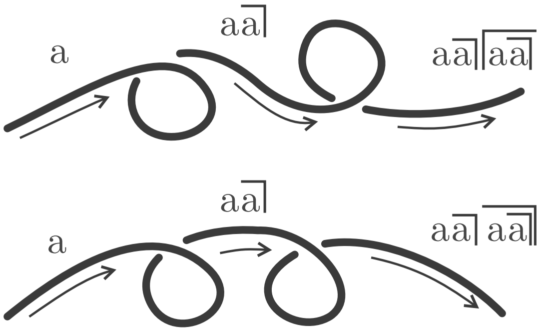

Figure 7: Two ways to twist an arc using two twists (as seen in the first Reidemeister move). Both can be simplified into smooth arcs using the first Reidemeister move twice in a row.

The top segment of a knot in figure 7 can be simplified as follows.

$$a\,\overline{a}\rceil\lceil\overline{a\,\overline{a}\rceil} = a\,\overline{a}\rceil\lceil\overline{a}\,\lceil\overline{a}\;\;\overline{a}\rceil = a\ .$$

The bottom segment of a knot in figure 7, however, cannot be simplified to that extent with just C1 and C2.

$$a\,\overline{a}\rceil\,\overline{a\,\overline{a}\rceil}\rceil = a\,\overline{a}\rceil\lceil\overline{a}\;\;\overline{a}\rceil\,\overline{a}\rceil = a\,\overline{a}\rceil\,\overline{a}\rceil\,.$$

Similarly,

$$a\overline{|a}\,\overline{|a\overline{|a}} = a\overline{|a}\;\overline{a|}\overline{|a}\,\overline{|a} = a\overline{|a}\,\overline{|a}\ .$$

By defining $a\,\overline{a|} = a \triangleright b$ and $a\overline{|b} = b \triangleleft a$ and utilizing the definitions of racks and quandles supplied with two operations, we see that a crystal with the two axioms C1 and C2 is actually the generlization of a rack. A third axiom C3 must be added in order to relate the crystal to the quandle:

C3 $a\,\overline{a|} = a = a\overline{|a}$ for all $a \in C(K)$.

2 Rack rings and algebras

The quandle ring was defined by Bardakov et al. as an analogy to the group ring [5]. The study of group rings dates back to Arthur Cayley's 1854 article on group theory [6].

With the added axiom of multiplicative commutativity and the existence of multiplicative inverses on the ring (making it a field), and finiteness on the group, the group ring becomes a group algebra: an algebra composed of the linear combinations of a finite group on a field.

While Elhamdadi et al. used quandles in place of the group in the group algebra structure [7], in this thesis the relevant results will be written in terms of rack algebras, a more general form of a quandle algebra.

This thesis uses the Einstein notation for summation: the summation symbol will be omitted when two elements have the same indices, with one index being a subscript and the other a superscript, such that

$$\sum_i a^i b_i = a^i b_i\ .$$

Definition 2.1. Let $(X, \triangleright)$ be a rack and R some associative ring. The *rack ring* $R[X]$ is a collection of elements formed through finite linear combinations of the ring R on the rack X:

$$R[X] = \left\{ a^i x_i \,:\, a^i \in R,\, x_i \in X \right\}\ ,$$

with multiplication defined as

$$x \cdot y = a^i b^j \, (x_i \triangleright y_j)\ ,$$

where $x = a^i x_i$ and $y = b^j y_j$.

Definition 2.2. Let $(R, \triangleright)$ be a rack. The *rack algebra* $\mathbb{K}[R]$, where $\mathbb{K}$ is a field, is a set of elements uniquely expressible as $a^i e_i$, with e_i, $(i \in R)$ representing the basis elements of $\mathbb{K}[R]$ equipped with addition and multiplication. Multiplication on the rack algebra $\mathbb{K}[R]$ is defined as

$$x \cdot y = x^i y^j e_{i \triangleright j} \ .$$

Note that both the rack ring and algebra are generally non-associative. The augmentation ideal I_R is defined as the kernel of the surjective algebra homomorphism

$$\varepsilon : \mathbb{K}[R] \to \mathbb{K} \ , \ \ \varepsilon(x) = \varepsilon(x^i e_i) = \sum x^i \ ,$$

as it is on group algebras, and following this analogy the basis of I_R is $\{e_i - e_j : i, j \in R,\ i \neq j\}$ where j is a fixed element of the rack R.

Proposition 2.1. *The set* $J_R := \langle e_{i \triangleright j} - e_{j \triangleright i},\ i, j \in R \rangle$ *generated by the basis* $e_{i \triangleright j} - e_{j \triangleright i} \in \mathbb{K}[R]$ *is a right ideal of the rack algebra* $\mathbb{K}[R]$

Proof. Let e_i, $i \in R$ be the basis of the rack algebra $\mathbb{K}[R]$. Then

$$(e_{i \triangleright j} - e_{j \triangleright i}) \cdot e_k = e_{(i \triangleright j) \triangleright k} - e_{(j \triangleright i) \triangleright k} = e_{(i \triangleright k) \triangleright (j \triangleright k)} - e_{(j \triangleright k) \triangleright (i \triangleright k)} \ .$$

□

3 Derivation of Rack Algebras

Derivations have been defined on various structures, such as group rings [3] and transposed Poisson algebras [4]. While the derivation on group rings could have been expanded to rack rings, the addition of a basis which enables the usage of matrices as a representation of the derivation allows us to more closely study the effects of the quandle's structure on the derivation algebra. All propositions and theorems presented in this chapter were proven by Elhamdadi et al. [7].

Definition 3.1. A linear map $D : A \to A$, where A is an algebra, is called a *derivation* on

A if it satisfies the Leibniz rule:

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y)\ ,\ x, y \in A\ .$$

Definition 3.2. A *Lie algebra* is a vector space V over a field $\mathbb{K}$ supplied with a bilinear operation called the Lie bracket $[\ ,\ ]$, such that

1. for any $x \in V$, $[x, x] = 0$ (alternativity),
2. the Jacobi identity holds: for any, $x, y, z \in V$ $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

The set of all derivations $\mathcal{D}(A)$ on an algebra A is called a *derivation algebra* and forms a Lie algebra. In the case of $\mathcal{D}(A)$ it is sufficient to define the Lie bracket as $[A, B] = A \circ B - B \circ A$, where $\circ$ is the composition operation.

