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Non-unital Morita equivalence in a bicategorical setting

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Non-unital Morita equivalence in a bicategorical setting

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Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Tartu, Estonia

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Supervisor:

Prof. Valdis Laan
University of Tartu
Tartu, Estonia

Opponents:

Gabriella Böhm, scientific advisor, D.Sc.
Wigner Research Centre for Physics
Budapest, Hungary

Laiachi El Kaoutit, Lecturer “Profesor Titular de Universidad”, PhD
University of Granada
Granada, Spain

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Contents

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Chapter 1

Introduction

1.1 Purpose

The purpose of this thesis is to study the Morita theory of structures such as non-unital rings or semigroups in settings of varying generality. Our goal is to prove Morita style theorems that would apply to at least these two examples and at a level of generality that seems most appropriate for the given result.

1.2 Overview

The tensor product structure on the bimodules between unital rings plays a central role in Morita theory of rings. It is therefore no surprise that Morita theory seems to have a natural home in the bicategory of rings and bimodules, which has the tensor product of bimodules as composition. Several results about the Morita theory of unital rings are simple consequences of the fact that the notion of Morita equivalence coincides with the notion of equivalence of rings as objects of the bicategory of bimodules.

Since non-unital rings and bimodules do not form a bicategory, the same approach does not directly work. While one can restrict attention to certain types of rings and bimodules, which do form a bicategory, it is worth trying to generalize the non-unital case to a more general setting. We do this by slightly relaxing the notion of a bicategory, by allowing the units of the composition to be lax. Various kinds of lax versions of bicategories and monoidal categories have been defined and studied in the past. They have differing structural maps that are chosen to be non-invertible and differing direction for the structural maps. From our perspective, the most notable might be the paper [5] by A. Burroni, where he defined pseudocategories which are essentially bicategories where the invertibility of all structural maps is omitted. The unitors in the case of pseudocategories go in the opposite direction compared to us.

For our purposes, we only want to relax the invertibility of the unitors and not the associators. When the associators are not invertible, Morita theory related matters are greatly complicated and it becomes very hard to prove anything general that would be useful in the applications that are our main motivations.

In lax-unital categories, we will study right wide Morita contexts, which were defined for bicategories by El Kaoutit in [8]. It is difficult to prove things for lax-unital bicategories in general. We usually want the unitors to be at least epimorphic. We will study the relationship between a lax-unital bicategory and its local full lax-unital subbicategory determined by the 1-cells for which the unitors are suitably good epimorphisms or isomorphisms. We will do so by constructing adjunction between the hom-categories of the respective lax-unital bicategories. We will also study how right wide Morita contexts act with respect to these constructions. We will call the 1-cells with invertible unitors firm and the 1-cells with unitors belong to a fixed class of epimorphisms unitary. This is motivated by usage in the case of non-unital rings and modules. The term 'firm' originates from Quillen, who took to calling such modules firm in [27], although the concept does not originate from him. Modules with this property seem to have been first introduced by Taylor in [32].

The strongest assumptions we will put on the lax-unital bicategory will be that the epimorphic and monomorphic 2-cells of each morphism category form an orthogonal factorization system and that the functor of tensoring with a 1-cell always has a right adjoint. This property of that right adjoint always existing is usually called the closedness of the bicategory. Under these assumptions we will prove that when there exists a right wide Morita context between two 1-cells with epimorphic maps, then we can construct an equivalence of categories between certain hom-categories.

After that we will explore our main method of constructing lax-unital bicategories. This will be the construction of the lax-unital bicategory of bimodules between semigroup objects in a monoidal category. We will show that this construction indeed results in a lax-unital category and study under what assumptions on the monoidal category are the prerequisites of the results of the preceding section, on lax-unital bicategories, satisfied. We will prove an Eilenberg-Watts theorem for firm bimodules between firm semigroup objects in a monoidal category. This proof is based on the work of Bodo Pareigis in his series of papers on the Morita theory of monoid objects in a monoidal category [23], [24], [25], where he proved a similar result in the unital case.

1.3 On notation

We note the following things about our notation for bicategories:

- we will write the composition of 1-cells of a (lax-unital) bicategory from left to right and the composition of 2-cells from right to left,
- as is customary, we will usually omit the subscripts from the natural 2-cells $a_{M,N,L}$, l_M and r_M,
- we will write $\mathcal{B}_0$ for the collection of objects of a (lax-unital) bicategory $\mathcal{B}$,
- we will often denote the composition of 2-cells f, g in $\mathcal{B}(A, B)$ by $f \circ g$,
- sometimes, especially when talking about the semigroups and modules, we will write ${}_A\mathcal{B}_B$ instead of $\mathcal{B}(A, B)$ for the hom-categories of $\mathcal{B}$ and use the tensor sign for the 1-cell composition to separate it from the abstract notion.

Below, $\mathcal{E}$ will denote a suitably nice class of epimorphic 2-cells of $\mathcal{B}$. The condition that this class should satisfy vary from section to section, but it should always satisfy at least the conditions given in section 2.4.

When talking about semigroups in a monoidal category $\mathcal{V}$, we will also extend the notation $\mathcal{E}$ to a class of epimorphisms in $\mathcal{V}$, since in the context of that section, a morphism between modules will belong to $\mathcal{E}$ if and only if it will belong to $\mathcal{E}$ as a morphism of $\mathcal{V}$.

Chapter 2

Lax-unital bicategories and Morita contexts

2.1 Lax-unital bicategories

In this section we will introduce Lax-unital bicategories as a slight generalization of the familiar notion of a bicategory. The contents of this section is based on the author's first paper [28]. Bicategories in some sense capture the abstract aspects of an associative tensor product with units. A typical example, and a one to which we will often refer, is the bicategory of bimodules over unital rings along with their tensor product. Bicategories were introduced by Jean Bénabou in 1967 in his seminal paper [2]. The notion of a bicategory generalizes (strict) 2-categories, which are essentially bicategories where all structural 2-cells are identities. In some sense, the two notions are equivalent, in that any bicategory is biequivalent to a strict 2-category. Bicategories however are often somewhat more natural to use, because one does not always want to "strictify" the naturally occurring examples of bicategories.

While there are many interesting examples of bicategories besides the bicategory of bimodules over unital rings, we will very rarely be using bicategories other than that. One of the main reasons is that Morita theory originates from this context and it will be the main motivating example for everything in this thesis. The bicategory of bimodules has also been an important example of a bicategory since the beginning and it was noticed early on that the Morita equivalence of rings is the same as the equivalence of rings as objects of the bicategory of bimodules. Since then, several aspects of the Morita theory of rings have been generalized and studied in the bicategorical setting. For example the Eilenberg-Watts theorem by Niels Johnson in his thesis [11] and Morita contexts by El Kaoutit in [8].

Bicategories are a nice context to define and prove results about various notions. One of these notions is the notion of a right wide Morita context which was introduced by L. El Kaoutit in his paper [8] in 2008. These generalize Morita contexts that are familiar to us from the Morita theory of rings with an identity. The definition of a Morita context for unital rings also makes sense for rings without an identity. Morita contexts are useful and serve a similar purpose in both cases.

While the theory of Morita contexts and the Morita equivalence of rings without an identity element is very similar to the one of rings with an identity and it might seem that the bicategorical approach to Morita theory would work well in this case, we are hindered by the fact that non-unital rings and bimodules do not actually form a bicategory. The tensor product of bimodules between non-unital rings is associative and coherent, it even has nice candidates for unit modules, but unfortunately, it actually does not in general have a unit for the tensor product and hence does not give us a bicategory.

Since one would still like to abstract the Morita theory, one approach would be to simply forget about tensor units altogether and use semibicategories, or"bicategories without unit 1-cells". It turns out however that the unit elements were quite useful for defining certain notions and the lack of these makes the developement of a good general theory difficult. Examining the situation more closely one can notice that the tensor product of bimodules between non-unital rings does have what might be called *lax units*, meaning a distinguished bimodule for each ring, such that there is a coherent natural transformation between the functors of tensoring with that module and the unit functor. In the case of a bicategory this natural transformation would be invertible, making these disinguished bimodules the unit bimodules, but a lot of definitions make sense even when these natural trasformations, called *unitors*, are not invertible.

With that motivation, we define *lax-unital bicategories* as "bicategories with non-invertible unitors" and study how well we can generalize the Morita theory of non-unital rings to this context.

2.2 Definition

A bicategory $\mathcal{B}$ is essentially an abstract composition structure

$$\mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$$

on a family of categories indexed by some class, along with some more structural data that determines its properties. For the bicategory of bimodules over unital rings, the composition functor is simply the tensor product functor

$$- \otimes - \colon {}_R\mathrm{Mod}_T \times {}_T\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S\,,\ \ ({}_RM_T, {}_TN_S) \mapsto {}_R(M \otimes_T N)_S\,.$$

Now we will give the exact definition of a *lax-unital bicategory.* Note that it will resemble the definition of a category quite closely, except sets are replaced by categories and properties are replaced by some distinguished 2-cells satisfying coherence conditions. For that reason structures of this type are sometimes called “higher categories”.

Definition 1 (Lax-unital bicategory)**.** A *lax-unital bicategory* $\mathcal{B}$ is given by the following data:

- a collection $\mathcal{B}_0$, which will be the collection of *objects of* $\mathcal{B}$,
- for each pair of objects $A, B \in \mathcal{B}_0$, a category $\mathcal{B}(A, B)$, the objects and morphisms of which are called the 1*-cells and* 2*-cells of* $\mathcal{B}$ respectively,
- for each triple of objects $A, B, C \in \mathcal{B}_0$, a composition functor

  $$\mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C),$$

  taking a pair of composable 1-cells (M, N) to a 1-cell denoted by MN and a pair of composable 2-cells (f, g) to a 2-cell denoted by fg,
- for each object $A \in \mathcal{B}_0$, a distinguished 1-cell $I_A \in \mathcal{V}(A, A)$, called the *lax unit of* A; we will occasionally just call these 1-cells *units*,
- for each quadruple of objects $A, B, C, D \in \mathcal{B}_0$ a collection of 2-cells

  $$a_{M,N,L} \colon (MN)L \to M(NL)$$

  natural in $M \in \mathcal{B}(A, B)$, $N \in \mathcal{B}(B, C)$, $L \in \mathcal{B}(C, D)$, called the *associators*,
- for each pair of objects $A, B \in \mathcal{B}_0$, two collections of 2-cells

  $$l_M \colon I_A M \to M, \quad r_M \colon M I_B \to M$$

  natural in $M \in \mathcal{B}(A, B)$, called the *left* and *right unitors* respectively.

The morphisms $a_{M,N,L}$ are required to be invertible, but the morphisms l_M and r_M are not. The natural morphisms $a_{M,N,L}$, l_M and r_M need to be such that the diagrams

$$\begin{tikzcd}
((MN)L)K \arrow[r, "a_{MN,L,K}"] \arrow[d, "a_{M,N,L}1_K"] & (MN)(LK) \arrow[dr, "a_{M,N,LK}"] & \\
 & & M(N(LK)), \\
(M(NL))K \arrow[r, "a_{M,NL,K}"'] & M((NL)K) \arrow[ur, "1_M a_{N,L,K}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(MI)N \arrow[d, "a_{M,I,N}"'] \arrow[dr, "r_M 1_N"] & \\
M(IN) \arrow[r, "1_M l_N"'] & MN
\end{tikzcd}$$

commute and the diagrams

$$\begin{tikzcd}
(MN)I \arrow[d, "a_{M,N,I}"'] \arrow[dr, "r_{MN}"] & \\
M(NI) \arrow[r, "1_M r_N"'] & MN,
\end{tikzcd}
\quad
\begin{tikzcd}
(IM)N \arrow[d, "a_{I,M,N}"'] \arrow[dr, "l_M 1_N"] & \\
I(MN) \arrow[r, "l_{MN}"'] & MN,
\end{tikzcd}
\quad
\begin{tikzcd}
II \arrow[r, bend left, "l_I"] \arrow[r, bend right, "r_I"'] & I
\end{tikzcd}$$

commute.

When l and r are invertible, this structure is called a *bicategory*.

Remark 1. While it might be considered bad form to have the notion of *bicategory* be a special case of the notion of *lax-unital bicategory*, it does seem to be the most natural name for the concept, since a *lax bicategory* would be one where additionally even the structural 2-cells a would not need to be invertible. There are also unbiased notions of a lax bicategory, for example the one in Leinster's [21]. Our notion of a lax-unital bicategory can actually be seen as a special case of the unbiased lax bicategory of [21], but that observation is not that helpful for our cause.

Remark 2. While in the definition above the definition of a lax-unital bicategory differs from that of a bicategory in only the invertibility of r and l, the last three diagrams are usually not included in the definition of a bicategory, since their commutativity follows from the first two when r and l are invertible.

Remark 3. We note that one could also define some kind of bicategory with lax units, where the direction of the structural 2-cells r and l is different. Our choice of direction is derived from our main example: the bicategory of the bimodules over non-unital rings. If the direction of both r and l was reversed in all definitions, the structure we get is in a sense dual to a lax-unital bicategory, and so we call it an *oplax-unital bicategory.* Therefore, the usual duality on a bicategory that reverses all 2-cells, when applied to a lax-unital bicategory, results in an oplax-unital bicategory. In other words, everything we define and prove for lax-unital bicategories will have a corresponding dual for oplax-unital bicategories.

Remark 4. Compared to the last remark one could also imagine a structure where the direction of only one of the natural structural 2-cells, l and r, is reversed. Such a situation for monoidal categories was studied by Kornél Szlachányi in [29]. We note that in his case, the associators a were not invertible. Since he called such structures *skew-monoidal categories*, we would suggest calling the bicategorical structures where the unitors r and l go in opposite directions *skew-unital bicategories.*

Example 1. Our motivating example of a lax-unital bicategory is the following one, denoted by Mod.

- The objects of Mod are the associative rings which do not have necessarily an identity element.
- For rings $R, S \in \mathrm{Mod}_0$, $\mathrm{Mod}(R,S) =: {}_R\mathrm{Mod}_S$ is the category of R-S-bimodules and bimodule homomorphisms.
- The composition functor ${}_R\mathrm{Mod}_S \times {}_S\mathrm{Mod}_T$ is given by

$$\begin{aligned}({}_RM_S, {}_SN_T) &\mapsto {}_R(M \otimes_S N)_T,\\ (f,g) &\mapsto f \otimes g.\end{aligned}$$

- The associators ${}_R((M \otimes_S N) \otimes_T L)_U \to {}_R(M \otimes_S (N \otimes_T L))_U$ are given by

$$(m \otimes n) \otimes x \mapsto m \otimes (n \otimes x).$$

  It is well known and easy to show that such maps are natural isomorphisms making the necessary diagrams commute.
- If $R, S \in \mathrm{Mod}_0$ and ${}_RM_S$ is an R-S-bimodule then $I_R = {}_RR_R$, $I_S = {}_SS_S$ and the unitors are given by

$$\begin{aligned}l_M &: {}_R(R \otimes_R M)_S \to {}_RM_S, \quad r \otimes m \mapsto rm,\\ r_M &: {}_R(M \otimes_S S)_S \to {}_RM_S, \quad m \otimes s \mapsto ms.\end{aligned}$$

  These homomorphisms of bimodules are natural in M, but not necessarily invertible, injective or surjective. It can be verified that the required diagrams commute. In particular, obviously $l_R = r_R$.

A non-additive analogue of this is the lax-unital bicategory of semigroups, where 1-cells are biacts and 2-cells are biact homomorphisms.

2.3 Coherence

The reason why three additional diagrams are included in the definition of a lax-unital bicategory is because we want the analogue of the bicategorical *coherence* theorem to hold. That is to say, we want any two 2-cells that are the results of composing the 2-cells l, r, a and 1 in various ways to coincide whenever their domains are formally the same and codomains are formally the same. The bicategorical version of the result, when originally proven [22], included the additional three diagrams in our definition. It was later shown by Kelly [13] that these diagrams were redundant in the bicategorical case.

One can check that MacLane's proof [22] of the coherence theorem works for lax-unital bicategories. Additionally one can check that Kelly's proof [13] of the redundancy of the last three diagrams holds when the composition functor of the lax-unital bicategory preserves epimorphic 2-cells and the unitors l and r are epimorphisms.

2.4 Unitarity

Until the end of this thesis, $\mathcal{E}$ will denote a class of epimorphic 2-cells of the lax-unital bicategory $\mathcal{B}$ under discussion. We will require $\mathcal{E}$ to satisfy:

E1. $\mathcal{E}$ is closed under the vertical composition of 2-cells,

E2. all invertible 2-cells belong to $\mathcal{E}$,

E3. if $f \circ g$ and g are in $\mathcal{E}$ then so is f,

E4. $\mathcal{E}$ is closed under the horizontal composition of 2-cells.

The last condition means that 1-cell composition functor of $\mathcal{B}$ maps members of $\mathcal{E}$ into $\mathcal{E}$, in other words if $f\colon M \to N$ is a 2-cell in $\mathcal{E}$, then the horizontal composites $f1$ and $1f$ are also in $\mathcal{E}$.

The main example for the class $\mathcal{E}$ is the class of all epimorphic 2-cells in the case of non-unital rings and also in the case of semigroups. It is well known that the epimorphisms between bimodules or biacts are precisely the surjective homomorphisms, and it is easy to verify that conditions **E1**–**E4** are satisfied for them. The reason we have a class $\mathcal{E}$ instead of just epimorphisms is because while in Set and Ab epimorphisms are surjective, this is not always the case. The class $\mathcal{E}$ is meant to model, if needed, some stronger kind of epimorphism, for example surjective morphisms, which are often what we want $\mathcal{E}$ to be.

Motivated by [1] in the case of rings and for example [20] in the case of semigroups, we will make the following definition

Definition 2. We will call a 1-cell M of a lax-unital bicategory $\mathcal{B}$ *left (right) unitary* if the left (right) unitor of M belongs to $\mathcal{E}$ and we call M *unitary* if it is both left and right unitary.

We will call an object A of a lax-unital bicategory $\mathcal{B}$ *(left/right) unitary* if its unit 1-cell I_A is (left/right) unitary.

We note that in the case of semigroups in Set, semigroups that are unitary in our sense are usually called factorizable semigroups and in the case of semigroups in Ab they are called idempotent rings. Therefore unitarity does not actually imply that the semigroup object has an identity element.

If $\mathcal{B}$ is a lax-unital bicategory we let $\mathcal{B}^U$ denote the full lax-unital subbicategory of $\mathcal{B}$ determined by the unitary objects. Additionally, let $LU\mathcal{B}$, $RU\mathcal{B}$ and $U\mathcal{B}$ be the lax-unital locally full subbicategories determined by left unitary, right unitary and unitary objects and 1-cells of $\mathcal{B}$ respectively. The fact that the collection of 1-cells of $RU\mathcal{B}$ is closed under composition can be seen from the following diagram, which commutes by coherence:

$$\begin{tikzcd}
(MN)I \arrow[r, "a"] \arrow[dr, "r_{MN}"'] & M(NI) \arrow[d, "1_M r_N"] \\
& MN.
\end{tikzcd}$$

In a similar fashion, one can show that the $LU\mathcal{B}$ is closed with respect to the composition of 1-cells and therefore the same for $U\mathcal{B}$ follows.

