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OPTIMAL CONTROL THEORY AND PORTFOLIO OPTIMIZATION

Actuarial and Financial Engineering

Master's Thesis (30 ECTS)

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Master's Thesis

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Abstract: The objective of the thesis is to use optimal control theory in order to optimize portfolios. More precisely, using principles from calculus of variation in order to define the portfolio problem with reasonable constraints to maximize the profit while minimizing the risk or vice versa. Theoretical cases would be solved with simple constrains, and real application part would be made in Tallinn stock market. The latter is still in development with sixteen companies listed, fourteen which are taken in the analysis. The Values at Risk (VaR) method was the most successful in generating profit but really affected by the randomness of the solution and the nature of the market. The most stable method was the Conditional Values at Risk (CVaR) growing the portfolio slowly but surely. The whole market seems to be suffering from the COVID-19 pandemic resulting in an sharp drop in the stocks making the future returns negative.

Keywords: Euler's equation, Dynamic programing, Stochastic optimal control, Markowitz portfolio, Value at Risk (VaR) model, Conditional Value at Risk (CVaR) model, Auto Regressive Integrated Moving Average (ARIMA) model.

CERCS research specialization: P160 Statistics, operation research, programming, actuarial mathematics; P140 Series, Fourier analysis, functional analysis; S181 Financial science.

Optimaalse kontrolli teooria ja portfelli optimeerimine

Magistritöö

Omar Setihe

Lühikokkuvõte: Selle magistritöö eesmärk on kasutada optimaalse kontrolli teooriat portfelli optimeerimiseks. Täpsemalt kasutatakse variatsiooniarvutuse põhimõtteid, et defineerida mõistlike piirangutega portfelli probleem eesmärgiga maksimeerida kasumit ja minimiseerida riske ning vastupidi. Teoreetilised juhud lahendatakse lihtsate piirangutega ning praktilises pooles kasutatakse Tallinna aktsiaturu andmeid. Viimane on veel arengujärgus – börsil on noteeritud kuusteist firmat, neljateist neist kasutatakse käesolevas analüüsis. VaR meetod oli kõige edukam kasumi genereerimises, kuid kergesti mõjutatav lahendi juhuslikkusest ja turu loomusest. Kõige stabiilsem meetod oli tinglik VaR, mis kasvatas portfelli aeglaselt, kuid kindlalt. Kogu turg paistab olema mõjutatud COVID-19 pandeemiast, mistõttu aktsiahinnad kukuvad ning tulevane rentaablus on negatiivne.

Märksõnad: Euleri võrrand, dünaamiline programmeerimine, stohhastiline optimaalne kontroll, Markowitzi portfoolio, VaR mudel, tinglik VaR mudel, ARIMA mudel.

CERCS teaduseriala: P160 Statistika, operatsioonanalüüs, programmeerimine, finants- ja kindlustusmatemaatika; P140 Jadad, Fourier analüüs, funktsionaalanalüüs; S181 Rahandus.

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1. Introduction

Optimization techniques have crossed many stages from calculating the greatest area of a given rectangle with a total length of edges solved by Plato (427BC , 347BC) to advanced optimization techniques first discussed in that formulation by Oskar Bolza (1857-1942). Thus leading to many applications in different fields [11, 9]. The main aim of this thesis is to discuss the relevant theories leading to financial optimization.

In the first five chapters, detailed explanation of theories and few mathematically solvable applications will be discussed. Starting from the theory of maxima and minima for functions Euler's equations for functionals are presented. The next step is introducing optimal control theory and dynamic programing to deal with the stochastic optimal control process. The final step would be discussing Merton's portfolio with an easy application.

The next subject would be discussing the portfolio theory with Markowitz in addition to dealing with the riskiness using the Value at Risk model and Conditional Value at Risk model. This will allow the base for applying those methods on fourteen stocks in Tallinn market.

An extra analysis is added by performing time series models on those stocks for the sake of understanding the current situation of each company before and after the time period used. All of that said let's start with the methods of calculus of variations as a base of the optimization theory.

2. Methods of Calculus of Variation

In the following one has to distinguish a function and a functional, respectively a function is a relation or expression transforming a set of numbers based on an expression, but a functional is a function which takes as an input a function thus leading us to more required transformations to get a result if it exists.

The method of variation is used to find the maximum or the minimum of a functional but it is not much different from finding the minimum or maximum of a function. Thus the author will start by recalling the theory of maxima and minima of the ordinary function then moving to functionals.

2.1. Maxima and minima of ordinary functions

Let's define a function $y = f(x)$ which is dependent on one variable x on an interval (a, b). Assume that the function is defined on all the values of x within that interval, and also is continuous. Often for every small change of the independent variable x in the interval $[a, b]$ will result in a small change in the dependent variable y.

Let's assume also that the function is differentiable on the interval (a, b). The differentiability means that $for\ each\ x \in (a. b), f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}\ exists.$

A function is said to have a relative maximum at $x = x^*$ in $[a, b]$, if $\exists \delta\ such\ \forall h\ \in (-\delta, \delta)$, $f(x^* + h) - f(x^*) \leq 0$ and the value of the difference vanished only once. This means that the value of , $\frac{f(x^*+h)-f(x^*)}{h} \geq 0\ for\ every\ h \in (-\delta, 0]$ .Consequently

$$\lim_{h \to 0^-} \frac{f(x^* + h) - f(x^*)}{h} \geq 0 \tag{1}$$

On the other hand, if $\frac{f(x^*+h)-f(x^*)}{h} \leq 0\ for\ every\ h \in [0, \delta)$, it means that:

$$\lim_{h \to 0^+} \frac{f(x^* + h) - f(x^*)}{h} \leq 0 \tag{2}$$

From both (1) and (2) and the definition of a limit one can say that:

$$\lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} = 0 = f'(x^*)$$

Thus, in order for a point to be called a relative maximum:

- The first derivative $f'(x^*) = 0$
- For a small number ϵ the difference $f(x^* + \epsilon) - f(x^*) < 0\ \cap f(x^* - \epsilon) - f(x^*) < 0$

The same analogies could be used to say that a function is said to have a relative minimum at x^* if

$\exists \delta \; such \; \forall h \; \in (-\delta, \delta), \; f(x^* + h) - f(x^*) \geq 0$ and the value of the difference is vanished only once. Changing the signs, one can safely say that a point is called a relative minimum if

- The first derivative $f'(x^*) = 0$
- For a small number ϵ the difference $f(x^* + \epsilon) - f(x^*) > 0 \; \cap f(x^* - \epsilon) - f(x^*) > 0$

From all of that, a differentiable function takes a minimum or a maximum at an internal point $x = x^*$ if $f'(x^*) = 0$ [6].

2.2. Maxima and minima for functionals

Let's define a functional F depending on the function $y(x)$, or $F = F(y(x))$. This means that for each function $y(x)$, from a family of functions, there correspond a number F which depends on that function. One says that a functional is continuous if a small variation of the function $y(x)$ within a certain class of functions results is a small variation of the functional F. However, the definition of closeness of a function should be defined.

Assume the existence of two functions $y_1(x)$ and $y_2(x)$. One can say that those functions are close if the difference of $y_1(x) - y_2(x)$ is really small for each x. This means that those two function are close based on their coordinates.

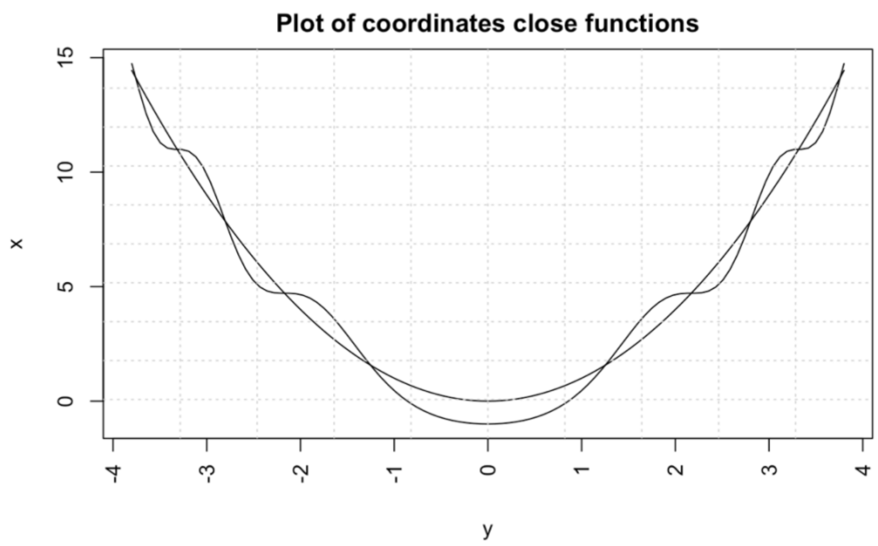

Figure 1: Function close on base zero

However, the change of slopes within those functions can really be significant and totally different. Consequently, if one set the first derivative of those functions too $y_1'(x) - y_2'(x)$ to be really close this could result in a more close shape-wise functions.

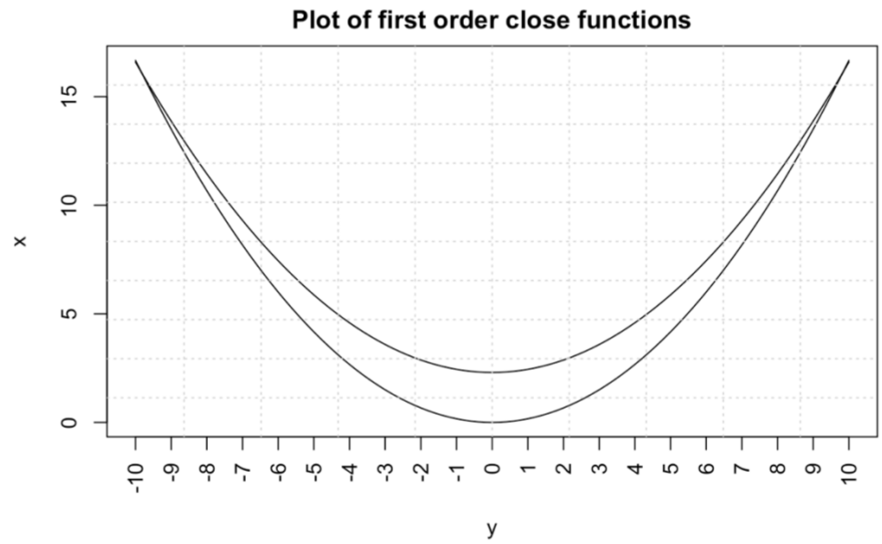

Figure 2: First order close functions

So one can do the same for other degrees of derivation leading us to the following definition:

Two curves $y_1(x)$ and $y_2(x)$ are neighboring in the sense of closeness of order zero if the difference $y_1(x) - y_2(x)$ is small.

Two curves $y_1(x)$ and $y_2(x)$ are neighboring in the sense of closeness of order one if the difference $y_1(x) - y_2(x)$ and $y_1'(x) - y_2'(x)$ respectively, are both small.

One can generalize the concept to two curves $y_1(x)$ and $y_2(x)$ are neighboring in the sense of closeness of order n if the difference $y_1(x) - y_2(x)$, $y_1'(x) - y_2'(x)$, $y_1^{(2)}(x) - y_2^{(2)}(x), \ldots, y_1^{(n)}(x) - y_2^{(n)}(x)$ are all small.

A functional is continuous in the sense of closeness of order n, if for an arbitrary number ε $\exists \delta > 0 \; such \; that \; \left|F\big(y_1(x)\big) - F\big(y_2(x)\big)\right| < \varepsilon \; whenever$

$$|y_1(x) - y_2(x)| < \delta \, , |y_1^{(2)}(x) - y_2^{(2)}(x), | < \delta, \ldots, |y_1^{(n)}(x) - y_2^{(n)}(x), | < \delta$$

2.3. Euler's equation derivation

Let's consider a functional

$$I = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)dx \tag{3}$$

where the boundaries conditions are $y(x_1) = y_1,\ y(x_2) = y_2$. Let's suppose that the function $z(x)$ makes the functional I stationary. Which means that for the first order change in I with respect to y(x) vanishes.

Let's introduce a function $\eta(x)\ which\ satisfies\ \eta(x_1) = \eta(x_2) = 0$ such that η is continuous and twice differentiable with respect to x.

Now let's introduce another new function $\bar{z}(x) = z(x) + \varepsilon\, \eta(x)$ such that ε does not depend on x , one can easily notice that $\bar{z}(x)$ satisfies the same boundary conditions as any other function y. Since the $\bar{z}(x)$ depends on the $\eta(x)\ and\ z(x)$ it represents a whole family of curves passing through the boundaries.

Now let's try to find the $\bar{z}(x)$ that makes the functional (3) stationary. Here I does depend only on ε, since all the other variables depend in the integral depend on x thus will no more exist after the integration. So the optimization problem is to find the $I(\varepsilon)$. This could be interpreted as a function problem just making the $\frac{d(I)}{d\varepsilon} = 0$.

The function is stationary in $z(x)$ by assumption, thus one can simply say that $\frac{d(I)}{d\varepsilon} = 0$ when $\varepsilon = 0$. Now let's try to expend the integral to reach the desired outcome.

From the equality

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0\ one\ can\ conclude\ that$$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{x_1}^{x_2} F\big(x, \bar{z}(x), \bar{z}'(x)\big) dx = 0.$$

Therefore,

$$\int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} F\big(x, \bar{z}(x), \bar{z}'(x)\big)\bigg|_{\varepsilon=0} dx = 0,$$

Or

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{z}'} \frac{\partial \bar{z}'}{\partial \varepsilon}\right]\bigg|_{\varepsilon=0} dx = 0$$

Let's try to develop the expression using $\bar{z}(x) = z(x) + \varepsilon\, \eta(x)$. First, $\bar{z}'(x) = z'(x) + \varepsilon\, \eta'(x)$. One can see that $\frac{\partial \bar{z}}{\partial \varepsilon} = \eta(x)$ and $\frac{\partial \bar{z}'}{\partial \varepsilon} = \eta'(x)$. Let's plug them back to the integral. It was shown above that at the stationary point

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{z}} \eta + \frac{\partial F}{\partial \bar{z}'} \eta'\right]\bigg|_{\varepsilon=0} dx = 0 \tag{4}$$

Let's use the integration by parts on the following term $\frac{\partial F}{\partial \bar{z}'}\eta'$. Knowing that the $\int \eta'\, dx = \eta$. Thus,

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{z}'} \eta' dx = \frac{\partial F}{\partial \bar{z}'} \int_{x_1}^{x_2} \eta' dx - \int_{x_1}^{x_2} \left(\int \eta'\right) \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{z}'}\right] dx = \frac{\partial F}{\partial \bar{z}'} [\eta]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta\, \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{z}'}\right] dx.$$

Since $\eta(x_1) = \eta(x_2) = 0$ the term $\frac{\partial F}{\partial \bar{z}'} [\eta]_{x_1}^{x_2} = 0$;

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{z}'} \eta' dx = -\int_{x_1}^{x_2} \eta\, \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{z}'}\right] dx$$

taking that to the integral (4) would lead to the results

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{z}} \eta - \eta \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{z}'}\right]\right]\bigg|_{\varepsilon=0} dx = 0$$

From the last equation one obtains,

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{z}} - \frac{d}{dx} \left[\frac{\partial F}{\partial \bar{z}'} \right] \right] \eta \Bigg|_{\varepsilon=0} dx = 0$$

Since $\bar{z} = z \mid_{\varepsilon=0}$ the last equation leads to the equality

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial z} - \frac{d}{dx} \left[\frac{\partial F}{\partial z'} \right] \right] \eta dx = 0$$

Now since $\eta(x)$ is an arbitrary function the only way to have the integral equal to zero is by making

$$\frac{\partial F}{\partial z} - \frac{d}{dx} \left[\frac{\partial F}{\partial z'} \right] = 0$$

This is called the Euler's equation or in some cases Euler-Lagrange equation.

Consequently, in order to find the function $y(x)$ which makes the functional $I = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big) dx$ with defined boundaries $y(x_1) = y_1,\ y(x_2) = y_2$ stationary one just need to solve the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left[\frac{\partial F}{\partial y'} \right] = 0 \tag{5}$$

One can see that Euler's equation is a second order differential equation since the second term is actually depending on tree variables so,

$$\frac{d}{dx} \left[\frac{\partial F}{\partial y'} \right] = \frac{dF_{y'}}{dx} = F_{y'x} + F_{y'y} \frac{dy}{dx} + F_{y'y'} \frac{dy'}{dx} = F_{y'x} + F_{y'y} y' + F_{y'y'} y''.$$

Thus Euler's equation can be written

$$F_y = F_{y'x} + F_{y'y} y' + F_{y'y'} y'' \tag{6}$$

Euler's equation was one the revolutionary discoveries that changed the perception of optimization theory. It was used in many fields including finance and investment. The function allowed the definition of an optimal path for consumption and utility. The following section would deal with a transformation of the second order differential equation to a simpler version if some conditions are met [7].

2.4. Hamiltonian

One way to simplify the order of Euler's equation is by breaking it down to two first order differential equation. This is known as the canonical form of Euler's equation. To do that let's start by defining a function $p(x)$ such that

$$p(x) = F_{y'}(x, y(x), y'(x)) \tag{7}$$

Note that the function $p(x)$ does not depend of the function y thus $y'(x)$ can be expressed depending on $x, y(x)\ and\ p(x)$ from the previous equation.. The next thing to do is to define the Hamiltonian $H(x, y, p(x))$ in the following form

$$H\big(x, y, p(x)\big) = -F\big(x, y(x), y'(x)\big) + p(x)y'(x) \tag{8}$$

The differential of the Hamiltonian can be expressed as

$$dH = -F_x dx - F_y dy - F_{y'} dy' + p(x)dy' + y'(x)dp(x)$$

One can see from the definition of the function $p(x)$ (7) that

$$F_{y'} dy' = p(x)dy'$$

Meaning that

$$dH = -F_x dx - F_y dy + y' dp$$

So one can conclude that

$$\frac{\partial H}{\partial y} = -F_y\,, \frac{\partial H}{\partial p} = y'$$

Now let's assume that the function y satisfies Euler's equation (5) meaning that

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = \frac{dp(x)}{dx} = p'(x)$$

Finally, the Euler's equation is equivalent to the following system of equation

$$p' = -\frac{\partial H}{\partial y}, y' = \frac{\partial H}{\partial p}. \tag{9}$$

Sometimes solving those two equation is way much more time efficient especially if the function p can easily be found from the expression of F [7].

2.4.1. Application in capital investment

The following section would deal with a simple application of Euler's equation is capital investment. The issue that would be to solve the problem of managing a portfolio with a consumption rate and a final values objective at a certain time in the future. So let's start by defining the variables concerning this problem.

Assume someone has a stock capital that depend on time, $S(t)$. Additionally, let's assume that a function $F(S(t))$ represents the value produced from the capital $S(t)$ which is continuous, twice differentiable, increasing and concave. One can use the value created from $F(S(t))$ to either, increase the capital by a rate $S'(t) = \frac{dS}{dt}$ leading to more value next time, or can be consumed at a rate $C(t)$. Consequently, the following relation can be written

$$F\big(S(t)\big) = \frac{dS}{dt} + C(t).$$

The objective is to maximize the utility $U(C(t))$ from consumption by choosing how much to invest at each time moment t. Not forgetting that the continuous discount rate of the utility r_1, one can write the performance index to be maximized as

$$\int_0^T e^{-r_1 t} U(C(t))\, dt,$$

$$or \int_0^T e^{-r_1 t} U\left(F\big(S(t)\big) - S'(t)\right) dt. \tag{10}$$

$$if\ the\ initial\ and\ terminal\ values\ of\ S\ or\ known\ then\ S(0) = S_0, S(T) = S_T \tag{11}$$

Solving this problem requires finding the function F, and initial and terminal values of the stock.

Let's $S(t)$ to be the wealth of a person. At time $t = 0$ the investment S_0 is growing at a rate r_2. Assume also that the person has a salary $w(t)$ which is payed continuously. One can write the values created by the person is

$$F\big(t, S(t)\big) = w(t) + r_2 S(t).$$

Assume also that the person wants to have a stock of capital S_T at $t = T$, and that the utility function is increasing and concave.

Using the previous equation in (10), one can present

$$I = \int_0^T e^{-r_1 t} U\left(F\big(S(t)\big) - S'(t)\right) dt\ , or$$

$$I = \int_0^T e^{-r_1 t} U(w(t) + r_2 S(t) - S'(t)) dt.$$

Let's find the values of S(t) that make the functional

$$I = \int_0^T e^{-r_1 t} U(w(t) + r_2 S(t) - S'(t)) dt \tag{12}$$

stationary using Euler's equation (5) assuming

$$F\big(t, S(t), S'(t)\big) = e^{-r_1 t} U(w(t) + r_2 S(t) - S'(t)),$$

One gets

$$\frac{\partial F}{\partial S} - \frac{d}{dt}\left[\frac{\partial F}{\partial S'}\right] = 0.$$

Let's calculate the partial derivatives

$$\frac{\partial F}{\partial S} = \frac{\partial F}{\partial C}\frac{\partial C}{\partial S} = r_2 e^{-r_1 t}\frac{\partial U}{\partial C},$$

And

$$\frac{d}{dt}\left[\frac{\partial F}{\partial S'}\right] = \frac{d}{dt}\left[-e^{-r_1 t}\frac{\partial U}{\partial C}\right] = r_1 e^{-r_1 t}\frac{\partial U}{\partial C} + e^{-r_1 t}\frac{\partial^2 U}{\partial^2 C}\frac{\partial C}{\partial t}.$$

Plugging them back in Euler's equation

$$r_2 e^{-r_1 t}\frac{\partial U}{\partial C} = r_1 e^{-r_1 t}\frac{\partial U}{\partial C} + e^{-r_1 t}\frac{\partial^2 U}{\partial^2 C}\frac{\partial C}{\partial t}.$$

This leads to

$$-\frac{\frac{\partial^2 U}{\partial^2 C}\frac{\partial C}{\partial t}}{\frac{\partial U}{\partial C}} = r_2 - r_1 \tag{13}$$

This equation describes that the change in the marginal utility is related to the difference between the growth rate r_2 and the rate of time difference r_1. According to the assumptions $-\frac{\frac{\partial^2 U}{\partial^2 C}}{\frac{\partial U}{\partial C}} > 0$. Which means that $\frac{\partial C}{\partial t}$ $is\ positive$ as long as $r_2 - r_1 > 0$.

Now let's assume that the utility function is $U(C) = \ln(C)$. This means that the more we take money out from F the less we can do something useful with it. Also assume that the $w(t) = 0$ which means that the investment is self-sufficient; no money will be added to the capital. Assume also that person wants to liquidate the whole investment in the end, $S(T) = 0$.In this case one can see from deriving U that

$$-\frac{\frac{\partial^2 U}{\partial^2 C}}{\frac{\partial U}{\partial C}} = -\frac{-\frac{1}{C^2}}{\frac{1}{C}} = \frac{1}{C}.$$

Plugging it in the result of Euler's equation (13) one gets

$$\frac{\frac{\partial C}{\partial t}}{C} = r_2 - r_1.$$

Integrating the equation from 0 to t, one gets

$$[\ln C(s)]_0^t = \int_0^t (r_2 - r_1) ds.$$

Evaluating the integral will give

$$\frac{C(t)}{C(0)} = e^{(r_2 - r_1)t},$$

or

$$C(t) = C(0)e^{(r_2 - r_1)t}. \tag{14}$$

The next step is to write $C(t)$ with respect to S. Knowing that

$$C(t) = w(t) + r_2 S(t) - S'(t),$$

And since $w(t) = 0$ by assumption one can get

$$C(t) = r_2 S(t) - S'(t),$$

Let's multiply the whole equation by $e^{-r_2 t}$

$$C(t)e^{-r_2 t} = S(t) r_2 e^{-r_2 t} - S'(t)e^{-r_2 t},$$

Using the expression of $C(t)$ (14), and integrating results in

$$\int_0^t C(0)e^{(r_2 - r_1)s} e^{-r_2 s} ds = \int_0^t S(s) r_2 e^{-r_2 s} - S'(s)e^{-r_2 s} ds.$$

Developing the expressions results in

$$\frac{C(0)}{-r_1}[e^{-r_1 s}]_0^t = [-S(s)e^{-r_2 s}]_0^t.$$

Evaluating the expression gives,

$$\frac{C(0)}{-r_1}(e^{-r_1 t} - 1) = S(0) - S(t)e^{-r_2 t},$$

or

$$S(t) = e^{r_2 t} S(0) + \frac{C(0)}{r_1} e^{r_2 t}(e^{-r_1 t} - 1). \tag{15}$$

Now let's use the boundary conditions for $S(0)$ and $C(0)$. $One\ can\ see$

$$S(0) = S_0, and\ S(T) = S_T = 0.$$

Evaluating the expression of S (15) at time $t = T$, one can get

$$S(T) = 0 = e^{r_2 T} S_0 + \frac{C(0)}{r_1} e^{r_2 T}(e^{-r_1 T} - 1),$$

Rearranging the arguments gives

$$C(0) = \frac{r_1 S_0}{1 - e^{-r_1 T}}.$$

Replacing the constants in the expression of $S(t)$ in (15) will give,

$$S(t) = S_0 e^{r_2 t}\left(1 - \frac{1 - e^{-r_1 t}}{1 - e^{-r_1 T}}\right). \tag{16}$$

For $C(t)$, one can get

$$C(t) = \frac{r_1 S_0 e^{(r_2 - r_1)t}}{1 - e^{-r_1 T}}. \tag{17}$$

Finally, one can say that the previous formulas makes the functional I stationary. If one investigates more using the second Euler's variation, which is not covered in this paper, one can say that, in order to maximize the utility function U, the consumption rate should be $C(t)$.

Assume $S(0) = 100, r_2 = 0.1\ , r_1 = 0.03\ , T = 35(time\ period)$ applying it gives, The following two plots one is for the utility function and the other one is for the overall values of wealth with respect to time.

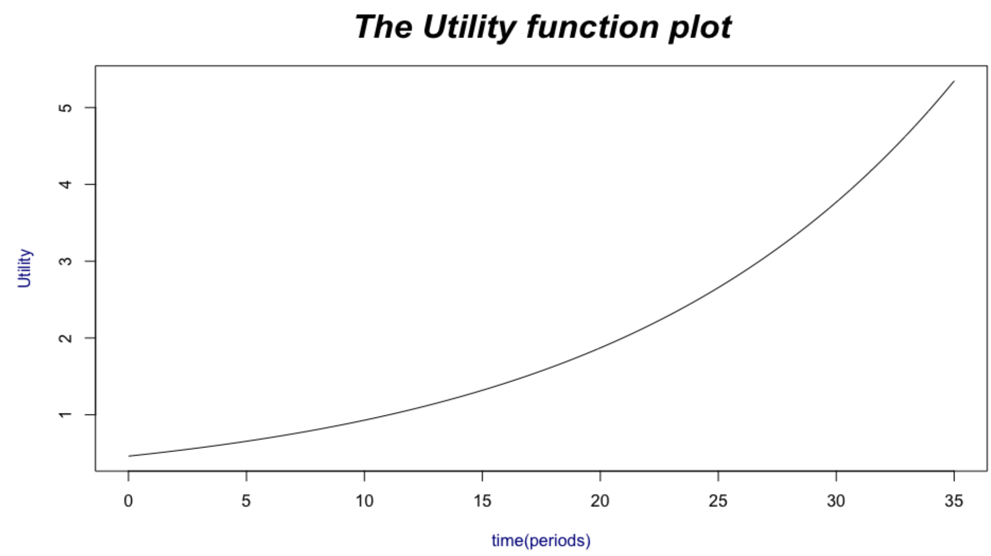

Figure 3:: Special solution of investment capital problem utility plot

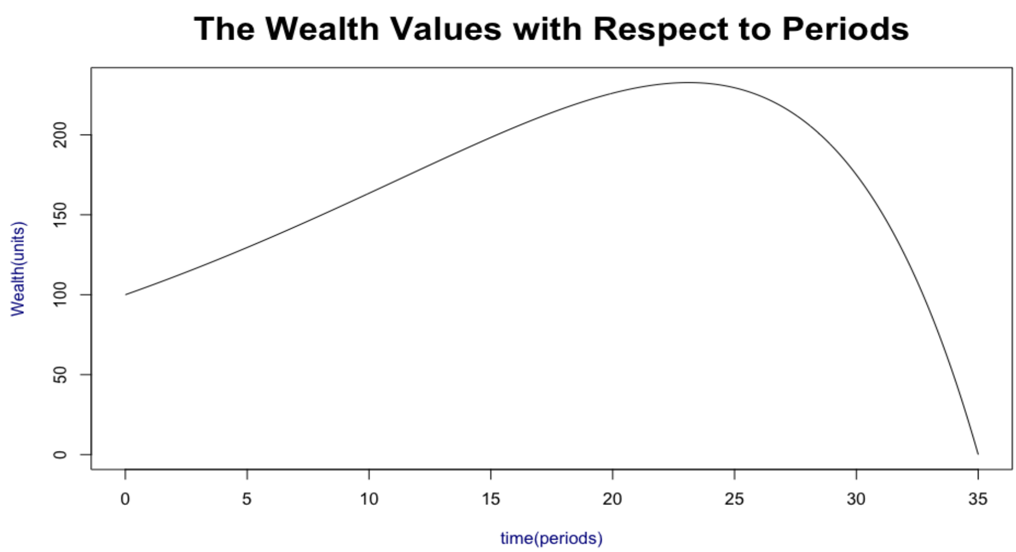

Figure 4: Special solution of investment capital problem wealth plot

Following that path insures maximizing the utility function. See **Appendix A** for details about the graph.