The derivation on a rack algebra $\mathbb{K}[R]$ is similarly defined; it is a linear map $D : \mathbb{K}[R] \to \mathbb{K}[R]$ that obeys the Leibniz rule on the basis vectors

$$D(e_i \cdot e_j) = D(e_i) \cdot e_j + e_i \cdot D(e_j)\ .$$

A linear map on a finite-dimensional vector space with a basis can be expressed as a matrix. As this applies to algebras over fields, the derivation D on a rack algebra can be represented as an n-dimensional matrix (c^i_j), such that

$$D(e_j) = c^i_j e_i\ . \tag{3}$$

Proposition 3.1. *Let $\mathbb{K}[R]$ be the rack algebra on a finite rack R. If D is a derivation on $\mathbb{K}[R]$, then the matrix of derivation (c^i_j) in the basis e_i $(i \in R)$ must satisfy*

$$c^k_{i \triangleright j} = c^{j \triangleleft k}_i + \sum_{l \in R:\, i \triangleright l = k} c^l_j\ , \tag{4}$$

for all $i, j, k \in R$

Proof. If c^i_j represents the derivation D in the basis e_i it must satisfy the Leibniz rule outlined in Definition 3.1. For any $i, j \in R$,

$$\begin{aligned} c^k_{i \triangleright j} e_k = D(e_i \cdot e_j) &= D(e_i) \cdot e_j + e_i \cdot D(e_j) \\ &= c^{k'}_i e_{k'} \cdot e_j + e_i \cdot c^{k''}_j e_{k''} = c^{k'}_i e_{k' \triangleright j} + c^{k''}_j e_{i \triangleright k''} \, . \end{aligned}$$

Due to e_k being the basis, this can be rewritten as

$$c^k_{i \triangleright j} e_k = c^{j \triangleleft k}_i e_k + \sum_{i \triangleright k'' = k} c^{k''}_j e_k \, ,$$

where we have set $k' \triangleright j = k$ and $i \triangleright k'' = k$ in order to match the basis element of each term. Since the left inverse exists and is unique (see rack axiom R2), $k' = j \triangleleft k$. On the other hand the right inverse is not unique, or may not exist at all, so it must be written as a sum. □

This proposition was presented by Elhamdadi et al. for quandle algebras [7], but, since it did not use the third quandle condition Q3, in this thesis it has been reworded to include rack algebra. The merit of this result lies in its simplification of the calculation process of the constraints on the derivation algebra of a rack algebra.

3.1 Application on a three dimensional quandle

The goal of this subsection is to compute the constraints and subsequent derivation algebra of a given 3-dimensional quandle. Elhhamdadi et al. outlined the three isomorphism classes of 3-dimensional quandles, out of which the first is a trivial quandle and the third a dihedral quandle [7]. The second quandle isomorphism class was chosen: let Q be a three dimensional quandle on the set $\{1, 2, 3\}$, with the products $a \triangleright b$ $(a, b \in \{1, 2, 3\})$ defined according to the following Cayley table.

Let D be a derivation on the quandle algebra $\mathbb{K}[Q]$, represented by the 3×3 matrix (c^i_j). Consider all combinations of $i, j, k \in Q$ in equation (4). First, let $i = j = k = 1$. Then

$$c^1_{1 \triangleright 1} = c^{1 \triangleleft 1}_1 + \sum_{1 \triangleright l = 1} c^l_1 \, . \tag{5}$$

Referring to the Cayley table presented in table 1, we see that $1 \triangleright 1 = 1$. Furthermore,

Table 1: The Cayley table of the second 3-dimensional quandle isomorphism class outlined in [7].

$\triangleright$	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

$g \triangleright 1 = 1$ if and only if $g = 1$, so the unique element $g = 1 \triangleleft 1 = 1$. As for the right inverse, $1 \triangleright l = 1$ when $l \in \{1, 2\}$. Therefore, equation (5) becomes

$$c_1^1 = c_1^1 + c_1^1 + c_1^2 \iff c_1^2 = -c_1^1 \ .$$

When referring to a Cayley table (in this case table 1), the right inverse operation $i \triangleleft j$ is the index of the row on which its value equals j in the i^{th} column. The left inverse $l : i \triangleright l = k$, which may yield multiple values or none at all, is the set of column indices in which the i^{th} row obtains the value k.

The following is a list of all combinations of $i, j, k \in Q$ (written in the form (i, j, k)) followed by the form equation (4) takes given those elements. The elements within the square brackets are the results of the right inverse: not all combinations of quandle elements yield in a right inverse.

$$\begin{array}{rc|rc|rc}
(1,1,1): & c_1^1 = c_1^1 + [c_1^1 + c_1^2] & (1,1,2): & c_1^2 = c_1^2 + [c_1^3] & (1,1,3): & c_1^3 = c_1^3 \\
(1,2,1): & c_1^1 = c_1^1 + [c_2^1 + c_2^2] & (1,2,2): & c_1^2 = c_1^2 + [c_2^3] & (1,2,3): & c_1^3 = c_1^3 \\
(1,3,1): & c_2^1 = c_1^2 + [c_3^1 + c_3^2] & (1,3,2): & c_2^2 = c_1^1 + [c_3^2] & (1,3,3): & c_2^3 = c_2^3 \\
(2,1,1): & c_2^1 = c_2^1 + [c_1^3] & (2,1,2): & c_2^2 = c_2^2 + [c_2^1 + c_2^2] & (2,1,3): & c_2^3 = c_2^3 \\
(2,2,1): & c_2^1 = c_2^1 + [c_2^3] & (2,2,2): & c_2^2 = c_2^2 + [c_2^1 + c_2^2] & (2,2,3): & c_2^3 = c_2^3 \\
(2,3,1): & c_1^1 = c_2^2 + [c_3^3] & (2,3,2): & c_1^2 = c_2^1 + [c_3^1 + c_3^2] & (2,3,3): & c_1^3 = c_2^3 \\
(3,1,1): & c_3^1 = c_3^1 & (3,1,2): & c_3^2 = c_3^2 & (3,1,3): & c_3^3 = c_3^3 + [c_1^1 + c_1^2 + c_1^3] \\
(3,2,1): & c_3^1 = c_3^1 & (3,2,2): & c_3^2 = c_3^2 & (3,2,3): & c_3^3 = c_3^3 + [c_2^1 + c_2^2 + c_2^3] \\
(3,3,1): & c_3^1 = c_3^2 & (3,3,2): & c_3^2 = c_3^1 & (3,3,3): & c_3^3 = c_3^3 + [c_3^1 + c_3^2 + c_3^3]
\end{array}$$

Additionally, out of the $3^3 = 27$ constraints, only 9 are necessary to solve the system of equations (for example: $(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(1,3,1)$, $(1,3,2)$, $(2,3,1)$, $(3,2,3)$, and $(3,3,1)$, the rest are linearly dependent on these equations).

We have now found, that every derivation D on the chosen quandle algebra can be represented as a matrix $C = (c_j^i)$, where

$$C = \begin{pmatrix} c_1^1 & -c_1^1 & 0 \\ -c_1^1 & c_1^1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.2 Derivation algebras of dihedral quandle algebras

Having defined the quandle algebra derivation and computed the characterizations of derivation algebras for 3- and 4- dimensional quandle isomorphism classes, Elhamdadi et al. fully characterized the derivation of dihedral quandle algebras [7]. The theorem has been included without its proof to show how a derivation might be described. Since the operation of the dihedral quandle can be summed up in a single equation using modular addition as seen in equation (1), it is possible to fully utilize the Leibniz rule to characterize its quandle algebra's derivation.