It is entirely possible that given a lax-unital bicategory, all of the lax-unital subbicategories that we listed are empty, meaning that there are no unitary objects. However in the examples that we are interested in, there always exists at least one unitary object. In the lax-unital bicategory of bimodules between semigroup objects in a monoidal category, the unit semigroup on the unit object of the monoidal category always has invertible unitors, which means that the unitors must belong to $\mathcal{E}$.

For 1-cells between unitary objects, there is an easy way of making them unitary. We simply compose the given 1-cell with the unit 1-cell on the side that we want to unitarize. Indeed, since given a 1-cell $M\colon A \to B$, the following diagram commutes by coherence

$$\begin{tikzcd}
(MI)I \arrow[d, "a"] \arrow[dr, "r_{MI}"] & \\
M(II) \arrow[r, "1_M r_I"'] & MI,
\end{tikzcd}$$

the right unitor of MI is in $\mathcal{E}$ if the lax unit of B is unitary. Indeed, by the assumptions on the collection $\mathcal{E}$, if r_I belongs to $\mathcal{E}$, then $1_M r_I$ must also belong to $\mathcal{E}$.

Therefore for each pair of unitary objects A, B of $\mathcal{B}$, we have a functor

$$-I\colon \mathcal{B}^U(A,B) \to RU\mathcal{B}(A,B)$$

that acts by composing with I on the right and turns 1-cells into right unitary 1-cells. Of course we also have functors

$$I-\colon \mathcal{B}^U(A,B) \to LU\mathcal{B}(A,B) \quad \text{and} \quad (I-)I\colon \mathcal{B}^U(A,B) \to U\mathcal{B}(A,B)\,,$$

which have similar effect. Because of coherence the functor $(I-)I$, which is the composition of $I-$ and $-I$, is isomorphic to the functor $I(-I)$, which is the composition of $-I$ and $I-$.

Unitary objects have a certain closure property with respect to so called "$\mathcal{E}$-images".

Proposition 1. *Suppose that M and N are 1-cells of a lax-unital bicategory $\mathcal{B}$ such that there exists a 2-cell $e\colon M \to N$ belonging to $\mathcal{E}$. Then whenever M is (left/right) unitary, N will also be (left/right) unitary.*

Proof. Let us suppose that M is right unitary and write out the naturality diagram of the right unitor for $e\colon M \to N$:

$$\begin{tikzcd} MI \arrow[r, "e1"] \arrow[d, "r_M"'] & NI \arrow[d, "r_N"] \\ M \arrow[r, "e"'] & N. \end{tikzcd}$$

Every 2-cell in the diagram except for r_M belongs to $\mathcal{E}$ by assumption. Therefore by using the property **E3** of $\mathcal{E}$, we get that r_N must also belong to $\mathcal{E}$, making N right unitary. □

2.5 Firmness

The unitarity of a bicategory is often not enough to prove what we want. One of strongest things that we can require of the unitors is their invertibility. Indeed, if all the unitors of a lax-unital bicategory are invertible, we are dealing with a bicategory. Following [27], we will call the property of a 1-cell having invertible unitors, firmness. We can make a definition analogous to the unitary case

Definition 3. We will call a 1-cell M of a lax-unital bicategory $\mathcal{B}$ *left (right) firm* if the left (right) unitor of M is invertible and we call M *firm* if it is both left and right firm.

We will call an object A of a lax-unital bicategory $\mathcal{B}$ (left/right) *firm* if its unit 1-cell I_A is (left/right) firm.

So we can see that when a lax-unital bicategory is such that all of its objects and 1-cells a firm, we get something familiar.

Proposition 2. *When all the objects and* 1*-cells of a lax-unital bicategory are firm, it is a bicategory.*

Again, following suit from the unitary case, if $\mathcal{B}$ is a lax-unital bicategory, $\mathcal{B}^F$ will denote the the full lax-unital subbicategory of $\mathcal{B}$ determined by the firm objects and we let $LF\mathcal{B}$, $RF\mathcal{B}$ and $F\mathcal{B}$ denote the lax-unital locally full subbicategories determined by left firm, right firm and firm objects and 1-cells respectively. As we noted, $F\mathcal{B}$, since all of its unitors are invertible, is a bicategory.

As in the unitary case, the listed lax-unital subbicategories could all be empty, but in the example of bimodules between semigroup objects in a monoidal category, the unit semigroup will always be firm.

Note that when for 1-cells between firm objects, we can turn them into firm 1-cells just as in the case with unitary modules. Because of the same digram

$$\begin{tikzcd} (MI)I \arrow[d, "a"] \arrow[dr, "r_{MI}"] & \\ M(II) \arrow[r, "1_M r_I"'] & MI, \end{tikzcd}$$

when A and B are firm objects of $\mathcal{B}$, we have the functors

$$I-\colon \mathcal{B}^F(A,B) \to LF\mathcal{B}(A,B)\,, \quad -I\colon \mathcal{B}^F(A,B) \to RF\mathcal{B}(A,B)$$

and

$$(I-)I\colon \mathcal{B}^F(A,B) \to F\mathcal{B}(A,B).$$

In this case however these functor have slightly better properties.

Proposition 3. *All of the functors listed above are right adjoint to the inclusion functors. Therefore the hom-categories determined by the firm* 1*-cells are coreflective subcategories of the full hom-categories.*

This fact is very useful in transfering data and properties between $\mathcal{B}^F$ and $F\mathcal{B}$.

To give a concrete description of these adjunctions, it suffices if we provide the unit and the counit. The adjunction

$$\begin{tikzcd} \mathcal{B}^F(A,B) \arrow[r, bend left, "-I"] \arrow[r, phantom, "\top"] & RF\mathcal{B}(A,B) \arrow[l, bend left, "\text{inclusion}"] \end{tikzcd}$$

has $r_A \colon AI \to A$ as its counit and $r_A^{-1} \colon A \to AI$ as its unit. The fact that they satify the triangle identities is easily checkable using coherence. The units and counits of the other adjunctions are analogous.

This can be seen as a consequence of the idempotence of the functor $-I_A$ for a firm object A. To be more precise, we have the following notion of idempotence for a functor:

Definition 4. An endofunctor F on a category $\mathcal{C}$ along with a natural map $\xi_A : A \to F(A)$, with $\xi_{F(A)} = F(\xi_A) \colon F(A) \to F(F(A))$ invertible is called an **idempotent pointed endofunctor** on $\mathcal{C}$. The dual notion is called an **idempotent copointed endofunctor**.

This definition and the proof of next Lemma below about (co)reflective subcategories can be found in Section 5.1 of [7]. We note that idempotent (co)pointed endofunctors are in a sense a concept equivalent to *idempotent (co)monads.* For idempotent monads see for example Proposition 4.2.3 of [4]. The condition $\xi_{F(A)} = F(\xi_A)$ for a pointed endofunctor is called *well-pointedness* and is actually implied by the idempotence condition of $\xi_{F(A)}$ and $F(\xi_A)$ being invertible (Lemma 4.1.2. of [12]).

If κ is a natural transformation between functors with domain $\mathcal{C}$, we say that an object A of $\mathcal{C}$ is fixed by κ when κ_A is invertible. Let $\mathrm{Fix}(\mathcal{C},\kappa)$ denote the full subcategory of $\mathcal{C}$ induced by the objects fixed by κ. Clearly it is a replete subcategory of $\mathcal{C}$. We then have the following lemma, the proof of which is trivial.

Lemma 1. *If (F,ξ) is an idempotent (co)pointed endofunctor on $\mathcal{C}$, then $\mathrm{Fix}(\mathcal{C},\xi)$ is a (co)reflective subcategory of $\mathcal{C}$ with (co)reflection given by the corestriction of F to $\mathrm{Fix}(\mathcal{C},\xi)$. The adjunction unit in the copointed case is given by $\xi_A^{-1} \colon A \to F(A)$.*

The abovementioned corestriction will be denoted by $F|^{\mathrm{Fix}(\mathcal{C},\xi)}$. Note that when we think of F as a monad, then the above lemma is essentially about the free-forgetful adjunction of the Eilenberg-Moore category $\mathrm{Fix}(\mathcal{C},\xi)$ of F.

Note that if we take F to be the functor $-I\colon \mathcal{B}^F(A,B) \to \mathcal{B}^F(A,B)$ and ξ to be r in the last lemma, then it does produce $RF\mathcal{B}(A,B)$ as a coreflective subcategory of $\mathcal{B}^F(A,B)$,

2.6 Right wide Morita contexts

The definition of a right wide Morita context in a lax-unital bicategory is quite similar to the definition of an adjoint pair in a bicategory, the difference being the direction of one of maps. Even diagrams that need to commute are the same up to the direction of maps. In that sense, one can think of right wide Morita contexts as a skew version of an adjoint pair.

Right wide Morita contexts were first defined in [8] by El Kaoutit in the context of bicategories. However it is easy to see, that while the definition includes unitors, it does not require the unitors to be invertible. Indeed, this observation was one the main motivations for studying Morita theory in lax-unital bicategories.

Definition 5 (Right wide Morita context)**.** Let A and B be objects in a lax-unital bicategory $\mathcal{B}$. A *right wide Morita context* from A to B is a quadruple $\Gamma = (P, Q, \theta, \phi)$, with 1-cells

$$P\colon A \to B, \qquad Q\colon B \to A$$

and 2-cells

$$\theta\colon PQ \to I_A, \qquad \phi\colon QP \to I_B\,,$$

such that the following two diagrams commute

$$\begin{tikzcd}
Q(PQ) \arrow[r, "1\theta"] & QI \arrow[dr, "r"] & \\
 & & Q, \\
(QP)Q \arrow[uu, "a"] \arrow[r, "\phi 1"'] & IQ \arrow[ur, "l"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
P(QP) \arrow[r, "1\phi"] & PI \arrow[dr, "r"] & \\
 & & P. \\
(PQ)P \arrow[uu, "a"] \arrow[r, "\theta 1"'] & IP \arrow[ur, "l"'] &
\end{tikzcd}$$

When θ and ϕ are invertible, we will call Γ an *adjoint equivalence* and call it firm when P and Q are firm. If Γ is a firm adjoint equivalence, we will call Γ a *Morita equivalence.* When P and Q are unitary 1-cells and the 2-cells θ and ϕ belong to $\mathcal{E}$, we will call Γ an $\mathcal{E}$*-Morita context.*

We can think of the maps θ and ϕ in the last definition as mixed multiplication maps and of the two commutative diagrams in the definition as mixed associativity laws for these multiplication maps.

As we noted, an adjoint pair in a bicategory differs from the definition we just gave in that one of the arrows is going in the opposite direction, so the definition of an adjoint pair also makes sense in a lax-unital bicategory. This means that when θ and ϕ in the last definition are invertible, this data does actually give us an adjoint pair between the same objects in the obvious way. Since an adjoint pair where the corresponding 2-cells are isomorphisms is called an *adjoint equivalence*, it is somewhat justified that we instead call a wide Morita context with θ and ϕ invertible an *adjoint equivalence*.

If we reverse the direction of both of the 2-cells θ and ϕ in the definition of a right wide Morita context, the commutativity of the diagrams in the definition starts to make less sense. To get something sensible out of that, we would also need to reverse the direction of the structural 2-cells r and l (or at least one of them), which would mean that a left wide Morita context wants to live in a oplax-unital bicategory (or skew-unital, if needed).

Remark 5. Pécsi in [26] noticed, that right wide Morita contexts, as just defined, can be viewed as normalized lax functors from Iso, the category with two objects and precisely one morphism between any two objects, into the bicategory $\mathcal{B}$. The consequences of this observation were later studied by Lack [18]. The definition of a lax functor will be recalled in section 2.10.

2.7 The lax-unital bicategory of right wide Morita contexts

It was observed by El Kaoutit in [8] that the right wide Morita contexts in a bicategory $\mathcal{B}$ form themselves a bicategory $\mathrm{RC}(\mathcal{B})$. Going over El Kaoutit's proof that they actually form a bicategory, one can easily observe that the only place where it was required that the unitors r and l be invertible was to show that the unitors of the bicategory of right wide Morita contexts were invertible. Therefore we have the following

Proposition 4. *When $\mathcal{B}$ is a lax-unital bicategory, there exists a lax-unital bicategory $\mathrm{RC}(\mathcal{B})$ of right wide Morita context in $\mathcal{B}$.*

Since the proof of that fact transfers essentially unchanged and is somewhat lengthy and routine, we will not reproduce it here. We will however list how the operations of the lax-unital bicategory $\mathrm{RC}(\mathcal{B})$ are defined.

Obviously the objects of the bicategory of right wide Morita contexts in $\mathcal{B}$ are the same as the objects of $\mathcal{B}$ and the 1-cells are the right wide Morita contexts between these objects. First there needs to be a notion of morphism between parallel right wide Morita contexts. This will define the 2-cells of the bicategory. If $\Gamma_1 = (P_1, Q_1, \theta_1, \phi_1)$ and $\Gamma_2 = (P_2, Q_2, \theta_2, \phi_2)$ are a parallel pair of right wide Morita contexts from A to B, then a morphism of from Γ_1 to Γ_2 consists of two 2-cells $p\colon P_1 \to P_2$ and $q\colon Q_1 \to Q_2$ making the diagrams

$$\begin{tikzcd} P_1Q_1 \arrow[rr, "pq"] \arrow[dr, "\theta_1"'] & & P_2Q_2 \arrow[dl, "\theta_2"] \\ & I & \end{tikzcd}, \qquad \begin{tikzcd} Q_1P_1 \arrow[rr, "qp"] \arrow[dr, "\phi_1"'] & & Q_2P_2 \arrow[dl, "\phi_2"] \\ & I & \end{tikzcd}$$

commute.

The vertical composition of these 2-cells is defined in the natural way, simply composing the respective components of the 2-cell. This definition makes it obvious that the components of the identity morphism of $\Gamma = (P, Q, \theta, \phi)$ should be the the identity morphims of P and Q.

Next we need a way to compose the 1-cells. Let A, B and C be objects of $\mathcal{B}$, let $\Gamma_1 = (P_1, Q_1, \theta_1, \phi_1)$ be a right wide Morita context from A to B and let $\Gamma_2 = (P_2, Q_2, \theta_2, \phi_2)$ be a right wide Morita context from B to C. Then we define the composite context $\Gamma_1\Gamma_2$ from A to C to be

$$\Gamma_1\Gamma_2 = (P_1P_2, Q_2Q_1, \theta_1 * \theta_2, \phi_2 * \phi_1)\,,$$

where the 2-cells $\theta_1 * \theta_2$ and $\phi_2 * \phi_1$ are defined as the following composites:

$$\begin{tikzcd} (P_1P_2)(Q_2Q_1) \arrow[r, "a"] \arrow[d, "\theta_1 * \theta_2"'] & P_1(P_2(Q_2Q_1)) \arrow[r, "1a^{-1}"] & P_1((P_2Q_2)Q_1) \arrow[d, "1(\theta_2 1)"] \\ I & P_1Q_1 \arrow[l, "\theta_1"] & P_1(IQ_1) \arrow[l, "1l"] \end{tikzcd}$$

and

$$\begin{tikzcd} (Q_2Q_1)(P_1P_2) \arrow[r, "a"] \arrow[d, "\phi_2 * \phi_1"'] & Q_2(Q_1(P_1P_2)) \arrow[r, "1a^{-1}"] & Q_2((Q_1P_1)P_2) \arrow[d, "1(\phi_1 1)"] \\ I & Q_2P_2 \arrow[l, "\phi_2"] & Q_2(IP_2). \arrow[l, "1l"] \end{tikzcd}$$

The lax unit 1-cell in $\mathrm{RC}(\mathcal{B})$ on an object A is the right Wide morita context (I_A, I_A, r_I, l_I). We will call this the *unit context* on A. The fact that it is a right wide Morita context follows from coherence. Of course r_I and l_I are equal. The horizontal composition is defined componentwise. For more details, one can consult [8].

2.8 Morita equivalence

In the case of unital rings, Morita equivalence has many different equivalent formulations. Here are some of the conditions of interest to us that are equivalent to Morita equivalence in the case of unital rings S and T:

1. there exists a Morita context with surjective θ and ϕ from S to T,
2. there exists a Morita context with bijective θ and ϕ from S to T,
3. there exist an S-T-bimodule P and a T-S-bimodule Q such that
$$P \otimes Q \cong S \ , \text{ and } \ Q \otimes P \cong T,$$
4. the categories of right S-modules and right T-modules are equivalent
$$\mathrm{Mod}_S \simeq \mathrm{Mod}_T \, .$$

However even in the case of rings without identity these conditions are not all equivalent for arbitrary rings S and T. See for example [9]. For now, we will only care about the first three conditions in that list. For objects A and B of a lax-unital bicategory these conditions become

1. there exists a right wide Morita context from A to B with θ and ϕ belonging to $\mathcal{E}$,
2. there exists an adjoint equivalence from A to B,
3. there exists an invertible 1-cell from A to B,
4. the categories $\mathcal{B}(X, A)$ and $\mathcal{B}(X, B)$ are equivalent for some objects X.

First let us examine how these conditions relate to each other when $\mathcal{B}$ is a bicategory. The second and third conditions are equivalent, because one states the equivalence of the objects A and B and the other their adjoint equivalence. It is well known that in a bicategory these two notions coincide. See for example Proposition 1.1. of [8].

The first and fourth follow from the other two. The first trivially and the equivalence of categories in the fourth can be thought of as the image under the bicategorical Yoneda embedding of the equivalences in the middle two.

In general the first condition does not imply the others. Theorem 1 that is proved below will tell us that under the assumption of the first condition, θ and ϕ will be monomorphisms. Whether monomorphisms belonging to $\mathcal{E}$ will be invertible will depend on the bicategory and the objects A and B.

The fourth condition by itself will not in general imply the others, but using the Yoneda lemma for bicategories, we can get an inverse implication by putting additional conditions on the equivalence in condition four. See for example Johnson's [11].

We defined firm objects A and B of a lax-unital bicategory to be Morita equivalent when there exists an adjoint equivalence from A to B, with P and Q firm. By what was mentioned above, this is a well behaved notion, since it lives in the bicategory $F\mathcal{B}$.

For arbitrary objects of a lax-unital bicategory, in general none of these condition will be equivalent, so it is not obvious how it would be best to define Morita equivalence in such a case. The trouble with the first three conditions is that they do not in general actually define an equivalence relation. The fourth one leads to a theory that is not very desirable, which for the case of non-unital rings was noted in [9].

In the case of non-unital rings it was discovered that for a fruitful Morita theory, one should either limit their attention to some smaller class of rings, or in the case of the fourth condition, to a smaller class of modules (see [9]). This has also turned out to be the case for the Morita theory of semigroups, where several different classes of semigroups and acts have been considered, for example [20] studied semigroups with local units and closed acts, which are actually the same as firm acts in our sense. In the generalizing the aspects of that theory to lax-unital bicategories, we do the same. The unitary/firm 1-cells and objects as defined above, make it easier to develop a Morita theory.

It might be worth mentioning that the mere existence of a right wide Morita context between some two objects can be seen as kind of a lax notion of equivalence for general objects of a lax-unital bicategory.

2.9 $\mathcal{E}$-Morita contexts

Now we will examine some properties of $\mathcal{E}$-Morita contexts. First we observe that they do determine an equivalence relation between the unitary objects of a lax-unital bicategory.

Definition 6. If there exists an $\mathcal{E}$-Morita context between objects A and B of a lax-unital bicategory, we will say that the objects A and B are *$\mathcal{E}$-equivalent.*

We note that if we translate this definition into the semigroup-theoretical context, we get the notion of *strong Morita equivalence* of semigroups [31].