2.5. The Beltrami identity

Let us consider the functional $F\big(y(x), y'(x)\big)$ which depends on x explicitly. In this case, if one wants to find the function $y(x)$ that makes $I = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)dx$ stationary, solving Euler's equation (5) is required. Thus

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$$

Now let's multiply both sides of the equation by $y^{'}$. This leads to the equation

$$y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0 \tag{18}$$

Let's try to find the derivative of F with respect to x. If we use the chain rule. This means

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial x} + y''\frac{\partial F}{\partial x}$$

Thus, one can define

$$y'\frac{\partial F}{\partial x} = \frac{dF}{dx} - \frac{\partial F}{\partial x} - y''\frac{\partial F}{\partial x}$$

If we use that in Euler's equation multiplied by y' (18) one could get

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - y''\frac{\partial F}{\partial x} - y'\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0.$$

Using the rule of differentiation of a product

$$-y''\frac{\partial F}{\partial x} - y'\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = -\left[y''\frac{\partial F}{\partial x} + y'\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right] = -\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

The equation becomes

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = 0,$$

or in the other form

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}.$$

Since F does not depend on x directly we can say that the partial derivative with respect to x is zero. This means that

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

Now let's integrate with respect to x, which results in

$$F - y'\frac{\partial F}{\partial y'} = C \tag{19}$$

Where C is a constant that depends on the boundary conditions.

To recapitulate, If the functional F does not depend on x explicitly, one can use the Beltrami Identity to find the function y(x) that makes I= $\int_{x_1}^{x_2} F\big(y(x), y'(x)\big)dx$ stationary. The problem leads to the equation (19).

2.5.1. Application: The Brachistochrone Problem

The problem is to find the path that minimizes time of transit between two points $A(x_1\,, y_1)$ and $B(x_2\,, y_2)$ under the influence of gravity. Intuitively speaking, one will say that the shortest path which is a line is the one that minimizes the time. However, as the particle moves down it gains more speed thus making it faster to travel the distance.

To solve the problem let's introduce some coordinates in the x and y axis thus making it easier to solve. Let $x_1 = 0$ and $y_2 = 0$ at the initial moment when the particle fall from a height y_1 to each the x-axis in x_2.

Since the objective is to minimize the time to transit, one needs to define all the functions needed for the analysis. Starting with the time need to be minimized, $T = \int_A^B dt$. Now, it is known that the time is the distance over the velocity, meaning $dt = \frac{dS}{V(x,y)}$. The dS can be rewritten using the Pythagorean theorem in terms of x and y in the form

$$dS = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \frac{(dy)^2}{(dx)^2}}dx.$$

The next step is to express the velocity of the object. In other words, deriving the kinetic energy and potential energy equations and relate them using the conservation of energy theorem.

Energy is the change of work in a system. Work is simply the force that is used to do the work times the distance it was applied in. One can say $W = \int Fds\ \ over\ the\ trajectory$. According to Newton laws, a force is equal to the product of mass of the object and it's acceleration.

In the case of potential energy of the object at height $y_1$has a force applied to it downward following the gravitational field g. Assuming the particle have a mass m, one can say $F_g = mg$ going down. The potential energy change between y_1 and a point with y as its vertical coordinate is

$$\Delta P_E = \Delta W = \int_{y_1}^{y} F_g ds = \int_{y_1}^{y} -mgdy,$$

where the minus comes from the direction of the force. If the gravitational acceleration does not change with respect to y, one can write

$$\Delta P_E = -mg \int_{y_1}^{y} dy = mgy_1 - mgy.$$

Moving now to the kinetic energy expression,

$$\Delta K_E = W = \int F(s)d\,s = \int mads = m \int \frac{d\,V}{dt} ds = m \int \frac{ds}{dt} dV = m \int -VdV.$$

This means that the change of kinetic energy depends on the velocity at the beginning and at the end. Thus $\Delta K_E = -\frac{m}{2}[V^2]_{v_0}^{v_1}$. In this case one can say that $v_0 = 0$ so $\Delta K_E = -\frac{mv_1^2}{2}$such that V varies over the path.

The conservation of energy theorem states that the change of energy between two states is always zero $\Delta E = 0$. So one can write $\Delta P_E + \Delta K_E = 0$.

This yields to the equation

$$mgy_1 - mgy - \frac{mv_1^2}{2} = 0.$$

From the last equation one obtains

$$v_1 = \sqrt{2g(y_1 - y)} = V(x, y).$$

Using that on the expression of time to find $y(x)$ that makes the functional

$$T = \int_A^B \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(y_1 - y)}} dx$$

minimal.

To solve that problem, one needs to find the $y(x)$ that makes T stationary. The Euler-Lagrange equation is used for that purpose.

Recall: if one wants to find $y(x)$ which makes the functional $I = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)dx$ with defined boundaries $y(x_1) = y_1,\ y(x_2) = y_2$ stationary, one just need to solve the equation (5). If the $F\big(x, y(x), y'(x)\big)$ does not depend on x, Euler's equation becomes the Beltrami identity (19).

In this problem

$$T = \int_A^B \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2g(y_1-y)}} dx,\ F\big(x, y(x), y'(x)\big) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2g(y_1-y)}}$$

does not depend on x explicitly. Thus the Beltrami identity would be used.

$$F - y'\frac{\partial F}{\partial y'} = C.$$

Let's find

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{2g(y_1 - y)}} \frac{1}{2\sqrt{1 + (y')^2}} (1 + (y')^2)' = \frac{1}{\sqrt{2g(y_1 - y)}} \frac{1}{2\sqrt{1 + (y')^2}} 2y'$$

$$= \frac{y'}{\sqrt{2g(y_1 - y)(1 + (y')^2)}}$$

Thus, the Beltrami identity equation yields

$$\sqrt{\frac{1 + (y')^2}{2g(y_1 - y)}} - y' \frac{y'}{\sqrt{2g(y_1 - y)(1 + (y')^2)}} = C,$$

or

$$\frac{1 + (y')^2 - (y')^2}{\sqrt{2g(y_1 - y)(1 + (y')^2)}} = C.$$

From the last relation one can easily conclude that

$$C\sqrt{2g(y_1 - y)(1 + (y')^2)} = 1,$$

or

$$C^2 2g(y_1 - y)\sqrt{1 + (y')^2} = 1.$$

Evidently, g is constant. So one can write

$$\frac{1}{C^2 2g} = (y_1 - y)(1 + (y')^2).$$

Defining a new constant $C_1 = \frac{1}{C^2 2g}$ means the equation becomes

$$(y_1 - y)(1 + (y')^2) = C_1.$$

From here one can express

$$(y')^2 = \frac{C_1}{(y_1 - y)} - 1,$$

and

$$y' = \sqrt{\frac{C_1}{(y_1 - y)} - 1} = \frac{dy}{dx}$$

We can see that this is a differential equation with separable variables. Thus;

$$dx = \sqrt{\frac{(y_1 - y)}{C_1 - (y_1 - y)}}\, dy$$

Now we can integrate both sides of the equation. This gives us the relation

$$x = \int \sqrt{\frac{(y_1 - y)}{C_1 - (y_1 - y)}}\, dy.$$

In order to solve this equation the change of variables is needed. Setting

$$y = y_1 - C_1 \sin^2\frac{\theta}{2} \text{ means that } dy = -C_1 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta.$$

Now let’s plug everything back in the integral obtaining

$$x = \int \sqrt{\frac{\left(y_1 - y_1 + C_1 \sin^2\frac{\theta}{2}\right)}{C_1 - \left(y_1 - y_1 + C_1 \sin^2\frac{\theta}{2}\right)}} - C_1 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta,$$

or

$$x = \int \sqrt{\frac{\left(C_1 \sin^2\frac{\theta}{2}\right)}{C_1 - \left(C_1 \sin^2\frac{\theta}{2}\right)}} - C_1 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

Canceling C_1 inside the square root and knowing that $1 - \sin^2\frac{\theta}{2} = \cos^2\frac{\theta}{2}$ gives ;

$$x = -C_1 \int \sqrt{\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta,$$

or

$$x = -C_1 \int \sin^2 \frac{\theta}{2} d\theta.$$

$Knowing\ that\ \sin^2 \frac{\theta}{2} = \frac{1 - \cos\theta}{2}, yields\ to$

$$x = -\frac{C_1}{2} \int 1 - \cos\theta \, d\theta.$$

After integration, one obtains

$$x = \frac{-C_1}{2}(\theta - \sin\theta) + K_2,$$

where that C_1 and K_2 are constants.

It is a difficult to express directly x as a function of y. So the equation would be presented in a parametric form,

$$\begin{cases} x = \dfrac{-C_1}{2}(\theta - \sin\theta) + K_2 \\ y = y_1 - C_1 \sin^2 \dfrac{\theta}{2} = y_1 - C_1 \left(\dfrac{1 - \cos\theta}{2}\right) \end{cases}.$$

Now let's use the boundary conditions. When $y = y_1, we\ have\ x = 0$, and

$$y = y_1 = y_1 - C_1 \left(\frac{1 - \cos\theta}{2}\right)$$

which means $\cos\theta = 1$. Thus, $\theta = 2k\,\pi \mid k \in \mathbb{Z}$. To simplify let's take $\theta = 0$ at the initial time instant. So

$$x = \frac{-C_1}{2}(\theta - \sin\theta) + K_2 = K_2 = 0.$$

Now let's use the second boundary condition when $y = 0$, $x = x_2$

$$\begin{cases} x = x_2 = \dfrac{-C_1}{2}(\theta - \sin\theta) \\ 0 = y_1 - C_1\left(\dfrac{1-\cos\theta}{2}\right) \end{cases}.$$

Finally one obtains

$$\begin{cases} 2x_2 = -C_1(\theta_2 - \sin\theta_2) \\ -2y_1 = -C_1(1-\cos\theta_2) \end{cases}. \tag{20}$$

Solving those equations will result in C_1 with respect to $x_2\ and\ y_1$.

In the end the solution would be;

$$\begin{cases} x = \dfrac{-C_1}{2}(\theta - \sin\theta) \\ y = y_1 - C_1\left(\dfrac{1-\cos\theta}{2}\right) \end{cases}$$

Let's assume $K_1 = -C_1$ which means

$$\begin{cases} x = \dfrac{K_1}{2}(\theta - \sin\theta) \\ y = y_1 + K_1\left(\dfrac{1-\cos\theta}{2}\right) \end{cases} \tag{21}$$

This parametric function happens to be the equation of a cycloid thus one can say, a cycloid makes the functional of the Brachistochrone problem stationary.

However, to prove that it is actually the minimum time one need to introduce the second variation. This will not be covered in this paper.

Taking A(0,2) and B(6,0), solving (20) and plugin it in (21), one gets the following graph showing the fastest descent. The result is depicted in **Figure *5***.

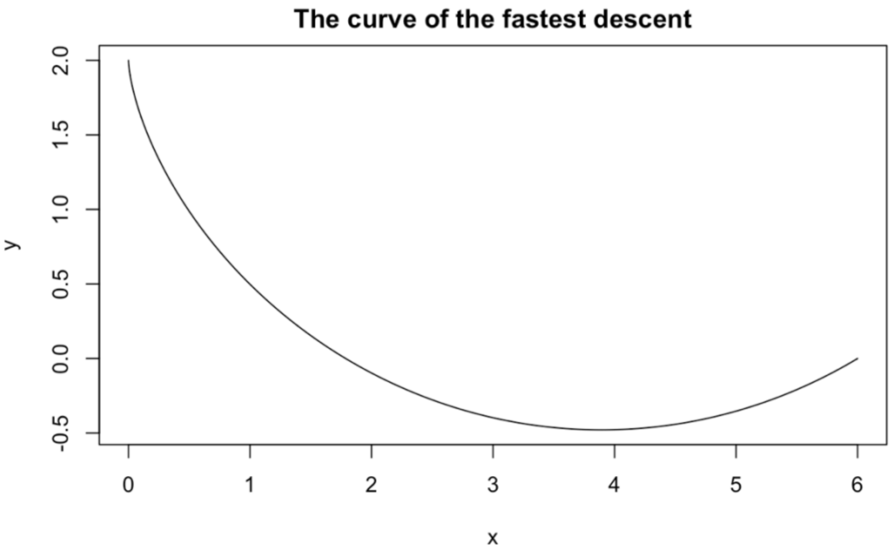

Figure 5: Case solution of Brachistochrone problem

Detailed solution can be found in **Appendix B**.

2.6. Movable boundaries

Up till now, the functional $I = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)dx$ that was considered has fixed boundaries $y(x_1) = y_1,\ y(x_2) = y_2$. In various problems usually the upper boundary is not given. Thus let's solve a functional problem with one of the boundaries not given.

Let's define a functional $I = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)dx$ with a defined boundary x_1 and an unfixed one say x_2. From the previous proof of Euler's equation δI should vanish in the extremum, but for it to hold one needs two conditions defining the boundaries. The first one will be given within the problem and the second one shall be derived in this section.

Let's define the set of curves in the form of $y = y(x, C_1, C_2)$, C_1, C_2 depend on the boundary conditions, for the solution to Euler's equation. An extremum can occur only in one of those curves. So the functional I will become a function $I(y(x, C_1, C_2))$ with two parameters $C_1 and\ C_2$. The function $I(y(x, C_1, C_2))$ is defined beyond the limit of integration x_0 and x_1.One can say that the variation of I will coincide with the derivative $I(y(x, C_1, C_2))$.

One knows, from the definition from closeness between two functions, that two curves $y = y(x)$ and $y = y(x) + \delta y$ are considered close if the variation $\delta y \ and \ \delta y'$ are small. Meaning that in the end of the curve at x_2, δx_2 and $\delta y_1$are also small. This will result in a smooth line ending the set of functions that satisfies Euler's Equation.

Since one boundary is defined, the function $I(y(x, C_1))$ will depend only on one variable C_1 that describes the boundaries in x_1 and x. One can see that some functions will not intersect with the boundary line thus should not be considered in this analysis.

Now let's calculate the variation of $I(y(x, C_1))$ with a change in the ending point (x_2, y_2). The variation of that point would be noted $(x_2 + \delta x_2, y_2 + \delta y_2)$. The variation of δI will be continuous along the line in the end of it thus will lead us to the following expressions

$$\delta I = \int_{x_1}^{x_2+\delta x_2} F(x, y(x) + \delta y, y'(x) + \delta y')dx - \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)dx.$$

This equality can be presented as

$$\begin{aligned}\delta I = &\int_{x_2}^{x_2+\delta x_2} F(x, y(x) + \delta y, y'(x) + \delta y')dx \\ &+ \int_{x_1}^{x_2} F(x, y(x) + \delta y, y'(x) + \delta y') - F\big(x, y(x), y'(x)\big)dx.\end{aligned} \tag{22}$$

Using the mean value theorem on the first part of (22) one can say that

$$\int_{x_2}^{x_2+\delta x_2} F(x, y(x) + \delta y, y'(x) + \delta y')dx = F(x, y(x) + \delta y, y'(x) + \delta y')|_{x=x_2+\alpha\delta x_2}\delta x_2$$

$such \ that$ $0 < \alpha < 1$.

Since F is continuous so one can say that

$$F(x, y(x) + \delta y, y'(x) + \delta y')|_{x=x_2+\alpha\delta x_2} = F(x, y(x) + \delta y, y'(x) + \delta y')|_{x=x_2} + \varepsilon$$

where $\varepsilon\ is\ smal$ number. Evidently,

$$\int_{x_2}^{x_2+\delta x_2} F(x, y(x)+\delta y, y'(x)+\delta y')dx = (F(x, y(x)+\delta y, y'(x)+\delta y')|_{x=x_2} + \varepsilon\ )\delta x_2$$

Ignoring the effect of $\varepsilon\delta x_2$ one can write

$$\int_{x_2}^{x_2+\delta x_2} F(x, y(x)+\delta y, y'(x)+\delta y')dx \approx F(x, y(x)+\delta y, y'(x)+\delta y')|_{x=x_2} dx_2 \quad (23)$$

Using the difference of Taylor's expressions for the two functionals $F(x, y(x)+\delta y, y'(x)+\delta y')$ and $F\big(x, y(x), y'(x)\big)$, one can cancel many terms from the second part of (22). Thus one can write,

$$\begin{aligned} &\int_{x_1}^{x_2} F(x, y(x)+\delta y, y'(x)+\delta y') - F\big(x, y(x), y'(x)\big)dx \\ &\qquad = \int_{x_1}^{x_2} F_y\big(x, y(x), y'(x)\big)\delta y - F_{y'}\big(x, y(x), y'(x)\big)\delta y'\ dx + C. \end{aligned} \quad (24)$$

Here the C is representing higher order terms than δy and $\delta y'$ and can be ignored.

Next, to the equation δI (22) and plugging (23) and (24). The arguments in each F are the same and the equation becomes

$$\delta I = F|_{x=x_2+\alpha\delta x_2}\delta x_2 + \int_{x_1}^{x_2} (F_y\delta y - F_{y'}\delta y')\ dx.$$

The next step would be integration by part of the second terms of this equation. This yields

$$\int_{x_1}^{x_2} (F_y\delta y - F_{y'}\delta y')\ dx = \left[F_{y'}\delta y\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} F_y - \frac{dF_{y'}}{dx}\ dx.$$

According to Euler's equation $F_y - \frac{dF_{y'}}{dx} \equiv 0$.

The value of δy at the boundary (x_1, y_1) would be zero since it is a fixed one. This means that

$$\int_{x_1}^{x_2} F_y\delta y - F_{y'}\delta y'\ dx = F_{y'}\delta y\Big|_{x=x_2}.$$

The variation of $\delta y|_{x=x_2}$ is actually the change of y_2 such that x_2 is constant. Thus, $\delta y|_{x=x_2} \neq \delta y_2$. Here δy_2 is the increment of y_2 where both, $x_2 and\ y_2$are changing. So $\delta y|_{x=x_2} = \delta y_2 - y'(x_2)\delta x_2$.

This means that

$$F_{y'}\delta y\Big|_{x=x_2} = F_{y'}\Big|_{x=x_2}(\delta y_2 - y'(x_2)\delta x_2).$$

The total increment of the functional ΔI becomes

$$\delta I = F|_{x=x_2}\delta x_2 + F_{y'}\Big|_{x=x_2}(\delta y_2 - y'(x_2)\delta x_2).$$

This means that

$$\delta I(x_2, y_2) = (F - F_{y'}y')\Big|_{x=x_2}\delta x_2 + F_{y'}\Big|_{x=x_2}\delta y_2.$$

Now let's recapitulate what was done. The $I(x_2, y_2)$ obtained in the previous equation is the result of making $I(x_2, y_2)$ follow the smooth line at the end of the second border defined with $y = y(x, C_1))$. Also $\delta x_2 = \Delta x_2 = dx_2$ since it represents the change at the end point. Thus, one can use the theory of maxima and minima that for I to be stationary, it must satisfy $\delta I = 0$.

Finally, the equation

$$(F - F_{y'}y')\Big|_{x=x_2}\delta x_2 + F_{y'}\Big|_{x=x_2}\delta y_2 = 0 \tag{25}$$

represents the second necessary condition for an extremum problem with unfixed boundary.

2.7. Lagrange Multiplier

Assume that we have the following problem with a constraint which consist in the maximization of $F = f(x, y)$ so that $g(x) = a, or\ g(x) - a = 0$.

The last equality is multiplied by a Lagrange multiplier λ_1, then subtracted from the original function thus making the problem the maximization of

$$L(x, y, \lambda_1) = f(x, y) - \lambda_1[g(x) - a].$$

Those problems are the same since $g(x) - a = 0$. However their derivatives are not. The same if one wants to generalize this to cover more objective function , one needs to add another Lagrange multiplier. If we have n constraints, then Lagrange function would be

$$L(\mathrm{x}, \mathrm{y}, \lambda_1, \dots, \lambda_n) = f(x, y) + \sum_{i=1}^{n} \lambda_i[g_i(x) - a_i]. \tag{26}$$

The question about the equivalence of those problems got it start already during the lifetime of Joseph-Louis Lagrange (1936 – 1813). Assuming two function, the maximum involving them would be a point in which both of their changes are perpendicular, thus their scalar product is zero. However the magnitude of those changes are not the same which it is corrected using the Lagrange multiplier.

3. Optimal control problem

From the previous sections, the constraints to the problems are always defining the function on some points or interval. There is also the assumption of the continuity and differentiability of the function we are searching for. Also one can note that constraints are defining the end points of the functional but not constraints on the behavior of the independent function during the transition from those boundaries. The optimal control gives a more dynamic way of dealing with extremum problems by dividing the variables into two different types, state variables and control variables. The state variable change can be expressed in terms of transition equation. Thus one can express an optimal control problem by setting a function $u(t)$ in a control space that is peace wise continuous in the interval $t_0 \leq t \leq T$, to determine the maximum of the functional

$$I = \int_{t_0}^{T} f\big(t, x(t), u(t)\big) dt$$

So that the differential constraint $x'(t) = g\big(t, x(t), u(t)\big)$ is satisfied at each $t \in [t_0, T]$ and,

$$x(t_0) = x_0,$$

$$x(T)\ is\ free.$$

The functions $f\ and\ g$ depend on three different variables $x, u\ and\ t$. Of course, both $f\ and\ g$ are assumed to be continuous differentiable functions of the three independent variables over the interval $[t_0, T]$. The function $u(t)$ must be piecewise continuous. It is defined on all the points, with a finite number of breaks, and it does not diverge in the interval $[t_0, T]$.

The solution to this system of equations can be found using the Lagrange multiplier (26) giving $u^*(t)\ and\ x^*(t)$ maximizing I. Thus the langrage multiplier $\lambda(t)$ must make the effect of the state equation vanished over the time. Thus satisfy the following equation,

$$\int_{t_0}^{T} f\big(t, x(t), u(t)\big)dt = \int_{t_0}^{T} \big[f\big(t, x(t), u(t)\big)dt + \lambda(t)g\big(t, x(t), u(t)\big) - \lambda(t)x'(t)\big]dt.$$

Integrating by part the last term of that equation would result in,

$$-\int_{t_0}^{T} \lambda(t)x'(t)dt = -[\lambda(t)x(t)]_{t_0}^{T} + \int_{t_0}^{T} \lambda'(t)x(t)dt$$

Replacing those terms on the equation and evaluating the first term of the second part would give

$$\int_{t_0}^{T} f\big(t, x(t), u(t)\big)dt$$

$$= \int_{t_0}^{T} \big[f\big(t, x(t), u(t)\big)dt + \lambda(t)g\big(t, x(t), u(t)\big) + \lambda'(t)x(t)\big]dt - \lambda(T)x(T)$$

$$+ \lambda(t_0)x(t_0).$$

The Lagrange parameter λ, $u(t)\ and\ x(t)$ should satisfy the state equation

$$x'(t) = g\big(t, x(t), u(t)\big), x(t_0) = x_0,$$

The adjoint equation has the form

$$\lambda'(t) = -[f_x + \lambda(t)g_x],$$

The optimality condition

$$f_u + \lambda(t)g_u = 0.$$

One can generate these conditions by defining the Hamiltonian

$$H\big(t, x(t), u(t)\big) = f\big(t, x(t), u(t)\big) + \lambda(t) g\big(t, x(t), u(t)\big)$$

Thus, from the equation

$$\frac{\partial H}{\partial u} = 0$$

Is equivalent to

$$f_u + \lambda(t) g_u = 0.$$

The adjoint system

$$\frac{\partial H}{\partial x} = \lambda'$$

Is equivalent to

$$\lambda'(t) = -[f_x + \lambda(t) g_x].$$

Finally,

$$\frac{\partial H}{\partial \lambda} = x'$$

Is equivalent to

$$x'(t) = g\big(t, x(t), u(t)\big).$$

Those equations represent the Hamiltonian equations for finding the optimal solution. Now moving to a more simplified version of an optimization problem.

4. Dynamic programing

In this section the Hamilton-Jacobi-Bellman equation would be introduced. To get to that concept, the author will start by introducing the principle of optimality. One can summarize that idea by thinking about the decision variable being optimal over the whole path whatever is the initial values or control. Thus one can express the problem as depending on the initial values but optimal throughout all the path.

To start deriving the relations let's consider the maximization of the functional

$$J = \int_0^T f\big(t, x(t), u(t)\big)dt + q(T, x(T)),$$

such that the state equation $x'(t) = g\big(t, x(t), u(t)\big)$, with the initial condition $x(0) = a$ satisfied. The $q(x(T), T)$ represents the terminal value condition of the system.

Assume that the function $J(t_1, x_1)$ is such that it represents the best values that can be found at time $t_1$in state x_1. The function defined should have values for all $0 \leq t_1 \leq T$ and for all the possible values of the state variable x_1 that could be generated at t_1. So one can write

$$J(t_1, x_1) = max_u \left(\int_{t_1}^T f\big(t, x(t), u(t)\big)dt + q(T, x(T)) \right),$$

At the same time the state equation and initial condition

$$x'(t) = g\big(t, x(t), u(t)\big), x(t_1) = x_1,$$

must be satisfied.

Thus, one can see that the final value of $J\big(T, x(T)\big) = q(T, x(T))$.

Let's take a Δt as a small positive number representing the change in time. One can change the previous integral to

$$J(t_1, x_1) = max_u \left(\int_{t_1}^{t_1+\Delta t} f\big(t, x(t), u(t)\big)dt + \int_{t_1+\Delta t}^T f\big(t, x(t), u(t)\big)dt + q(T, x(T)) \right),$$

Based on the dynamic programing principle that $u(t) for, t_1 + \Delta t \leq t \leq T$, should also give the optimal value for that time interval with a starting state variable

$$x(t_1 + \Delta t) = x_1 + \Delta x_1,$$

Thus one can see the previous problem as

$$J(t_1, x_1) = \max_{\mathrm{u}, t_1 \leq t \leq t_1+\Delta t} \left(\int_{t_1}^{t_1+\Delta t} f dt + \max_{u, t_1+\Delta t \leq t \leq T} \left(\int_{t_1+\Delta t}^T f dt + +q(T, x(T)) \right) \right),$$

Here the state equation and initial condition are

$$x'(t) = g\big(t, x(t), u(t)\big),$$

$$x(t_1 + \Delta t) = x_1 + \Delta x_1.$$

One can express the following formulas as

$$J(t_1, x_1) = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(\int_{t_1}^{t_1+\Delta t} f dt + J(t_1 + \Delta t, x_1 + \Delta x_1) \right),$$

Now using the mean value theorem, one can express the inner integral in the following form

$$\int_{t_1}^{t_1+\Delta t} f(t, x(t), u(t)) dt = F\big(t, x(t), u(t)\big)|_{t=t_1} \Delta t.$$

Assuming the J is twice differentiable and using Taylor's expansion one can write

$$J(t_1 + \Delta t, x_1 + \Delta x_1) = J(t_1, x_1) + J_t(t_1, x_1)\Delta t + J_x(t_1, x_1)\Delta x + C,$$

The constant C represents higher order terms of the Taylor's expansion and can be ignored thus

$$J(t_1, x_1) = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(F\big(t_1, x(t_1), u(t_1)\big)\Delta t + J(t_1, x_1) + J_t(t_1, x_1)\Delta t + J_x(t_1, x_1)\Delta x \right),$$

Now let's subtract $J(t_1, x_1)$ from both sides, thus

$$0 = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(F\big(t_1, x(t_1), u(t_1)\big)\Delta t + J_t(t_1, x_1)\Delta t + J_x(t_1, x_1)\Delta x \right),$$

Next, let's divide over Δt,

$$0 = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(F\big(t_1, x(t_1), u(t_1)\big) + J_t(t_1, x_1) + J_x(t_1, x_1)\frac{\Delta x}{\Delta x} \right),$$

Setting $\Delta x \to 0$ will result in,

$$0 = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(F\big(t_1, x(t_1), u(t_1)\big) + J_t(t_1, x_1) + J_x(t_1, x_1)x' \right),$$

Using the state function one can find

$$0 = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(F\big(t_1, x(t_1), u(t_1)\big) + J_t(t_1, x_1) + J_x(t_1, x_1) g\big(t_1, x(t_1), u(t_1)\big) \right),$$

Since $J_t(t_1, x_1)$ does not depend on u , one can take it to the other side, thus

$$-J_t(t_1, x_1) = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \left(F\big(t_1, x(t_1), u(t_1)\big) + J_x(t_1, x_1) g\big(t_1, x(t_1), u(t_1)\big) \right).$$

Finally one can drop the subscript in $t_1$since it was used only to avoid the confusion in the intervals within the proof. Thus leading to

$$-J_t(t, x) = \max_{\mathrm{u}, t_1 \le t \le t_1 + \Delta t} \big(F(t, x, u) + J_x(t, x) g(t, x, u) \big). \tag{27}$$

Note that $J_t(t,x)$ represents the partial derivative of J with respect to time and t.

This partial differential equation which assumes the optimal value $J(t,x)$ is referred to as the Hamilton-Jacobi-Bellman (HJB) equation.

Using to solve the system one needs to maximize u interms of $t \ and \ x$ and the unknow parameter J_x. Then substituting everything in the HJB equation and get the partial differential equation to be solved.

5. Stochastic optimal control problem

The aim of this section, is to provide a mathematical model for stochastic control problems. An exact definition would be provided later with more detailed proof of the results.

Suppose the stochastic differential equation defined in $t_0 \leq t \leq T$ such that

$$\begin{cases} dx(t) = g\big(t, x(t), u(t)\big)dt + \sigma\big(t, x(t), u(t)\big)dB_t, \\ x(t_0) = x \end{cases} \tag{28}$$

$u(t)$ is a control function, B_t is a Brownian motion, σ is the volatility, and T is the terminal time. Let x be the state variable and u the control.