Theorem 3.2. *Let $A = \mathbb{K}[Z_n]$ be a quandle algebra on a dihedral quandle and $\mathbb{K}$ a field where char $\mathbb{K} = 0$.*

1. *If n is odd the derivation algebra $\mathcal{D}(A)$ is trivial.*
2. *If n is divisible by 4, a derivation matrix $(c_i^j) \in \mathcal{D}(A)$ must satisfy the symmetries $c_{j+2k}^i = c_j^{2j+2k-i}$, $c_j^i = -c_j^{i+2k}$, and $c_j^i = c_{j+2k}^{i+2k}$ for all $i, j, k \in Z_n$.*
3. *If $n = 2k$ for some odd k the derivation matrix $(c_i^j) \in \mathcal{D}(A)$ must satisfy the symmetries $c_{j+2k}^i = c_j^{2j+2k-i}$ and $c_j^i = -c_j^{i+2k}$.*

In particular $c_j^{i+j} = 0$ for all j.

3.3 Derivation algebras of trivial quandle algebras

The following proposition coincides with proposition 3.4 in [7]. The result has been reworded to display the form a derivation matrix on a trivial quandle takes.

Proposition 3.3. *Let $\mathbb{K}[T]$ be the quandle algebra on an n-dimensional trivial quandle and field $\mathbb{K}$ where char $\mathbb{K} = 0$. Then the derivation matrix takes the form*

$$\begin{pmatrix} \sum_{l=2}^{n} c_1^l & \cdots & \sum_{l=2}^{n} c_n^l \\ c_1^2 & \cdots & c_n^2 \\ \vdots & \ddots & \vdots \\ c_1^n & \cdots & c_n^n \end{pmatrix} .$$

Proof. Let $\mathbb{K}[T]$ be the quandle algebra of an n-dimensional trivial quandle T. Let D be a derivation on $\mathbb{K}[T]$. Using Proposition 3.1 the derivation D can be characterized by

$$c_{i \triangleright j}^k = c_i^{j \triangleleft k} + \sum_{l \in R \colon i \triangleright l = k} c_j^l \,, \tag{6}$$

for every $i, j, k \in T$. Since T is the trivial quandle, $i \triangleright j = i$ and $j \triangleleft k = k$. Furthermore, $i \triangleright l = k$ yields a right inverse if and only if $i = k$, in which case the right inverse l is the set of all elements of T. Therefore, for any $i, j, k \in T$ equation (6) becomes one of the following.

$$\begin{aligned} -\sum_{l \in T} c_j^l = 0 \ , \ \ i = k \ , \\ c_i^k = c_i^k \ , \ \ i \neq k \ . \end{aligned}$$

□

4 Symmetric conjugation quandle

Instead of constructing a quandle directly, one can define a quandle on a group, as was outlined in Definition 1.5. This section discusses conjugation quandles on symmetric groups and investigates the characterization of the derivation algebra on a quandle with conjugacy classes.

The Python package SymPy [13] was used to compute the derivation algebras as well as the Cayley tables of quandles. The code can be accessed in [1].

In this thesis, it was found that while the 2-dimensional symmetric conjugation quandle is the 2-dimensional trivial quandle and its quandle algebra's derivation algebra follows Proposition 3.3, the derivation algebras of 3-, 4-, and 5-dimensional symmetric conjugation quandle algebras are trivial (null matrices). The Cayley tables and derivation algebra matrices of the conjugation quandles on subgroups of the 4-dimensional symmetric group S^4 are included in appendix 1.

4.1 Symmetric groups

Definition 4.1. The symmetric group S^n is a group comprised of all bijections on the set $\{1, ..., n\}$ supplied with the composition operation. An element of this group is called a *permutation*.

Permutations can be expressed in multiple ways. In this thesis they will be written using the disjoint notation: for example, the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix} ,$$

is expressed as (12)(456) using disjoint notation. Using disjoint notation is useful when discussing conjugacy classes on the symmetric group. First, we define the concept of conjugacy.

Definition 4.2. Let $(G, \cdot)$ be a group. Two elements $a, b \in G$ are conjugates if and only if $g^{-1} \cdot a \cdot g = b$ for some $g \in G$.

Proposition 4.1. *A conjugacy class on the n-dimensional symmetric group S^n is composed of permutations with the same amount of disjoint cycles of the same length.*

Proof. Without loss of generality, let α be a permutation in the n-dimensional symmetric group S^n composed of a single disjoint cycle $(a_1 \dots a_k)$, and let $\sigma \in S^n$. From this $\alpha(a_i) = a_{i+1 \pmod k}$. Let each element $j \in \{1, \dots, n\}$ be represented as $\sigma^{-1}(a_i)$. First let $\sigma^{-1}(a_i) \notin \{a_1, ..., a_k\}$. Then $\sigma^{-1}\alpha\sigma(\sigma^{-1}(a_i)) = \sigma^{-1}(a_i)$, because $\sigma^{-1}(a_i)$ is not a part of the cycle of α. Then let $\sigma^{-1}(a_i) \in \{a_1, ..., a_k\}$, from which

$$\sigma^{-1}\alpha\sigma(\sigma^{-1}(a_i)) = \sigma^{-1}(a_{i+1 \pmod k}) ,$$

meaning that $\sigma^{-1}\alpha\sigma = (\sigma^{-1}(a_1) \dots \sigma^{-1}(a_k))$. □

Each conjugacy class of S^n has

$$\frac{n!}{\prod_i m_i! m_i^{k_i}}$$

elements, where m_i is the amount of k_i-length disjoint cycles. It is clear from Proposition 4.1 that the permutation σ in $\sigma^{-1}\alpha\sigma = \beta$ is not unique: as long as $\sigma(a_{i_j}) = b_{l_m}$ the equality holds.

4.2 Cyclic groups

Definition 4.3. A group $(C, \cdot)$ is a (finite) n-th order cyclic group if its elements can be ordered $(e = g^0, g^1, ..., g^{n-1})$, so that $g^i = g^j$ whenever $i = j \pmod n$.

Cyclic groups are isomorphic to certain subgroups of symmetric groups, such as the subgroup of S^4 generated by the permutation (1234). For any $g^i, g^j \in C$

$$g^j \triangleright g^i = g^{-i} g^j g^i = g^{-i+j+i} = g^j \ ,$$

meaning that the cyclic group quandle is the trivial quandle. It follows, that the derivation algebra of a conjugation quandle algebra on a cyclic group must take the form outlined in Proposition 3.3.

4.3 Derivation algebras on conjugation quandle algebras with conjugacy classes

Removing the intrinsic structure of the symmetric conjugation quandle we obtain from the interaction between permutations, we are left with an n-dimensional quandle $(G, \triangleright)$ with m conjugacy classes $\{K_1, ..., K_m\}$. At least one conjugacy class (K_1) contains a single element: the identity element of the group G. In case of a quandle that has not been derived from a group there may not be a conjugacy class with a single element.