The only thing stopping $\mathcal{E}$-equivalence from being an equivalence relation on all objects of an lax-unital bicategory is the potential lack of reflexivity.

Proposition 5. *The relation of $\mathcal{E}$-equivalence is a transitive and symmetric relation on the objects of a lax-unital bicategory.*

Proof. Let $\mathcal{B}$ be a lax-unital bicategory. First observe that the relation is symmetric by definition. To see that it is transitive, let A, B and C be objects of $\mathcal{B}$, let $\Gamma_1 = (P_1, Q_1, \theta_1, \phi_1)$ be an $\mathcal{E}$-Morita context from A to B and let $\Gamma_2 = (P_2, Q_2, \theta_2, \phi_2)$ be an $\mathcal{E}$-Morita context from B to C. Recall that since right wide Morita contexts in $\mathcal{B}$-form a lax-unital bicategory, we can compose Γ_1 and Γ_2 with respect to its composition to get a right wide Morita context

$$\Gamma_1\Gamma_2 = (P_1P_2, Q_2Q_1, \theta_1 * \theta_2, \phi_2 * \phi_1)$$

from A to C. Since the unitary 1-cells are closed with respect to composition, P_1P_2 and Q_2Q_1 are unitary. Now we just need $\theta_1 * \theta_2$ and $\phi_2 * \phi_1$ to belong to $\mathcal{E}$. Recall that $\theta_1 * \theta_2$ was defined as the composite

$$\begin{tikzcd}
(P_1P_2)(Q_2Q_1) \arrow[r, "a"] \arrow[d, "\theta_1 * \theta_2"'] & P_1(P_2(Q_2Q_1)) \arrow[r, "1a^{-1}"] & P_1((P_2Q_2)Q_1) \arrow[d, "1(\theta_2 1)"] \\
I & P_1Q_1 \arrow[l, "\theta_1"] & P_1(IQ_1). \arrow[l, "1l"]
\end{tikzcd}$$

Note that $\theta_1 * \theta_2$ is a composite 2-cells belonging to $\mathcal{E}$, and therefore belongs to $\mathcal{E}$ itself, making $\Gamma_1\Gamma_2$ an $\mathcal{E}$-Morita context. □

We have however the following observation

Proposition 6. *Let A and B be arbitrary objects of a lax-unital bicategory and suppose that there exists an $\mathcal{E}$-Morita context from A to B. Then A and B are unitary.*

Proof. Suppose that $\Gamma = (P, Q, \theta, \phi)$ is an $\mathcal{E}$-Morita context from A to B. This means that P and Q are unitary, so PQ and QP are also unitary. Therefore θ is a 2-cell belonging to $\mathcal{E}$ from a unitary object to A. Therefore by Proposition 1 the object A is also unitary. Similarly we get that B must be unitary. □

For unitary objects A we have a canonical choice of $\mathcal{E}$-Morita context from A to itself.

Proposition 7. *The relation of $\mathcal{E}$-equivalence is an equivalence relation precisely between the unitary objects of a lax-unital bicategory.*

Proof. Compared to the last proposition we are just missing transitivity. For any object A of $\mathcal{B}$, let $\Gamma = (I_A, I_A, r_I, l_I)$, be the unit right wide Morita context from A. This is an $\mathcal{E}$-Morita context precisely when A is unitary. This proves that $\mathcal{E}$-equivalence is reflexive for unitary A. □

Lemma 2. *Let (P, Q, θ, ϕ) be a right wide Morita context in a lax-unital bicategory $\mathcal{B}$. Then the following diagrams commute*

$$\begin{tikzcd}
(PQ)(PQ) \arrow[r, "1\theta"] \arrow[d, "\theta 1"'] & (PQ)I \arrow[d, "r"] \\
I(PQ) \arrow[r, "l"'] & PQ,
\end{tikzcd}
\qquad
\begin{tikzcd}
(QP)(QP) \arrow[r, "1\phi"] \arrow[d, "\phi 1"'] & (QP)I \arrow[d, "r"] \\
I(QP) \arrow[r, "l"'] & QP.
\end{tikzcd}$$

Proof. In the diagram

$$\begin{tikzcd}
(PQ)(PQ) \arrow[rrr, "\theta(1)", "\theta(11)"'] \arrow[dr, "a^{-1}"] \arrow[ddddr, "a"] \arrow[ddddd, "(1)\theta"', "(11)\theta"] & & & I(PQ) \arrow[dl, "a^{-1}"'] \arrow[ddddd, "l"] \\
& ((PQ)P)Q \arrow[r, "(\theta 1)1"] \arrow[d, "a1"'] & (IP)Q \arrow[ddddr, "l1"] & \\
& (P(QP))Q \arrow[r, "(1\phi)1"] \arrow[d, "a"'] & (PI)Q \arrow[d, "a"'] \arrow[dddr, "r1"] & \\
& P((QP)Q) \arrow[r, "1(\phi 1)"] \arrow[d, "1a"'] & P(IQ) \arrow[ddr, "1l"] & \\
& P(Q(PQ)) \arrow[r, "1(1\theta)"] & P(QI) \arrow[dr, "1r"] & \\
(PQ)I \arrow[uurr, "a"] \arrow[rrr, "r"'] & & & PQ,
\end{tikzcd}$$

all the parts commute either because of naturality, coherence or the axioms of a right wide Morita context.

That proves that the first diagram in the lemma commutes. In a similar way we can prove that the second diagram commutes. □

The following fact is very easy to prove, but is actually quite useful in may cases.

Theorem 1. *Suppose that (P, Q, θ, ϕ) is a right wide Morita context in a lax-unital bicategory $\mathcal{B}$, where either all left unitors or all right unitors are epimorphisms. Then, if θ (resp. ϕ) is in $\mathcal{E}$, it is a monomorphism.*

Proof. Suppose all left unitors are epimorphisms, the proof is similar if all right unitors are epimorphisms. Also suppose that $\theta\colon PQ \to I$ is in $\mathcal{E}$. We will show that it is a monomorphism. Let $u, v\colon X \to PQ$ be such that $\theta \circ u = \theta \circ v$. If we apply the functor $(PQ) \cdot -$ to that equality, we get

$$(1\theta) \circ (1u) = (1\theta) \circ (1v).$$

We have the diagram

$$\begin{tikzcd}
(PQ)X \arrow[r, shift left, "1u"] \arrow[r, shift right, "1v"'] \arrow[d, "\theta 1"'] & (PQ)(PQ) \arrow[d, "\theta 1"'] \arrow[dr, "1\theta"] & \\
IX \arrow[r, shift left, "1u"] \arrow[r, shift right, "1v"'] \arrow[d, "l"'] & I(PQ) \arrow[d, "l"'] & (PQ)I \arrow[dl, "r"] \\
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & PQ &
\end{tikzcd}$$

which is commutative with respect to the upper (lower) morphisms of the parallel pairs of 2-cells. The squares with horizontal morphisms $1u$ and horizontal morphisms $1v$ commute because of the functoriality of the multiplication, while the lower squares commute because of naturality. The right part of the diagram commutes because of Lemma 2. From that we get

$$u \circ l \circ (\theta 1) = v \circ l \circ (\theta 1),$$

which implies $u = v$, since l and $\theta 1$ are epimorphisms. Therefore θ is a monomorphism.

Similar arguments work for ϕ. □

Therefore we have the following fact for $\mathcal{E}$-Morita contexts.

Corollary 1. *Suppose that (P, Q, θ, ϕ) is an $\mathcal{E}$-Morita context in LU$\mathcal{B}$ or RU$\mathcal{B}$. Then, θ and ϕ are monomorphic 2-cells in LU$\mathcal{B}$ or RU$\mathcal{B}$ respectively.*

One useful thing to notice is that there is no point in defining a weaker version of $\mathcal{E}$-equivalence where the 1-cells of the contexts are only required to be unitary on one side.

Proposition 8. *Let (P, Q, θ, ϕ) be a right wide Morita context in a lax-unital bicategory $\mathcal{B}$ with θ and ϕ in $\mathcal{E}$. Then P (resp. Q) is left unitary if and only if P (resp. Q) is right unitary.*

Proof. The proof becomes evident when we consider the properties of $\mathcal{E}$ while looking at the diagram

$$\begin{tikzcd} P(QP) \arrow[r, "1\phi"] & PI \arrow[dr, "r"] & \\ & & P. \\ (PQ)P \arrow[uu, "a"] \arrow[r, "\theta 1"'] & IP \arrow[ur, "l"'] & \end{tikzcd}$$

□

In a similar fashion we can prove an analogous result for adjoint equivalences.

Proposition 9. *Let (P, Q, θ, ϕ) be an adjoint equivalence in a lax-unital bicategory $\mathcal{B}$. Then P (resp. Q) is left firm if and only if P (resp. Q) is right firm.*

2.10 Lax functors

A notion of morphism between lax-unital bicategories that has multiple occurrences in different places of this thesis is the lax functor. A lax functor is essentially a relaxed notion of a 2-functor. It need not preserve the unit 1-cells or the composition of 1-cells, but it instead has natural comparison maps $F(A)F(B) \to F(AB)$ and $I \to F(I)$. This notion was introduced along with the notion of a bicategory by Bénabou in [2], where he called them morphisms of bicategories. One of the motivations of defining them was that some categorical structures could viewed as lax functors between specific bicategories. For example, he noticed that a monad is just a lax functor from category with one object and one morphism to the bicategory in which the monad lives.

Definition 7. Let C and D be lax-unital bicategories. A *lax functor F* from C to D consists of

- for each object A of C, an object $F(A)$ of D,
- for each pair $A, B \in C_0$, a functor $F_{A,B} \colon C(A, B) \to D(F(A), F(B))$, which we will refer to as the local part of the lax functor,
- natural comparison 2-cells $\Phi_{M,N} \colon F(M)F(N) \to F(MN)$,
- comparison 2-cells $\Phi^0_A \colon I_{F(A)} \to F(I_A)$.

The comparison 2-cells need to be such that the following diagrams commute:

$$\begin{tikzcd}
 & (F(M)F(N))F(K) \arrow[r, "a"] \arrow[dl, "\Phi 1"'] & F(M)(F(N)F(K)) \arrow[dr, "1\Phi"] & \\
F(MN)F(K) \arrow[dr, "\Phi"'] & & & F(M)F(NK) \arrow[dl, "\Phi"] \\
 & F((MN)K) \arrow[r, "F(a)"'] & F(M(NK)), &
\end{tikzcd}$$

$$\begin{tikzcd}
F(M)I_{F(B)} \arrow[r, "1\Phi^0_B"] \arrow[d, "r_{F(M)}"'] & F(M)F(I_B) \arrow[d, "\Phi_{M,I}"] \\
F(M) & F(MI_B), \arrow[l, "F(r)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{F(A)}F(M) \arrow[r, "\Phi^0_A 1"] \arrow[d, "l_{F(M)}"'] & F(I_A)F(M) \arrow[d, "\Phi_{I,M}"] \\
F(M) & F(I_A M). \arrow[l, "F(l)"]
\end{tikzcd}$$

Note that since a category can be seen as special case of a bicategory, the preceding definition also gives us the notion of a lax functor from a category into a lax-unital bicategory, which is sometimes useful.

We call a lax functor normal, when the comparison maps $I \to F(I)$ are identities and call it a *pseudofunctor* when the comparison maps

$$I \to F(I) \quad \text{and} \quad F(M)F(N) \to F(MN)$$

are invertible.

As was noticed by Pécsi in [26], it is possible to describe right wide Morita contexts in a bicategory $\mathcal{B}$ as normal lax functors from Iso to $\mathcal{B}$, but of course this also works for lax-unital bicategories. Here Iso denotes the category with two objects and precisely one morphism between any two objects. This means that Iso is the smallest category containing two isomorphic objects. Clearly for any category $\mathcal{C}$ there is a bijection (of classes, possibly, or large enough sets) between the isomorphisms of $\mathcal{C}$ and functors from Iso to $\mathcal{C}$. Since right wide Morita contexts are a lax version of this same concept, it supports the view that a right wide Morita context are a kind of a lax notion of isomorphism between objects of a lax-unital bicategory.

2.11 Improving contexts

When dealing with a right wide Morita equivalence $\Gamma = (P, Q, \theta, \phi)$, the better properties P and Q have, the easier it will be to work with Γ. For example, when we were proving the transitivity of $\mathcal{E}$-equivalence, we needed the unitarity of P and Q.

In this section we will give a few results that allow us to improve right wide Morita contexts, such as to give P and Q better properties.

First of all we will show that we could have defined $\mathcal{E}$-equivalence without the requirement that P and Q should be unitary in the $\mathcal{E}$-Morita context.

Proposition 10. *Let A and B be unitary objects of a lax-unital bicategory $\mathcal{B}$. Let $\Gamma = (P, Q, \theta, \phi)$ be a right wide Morita context from A to B such that θ and ϕ are in $\mathcal{E}$. Then there exists an $\mathcal{E}$-Morita context $\Gamma' = (P', Q', \theta', \phi')$ from A to B. Furthermore, if A and B are firm then P' and Q' will be right firm and if θ and ϕ are invertible or if P and Q are left firm, P' and Q' will also be left firm.*

Proof. We define the right wide Morita context $\Gamma' = (P', Q', \theta', \phi')$ by setting $P' = PI$, $Q' = QI$ and defining θ' and ϕ' using the diagrams

$$\begin{tikzcd} (PI)(QI) \arrow[r, "rr"] \arrow[dr, "\theta'"'] & PQ \arrow[d, "\theta"] \\ & I, \end{tikzcd} \qquad \begin{tikzcd} (QI)(PI) \arrow[r, "rr"] \arrow[dr, "\phi'"'] & QP \arrow[d, "\phi"] \\ & I. \end{tikzcd}$$

Now we will show that Γ' is indeed a right wide Morita context. That means the diagrams in the definition of a right wide Morita context have to commute. We will only check that one of the diagrams commutes, since the other one can be shown to commute in a similar way. We have the following commutative diagram.

$$\begin{tikzcd}
((PI)(QI))(PI) \arrow[rrrr, "a"] \arrow[d, "(1r)1"] & & & & (PI)((QI)(PI)) \arrow[d, "1(r1)"] \\
((PI)Q)(PI) \arrow[rrrr, "a"] \arrow[dd, "(r1)1"'] & & & & (PI)(Q(PI)) \arrow[dl, "a^{-1} \circ (1a^{-1})"'] \arrow[dd, "1(1r)"] \\
& ((PQ)P)I \arrow[d, "(\theta 1)1"] \arrow[dr, "a1"] & & ((PI)(QP))I \arrow[dl, "(r1)1"'] \arrow[d, "(1\phi)1"] \arrow[dr, "r"] & \\
(PQ)(PI) \arrow[ur, "a^{-1}"] \arrow[d, "\theta 1"', "\theta(11)"] & (IP)I \arrow[dr, "l1"] & (P(QP))I \arrow[drr, "(1\phi)1"'] & ((PI)I)I \arrow[dr, "r", "(r1)1"'] & (PI)(QP) \arrow[d, "1\phi"] \\
I(PI) \arrow[ur, "a^{-1}"'] \arrow[rr, "l"'] & & PI & & (PI)I. \arrow[ll, "r1"', "r"]
\end{tikzcd}$$

All the inner diagrams commute either by naturality, coherence or because Γ is a right wide Morita context. Therefore the outer rectangle commutes. The composition of the left (right) edge of the outer rectangle is $\theta'1$ (resp. $1\phi'$). Therefore the second diagram in the definition of a right wide Morita context commutes.

Next let us check that θ' and ϕ' are in $\mathcal{E}$. We will check that for θ' since the proof for ϕ' is similar. Let Δ be the coherent natural transformation $\Delta_X \colon X(II) \to X$. Clearly Δ_I is in $\mathcal{E}$. We have the following commutative diagram, where $\sim$ represent the various coherent combinations of associators.

$$\begin{tikzcd}
(P(QP))(Q(PQ)) \arrow[rrr, "(1\phi)(1\theta)"] \arrow[dr, "(11)(1\theta)"] \arrow[dddd, "\sim"'] & & & (PI)(QI) \arrow[dl, "\sim"] \arrow[dddd, "rr"] \\
& (P(PQ))(QI) \arrow[urr, "(1\phi)(11)"] \arrow[d, "\sim"'] & (P(IQ))I \arrow[d, "(1l)1"] & \\
& (P((QP)Q))I \arrow[ur, "(1(\phi 1))1"] \arrow[d, "(1a)1"'] & (PQ)I \arrow[ddr, "r"] & \\
& (P(Q(PQ)))I \arrow[r, "1((1\theta)1)"'] \arrow[d, "\sim"'] & (P(QI))I \arrow[u, "(1r)1"'] \arrow[d, "\sim"'] & \\
(PQ)((PQ)(PQ)) \arrow[r, "1(1\theta)"] \arrow[d, "\theta(\theta\theta)"'] & (PQ)((PQ)I) \arrow[r, "1(\theta 1)"] & (PQ)(II) \arrow[r, "\Delta"'] \arrow[dll, "\theta(11)"'] & PQ \arrow[d, "\theta"] \\
I(II) \arrow[rrr, "\Delta"'] & & & I.
\end{tikzcd}$$

The diagram commutes since every small diagram in the interior commutes either because of coherence, naturality or because Γ is a right wide Morita context. Once again we can use the property **E3** of $\mathcal{E}$ to deduce that the right edge of the outer rectangle is in $\mathcal{E}$, but the right edge is just θ'.

Now we need to check that the unitors of P' and Q' are in $\mathcal{E}$. For the right unitors this is follows from Section 2.4, since $P' = PI$ and $Q' = QI$ and since P and Q are 1-cells between unitary objects. Since we P' and Q' are part of a right wide Morita context with 1-cells in $\mathcal{E}$, Proposition 8 implies that the left unitors of P' and Q' must also be in $\mathcal{E}$.

In the firm case, the proof is analogous. We just need to add that when P and Q are left firm then from diagrams

$$\begin{tikzcd} I(PI) \arrow[d, "a^{-1}"] \arrow[dr, "l_{PI}"] & \\ (IP)I \arrow[r, "l_P 1_I"'] & PI, \end{tikzcd} \qquad \begin{tikzcd} I(QI) \arrow[d, "a^{-1}"] \arrow[dr, "l_{QI}"] & \\ (IQ)I \arrow[r, "l_Q 1_I"'] & QI, \end{tikzcd}$$

we can see that PI and QI will also be left firm. □

Essentially the last proposition allows us to transfer right Wide Morita contexts with θ and ϕ in $\mathcal{E}$ from $\mathcal{B}^U$ to $U\mathcal{E}$. While it may seem that this would allow us to extend the use of Theorem 1 from $U\mathcal{E}$ to $\mathcal{B}^U$, and although to an extent it does, the monomorphic 2-cells of $U\mathcal{E}$ might not be the same as the monomorphic 2-cells of $U\mathcal{E}$.

The relation between $U\mathcal{E}$ and $\mathcal{B}^U$ is in general not that clear. We need a better way to transfer information between these two lax-unital bicategories. The process described above is clearly not ideal for unitary objects, since the construction is not idempotent in the sense that it need not give us an isomorphic context when the context we are applying it to is already an $\mathcal{E}$-Morita context.

Note that it does fix contexts between firm objects when θ and ϕ are isomorphisms, which is to say that it takes adjoint equivalences between firm objects to Morita equivalences between firm objects in a way that is in a sense the most optimal.