5.1. Brownian motion characteristics

The Brownian motion of the wiener process is characterized with some properties that will be used in the analysis. The first one is concerning the initial value of the Brownian motion

$$B_0 = 0.$$

The increment in the Brownian motion is independent from previous values, that is the memory less property. For $t > 0$ the values of $B_{t+\Delta t} - B_t$ such that $\Delta t > 0$, does not depend on any previous value of the Brownian motion.

The increment of the Brownian motion is normally distributed with mean 0 and standard deviation Δt. One can say

$$B_{t+\Delta t} - B_t \sim \mathcal{N}(0, \Delta t).$$

One can say that the Brownian motion is continuous in t.

5.2. Ito's Formula

This formulas will help later to break the stochastic problem to more easier terms to deal with.

Assume that the function $y = F(t, B)$ is twice differentiable such that B is the wiener process. Expending the differential of the function using Taylor's series will give

$$dy = F_t dt + F_B dB + \frac{1}{2} F_{tt} (dt)^2 + \frac{1}{2} F_{t,B} dt\, dB + \frac{1}{2} F_{B,t} dB dt + \frac{1}{2} F_{B,B} (dB)^2 + C,$$

The C represents higher order derivatives than the second order and can be ignored.

Using the properties of the Brownian motion, $dt\, dB = 0$,$(dt)^2 = 0$, and $(dB)^2 = dt$, one can find

$$dy = (F_t + \frac{1}{2} F_{B,B}) dt + F_B dB,$$

Now let's assume that $y = F(t, x)$ such that

$$dx(t) = g\big(t, x(t), u(t)\big) dt + \sigma\big(t, x(t), u(t)\big) dB_t.$$

This means that y would be also stochastic since x is a stochastic differential equation. Thus using Taylor's series to find

$$dy = F_t dt + F_x dx + \frac{1}{2} F_{tt} (dt)^2 + \frac{1}{2} F_{t,x} dt\, dx + \frac{1}{2} F_{x,t} dx dt + \frac{1}{2} F_{x,x} (dx)^2 + C$$

Using the same rules as before, that expression will become

$$dy = F_t dt + F_x dx + \frac{1}{2} F_{x,x} (dx)^2,$$

Since dx is derived, one can write

$$dy = F_t dt + F_x (g dt + \sigma dB_t) + \frac{1}{2} F_{x,x} (g dt + \sigma dB_t)^2,$$

expanding gives,

$$dy = (F_t + F_x g) dt + F_x \sigma dB_t + \frac{1}{2} F_{x,x} (g (dt)^2 + 2 g \sigma dt dB + (\sigma dB_t)^2),$$

Taking into consideration the assumption about the Brownian motion's relations about the derivatives one can find,

$$dy = \left(F_t + F_x g + \frac{1}{2} F_{x,x} \sigma^2\right) dt + F_x \sigma dB_t. \tag{29}$$

This formula is the Ito's formula.

One can even extend Ito's process to many variables. This result will not be proven.

Assume $x = [x_1, x_2, \dots, x_n]$, meaning n stochastic equations in the following form.

$$dx(t) = g_i\big(t, x(t), u(t)\big)dt + \sum_{j=1}^{n} \sigma_{i,j}\big(t, x(t), u(t)\big)dB_j, \qquad i = 1,2, \dots, n$$

Let's expressing the correlation coefficient dB_i and dB_j as ρ_{ij} , and expression $dB_i dB_j = \rho_{ij} dt$.

If one assumes the previous equality, and the existence of a function $F(t, x)$. Then Ito's lemma would be

$$dy = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i dx_j.$$

The previous section is a derivation of the computational finance class offered be the University of Tartu. In the next section a general solution of a stochastic optimal control problem would be derived.

5.3. Stochastic optimal control optimization

Consider the optimal control problem with a stochastic differential equation as a constraint. The problem consists in the maximization of

$$J = E\left(\int_0^T f\big(t, x(t), u(t)\big)dt + q(T, x(T))\right),$$

So that the state equation and initial condition are

$$x'(t) = g\big(t, x(t), u(t)\big)dt + \sigma\big(t, x(t), u(t)\big)dB_t, \qquad x(0) = x_0,$$

Assume that the $J(t_1, x_1)$ is the maximum expected value that one can obtain in that problem.

Following the same analogies as in the dynamic programming section one can find,

$$J(t_1, x_1) = \max_{\mathrm{u}} E\left(\int_{t_0}^{T} f dt + q\big(T, x(T)\big)\right),$$

subject to the same constraints with $x(t_1) = x_1$. Thus applying the same procedure and using the mean value theorem while dropping the subscript, one can write

$$J(t, x) = \max_{\mathrm{u}} E\big(f(t, x, u)\Delta t + J(t + \Delta t, x + \Delta x_1)\big),$$

Since the function $J(t, x)$ is twice differentiable one can write Taylor's expansion as in Ito's formulas as

$$J(t + \Delta t, x + \Delta x_1) - J(t, x) = J_t \Delta t + J_x \Delta x + \frac{1}{2} J_{x,x} (\Delta x)^2,$$

Using the same analogies and condition used for Ito's formulas not forgetting the state equation one gets

$$J(t + \Delta t, x + \Delta x_1) = J + J_t \Delta t + J_x g \Delta t + J_x \sigma \Delta B + \frac{1}{2} J_{xx} \sigma^2 \Delta t,$$

Plugging the values in the maximization formulas to get

$$J(t, x) = \max_{\mathrm{u}} E\left(f(t, x, u)\Delta t + J + J_t \Delta t + J_x g \Delta t + J_x \sigma \Delta B + \frac{1}{2} J_{xx} \sigma^2 \Delta t\right),$$

Using the expected value on the Brownian motion will result to its mean zero and taking off $J(t, x)$ from both sides of the equation then dividing by Δt one obtains

$$-J_t(t, x) = \max_{\mathrm{u}} \left(f(t, x, u) + J_x(t, x) g(t, x, u) + \frac{1}{2} \sigma^2 J_{xx}(t, x)\right), \tag{30}$$

This equation is needed to define a stochastic optimal control with a boundary condition

$$J(T, x(T)) = q\big(T, x(T)\big).$$

Those are the same conditions found in the HJB (27) equation derivation but with an extra term extracted from the Taylor's series.

5.4. Market model

The purpose of this section is to define a market model and apply all the previously stated theory to solve one problem.

5.4.1. Black-Sholes model

Let's assume that the stock price of asset i, P_i, moves according the following equation

$$dP_i(t) = P_I(t)(\mu_i(P,t)dt + \sigma_i(P,t)dB_i), \tag{31}$$

such that $\mu_i(P,t)$ represents the average growth of the stock per each unit of time and $\sigma_i(P,t)$ is the volatility of the stock price. Of course B_i is the Brownian motion. This equation describes intuitively the change of the stock price by adding an expected potion of the mean value of the stock linearly and a random portion characterized by a normal distribution.

5.4.2. Additional assumption

Let's assume that the market has the following properties:

- It is possible to trade continuously in the market with fraction number of stocks.
- The market has no arbitrage opportunities meaning that one cannot make money without taking any risk.
- There is not transaction cost.
- The risk free rate is known r and is the same for lending and borrowing.

Some additional assumption would be made later to simplify the calculations and would be stated clearly in this corresponding section.

5.4.3. Budget equation

To get to the continuous time model one needs to examine the discreate-time formulations since it is more intuitive. Then one can set the change of the time to zero and get the continuous model.

- Since it is a discrete time approach, let's define h and the small change in time.
- Let's define the variables $N_i(t)$ as the number of shares of asset i at time t. Meaning, in a discrete time approach, $N_i(t)$ is the number of share between $t \ and \ t + h$.
- Let's define the consumption function $C(t)$ as the amount needed at time t.

- The wealth invested at time t would be defined from previous number of stocks owned times the current price of those stocks. Assuming n number of securities in the portfolio one can write

$$W(t) = \sum_{i}^{n} N_i(t-h)P_i(t).$$

- Assume $y(t)$ to representing the addition to the capital coming from sources other than gain or loss (portion of the salary, bank transfer etc.).

The next thing is to determine the amount of consumption per each instance period. For that the change of the number of stocks from $t - h\ to\ t$ should be multiplied by the price at time t. This will represent the consumption accrued in time t and one can write

$$y(t) - C(t)h = \sum_{i}^{n} [N_i(t) - N_i(t-h)]P_i(t).$$

After one time instance h one can write the following equations representing the consumption and the wealth at time $t + h$.

$$y(t+h) - C(t+h)h = \sum_{i}^{n} [N_i(t+h) - N_i(t)]P_i(t+h)$$

$$= \sum_{i}^{n} [N_i(t+h) - N_i(t)][P_i(t+h) - P_i(t)] + \sum_{i}^{n} [N_i(t+h) - N_i(t)]P_i(t),$$

$$W(t+h) = \sum_{i}^{n} N_i(t)P_i(t+h).$$

Now setting $h \to 0$ gives

$$dy(t) - C(t)dt = \sum_{i}^{n} [dN_i(t)][dP_i(t)] + \sum_{i}^{n} [dN_i(t)]P_i(t),$$

$$W(t) = \sum_{i}^{n} N_i(t)P_i(t).$$

Since $P_i(t)$ follows the Black-Sholes equation (31), one can derive the wealth as

$$dW(t) = \sum_i^n N_i(t)dP_i(t) + \sum_i^n dN_i(t)P_i(t) + \sum_i^n dN_i(t)dP_i(t),$$

When one looks at this equation one can see that the additional sources to the wealth comes from the last two last terms which are representing the consumption plus the non-capital gains of the portfolio showed previously. Thus

$$dW(t) = \sum_i^n N_i(t)dP_i(t) + y(t)dt - C(t)dt.$$

Replacing the $dP_i(t)$ by the Black-Sholes equation gives

$$dW(t) = \sum_i^n N_i(t)P_I(t)[\mu_i(P,t)dt + \sigma_i(P,t)dB_i] + y(t)dt - C(t)dt,$$

In order not to work with N_i and generalize the equation, one can define the weight of each stock as

$$w_i(t) = \frac{N_i(t)P_i(t)}{\sum_i^n N_i(t)P_i(t)} = \frac{N_i(t)P_i(t)}{W(t)}.$$

One can see that

$$\sum_i^n w_i(t) = 1.$$

Note that short selling would be allowed by letting $w_i(t) < 0$.

Replacing $N_i(t)P_i(t)$ by $w_i(t)W(t)$ one gets

$$dW(t) = \sum_i^n w_i(t)W(t)[\mu_i(P,t)dt + \sigma_i(P,t)dB_i] + dy(t) - C(t)dt.$$

Assume that asset k is risk risk-free meaning that $\sigma_k(P,t) = 0$. The return on that asset would be $\mathrm{r}w_k(t)W(t)$. Thus one can write the equation as

$$dW(t) = \sum_{i\,(i\neq k)}^n w_i(t)W(t)[\mu_i(P,t)dt + \sigma_i(P,t)dB_i] + \big(\mathrm{r}w_k(t)W(t) - C(t)\big)dt + dy(t)$$

For the sake of simplicity assume $k = n$ and taking out the subscripts,

$$dW(t) = \sum_{i}^{n-1} w_i W[\mu_i - r]dt + (\mathrm{r}W - C)dt + dy + \sum_{i}^{n-1} w_i W \sigma_i dB_i. \qquad (32)$$

This equation represents the change in the wealth and was derived first by Merton.

5.4.4. Optimal portfolio under consumption.

The problem of an optimal portfolio under consumption for a time horizon of T is formulated in the following way:

Find the maximum of

$$E\left[\int_0^T U(C(t),t)dt + D(W(T),T)\right],$$

$$subject\ to\ \ (32), W(0) = W_0.$$

It is near impossible to solve the following problem without additional assumption about all the parameters within the problem. Let's start by expanding the parameters and making some reasonable assumptions.

- Assume that $U(C(t),t)$ is the utility function and that it is strictly concave in $C(t)$ meaning that the derivative of this function does not change its sign over the domain definition of that variable.
- The wealth equation (32) assumes the existence of a risk free asset in the portfolio.
- $D(W(T),T)$ is the terminal value of the portfolio is also assumed to be concave on $W(T)$.

To solve the problem stochastic dynamic programing would be used. This means the definition of a function

$$J(W(t),P,t) = \max_{\{\mathrm{C,W}\}} E_t\left(\int_t^T U(C(s),s)ds + D(W(T),T)\right)$$

Since the problem consists in the maximization over two variables, the solution of this problem would require advanced mathematical expressions including the Dynkin operator. However, expanding this paper to cover those subjects would be out of context. More in depth covering of this could be found in the book by Merton [12]. The solution to this problem is derived from creating a function describing the utility and the Dynkin operator. Then later describing the constraints on the weights of the portfolio and adding the condition using the Lagrange multiplier. Finally use (26) to solve for the optimal solution.

The solution presented using the risk free asset is to solve the following equation for m assets to get $J(W(t), P, t)$

$$0 = U\big(G(P,T)\big) + J_t + J_W[rW - G] + \sum\nolimits_{i=1}^{m} J_i \alpha_i P_i + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} J_{iJ}\sigma_{ij}P_iP_j$$
$$- \frac{J_W}{J_{WW}}\sum\nolimits_{j=1}^{m} J_{jW}P_j\big(\alpha_j - r\big) + \frac{J_W^2}{2J_{WW}}\sum_{i=1}^{m}\sum_{j=1}^{m} v_{iJ}(\sigma_i - r)\big(\sigma_j - r\big)$$
$$- \frac{1}{2J_{WW}}\sum_{i=1}^{m}\sum_{j=1}^{m} J_{iW}J_{jW}\sigma_{ij}P_iP_j$$

Here G is the inverse function of the first derivative of U with respect to C. Namely, $G = [U_C]^{-1}$ such that $U_C = \frac{\partial U}{\partial C}$.

The derivatives are denoted as follows

$$J_t = \frac{\partial J}{\partial t}, J_W = \frac{\partial J}{\partial W}, J_i = \frac{\partial J}{\partial P_i}, J_{ij} = \frac{\partial^2 J}{\partial P_i \partial P_j}, J_{iW} = \frac{\partial^2 J}{\partial P_i \partial W}.$$

Here $\left[v_{iJ}\right] \equiv \Omega^{-1}$ is an $n\ x\ n$ matrix defined as the inverse of the variance covariance matrix $\Omega \equiv [\sigma_{ij}]$.

After the function $J(W(t), P, t)$ is derived the optimal solution can be found by using the following formulas

$$w_k^* = -\frac{J_W}{J_{WW}^W}\sum_{i}^{m} v_{kj}\big(\alpha_j - r\ \big) - \frac{J_{kW}P_K}{J_{WW}^W}, k = 1,2,..,m,$$

$$C^* = G(J_W, t).$$

The following equation were obtained in the previously stated source and were not derived by the author.

5.4.5. Application of portfolio under consumption

The following section will deal with a basic portfolio problem composed from two different assets. One of these is riskless and the other one is risky asset. Assuming there is no transaction cost we define the following functions and variables as the problems given statements.

- W(t) is the function defining the total wealth would be W .
- w is the proportion of the wealth invested in the risky asset.
- r is the risk free rate or the return of the riskless asset.
- $R_w$is expected return of the risky asset.
- σ^2 fixed variance of the return of the risky asset.
- c is a constant consumption rate of the asset.
- $U(c) = \frac{c^b}{b}, b < 1$ which is a special case of the King–Plosser–Rebelo preferences utility function.

Keep in mind that the equation describing the change in wealth is (32). Since we have two assets the equation becomes

$$dW(t) = wWR_w dt + (\mathrm{r}(1 - \mathrm{w})W - C)dt + dy + wW\sigma dB,$$

or in another form

$$dW(t) = (wWR_w + \mathrm{r}(1 - \mathrm{w})W - C)dt + dy + wW\sigma dB. \qquad (33)$$

The optimization problem would be to maximize the expected discounted utility steam as discussed in over an infinite time horizon see page 9.

$$\max E\left[\int_0^\infty e^{-rt} U(c)\, dt\right],$$

$$subject\ to\ (33), W(0) = W_0.$$

This is a stochastic optimal problem that can be solved using the HJB equation since the wealth equation had the form of a Black Sholes equation. Solving this problem directly would lead to the assumption of a function

$$J(t_0, W) = \max_{c,w} E\left[\int_{t_0}^{\infty} e^{-rt} U(c)\, dt\right]$$

That is the optimal value of the maximization.

This problem can be broken down into sub problems since the starting time does not affect the end of the period. Thus one can introduce a similar problem in the following way to make the math a bit easier. This consist sin the maximization of

$$F(W_o) = E\left[\int_{t_0}^{\infty} e^{-r(t-t_0)} U(c)\, dt\right]$$

$$subject\ to\ (33), W(t_0) = W_0.$$

One can see that the

$$J(t, W) = e^{-rt} F(W)$$

While F does not depend on t.

Thus using the equation (30)

$$-J_t(t, W) = \max_{\mathrm{c,w}}\left(U(c) + J_W(t, W) g(W) + \frac{1}{2}\sigma^2 J_{WW}(t, W)\right),$$

would require a definition of the function $g(W)$ which is a cording to (33) is

$$g(W) = wWR_w + \mathrm{r}(1 - \mathrm{w})W - C$$

Other functions would be

$$\sigma^2 = (wW\sigma)^2$$

$$U(c) = \frac{c^b}{b}$$

$$-J_t(t, W) = re^{-rt} F(W)$$

$$J_W(t, W) = e^{-rt} F'(W)$$

$$J_{WW}(t, W) = e^{-rt} F''(W)$$

Plugging the results in the equation would give

$$re^{-rt}F(W) = \max_{\mathrm{c,w}}\left(\frac{c^b}{b} + e^{-rt}F'(W)\ (wWR_w + \mathrm{r}(1-\mathrm{w})W - C) + \frac{1}{2}(wW\sigma)^2 e^{-rt}F''(W)\right),$$

Thus

$$rF(W) = \max_{\mathrm{c,w}}\left(\frac{c^b}{b} + F'(W)(wWR_w + \mathrm{r}(1-\mathrm{w})W - c) + \frac{1}{2}(wW\sigma)^2 F''(W)\right),$$

Deriving the following linear equation over the optimization variables would give

$$0 = c^{b-1} - F'(W),$$

$$0 = (WR_w - rW)F'(W) + wW^2\sigma^2F''(W)$$

Thus

$$c = [F'(W)]^{\frac{1}{b-1}},$$

$$w = \frac{F'(W)(r - R_w)}{W\sigma^2F''(W)}$$

Let's substitute the results in the HJB equation development

$$\begin{aligned} rF(W) = {} & \frac{[F'(W)]^{\frac{b}{b-1}}}{b} \\ & + F'(W)\left(\frac{F'(W)(r-R_w)}{W\sigma^2F''(W)}WR_w + \mathrm{r}\left(1 - \frac{F'(W)(r-R_w)}{W\sigma^2F''(W)}\right)W - [F'(W)]^{\frac{1}{b-1}}\right) \\ & + \frac{1}{2}\left(\frac{F'(W)(r-R_w)}{W\sigma^2F''(W)}W\sigma\right)^2 F''(W), \end{aligned}$$

which is equivalent to

$$\begin{aligned} rF(W) = {} & [F'(W)]^{\frac{b}{b-1}}\left(\frac{1-b}{b}\right) + F'(W)rW + F'(W)\frac{F'(W)(r-R_w)}{\sigma^2F''(W)}(R_w - r) \\ & + \frac{1}{2}\frac{F'(W)^2(r-R_w)^2}{\sigma^2F''(W)}, \end{aligned}$$

Developing more gives

$$rF(W) = [F'(W)]^{\frac{b}{b-1}}\left(\frac{1-b}{b}\right) + F'(W)rW + \frac{F'(W)^2}{2\sigma^2F''(W)}((r-R_w)^2 - 2(r-R_w)^2),$$

leading to,

$$rF(W) = [F'(W)]^{\frac{b}{b-1}}\left(\frac{1-b}{b}\right) + F'(W)rW + \frac{F'(W)^2}{2\sigma^2 F''(W)}(-(r-R_w)^2),$$

Thus

$$rF(W) = [F'(W)]^{\frac{b}{b-1}}\left(\frac{1-b}{b}\right) + F'(W)rW - \frac{F'(W)^2(r-R_w)^2}{2\sigma^2 F''(W)}.$$

This is a second order differential equation that cannot be solved easily. This part of the analysis would not be covered and thus now moving to a more easy way of dealing with multiple assets within the same portfolio.

6. The portfolio selection problem

Before introducing the portfolio selection problem, one needs to define the variables used in the analysis.

6.1. Definition of variables

Assume a portfolio with n assets, and each asset i has a weight assigned to it w_i. Obviously,

$$\mathbf{w} = (w_1, w_2, \dots, w_n) \ such\ that \sum_{i=1}^{n} w_i = 1.$$

The rates of return on each of those assets are random variables

$$\mathbf{R} = (R_1, R_2, \dots, R_n)^T.$$

This shows that the portfolio is also a random variable

$$R_p = \mathbf{w}^{\mathbf{T}}\mathbf{R} = \sum_{i=1}^{n} w_i R_i.$$

Let's assume that the expected rate of return of each asset is

$$E[\mathbf{R}] = \boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)^T,$$

and the variance of each asset i is represented as

$$\sigma_i^2 = \mathrm{VAR}(R_i) = E[(R_i - \mu_1)^2] = E[R_i^2] - \mu_1^2.$$

Let's define the variance-covariance matrix of $\mathbf{R}$ with entries i, j as follow:

$$\Sigma_{i,j} = cov[R_i, R_j] = E[(R_i - \mu_1)(R_j - \mu_j)],$$

Generalizing this into a matrix will give,

$$\mathrm{COV}[\mathbf{R}] = \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{1,1}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{1,n}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_{\mathbf{n,1}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{n,n}} \end{bmatrix}.$$

Note that the diagonal vector the previous matrix is the variance vector.

The next step is to calculate the mean and variance of the portfolio. The mean would be easily found in the following way

$$\mu_p = E[R_p] = \sum_{i=1}^{n} w_i \mu_i = \mathbf{w}^{\mathbf{T}} \boldsymbol{\mu}.$$

The variance of the portfolio would be

$$\sigma_p^2 = E\left[(R_p - \mu_p)^2\right] = E\left[\left(\sum_{i=1}^{n} w_i R_i - \sum_{i=1}^{n} w_i \mu_i\right)^2\right] = E\left[\left(\sum_{i=1}^{n} w_i (R_i - \mu_i)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i (R_i - \mu_i)\right)\left(\sum_{i=1}^{n} w_i (R_i - \mu_i)\right)\right]$$

$$= E\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (R_i - \mu_i)(R_j - \mu_j)\right)^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \Sigma_{i,j}$$

This can be expressed in the matrix form as

$$\sigma_p^2 = \mathbf{w}^{\mathbf{T}} \boldsymbol{\Sigma} \mathbf{w}.$$

Later in the analysis a notation $\mathbf{1}_{\boldsymbol{n}}$ would represent a vector of n values made from ones. The same is for $\mathbf{0}_{\mathbf{n}}$ but it is made from zeros.

6.2. The Markowitz problem

The problem is to find the a portfolio with a low variance and maximum return on a specific number of assets. The expected return from the portfolio would be given as μ_0. The problem would be

$$\text{Minimize: } \frac{1}{2}\mathbf{w}^{\mathbf{T}}\mathbf{\Sigma}\mathbf{w}$$
$$\text{Subject to: } \mathbf{w}^{\mathbf{T}}\boldsymbol{\mu} = \mu_0, \mathbf{w}^{\mathbf{T}}\mathbf{1}_{\boldsymbol{n}} = 1. \tag{34}$$

To solve this problem, one needs to define Lagrange function (26)

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}^{\mathbf{T}}\mathbf{\Sigma}\mathbf{w} + \lambda_1(\mu_0 - \mathbf{w}^{\mathbf{T}}\boldsymbol{\mu}) + \lambda_2(1 - \mathbf{w}^{\mathbf{T}}\mathbf{1}_{\mathbf{n}}).$$

As known, the solution to the system needs the first-order conditions

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{0}_{\mathbf{n}} = \mathbf{\Sigma}\mathbf{w} - \lambda_1\boldsymbol{\mu} - \lambda_2\mathbf{1}_{\mathbf{n}},$$

$$\frac{\partial L}{\partial \lambda_1} = 0 = \mu_0 - \mathbf{w}^{\mathbf{T}}\boldsymbol{\mu},$$

$$\frac{\partial L}{\partial \lambda_2} = 0 = 1 - \mathbf{w}^{\mathbf{T}}\mathbf{1}_{\mathbf{n}}.$$

Then expressing $\mathbf{w}$ with respect to λ_1 and λ_2 leads to,

$$\mathbf{\Sigma}\mathbf{w}^* = \lambda_1\boldsymbol{\mu} + \lambda_2\mathbf{1}_{\mathbf{n}},$$

keep in mind that the inverse of a symmetric matrix always exists which is the case of $\mathbf{\Sigma}$. Thus

$$\mathbf{w}^* = \lambda_1\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \lambda_2\mathbf{\Sigma}^{-1}\mathbf{1}_{\mathbf{n}},$$

Using the other first order conditions gives

$$\mu_0 = \mathbf{w}^{*\mathbf{T}}\boldsymbol{\mu} = \lambda_1(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) + \lambda_2(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}_{\mathbf{n}}),$$

$$1 = \mathbf{w}^{*\mathbf{T}}\mathbf{1}_{\mathbf{n}} = \lambda_1(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}_{\mathbf{n}}) + \lambda_2(\mathbf{1}_{\mathbf{n}}^{T}\mathbf{\Sigma}^{-1}\mathbf{1}_{\mathbf{n}}).$$

Let's set $x = (\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu})$, $y = (\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1}_{\mathbf{n}})$ and $z = (\mathbf{1}_{\mathbf{n}}^{T}\mathbf{\Sigma}^{-1}\mathbf{1}_{\mathbf{n}})$.

Those two equation could be written in the form

$$\begin{bmatrix} \mu_0 \\ 1 \end{bmatrix} = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{35}$$

This is a system of equation that can be solved easily using Cramer's rule

$$\lambda_1 = \frac{det\begin{bmatrix} \mu_0 & y \\ 1 & z \end{bmatrix}}{det\begin{bmatrix} x & y \\ y & z \end{bmatrix}} = \frac{\mu_0 z - y}{xz - y^2},$$

$$\lambda_2 = \frac{det\begin{bmatrix} x & \mu_0 \\ y & 1 \end{bmatrix}}{det\begin{bmatrix} x & y \\ y & z \end{bmatrix}} = \frac{x - \mu_0 y}{xz - y^2}.$$

So the mean of the portfolio is

$$\mathbf{w}^* = \lambda_1 \mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \lambda_2 \mathbf{\Sigma}^{-1}\mathbf{1_n}, \tag{36}$$

where

$$\lambda_1 = \frac{\mu_0 z - y}{xz - y^2} \; and \; \lambda_2 = \frac{x - \mu_0 y}{xz - y^2}.$$

The variance of the portfolio will need some calculation

$$\sigma_p^{*2} = \mathbf{w}^{*\mathbf{T}}\mathbf{\Sigma}\mathbf{w}^* = \lambda_1^2(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) + 2\lambda_1\lambda_2(\boldsymbol{\mu}^{\mathbf{T}}\mathbf{\Sigma}^{-1}\mathbf{1_n}) + \lambda_2^2(\mathbf{1_n^{\mathit{T}}}\mathbf{\Sigma}^{-1}\mathbf{1_n}),$$

One can write that in a quadratic matrix form

$$\sigma_p^{*2} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

From (35), one gets

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}^{-1} \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}.$$

Substituting it in the σ_p^{*2} expression gives

$$\sigma_p^{*2} = \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x & y \\ y & z \end{bmatrix}^{-1} \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix}^{-1} \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x & y \\ y & z \end{bmatrix}^{-1} \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix}^{-1} = \frac{1}{det\begin{bmatrix} x & y \\ y & z \end{bmatrix}} \begin{bmatrix} z & -y \\ -y & x \end{bmatrix},$$

the expression will become

$$\sigma_p^{*2} = \frac{1}{det\begin{bmatrix} x & y \\ y & z \end{bmatrix}} [\mu_0 z - y \quad -\mu_0 y + x] \begin{bmatrix} \mu_0 \\ 1 \end{bmatrix} = \frac{\mu_0^2 z - 2\mu_0 y + x}{xz - y^2}.$$

This finishes the optimization.

6.3. Expected Return Maximization

Now consider the opposite of the previous case. With an expected risk objective let's find the maximum return. The problem would be

$$\text{Maximize: } \mathbf{w^T \mu}$$

$$\text{Subject to: } \mathbf{w^T \Sigma w} = \sigma_0^2, \mathbf{w^T 1}_{\boldsymbol{n}} = 1.$$

It has been shown by Markowitz that this problem (34) is equivalent to the previous one with a right change of variables one can get the same solution.

6.4. Risk Aversion Optimization

The Arrow-Pratt risk aversion index would be introduced in this section. In short words, the risk aversion index is defined as the additional reward one requires for such additional risk. Thus writing

$$A = \frac{dE(R_p)}{d\sigma_p},$$

The problem would be to maximize the difference between our expected return and the expected return resulting from taking the risk. The problem would be

$$\text{Maximize: } \mathbf{w^T \mu} - \frac{\mathbf{1}}{\mathbf{2}} \boldsymbol{A} \mathbf{w^T \Sigma w}$$

$$\text{Subject to: } \mathbf{w^T 1}_{\boldsymbol{n}} = 1.$$

It has also been proven that the previously stated problem is equivalent to the mean-variance model solved in the Markowitz problem section (34).