Theorem 4.2. *Let G be a group with n elements and m conjugacy classes and Q its conjugation quandle. Then the derivation algebra of the quandle algebra $\mathbb{K}[Q]$ must satisfy the following symmetry*

$$c_j^i = c_{j \triangleright k}^{i \triangleright k} \ ,$$

as well as the sums

$$\sum_{i \triangleright l = k} c_1^l = 0 \ \textit{and} \ \sum_{l \in Q} c_j^l = 0 \ ,$$

for any $i, j, k \in Q$, with 1 being the identity element of G.

Proof. Let Q be the n-dimensional conjugation quandle of a group G with m conjugacy classes $\{K_1, ..., K_m\}$. Let the quandle elements be ordered according to their conjugacy classes, and let the first conjugacy class K_1 be the identity element 1 of the group G.

The characterization of derivations found in preposition 3.1 for the product $e_i \cdot e_j$ takes the form

$$c_{i \triangleright j}^k = c_i^{j \triangleleft k} + \sum_{i \triangleright l = k} c_j^l \ .$$

Let $i \in K$ and $k \in K'$, where K, K' are conjugacy classes and $K \neq K'$, then

$$c^k_{i \triangleright j} = c^{j \triangleleft k}_i \ ,$$

because there is no right inverse in case of different conjugacy classes. This can be rewritten by defining $s = j \triangleleft k$, which is equivalent to $k = s \triangleright j$, resulting in

$$c^{s \triangleright j}_{i \triangleright j} = c^s_i \ .$$

Since 1 is the identity element of the group G, then for any $i \in S$, $i \triangleright 1 = i$ and $1 \triangleright i = 1$, as well as $1 \triangleleft i = i$ and $i \triangleleft 1 = 1$. Let $j = 1$. Then

$$c^k_{i \triangleright 1} = c^{1 \triangleleft k}_i + \sum_{i \triangleright l = k} c^l_1 \implies c^k_i = c^k_i + \sum_{i \triangleright l = k} c^l_1 \implies \sum_{i \triangleright l = k} c^l_1 = 0 \ ,$$

for any $i, k \in Q$.

Now let $k = i = 1$ instead:

$$c^1_{1 \triangleright j} = c^{j \triangleleft 1}_i + \sum_{1 \triangleright l = 1} c^l_j \implies c^1_1 = c^1_i + \sum_{l \in Q} c^l_j \implies \sum_{l \in Q} c^l_j = 0 \ .$$

□

5 Dihedral conjugation quandle

The $2n$-dimensional dihedral group is referred to as either D_n or D_{2n} depending on the field (geometry and abstract algebra respectively). In this thesis it will be referred to as D_n. The description of the dihedral quandle algebra is followed by an original theorem proving, that in case of an odd n the derivation algebra of the dihedral conjugation quandle algebra is trivial.

The derivation algebras of several dihedral conjugation quandle algebras were computed using [1] and can be seen in appendix 1.

Definition 5.1. The $2n$-dimensional *dihedral group* D_n is the group of symmetries on a regular n-polygon, made up of n rotations and n reflections. The group operation is composition, which is defined by the formulae

$$r_i r_j = r_{i+j} \ , \ \ r_i s_j = s_{i+j} \ , \ \ s_i r_j = s_{i-j} \ , \ \ s_i s_j = r_{i-j} \ ,$$

where $\{r_0, ..., r_{n-1}\}$ is the set of rotations on an n-polygon and $\{s_0, ..., s_{n-1}\}$ the set of possible reflections (symmetries). All addition and subtraction operations in the subscripts are performed with modulus n.

Due to the usage of modular arithmetic, the derivation algebra of a dihedral conjugation quandle algebra can be studied in a way similar to that of the dihedral quandle. As the element r_0 is the unit element, the inverse of x_i is x_{-i}, because $r_i r_{-i} = r_{-i} r_i = r_{i-i} = r_0$. Similarly, the inverse of s_i is s_i: $s_i s_i = r_{i-i} = r_0$. Knowing the inverses, we can now describe the conjugation operation used as the quandle operation $\triangleright$:

$$\begin{aligned}
&r_i \triangleright r_j = r_{-i} r_j r_i = r_{j-i} r_i = r_i \ , \\
&r_i \triangleright s_j = s_j r_i s_j = s_{j-i} s_j = r_{j-i-j} = r_{-i} \ , \\
&s_i \triangleright r_j = r_{-j} s_i r_j = s_{i-j} r_j = s_{i-j-j} = s_{i-2j} \ , \\
&s_i \triangleright s_j = s_j s_i s_j = r_{j-i} s_j = s_{j-i+j} = s_{2j-i} \ .
\end{aligned}$$

Similarly, we can find the right inverse values represented by $\triangleleft$:

$$\begin{aligned}
&r_j \triangleleft r_i = r_j r_i r_{-j} = r_i \ , \\
&r_j \triangleleft s_i = r_j s_i r_{-j} = s_{j+i} r_{-j} = s_{j+i+j} = s_{i+2j} \ , \\
&s_j \triangleleft r_i = s_j r_i s_j = s_{j-i} s_j = r_{j-i-j} = r_{-i} \ , \\
&s_j \triangleleft s_i = s_j s_i s_j = r_{j-i} s_j = s_{j-i+j} = s_{2j-i} \ .
\end{aligned}$$

The dihedral conjugation quandle has two conjugacy classes: $\{r_i\}$ and $\{s_i\}$.

Theorem 5.1. *Let $A_D = \mathbb{K}[D_n]$ be a quandle algebra on a dihedral group conjugation quandle and $\mathbb{K}$ a field where char $\mathbb{K} = 0$. Let n be odd. Then the derivation algebra of A_D is trivial.*

Proof. Let D_n be the dihedral group and $A_D = \mathbb{K}[D_n]$ its conjugation quandle algebra, where $\mathbb{K}$ is a field where char $\mathbb{K} = 0$. Let D be some derivation on A_D. Since A_D is finite we can use

Proposition 3.1 to outline the constraints on the derivation D using all possible three element combinations of the basis elements of A_D. The dihedral conjugation quandle is composed of two conjugacy classes, so there are $2^3 = 8$ possible variations of the characterization of derivations from preposition 3.1. Note, that for an odd n the set $\{\pm i \pm 2j \pmod{n} : j = 0, ..., n-1\}$ is equal to $\mathbb{Z}_n$ for any $i \in \mathbb{Z}_n$. First are the four cases in which there are no right inverses: i and k are from different conjugacy classes.

Let $i \to r_i$ represent replacing the index i in equation (3) with the i-th rotation, and $i \to s_i$ with the i-th reflection. Since addition and multiplication for the indices of the dihedral group's elements are modular, when choosing a general index i for a dihedral group element r_i or s_i we will write $i \in \mathbb{Z}_n$.