Chapter 3

Unitarization

We can use a different method to improve 1-cells and contexts with respect to their unitarity. In the paper [20] for example, the method used to make the unitors of acts surjective used the assignment

$$M_S \mapsto \{ms \mid m \in M,\, s \in S\},$$

which maps a right act M_S to the image of the act's unitor $M \otimes S \to M$, $m \otimes s \mapsto ms$. We can generalize that construction to the our situation, but we have to make more assumptions. We will need to use orthogonal factorization systems.

3.1 Orthogonal factorization systems

As the name suggests, an orthogonal factorization system allows us to split morphisms into factors. Intuitively we can think of it acting like the homomorphism theorem, splitting a morphism into the surjective part and the injective part.

Definition 8. Let $\mathcal{C}$ be a category and let $\mathcal{E}$ be a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms belonging to that category. Morphisms $e\colon A \to B$ and $m\colon C \to D$ are said to be *orthogonal*, a situation expressed by writing $e \perp m$, when for each commuting square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d] & B \arrow[d] \arrow[dl, dashed, "s"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

there exists a unique diagonal fill-in $s\colon B \to C$ making the whole diagram commute. Let $\mathcal{E}$ consist of precisely those morphisms e of $\mathcal{C}$ for which $e \perp m$ for each $m \in \mathcal{M}$ and let $\mathcal{M}$ consist of precisely those morphisms m of $\mathcal{C}$ for which $e \perp m$ for each morphism e in $\mathcal{E}$. If each morphism f of $\mathcal{C}$ factors as $f = me$, we say that $(\mathcal{E}, \mathcal{M})$ is an *orthogonal factorization system* on $\mathcal{C}$.

For more information about factorization systems, see for example section 5.5 of [3]. We will now assume that on each morphism category of our lax-unital bicategory $\mathcal{B}$, the 2-cells in $\mathcal{E}$ and the monomorphic 2-cells constitute an orthogonal factorization system on that category. This means that the collection $\mathcal{E}$ is precisely the collection of all strongly epimorphic 2-cells. For convinience, we will make the following definition in order to refer to lax-unital bicategories with this property later.

Definition 9. We will say that a lax-unital bicategory has *2-cell factorizations*, when the 2-cells in $\mathcal{E}$ and the monomorphic 2-cells constitute an orthogonal factorization system on each of the hom-categories.

Remark 6. We now have a slightly easier way of checking whether a given class $\mathcal{E}$ satisfies the conditions we required of it in the beginning. Let us assume we are given a random lax-unital bicategory $\mathcal{B}'$, such that each morphism category has $(\text{StrongEpis}, \text{Monos})$ as an orthogonal factorization system. Because of some well known properties of factorization systems, the class of all strongly epimorphic 2-cells automatically satisfies all but the last condition required of $\mathcal{E}$, which is **E4**, the requirement that $\mathcal{E}$ is closed with respect to the horizontal composition.

The first of the two can often be deduced from other properties of $\mathcal{B}'$. For example, if the functor of composing with a 1-cell always has a right adjoint, it automatically preserves regular epimorphisms, which in nice enough categories coincide with strong epimorphisms. Such is the case in the lax-unital bicategory of non-unital rings and modules and the lax-unital bicategory of semigroups and two sided actions.

3.2 The unitarization lax functor

We can now define an alternate way of improving right wide Morita contexts and the unitors of 1-cells. First we will do it for right unitarity. We will do this in a way that is in a sense the best possible, since it will be locally right adjoint to the inclusion $RU\mathcal{B} \to \mathcal{B}^U$.

Let us fix for each 1-cell M in $\mathcal{B}^U$, an ($\mathcal{E}$,mono)-factorization (e_M, m_M) of r_M. Let A and B be unitary objects of $\mathcal{B}$. We will now define a functor

$$R_{A,B}\colon \mathcal{B}^U(A, B) \to RU\mathcal{B}(A, B).$$

This functor will depend on the choice of the factorizations (e_M, m_M), but will be unique up to isomorphism. Since $R_{A,B}$ will later turn out to be the local part of a lax functor, we will omit the subscripts A and B when applying the functor. Let $M\colon A \to B$. We define $R(M)$ as the 1-cell through which r_M factors, as seen in the diagram

$$\begin{tikzcd} MI \arrow[r, "e_M"'] \arrow[rr, bend left, "r_M"] & R(M) \arrow[r, "m_M"'] & M. \end{tikzcd}$$

We need to check that $R(M)$ is actually in $RU\mathcal{B}(A,B)$. To show that, we will use the following diagram, which commutes because of coherence and the naturality of r.

$$\begin{tikzcd} & (MI)I \arrow[r, "e1"] \arrow[dl, "a"'] \arrow[d, "r_{MI}"] & R(M)I \arrow[d, "r_{R(M)}"] \\ M(II) \arrow[r, "1r_I"'] & MI \arrow[r, "e"'] & R(M). \end{tikzcd}$$

The right composite $r_{R(M)} \circ e1$ is in $\mathcal{E}$, since the left side composite is the composite of 2-cells in $\mathcal{E}$. This implies that $r_{R(M)}$ is in $\mathcal{E}$, which means that $R(M)$ is a 1-cell in $RU\mathcal{B}$.

Now suppose that $f\colon M \to N$ is a 1-cell in $\mathcal{B}^U$. Then we can define $R(f)$ to be the unique 2-cell $R(M) \to R(N)$ that makes the following diagram commute and exists because of the diagonal fill-in property of a factorization system.

$$\begin{tikzcd} MI \arrow[r, "e_M"'] \arrow[rr, bend left, "r_M"] \arrow[d, "f1"'] & R(M) \arrow[r, "m_M"'] \arrow[d, dashed, "R(f)"] & M \arrow[d, "f"] \\ NI \arrow[r, "e_N"] \arrow[rr, bend right, "r_N"'] & R(N) \arrow[r, "m_N"] & N \end{tikzcd}$$

It is clear that we can vertically paste the defining diagrams of $R(f)$ and $R(g)$ of a composable pair of 2-cells f and g. It is also clear that the resulting diagram will be the defining diagram of $R(g \circ f)$. Therefore, since $R(g \circ f)$ is the unique 2-cell making the diagram commute, it must equal $R(g) \circ R(f)$. Since R also clearly takes unit 2-cells to unit 2-cells, we have shown that $R_{A,B}$ is a functor.

Now we will construct an identity-on-objects lax functor $R\colon \mathcal{B}^U \to RU\mathcal{B}$ with the functors $R_{A,B}$ as the local components. The only data missing is the comparison 2-cells. We will define the comparison 2-cell

$$\Phi_{M,N} \colon R(M)R(N) \to R(MN)$$

as the unique 2-cell that makes the following diagram commute

$$\begin{tikzcd}
(MI)(NI) \arrow[r, "e_M e_N"] \arrow[d, "a^{-1} \circ (r1)"'] & R(M)R(N) \arrow[d, dashed, "\Phi_{M,N}"'] \arrow[dr, "m_M m_N"] & \\
(MN)I \arrow[r, "e_{MN}"'] & R(MN) \arrow[r, "m_{MN}"'] & MN.
\end{tikzcd}$$

The unique 2-cell exists because of the properties of a factorization system, since ee is in $\mathcal{E}$, m is a monomorphism and the outer composites are equal. To see that the outer composites are equal, we simply need to remember that $m_M \circ e_M = r_M$ and use coherence.

It might be worth noting that from the definition of $\Phi_{M,N}$ we can see that it is always a monomorphism.

To define Φ^0_A for an object A of $\mathcal{B}$, we need to remember that $r_I \colon II \to I$ is in $\mathcal{E}$. This means that m_I, the monomorphic part of the ($\mathcal{E}$,mono)-factorization of r_I, is an isomorphism. Therefore we can define $\Phi^0_A \colon I \to R(I)$ to be $m_I^{-1} \colon I \to R(I)$. Finally we need to check that Φ and Φ^0 satisfy the conditions required of comparison maps and that Φ is natural.

For the first condition, observe that all the small parts of the following diagram commute either because of naturality, functoriality, the definition of Φ or the definition of R.

$$\begin{tikzcd}
 & (R(M)R(N))R(K) \arrow[r, "a"] \arrow[d, "(mm)1"'] \arrow[ddl, "\Phi 1"'] \arrow[ddd, bend left, "(mm)m"] & R(M)(R(N)R(K)) \arrow[d, "1(mm)"] \arrow[ddr, "1\Phi"] \arrow[ddd, bend right, "m(mm)"'] & \\
 & (MN)R(K) \arrow[dd, "1m"', "(11)m"] & R(M)(NK) \arrow[dd, "m(11)"', "m1"] & \\
R(MN)R(K) \arrow[ur, "m1"] \arrow[dr, "mm"'] \arrow[ddr, "\Phi"'] & & & R(M)R(NK) \arrow[ul, "1m"'] \arrow[dl, "mm"] \arrow[ddl, "\Phi"] \\
 & (MN)K \arrow[r, "a"'] & M(NK) & \\
 & R((MN)K) \arrow[u, "m"'] \arrow[r, "R(a)"'] & R(M(NK)). \arrow[u, "m"] &
\end{tikzcd}$$

One can easily check that the left composite and the right composite of the outer hexagon are equal under m. This means that they are equal, because m

is a monomorphism. Therefore the first condition holds. In a similar fashion, we can see that the second condition holds by looking at the diagram

$$\begin{tikzcd}
R(M)I \arrow[rrrr, "1\Phi^0", "1m^{-1}"'] \arrow[ddd, "r"'] \arrow[drr, "m1"] & & & & R(M)R(I) \arrow[dll, "mm"'] \arrow[ddd, "\Phi_{M,I}"] \\
& M & MI \arrow[l, "r"] & & \\
& & & & \\
R(M) \arrow[uur, "m"] & & & & R(MI). \arrow[uull, "m"] \arrow[llll, "R(r)"]
\end{tikzcd}$$

One can use a diagram very similar to the previous one to check that the third condition holds. To check the naturality of Φ, let $f\colon M \to U$ and $g\colon N \to V$ be 2-cells between suitably composable 1-cells. Then the commutativity of the following diagram is checked by using the monomorphicity of m, as we did before.

$$\begin{tikzcd}
R(M)R(N) \arrow[rrr, "R(f)R(g)"] \arrow[dd, "\Phi"'] \arrow[dr, "mm"] & & & R(U)R(V) \arrow[dl, "mm"'] \arrow[dd, "\Phi"] \\
& MN \arrow[r, "fg"] & UV & \\
R(MN) \arrow[ur, "m"] \arrow[rrr, "R(fg)"'] & & & R(UV). \arrow[ul, "m"']
\end{tikzcd}$$

We have now shown everything we need to prove the first part of the following theorem.

Theorem 2. *If for the hom-categories of a lax-unital bicategory $\mathcal{B}$, the 2-cells belonging to $\mathcal{E}$ and the monomorphic 2-cells form an orthogonal factorization system, then R as constructed above is an identity on objects lax functor from $\mathcal{B}^U$ to $RU\mathcal{B}$. Additionally, this lax functor is locally right adjoint to the inclusion*

$$RU\mathcal{B} \to \mathcal{B}^U\,.$$

Proof. We will now prove that $R\colon \mathcal{B}^U \to RU\mathcal{B}$ is locally right adjoint to the inclusion $RU\mathcal{B} \to \mathcal{B}^U$. We want to use Lemma 1 to do so. Let us look at R as an endofunctor on $\mathcal{B}^U(A, B)$. The copointed structure on R will be provided by $m\colon R(M) \to M$. Indeed, $m_{R(M)}$ will be an isomorphism since $R(M)$ is already left unitary and if we use the naturality of m (see the following lemma) for m_M, we get a commutative diagram

$$\begin{tikzcd} RR(M) \arrow[r, "R(m_M)"] \arrow[d, "m_{R(M)}"'] & R(M) \arrow[d, "m_M"] \\ R(M) \arrow[r, "m_M"'] & M , \end{tikzcd}$$

which implies that $R(m_M)$ and $m_{R(M)}$ coincide, since m_M is a monomorphism. □

The following is a simple observation, which follows directly from the definition of e_M and m_M.

Lemma 3. *The* 2*-cells*

$$MI \xrightarrow{e_M} R(M) \xrightarrow{m_M} M$$

defined in the construction of the lax functor R are natural in M. Also e_M always belongs to $\mathcal{E}$ and m_M is always a monomorphism.

Remark 7. Note that m is essentially a familiy of natural transformations indexed by A and B from $R_{A,B}$ to the identity functors of the respective categories. Such families (not necessarily into the unit functor) of natural transformations were studied by Stephen Lack in [17], where, when satisfying some additional axioms, they were called *icons*. Indeed, m satisfies the axioms of an icon $R \to 1$ by the definition of the comparison maps of R, if we view R as a lax endofunctor of $\mathcal{B}^U$. If $-I$ were a lax functor, then e would also be an icon $-I \to R$, but $-I$ is only a lax functor on firm objects. We will not go into any more details about icons.

What we have shown so far in this section is that in addition to R being a lax functor, $RU\mathcal{B}$ is a locally coreflextive lax-unital subbicategory of $\mathcal{B}^U$. This connection allows us to transfer useful properties between the two lax-unital bicategories.

We could go through a similar process for the left unitors, constructing a lax functor $L\colon \mathcal{B}^U \to LU\mathcal{B}$ and showing that $LU\mathcal{B}$ is a locally coreflective lax-unital subbicategory of $\mathcal{B}^U$.

$$\begin{tikzcd} & \mathcal{B}^U \arrow[dl, "L"'] \arrow[dr, "R"] & \\ LU\mathcal{B} \arrow[dr, "R"'] & & RU\mathcal{B} \arrow[dl, "L"] \\ & U\mathcal{B} & \end{tikzcd}$$

Composing these lax functors we now have two lax functors RL and LR from $\mathcal{B}^U$ to $U\mathcal{B}$. However since they are both identity on objects and both locally right adjoint to the inclusion $U\mathcal{B} \to \mathcal{B}^U$, they must be locally isomorphic. This local isomorphism can be shown to be an icon in the sense of [17].

The following observation about the lax functor R will prove useful in the future

Proposition 11. *Suppose that $f\colon M \to N$ is a monomorphic 2-cell in $\mathcal{B}^U$. Then $R(f)\colon R(M) \to R(N)$ is monomorphic both as a 2-cell of $\mathcal{B}^U$ and as a 2-cell of $RU\mathcal{B}$.*

Proof. Since R is locally right adjoint to the inclusion $RU\mathcal{B} \to \mathcal{B}^U$, it will take 2-cells monomorphic in $\mathcal{B}^U$ to 2-cells monomorphic in $RU\mathcal{B}$.

To see that $R(f)$ is also monomorphic as a 2-cell of $\mathcal{B}^U$, let us write out the naturality diagram of $m\colon R(A) \to A$ for f:

$$\begin{tikzcd} R(M) \arrow[r, "R(f)"] \arrow[d, "m_M"'] & R(N) \arrow[d, "m_N"] \\ M \arrow[r, "f"'] & N\,. \end{tikzcd}$$

In the diagram, clearly everything besides $R(f)$ is monomorphic as a 2-cell of $\mathcal{B}^U$. Therefore $m_N R(f)$ is monomorphic as a 2-cell of $\mathcal{B}^U$, which implies that $R(f)$ must also be monomorphic as a 2-cell of $\mathcal{B}^U$. $\square$

Example 2. Let us see what the unitarization process does in the lax-unital bicategory of semigroups and biacts. We recall that in that case the composition of 1-cells, the two sided actions, or biacts as they are sometimes called, is by tensor product. The unit 1-cell on S is ${}_S S_S$, which is S acting on itself. The unitors in a sense carry information about the structure of the actions of the semigroups on the biacts. But only in a sense, because we can not really define the actions of the semigroups in terms of the unitors, since the unitors depend on the tensor product structure, which itself depends on the actions of the semigroups. For right actions we have

$$r_M \colon M \otimes S \to M\,, \quad m \otimes s \mapsto ms\,.$$

The orthogonal factorization system on the categories of biacts separates morphisms $f\colon M \to N$ into the mapping onto their image $f(M)$ and the inclusion of the image. Therefore the orthogonal factorization system acts on M by separating $R(M)$ as the image of r_M, giving us the decomposition

$$\begin{tikzcd} M \otimes S \arrow[r, "e_M"'] \arrow[rr, bend left, "r_M"] & R(M) \arrow[r, "m_M"'] & M, \\ M \otimes S \arrow[r, "m \otimes s \mapsto ms"'] & \{ms\colon m \in M, s \in S\} \arrow[r, "\text{inclusion}"'] & M. \end{tikzcd}$$

For semigroups it is customary to denote

$$MS = \{ms\colon s \in S, m \in M\}$$

for the set of all the products of the elements of M and S. Supposing that S is a factorizable semigroup, meaning that it is unitary in our sense, MS will be right unitary.

There is an interesting point to make about the action of the lax functor R on morphisms between biacts. Let us look at the defining diagram of $R(f)$ for a morphism $f\colon M \to N$ of biacts for M and N between factorizable semigroups:

$$\begin{tikzcd} M \otimes S \arrow[r, "m \otimes s \mapsto ms"'] \arrow[d, "f \otimes 1"'] \arrow[rr, bend left, "r_M"] & MS \arrow[r, "inclusion"'] \arrow[d, dashed, "R(f)"] & M \arrow[d, "f"] \\ N \otimes S \arrow[r, "n \otimes s \mapsto ns"] \arrow[rr, bend right, "r_N"'] & NS \arrow[r, "inclusion"] & N\,. \end{tikzcd}$$

We could equivalently define $R(f)$ as the unique morphism making the left square commute or the unique morphism making the right square commute. The left square would define $R(f)$ using the relation

$$R(f)(ms) = f(m)s\,.$$

Using the right square defines $R(f)$ as the restriction of f to MS. Both of these are possiblities that come to mind when a person working with semigroups and acts is trying to make $-S$ into a functor. This approach makes it very clear that these two definitions are equivalent.

3.3 Transfering contexts

In the paper [8] it is shown that given a right wide Morita context Γ from A to B in a bicategory $\mathcal{C}$, a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ will induce a right wide Morita context from $F(A)$ to $F(B)$ in $\mathcal{D}$. One can follow that proof to easily check that it also works for lax-unital bicategories and that the comparison 2-cells $F(A)F(B) \to F(AB)$ do not have to be invertible for the construction to be valid. This means that the lax functors L and R can be used to improve right wide Morita contexts, just as we did in Proposition 10.

According to the results in [8], a pseudofunctor F will take a right wide Morita context $\Gamma = (P, Q, \theta\, \phi)$ to the context $\Gamma_F(F(P), F(Q), \theta^*, \phi^*)$, where θ^* is the composite

$$F(P)F(Q) \xrightarrow{\Phi} F(PQ) \xrightarrow{F(\theta)} F(I) \xrightarrow{\Phi^{0-1}} I$$

and ϕ^* is the obvious counterpart. If we apply that to the lax functor R, we see that the 1-cells in Γ_R are given by $P_R = R(P)$, $Q_R = R(Q)$, the 2-cell θ_R is the composite

$$P_R Q_R = R(P)R(Q) \xrightarrow{\Phi} R(PQ) \xrightarrow{R(\theta)} R(I) \xrightarrow{m} I$$

and the 2-cell ϕ_R is a similar composite. To see that θ_R is in $\mathcal{E}$ if θ is in $\mathcal{E}$, observe that the following diagram commutes because of the definitions of R and Φ.

$$\begin{tikzcd}
(PI)(QI) \arrow[r, "a \circ (r1)"] \arrow[d, "ee"'] & (PQ)I \arrow[r, "\theta 1"] \arrow[d, "e"'] & II \arrow[d, "e"'] \arrow[dr, "r"] & \\
R(P)R(Q) \arrow[r, "\Phi"'] & R(PQ) \arrow[r, "R(\theta)"'] & R(I) \arrow[r, "m"'] & I.
\end{tikzcd}$$

By using naturality and coherence, one can simply verify that the upper composite of the diagram above is just θ' as constructed in Proposition 10. This means that $\theta_R \circ ee = \theta'$ is in $\mathcal{E}$. Therefore θ_R is in $\mathcal{E}$.