6.5. Mean Value maximization with risk free asset

This section will cover the portfolio theory maximization with a risk free asset existence. As one knows the risk free asset has an expected return $E(R_0) = r_0$ and the variance of that return turn out to be zero. Assume that one has m risky assets and a risk free one. Meaning one has $w^T \mathbf{1}_{\boldsymbol{m}} = \sum_1^m w_i$ from the capital invested in risky assets and $1 - \sum_1^m w_i$ invested in risk free asset. If one allows the previous quantity to be negative it means allowing to borrow with r_0 rate.

The return of the portfolio would be

$$R_p = w^T \boldsymbol{R} + (1 - w^T \mathbf{1}_{\boldsymbol{m}}) R_0,$$

Thus the expected mean would be

$$\mu_p = w^T \boldsymbol{\mu} + (1 - w^T \mathbf{1}_{\boldsymbol{m}}) r_0,$$

Since the variance of the risk free asset is zero it means that the variance of the portfolio would be

$$\sigma_p^2 = \mathbf{w^T \Sigma w}.$$

Thus the risk minimalization problem will become

$$\text{Minimize: } \frac{1}{2}\mathbf{w^T \Sigma w}$$
$$\text{Subject to: } w^T \boldsymbol{\mu} + (1 - w^T \mathbf{1}_{\boldsymbol{m}}) r_0 = \mu_0. \tag{37}$$

Using Lagrange multiplier gives

$$L(\mathbf{w}, \lambda_1) = \frac{1}{2}\mathbf{w^T \Sigma w} + \lambda_1 (w^T \boldsymbol{\mu} + (1 - w^T \mathbf{1}_{\boldsymbol{m}}) r_0 - \mu_0)),$$

Solving the system with the first order condition one gets,

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{0_n} = \mathbf{\Sigma w} - \lambda_1 (\boldsymbol{\mu} - \mathbf{1_m} r_0),$$

$$\frac{\partial L}{\partial \lambda_1} = 0 = w^T \boldsymbol{\mu} + (1 - w^T \mathbf{1}_{\boldsymbol{m}}) r_0 - \mu_0 = (r_0 - \mu_0) - w^T (\boldsymbol{\mu} - \mathbf{1}_{\boldsymbol{m}} r_0).$$

From the first equation one gets

$$\mathbf{w}^* = \lambda_1 \mathbf{\Sigma^{-1}} (\boldsymbol{\mu} - \mathbf{1_m} r_0), \tag{38}$$

Using the second condition one can get the value of λ_1 and follow

$$\lambda_1 (\boldsymbol{\mu} - \mathbf{1_m} r_0)^{\boldsymbol{T}} \mathbf{\Sigma^{-1}} (\boldsymbol{\mu} - \mathbf{1}_{\boldsymbol{m}} r_0) = (r_0 - \mu_0),$$

Thus

$$\lambda_1 = \frac{(r_0 - \mu_0)}{(\boldsymbol{\mu} - \mathbf{1_m} r_0)^{\boldsymbol{T}} \mathbf{\Sigma^{-1}} (\boldsymbol{\mu} - \mathbf{1}_{\boldsymbol{m}} r_0)}.$$

This end the optimization. Note that the variance of the portfolio would be

$$\sigma_p^{*2} = \mathbf{w}^{*\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}^* = \frac{(r_0 - \mu_0)^2}{(\boldsymbol{\mu} - \mathbf{1_m}r_0)^{\boldsymbol{T}}\boldsymbol{\Sigma}^{-\mathbf{1}}(\boldsymbol{\mu} - \mathbf{1}_{\boldsymbol{m}}r_0)}.$$

6.6. Value at risk

The Markowitz portfolio treats the variance of the portfolio symmetrically, Thus not leaving a chance to include the risk aversion of the investor. One way to take it into consideration is to include the investors risk aversion within the variance of the return. Since the return of the portfolio is a random variable depending on the weights and variance of the portfolio. One can write the change in the portfolio as $g(w, \sigma)$. Meaning the distribution of the change depends on the variance return on those securities. Assume for convenience that variance of the returns has a probability distribution function $f(\sigma)$ with a density thus allowing the modeling $g(w, \sigma)$. Thus one can write

$$\Psi(w, \sigma^*) = \int f(\sigma) d\sigma$$

The integral is calculated over the region $g(w, \sigma) < \sigma^*$.

- σ^*represents the threshold set a standard deviation for the portfolio.
- $\Psi(w, \sigma^*)$ is the cumulative distribution of the change and depends on w the weights of the portfolio.

The VaR for a specific set of weights could be expressed in general in the following way.

$$\sigma_\alpha(w) = \min\{\sigma \in \mathbb{R} : \Psi(w, \sigma) \geq \alpha\} \tag{39}$$

- Such that α represents the risk aversion parameter of the investor.

To make the problem a bit easier and assuming the returns are normally distributed with mean μ_p.

$$profit\&loss \sim N(0, \mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w})$$

Thus one can calculate the value at risk by using the risk aversion corresponding percentile

$\mathcal{Z}_\alpha, \alpha\%$ and the mean of the portfolio, say

$$VaR_\alpha = -(\mathbf{w}^{\mathbf{T}}\boldsymbol{\mu} + \mathcal{Z}_\alpha\sqrt{\mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}}\,)$$

The following can constructed using the values of risk such that β is the proportion from the mean taken as risk.

$$\text{Maximize} : -2\beta\mu_p - VaR_\alpha$$

$$\text{Subject to:} \mathbf{w}^{\mathbf{T}}\mathbf{1}_{\boldsymbol{n}} = 1$$

The problem is equivalent to solving

$$\begin{gathered} \text{Minimize} : (2\beta + 1)\mathbf{w}^{\mathbf{T}}\boldsymbol{\mu} + \mathcal{Z}_\alpha\sqrt{\mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}} \\ \text{Subject to:} \mathbf{w}^{\mathbf{T}}\mathbf{1}_{\boldsymbol{n}} = 1 \end{gathered} \tag{40}$$

To solve this problem one are going to start with defining Lagrangian function

$$L(\mathbf{w}, \lambda_1, \lambda_2) = (2\beta + 1)\mathbf{w}^{\mathbf{T}}\boldsymbol{\mu} + \mathcal{Z}_\alpha\sqrt{\mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}} + \lambda(\mathbf{w}^{\mathbf{T}}\mathbf{1}_{\mathbf{n}} - 1).$$

Using the first-order equalities one gets

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{0}_{\mathbf{n}} = (2\beta + 1)\boldsymbol{\mu} + \frac{\mathcal{Z}_\alpha\boldsymbol{\Sigma}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}}} + \lambda\mathbf{1}_{\mathbf{n}},$$

$$\frac{\partial L}{\partial \lambda} = 0 = \mathbf{w}^{\mathbf{T}}\mathbf{1}_{\mathbf{n}} - 1,$$

The first derivative will give,

$$\mathbf{0}_{\mathbf{n}} = (2\beta + 1)\boldsymbol{\mu} + \frac{\mathcal{Z}_\alpha\boldsymbol{\Sigma}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}}} + \lambda\mathbf{1}_{\mathbf{n}},$$

The solution for $\beta > 0$ is

$$\lambda = \frac{-B + (B^2 - 4AC)^{\frac{1}{2}}}{2A},$$

$$\text{such that: } A = 1_n^T\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}_{\mathbf{n}},$$

$$B = (2\beta + 1)(\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}_{\mathbf{n}} + \mathbf{1}_{\boldsymbol{n}}\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\mu}),$$

$$\text{and } \mathbf{C} = (2\beta + 1)^2\boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\mu} - \mathcal{Z}_\alpha$$

If the solution of $B^2 - 4AC \geq 0$ one can say that

$$\mathbf{w}^* = \frac{(2\beta + 1)\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\mu} + \lambda\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}_{\mathbf{n}}}{(2\beta + 1)\mathbf{1}_{\mathbf{n}}\boldsymbol{\Sigma}^{-\mathbf{1}}\boldsymbol{\mu} + \lambda 1_n^T\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}_{\mathbf{n}}} \tag{41}$$

The means and the VaR of the portfolio can be calculated using the previous relations in (40).

6.7. Conditional Value at risk

The VaR method is returning the worst case scenario associated with the a certain probability, however sometimes this could be seen as being too optimistic. Thus the idea of having an expected loss when the Var threshold is crossed could make the portfolio lose even more. Thus the CVaR method was introduced as a way to quantify the threshold beyond the VaR portfolio. Assuming that the distributions of the VaR value (39) is continuous and non-decreasing. One can write that CVaR is

$$\phi_\alpha(w) = (1-\alpha)^{-1} \int g(w,\sigma)p(\sigma)d\sigma$$

The integral is calculated over the range $g(w,\sigma) \geq \sigma_\alpha(w)$.

The previously stated equation comes to describe the conditional expectation of the loss associated with the weights w.

The main assumption about the random variable $g(w,\sigma)$ is that it has a distribution in $\mathbb{R}$ thus allowing us to have define characteristic function defining the CVaR

$$G_\alpha(w) = \alpha + (1-\alpha)^{-1} \int [g(w,\sigma) - \alpha]^+ p(\sigma)d\sigma$$

- The integral is calculated over such that $\sigma \in \mathbb{R}^m$.
- Such that m is the dimension of the weight or the number of securities in the portfolio.
- The operator $[\quad]^+$ is defined as

$$[s]^+ = \begin{cases} s, s > 0, \\ 0, s \leq 0, \end{cases}$$

Note that the whole analysis could be found with more details in the following source [18].

This means that the α_CVaR of the loss can be found by solving the following problem

$$\phi_\alpha(w) = \underset{\alpha \in \mathbb{R}}{Min} F_\alpha(w).$$

Thus the following constraint would be added to the VaR problem and solved for an optimal portfolio.

6.8. Using the CAPM formula and an Index

This is an easy way to perceive the markets based on indexes, The problem is to maximize the return

$$\mu_p = \sum_{n=1}^{n} \mu_i w_i.$$

subject to linear constrains

$$\sum_{n=1}^{n} b_i w_i \leq \ b_{Index}, \mathbf{w}^{\mathbf{T}} \mathbf{1}_{\boldsymbol{n}} = 1, \mathbf{w} \geq \mathbf{0}_{\mathbf{n}}.$$

The way to get the beta (b_i) is by creating a linear regression model between the independent variable that would be the index and the security for which one wants to find the beta. The beta represents the volatility of the stock with respect to the index which is a measurement of the riskiness of the company with respect to the market chosen.

The index is the reference of the whole market. The most important one in the Tallinn stock Exchange is the OMX Tallinn (^OMXT). The index is a weighted chain-linked total return of stocks issued by all the companies traded in Tallinn stock market. It was first initiated to 100 in 3rd June 1996. It is also a good measure of liquidity, return and market size of the Estonian economy.

7. Application of the Models in R

The application of the portfolio optimization was made in Tallinn Exchange stock market. The data used in the analysis is for two years one month starting from 1st February 2018 till the 29th February 2020 taking out the effect of the COVID-19 on the economy. Other IPO's (Initial public offering) happening after the 1st February 2018 were not taken into consideration. Meaning that fourteen companies were taken as the population for the application. A small introduction about those companies can be found in the following table:

Table 1: List of companies in Tallinn stock exchange with historic values more than two year before "28-02-2020"

Names of the company (Index)	Number of shares	Description
Acro Vara (ARC1T.TL)	8.998M	The company is a public limited liability operating in the real estate business. It has two main services, the first one is a real estate service segment which translates to estate advisory, brokerage, and appraisal services. The second main business is a real estate development segment focused on making new residential areas.
Baltika (BLT1T.TL)	54M	Baltika is a public limited liability company operating in fashion retail. The company design, manufactures, distribute and sells its products. Operating under four main brands Monton, Mosaic, Baltman and Ivo Nikkolo. The company was having some hard times since 2017 and it is trying to restructure with the aim of generating revenue in the future.
EfTEN Real Estate Fund III AS (EFT1T.TL)	4.222M	The company is a closed alternative investment fund. Aimed for retail investors since it invests in commercial, storage, retail and logistic premises. Thus providing a constant income. The company operates all over the Baltic region mainly Lithuania.
Ekspress Group (EEG1T.TL)	29.796M	Ekspress Group is Estonian-based media company operating in media and printing services. It operates many online websites providing online advertisement. The company also provides outdoor digital advertisement. The second main revenue point for the company consist of publishing magazines, books, and newspapers in Estonia. The main source of income is from the media sector.
Harjo Elekter (HAE1T.TL)	17.739M	The company operates in three segments production, real estates and export. The production is mainly electric power distribution and systems. The real estate segment is related to providing development, maintenance, and advisory about properties. The third main activity is related to exporting manufactured goods to other countries mainly Finland.
LHV Group (LHV1T.TL)	28.819M	LHV is a holding company providing banking, security breakage, and financial services in Estonia. It operates in retail banking, financial intermediate and corporate banking. The company provide services for private individuals and small entities in the retail banking section. The corporate banking is for legal entities and corporate customers. The financial intermediate is mainly for fintech companies with large payments in Estonia and UK. The main revenue of the company comes from corporate banking segment.

Merko Ehitus (MRK1T.TL)	17.7M	Merko Ehitus is a construction company operating in two segments. The first segment is construction services related to taking state, corporate and private contraction deals. The second one is real estate development. The company operates mainly in the Baltic region and Norway.
Nordecon (NCN1T.TL)	32.375M	Nordecon operates in the construction sector. The main activities is making residential and non-residential buildings, roads and utility projects. Civil engineering is the main focus of the company. It provides maintenance project too.
PRfoods (PRF1T.TL)	38.682M	PRFoods is engaged in the business of food processing and distribution. The company is producing fish products in the UK, Finland, Estonia. However the farming is done in Finland and Sweden only. The majority of the revenues of the company comes from Finland.
Pro Kapital Grupp (PKG1T.TL)	56.687M	The company operates in the real estate industry. It mainly provides buying, selling and renting of its own and legally acquired real estate. It mainly focuses in the development, management and sales of modern real estate in the Baltic region and Germany.
Silvano Fashion Group (SFG1T.TL)	36M	The company operates in the retail sector. Mainly designing, manufacturing and marketing of women's lingerie. The brands made by the company include Milavitsa, Alisee, Aveline, Lauma Lingerie, Laumelle, and Hidalgo. The main source of revenue of the company is the lingerie wholesale channel.
Tallink Grupp (TAL1T.TL)	669.882M	The company operates in the marine shipping industry providing transport from and to Estonia, Latvia, Finland, and Sweden. It also owns and operates four hotel in Tallinn and one in Riga. The company has fourteen different vessels that ensures its operations.
Tallinn Kaubamaja Grupp (TKM1T.TL)	40.729M	Tallinn Kaubamaja operates in the department stores industry. It is engaged in retail trade and provisions of related services. The activity varies from department stores, supermarkets and footwear to real estate and car trade. The real estate part of the company deals with the properties owned by the company and their management.
Tallinna Vesi (TVEAT.TL)	20M	The company operates in the utility industry more specifically water. The firm provides water in Harju county in Estonia. Additionally, it provides construction services related to water pipelines and related issues. The revenue of the company comes mainly from water supply services.

The companies that were not included in the analysis are Coop Bank which started to be traded on the secondary market the 9th of December 2019 and Tallinn Sadam that started to be traded the 11th of June 2018. Those companies where taken out from the analysis to make it easier to implement. All the information presented above has been extracted from the Nasdaq website regulating the Baltic market [14].

7.1. Data Collection

The data used in the modeling of the portfolio is collected using two main libraries in R. The first one is “gdata” library containing the function getSymbols that returns the data in the form needed. The data about the stock closing prices has been extracted from Yahoo server which returns adjusted closing prices. By adjusted, the author means dividend adjustments of the stock prices. The second source of data was used since Yahoo does not have the historical data for the OMXT index. The package is called “Quandl” and requires a key which was generated after the subscription to the their website. A plot of the closing prices, after some NA “Not Available” entries have been treated by replacing them with the previous values in the data, is as follow

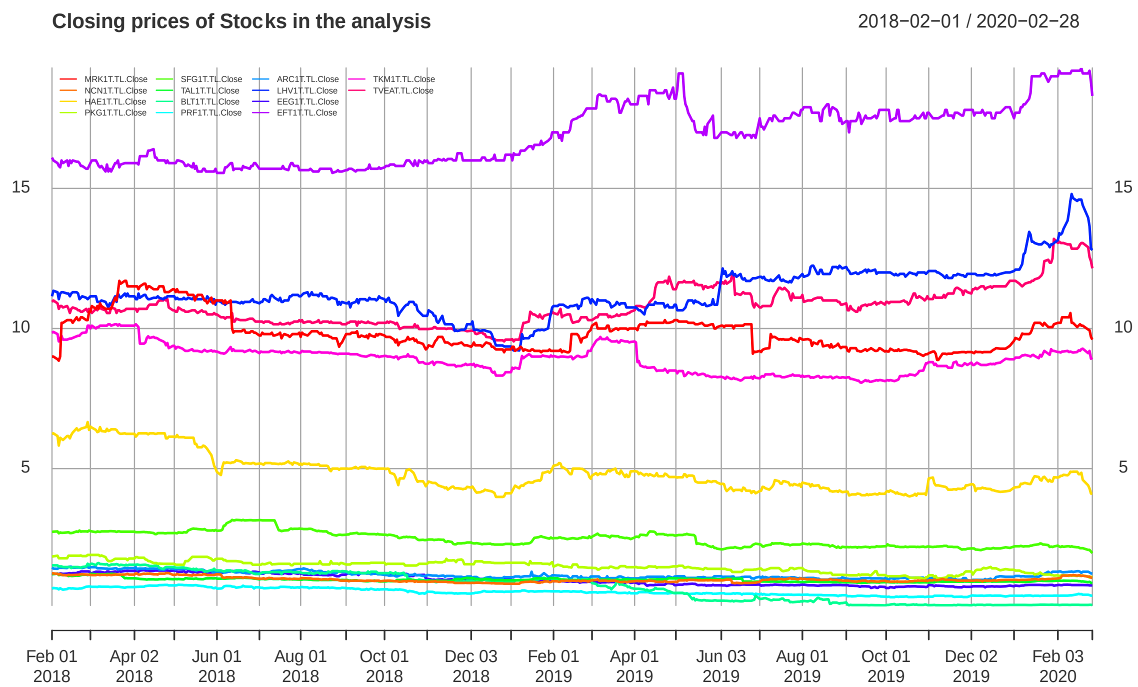

Figure 6: Closing price of all the stocks

The code used to generated the data and the plot could be found in **Appendix C**.

7.2. Time Series Analysis

The following analysis was based on the times series class offered by the university. Small discussion about the models and test used would be introduced.

7.2.1. Theory Used in the Analysis

A time series is a vector of data points measured with a specific time interval in between. Mathematically one can define the observation a discrete series as $z_t$such that $t = 1,2,3$, ... The main way to understand the time series is to start by plotting it to investigate its properties and different changes defining its path. In this paper the time series is based on daily stock market prices which is defined over the trading days between the 1st February 2018 till 28th February 2020. The most used and popular stochastic time series is Autoregressive Integrated Moving Average (ARIMA) model. The main assumption about the model is that it is linear and follows a particular distribution. Of course this is not usually the case in a time series, but advanced methods would require complicated implementations. Thus the ARIMA model straight forward application was chosen for the sake of its simplicity.

7.2.1.1. Component of the Time Series

The time series is usually affected by four components. The first one is the trend which describes the general change of the series in the long term. Meaning that a change that does not seem to be periodic is considered to be the trend of the series. The second component of the series is seasonal component describing a change happening on fixed and known time intervals. The third component is a cyclic component which describes a change in the time series happening over a cycle usually over two years. The last component of the time series is the irregular component which is describing the randomness of the time series after all the other components have been removed.

There are two decompositions of the times series that arises from the later description. One which is additive that sums up all those components thus assumes the independence of those components. The other type which is a multiplicative model assumes that the components are not usually independent and can affect each other's.

7.2.1.2. Stationarity of the process

The time series represent a vector of element time dependent of each other's. Assume that the time series $z_t$such that $t \in \mathbb{Z}$ is the realization of a random process $Z_t$such that $t \in \mathbb{Z}$. Thus a process is called 2nd weakly stationary if for every integer $m \in \mathbb{N}$, $q \in \mathbb{Z}$ and for each $t \in \mathbb{Z}$

$$E(Z_t) = \mu$$

$$cov(Z_t, Z_{t+p}) = \gamma(p), p \in \mathbb{N}$$

$$\gamma(0) = \sigma^2.$$

Here μ and σ^2 are constants.

In this coding part the functions acf and pacf would be used to check if the time series is stationary. The decisions is based that there should be no visible trend in the plot of those functions. The augmented Dickey–Fuller test (ADF) would be used in addition to check for the stationarity of the data. The null hypothesis if the test is that the data is not stationary. Meaning if the p-value<0.05 the data is stationary.

The second main assumption check in this section is that if the time series is generated from a stochastic process with identical independent random variables. For that we check if the residuals of the model are normally distributed. The function checkresiduals in the program would provide a distribution plot with Ljung-Box test.

Let's define the estimated autocorrelation of the series as $r_p = cor(Z_t, Z_p)$ with $p = 1,2, \dots, m$ with m fixed but arbitrary. For N observations The Ljung-Box test is a Q test defined as

$$Q_{LB} = N(N+2) \sum_{p=1}^{m} \frac{r_p^2}{N-P}$$

The distribution is approximated by a chi-squared distribution with m degrees of freedom.

The decision mainly would be made based on the test. The null hypothesis of the test is that the data is independently distributed. Thus is the p-value > 0.05 (confident interval of 95%) one can say the residual are not normally distributed thus the model exhibits lack of fit.

7.2.1.3. ARIMA Models

The time series in practice are sometimes not stationary. Thus one can transform the time series by introducing the difference between the observations of the time series that making it stationary. Then apply normal techniques of the difference to estimate it's time series. Using those differences and one of the boundaries one can retrace the whole time series in the required form. Assume $Z_t$to be the realization of a time series in time $t = 1,2, \dots, m$ an ARIMA(p,d,q) has the form:

$$W_t = \sum_{i=1}^{p} \phi_i W_{t-i} + A_t + \sum_{i=1}^{q} \theta_i A_{t-i}$$

Such that $W_t = (1 - B)^d Z_t$ and the process $A_t$is an uncorrelated process with zero mean and variance σ^2. The function B is a backtracking function that defines the previous observation in the series. The ARIMA model has two other main component the moving average one and the autocorrelated one which can be seen in the equation.

7.2.1.4. Comparison of the Models

The comparison of ARIMA models would be mainly done using the residuals of the models. For models with the same number of parameters the Akaike Information Criterion (AIC) would be used. The AIC is defined as

$$AIC = \log \frac{SEE}{n} + \frac{n + 2k}{n},$$

where SSE is the residual sum of squared with the model with k coefficients and n observations. In the case of different number of parameters the Bayesian Information Criterion (BIC) would be used for comparing the models since it penalizes more the number of parameters. The BIC is defined as

$$BIC = \log \frac{SSE}{n} + \frac{k \log n}{n}$$

The parameters are defined the same way for the AIC.

The decision criteria is the lower the AIC or BIC the better.

7.2.2. Observations from the Time Series Analysis

The full time series analysis could be found in Appendix D with stationarity, residuals check and forecast checks. The forecast was made forty trading days after the 28th February 2020 and compared with actual values that happened.

Table 2: Time series analysis

Names of the company (Index)	Model fitted	Forecast Goodness	Additional Comments
Acro Vara	ARIMA(0,1,3)	NOT good	Huge COVID-19 influence. The company is suffering since mid 2018
Baltika	ARIMA(1,1,0)	Really good	The company's price have dropped drastically over the past two years. The company is going through restructuring.
EfTEN Real Estate Fund III AS	ARIMA(3,1,0)	Not good	Real estate company suffering drastically in the COVID-19 crisis. The company seems to be generating revenue before the crisis hit.
Ekspress Group	ARIMA(0,1,2)	Not good	A big portion of the company's revenues comes from selling printable goods thus reducing the income of the company. The company seems to be losing money over the past two years not meeting the investors expectations.
Harjo Elekter	ARIMA(6,1,0)	Good	The company's sales over the last year have been volatile thus affecting the stock market over each quarter results.
LHV Group	ARIMA(1,1,0)	Average	The bank seems to generate more revenue over the past years attracting new investors. The crisis hit the stock price making nearly hit the lowest in the past two years.
Merko Ehitus	ARIMA(1,1,0)	Not good	The company is in the construction sector. The crisis hit the company's price really bad.
Nordecon	ARIMA(0,1,1)	Not good	The company is in the construction sector. The crisis drove the price to the lowest value in two years.

PRfoods	ARIMA(4,1,0)	Good	The company seems not to be affected much with the crisis. The overall performance of the company for the past two years have decreased.
Pro Kapital Grupp	ARIMA(0,1,2)	Not good	The company operate in the real estate industry thus being penalized severely by the crisis.
Silvano Fashion Group	ARIMA(0,1,1)	Average	The company has been hit by the crisis. The overall performance of the company for the past year seems to be constant.
Tallink Grupp	ARIMA(5,1,0)	Not good	The company suffered from the travel restriction set by countries. Before the crisis the company had a stable revenue for the past year and half meeting the investors’ expectations.
Tallinn Kaubamaja Grupp	ARIMA(1,1,0)	Not good	Since the stores had to close after the Covid-19 crisis, obviously the stock price needed to react to the situation. It drove down the price below the lowest for last two years. In general the company’s stock price seems to be affected greatly by investors speculation about the quarterly returns.
Tallinna Vesi	ARIMA(0,1,0)	Good	The company seems not to be greatly affected by the crisis since it belongs to the utilities sector. Overall the company seems to be growing in the past year.

Overall the Estonian market seems to be really losing its investors trust in the past two years. The pandemic seems to be dragging the prices down especially in the real estate and construction sectors. The next step is to apply the previously proven optimal portfolio techniques on the fourteen companies.

7.3. Trading Platform

The trading strategy made is fairly a basic strategy that assumes a highly liquid market allowing the trader theoretically to liquidate the portfolio in end of each trading day. This is fairly an unrealistic assumption looking at the volume traded each day in the Tallinn stock market, but could be addressed later as a separate issue in another paper. The first thing to consider is which part of the day one wants to update the portfolio weights. The author chose the end of the day as time to update the number of shares acquired.

The function updateIC created updates the working capital based on the number of shares acquired from the last optimization period. It takes as input the past capital estimation, last periods closing prices, the current period closing prices and the weights of the optimal portfolio of the last period. The number of securities in the portfolio before the end of the current trading period is simply the past capital estimation multiplied by the weights and divided by the past prices of the securities. Thus the current capital estimation would be the dote product of the vector of number of securities and the vector of current market prices.

The next thing that was part of the analysis is to create the models and get the results. The author chose to train the first model on 75% of the data and test it on 25% of the data. Thus the first optimization using any of the algorithms would start 16th August 2019 assuming that the observation obtained on that date is unknown. Then running the algorithm on the previous data getting the new weights then updating the initial capital and adding that observation to the train data and running the program again to get a vector describing the capital observations in the end of each trading day using the same algorithm till the 28th February 2020.

The function Capitalvector was created with the aim of applying the following steps and returning a list composed of the weights and capital in each trading day. It takes as input the train data values, the test data values, the algorithm, and the initial capital in the end period of the train data. Then perform the algorithm on the train data while adding already computed variables from the test data to model the next iteration.

The functions created could be found in the Appendix E.

7.4. PortfolioAnalytics Library in R

The library was used in this thesis to model the previously stated methods using numerical approaches and already existing functions. The aim is to compare those methods and came to the conclusion which is computationally extensive and of course which method did the best in the

Tallinn stock market within the time frame taken only daily data. The package offer many options to deal with complex and objective sets. The portfolio is created first using the function portfolio.spec that takes the name of the assets as an argument. The next step is to set constraints on the portfolio using the function add.constraint. Many types of constraints could be added that main ones would be described as follows

- “weight” which is a constraint on the sum of weights.
- “box” which is a constrain on the individual weights.
- “return” that describes a specific target mean return.
- “factor_exposure” that describes a specific risk factor exposure.
- “leverage_exposure” that specify the maximum leverage of the portfolio.’

The next step is setting up the portfolio is to add an objective using add.objective function which specifies the type of portfolio the user wants to optimize. Some of the objective types could be found as:

- “return” which optimize to get a specific return
- “risk” which specifies a type of risk that we are trying to minimize or to set to a specific level
- “quadratic utility” that use the quadratic utility function maximized

The last step is to solve for the optimal solution, and for that the function optimize.portfolio is used. It offers many types of methods that could be used for the optimization of the portfolio. The main ones are as follow:

- “random” which creates random portfolios and return the one with the corresponding constrains.
- “DEoptim” perform an evolutionary optimization using the differential evolution algorithm.

For more information about the library one should consult the publication about the library found in the following source [15].

The set of argument passed to the optimization problem used in this paper is to optimize the return of the portfolio with a target of 8% per year. The constraints on the problem is a box constraint with a maximum value of 0%, not allowing shortselling, and max value of 40% per each asset. Of course the sum of weight should be equal to 1.

7.5. PortfolioOptim Library in R

The library offers the solution to modeling portfolios based a risk measure. The one of interest for the author is the CVaR measure that was implemented in the package. The problem solved is as follows find the minimum of

$$F(w^T r),$$

subject to

$$w^T E(r) \geq r_p, Lowerbound \leq w \leq \ uperbound, Aw \leq B$$

The function F represents the risk measure used in this case it would be the CVaR measure of risk, the lower bound and upper bound would set constraints on the weights for diversification and short selling purposes, and the last constraint is to add other restriction if needed in the form of a matrix [2].

The method requires a risk threshold and a distribution of the portfolio's return. In the application a uniform distribution was used. In other word the probability of each of the returns of the training data are set to be equal. A target portfolio return of 8% is set and no short selling was allowed.