(1.1) $i, j \to r_i, r_j$ and $k \to s_k$

$$c^{s_k}_{r_i \triangleright r_j} = c^{r_j \triangleleft s_k}_{r_i} + \sum_{l : r_i \triangleright l = s_k} c^{l}_{r_j} \iff c^{s_k}_{r_i} = c^{s_{2j+k}}_{r_i}$$

(1.2) $i \to r_i$ and $j, k \to s_j, s_k$

$$c^{s_k}_{r_i \triangleright s_j} = c^{s_j \triangleleft s_k}_{r_i} + \sum_{l : r_i \triangleright l = s_k} c^{l}_{s_j} \iff c^{s_k}_{r_{-i}} = c^{s_{2j-k}}_{r_i}$$

(1.3) $j, k \to r_j, r_k$ and $i \to s_i$

$$c^{r_k}_{s_i \triangleright r_j} = c^{r_j \triangleleft r_k}_{s_i} + \sum_{l : s_i \triangleright l = r_k} c^{l}_{r_j} \iff c^{r_k}_{s_{i-2j}} = c^{r_k}_{s_i}$$

(1.4) $k \to r_k$ and $i, j \to s_j, s_i$

$$c^{r_k}_{s_i \triangleright s_j} = c^{s_j \triangleleft r_k}_{s_i} + \sum_{l : s_i \triangleright l = r_k} c^{l}_{s_j} \iff c^{r_k}_{s_{2j-i}} = c^{r_{-k}}_{s_i}$$

Then we consider the cases in which i and k are both from the rotation conjugacy class $\{r_i\}$. For both equation (2.1) and (2.2) if $i = k$ then $l = \{r_0, ..., r_{n-1}\}$, if $i = -k$ then $l = \{s_0, ..., s_{n-1}\}$. Otherwise, the sum is trivial.

(2.1) $i, j, k \to r_i, r_j, r_k$

$$c^{r_k}_{r_i \triangleright r_j} = c^{r_j \triangleleft r_k}_{r_i} + \sum_{l : r_i \triangleright l = r_k} c^{l}_{r_j} \iff c^{r_k}_{r_i} = c^{r_k}_{r_i} + \sum_{l : r_i \triangleright l = r_k} c^{l}_{r_j}$$

(2.2) $i, k \to r_i, r_k$ and $j \to s_j$

$$c^{r_k}_{r_i \triangleright s_j} = c^{s_j \triangleleft r_k}_{r_i} + \sum_{l : r_i \triangleright l = r_k} c^{l}_{s_j} \iff c^{r_k}_{r_{-i}} = c^{r_{-k}}_{r_i} + \sum_{l : r_i \triangleright l = r_k} c^{l}_{s_j}$$

Finally, we consider the two cases in which i and k are from the reflection conjugacy class $\{s_i\}$. For equations (3.1) and (3.2) there are two terms in the sum, because the right inverse can be either $s_i \triangleright r_l = s_{i-2l} = s_k$ from which $i - 2l = k \pmod{n}$, or $s_i \triangleright s_l = s_{2l-i} = s_k$, from which $2l - i = k \pmod{n}$.

(3.1) $j \to r_j$ and $i, k \to s_i, s_k$

$$c^{s_k}_{s_i \triangleright r_j} = c^{r_j \triangleleft s_k}_{s_i} + \sum_{l : s_i \triangleright l = s_k} c^{l}_{r_j} \iff c^{s_k}_{s_{i-2j}} = c^{s_{2j+k}}_{s_i} + \sum_{l : s_i \triangleright l = s_k} c^{l}_{r_j}$$

(3.2) $i, j, k \to s_i, s_j, s_k$

$$c^{s_k}_{s_i \triangleright s_j} = c^{s_j \triangleleft s_k}_{s_i} + \sum_{l : s_i \triangleright l = s_k} c^{l}_{s_j} \iff c^{s_k}_{s_{2j-i}} = c^{s_{2j-k}}_{s_i} + \sum_{l : s_i \triangleright l = s_k} c^{l}_{r_j}$$

First, consider the non-diagonal blocks of the derivation matrix ($(c^{s_i}_{r_j})$ and $(c^{r_i}_{s_j})$): equations (1.1) through (1.4) give the non-diagonal blocks a layered structure. It follows from (1.1) and (1.3), that

$$c^{s_j}_{r_i} = c^{s_0}_{r_i} \text{ and } c^{r_i}_{s_j} = c^{r_i}_{s_0} \; . \tag{7}$$

Furthermore, from setting $k = j = 0$ in equation (1.2) and $j = i = 0$ in equation (1.4), we get the additional symmetries

$$c^{s_0}_{r_{-i}} = c^{s_0}_{r_i} \text{ and } c^{r_k}_{s_0} = c^{r_{-k}}_{s_0} \; . \tag{8}$$

This means, that for every $i \in \mathbb{Z}_n$

$$c^{s_0}_{r_i} = c^{s_j}_{r_{-i}} = c^{s_j}_{r_i} \text{ and } c^{r_j}_{s_0} = c^{r_j}_{s_i} = c^{r_{-j}}_{s_i} \ .$$

Equations (2.1) and (2.2) provide us with four possible sums. As discussed, the sum terms of both (2.1) and (2.2) can be

$$\sum_{l=0}^{n-1} c^{r_l}_{v} \text{ and } \sum_{l=0}^{n-1} c^{s_l}_{v} \ , \tag{9}$$

where $v \in D_n$ and the former when $i = k$, the latter when $i = -k$. Since n is odd the two do not exclude each other only when $i = k = 0$.

The two $c^{r_k}_{r_i}$ terms in equation (2.1) cancel each other

$$c^{r_k}_{r_i} = c^{r_k}_{r_i} + \sum_{l:r_i \triangleright l = r_k} c^{l}_{r_j} \implies \sum_{l=0}^{n-1} c^{r_l}_{r_j} = 0 \text{ and } \sum_{l=0}^{n-1} c^{s_l}_{r_j} = 0 \ . \tag{10}$$

By combining this with $c^{s_j}_{r_i} = c^{s_0}_{r_i}$, we find that $c^{s_i}_{r_j} = 0$, because

$$0 = \sum_{l=0}^{n-1} c^{s_l}_{r_j} = \sum_{l=0}^{n-1} c^{s_0}_{r_j} = n c^{s_0}_{r_j} \ . \tag{11}$$

For equation (2.2), let us first consider the three main cases: $i = k = 0$, $i = k, i \neq 0$, and $i = -k, i \neq 0$. The first case gives us the equation

$$c^{r_0}_{r_0} = c^{r_0}_{r_0} + \sum_{l=0}^{n-1} c^{r_l}_{s_j} + c^{s_l}_{s_j} \iff \sum_{l=0}^{n-1} c^{r_l}_{s_j} + c^{s_l}_{s_j} = 0 \ . \tag{12}$$

When $i = k$ and $i \neq 0$, then

$$c^{r_i}_{r_{-i}} = c^{r_{-i}}_{r_i} + \sum_{l=0}^{n-1} c^{r_l}_{s_j} \ , \tag{13}$$

and when $i = -k$ and $i \neq 0$ instead,

$$c^{r_{-i}}_{r_{-i}} = c^{r_i}_{r_i} + \sum_{l=0}^{n-1} c^{s_l}_{s_j} \ . \tag{14}$$