Remark 8. Note that the above provides an alternate proof for the part of Proposition 10 that deals with unitary objects. However, this new construction is idempotent in the sense that it will produce an isomorphic context when applied to an $\mathcal{E}$-Morita context.

If we are dealing with firm objects instead, then by coherence, $-I$ clearly satisfies the axioms of a lax functor $\mathcal{B}^F \to RF\mathcal{B}$ with the obvious comparison maps. If we use the property of lax functors mapping Morita contexts to Morita contexts here, we can derive the part of Proposition 10 that is about firm objects.

All of the above can of course also be done for the lax functor L.

Chapter 4

Closed lax-unital bicategories

This chapter is in part a generalization of some of the results in the author's joint paper [16], with V. Laan and L. Márki, from semigroups to objects in a right closed lax-unital bicategory. In particular, Theorem 3 below generalizes the fact that if two firm semigroups are strongly Morita equivalent then they are Morita equivalent.

4.1 Definition

For a 2-categorical structure, left (right) closedness means that for any 1-cell N, the functor of composing with it on left (right) has a right adjoint.

In this chapter we will assume that **the lax-unital bicategory $\mathcal{B}$ is such that for all objects A, B, C of $\mathcal{B}$ and 1-cells $N\colon B \to C$, we have the following adjunction**

$$\begin{tikzcd} \mathcal{B}(A,C) \arrow[r, bend left, "N\triangleright -"] \arrow[r, phantom, "\top"] & \mathcal{B}(A,B) \arrow[l, bend left, "-N"] \end{tikzcd}$$

for some fixed family of functors $M\triangleright -$, called the *right closed structure of $\mathcal{B}$*. In other words, we will assume that $\mathcal{B}$ is right closed.

This means the existence of the following isomorphism natural in M and L:

$$\mathcal{B}(A,C)(MN,L) \cong \mathcal{B}(A,B)(M, N\triangleright L)\,.$$

We denote the unit of this adjunction by

$$\eta^M_N : N \to M\triangleright(NM)$$

and the counit by

$$\varepsilon_N^M \colon (M \triangleright N)M \to N\,.$$

As per usual in this case, we can make $M \triangleright -$ into a contravariant functor with respect to M by using the universality of the counit. If $f \colon M \to M'$ is a 2-cell of $\mathcal{B}$, then we define $f \triangleright 1_N \colon M' \triangleright N \to M \triangleright N$ as the unique 2-cell making the following diagram commute:

$$\begin{tikzcd}
(M' \triangleright N)M \arrow[r, "(f \triangleright 1_N)1_M"] \arrow[d, "1f"'] & (M \triangleright N)M \arrow[d, "\varepsilon_N^M"] \\
(M' \triangleright N)M' \arrow[r, "\varepsilon_N^{M'}"'] & N\,.
\end{tikzcd}$$

The commutativity of this diagram is sometimes called the *extranaturality* of ε_N^M in the variable M.

4.2 The closedness of $RU\mathcal{B}$

An important application of the results of the last section is that we can extend the closedness of $\mathcal{B}$ to $RU\mathcal{B}$.

Proposition 12. *Suppose that the lax-unital bicategory $\mathcal{B}$ is closed and has 2-cell factorizations. Then $RU\mathcal{B}$ is also closed. We will let $\blacktriangleright$ denote the closed structure on $RU\mathcal{B}$. The counit of this closed structure will be denoted by ε' and the unit η'.*

Proof. Let $N \colon B \to C$ be a right unitary 1-cell of $\mathcal{B}$. Under these assumptions we can compose the adjunction from Theorem 2 and the closedness adjunction of $\mathcal{B}$

$$\begin{tikzcd}
\mathcal{B}^U(A,C) \arrow[r, bend left, "N \triangleright -"] \arrow[r, phantom, "\top"] & \mathcal{B}^U(A,B) \arrow[l, bend left, "-N"] \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & RU\mathcal{B}(A,B)\,, \arrow[l, bend left, "\text{inclusion}"]
\end{tikzcd}$$

which will result in the adjunction

$$\begin{tikzcd}
\mathcal{B}^U(A,C) \arrow[r, bend left, "R(N \triangleright -)"] \arrow[r, phantom, "\top"] & RU\mathcal{B}(A,B)\,. \arrow[l, bend left, "-N"]
\end{tikzcd}$$

Finally we need to notice that $-N$ actually maps into $RU\mathcal{B}(A, C)$, since, by coherence, the diagram

$$\begin{tikzcd}
(MN)I \arrow[r, "a"] \arrow[dr, "r_{MN}"'] & M(NI) \arrow[d, "1_M r_N"] \\
& MN
\end{tikzcd}$$

commutes and N is right unitary. □

4.3 Non-singular right unitary 1-cells

When idempotent pointed or copointed endofunctor on $\mathcal{C}$ has a right adjoint, we get several equivalences of categories of free. These categories will be an essential colocalization of $\mathcal{C}$ and and essential localization of $\mathcal{C}$. An *essential (co)localization* is a (co)reflective subcategory such that the (co)reflection has a (right) left adjoint. See for example [3].

Lemma 4. *Let (F, ξ) be an idempotent copointed endofunctor on a category $\mathcal{C}$ and let $G\colon \mathcal{C} \to \mathcal{C}$ be right adjoint to F with adjunction unit η and counit ε. Then*

1. *the natural transformation $\zeta\colon 1 \to G$ corresponding to $\xi\colon F \to 1$ under the adjunction, defined componentwise as $\zeta_A = G(\xi_A)\eta_A$, makes G into an idempotent pointed endofunctor on $\mathcal{C}$;*

2. *we have*
$$\mathrm{Fix}(\mathcal{C}, \xi) = \mathrm{Fix}(\mathcal{C}, \varepsilon)$$
and it is an essential colocalization of $\mathcal{C}$ with a coreflection $F|^{\mathrm{Fix}(\mathcal{C},\xi)}$;

3. *we have*
$$\mathrm{Fix}(\mathcal{C}, \zeta) = \mathrm{Fix}(\mathcal{C}, \eta)$$
and it is an essential localization of $\mathcal{C}$ with a reflection $G|^{\mathrm{Fix}(\mathcal{C},\zeta)}$;

4. *the adjunction between F and G restricts to an adjoint equivalence*

$$\begin{tikzcd}
\mathrm{Fix}(\mathcal{C}, \xi) \arrow[r, bend left, "F"] \arrow[r, phantom, "\top"] & \mathrm{Fix}(\mathcal{C}, \zeta) \arrow[l, bend left, "-G"]
\end{tikzcd}.$$

Proof. (1) Recall (beginning of Section 1 of [6]) that if $L \dashv R$ and $L' \dashv R'$ are two adjunctions on a category $\mathcal{D}$, then there is a bijection between natural transformations $u\colon L \to L'$ and natural transformations $v\colon R' \to R$ compatible with the vertical and horizontal composition of natural transformations. Since ζ corresponds to ξ and 1_F to 1_G, by compatibility $F\xi = \xi F$ being isomorphisms implies $\zeta G = G\zeta$ being isomorphisms.

(2) By Lemma 1 we have a coreflective subcategory $\mathrm{Fix}(\mathcal{C}, \xi)$ with the coreflection functor $F|^{\mathrm{Fix}(\mathcal{C},\xi)}$. Clearly $G|_{\mathrm{Fix}(\mathcal{C},\xi)}$ is right adjoint to the coreflection, making $\mathrm{Fix}(\mathcal{C}, \xi)$ an essential colocalization of $\mathcal{C}$, meaning that $G|_{\mathrm{Fix}(\mathcal{C},\xi)}$ is full and faithful according to Proposition 3.4.2 of [3]. Therefore ε_A is invertible for objects A from $\mathrm{Fix}(\mathcal{C}, \xi)$ (Proposition 3.4.1 of [3]). This gives us $\mathrm{Fix}(\mathcal{C}, \xi) \subseteq \mathrm{Fix}(\mathcal{C}, \varepsilon)$. To get the reverse inclusion, consider $A \in \mathrm{Fix}(\mathcal{C}, \varepsilon)$ and calculate

$$\xi_A F(\varepsilon_A) = \varepsilon_A \xi_{F(G(A))}$$

using the naturality of ξ. Since the morphisms other than ξ_A in the last equality are invertible, ξ_A must be invertible, meaning that $A \in \mathrm{Fix}(\mathcal{C}, \xi)$.

(3) Analogous to (2).

(4) Note that any adjunction restricts to an adjoint equivalence between $\mathrm{Fix}(\mathcal{C}, \eta)$ and $\mathrm{Fix}(\mathcal{C}, \varepsilon)$ due to the triangle identities (Section 0.4 of [19]). $\square$

Let A and B be firm objects of the lax-unital bicategory $\mathcal{B}$. We will now apply this lemma to the functor

$$-I\colon RU\mathcal{B}(A, B) \to RU\mathcal{B}(A, B)$$

with copointed structure provided by $r\colon -I \to 1$. According to the last section this functor has a right adjoint

$$I{\blacktriangleright}-\colon RU\mathcal{B}(A, B) \to RU\mathcal{B}(A, B)\,.$$

We will denote the idempotent pointed structure that will exist on $I{\blacktriangleright}-$ according to the lemma by $i\colon 1 \to I{\blacktriangleright}-$ and we will denote the category $\mathrm{Fix}(RU\mathcal{B}(A, B), i)$ by $RN\mathcal{B}(A, B)$. Then, according to the lemma, we have the following proposition, since $\mathrm{Fix}(RU\mathcal{B}(A, B), r)$ is just $RF\mathcal{B}(A, B)$.

Proposition 13. *The category $RN\mathcal{B}(A, B)$ that we just defined is a reflective subcategory of $RU\mathcal{B}(A, B)$ and we have an equivalence of categories*

$$-I\colon RN\mathcal{B}(A, B) \to RF\mathcal{B}(A, B)\,.$$

The reason we are interested in $RN\mathcal{B}(A, B)$ is that it combines the invertibility of the natural transformation $i\colon 1 \to I{\blacktriangleright}-$ with the fact that the epimorphisms in $RN\mathcal{B}(A,B)$ are the same as the ones in $\mathcal{B}^U(A,B)$ – a fact that we will prove in the next section.

We call 1-cells of $\mathcal{B}$ which lie in $RN\mathcal{B}(A,B)$ *non-singular unitary* 1*-cells*. We take this name from semigroup theory, where the unitary acts characterized by the invertibility of i are called non-singular unitary acts and the category they form is denoted by NAct_S. From the last proposition it follows that for a firm semigroup S, the epimorphisms in the category NAct_S of non-singular unitary S-acts are surjective.

Remark 9. We are interested in the categories $RN\mathcal{B}(A,B)$ for firm A and C. Unfortunately that stops us from talking about $RN\mathcal{B}$ as a lax-unital subbicategory of $\mathcal{B}$, since in general it will not be one.

4.4 On the epimorphicity of 2-cells

Let A and B be firm objects of a lax-unital bicategory $\mathcal{B}$. We need to study how the categories $\mathcal{B}(A,B)$, $RU\mathcal{B}(A,B)$, $RF\mathcal{B}(A,B)$ and $RN\mathcal{B}(A,B)$ relate to each other with respect to the epimorphicity of their morphisms. We want to show that a morphism of $RN\mathcal{B}(A,B)$ is also an epimorphism as a morphism of $\mathcal{B}(A,B)$.

Since $RF\mathcal{B}(A,B)$ is a coreflective subcategory of $\mathcal{B}(A,B)$, we have the following proposition.

Proposition 14. *Morphisms of $RF\mathcal{B}(A,B)$ are epimorphic precisely when they are epimorphic as morphisms in $\mathcal{B}(A,B)$.*

Proposition 15. *A morphism f of $RN\mathcal{B}(A,B)$ is an epimorphism precisely when it is an epimorphism as a morphism of $\mathcal{B}(A,B)$.*

Proof. Suppose that $f\colon M \to N$ is a morphism of $RN\mathcal{B}(A,B)$. Since

$$-I\colon RN\mathcal{B}(A,B) \to RF\mathcal{B}(A,B)$$

is an equivalence functor, f is an epimorphism in $RN\mathcal{B}(A,B)$ if and only if $f1_I\colon MI \to NI$ is an epimorphism in $RF\mathcal{B}(A,B)$, which by Proposition 14 occurs pecisely when $f1_I$ is epimorphic in $\mathcal{B}(A,B)$. If we have a look at the naturality diagram of r with respect to f

$$\begin{array}{ccc} MI & \xrightarrow{f1_I} & NI \\ {\scriptstyle r_M}\downarrow & & \downarrow{\scriptstyle r_N} \\ M & \xrightarrow[f]{} & N\,, \end{array}$$

and consider that every morphism there besides f is epimorphic in $\mathcal{B}(A, B)$, we can see that f must also be epimorphic in $\mathcal{B}(A, B)$. □

We can also use these adjunctions to study monomorphisms in the various hom-categories, as was done in the author's joint paper with V. Laan [15].

4.5 On $\mathcal{E}$-equivalence in right closed lax-unital bicategories

The following theorem is one of the main results of this work and says that in the setting of this chapter, when epimorphisms have nice enough properties, the $\mathcal{E}$-equivalence of firm objects implies equivalence of the hom-categories of right firm 1-cells.

Theorem 3. *Let $\mathcal{B}$ be a right closed lax-unital bicategory in which the 2-cell factorizations in $\mathcal{B}$ are given by the epimorphic and the monomorphic 2-cells. Then, if two firm objects A and B of $\mathcal{B}$ are $\mathcal{E}$-equivalent, the categories $RF\mathcal{B}(C, A)$ and $RF\mathcal{B}(C, B)$ are also equivalent for any firm object C of $\mathcal{B}$.*

Proof. The $\mathcal{E}$-equivalence of A and B means that there exists an $\mathcal{E}$-Morita context (P, Q, θ, ϕ) from A to B. By applying Proposition 10 on the right and its analogue on the left, we can assume without loss of generality that P and Q are firm. First let us show that for any 1-cell M in $\mathcal{B}(C, A)$, the 2-cell

$$\theta \triangleright 1_M \colon I_M \triangleright M \to PQ \triangleright M$$

is a monomorphism in $\mathcal{B}(C, A)$. Let X be a 1-cell in $\mathcal{B}(C, A)$ and let

$$u, v \colon X \to I_A \triangleright M$$

be 2-cells such that

$$(\theta \triangleright 1)v = (\theta \triangleright 1)u\,.$$

Since the diagram

$$\begin{tikzcd}
X(PQ) \arrow[r, shift left, "u1"] \arrow[r, shift right, "v1"'] \arrow[d, "1\theta"'] & (I \triangleright M)(PQ) \arrow[r, "(\theta \triangleright 1)1"] \arrow[d, "1\theta"'] & ((PQ) \triangleright M)(PQ) \arrow[d, "\varepsilon_M^{PQ}"] \\
XI \arrow[r, shift left, "u1"] \arrow[r, shift right, "v1"'] & (I \triangleright M)I \arrow[r, "\varepsilon_M^I"'] & M
\end{tikzcd}$$

commutes and since 1θ is an epimorphism, the two composites in the bottom row coincide. Since these composites are the 2-cells corresponding to u and

v under the adjunction defining $\triangleright$, the 2-cells u and v must also coincide. This implies that $\theta \triangleright 1_M$ is a monomorphic 2-cell in $\mathcal{B}(C, A)$. Using Proposition 11, we see that $L(\theta \triangleright 1_M)$ is also a monomorphic 2-cell in $\mathcal{B}(C, A)$. But $L(\theta \triangleright 1_M)$ is equal to $\theta \blacktriangleright 1_M$, so in conclusion we have shown that $\theta \blacktriangleright 1_M$ is a monomorphic 2-cell in $\mathcal{B}(C, A)$. It is however, by the same proposition, also monomorphic in $RU\mathcal{B}(C, A)$.

From now on let us assume that M lies in $RN\mathcal{B}(C, A)$. Let

$$p \colon (LU) \blacktriangleright V \to L \blacktriangleright (U \blacktriangleright V)$$

be the natural invertible 2-cell that corresponds to the associator under the adjunction that defines $\blacktriangleright$. We define

$$\bar{\theta}_M \colon M \to P \blacktriangleright (Q \blacktriangleright M)$$

to be the composite

$$M \xrightarrow{i_M} I \blacktriangleright M \xrightarrow{\theta \blacktriangleright 1} (PQ) \blacktriangleright M \xrightarrow{p} P \blacktriangleright (Q \blacktriangleright M)$$

and $\bar{\phi}_M \colon M \to Q \blacktriangleright (P \blacktriangleright M)$ to be the composite

$$M \xrightarrow{i_M} I \blacktriangleright M \xrightarrow{\phi \blacktriangleright 1} (QP) \blacktriangleright M \xrightarrow{p} Q \blacktriangleright (P \blacktriangleright M).$$

Since $\bar{\theta}$ is a composite of natural transformations, it must itself be natural. Also, since p is always an isomorphism and i_M is an isomorphism for M in $RN\mathcal{B}(C, A)$, the morphisms $\bar{\theta}$ and $\bar{\phi}$ are both monomorphisms in $\mathcal{B}(C, A)$. Let us show that

$$P \blacktriangleright (Q \blacktriangleright \bar{\theta}_M) = \bar{\theta}_{P \blacktriangleright (Q \blacktriangleright M)} \,.$$

We have the following diagram:

$$\begin{tikzcd}
P \blacktriangleright (Q \blacktriangleright (P \blacktriangleright M)) & & & (PQ) \blacktriangleright (P \blacktriangleright M) \arrow[lll, "p"'] \\
P \blacktriangleright ((QP) \blacktriangleright M) \arrow[u, "1 \blacktriangleright p"] & (P(QP)) \blacktriangleright M \arrow[l, "p"] \arrow[r, "a \blacktriangleright 1"] & ((PQ)P) \blacktriangleright M \arrow[ur, "p"] & \\
P \blacktriangleright (I \blacktriangleright M) \arrow[u, "1 \blacktriangleright (\phi \blacktriangleright 1)"] & (PI) \blacktriangleright M \arrow[l, "p"'] \arrow[u, "(1\phi) \blacktriangleright 1"] & (IP) \blacktriangleright M \arrow[u, "(\theta 1) \blacktriangleright 1"] \arrow[r, "p"] & I \blacktriangleright (P \blacktriangleright M) . \arrow[uu, "\theta \blacktriangleright 1"'] \\
& P \blacktriangleright M \arrow[ul, "1 \blacktriangleright i_M"] \arrow[u, "r_P \blacktriangleright 1"] \arrow[ur, "l_P \blacktriangleright 1"] \arrow[urr, "i_{P \blacktriangleright M}"'] & &
\end{tikzcd}$$

The center part of this diagram commutes, since (P, Q, θ, ϕ) is a Morita context and the left and right squares commute by the naturality of p. The upper pentagon corresponds to the associativity pentagon under the adjunction that defines $\blacktriangleright$, so it commutes. See, for example, Proposition 3.12 in [23]. The two lower triangles correspond under the adjunction to two of the coherence axioms involving the lax unit.