7.6. Other methods

Some other methods have been implemented with other solvers. The first one is the Markowitz portfolio problem munched in page 45 using a quadratic solver. The solver used is solve.qp is made for solving a quadratic routine using the dual method of Goldfarb and Idnani as specified in the following source [17].

The last method implemented is the one discussed in page 53. The function lm creates a linear model and the second coefficient of the model is taken as a beta. The model later is solved using the simplex method since it is a linear programing problem. Short selling was not allowed in the model and a beta coefficient of 1.5 was set to be used as 0.5 more risker than the index. A diversification constraints was added not allowing to invest more than 40% of the portfolio in one security.

7.7. Results

A function was created for each method returning the weights after running the algorithm. The analysis could be found in Appendix E along with the functions in R. The change of weights of each method was then plotted around the test data's dates to visualize how the portfolio changed from one day to the other. Later the portfolio's values were plotted along the same period to visualize the daily impact of the algorithm. The results of the analysis could be found separately for each method in the next sections.

7.7.1. Markowitz Portfolio Estimation

The quadratic solution problem corresponds to the Markowitz portfolio. Since the method requires finding one solution which could be hard to find without allowing some short-selling. Thus the author decided to allow short-selling to a degree.

The weights pf the portfolio could be found in the following figure.

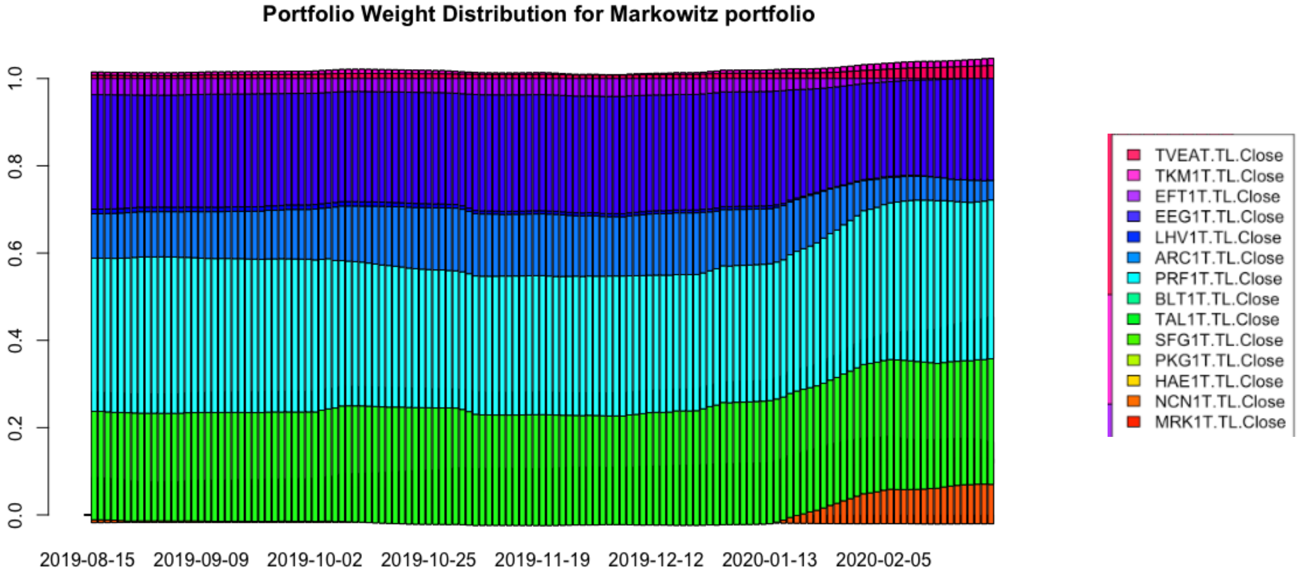

Figure 7: Weight Change Over Time for Markowitz Portfolio

The weights seems to be mainly made from four stocks. The companies are Tallink Grupp, Acro Vara, PRfoods, and Express Group. Seems that the algorithm takes into consideration the correlation of the stocks. Thus it took uncorrelated stocks as the main ones for the portfolio. Anyone could see that those companies operates in different industries thus the change in one of those could would not affect the others.

The results about the change of capital would be discussed in the section 7.8.

7.7.2. The VaR Method

As shared previously the DEoptim function would be used to find the global minimum of the VaR. The algorithm work by generating a random set of weights and then choosing the best one then repeating the same process while going through iterations to find the best set of weights. The randomness in the first portfolio could lead for the algorithm to be stuck in a local minimum since the beginning. Thus could make the portfolio a bit random. The results of the weights could be found in the following graph:

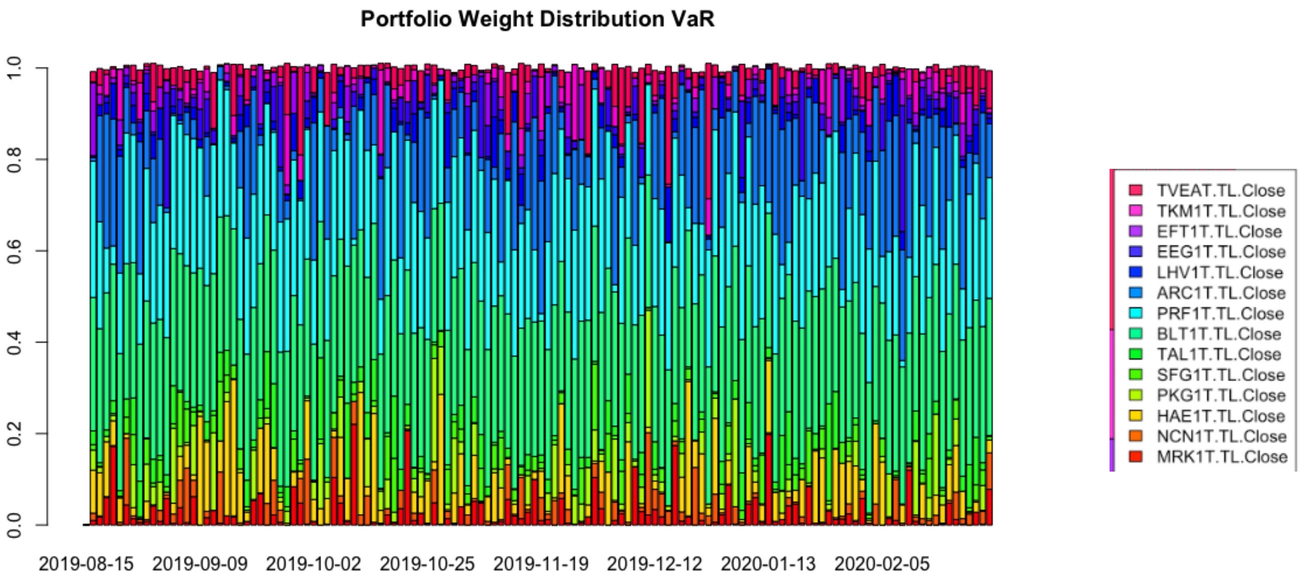

Figure 8: Change of Weights for the VaR Method using DEoptim.

The algorithm chose to invest mainly in PRfoods, Acro Vara, and Baltika. The change of weight form one day to the other is really huge thus requiring great liquidity from the market. Is seems the algorithm chose to invest in Baltika for the latest stability of its stock price. A default risk measure should be add to the algorithm to make sure that it takes into consideration the fact that the company is having troubles. This and the randomness of the portfolios change from one day to the other resulted in a big variance in the total change of the portfolio. This would be discussed in the last section of this chapter.

7.7.3. The CVaR Method

The algorithm is the result of a 95% confidence interval with a generated stepwise uniform distribution that was included as the last column of the data before using the DBportfolio_optim function. the distribution give more probabilities to the latest observations than previous ones. The result of the optimization is the daily change of weights could be found in the following figure:

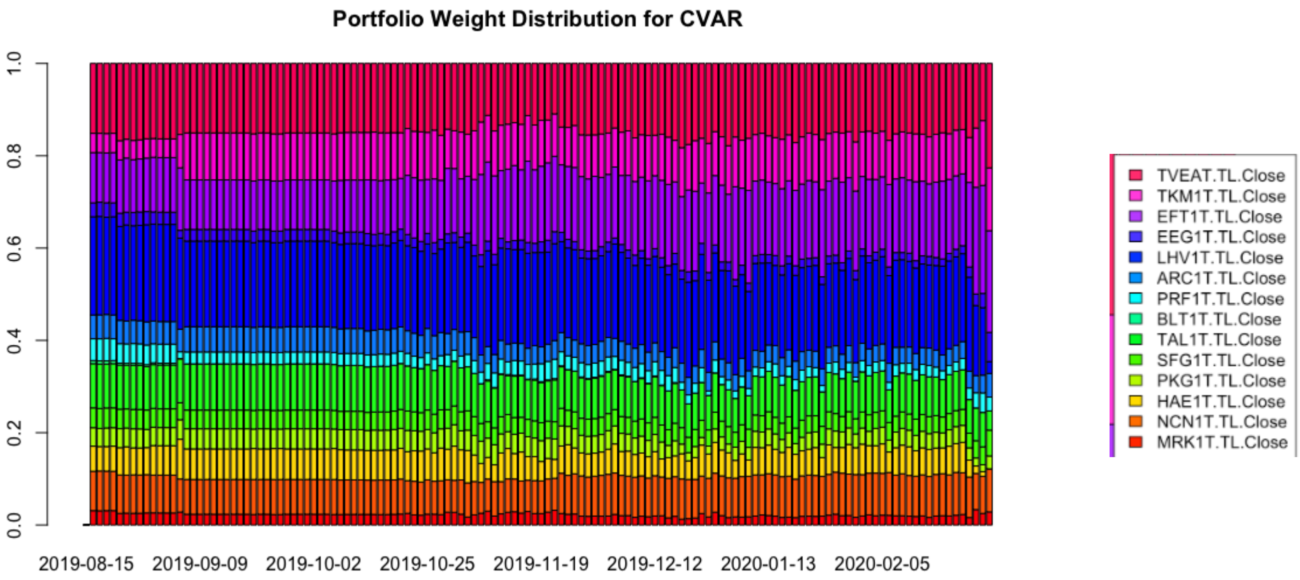

Figure 9: Change of Weights for the CVaR Method using DB_portfolio_optim.

The algorithm seems to take into consideration new observations by slowly changing the weights of the portfolio from one day to another. This method would not require huge liquidity of the market especially which good for the Estonian market. The algorithm responded quickly to the crisis by changing the weights in the end of February 2020. The change of overall portfolio could be found in the last section.

7.7.4. The CAPM Linear Problem

The algorithm strictly invests in stocks with positive return over the indicated period while including a basic risk measure following the overall performance of the market. A 40% diversification condition was added and a no short selling was allowed. The following figure shows the change of the portfolio.

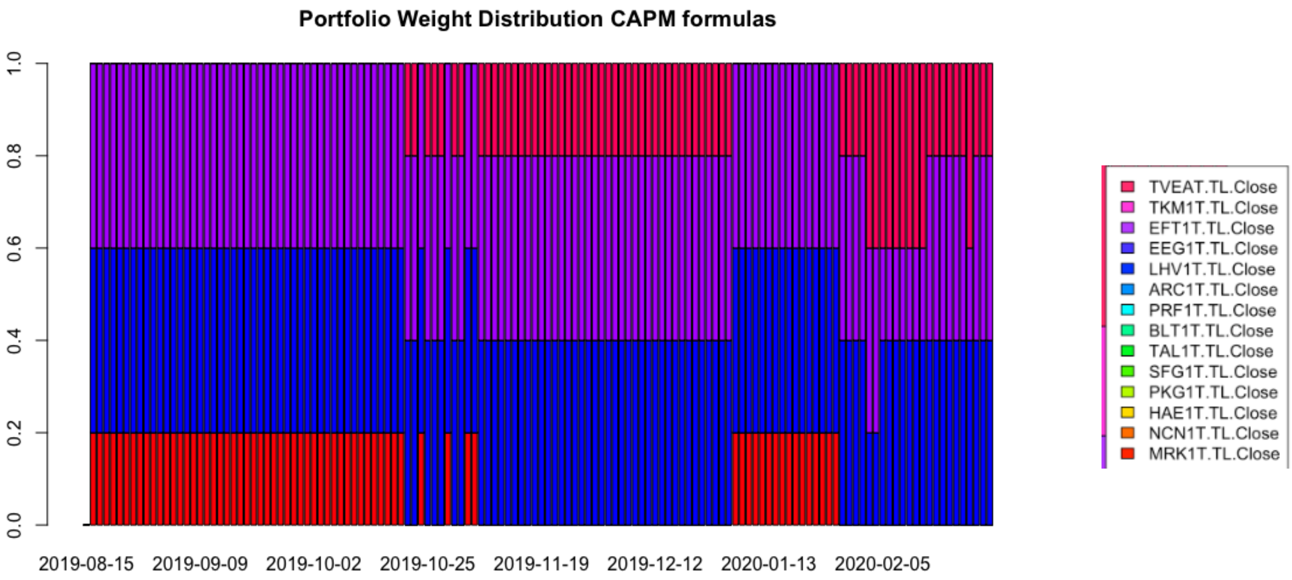

Figure 10: Change of weights for linear problem including the Beta of the companies

The result seems to be strictly not changing and greatly bounded by the diversification condition. Meaning that the optimal solution exists in the end points of the problem. The algorithm seems to be investing in four main stocks, LHV Group, Merko Ehitus, Eften Real Estate, and Tallinn Vesi. The algorithm requires great liquidity in the days in which the portfolio changes thus would be hard to apply in the Estonian market.

7.8. Comparison of the Results

In the following section, the change of the portfolio would in each method would be evaluated then compared in the end. The reference strategy to which each algorithm would be compared is a buy and hold strategy. Later all the results of the companies would be shown and discussed.

7.8.1. Buy and hold strategy

The buy and hold strategy would be performed by buying stocks in the beginning of the testing period and not sell them till the end of the trading period. The buying part would be done by using the algorithm once in the beginning, and the selling part would use the last entre in the test data. The return of the portfolio would be defined as

$$r_{buyandhold} = \frac{C_e}{C_i} - 1$$

- C_e is the new capital in the end of the period after selling the securities.
- C_i is the initial capital which set to 10000 in the coding.

7.8.2. Comparison of Individual Algorithm's Performance

The comparison would be done to the buy and hold strategy. Results and calculations would be found in Appendix E.

Definition of the variables in the table:

- $R_{buyandhold}$ is the return of the method using the buy and hold strategy.
- R is the last values of the portfolio after updating the weights daily over the initial capital.
- $P = \frac{R - R_{buyandhold}}{R}$ is a ratio describing how much better did the algorithm do with respect to the buy and hold strategy.

Table 3: Returns of the Methods.

Method	R	$R_{buyandhold}$	P	Comments
Markowitz	5.70%	3.82%	33.07%	The Markowitz portfolio seems to be doing good with 5.7% on 6 months period. It seems that updating the portfolio works well for the method.
VaR	8.64%	-20.34%	335%	The VaR portfolio seems to be doing really good in the end but a buy and hold strategy would have resulted in a great loss. Certainly the update of the portfolio is crucial for the method.
CVaR	4.64%	3.82%	36.7%	The CVaR method seems to be doing good with both methods. However making a bit more money with the daily updating version. Keep in mind that this is the most close

				version to the Estonian market since updating the portfolio does not require much liquidity.
CAPM	3.3%	4.5%	-20.7%	The method seems to work less better with updating the portfolio. However it is understandable since the solution are in the endpoints of the algorithm as stated previously.

Over all the methods seems to be generating return on the Estonian market in the end of the period. However, one should have a look at the change of the portfolios values around the six months period.

7.8.3. Comparison of the Algorithms

All the methods used in the analysis seems to be generating return in the end of the algorithm let's now have a look at the daily change of those algorithms.

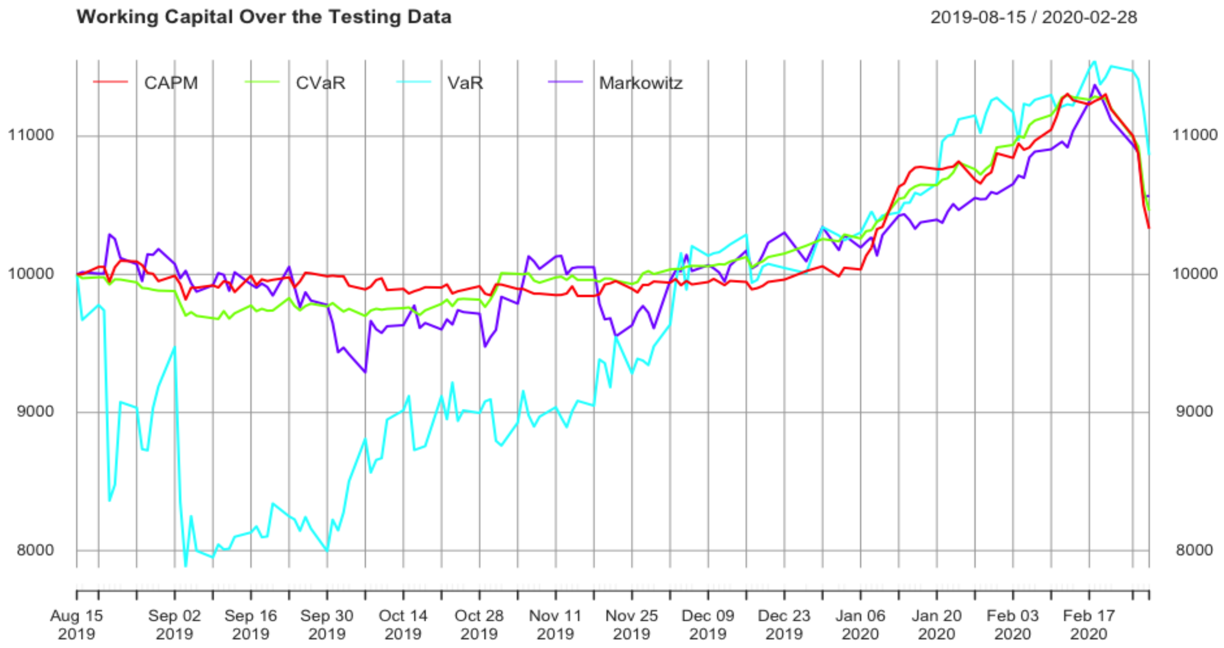

Figure 11: Comparison of the Methods

The model generated from the Markowitz algorithm seems to be affected by the volatility of the market and seems to be volatile for the first four months then later it started generating return in the end of November 2019. The method seems not to be stable. The method also allows short-selling which not the case in the Estonian market.

The model generated from the VaR seems to be losing so much money in the beginning and then starts to gain back the money slowly. This is really a risky method but it is understandable due to the nature of the algorithm which takes the risk estimation based on one threshold. This is not conservative enough in the Estonian market. The model is generated from a first random generation which could be somehow misleading thus be stuck in a local minimum. The method needs some refining and a choice of a better optimizer.

The CVaR method seems to be stable and conservative not changing much. Which is understandable from the algorithm since it is based on the 95 percentile after the VaR threshold is crossed. The value of the portfolio seems to be steadily increasing and not volatile at all. The method also does not allow shot-selling which is also the reality in the Estonian stock market.

The linear programing problem seems to be following the trend of the index however it is strictly affected by some stocks and not diversified enough. The Estimation seems to be falling in the endpoint of the problem which means that other constrains have not been met correctly. The problem need some development and making it a quadratic problem since that's the nature of risk.

8. Conclusion

This thesis gave an overview about how to solve the portfolio problem using mathematics and optimization techniques. As was shown solving the problem could be challenging using only mathematics since it could result in a second degree differential equation that needs a lot of development. However, the numerical methods showed to be efficient and working well with the great amount of data.

The application part showed that Tallinn stock exchange is still in development and suffering during the COVID-19 pandemic. Especially the real estate and construction sectors. The second main thing shown is that the VaR method is volatile although it is having a good return in the end of the period. The CVaR method proved to be the most stable insuring a constant return without much volatility. The Markowitz portfolio generate return but no solution could be found without short-selling which is not allowed in the Estonian market.

Future work could be done by including a liquidity analysis about the Estonian market while taking inconsideration that fractions of stocks could not be sold. Also a direct application of the methods could be programed without using already existing functions and libraries. Note that the portfolio frontier theory was not covered since a risk free investment would require a significant market analysis. The seasonal component of the ARIMA model could also be analyzed for each company.

Kokkuvõte

See magistritöö andis ülevaate, kuidas lahendada portfelli probleemi kasutades matemaatilise optimeerimise tehnikaid. Nagu eelnevalt näidatud, võib probleemi lahendamine ainult matemaatikat kasutades olla keeruline, sest võib nõuda teise astme diferentsiaalvõrrandi lahendamist. Siiski, numbrilised meetodid olid efektiivsed ning töötasid hästi suurte andmemahtude peal.

Praktiline osa näitas, et Tallinna aktsiaturg on veel arenemisjärgus ning ei anna häid tulemusi COVID-19 pandeemia ajal, eriti kinnisvara- ja ehitussektoris. Teine peamine tulemus näitas, et VaR meetod on volatiilne, kuigi annab lõpuks hea rentaabluse. Tinglik VaR meetod oli kõige stabiilsem ning andis konstantse rentaabluse ilma suure volatiilsuseta. Markowitzi portfell genereeris rentaabluse, kuid ei andnud lahendust ilma lühikeseks müümiseta, mis ei ole Eesti turul lubatud.

Edaspidises töös saaks lisada Eesti turu likviidsusanalüüsi ning arvestada, et aktsiad ei saa müüa murdosades. Lisaks võiks programmeerida meetodite otsese kasutuse ilma juba olemasolevaid funktsioone ja pakette kasutamata. Selles töös ei kaetud portfelli piiri teooriat, sest riskivaba investeerimise uurimine nõuaks sügavamat turuanalüüsi. Lisaks võiks uurida ARIMA mudeli sesoonset komponenti iga firma jaoks.

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Appendix A: Implementation of the Utility and Wealth Function

First Application

Omar Setihe

5/7/2020

from the result we can see that:

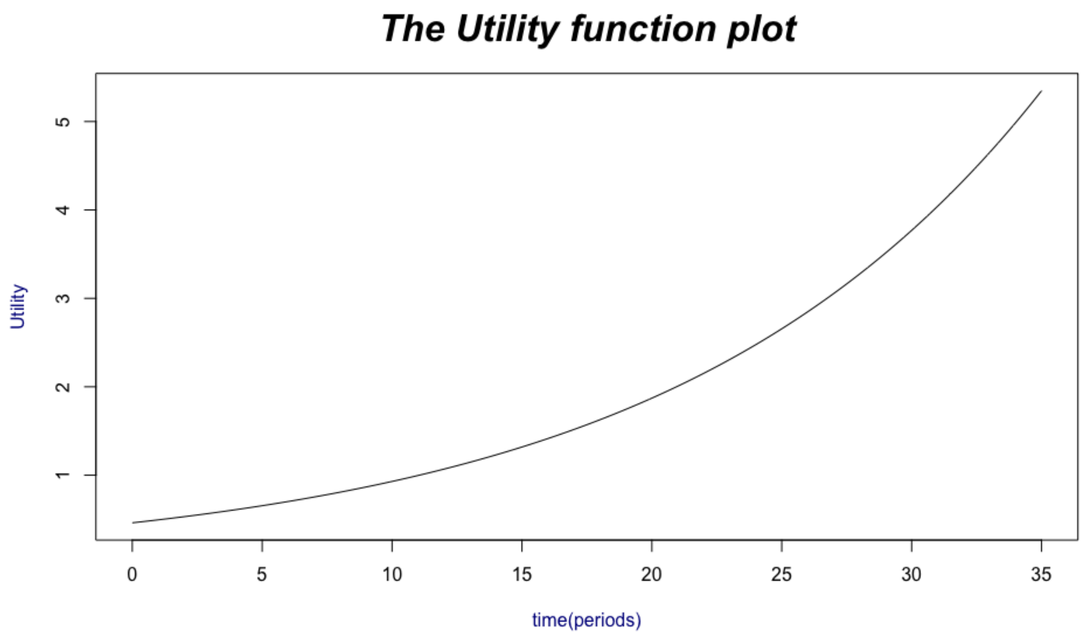
$$C(t) = \frac{r_1 S_0 e^{(r_2 - r_1)t}}{1 - e^{-r_1 T}}$$

Assume

$$S(0) = 100, r_2 = 0.1, r_1 = 0.03, T = 35$$

and let's have a look at the function.

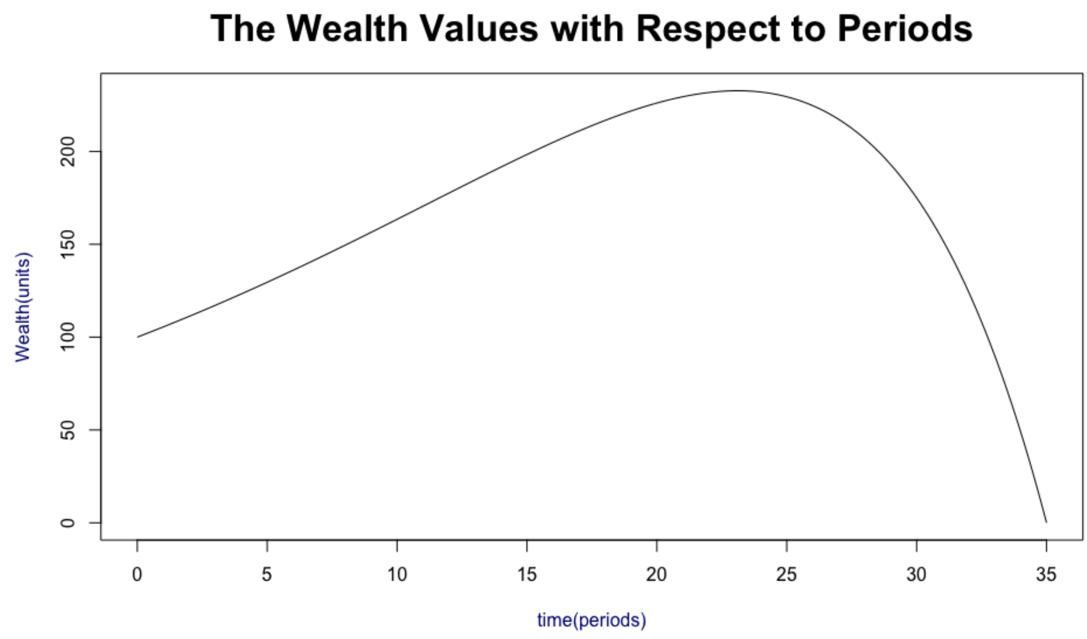
```
T=35
S0=100
r_1=0.03
r_2=0.1

C_function= function(t) {
  return(r_1*r_2*S0*exp((r_2-r_1)*t)/(1-exp(-r_1*T)))
}
t=seq(0,T,0.1)
Ct <- C_function(t)

plot(t,Ct,type="l", main = "", xlab="", ylab="")
title(main = "The Utility function plot",
      xlab = "time(periods)", ylab = "Utility",
      cex.main = 2,   font.main= 4,
      col.lab ="darkblue"
      )
```



Now let's have a look at our wealth S(t)

$$S(t) = S_0 e^{r_2 t}(1 - \frac{1 - e^{-r_1 t}}{1 - e^{-r_1 T}})$$

```
S_function = function(t) {
  return(S0* exp(r_2*t) * (1- (1-exp(-r_1*t)) / (1- exp(-r_1*T) )) )
}
St<-S_function(t)

plot(t,St,type="l", main = "", xlab="", ylab="")
title(main = "The Wealth Values with Respect to Periods",
      xlab = "time(periods)", ylab = "Wealth(units)",
      cex.main = 2,   font.main= 2,
      col.lab ="darkblue"
      )
```



NB: Please note that this is an R-markdown output that can be obtained from the .Rmd attached to this document.

Appendix B: Implementation of the Brachistochrone Problem

The Brachistochrone Problem

Omar Setihe

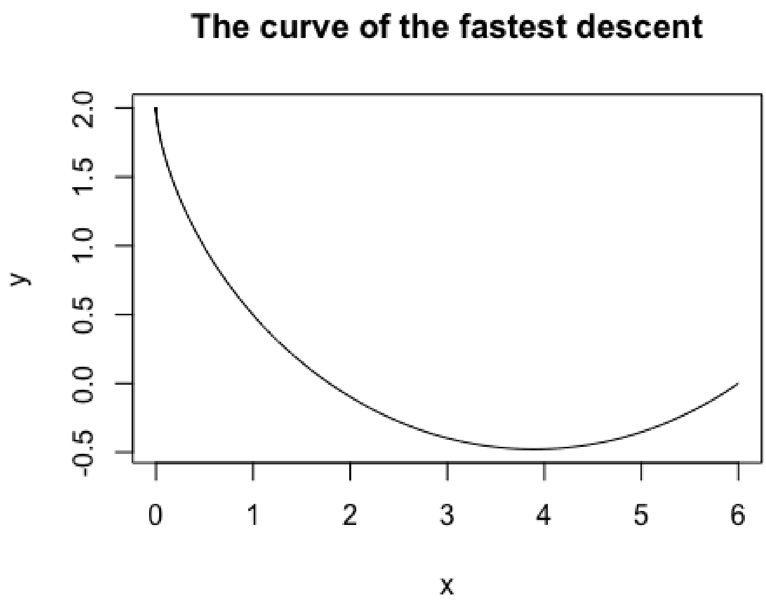
4/20/2020

Solving The Brachistochrone Problem.