For all other i, k that are unrelated

$$c^{r_k}_{r_{-i}} = c^{r_{-k}}_{r_i} \ . \tag{15}$$

Finally, equations (3.1) and (3.2) can be written as

$$c^{s_k}_{s_{i-2j}} = c^{s_{2j+k}}_{s_i} + c^{s_l}_{r_j} + c^{r_{l'}}_{r_j} \text{ and } c^{s_k}_{s_{2j-i}} = c^{s_{2j-k}}_{s_i} + c^{s_l}_{s_j} + c^{r_{l'}}_{s_j} \ , \tag{16}$$

respectively, where $i - 2l' = k$ and $2l - i = k$. Since $c^{s_i}_{r_j} = 0$. Furthermore, by transforming i, k into $2i, 2k$, we can isolate l, l': $2l = 2k + 2i$ and $l' = 2i - 2k$, and taking into account the fact that $c^{s_i}_{r_j} = 0$, equations (3.1) and (3.2) can be written as

$$c^{s_{2k}}_{s_{2i-2j}} = c^{s_{2j+2k}}_{s_{2i}} + c^{r_{i-k}}_{r_j} \ , \tag{17}$$

$$c^{s_{2k}}_{s_{2j-2i}} = c^{s_{2j-2k}}_{s_{2i}} + c^{s_{k+i}}_{s_j} + c^{r_{i-k}}_{s_j} \ . \tag{18}$$

When $i - k = j$ in equation (17),

$$c^{s_{2k}}_{s_{2k}} = c^{s_{2i}}_{s_{2i}} + c^{r_{i-k}}_{r_{i-k}} \ .$$

further defining $2k = 2i - 2m$ enables us to investigate the diagonal elements $c^{r_j}_{r_j}$. For any $i, m \in \mathbb{Z}_n$

$$c^{s_{2i-2m}}_{s_{2i-2m}} = c^{s_{2i}}_{s_{2i}} + c^{r_m}_{r_m} \ . \tag{19}$$

Fixing $i = m = 0$ we get $c^{r_0}_{r_0} = 0$. Now let us fix $i = 0$, then

$$c^{s_{2m}}_{s_{2m}} = c^{s_0}_{s_0} + c^{r_m}_{r_m} \ . \tag{20}$$

At the same time

$$c^{s_{2m}}_{s_{2m}} = c^{s_{2(m-1)+2}}_{s_{2(m-1)+2}} = c^{s_{2(m-1)}}_{s_{2(m-1)}} + c^{r_1}_{r_1} \ , \tag{21}$$

from which

$$c^{s_0}_{s_0} + m c^{r_1}_{r_1} = c^{s_0}_{s_0} + c^{r_m}_{r_m} \ ,$$

since equation (21) can be nested into the left side of equation (20). As a result, $c^{r_j}_{r_j} = j c^{r_1}_{r_1}$.

This can be further inserted into equation (20), such that

$$c_{s_{2m}}^{s_{2m}} = c_{s_0}^{s_0} + m c_{r_1}^{r_1} \ . \tag{22}$$

This can be used in equation (18) by fixing $i + k = j$. Then

$$c_{s_{2j-2i}}^{s_{2(j-i)}} = c_{s_{2i}}^{s_{2i}} + c_{s_j}^{s_j} + c_{s_j}^{r_{2i-j}} \ , \tag{23}$$

which can be simplified using equation (22), giving us

$$c_{s_0}^{s_0} + (j-i) c_{r_1}^{r_1} = c_{s_0}^{s_0} + i c_{r_1}^{r_1} + c_{s_0}^{s_0} + j c_{r_1}^{r_1} + c_{s_j}^{r_{2i-j}} \ . \tag{24}$$

From this, every element $c_{s_j}^{r_k}$ can be expressed as

$$c_{s_j}^{r_{i-j}} = -c_{s_0}^{s_0} - i c_{r_1}^{r_1} \ , \tag{25}$$

where $i = k + j$.

Looking back at equation (2.2), specifically the case in which $i = -k$ presented in equation (14), the equation $c_{r_j}^{r_j} = j c_{r_1}^{r_1}$ can be used to simplify it, such that

$$-i c_{r_1}^{r_1} = i c_{r_1}^{r_1} + \sum_{l=0}^{n-1} c_{s_j}^{s_l} \implies -2i c_{r_1}^{r_1} = \sum_{l=0}^{n-1} c_{s_j}^{s_l} \ , \tag{26}$$

for all $i, j \in D_n$. Focusing on the value of i, we see that for a fixed $j \in D_n$ and $i, i' \in D_n$ such that $i \neq i'$,

$$-2i c_{r_1}^{r_1} = \sum_{l=0}^{n-1} c_{s_j}^{s_l} = -2i' c_{r_1}^{r_1} \ . \tag{27}$$

This only holds when $c_{r_1}^{r_1} = 0$, so

$$\sum_{l=0}^{n-1} c_{s_j}^{s_l} = 0 \ . \tag{28}$$

Due to equation (12), this implies, that

$$\sum_{l=0}^{n-1} c_{s_j}^{r_l} = 0 \ , \tag{29}$$

as well as

$$\sum_{l'=0}^{n-1} c_{s_j}^{r'_l - j} = -n c_{s_0}^{s_0} - \sum_{l'=0}^{n-1} l' c_{r_1}^{r_1} = -n c_{s_0}^{s_0} = 0 \tag{30}$$

based on equation (25).

Since all elements in the form of $c_{r_j}^{s_i}$ or $c_{s_j}^{r_i}$ are equal to zero, equation (18) can be used to study the second main diagonal block ($c_{s_j}^{s_i}$). First we will transform it by defining $a = i + k$, such that

$$c_{s_{2j-2i}}^{s_{2(a-i)}} = c_{s_{2i}}^{s_{2j-2(a-i)}} + c_{s_j}^{s_a} \ . \tag{31}$$

Let us fix $i = 0$, then

$$c_{s_j}^{s_a} = c_{s_{2j}}^{s_{2a}} - c_{s_0}^{s_{2j-2a}} \ . \tag{32}$$

This equation can be inserted into itself by transforming a, j to $2a, 2j$, until we attain $2na, 2nj$ which are both equal to zero, as was done with equations (20) and (21),

$$c_{s_j}^{s_a} = c_{s_0}^{s_0} - \sum_{m=1}^{n} c_{s_0}^{s_{2m(j-a)}} \ . \tag{33}$$

Since n is odd (from which the sets $\{2m(j-a),\ m \in \mathbb{Z}_n\}$ and $\mathbb{Z}_n$ are equivalent), and $c_{s_0}^{s_0} = 0$, this is equivalent to

$$c_{s_j}^{s_a} = - \sum_{m'=1}^{n-1} c_{s_0}^{s_{m'}} \ . \tag{34}$$

but putting this into equation (14), while keeping in mind that $c_{r_i}^{r_i} = 0$ for any $i \in \mathbb{Z}_{\ltimes}$, results in the sum being equal to 0. Therefore, all elements $c_{s_j}^{s_i}$ in the form of, and by extension $c_{r_j}^{r_i}$ thanks to equation (18), are equal to zero. $\square$

Conclusion

This thesis provides an overview on quandles, their relation to knots, a partial summary of the article by Elhamdadi et al. on quandle algebra derivations [7], and two main results: one concerning conjugation quandles with conjugacy classes and the other dihedral conjugation quandles specifically.