Note that left composite of this diagram is $1_P \blacktriangleright \bar{\phi}_M$ and the right composite is $\bar{\theta}_{P\blacktriangleright M}$ giving us

$$1_P \blacktriangleright \bar{\phi}_M = \bar{\theta}_{P\blacktriangleright M}\,.$$

Doing the same while swapping θ and ϕ results in

$$1_Q \blacktriangleright \bar{\theta}_M = \bar{\phi}_{Q\blacktriangleright M}\,.$$

Now we have

$$P\blacktriangleright(Q\blacktriangleright\bar{\theta}_M) = P\blacktriangleright(\bar{\phi}_{Q\blacktriangleright M}) = \bar{\theta}_{P\blacktriangleright(Q\blacktriangleright M)}\,.$$

Let X be in $RN\mathcal{B}(C, A)$ and let

$$u, v\colon P\blacktriangleright(Q\blacktriangleright M) \to X$$

be such that $u\bar{\theta}_M = v\bar{\theta}_M$. Then we can use the naturality of $\bar{\theta}$ to calculate

$$\begin{aligned}
\bar{\theta}_X u &= (P\blacktriangleright(Q\blacktriangleright u))\bar{\theta}_{P\blacktriangleright(Q\blacktriangleright M)} \\
&= (P\blacktriangleright(Q\blacktriangleright u))(P\blacktriangleright(Q\blacktriangleright\bar{\theta}_M)) \\
&= (P\blacktriangleright(Q\blacktriangleright u\bar{\theta}_M)) \\
&= (P\blacktriangleright(Q\blacktriangleright v\bar{\theta}_M)) \\
&= \bar{\theta}_X v\,.
\end{aligned}$$

Since $\bar{\theta}_M$ is a monomorphism in $\mathcal{B}(C, A)$, it is also a monomorphism in $RN\mathcal{B}(C, A)$, because it is a reflective subcategory of $RU\mathcal{B}(C, A)$, in which $\bar{\theta}_M$ is a monomorphism.

Therefore we have $u = v$, meaning that $\bar{\theta}_M$ is an epimorphism in the category $RN\mathcal{B}(C, A)$. By Proposition 15, $\bar{\theta}$ will also be an epimorphism in the category $\mathcal{B}(C, A)$. Since by our assumption epimorphisms and monomorphisms form an orthogonal factorization system on $\mathcal{B}(C, A)$, we have shown that $\bar{\theta}_M$ must be an isomorphism. Similarly we can show that $\bar{\phi}_M$ is an isomorphism. Therefore we have an equivalence of categories

$$P\blacktriangleright -\colon RN\mathcal{B}(C, B) \leftrightarrows RN\mathcal{B}(C, A) : Q\blacktriangleright -\ .$$

This means, because of Proposition 13, that $RF\mathcal{B}(C, B)$ and $RF\mathcal{B}(C, A)$ must also be equivalent. $\square$

Therefore we have shown that firm $\mathcal{E}$-equivalence between firm objects implies the equivalence of hom-categories of firm 1-cells. Unfortunately that was under quite strict conditions. Many cases of interest do not satisfy the condition that epimorphisms and monomorphisms in hom-categories form an orthogonal factorization system. We note however that the lax-unital bicategory of non-unital rings and bimodules and the lax-unital bicategory of semigroups and two-sided actions do satisfy the condition.

Theorem 3 gives a bicategorical proof of the fact that, using the terminology of the paper [16], the strong Morita equivalence of firm semigroups implies their Morita equivalence.

Remark 10. In the last proof we essentially constructed a left wide Morita context $(P\blacktriangleright -, Q\blacktriangleright -, \bar{\theta}, \bar{\phi})$ between $RN\mathcal{B}(C,A)$ and $RN\mathcal{B}(C,B)$ in the (quite big) 2-category CAT of sufficiently large categories.

The calculation done to deduce the epimorphicity of $\bar{\theta}$ from its monomorphicity was essentially the application of the dual of Theorem 1 to that left wide Morita context in CAT, which as a 2-category is of course also an oplax-unital bicategory.

Chapter 5

The lax-unital bicategory of modules between semigroups in a monoidal category

In the section we will introduce our main source of lax-unital bicategories – the tensor product structure on bimodules between semigroup objects in a monoidal category. The main two examples of this construction that one can keep in mind are the case of non-unital rings and bimodules, in which case the monoidal category in question is the category of Abelian groups with the tensor product of Abelian groups, and the case of semigroups and two-sided semigroup actions (sometimes called biacts), in which case the monoidal category is the category of sets with the cartesian product.

We do not need to assume anything of the monoidal category to be able to define semigroup objects in it and bimodules between the semigroup objects. In order to define the tensor products of bimodules, we just need the monoidal category have coequalizers and that the monoidal product preserves these coequalizers in both variables. Of course we will later make additional assumptions like the existence of a right adjoint for the monoidal product, meaning the closedness of the monoidal category. This construction is quite standard.

Until the end of this thesis, $\mathcal{V}$ will denote a *monoidal category*. We will denote the monoidal product of objects A and B by their juxtaposition AB, the unit object of $\mathcal{V}$ by I and the structure maps by

$$\mathfrak{a}_{A,B,C}\colon (AB)C \to A(BC)\,, \quad \mathfrak{r}_A\colon AI \to A, \quad \mathfrak{l}_A\colon IA \to A\,.$$

We recall that the structure maps are required to be isomorphisms and satisfy the coherence property, meaning that any two morphisms constructed from $\mathfrak{a}$, $\mathfrak{r}$, $\mathfrak{l}$, their inverses and identity morphisms using morphism composition and the monoidal product must coincide when they have the same source and target objects.

Note that we can think of monoidal categories with bicategories with a single object. Their relationship is the same as the relationship between a monoid and a category.

In most cases the coherence property allows us to act as if the maps $\mathfrak{a}$, $\mathfrak{r}$ and $\mathfrak{l}$ were identity morphisms.

5.1 Semigroups and modules in $\mathcal{V}$

In this section we will give the definition of semigroups in $\mathcal{V}$ and related constructions. The constructions given are the same as in the case of monoids in $\mathcal{V}$ [23], except for the lack of identities and identity related axioms.

There is a natural notion of a semigroup in the monoidal category $\mathcal{V}$. Likewise, given two semigroups in $\mathcal{V}$, there is a natural notion of a one or two sided module between these semigroups.

Note that these definitions also make sense for semimonoidal $\mathcal{V}$, as do a lot of the proofs, but the existence of a unit object in $\mathcal{V}$ makes things easier for us.

Definition 10. A *semigroup* in $\mathcal{V}$ is an object of S of $\mathcal{V}$ along with an associative operation $m\colon SS \to S$. In other words, a semigroup in $\mathcal{V}$ is a pair (S, m), where S is an object of $\mathcal{V}$ and m is a morphism which makes the diagram

$$\begin{tikzcd}
(SS)S \arrow[rr, "\mathfrak{a}"] \arrow[d, "m1"'] & & S(SS) \arrow[d, "1m"] \\
SS \arrow[r, "m"'] & S & SS \arrow[l, "m"]
\end{tikzcd}$$

commute. Since outside of examples we will only be considering semigroups in some fixed monoidal category $\mathcal{V}$, we will not always explicitly mention $\mathcal{V}$ when talking about semigroups.

Example 3. In any monoidal category $\mathcal{V}$ there always exists the unit semigroup $(I, \mathfrak{r}_I)$ on the unit object I with the structure map $\mathfrak{r}_I = \mathfrak{l}_I$.

Definition 11. Let S be a semigroup in $\mathcal{V}$, then a *right S-module* is an object A of $\mathcal{V}$ together with a right action $r_A\colon AS \to A$ of S on A compatible with the semigroup operation of T, in other words, r_A needs to make the diagram

$$\begin{tikzcd}
(AS)S \arrow[rr, "\mathfrak{a}"] \arrow[d, "r_A 1"'] & & A(SS) \arrow[d, "1m"] \\
AS \arrow[r, "r_A"'] & A & AS \arrow[l, "r_A"]
\end{tikzcd}$$

commute. Similarly, for a semigroup T in $\mathcal{V}$, a *left T-module* is an object A of $\mathcal{V}$ together with a left action $l_A\colon TA \to A$ of T on A, making the diagram

$$\begin{tikzcd} (TT)A \arrow[rr, "\mathfrak{a}"] \arrow[d, "m1"'] & & T(TA) \arrow[d, "1r_A"] \\ TA \arrow[r, "l_A"'] & A & TA \arrow[l, "l_A"] \end{tikzcd}$$

commute. Finally, given semigroups S and T in $\mathcal{V}$ a *T-S-bimodule* will be an object A of $\mathcal{V}$, which carries a right S-module structure $r_A\colon AS \to S$ and a left T-module structure $l_A\colon TA \to A$ which are compatible with each other, meaning that the diagram

$$\begin{tikzcd} (TA)S \arrow[rr, "\mathfrak{a}"] \arrow[d, "l_A 1"'] & & T(AS) \arrow[d, "1r_A"] \\ AS \arrow[r, "r_A"'] & A & TA \arrow[l, "l_A"] \end{tikzcd}$$

commutes.

Example 4. If (S, m) is a semigroup in a monoidal category $\mathcal{V}$, then we always have a canonical S-S-bimodule structure on S with $l_S = r_S = m$.

We can define a *morphism of modules* in the obvious way as a morphism in $\mathcal{V}$ compatible with the right modules stucture.

Definition 12. Let S be a semigroup in $\mathcal{V}$ and let A and B be right S modules. A morphism $f\colon A \to B$ in $\mathcal{V}$ will be called a *morphism of right S-modules*, if the following diagram commutes

$$\begin{tikzcd} AS \arrow[r, "f1"] \arrow[d, "r_A"'] & BS \arrow[d, "r_B"] \\ A \arrow[r, "f"'] & B\,. \end{tikzcd}$$

Morphisms of left modules are defined in a similar fashion. A *morphism of bimodules* is a morphism compatible with both the left and right module structures.

Given a semigroups S and T in a monoidal category $\mathcal{V}$, we define, using the preceding definitions, the *category* Mod_S *of right S-modules*, the *category* ${}_T\mathrm{Mod}$ *of left T-modules* and the *category* ${}_T\mathrm{Mod}_S$ *of T-S-bimodules*. Note that ${}_I\mathrm{Mod}_S$ and ${}_T\mathrm{Mod}_I$ are clearly isomorphic to Mod_S and ${}_T\mathrm{Mod}$, respectively, which is quite handy, since we can define and prove things for bimodules and the corresponding results for one sided modules follow.

To define tensor products, we need coequalizers to exist in ${}_T\mathrm{Mod}_S$. Therefore we need to study colimits in that bicategory.

If $\mathcal{D}$ and $\mathcal{C}$ are categories, we will call a functor $\mathcal{D} \to \mathcal{C}$ a *diagram of shape* $\mathcal{D}$ and a (co)limit of such a diagram a *(co)limit of shape* $\mathcal{D}$. The following results about colimits and limits are proved almost exactly as they are proved for the case of monoids in monoidal categories. The only difference is the lack of units.

Proposition 16. *Let S and T be semigroups in a monoidal category $\mathcal{V}$. Let us suppose that colimits of shape $\mathcal{D}$ exist in $\mathcal{V}$ and that the monoidal product functors $X \mapsto XS$ and $X \mapsto TX$ of $\mathcal{V}$ preserve all such colimits. Then ${}_T\mathrm{Mod}_S$ has colimits of shape $\mathcal{D}$. To be more precise, the forgetful functor $U\colon {}_T\mathrm{Mod}_S \to \mathcal{V}$ creates colimits of shape $\mathcal{D}$.*

Proof. Let $D\colon \mathcal{D} \to {}_T\mathrm{Mod}_S$ be a diagram of shape $\mathcal{D}$ in ${}_T\mathrm{Mod}_S$. Let $\big(L, (\theta_i)_{i\in\mathcal{D}}\big)$ be the colimit of UD. We can define a right S-module structure on L using the universal property of colimits as the unique map $r_L\colon LS \to L$ making the following diagram

$$\begin{tikzcd}
D(i)S \arrow[rr, "D(x)1"] \arrow[drr, "r_{D(i)}"'] \arrow[dr, "\theta_i 1"'] & & D(j)S \arrow[drr, "r_{D(j)}"] \arrow[dl, "\theta_j 1"] & & \\
& LS \arrow[drr, dashed, "r_L"'] & D(i) \arrow[rr, "D(x)"'] \arrow[dr, "\theta_i"] & & D(j) \arrow[dl, "\theta_j"] \\
& & & L &
\end{tikzcd}$$

commute for any objects i, j of $\mathcal{D}$ and all morphisms $x\colon i \to j$. This definition makes all $\theta_i\colon D(i) \to L$ morphisms of right S-modules.

To see that r_L indeed makes L into a right S-module, let i be any element of $\mathcal{D}$ and behold the following diagram.

$$\begin{tikzcd}
(D(i)S)S \arrow[rrrr, "\mathfrak{a}"] \arrow[dr, "(\theta_i 1)1"'] \arrow[ddd, "r_{D(i)}1"'] & & & & D(i)(SS) \arrow[dl, "\theta_i(11)"] \arrow[ddd, "1m"] \\
& (LS)S \arrow[rr, "\mathfrak{a}"] \arrow[d, "r_L 1"'] & & L(SS) \arrow[d, "1m"] & \\
& LS \arrow[dr, "r_L"'] & & LS \arrow[dl, "r_L"] & \\
D(i)S \arrow[ur, "\theta_i 1"] \arrow[drr, "r_{D(i)}"'] & & L & & D(i)S \arrow[ul, "\theta_i 1"'] \arrow[dll, "r_{D(i)}"] \\
& & D(i) \arrow[u, "\theta_i"] & &
\end{tikzcd}$$

Observe that the outer pentagon and every quadrilateral commutes. By using diagram chasing, we can see that the inner pentagon commutes under $(\theta_i 1)1$. However, since i was an arbitrary element of $\mathcal{D}$ and $(\theta_i 1)1$, $i \in \mathcal{D}$ constitute a colimiting cocone, thus being jointly epimorphic, we can conclude that the inner pentagon commutes.

Similarily, we can define a left T-module structure $l_L \colon TL \to L$ on L. To see that the right module structure and the left module structure on L are compatible, consider the diagram

$$\begin{tikzcd}
(TD(i))S \arrow[rrrr, "\mathfrak{a}"] \arrow[dr, "(1\theta_i)1"'] \arrow[ddd, "l_{D(i)}1"'] & & & & T(D(i)S) \arrow[dl, "1(\theta_i 1)"] \arrow[ddd, "1r_{D(i)}"] \\
& (TL)S \arrow[rr, "\mathfrak{a}"] \arrow[d, "l_L 1"'] & & T(LS) \arrow[d, "1r_L"] & \\
& LS \arrow[dr, "r_L"'] & & TL \arrow[dl, "l_L"] & \\
D(i)S \arrow[ur, "\theta_i 1"] \arrow[drr, "r_{D(i)}"'] & & L & & TD(i) \arrow[ul, "1\theta_i"'] \arrow[dll, "l_{D(i)}"] \\
& & D(i) \arrow[u, "\theta_i"] & &
\end{tikzcd}$$

and reason in the same way as for the digram before that.

Next let us check that L is the colimit of D. Suppose that we have another T-S-bimodule Q along with a cocone $\kappa_i \colon D(i) \to Q$, $i \in \mathcal{D}$ on D. Using the fact that L is a colimit of D in $\mathcal{V}$, we get a morphism $m \colon L \to Q$ in $\mathcal{V}$. We need to show that m is a morphism of T-S-bimodules. Let $x \colon i \to j$ be an arbitrary morphism in $\mathcal{D}$ and behold the following diagram.

$$\begin{tikzcd}
D(i)S \arrow[rr, "D(x)1"] \arrow[drr, "r_{D(i)}"] \arrow[ddr, "\theta_i 1"] \arrow[ddddr, "\kappa_i 1"'] & & D(j)S \arrow[drr, "r_{D(j)}"] \arrow[ddl, "\theta_j 1"'] \arrow[ddddl, "\kappa_j 1"] & & \\
& & D(i) \arrow[rr, "D(x)"'] \arrow[ddr, "\theta_i"] \arrow[ddddr, "\kappa_i"'] & & D(j) \arrow[ddl, "\theta_j"'] \arrow[ddddl, "\kappa_j"] \\
& LS \arrow[drr, "r_L"] \arrow[dd, dashed, "m1"'] & & & \\
& & & L \arrow[dd, dashed, "m"] & \\
& QS \arrow[drr, "r_Q"'] & & & \\
& & & Q\,. &
\end{tikzcd}$$

In the same way as we showed the commutativity of the inner pentagon in the last diagram, we can show the commutativity of the lower quadrilateral by using the fact that the other parts of the diagram commute and the fact that $\theta_i 1$ constitute a colimiting cocone. This shows that m is a morphism of right S-modules. Similarily we can show that m is a morphism of left T-modules. The uniqueness of m as a T-S-module morphism comes from its uniqueness as a morphism in $\mathcal{V}$. This concludes the proof that the forgetful functor U creates colimits of shape $\mathcal{D}$. □

There is a similar result for limits, with a proof along the same lines, which does not require anything from the monoidal product functor of $\mathcal{V}$.

Proposition 17. *Let S and T be semigroups in a monoidal category $\mathcal{V}$. Let us suppose that limits of shape $\mathcal{D}$ exist in $\mathcal{V}$. Then ${}_T\mathrm{Mod}_S$ has limits of shape $\mathcal{D}$ and the forgetful functor $U\colon {}_T\mathrm{Mod}_S \to \mathcal{V}$ creates limits of shape $\mathcal{D}$.*

We will need the following construction to define tensor products

Construction 1. Let S, T and R be semigroups in $\mathcal{V}$, let A be an R-T-bimodule and let B be a T-S-bimodule. Then we define an R-S-bimodule structure on AB with the following structure maps

$$l_{AB} = (l_A 1_B) \circ \mathfrak{a}^{-1} \colon T(AB) \to AB, \quad r_{AB} = (1_A r_B) \circ \mathfrak{a} \colon (AB)S \to AB\,.$$

This defines a functor ${}_R\mathrm{Mod}_T \times {}_T\mathrm{Mod}_R \to {}_R\mathrm{Mod}_S$, which acts on the object part of modules and on morphisms as the monoidal product functor of $\mathcal{V}$, meaning that $f\colon A \to B$ and $g\colon A' \to B'$ will map to $fg\colon AB \to A'B'$ for f a morphism in ${}_R\mathrm{Mod}_T$ and g a morphism in ${}_S\mathrm{Mod}_T$. The details are easy to verify and the verification is in part done in Proposition 2.04 of [14].

Now that we have this functor, we can note that structure maps of the modules and the associator of $\mathcal{V}$ induce maps of bimodules, as noted by the next proposition.

Proposition 18. *Let S, T, R and W be semigroups in $\mathcal{V}$. Let A be a W-R-bimodule, B an R-T bimodule, and C a T-S-bimodule. Then the following maps are morphisms of right S-modules natural in A (and B, C in the case of $\mathfrak{a}$).*

1. $\mathfrak{a}\colon (AB)C \to A(BC)$
2. $r_A\colon AR \to A$
3. $l_A\colon WA \to A$
4. $m\colon SS \to S$.