A(0, y_1) and B(x_2, 0)

y_1=2 x_2=5

```
y1=2
x2=6
boundry = function(theta) {y1/x2 - (1-cos(theta))/(theta-sin(theta))}

library(pracma)
theta2 <- newtonRaphson(boundry, pi/2)
K1 = 2*y1 / (1 - cos(theta2$root))

theta = seq(0, theta2$root, (theta2$root)/1000)
x = K1 * (theta - sin(theta))/2
y = -K1 * (1 - cos(theta))/2 +y1

plot(x,y,type="l",xlab="x",
     ylab="y",
     main="The curve of the fastest descent")
```



NB: Please note that this is an R-markdown output that can be obtained from the .Rmd attached to this document.

Appendix C: Data Collection Process

Extraction of the Stock Prices

Omar Setihe

8/01/2020

The following Document would show the Data collection process.

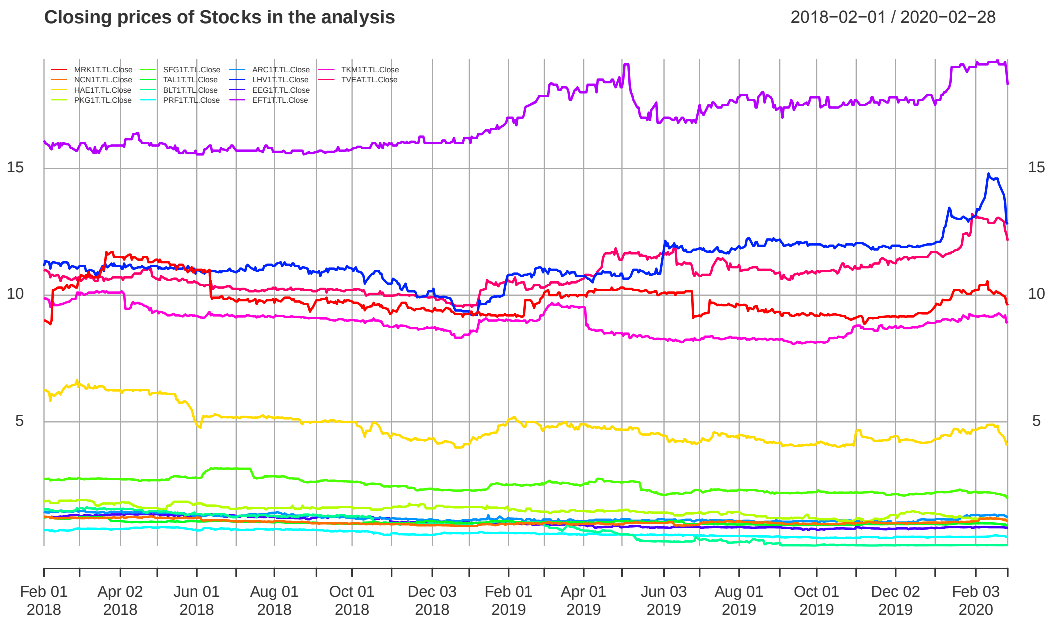
Collecting the data.

```
library("quantmod")
library(Quandl)
library(gdata)

rm(list=ls()) # clears the enviroment from the data
tickers <- c("ARC1T.TL","BLT1T.TL","EFT1T.TL","EEG1T.TL","HAE1T.TL","LHV1T.TL
","MRK1T.TL","NCN1T.TL","PRF1T.TL","PKG1T.TL","SFG1T.TL","TAL1T.TL","TKM1T.TL
","TVEAT.TL") # indexes of stocks in Yahoo
data_env <- new.env() # creates an environment for not having all the stocks
in the enviroments
getSymbols(tickers,from = "2018-02-01",to = "2020-02-29",env= data_env) # not
the getsymbols function takes the last time excluded form the data that's why
it's 29 in this line of code.

##  [1] "ARC1T.TL" "BLT1T.TL" "EFT1T.TL" "EEG1T.TL" "HAE1T.TL" "LHV1T.TL"
##  [7] "MRK1T.TL" "NCN1T.TL" "PRF1T.TL" "PKG1T.TL" "SFG1T.TL" "TAL1T.TL"
## [13] "TKM1T.TL" "TVEAT.TL"

close_data <- do.call(merge, eapply(data_env, Cl)) # get the close price form
the data and merge them
#Cl is the close data extraction fucntion
#do.cal allow to merge the data in an enviroment.
index<- Quandl("NASDAQOMX/OMXTGI", api_key="Zf_29iNsDg7Qm3r5zUBN",start_date=
"2018-02-01" , end_date= "2020-02-28",type= "xts") # Extracting the index fro
m the Quandal server the function includes the ending date so no need to put
the 29 there.
#index[,1]
close_data_all <- merge(close_data,index[,1])
close_data_all <- na.locf(close_data_all)

Untact<-data.matrix(as.data.frame(close_data_all)) # get the numeric values f
rom the Ts

tickers <- colnames(Untact)
```

Plotting the data using the plot.xts() function:

```
Stock_Data<- close_data_all[,-ncol(close_data_all)]
Rainbow_colours = rainbow(ncol(as.zoo(Stock_Data)))
invisible(plot.xts(Stock_Data,main= "Closing prices of Stocks in the analysis
", ylab = "Price", col=Rainbow_colours ))
addLegend("topleft",
```

```
        legend.names=colnames(Stock_Data),
        lty=rep(1,14),
        cex=0.5,
        ncol = 4)
```



NB: Please note that this is an R-markdown output that can be obtained from the .Rmd attached to this document.

Appendix D: Time Series Analysis

Time Series Analysis

Omar Setihe

8/10/2020

In the following Document a time series analysis made for fourteen Stocks in the Tallinn stock exchange market.

Importing the data and making it in the required form as specified in the data collection part of the thesis.

```
rm(list=ls())
library("quantmod")
library(Quandl)
library(gdata)
tickers <- c("ARC1T.TL","BLT1T.TL","EFT1T.TL","EEG1T.TL","HAE1T.TL","LHV1T.TL
","MRK1T.TL","NCN1T.TL","PRF1T.TL","PKG1T.TL","SFG1T.TL","TAL1T.TL","TKM1T.TL
","TVEAT.TL") # indexes of stocks in Yahoo
data_env <- new.env() # creates an environment for not having all the stocks
in the environments
getSymbols(tickers,from = "2018-02-01",to = "2020-02-29",env= data_env) # not
the  function takes the last time excluded form the data that's why it's 29 i
n this line of code.

##  [1] "ARC1T.TL" "BLT1T.TL" "EFT1T.TL" "EEG1T.TL" "HAE1T.TL" "LHV1T.TL"
##  [7] "MRK1T.TL" "NCN1T.TL" "PRF1T.TL" "PKG1T.TL" "SFG1T.TL" "TAL1T.TL"
## [13] "TKM1T.TL" "TVEAT.TL"

close_data_all <- do.call(merge, eapply(data_env, Cl)) # get the close price
form the data and merge them
close_data_all <- na.locf(close_data_all)
tickers_all <- colnames(close_data_all)

data_env <- new.env()
getSymbols(tickers,from = "2020-02-29",to = "2020-04-30",env= data_env)

##  [1] "ARC1T.TL" "BLT1T.TL" "EFT1T.TL" "EEG1T.TL" "HAE1T.TL" "LHV1T.TL"
##  [7] "MRK1T.TL" "NCN1T.TL" "PRF1T.TL" "PKG1T.TL" "SFG1T.TL" "TAL1T.TL"
## [13] "TKM1T.TL" "TVEAT.TL"

close_data_all_future <- do.call(merge, eapply(data_env, Cl)) # get the close
price form the data and merge them
close_data_all_future <- na.locf(close_data_all_future)
tickers_future <- colnames(close_data_all_future)
```

The following function was used in the Times Series Analysis class provided by the University of Tartu. Creating the functions is as follow:

```
library(forecast)

## Warning: package 'forecast' was built under R version 4.0.2
```

```r
library(tseries)

## Warning: package 'tseries' was built under R version 4.0.2

library(quantmod)
TSgraph=function(series,nlag=30){
  layout(1:3)
  plot(series) # plots the data
  acf(series,nlag) # auto-correlation coefficient function
  pacf(series,nlag) # partial auto-correlation
  layout(1)
}
```

Acro Vara

The first company is the analysis would be Acro Vara (ARC1T.TL).

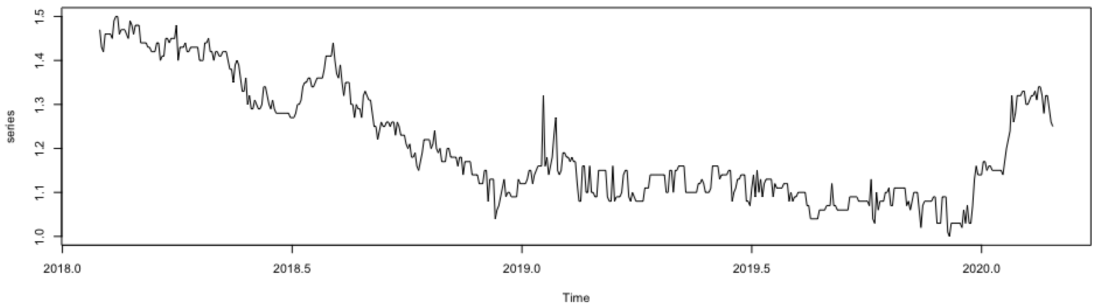
The original data is an XTS object with index values of 1-end thus some transformation of the data is needed to make it a time series that could be modeled.

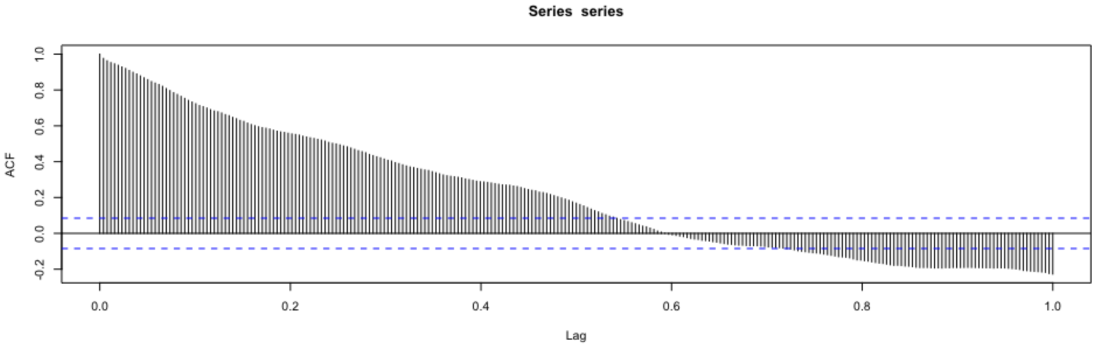
The frequency of the data in a year would be averaged to the number of observation we have within a year.

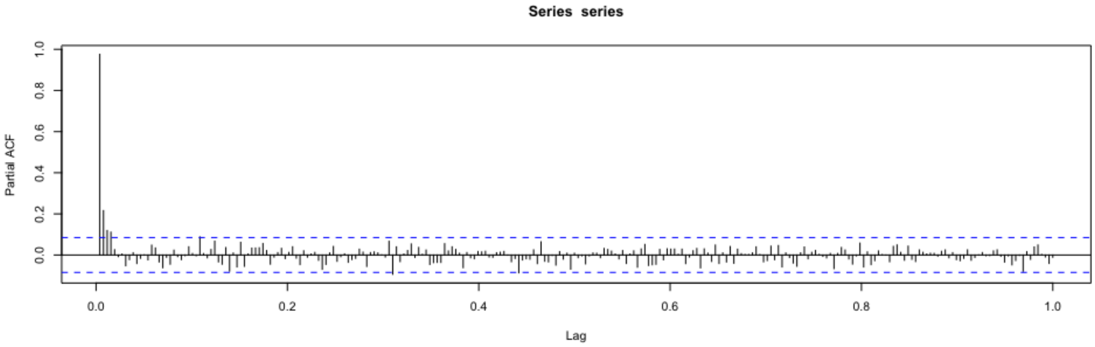
```r
#9
daysintimeperiod <-as.numeric(difftime(as.Date("2020/02/28",format="%Y/%m/%d"
),as.Date("2018/02/01", format="%Y/%m/%d"),units = "days"))
daysindata<- nrow(close_data_all)

n<- floor((daysindata*365)/daysintimeperiod) # this results to 258 trading da
ys on average in the time period
Z9 <- as.numeric(close_data_all[,9])
ts9 <- ts(Z9,  start=c(2018, 22), frequency = n) # creating the time series
```

The next step is to check in the data is stationary

```r
TSgraph(ts9, nlag = n)
```








```r
adf.test(ts9)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  ts9
## Dickey-Fuller = -1.0024, Lag order = 8, p-value = 0.9383
## alternative hypothesis: stationary
```

The data seems to be not stationary since the p-value>0.05of the ADF test (the confidence interval set by the author).

Next step is to try the first difference

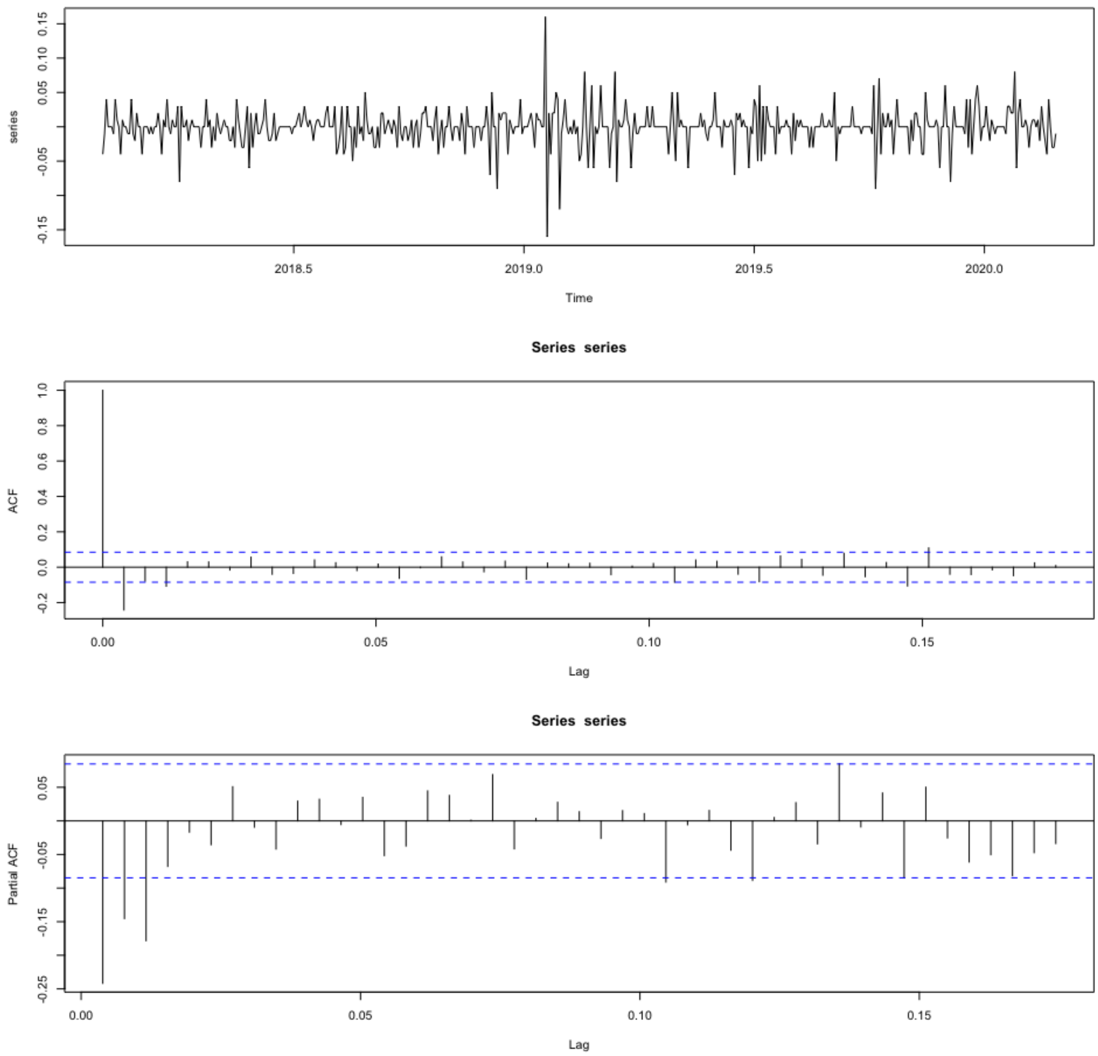
```r
adf.test(diff(ts9))
```

```
## Warning in adf.test(diff(ts9)): p-value smaller than printed p-value
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  diff(ts9)
```

```
## Dickey-Fuller = -9.0571, Lag order = 8, p-value = 0.01
## alternative hypothesis: stationary
```

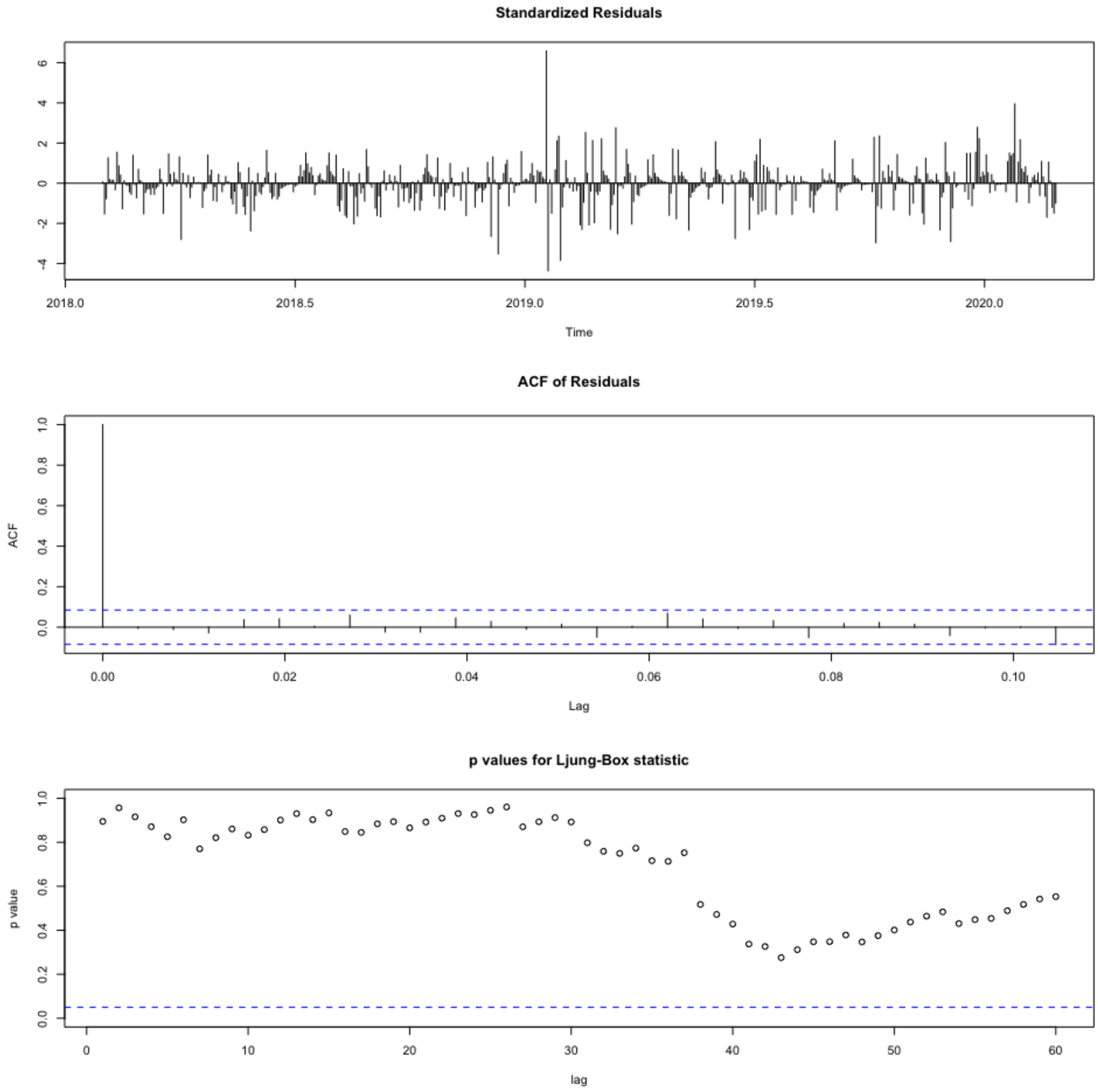
```r
TSgraph(diff(ts9), nlag = 45)
```



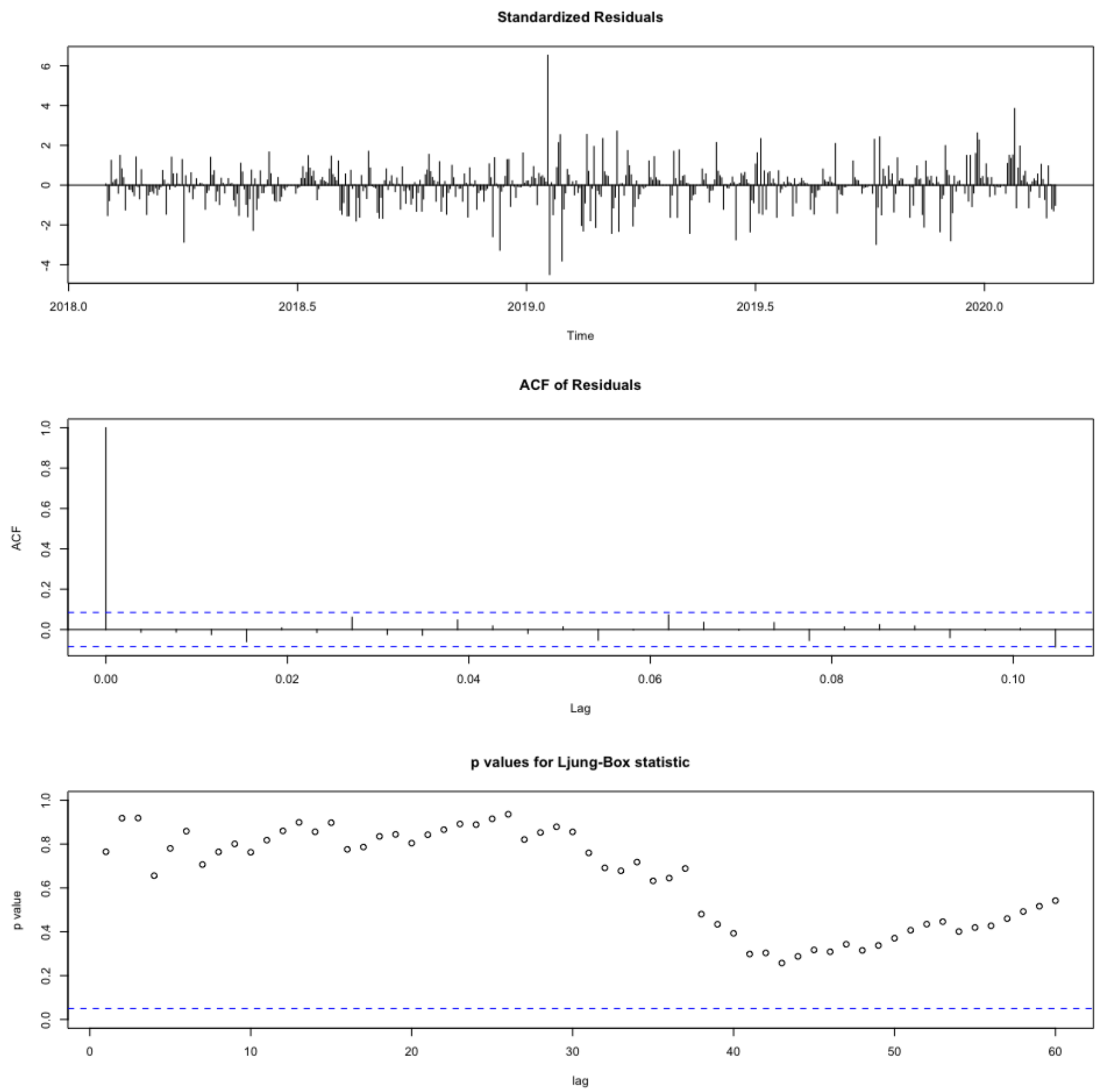
The test seems to be good and the data seems to be stationary since the p-value<0.05.

The model that seems to be fitting in the model from the ACF and PACF are ARIMA(3,1,0) and ARIMA(0,1,3)

```r
m9.1 <- arima(ts9,order = c(0,1,3))
m9.2 <- arima(ts9,order = c(3,1,0))
tsdiag(m9.1, 60)
```




```
tsdiag(m9.2, 60)
```



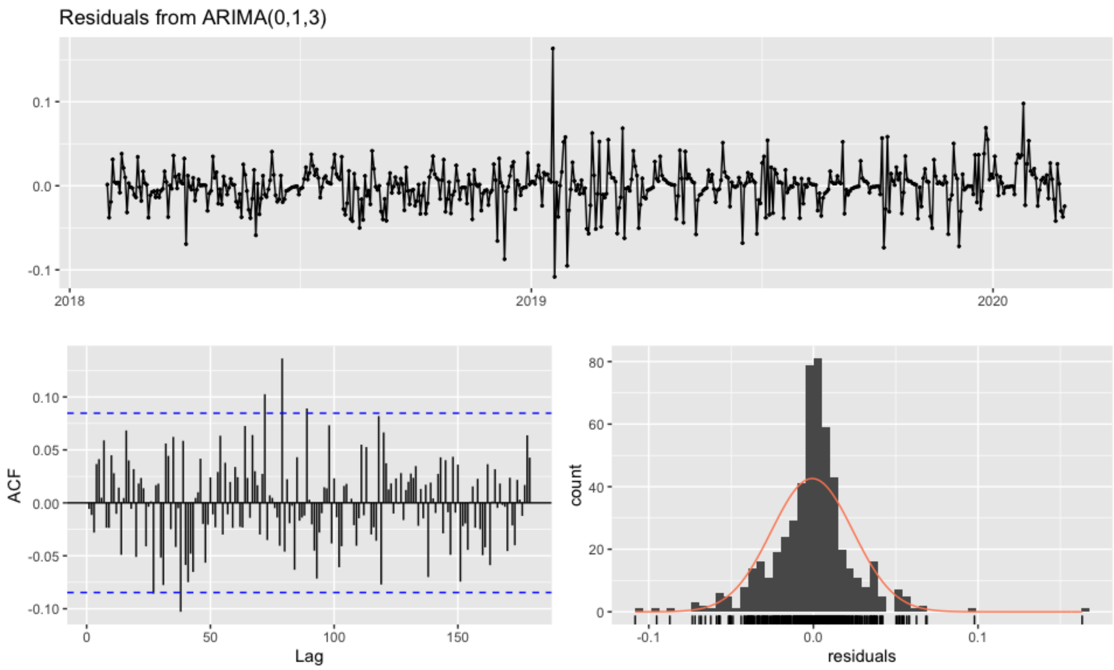
Both models seems to be fitting well the decision would be left to the AIC of the models

```
AIC(m9.1)

## [1] -2429.665

AIC(m9.2)

## [1] -2429.383
```

Since AIC(ARIMA(0,1,3))< AIC(ARIMA(3,1,0)) the model ARIMA(0,1,3) is taken to be the best.

Let’s check of the residuals are normally distributed

```
checkresiduals(m9.1)
```




```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(0,1,3)
## Q* = 112.98, df = 104, p-value = 0.2574
##
## Model df: 3.   Total lags used: 107
```

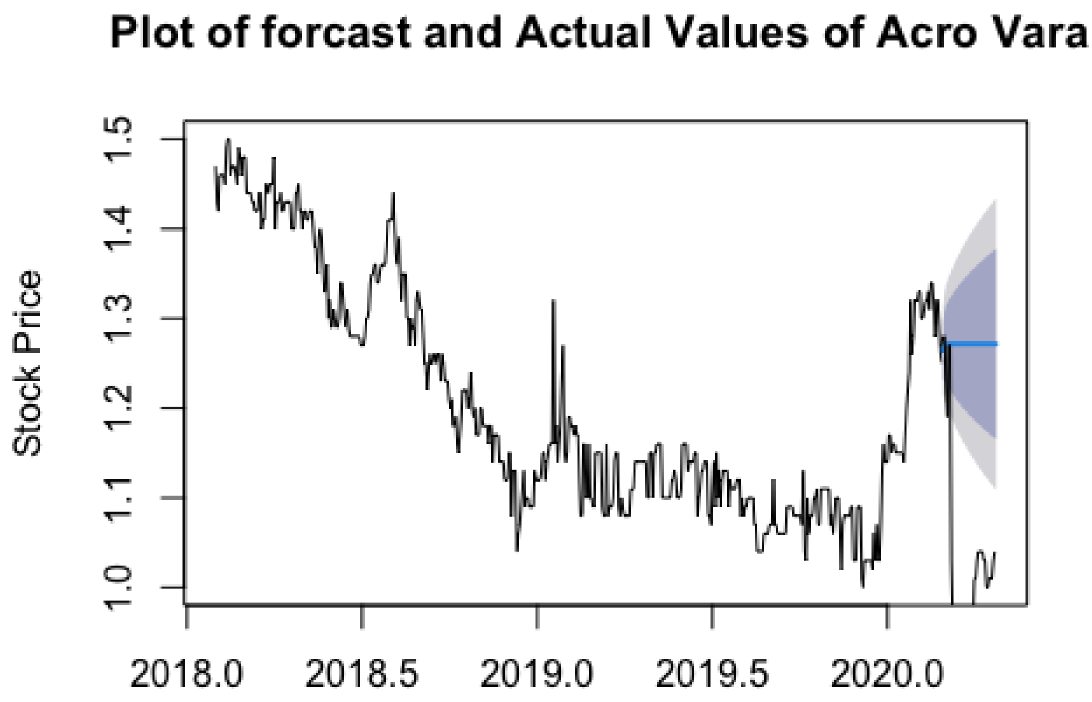
from Ljung-Box test p>0.05 meaning that the test failed to reject the hypothesis that the residuals are normally distributed.