The 2-dimensional symmetric group S^2 results in the 2-dimensional trivial quandle; its derivation algebra follows from proposition 3.3. The three subsequent symmetric groups S^3, S^4, S^5 yield non-trivial quandles and trivial derivation algebras. The derivation algebras of conjugation quandle algebras on subgroups of S^4 and 4- to 16-dimensional dihedral groups were computed as well, with the resulting quandle Cayley tables (only for the subgroups of S^4) and derivation algebra representations outlined in appendix 1. This was done using [1].

Theorem 4.2 characterized the derivation algebras of conjugation quandles on groups with conjugacy classes with one symmetry and two sums. Theorem 5.1 characterized the derivation algebras of n-dimensional dihedral conjugation quandle algebras for an odd n, proving that the derivation algebra is trivial.

The derivation algebras of quandles similar to the dihedral quandle can be considered for further study - specifically those that can be defined using general equations (unlike the symmetric group, for which one must rely on the Cayley table). For example, one such possibility is the class of dicyclic groups Q_{4n}, of which Q_8 is isomorphic to the quaternion group Q.

The characterization of derivations in equation (3) cannot be easily extended to quandles that are knot invariants unless the quandle is specified: the quandle invariant of a knot with n crossings only defines up to n relations (in the form of $a \triangleright b = c$) and the presence of these undefined $\triangleright$ products greatly raises the complexity of solving the system of equations defining the derivation algebra. One potential solution is using the dihedral quandle, relating the derivation algebra of the (n-dimensional dihedral) knot quandle to all n-colorable knots. Alternatively the undefined products could be filled: one could start out with a trivial quandle and redefine the products $a \triangleright b$ which are defined on the knot at some crossing. In order to preserve the second quandle axiom R2, for every crossing $a \triangleright b = c$ one would also have to redefine the product $a \triangleright c = b$.

Using this method of defining the full structure of a knot quandle and studying the derivation algebra of its quandle algebra could provide insight into the applications of the derivation algebra.

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Appendix 1. Computational results

The code used to perform all computations for derivation algebra on conjugation quandles can be found in [1]. This was done using the SymPy package [13].

5.1 S^4 subgroups

5.1.1 2-dimensional subgroups

The only 2-dimensional subgroup of S^4 is S^2 in S^4, a subgroup generated by double transposition. There is only one possible 2-dimensional quandle: the trivial quandle. The cyclic group C_2 is trivial as well. The matrix of the Cayley table is as follows

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

and the resulting derivation algebra contains elements in the form of

$$\begin{pmatrix} -c_1^2 & -c_2^2 \\ c_1^2 & c_2^2 \end{pmatrix}.$$

5.1.2 3-dimensional subgroups

The alternating group A_3 in S_4 is isomorphic to the cyclic group C_3. Their Cayley table is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

and derivation

$$\begin{pmatrix} -c_1^2 - c_1^3 & -c_2^2 - c_2^3 & -c_3^2 - c_3^3 \\ c_1^2 & c_2^2 & c_3^2 \\ c_1^3 & c_2^3 & c_3^3 \end{pmatrix}.$$

5.1.3 4-dimensional subgroups

The cyclic group C_4 in S^4, normal Klein four-subgroup of S^4, non-normal Klein four-subgroup of S^4 are all isomorphic to the cyclic group C_4, with the Cayley table

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

and derivation

$$\begin{pmatrix} -c_1^2 - c_1^3 - c_1^4 & -c_2^2 - c_2^3 - c_2^4 & -c_3^2 - c_3^3 - c_3^4 & -c_4^2 - c_4^3 - c_4^4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \\ c_1^4 & c_2^4 & c_3^4 & c_4^4 \end{pmatrix}.$$

5.1.4 6-dimensional subgroups

The symmetric group S^3 in S^4 is isomorphic to S^3. As a result, the Cayley table is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 2 & 2 & 2 \\ 4 & 6 & 5 & 4 & 6 & 5 \\ 5 & 4 & 6 & 6 & 5 & 4 \\ 6 & 5 & 4 & 5 & 4 & 6 \end{bmatrix}$$

and the derivation algebra is trivial.

5.1.5 8-dimensional subgroups

The dihedral group D_4 in S_4 has the Cayley table

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 8 & 8 & 8 & 8 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 6 & 4 & 4 & 6 & 4 & 6 & 6 \\ 5 & 7 & 5 & 7 & 5 & 7 & 5 & 7 \\ 6 & 4 & 6 & 6 & 4 & 6 & 4 & 4 \\ 7 & 5 & 7 & 5 & 7 & 5 & 7 & 5 \\ 8 & 8 & 8 & 2 & 2 & 2 & 2 & 8 \end{bmatrix}$$

and derivation

$$\begin{pmatrix} -c_1^3 & -c_8^3 & -c_3^3 & -c_6^3 & -c_5^3 & -c_6^3 & -c_5^3 & -c_8^3 \\ 0 & c_8^8 & 0 & 0 & 0 & 0 & 0 & -c_8^8 \\ c_1^3 & c_8^3 & c_3^3 & c_6^3 & c_5^3 & c_6^3 & c_5^3 & c_8^3 \\ 0 & 0 & 0 & c_6^6 & 0 & -c_6^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_7^7 & 0 & -c_7^7 & 0 \\ 0 & 0 & 0 & -c_6^6 & 0 & c_6^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_7^7 & 0 & c_7^7 & 0 \\ 0 & -c_8^8 & 0 & 0 & 0 & 0 & 0 & c_8^8 \end{pmatrix},$$

which is different from the one seen in the next section, as the elements of D_4 in S_4 have not been ordered in the same manner as the elements in the later D_4.

5.1.6 12-dimensional subgroups

The final, and largest, subgroup of S^4 is the alternating group A_4 in S_4. Its derivation algebra is trivial.

5.2 Dihedral conjugation quandles

The Cayley tables have been omitted, as the multiplication rules were outlined in section 5.

The $2{\cdot}2 = 4$-dimensional dihedral conjugation quandle is isomorphic to the 4-dimensional trivial quandle, and therefore has the same derivation algebra

$$\begin{pmatrix} c_1^1 & c_2^1 & c_3^1 & c_4^1 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \\ -c_1^1 - c_1^2 - c_1^3 & -c_2^1 - c_2^2 - c_2^3 & -c_3^1 - c_3^2 - c_3^3 & -c_4^1 - c_4^2 - c_4^3 \end{pmatrix}.$$

The 6-, 10-, and 12-dimensional dihedral conjugation quandle algebra $\mathbb{K}[D_3], \mathbb{K}[D_5], \mathbb{K}[D_7]$ has a trivial derivation algebra.