Proof. For $\mathfrak{a}$, note that up to $\mathfrak{a}$ the left action of W on $(AB)C$ and $A(BC)$ is $l_A 11$ and similarly for the right action.

For r_A, being a map of left W-modules is basically equivalent to the compatibility between the left and right module structures of A and being a map of right R-modules is equivalent to r_A giving A a right R-module structure. For l_A we have an analogous explanation and m is just a special case, since $m = r_S = l_S$. □

5.2 Tensor products

Starting from this section we will require that $\mathcal{V}$ **has coequalizers and that the monoidal product of $\mathcal{V}$ preserves them in both variables.** Some authors express this situation by saying that the monoidal category $\mathcal{V}$ *has stable coequalizers.*

The definition of tensor products of bimodules we use is standard and the same as the one used in the monoidal case. For some proofs regarding tensor products we will refer to [14], which, while written in a slightly different and more general setting and using somewhat different terminology, still has proofs that apply to our situation.

Proposition 19. *Suppose that colimits of shape $\mathcal{D}$ exist in $\mathcal{V}$ and the monoidal product functor of $\mathcal{V}$ preserves them in both variables. Then for any semigroups S, T, R in $\mathcal{V}$ the functor*

$$_R\mathrm{Mod}_T \times {}_T\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S, \quad (A, B) \mapsto AB$$

preserves colimits of shape $\mathcal{D}$.

Proof. Let us show that the functor in question preserves colimits of shape $\mathcal{D}$ in the right variable. The colimit preservation in the left variable can be shown in the same fashion. Let A be an R-T-bimodule and let us reason about the functor

$$A-\colon {}_T\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S, \quad B \mapsto AB\,.$$

We have the following diagram of categories, which clearly commutes:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "D"] & {}_T\mathrm{Mod}_S \arrow[r, "A-"] \arrow[d, "U"'] & {}_R\mathrm{Mod}_S \arrow[d, "U"] \\ & \mathcal{V} \arrow[r, "A-"'] & \mathcal{V}\,. \end{tikzcd}$$

Since U creates colimits of shape $\mathcal{D}$ and $X-\colon \mathcal{V} \to \mathcal{V}$ preserves colimits of shape $\mathcal{D}$, we can easily deduce that the functor $A-\colon {}_T\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S$ preserves colimits of shape D as follows:

$$\begin{aligned} U(A-)\mathrm{colim}D &= (A-)U\mathrm{colim}D \\ &= (A-)\mathrm{colim}(UD) \\ &= \mathrm{colim}((A-)UD) \\ &= \mathrm{colim}(U(A-)D) \\ &= U\mathrm{colim}((A-)D)\,. \end{aligned}$$

□

Construction 2 (Tensor product of bimodules)**.** Let S, T and R be semigroups in $\mathcal{V}$, let A be a right R-T-bimodule and B be an T-S-bimodule. We define the tensor product $A \otimes B$ of A and B using the following coequalizer diagram in ${}_R\mathrm{Mod}_S$:

$$\begin{tikzcd} (AT)B \arrow[rr, "r_A 1"] \arrow[dr, "\mathfrak{a}"'] & & AB \arrow[r, "\omega_{A,B}"] & A \otimes B\,. \\ & A(TB) \arrow[ur, "1 l_B"'] & & \end{tikzcd}$$

Proposition 20. *Let S, T and R be semigroups in $\mathcal{V}$. Then we get a functor*

$$- \otimes -\colon {}_R\mathrm{Mod}_T \times {}_T\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S$$

such that for a morphisms $f\colon A \to B$ in ${}_R\mathrm{Mod}_T$ and $g\colon A' \to B'$ in ${}_T\mathrm{Mod}_S$, the morphism $f \otimes g\colon A \otimes A' \to B \otimes B'$ is the unique morphism making the *diagram*

$$\begin{tikzcd}
AA' \arrow[r, "fg"] \arrow[d, "\omega_{A,A'}"'] & BB' \arrow[d, "\omega_{B,B'}"] \\
A \otimes A' \arrow[r, dashed, "f \otimes g"'] & B \otimes B'
\end{tikzcd}$$

commute.

Proof. It is evident that $- \otimes -$ is a functor when $f \otimes g$ is defined using the universal property of being the unique morphism making the preceding diagram commute. To show that such an universal property is satisfied, we can simply note that we have the following morphism of coequalizer cocones

$$\begin{tikzcd}
(AS)A' \arrow[r, "\mathfrak{a}"] \arrow[dd, "(f1)g"'] & A(SA') \arrow[r, "1l_{A'}"] \arrow[d, "f(1g)"'] & AA' \arrow[r, "\omega_{A,A'}"] \arrow[dd, "fg"] & A \otimes A' \arrow[dd, dashed, "f \otimes g"] \\
 & B(SB') \arrow[dr, "1l_{B'}"] & & \\
(BS)B' \arrow[ur, "\mathfrak{a}"] \arrow[rr, "r_B 1"'] & & BB' \arrow[r, "\omega_{B,B'}"] & B \otimes B' \\
(AS)A' \arrow[u, "(f1)g"] \arrow[rr, "r_A 1"'] & & AA' \arrow[u, "fg"'] \arrow[r, "\omega_{A,A'}"'] & A \otimes A', \arrow[u, dashed, "f \otimes g"']
\end{tikzcd}$$

which does give us the univeral property we need. □

Proposition 21. *The tensor product of bimodules as we just defined is associative up to a coherent natural isomorphism. The associator of the tensor product will henceforth be denoted by*

$$\alpha \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C).$$

It makes the diagram

$$\begin{tikzcd}
(AB)C \arrow[r, "\omega 1"] \arrow[d, "\mathfrak{a}"'] & (A \otimes B)C \arrow[r, "\omega"] & (A \otimes B) \otimes C \arrow[d, "\alpha"] \\
A(BC)' \arrow[r, "1\omega"'] & A(B \otimes C) \arrow[r, "\omega"'] & A \otimes (B \otimes C)
\end{tikzcd}$$

commute and is coherent.

Proof. For the proof we refer to Proposition 2.09. of [14]. □

If A is a T-S-bimodule, then the structure map $r_A\colon AS \to A$ induces the map $\rho_A\colon A \otimes S \to A$ in ${}_T\mathrm{Mod}_S$, defined using the universal property of coequalizers using the diagram

$$\begin{tikzcd}
(AS)S \arrow[rr, "r_A 1"] \arrow[dr, "a"'] & & AS \arrow[rr, "\omega_{A,S}"] \arrow[dr, "r_A"'] & & A \otimes S \arrow[dl, dashed, "\rho_A"] \\
& A(SS) \arrow[ur, "1m"'] & & A &
\end{tikzcd} .$$

Similarly the left S-module structure on A induces a map $\lambda_A\colon S \otimes A \to A$.

A special case of this is the map $m\colon SS \to S$, viewed as the structure map of a right S-module, inducing the map $\mu\colon S \otimes S \to S$.

Proposition 22. *Let S be a semigroup in a monoidal category $\mathcal{V}$. Then the morphisms ρ_A, λ_A as defined above form natural transformations*

$$\rho\colon -\otimes S \to 1\colon {}_T\mathrm{Mod}_S \to {}_T\mathrm{Mod}_S\,,$$

$$\lambda\colon T \otimes - \to 1\colon {}_T\mathrm{Mod}_S \to {}_T\mathrm{Mod}_S$$

and they make the diagrams

$$\begin{tikzcd}
(A \otimes T) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho_A \otimes 1"'] & & A \otimes (T \otimes B) \arrow[dl, "1 \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}, \quad
\begin{tikzcd}
(A \otimes B) \otimes S \arrow[rr, "\alpha"] \arrow[dr, "\rho_{A \otimes B}"'] & & A \otimes (B \otimes S) \arrow[dl, "1 \otimes \rho_B"] \\
& A \otimes B &
\end{tikzcd}$$

commute. The symmetric analogue of the second triangle also commutes.

Proof. Since every map besides ρ_A in the diagram defining ρ_A is natural in A, so will be ρ_A. Same for λ_A.

To see that the first triangle in the statement of the theorem commutes, observe that the outer composition and all the small parts besides the triangle commute in the following diagram

$$\begin{tikzcd}
(AT)B \arrow[rr, "\omega 1"] \arrow[dr, "r_A 1"] \arrow[dd, "\mathfrak{a}"'] & & (A \otimes T)B \arrow[rr, "\omega"] \arrow[dl, "\rho_A 1"'] & & (A \otimes T) \otimes B \arrow[dl, "\rho_A \otimes 1"] \arrow[dd, "\alpha"] \\
& AB \arrow[rr, "\omega"] & & A \otimes B & \\
A(TB) \arrow[rr, "1\omega"'] \arrow[ur, "1 l_B"] & & A(T \otimes B) \arrow[rr, "\omega"'] \arrow[ul, "1\lambda_B"'] & & A \otimes (T \otimes B) \arrow[ul, "1 \otimes \lambda_B"']
\end{tikzcd},$$

which implies that that the triangle also commutes, since ω is a regular epimorphism and so is $\omega 1$, since the functor $-B\colon {}_R\mathrm{Mod}_T \to {}_R\mathrm{Mod}_S$ preserves colimits.

The commutativity of the second triangle follows much in the same fashion from the diagram

$$\begin{tikzcd}
(AB)T \arrow[rr, "\omega 1"] \arrow[dr, "r_{AB}"] \arrow[dd, "\mathfrak{a}"'] & & (A \otimes B)T \arrow[rr, "\omega"] \arrow[dr, "r_{A\otimes B}"] & & (A \otimes B) \otimes T \arrow[dl, "\rho_{A\otimes B}"'] \arrow[dd, "\alpha"] \\
& AB \arrow[rr, "\omega"] & & A \otimes B & \\
A(BT)' \arrow[rr, "1\omega"'] \arrow[ur, "1r_B"] & & A(B \otimes T) \arrow[rr, "\omega"'] \arrow[ul, "1\rho_B"'] & & A \otimes (B \otimes T)\,. \arrow[ul, "1 \otimes \rho_B"']
\end{tikzcd}$$

□

We have now constructed all the components of the lax-unital bicategory of Mod of semigroups and bimodules in $\mathcal{V}$ and almost proved that all the necessary diagrams commute. The last thing we need is that $\rho_S = \lambda_S$, but this holds trivially, since $r_S = l_S$. Thus we have the following theorem:

Theorem 4. *The semigroups and bimodules in a monoidal category $\mathcal{V}$ with respect to the tensor product of bimodules make up a lax-unital bicategory* Mod. *The unitors of this lax-unital bicategory are ρ and λ and the associator is α.*

The notable difference with the case of monoids is that in the semigroup case ρ and λ are not always be invertible. Just as we did for lax-unital bicategories in general, we can restrict our attention to *firm* semigroups in $\mathcal{V}$ and firm modules between them, which will give us a bicategory FMod of firm semigroups and firm bimodules.

5.3 Firm modules

It is difficult to prove nice Morita type results for arbitrary semigroups and modules in $\mathcal{V}$ other than the ones that derive from the fact that they form a lax-unital bicategory. In this chapter firm semigroups and modules will be one of our main objects of interest. First let us just recap what firmness means in terms of semigroups and bimodules in $\mathcal{V}$.

Definition 13. Let S and T be semigroups in a monoidal category $\mathcal{V}$. We say that a T-S-bimodule A is *firm* if the maps $\rho_A \colon A \otimes S \to A$ and $\lambda_A \colon A \otimes T \to A$ are isomorphisms. If only λ_A or ρ_A is an isomorphism we say that A is firm as a left T-module or a right S-module respectively. Same for one sided modules.

Definition 14. We say that a semigroup S in $\mathcal{V}$ is *firm* if the associated map $\mu\colon S \otimes S \to S$ is an isomorphism. In other words S is firm if the bimodule associated to S is firm.

If S and T are firm semigroups in $\mathcal{V}$, then we will use ${}_T\mathrm{FMod}_S$ to denote the full subcategory of ${}_T\mathrm{Mod}_S$ induced by all the firm T-S-bimodules by ${}_T\mathrm{FMod}_S$. Similarly ${}_T\mathrm{FMod}$ and FMod_S will denote the full subcategories of ${}_T\mathrm{Mod}$ and Mod_S respectively, determined by the firm modules.

Proposition 23. *Let S and T be firm semigroups in $\mathcal{V}$ and let A be a T-S-biimodule. Then the following statements are equivalent:*

1. *A is a firm right S-module,*
2. *the following is a coequalizer diagram in ${}_T\mathrm{Mod}_S$.*

$$\begin{tikzcd}
(AS)S \arrow[rr, "r_A 1"] \arrow[dr, "a"'] & & AS \arrow[r, "r_A"] & A\,. \\
& A(SS) \arrow[ur, "1m"'] & &
\end{tikzcd}$$

A similar statement holds for the left module structure on A.

Proof. Since ρ_A was defined using the universal property of the coequalizer as the unique map such that $\rho_A\omega_{A,S} = r_A$, it will only be an isomorphism if r_A also coequalizes the same diagram. □

The last proposition says that our firm semigroups and firm modules are essentially the same as the interpolads and i-modules of [14]. Rephrasing some results about firmness in lax-unital bicategories we get:

Theorem 5.

- *Let S and T be semigroups in $\mathcal{V}$. Let A be a right S-module and let B be an T-S-bimodule which is firm as a right S-module. Then $A \otimes B$ is a firm right S-module. A similar statement holds for left modules and bimodules.*

- *Let S and T be firm semigroups in $\mathcal{V}$. The inclusion functor*

$$\mathcal{J}\colon {}_T\mathrm{FMod}_S \to {}_T\mathrm{Mod}_S$$

is left adjoint to the functor

$$(T \otimes -) \otimes S\colon {}_T\mathrm{Mod}_S \to {}_T\mathrm{FMod}_S\,, \quad A \mapsto (T \otimes A) \otimes S\,,$$

so ${}_T\mathbf{FMod}_S$ is a locally coreflective subcategory of ${}_T\mathrm{Mod}_S$.

- *Firm semigroups and firm bimodules in $\mathcal{V}$ form a bicategory.*

The last statement is essentially Theorem 3.02. of [14], although Koslowski used a different approach for proving it.

Proposition 24. *Let S, T and R be semigroups, let A be an R-T-bimodule and B a T-S-bimodule. If A is firm as a left R-module then so is AB. If B is firm as a right S-module, then so is AB.*

Proof. Suppose that B is firm as a right S-module. Then the diagram

$$\begin{tikzcd} (BS)S \arrow[rr, "r_B 1"] \arrow[dr, "\mathfrak{a}"'] & & BS \arrow[r, "r_B"'] & B\,. \\ & B(SS) \arrow[ur, "1m"'] & & \end{tikzcd}$$

is a coequalizer diagram in ${}_R\mathrm{Mod}_S$ by Proposition 23. Since the functor $A-$ preserves colimits in Mod_S, the following is also a coequalizer diagram in Mod_S

$$\begin{tikzcd} A((BS)S) \arrow[rr, "1(r_B 1)"] \arrow[dr, "1\mathfrak{a}"'] & & A(BS) \arrow[r, "1r_B"'] & AB\,, \\ & A(B(SS)) \arrow[ur, "1(1m)"'] & & \end{tikzcd}$$

which modulo associativity tells us that AB is firm as a right S-module.

The proof of the other part follows by symmetry. □

5.4 The Eilenberg-Watts theorem

Now we will prove the Eilenberg-Watts theorem for one-sided modules in $\mathcal{V}$.

Note that when we take $R = T = I$ for the functor,

$${}_R\mathrm{Mod}_T \times {}_T\mathrm{Mod}_S \to {}_R\mathrm{Mod}_S, \quad (A, B) \mapsto AB$$

we get a functor

$$\mathcal{V} \times \mathrm{Mod}_S \to \mathrm{Mod}_S, \quad (X, B) \mapsto XB\,,$$

which defines an action of the monoidal category $\mathcal{V}$ on the category Mod_S. Equivalently we could have taken $R = T = I$ in the tensor product functor.

To prove a version of the Eilenberg-Watts theorem for semigroups in $\mathcal{V}$, we need the functors between the module categories to respect the structure of these categories in the following sense.

Definition 15. Let S and T be semigroups in $\mathcal{V}$. We will call a functor $F\colon \mathrm{Mod}_S \to \mathrm{Mod}_T$ along with a natural transformation $\xi\colon XF(A) \to F(XA)$ a *$\mathcal{V}$-functor* if the natural transformation ξ makes the following diagram commute for all $X, Y \in \mathcal{V}$ and A in Mod_S.

$$\begin{tikzcd}
(XY)F(A) \arrow[rr, "\xi"] \arrow[d, "a"'] & & F((XY)A) \arrow[d, "F(a)"] \\
X(YF(A)) \arrow[r, "1\xi"'] & XF(YA) \arrow[r, "\xi"'] & F(X(YA))
\end{tikzcd}$$

This definition can be found for example in [24] or [10].

Example 5.
- In the case that $\mathcal{V}$ is Set with the cartesian product, XA is the X-fold coproduct in Mod_S of A with itself. Coproduct is in this case the disjoint union with the obvious induced actions. All functors are $\mathcal{V}$-functors and the map ξ is induced by the universal property of the coproduct. The map ξ is an isomorphism precisely when F preserves coproducts.
- In the case that $\mathcal{V}$ is Ab, the category of Abelian groups with the tensor product, $\mathcal{V}$-functors are precisely the additive functors.

Now we are ready to prove a version of the Eilenberg-Watts theorem.