The coefficients of the model are:

```
m9.1

##
## Call:
## arima(x = ts9, order = c(0, 1, 3))
##
## Coefficients:
##           ma1      ma2      ma3
##       -0.3104  -0.1051  -0.0809
## s.e.   0.0438   0.0428   0.0444
##
## sigma^2 estimated as 0.0006145:  log likelihood = 1218.83,  aic = -2429.66
```

Let's have a forecast of the future values and compare them to the actual ones

```
fore <- forecast(m9.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Acro Vara ", ylab = "Sto
ck Price")
lines(ts(as.numeric(close_data_all_future[,9]), start=c(2020, 40), frequency
=258 ))
```



The stock of Acro Vara (ARC1T.TL) seems to be going down since mid 2018 and it kept going down till their good performance of the last quarter of 2019 then the corona virus hit the market and thus the next months were really down.

The forested results seems to be good for 1 week falling in the 95% confidence interval but later really went down which requires additional analysis not covered is this paper.

Baltika

After a detailed analysis is showed in the first, the author decided not to include all the graphs. to not make the report too long.

After making the data stationary by the first difference the possible models that were seen from the ACF and PACF are ARIMA(1,1,0) and ARIMA(0,1,1).

```
Z7 <- as.numeric(close_data_all[,7])
ts7 <- ts(Z7, start=c(2018, 22), frequency = n)
TSgraph(ts7, nlag = n)
adf.test(ts7) # the data is not stationary
adf.test(diff(ts7)) # the data is stationary after 1st difference

## Warning in adf.test(diff(ts7)): p-value smaller than printed p-value

TSgraph(diff(ts7), nlag = 45)
m7.1 <- arima(ts7,order = c(1,1,0))
m7.2 <- arima(ts7,order = c(0,1,1))
tsdiag(m7.1, 60)#fit well
tsdiag(m7.2, 60)# fit well
```

```
AIC(m7.1) # best model
AIC(m7.2)

checkresiduals(m7.1)# test negative fail to reject
```

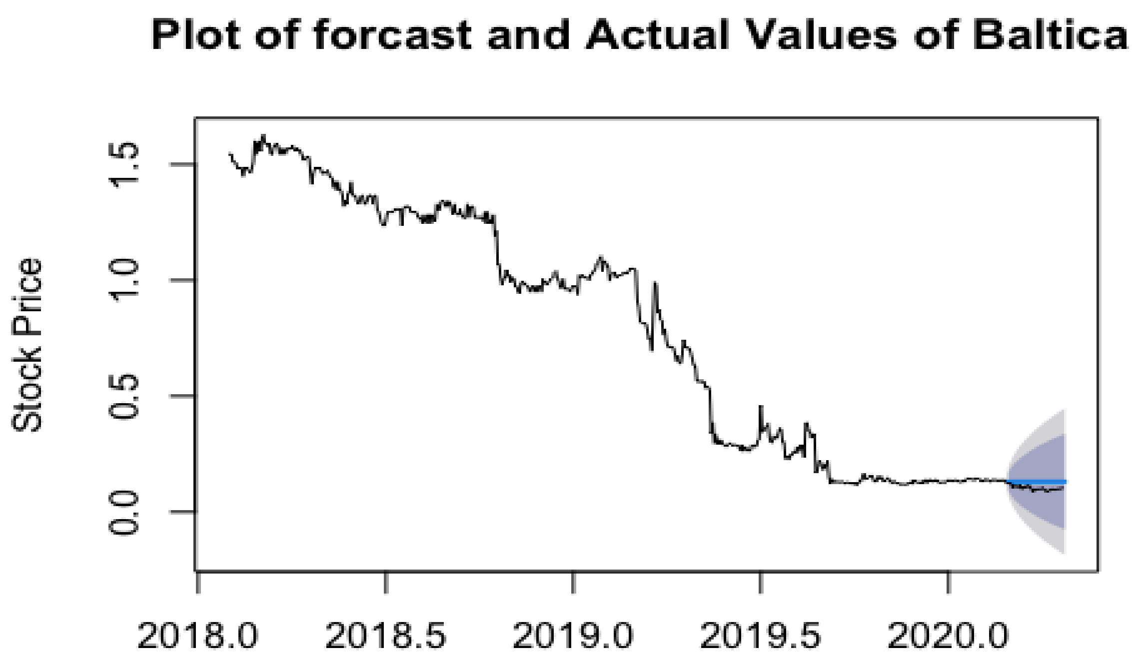
Based on the AIC the model ARIMA(1,1,0) takes as it was having lower AIC.

The coefficients of the model are:

```
m7.1

##
## Call:
## arima(x = ts7, order = c(1, 1, 0))
##
## Coefficients:
##           ar1
##       -0.1751
## s.e.   0.0425
##
## sigma^2 estimated as 0.0008909:  log likelihood = 1119.59,  aic = -2235.18
```

Let's plot the forecast with 40 days and plot it with the actual values

```
fore <- forecast(m7.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Baltica ", , ylab = "Sto
ck Price")
lines(ts(as.numeric(close_data_all_future[,7]), start=c(2020, 40), frequency
=258 ))
```



The forecast seems to be good in the 95% confidence interval.

The company seems to have problems since a long time with their stock dropping from 1.5Euros to 0.15 in 2 years. As it is share in the media the company undergoes changes in the management and restructuring in order to overcome the current situation.

EfTEN Real Estate Fund III AS

Make sure to run the code and remove echo=TRUE,results='hide' to have the full document if needed by the reader.

After making the data stationary the models that seems to fit are ARIMA(3,1,0) and ARIMA(0,1,3).
Let's try them and see the results of the AIC.

```
#12
Z12 <- as.numeric(close_data_all[,12])
ts12 <- ts(Z12,  start=c(2018, 22), frequency = n)
TSgraph(ts12, nlag = n)
adf.test(ts12) # the data seems to be not stationary
adf.test(diff(ts12)) # the data is stationary for first difference

## Warning in adf.test(diff(ts12)): p-value smaller than printed p-value

TSgraph(diff(ts12), nlag = 45)

m12.1 <- arima(ts12,order = c(3,1,0))
m12.2 <- arima(ts12,order = c(0,1,3))

tsdiag(m12.1, 60) #both models seems to fit
tsdiag(m12.2,60)

AIC(m12.1) # best AIC
AIC(m12.2)
checkresiduals(m12.1) # Test negative fail to reject that the residuals are n
ot normally distributed
```

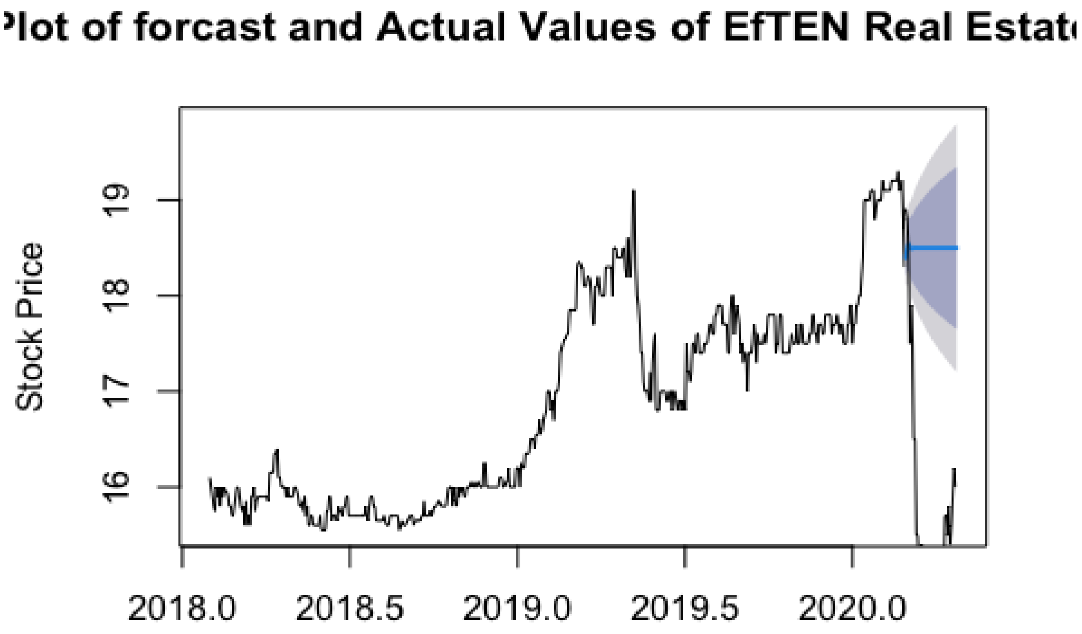
The coefficients of the model are:

```
m12.1

##
## Call:
## arima(x = ts12, order = c(3, 1, 0))
##
## Coefficients:
##           ar1      ar2      ar3
##       -0.1361  -0.1006  -0.1453
## s.e.   0.0434   0.0440   0.0437
##
## sigma^2 estimated as 0.02026:  log likelihood = 283.77,  aic = -559.54
```

Let's plot the forecast and have a look at the actual values.

```
fore <- forecast(m12.1, h = 40)
```

```
plot(fore,main="Plot of forcast and Actual Values of EfTEN Real Estate AS ", 
ylab = "Stock Price")
lines(ts(as.numeric(close_data_all_future[,12]), start=c(2020, 40), frequency
=258 ))
```



The forecast seems to be correct for some dates bu didn’t expect the fall of the COVID-19. Since the company operates on the real estate market and the investors would have penalized the company hardly.

The company seems to be doing good before the crisis hit.

Ekspress Group

The company operas on the media sector.

After making the data stationary the models that would be tested are ARIMA(0,1,2) and ARIMA(2,1,0).

```
#11
Z11 <- as.numeric(close_data_all[,11])
ts11 <- ts(Z11,   start=c(2018, 22), frequency = n)
TSgraph(ts11, nlag = n)
adf.test(ts11) # not passing
adf.test(diff(ts11)) # passing data stationary

## Warning in adf.test(diff(ts11)): p-value smaller than printed p-value

TSgraph(diff(ts11), nlag = 45)

m11.1 <- arima(ts11,order = c(0,1,2))
m11.2<-arima(ts11,order = c(2,1,0))
```

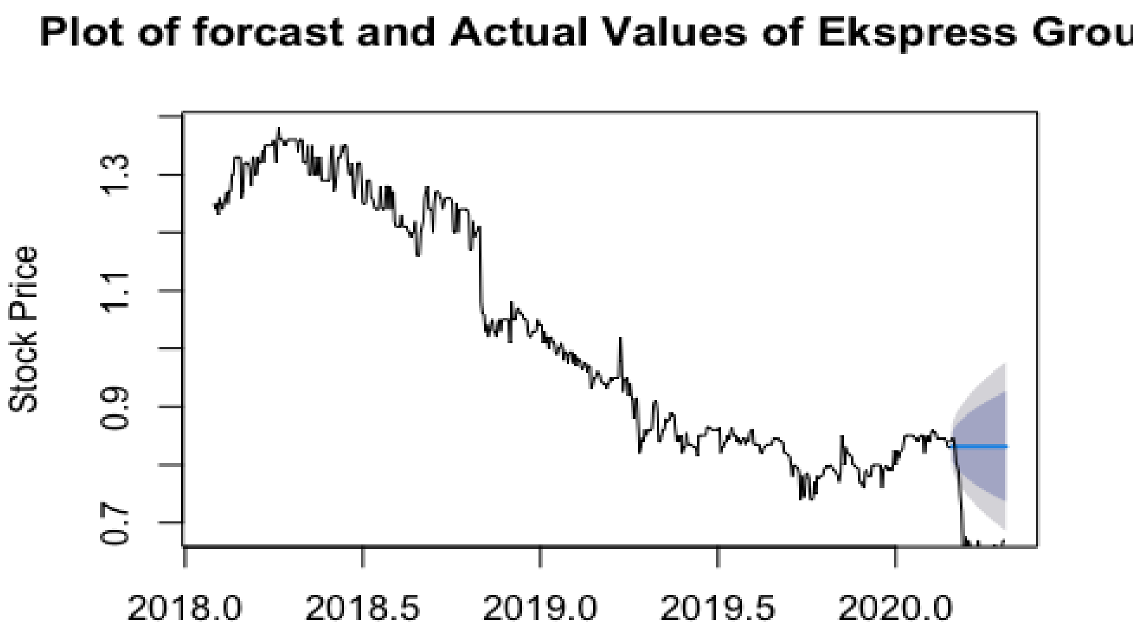
```r
tsdiag(m11.1, 60)# fits well
tsdiag(m11.2, 60)# fits well kinda

AIC(m11.1)# best
AIC(m11.2)
checkresiduals(m11.1) # failed to pass with 95% interval meaning data is good
```

The coefficients or the model chosen are:

```r
m11.1

##
## Call:
## arima(x = ts11, order = c(0, 1, 2))
##
## Coefficients:
##           ma1      ma2
##       -0.2560  -0.1419
## s.e.   0.0428   0.0432
##
## sigma^2 estimated as 0.000357:  log likelihood = 1364.16,  aic = -2722.33
```

Let's plot the models forecast and actual future values:

```r
fore <- forecast(m11.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Ekspress Group ", ylab =
"Stock Price")
lines(ts(as.numeric(close_data_all_future[,11]), start=c(2020, 40), frequency
=258 ))
```



The forecast seems to be correct for a couple of days with a confident interval of 95%.

Since a main part of the revenues of the company comes from selling newspapers and printable. It seems that the stock market penalized the expected return of the company early within the end of the first quarter.
In general the company seems to be doing bad in the past two years.

Harjo Elekter

The company operates mainly in the electric sector. After making the data stationary using the first difference, The models ARIMA(6,1,0) seemed to be the only one that would be fitting let's check the model.

One can observe some seasonality but this is not covered by this paper.

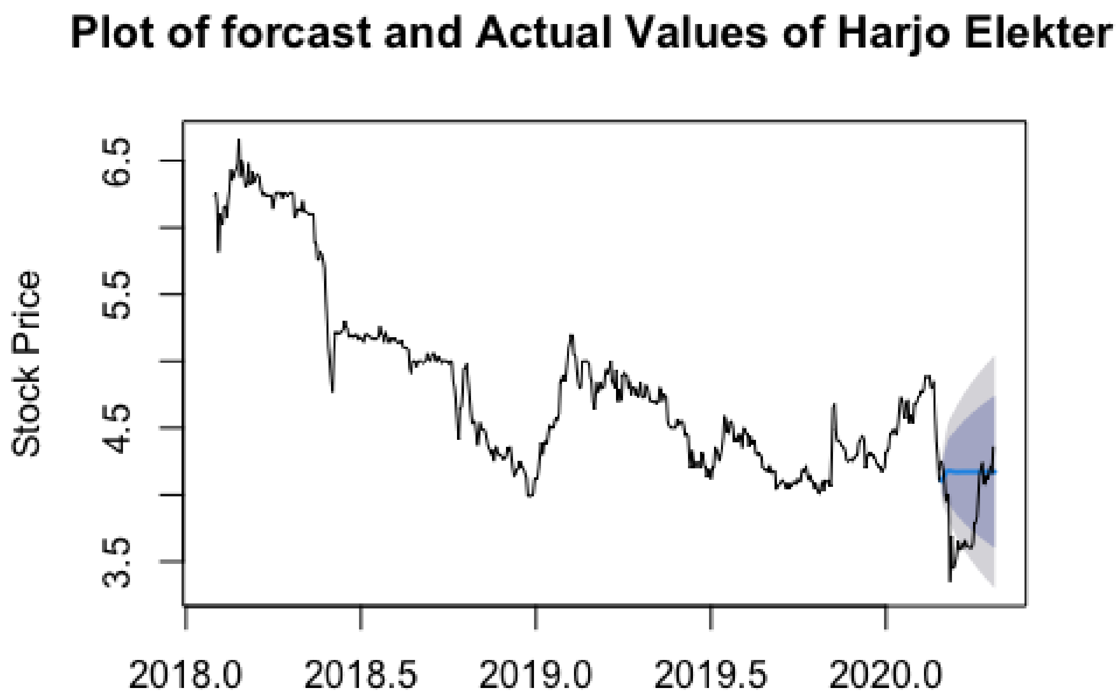
```
#3
Z3 <- as.numeric(close_data_all[,3])
ts3 <- ts(Z3,   start=c(2018, 22), frequency = n)
TSgraph(ts3, nlag = n)
adf.test(ts3)# not stationary
adf.test(diff(ts3)) # stationary

## Warning in adf.test(diff(ts3)): p-value smaller than printed p-value

TSgraph(diff(ts3), nlag = 45)
m3.1 <- arima(ts3,order = c(6,1,0))
tsdiag(m3.1, 60) #fits well

checkresiduals(m3.1) # normally distributed
```

The coefficients of the chosen model are:

```
m3.1

##
## Call:
## arima(x = ts3, order = c(6, 1, 0))
##
## Coefficients:
##          ar1      ar2     ar3      ar4      ar5      ar6
##       0.0297  -0.0028  0.0521  -0.0401  -0.0236  -0.0769
## s.e.  0.0431   0.0433  0.0443   0.0452   0.0457   0.0457
##
## sigma^2 estimated as 0.005422:  log likelihood = 636.46,  aic = -1258.91
```

The plot of the forecast and the actual values for future data.

```
fore <- forecast(m3.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Harjo Elekter  ", ylab =
"Stock Price")
lines(ts(as.numeric(close_data_all_future[,3]), start=c(2020, 40), frequency
=258 ))
```



The forecast seems to catch later values rather than current ones. Again one can say that the volatility in a pandemic situation seems to be big.

In the general the company's general values seems to be going down and not generating enough for its investors.

LHV Group

LHV Group is a holding company operating mainly as a Bank.

After making the data stationary, the following models would be analyzed ARIMA(1,1,0) ,ARIMA(2,1,0).
Since the number of parameters seems to affecting the goodness of the first the BIC criteria would be used in this case.

```
#10
Z10 <- as.numeric(close_data_all[,10])
ts10 <- ts(Z10,    start=c(2018, 22), frequency = n)
TSgraph(ts10, nlag = n)
adf.test(ts10) # not stationary
adf.test(diff(ts10)) # stationary

## Warning in adf.test(diff(ts10)): p-value smaller than printed p-value

TSgraph(diff(ts10), nlag = 45)
m10.1 <-arima(ts10,order = c(1,1,0))
m10.2 <-arima(ts10,order = c(2,1,0))
tsdiag(m10.1, 60)# fits well
```

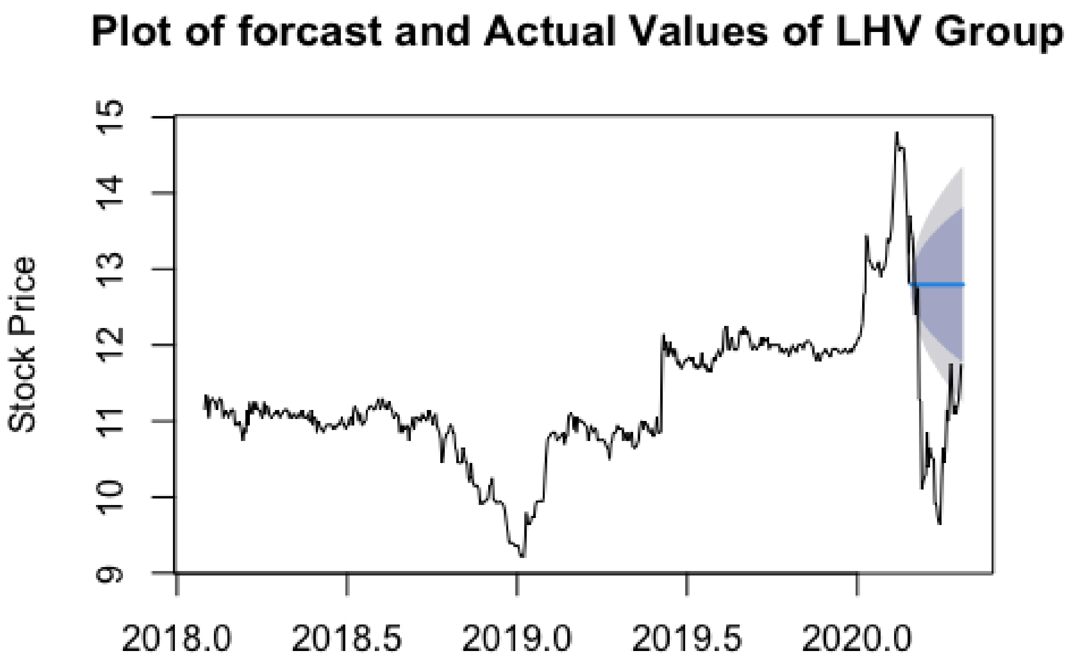
```r
tsdiag(m10.2, 60) # fits well

BIC(m10.1) # better
BIC(m10.2)

checkresiduals(m10.2) # normally distributed
```

The coefficients of the model are

```r
m10.1

##
## Call:
## arima(x = ts10, order = c(1, 1, 0))
##
## Coefficients:
##          ar1
##       0.0647
## s.e.  0.0432
##
## sigma^2 estimated as 0.01388:  log likelihood = 385.02,  aic = -766.04
```

Plotting the forecast and Actual future values.

```r
fore <- forecast(m10.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of LHV Group ", ylab = "Sto
ck Price")
lines(ts(as.numeric(close_data_all_future[,10]), start=c(2020, 40), frequency
=258 ))
```



The forecast seems to be off. Again due to COVID19 impact of the market.

In the general the bank's stock price seems not to impacted other than before and after the quarterly return declarations. This insures that the Estonian market is efficient and transparent.

Merko Ehitus

The company operates in the construction sector. After applying the first difference the model fitting seemed to be ARIMA(1,1,0) let's check the model.

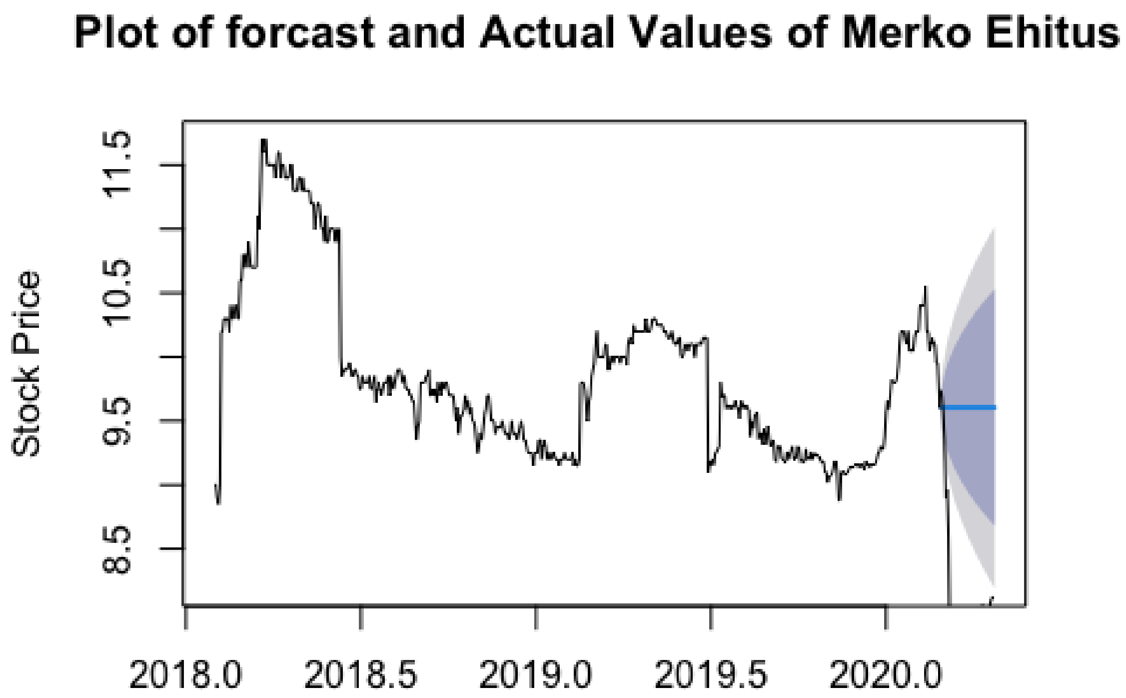
```
Z <- as.numeric(close_data_all[,1])
ts1 <- ts(Z,  start=c(2018, 22), frequency = n)
TSgraph(ts1, nlag = 45)
adf.test(ts1) # not stationary
adf.test(diff(ts1)) # stationary

## Warning in adf.test(diff(ts1)): p-value smaller than printed p-value

TSgraph(diff(ts1), nlag = 45 )
m1.1 <- arima(ts1,order = c(1,1,0))
tsdiag(m1.1, 100) # fits well
checkresiduals(m1.1) # the test seems to be failing thus the model is good
```

Models are:

```
m1.1

##
## Call:
## arima(x = ts1, order = c(1, 1, 0))
##
## Coefficients:
##           ar1
##       -0.0496
## s.e.   0.0432
##
## sigma^2 estimated as 0.01427:  log likelihood = 377.7,  aic = -751.4
```

Plot the forecast and the actual values

```
fore <- forecast(m1.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Merko Ehitus ", ylab = "
Stock Price")
lines(ts(as.numeric(close_data_all_future[,1]), start=c(2020, 40), frequency
=258 ))
```



The company seems to be strongly affected by the crisis driving its price bellow the lowest value in 2 years.
The company seems to be having changes on the quarterly return and independent events about deals that it is performing.

Nordecon

Nordecon operates in the construction sector.

After making the data stationary the models to be checked are ARIMA(3,1,0) and ARIMA(0,1,1). BIC would be used since it penalizes for the number of parameters.

```
Z2 <- as.numeric(close_data_all[,2])
ts2 <- ts(Z2,   start=c(2018, 22), frequency = n)
TSgraph(ts2, nlag = n)
adf.test(ts2) # not stationary
adf.test(diff(ts2))# stationary

## Warning in adf.test(diff(ts2)): p-value smaller than printed p-value

TSgraph(diff(ts2), nlag = 45)
m2.1 <- arima(ts2,order = c(3,1,0))
m2.2<-arima(ts2,order = c(0,1,1))

tsdiag(m2.1, 60)# fits well
tsdiag(m2.2, 60)# fits well

BIC(m2.1)
```

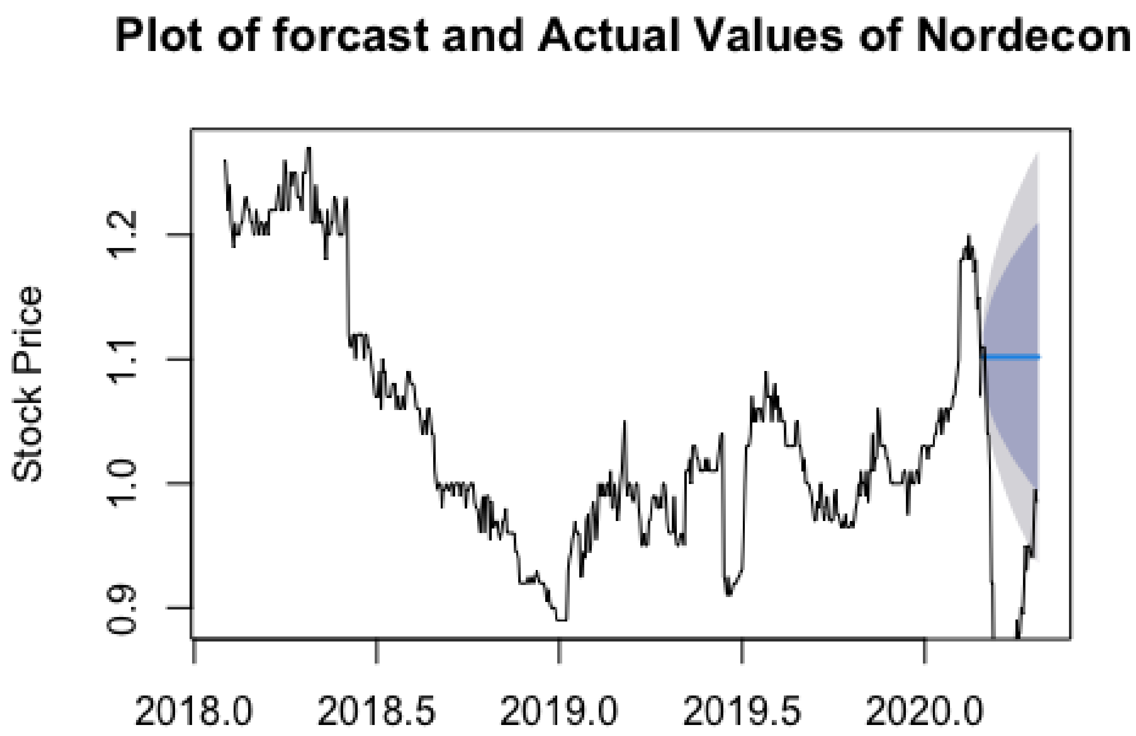
```r
BIC(m2.2)# better
checkresiduals(m2.2) # failing thus the model is good.
```

The model is:

```r
m2.2

## 
## Call:
## arima(x = ts2, order = c(0, 1, 1))
## 
## Coefficients:
##           ma1
##       -0.1194
## s.e.   0.0448
## 
## sigma^2 estimated as 0.0002298:  log likelihood = 1482.1,  aic = -2960.21
```

Plotting the model

```r
fore <- forecast(m2.2, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Nordecon ", ylab = "Stoc
k Price")
lines(ts(as.numeric(close_data_all_future[,2]), start=c(2020, 40), frequency
=258 ))
```



As the construction sector got a big hit because of COVID19, Nordecon was no exception.