The derivation algebra of $\mathbb{K}[D_4]$ is nontrivial, but the off-diagonal block $(C_{r_j}^{s_i})$ is equal to a zero-matrix

$$\begin{pmatrix} c_1^1 & c_2^1 & c_3^1 & c_2^1 & c_5^1 & c_6^1 & c_5^1 & c_6^1 \\ 0 & c_2^2 & 0 & -c_2^2 & 0 & 0 & 0 & 0 \\ -c_1^1 & -c_2^1 & -c_3^1 & -c_2^1 & -c_5^1 & -c_6^1 & -c_5^1 & -c_6^1 \\ 0 & -c_2^2 & 0 & c_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5^5 & 0 & -c_5^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6^6 & 0 & -c_6^6 \\ 0 & 0 & 0 & 0 & -c_5^5 & 0 & c_5^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_6^6 & 0 & c_6^6 \end{pmatrix}.$$

The derivation algebra of $\mathbb{K}[D_6]$ is

$$\begin{pmatrix}
c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_3^1 & c_2^1 & c_7^1 & c_8^1 & c_7^1 & c_8^1 & c_7^1 & c_8^1 \\
c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & c_6^2 & c_7^2 & c_8^2 & c_7^2 & c_8^2 & c_7^2 & c_8^2 \\
-c_1^2 & -c_6^2 & -c_5^2 & -c_4^2 & -c_3^2 & -c_2^2 & -c_7^2 & -c_8^2 & -c_7^2 & -c_8^2 & -c_7^2 & -c_8^2 \\
-c_1^1 & -c_2^1 & -c_3^1 & -c_4^1 & -c_3^1 & -c_2^1 & -c_7^1 & -c_8^1 & -c_7^1 & -c_8^1 & -c_7^1 & -c_8^1 \\
-c_1^2 & -c_2^2 & -c_3^2 & -c_4^2 & -c_5^2 & -c_6^2 & -c_7^2 & -c_8^2 & -c_7^2 & -c_8^2 & -c_7^2 & -c_8^2 \\
c_1^2 & c_6^2 & c_5^2 & c_4^2 & c_3^2 & c_2^2 & c_7^2 & c_8^2 & c_7^2 & c_8^2 & c_7^2 & c_8^2 \\
c_1^7 & c_2^7 & c_3^7 & c_4^7 & c_3^7 & c_2^7 & c_7^7 & c_8^7 & c_9^7 & c_{10}^7 & c_9^7 & c_8^7 \\
-c_1^7 & -c_2^7 & -c_3^7 & -c_4^7 & -c_3^7 & -c_2^7 & -c_9^7 & -c_{10}^7 & -c_9^7 & -c_8^7 & -c_7^7 & -c_8^7 \\
c_1^7 & c_2^7 & c_3^7 & c_4^7 & c_3^7 & c_2^7 & c_9^7 & c_8^7 & c_7^7 & c_8^7 & c_9^7 & c_{10}^7 \\
-c_1^7 & -c_2^7 & -c_3^7 & -c_4^7 & -c_3^7 & -c_2^7 & -c_7^7 & -c_8^7 & -c_9^7 & -c_{10}^7 & -c_9^7 & -c_8^7 \\
c_1^7 & c_2^7 & c_3^7 & c_4^7 & c_3^7 & c_2^7 & c_9^7 & c_{10}^7 & c_9^7 & c_8^7 & c_7^7 & c_8^7 \\
-c_1^7 & -c_2^7 & -c_3^7 & -c_4^7 & -c_3^7 & -c_2^7 & -c_9^7 & -c_8^7 & -c_7^7 & -c_8^7 & -c_9^7 & -c_{10}^7
\end{pmatrix}$$

The derivation algebra of $\mathbb{K}[D_8]$ is

$$\begin{pmatrix}
c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 & c_4^1 & c_3^1 & c_2^1 & c_9^1 & c_{10}^1 & c_9^1 & c_{10}^1 & c_9^1 & c_{10}^1 & c_9^1 & c_{10}^1 \\
c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & c_6^2 & c_7^2 & c_8^2 & c_9^2 & c_{10}^2 & c_9^2 & c_{10}^2 & c_9^2 & c_{10}^2 & c_9^2 & c_{10}^2 \\
0 & c_2^3 & c_3^3 & c_4^3 & 0 & -c_4^3 & -c_3^3 & -c_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1^2 & -c_8^2 & -c_7^2 & -c_6^2 & -c_5^2 & -c_4^2 & -c_3^2 & -c_2^2 & -c_9^2 & -c_{10}^2 & -c_9^2 & -c_{10}^2 & -c_9^2 & -c_{10}^2 & -c_9^2 & -c_{10}^2 \\
-c_1^1 & -c_2^1 & -c_3^1 & -c_4^1 & -c_5^1 & -c_4^1 & -c_3^1 & -c_2^1 & -c_9^1 & -c_{10}^1 & -c_9^1 & -c_{10}^1 & -c_9^1 & -c_{10}^1 & -c_9^1 & -c_{10}^1 \\
-c_1^2 & -c_2^2 & -c_3^2 & -c_4^2 & -c_5^2 & -c_6^2 & -c_7^2 & -c_8^2 & -c_9^2 & -c_{10}^2 & -c_9^2 & -c_{10}^2 & -c_9^2 & -c_{10}^2 & -c_9^2 & -c_{10}^2 \\
0 & -c_2^3 & -c_3^3 & -c_4^3 & 0 & c_4^3 & c_3^3 & c_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_1^2 & c_8^2 & c_7^2 & c_6^2 & c_5^2 & c_4^2 & c_3^2 & c_2^2 & c_9^2 & c_{10}^2 & c_9^2 & c_{10}^2 & c_9^2 & c_{10}^2 & c_9^2 & c_{10}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_9^9 & c_{10}^9 & 0 & -c_{10}^9 & -c_9^9 & -c_{10}^9 & 0 & c_{10}^9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_9^{10} & c_{10}^{10} & c_9^{10} & 0 & -c_9^{10} & -c_{10}^{10} & -c_9^{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{10}^9 & c_9^9 & c_{10}^9 & 0 & -c_{10}^9 & -c_9^9 & -c_{10}^9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_9^{10} & 0 & c_9^{10} & c_{10}^{10} & c_9^{10} & 0 & -c_9^{10} & -c_{10}^{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_9^9 & -c_{10}^9 & 0 & c_{10}^9 & c_9^9 & c_{10}^9 & 0 & -c_{10}^9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_9^{10} & -c_{10}^{10} & -c_9^{10} & 0 & c_9^{10} & c_{10}^{10} & c_9^{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{10}^9 & -c_9^9 & -c_{10}^9 & 0 & c_{10}^9 & c_9^9 & c_{10}^9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_9^{10} & 0 & -c_9^{10} & -c_{10}^{10} & -c_9^{10} & 0 & c_9^{10} & c_{10}^{10}
\end{pmatrix}.$$

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