Theorem 6. *Let S and T be firm semigroups in $\mathcal{V}$. Let F be a coequalizer preserving $\mathcal{V}$-functor $F\colon \mathrm{FMod}_S \to \mathrm{FMod}_T$ with an invertible ξ. Then the functor F is isomorphic to the functor $- \otimes F(S)$, where the left S-module structure on $F(S)$ is defined as the composition*

$$\begin{tikzcd}
SF(S) \arrow[rr, "l_{F(S)}"] \arrow[dr, "\xi"'] & & F(S)\,. \\
& F(SS) \arrow[ur, "F(m)"'] &
\end{tikzcd}$$

Proof. First we will define the morphisms $\tau_A\colon A \otimes F(S) \to F(A)$. Since A is a firm right S-module, the following is a coequalizer diagram in FMod_S

$$\begin{tikzcd}
(AS)S \arrow[rr, "r_A 1"] \arrow[dr, "a"'] & & AS \arrow[r, "r_A"] & A\,. \\
& A(SS) \arrow[ur, "1m"'] & &
\end{tikzcd}$$

The next diagram shows that ξ provides us with an isomorphism of coequalizer cocones between the image under F of the preceding coequalizer and the coequalizer defining the tensor product of A and $F(S)$. This isomorphism of coequalizer cocones in FMod_T defines the morphism τ_A of right T-modules.

$$
\begin{tikzcd}
(AS)F(S) \arrow[r, "a"] \arrow[ddd, "\xi"'] & A(SF(S)) \arrow[r, "1l_{F(S)}"] \arrow[d, "1\xi"'] & AF(S) \arrow[r, "\omega_{A,F(S)}"] \arrow[ddd, "\xi"] & A \otimes F(S) \arrow[ddd, dashed, "\tau_A"] \\
& AF(SS) \arrow[ur, "1F(m)"'] \arrow[d, "\xi"'] & & \\
& F(A(SS)) \arrow[dr, "F(1m)"] & & \\
F((AS)S) \arrow[ur, "F(a)"] \arrow[rr, "F(r_A 1)"'] & & F(AS) \arrow[r, "F(r_A)"] & F(A) \\
(AS)F(S) \arrow[u, "\xi"] \arrow[rr, "r_A 1"'] & & AF(S) \arrow[u, "\xi"'] \arrow[r, "\omega_{A,F(S)}"'] & A \otimes F(S) \arrow[u, dashed, "\tau_A"']
\end{tikzcd}
$$

Now we just need to show that $\tau_A \colon A \otimes F(S) \to F(A)$ is natural in A. Behold the following diagram:

$$
\begin{tikzcd}
AF(S) \arrow[rr, "f1"] \arrow[dr, "\omega_{A,F(S)}"] \arrow[dd, "\xi"'] & & BF(S) \arrow[dr, "\omega_{B,F(S)}"] \arrow[dd, dashed, "\xi"' near end] & \\
& A \otimes F(S) \arrow[rr, "f \otimes F(S)" near start] \arrow[dd, "\tau_A"' near start] & & B \otimes F(S) \arrow[dd, "\tau_B"] \\
F(AS) \arrow[rr, dashed, "F(f1)"' near end] \arrow[dr, "F(r_A)"'] & & F(BS) \arrow[dr, dashed, "F(r_B)"] & \\
& F(A) \arrow[rr, "F(f)"'] & & F(B)\,.
\end{tikzcd}
$$

Since every face of the cube other than the front face clearly commutes, the front face commutes under $\omega_{A,F(S)}$. Since $\omega_{A,F(S)}$ is a regular epimorphism, the front face of the cube also commutes, which means that τ is indeed natural. $\square$

5.5 Orthogonal factorizaton systems on $\mathcal{V}$

Now let us show that an orthogonal factorization system on $\mathcal{V}$ will extend virtually unchanged to an orthogonal factorization system on each of the hom-categories of the lax-unital bicategory Mod.

Proposition 25. *Suppose that $(\mathfrak{E}, \mathfrak{M})$ is an orthogonal factorization system on $\mathcal{V}$ such that the monoidal product of $\mathcal{V}$ maps morphims in $\mathfrak{E}$ to morphisms* *in $\mathfrak{E}$ in both variables. Then each category ${}_T\mathrm{Mod}_S$, where S and T are semigroups in $\mathcal{V}$, has a factorization system such that all morphism of bimodules are factorized as morphisms of $\mathcal{V}$ and the left and right actions on the middle object M of the factorization*

$$\begin{tikzcd} A \arrow[r, "e"'] \arrow[rr, bend left, "f"] & M \arrow[r, "m"'] & B. \end{tikzcd}$$

of $f\colon A \to B$ are defined as the unique morphisms making the diagrams

$$\begin{tikzcd} AS \arrow[r, "e1"] \arrow[d, "r_A"'] & MS \arrow[r, "m1"] \arrow[d, dashed, "r_M"] & BS \arrow[d, "r_B"] \\ A \arrow[r, "e"'] & M \arrow[r, "m"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TA \arrow[r, "1e"] \arrow[d, "l_A"'] & TM \arrow[r, "1m"] \arrow[d, dashed, "l_M"] & TB \arrow[d, "l_B"] \\ A \arrow[r, "e"'] & M \arrow[r, "m"'] & B \end{tikzcd}$$

commute.

Proof. To show that factorizations in ${}_T\mathrm{Mod}_S$ are inherited from $\mathcal{V}$, first we will to check that the diagonal fill-in morphism s of a square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dashed, "s"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

of T-S-bimodule morphisms is a morphism of T-S-bimodules. Let us check that the fill-in is compatible with the right S-module structure. We have the following commutative diagram:

$$\begin{tikzcd} AS \arrow[rr, "e1"] \arrow[dr, "r_A"] \arrow[dd, "u1"'] & & BS \arrow[dr, "r_B"] \arrow[dd, "v1"] \arrow[ddll, dashed, "s1"] & \\ & A \arrow[rr, "e"] \arrow[dd, "u"] & & B \arrow[dd, "v"] \arrow[ddll, dashed, "s"] \\ CS \arrow[rr, "m1"] \arrow[dr, "r_C"'] & & DS \arrow[dr, "r_D"] & \\ & C \arrow[rr, "m"'] & & D\,. \end{tikzcd}$$

We need the square with the dashed sides to commute, which it does, since everything else commutes and since $e1$ is an epimorphism and m is a monomorphism. Similarly we get the compatibility of s with the left T-module structure. Notice that we did not use the fact that any of the left T-module and right S-module structure maps of A,B,C and D was actually compatible with the respective semigroup structure of T and S. This observation makes the next part of the proof easier.

Next we need to check that when $f\colon A \to B$ is a morphism of T-S-bimodules and

$$A \xrightarrow{e} M \xrightarrow{m} N$$

is a factorization of f, then M carries a T-S-bimodule structure and m and e are morphisms of bimodules. The right S-module structure on M is defined as the unique morphism r_M making the diagram

$$\begin{tikzcd}
AS \arrow[r, "e1"] \arrow[d, "r_A"'] & MS \arrow[r, "m1"] \arrow[d, dashed, "r_M"] & BS \arrow[d, "r_B"] \\
A \arrow[r, "e"'] & M \arrow[r, "m"'] & B
\end{tikzcd}$$

commute. Note that this definition makes e and m into maps of right S-modules. Since M being being a right S-module is equivalent to $r_M\colon MS \to M$ satisfying the morphism of modules diagram, we have that M is a right S-module because of r_M being defined as a diagonal fill-in. In a similar fashion we will define the left T-module structure $l_M\colon TM \to M$ on M and observe that it makes e and m into morphisms of left T-modules.

Now we just need the left and right module structures of M to be compatible, but this is equivalent to l_M being a map of right M-modules, which also holds, since when defining $l_M : TM \to M$ using diagonal fill-in property, we can think of all of the modules involved as right S-modules. □

We also want the tensor product of bimodules to map morphisms in $\mathfrak{E}$ to morphisms in $\mathfrak{E}$. If $\mathfrak{M}$ consists of precisely the monomorphisms, as is the assumption in Chapter 3, then indeed, the tensor product will satisfy that property.

To see that, let us take a look at the diagram

$$\begin{tikzcd}
AA' \arrow[r, "fg"] \arrow[d, "\omega_{A,A'}"'] & BB' \arrow[d, "\omega_{B,B'}"] \\
A \otimes A' \arrow[r, dashed, "f \otimes g"'] & B \otimes B'
\end{tikzcd}$$

that defines the tensor product on morphisms. Note that the maps ω are regular epimorphisms, which means that they are also strong epimorphisms and must belong to $\mathfrak{E}$. This should make it clear, that in this case, if f and g are in $\mathfrak{E}$, the morphism $f \otimes g$ must also lie in $\mathfrak{E}$.

5.6 Closed monoidal $\mathcal{V}$

Starting with this section we will assume that $\mathcal{V}$ **is right closed**, or in other words, that the functor

$$-Y : \mathcal{V} \times \mathcal{V} \to \mathcal{V}, \quad X \mapsto XY$$

has a right adjoint

$$[Y, -] : \mathcal{V} \times \mathcal{V} \to \mathcal{V}, \quad Z \mapsto [Y, Z]$$

for all objects Y of $\mathcal{V}$.

Additionally we will assume that the category $\mathcal{V}$ has equalizers. This allows for the existence of right adjoints for various functors we have considered. We will not prove that these adjoints exist or show how they are constructed, for that we refer the reader to Section 3 of [23] and Section 5 of [14].

From now on, for semigroups R, T, S in $\mathcal{V}$ and a T-S-bimodule B, let

$$\mathrm{Hom}_S(B, -) : {}_R\mathrm{Mod}_S \to {}_R\mathrm{Mod}_T$$

denote the right adjoint of

$$- \otimes B : {}_R\mathrm{Mod}_T \to {}_R\mathrm{Mod}_S \,.$$

The counit and unit of this adjunction will be denoted by

$$\varepsilon^B_A : \mathrm{Hom}_S(B, A) \otimes B \to A \quad \text{and} \quad \eta^B_A : A \to \mathrm{Hom}_S(B, A \otimes B)\,.$$

Therefore we have a right closed structure on the bicategory of modules and we can apply all the results from the previous chapter. In the specific case of the lax unital bicategory of bimodules between semigroups in $\mathcal{V}$ The closedness has other nice uses.

If we take $T = R = I$ in the preceding adjunction, we get an adjunction

$$-B : \mathcal{V} \leftrightarrows \mathrm{Mod}_S : \mathrm{Hom}_S(B, -)\,.$$

We note that if $\mathcal{V}$ is a symmetric closed monoidal category, then we can use the nice theory of $\mathcal{V}$-categories and $\mathcal{V}$-functors (see [13]) to study our module categories and $\mathcal{V}$-functors, since the adjunction above equips the hom-categories of Mod with a $\mathcal{V}$-category structure (see [10]). For example, we can get functors that satisfy our Eilenberg-Watts theorem from enriched adjunctions and equivalences.

Proposition 26. *Suppose that the functors $F\colon \mathrm{Mod}_S \leftrightarrows \mathrm{Mod}_T : G$ are in an adjunction*

$$\mathrm{Mod}_T(F(A), B) \cong \mathrm{Mod}_S(A, G(B)),$$

such that we also have a natural isomorphism in $\mathcal{V}$

$$\mathrm{Hom}_T(F(A), B) \cong \mathrm{Hom}_S(A, G(B)).$$

Then F is a $\mathcal{V}$-functor with an invertible ξ.

Proof. We have natural isomorphisms

$$\begin{aligned}\mathrm{Mod}_S(XF(A), B) &\cong \mathcal{V}(X, \mathrm{Hom}_S(F(A), B)) \\ &\cong \mathcal{V}(X, \mathrm{Hom}_S(A, G(B))) \\ &\cong \mathrm{Mod}_S(XA, G(B)) \\ &\cong \mathrm{Mod}_S(F(XA), B).\end{aligned}$$

Therefore by Yoneda lemma we have an isomorphism

$$\xi\colon XF(A) \to F(XA).$$

It can be checked that it is natural in X and A and satisfies the conditions needed. □

5.7 Conclusion

Let us summarise what we can we conclude from the results in this thesis:

- lax-unital bicategories seem to be a natural environment for right wide Morita contexts,
- to get useful results, we should restrict our attention to 1-cells whose unitors are nice enough epimorphisms,
- to get results resembling the classical Morita theory, we need to put increasingly strict conditions on the epimorphic and monomorphic 2-cells of the lax-unital bicategory,

- the lax-unital bicategory Mod of semigroups and bimodules in a monoidal category provides a good source of examples of situations where the more general results can be used,

- the prerequisites of the general results for lax-unital bicategories can, in the case of the lax-unital bicategory Mod, be derived in quite a straightforward way from the corresponding properties on $\mathcal{V}$.

In terms of future work, we hope that it is possible to improve Theorem 3 to include situations where the epimorphic 2-cells need not be strong, which is what we originally wanted to achieve.

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Ühikuteta Morita ekvivalentsus bikategoorses keskkonnas

Kokkuvõte

Selles töös on uuritud Morita teooriat lõtvade ühikutega bikategooriate kontekstis. Peamiseks motivatsiooniks on ühikuta multiplikatiivsete struktuuride uurimine. Ühikuta multiplikatiivne struktuur on mõiste, mis võib elada mitmes keskkonnas ning selle kõige puhtamaks vormiks, vähemalt assotsiatiivsust eeldades, võib lugeda poolrühma. Sellega aga asi ei piirdu, multiplikatiivne struktuur võib asuda ka teise struktuuri peal. Ringid, matemaatikas ühed uurituimad algebralised struktuurid, saab lugeda ühiku puudumisel samuti ühikuta multiplikatiivseks struktuuriks.

Üldises mõttes on oluline see, et on antud mingi matemaatiline objekt, nimetame seda hetkel alusobjektiks, koos korrutamistehtega, kusjuures see korrutamistehe peaks olema selle objekti struktuuriga kooskõlas. Kooskõlas olemine tähendab seda, et multiplikatiivne struktuur peaks olema alusobjekti tüüpi objektide teisendus. Probleemi taoline sõnastus viitab sellele, et antud küsimust võiks olla kõige sobilikum uurida kategooriateooria kontekstis. Kategooriateooria uuribki ju objekte ning nendevahelisi teisendusi.

Üks eelnevalt kirjeldatud kategoorsetest esitustest on mõiste "poolrühmobjekt monoidkategoorias". Poolrühmobjekt viitab siin eelneva konteksti mõistes lihtsalt abstraktsele ühikuta multiplikatiivsele struktuurile ning monoidkategooria on keskkond kus sellised struktuurid loomulikul viisil elavad. Seda mõistet on uuritud antud töö viiendas peatükis, kuid see on töö peamine motivatsioon, mitte eesmärk.

Mainitud "poolrühmobjekte monoidkategoorias", edaspidi lihtsalt "poolrühmobjekte", on võimalik omavahel võrrelda mitmel viisil. Üks neist viisidest on tavaline poolrühmobjektide teisendus, mõiste, mis ringide ja poolrühmade korral annab homomorfismi mõiste. Antud töö seisukohalt eelistatav seose mõiste on bimoodul. Esmapilgul võib-olla ei kipu bimoodulitest mõtlema, kui "struktuuri säilitavatest teisendustest", kuid bimoodulid käituvad mitmeti nagu need oleksid seda. Võtmeks on siin bimoodulite tensorkorrutis, mis võimaldab neid järjestikku komponeerida.

Tensorkorrutis on mingis mõttes töö peamine uurimisobjekt. Nimelt on huvi pakkuv tensorkorrutise abstraktne analoog, bikategooria – kategooriat meenutav struktuur mis koosneb objektidest, nendevahelistest teisendustest ning teisenduste vahelistest teisendustest. Morita teooria ja selle teoreemid on sõnastatavad mitmeti, kuid üldjuhul on need seotud mingite struktuuride ning nendevaheliste moodulite või bimoodulitega. Bikategooria on seetõttu hea keskkond, kus Morita teooriat abstraktselt uurida.

Täpsemalt on huviobjektiks lõtvade ühikutega bikategooria. Tavaliste bikategooriate kontekstis on mitmed autorid Morita teooriat uurinud, kuid siinne peamine motivatsioon, ühikuta multiplikatiivsed struktuurid, tavaliselt bikategooriat ei moodusta. Seega tuleb lasta bikategooria definitsiooni nõrgemaks, nii et ka poolrühmad ja ühikuta ringid arendatava teooria alla jääks, ning ühikute lõdvaks laskmine on just see, mis seda võimaldab.

Teises peatükis on defineeritud lõtvade ühikutega bikategooria ja Morita kontekstid, selle töö tähtsuselt teine mõiste. Siin peatükis on uuritud nende mõistete omadusi ning sisu põhineb autori vastava pealkirjaga artiklil. Tulemused on suures osas inspireeritud analoogiliste tulemuste poolt poolrühmade ning ka ringide Morita teoorias. Nendel juhtudel on teoorial mitmed head omadused ning on üritatud nendele omadustele võimalikult lähedale jõuda. Samuti uuritakse kuidas Morita kontekstide kaudu defineeritud ekvivalentsus käitub ning kuidas kontekstide omadusi parandada.

Kolmas peatükk põhineb samal artiklil ning seal on uuritud kontekstide parandamist lisaeeldusel et ühikutega bikategooria 2-nooled, ehk teisenduste vahelised teisendused, oleksid tegurduvad homomorfismiteoreemile analoogilisel viisil – läbi teisenduse kujutise. See annab uue meetodi kontekstide parandamiseks ning võimaldab lihtsamalt uurida eelnevas peatükis defineeritud unitaarsuse mõistet.

Neljandas peatükis on uuritud juhtu, kus lõtvade ühikutega bikategooria on kinnine. Kinnisuse mõistet on kõige kergem ette kujutada ringide näitel: kui on antud kaks bimoodulit, millel on kas ühine vasak või parem ring, siis moodustavad nende bimoodulite vahelised teisendused omakorda bimooduli. See eeldus võimaldab tõestada selle peatüki põhitulemuse: kui kahe püsiva objekti vahel leidub epimorfsete kujutustega Morita kontekst, saame konstrueerida ekvivalentsuse teatud teisenduste kategooriate vahel. Mitmed siinsed tulemused on Valdis Laane, László Márki ja autori püsivate poolrühmade teemalise artikli analoogid bikategoorses kontekstis.

Viiendas peatükis on uuritud põhinäidet, poolrühmobjekte monoidkategoorias. On näidatud, mida peab monoidkategoorialt eeldama, et monoidobjektide ning bimoodulite lõtvade ühikutega bikategoorias oleksid täidetud esimeses neljas peatükis tehtud eeldused.

Tehtud töö järel kokkuvõttena võib öelda, et kuigi mitmed ühikuteta struktuuride Morita teoorias kasutatavad konstruktsioonid ja meetodid on üldistatavad lõtvade ühikutega bikategooriate konteksti ning Morita teooria arendamise keskkonnana paistab see struktuuri mõttes täiuslik, ei õnnestunud autoril tõestada esialgselt loodetud üldisusega tugevaid Morita ekvivalentsuse kohta käivaid tulemusi. Autori intuitsioon ütleb, et paar tulemust pole optimaalsed ning on optimistlik, et siinne lähenemine võib ka tulevikus vilja kanda.

Curriculum vitae

NAME: Ülo Reimaa
DATE AND PLACE OF BIRTH: May 5, 1988, Kohtla-Järve, Estonia
NATIONALITY: Estonian
ADDRESS: Institute of Mathematics, J. Liivi 2, 50409 Tartu, Estonia
E-MAIL: ulo.reimaa@gmail.ee

EDUCATION:

1995-2007	Jõhvi Gymnasium,
2007-2011	University of Tartu, bachelor studies in mathematics, *baccalaureus scientiarum* in mathematics 2004
2011-2013	University of Tartu, master studies in mathematics, *magister scientiarum* in mathematics 2005
2013-	University of Tartu, doctoral studies in mathematics

SPECIAL COURSES:

March 2017	Estonian Winter Schools in Computer Science, Palmse, Estonia

FOREIGN LANGUAGES: English

SCIENTIFIC INTERESTS: semigroup theory, category theory, type theory

Elulookirjeldus

NIMI: Ülo Reimaa
SÜNNIAEG JA -KOHT: 5. mai, 1988, Kohtla-Järve, Eesti
KODAKONDSUS: Eesti Vabariik
AADRESS: Tartu Ülikooli matemaatika instituut, J. Liivi 2, 50409 Tartu, Eesti
E-POST: ulo.reimaa@gmail.com

HARIDUS:

1995-2007	Jõhvi Gümnaasium
2007-2011	Tartu Ülikool, matemaatika bakalaureuseõpe, *baccalaureus scientiarum* matemaatika erialal 2004
2011-2013	Tartu Ülikool, matemaatika magistriõpe, *magister scientiarum* matemaatika erialal 2005
2013-	Tartu Ülikool, matemaatika doktoriõpe

ERIALANE ENESETÄIENDUS:

2017 märts	Eesti Arvutiteaduse Talvekool, Palmse, Eesti

VÕÕRKEELTEOSKUS: inglise

TEADUSLIKUD HUVID: poolrühmateooria, kategooriateooria, tüübiteooria

List of original publications

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