The companies stock price seems to be volatile and independent deals affects the companies stock price greatly.

PRfoods

The company is engaged in the food processing sector.

After making the data stationary the models that seems to be fitting are ARIMA(4,1,0) and ARIMA(0,1,5).

Since the number of parameters seems to be different BIC would be used.

```
#8
Z8 <- as.numeric(close_data_all[,8])
ts8 <- ts(Z8,   start=c(2018, 22), frequency = n)
TSgraph(ts8, nlag = 45)
adf.test(ts8) # sot stationary
adf.test(diff(ts8)) # stationary

## Warning in adf.test(diff(ts8)): p-value smaller than printed p-value

TSgraph(diff(ts8), nlag = 45)
m8.1 <- arima(ts8,order = c(4,1,0))
m8.2<- arima(ts8,order = c(0,1,5))
tsdiag(m8.1, 60)# fits well
tsdiag(m8.2, 60)# fits well

BIC(m8.1)# better
BIC(m8.2)

checkresiduals(m8.1) # failing model is good
```

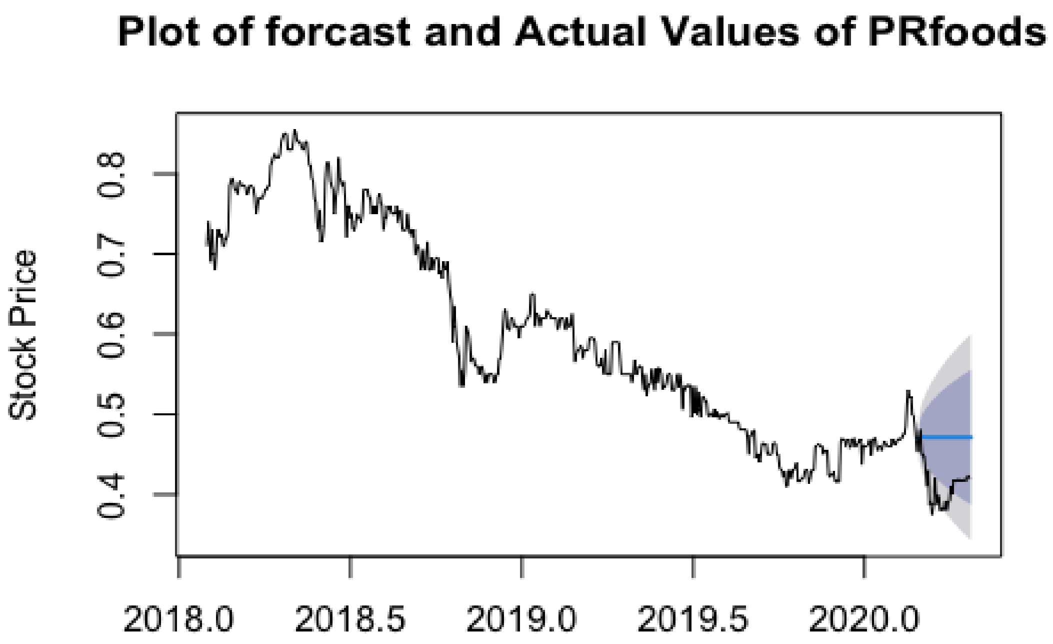
The model is:

```
m8.1

##
## Call:
## arima(x = ts8, order = c(4, 1, 0))
##
## Coefficients:
##           ar1      ar2      ar3      ar4
##       -0.1823  -0.0406  -0.0050  -0.1084
## s.e.   0.0431   0.0439   0.0441   0.0437
##
## sigma^2 estimated as 0.0001864:  log likelihood = 1538.03,  aic = -3066.05
```

plotting the model's forecast and actual values:

```
fore <- forecast(m8.1, h = 40)

plot(fore,main="Plot of forcast and Actual Values of PRfoods ", ylab = "Stock
Price")
```

```
lines(ts(as.numeric(close_data_all_future[,8]), start=c(2020, 40), frequency
=258 ))
```



The food sector seems not too affected by the crisis and the actual values seems to fall in the 85 percentile of the forecast.

PRfood's stock price seems to be declining over the years with the lowest value hitting during COVID-19. However the company seems to have some issues in generating revenue.

Pro Kapital Grupp

The company operates in the real estate industry.

The time series needed the first difference. The model ARIMA(0,1,2) seems to be a good fit let's check that.

```
#4
Z4 <- as.numeric(close_data_all[,4])
ts4 <- ts(Z4,  start=c(2018, 22), frequency = n)
TSgraph(ts4, nlag = 45)
TSgraph(diff(ts4), nlag = 45)
m4 <- arima(ts4,order = c(0,1,2))

tsdiag(m4, 60) # fits well
checkresiduals(m4) # failed so model is good
```

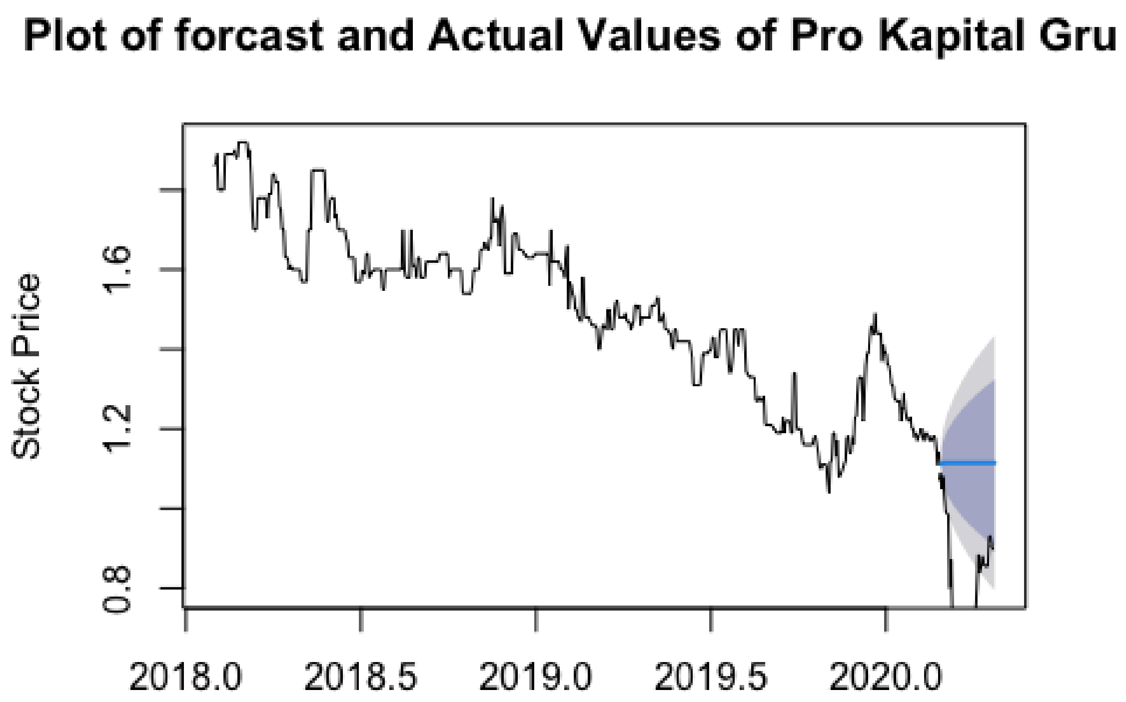
The model is

```
m4
```

```
##
## Call:
## arima(x = ts4, order = c(0, 1, 2))
##
## Coefficients:
##           ma1      ma2
##       -0.1166  -0.0935
## s.e.   0.0433   0.0445
##
## sigma^2 estimated as 0.001046:  log likelihood = 1076.55,  aic = -2147.1
```

LEt's plot the forecast with the actual values.

```
fore <- forecast(m4, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Pro Kapital Grupp ", yla
b = "Stock Price")
lines(ts(as.numeric(close_data_all_future[,4]), start=c(2020, 40), frequency
=258 ))
```



The real estate sector got a big hit with the covid-19 thus no surprises in seeing that decline.

The stock price of the company seems to be declining over the last year the reasons seems to be lack revenue generation and losing the investors trust.

Silvano Fashion Group

The company operates in the retail sector.

After making the data stationary the Models that seems to be good are ARIMA(0,1,2) or ARIMA(2,1,0)

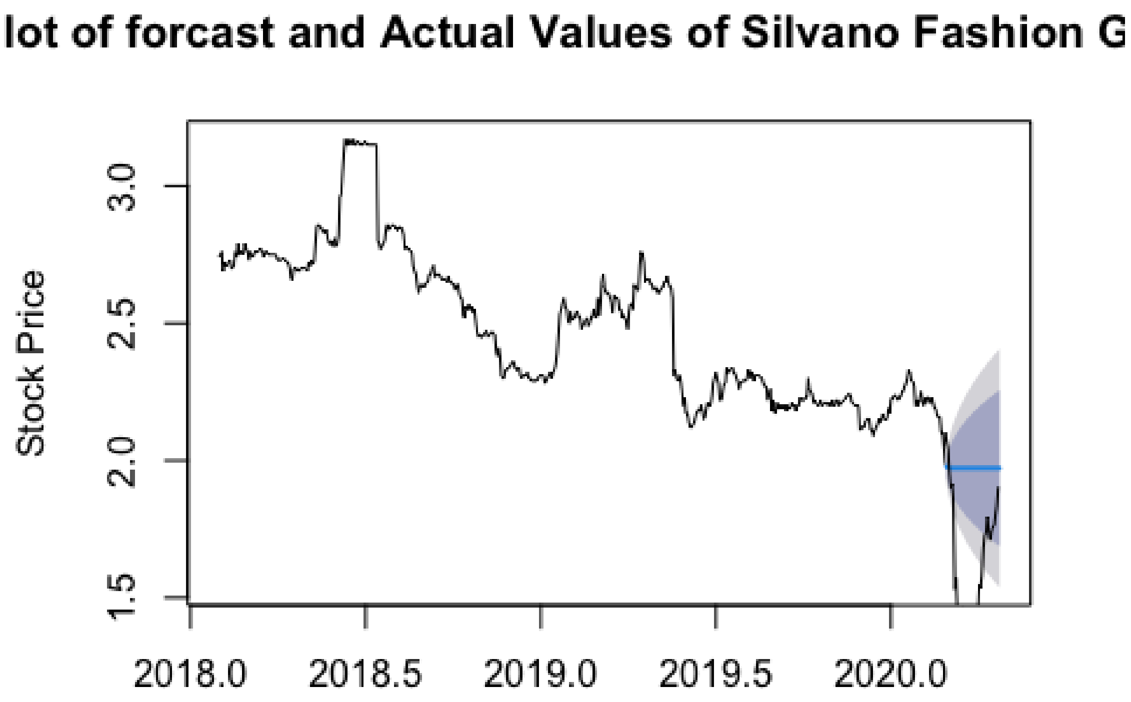
```
#5
Z5 <- as.numeric(close_data_all[,5])
ts5 <- ts(Z5,  start=c(2018, 22), frequency = n)
TSgraph(ts5, nlag = 45)
adf.test(ts5) # not passing
adf.test(diff(ts5)) # stationary

## Warning in adf.test(diff(ts5)): p-value smaller than printed p-value

TSgraph(diff(ts5), nlag = 45)
m5.1 <- arima(ts5,order = c(0,1,2))
m5.2<- arima(ts5,order = c(2,1,0))
tsdiag(m5.1, 60) #
tsdiag(m5.2, 60) # both models fit well
AIC(m5.1)
AIC(m5.2) # slightly better

checkresiduals(m5.2) # failing thus the model is good
```

the best model is:

```
m2.2

##
## Call:
## arima(x = ts2, order = c(0, 1, 1))
##
## Coefficients:
##           ma1
##       -0.1194
## s.e.   0.0448
##
## sigma^2 estimated as 0.0002298:  log likelihood = 1482.1,  aic = -2960.21
```

plotting the model:

```
fore <- forecast(m5.2, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Silvano Fashion Group ",
ylab = "Stock Price")
lines(ts(as.numeric(close_data_all_future[,5]), start=c(2020, 40), frequency
=258 ))
```



The forecast not to capture the bad fall but gets later values. The pandemic seems to severely affecting stocks volatility.

The company in overall seems to have a stable performance over the last year but the crisis drove really down the stock price.

Tallink Grupp

The company operates in the marine shipping industry.

After applying the first difference the data seems to be stationary. Models ARIMA(5,1,0) and ARIMA(0,1,5) seems to be the candidates for this time series.

```
#6
Z6 <- as.numeric(close_data_all[,6])
ts6 <- ts(Z6,   start=c(2018, 22), frequency = n)
TSgraph(ts6, nlag =45)
adf.test(ts6) # not passing with 0.05 confidence interval
adf.test(diff(ts6)) # data stationary

## Warning in adf.test(diff(ts6)): p-value smaller than printed p-value

TSgraph(diff(ts6), nlag = 45)
m6.1 <- arima(ts6,order = c(5,1,0))
m6.2 <- arima(ts6,order = c(0,1,5))
tsdiag(m6.1, 60)# fits well
tsdiag(m6.2, 60)#fits well
```

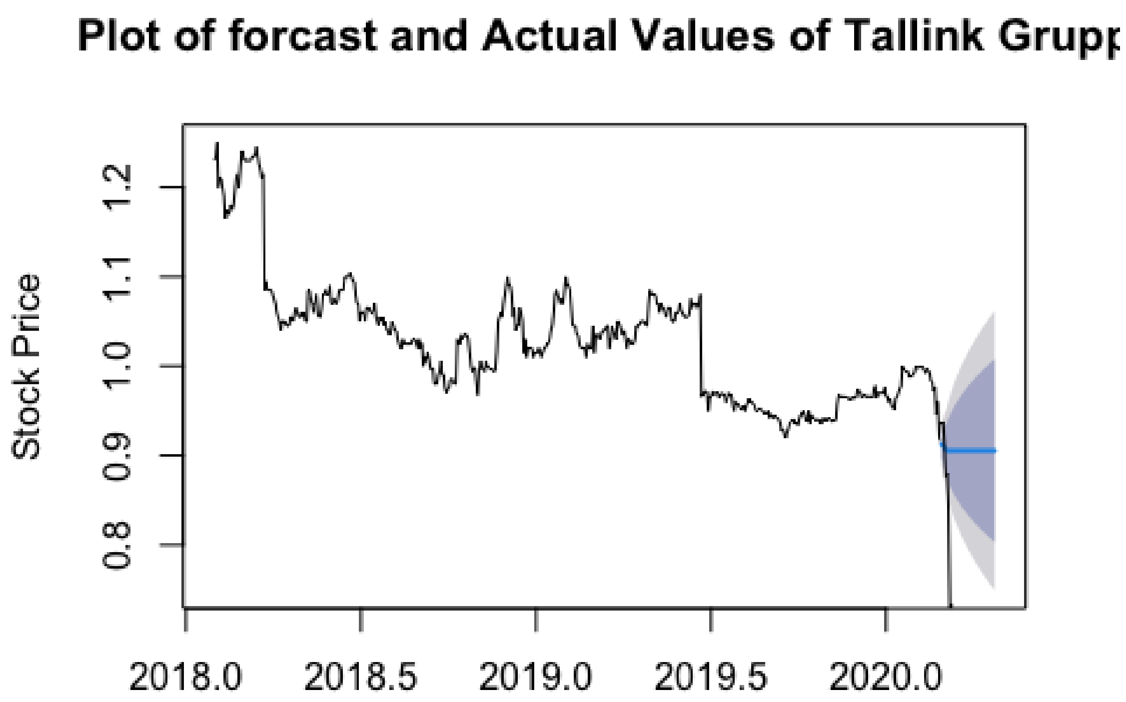
```r
AIC(m6.1)# slightly better
AIC(m6.2)
checkresiduals(m6.1)
```

The model is:

```r
m6.1

##
## Call:
## arima(x = ts6, order = c(5, 1, 0))
##
## Coefficients:
##           ar1     ar2      ar3     ar4     ar5
##       -0.1048  0.0390  -0.0010  0.0757  0.0967
## s.e.   0.0431  0.0435   0.0443  0.0445  0.0444
##
## sigma^2 estimated as 0.0001334:  log likelihood = 1627.47,  aic = -3242.94
```

Plotting the forecast and the actual values:

```r
fore <- forecast(m6.2, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Tallink Grupp", ylab = "
Stock Price")
lines(ts(as.numeric(close_data_all_future[,6]), start=c(2020, 40), frequency
=258 ))
```



The companies got really affected by the travel restriction in place from Estonia to other places thus the price of the stock got really driven down.

The company in general seems to have a stable revenue with doing a bit not good for the last year.

Tallinn Kaubamaja Grupp

Tallinn Kaubamaja operates in the department stores industry.

After making the data stationary the models that seems to be representing the series are ARIMA(1,1,0) or ARIMA(0,1,1)

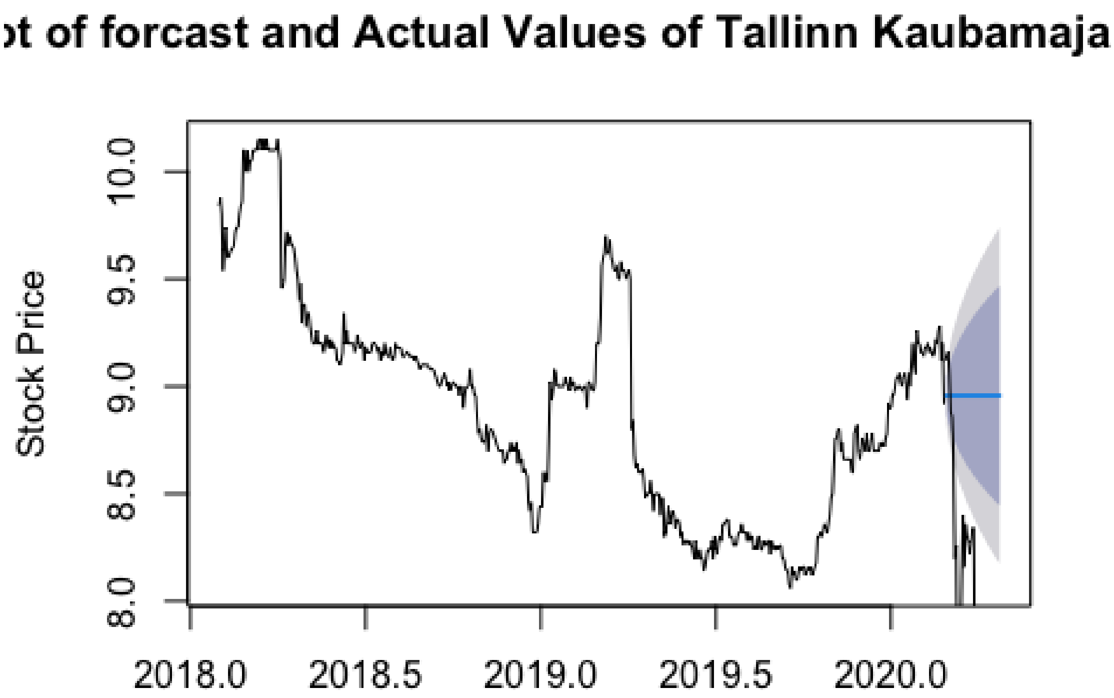
```
#13
Z13 <- as.numeric(close_data_all[,13])
ts13 <- ts(Z13,  start=c(2018, 22), frequency =n)
TSgraph(ts13, nlag = 45)
adf.test(ts13) # not stationary
adf.test(diff(ts13)) # stationary

## Warning in adf.test(diff(ts13)): p-value smaller than printed p-value

TSgraph(diff(ts13), nlag = 45)
m13.1 <- arima(ts13,order = c(0,1,1))
m13.2<- arima(ts13,order = c(1,1,0))
tsdiag(m13.1, 60) # fits well
tsdiag(m13.2, 60)# fits well

AIC(m13.1)
AIC(m13.2)# slightly better

checkresiduals(m13.2)
```

The best model is:

```
m13.2

##
## Call:
## arima(x = ts13, order = c(1, 1, 0))
##
## Coefficients:
##          ar1
##       -0.090
## s.e.   0.043
##
## sigma^2 estimated as 0.004717:  log likelihood = 673.77,  aic = -1343.55
```

The plot of the models forecast seems to be:

```
fore <- forecast(m13.2, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Tallinn Kaubamaja Grupp
", ylab = "Stock Price")
lines(ts(as.numeric(close_data_all_future[,13]), start=c(2020, 40), frequency
=258 ))
```



The forecast seems to be off. Since the stores had to close after the Covid19 hit the company seems to be having issues keeping the trust of the investors.

The companies stock price in the past seems to be volatile due to a lot of speculation from investors.

Tallinna Vesi

The company operates in the utility industry more specifically water.

The first difference model seems to be good. ARIMA(0,1,0)

```
#14
Z14 <- as.numeric(close_data_all[,14])
ts14 <- ts(Z14, start=c(2018, 22), frequency = n)
TSgraph(ts14, nlag = 45)

adf.test(ts14)# not stationary
adf.test(diff(ts14)) # stationary

## Warning in adf.test(diff(ts14)): p-value smaller than printed p-value

TSgraph(diff(ts14), nlag = 45)
m14 <- arima(ts14,order = c(0,1,0))
tsdiag(m14, 60)# fits well
checkresiduals(m14) # seems the model is good
```

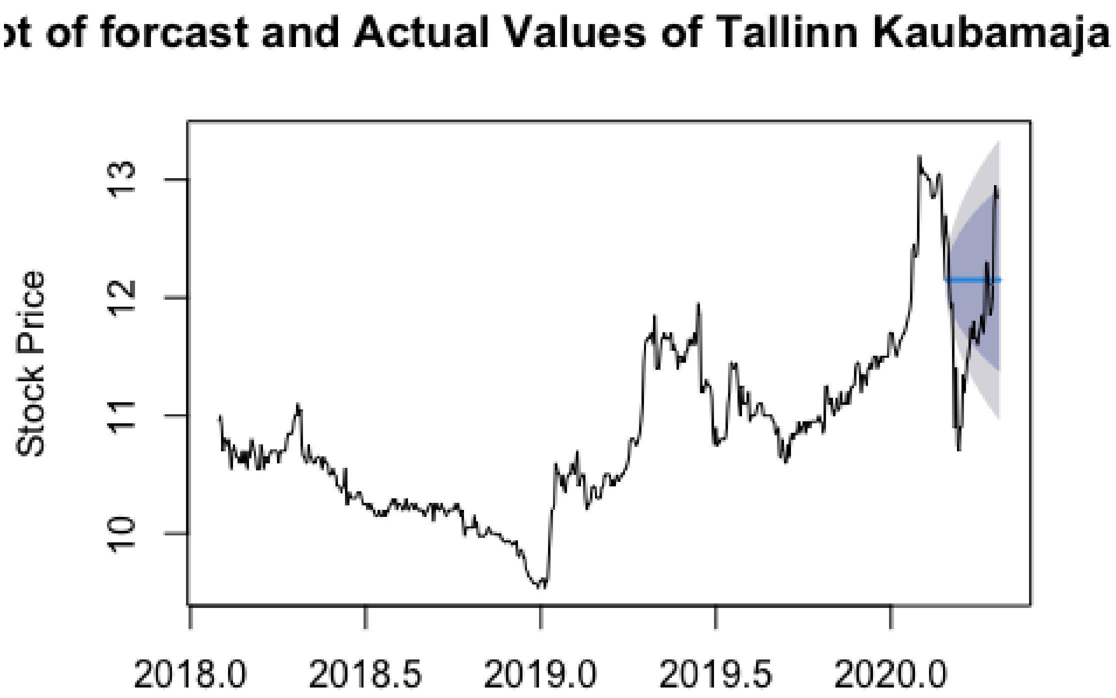
The model:

```
m14
```

```
##
## Call:
## arima(x = ts14, order = c(0, 1, 0))
##
##
## sigma^2 estimated as 0.009249:  log likelihood = 493.62,  aic = -985.25
```

The plot of the data:

```
fore <- forecast(m14, h = 40)

plot(fore,main="Plot of forcast and Actual Values of Tallinn Kaubamaja Grupp
", ylab = "Stock Price")
lines(ts(as.numeric(close_data_all_future[,14]), start=c(2020, 40), frequency
=258 ))
```



Seems the forecast was a bit off in the beginning however later predictions falls in the 95% confidence interval.

The company seems to be growing in the past year with new deals of providing water in Harju county.

General comments

The COVID 19 had a major impact on the companies stock price especially the construction, real estate and transport companies.

Most of the companies in the Estonian market seems to be having a declining stock price over the last 2 years.

THE END

NB: Please note that this is an R-markdown output that can be obtained from the .Rmd attached to this document or could be asked from the author.

Appendix E: Portfolio Optimization Code

Thesis

Omar Setihe

4/20/2020

Quandalkey:(Zf_29iNsDg7Qm3r5zUBN) Importing the symbols

```
library("quantmod")
library(Quandl)
library(gdata)

tickers <- c("ARC1T.TL","BLT1T.TL","EFT1T.TL","EEG1T.TL","HAE1T.TL","LHV1T.TL
","MRK1T.TL","NCN1T.TL","PRF1T.TL","PKG1T.TL","SFG1T.TL","TAL1T.TL","TKM1T.TL
","TVEAT.TL")
data_env <- new.env()
getSymbols(tickers,from = "2018-02-01",to = "2020-02-29",env= data_env) # not
ethe getsymboleufnction takes the function excluding the last time that's why
it's 29

##  [1] "ARC1T.TL" "BLT1T.TL" "EFT1T.TL" "EEG1T.TL" "HAE1T.TL" "LHV1T.TL"
##  [7] "MRK1T.TL" "NCN1T.TL" "PRF1T.TL" "PKG1T.TL" "SFG1T.TL" "TAL1T.TL"
## [13] "TKM1T.TL" "TVEAT.TL"

close_data <- do.call(merge, eapply(data_env, Cl)) # get the close price form
the data and merge them

index<- Quandl("NASDAQOMX/OMXTGI", api_key="Zf_29iNsDg7Qm3r5zUBN",start_date=
"2018-02-01" , end_date= "2020-02-28",type= "xts")
close_data_all <- merge(close_data,index[,1])
close_data_all <- na.locf(close_data_all)

Untact<-data.matrix(as.data.frame(close_data_all)) # get the numeric values f
rom the Ts

tickers <- colnames(Untact)
# consider to calculate the beta based on weekly returns for T=3 years ( Bloo
mberg )
# consider making a portfolio at the beginning of the assets
# check the COVID_19 situation
```

the Least square method to solve the problem of CAPM

```
Data2 <- Untact

LPportfolio <- function(traindata) {
  library(linprog)
  n<- tickers

  indexvalue <- Data2[1:nrow(traindata), length(n)]

  for(i in 1:(length(n)-1)) {
```

```
    l <- lm(traindata[,i] ~ indexvalue) #create the linear model
    if(i==1)
    {
      y<-coefficients(l) # get the Coef
      r<-exp(sum(diff(log(traindata[,i]))))-1 # get the return
    }
    else
    {
      y<-rbind(y,coefficients(l)) # get the coef
      r<-c(r,exp(sum(diff(log(traindata[,i]))))-1) # get the return
    }
    }

  rownames(y)<-n[-length(n)]# take off the index name from tickers
  colnames(y)<-c("b0","b1") # add the beta 0 and beta 1


  c<-r[] # create the retun vector
  y<-y[,2] # take the beta 1

  b<- c(1.4,0,1,-1,rep(0.4,length(n)-1)) # creates the b vecrot
  A<- rbind(y,-y,rep(1,length(n)-1),rep(-1,length(n)-1),diag(length(n)-1)) #
create the contraints vector
  res <- solveLP(c, b, A, maximum=TRUE) # solve using linear method
  return(res$solution) # send back the weights
}
```

Minimum variance portfolio we are minimizing the variance of the portfolio meaning w covariance w This is a quadratic optimization problem so we gona be using solve.QP

```
MKportfolio <- function (traindata){
  library("quadprog") # call the quadratic solver library
  num<- ncol(traindata) # get the number of securites
  dvec<- array(0, dim = c(num,1)) #objective function

  Amat<- t(rbind(rep(1,num),rep(-1,num))) # the constraint matrix

  bvec<- t(c(1,-1)) # the b vector of the constrain matrix

  sol<- solve.QP(cov(traindata),dvec,Amat,bvec) # use the quadratic solver

  return(as.numeric(sol$solution)) # return the weights
}
```

PortfolioAnalysis function for solving the VAR portfolio.

```
SDVportfolio <- function(traindata) {
  library(PortfolioAnalytics) # call the library
  R <- diff(traindata)/traindata[-1,] # calculate the change of the stock pri
ce

  p <- portfolio.spec(assets = colnames(traindata)) # creates the portfolio
  p <- add.constraint(portfolio = p, type = 'weight_sum',
                      min_sum = 0.99, max_sum = 1.01)  # sum of weights equal t
o zero
```

```
  p <- add.constraint(portfolio = p, type = 'box',
                      min = 0, max = 0.3) # add a box constraints min value 0 m
ax value 30%
  p<- add.objective(portfolio=p,type='return', name='VaR') # objective of the
portfolio

  opt <- optimize.portfolio(R, portfolio=p, optimize_method="DEoptim",search_
size=2000) # search size 2000 creates 2000 portfolios in each itration
  return(as.numeric(opt$weights)) # return the weights
}
```

PortfolioOptim for solving the CVAR problem

```
CVARPortfolio <- function (traindata){
  library (Rsymphony)
  library(Rglpk)
  library(mvtnorm)
  library(PortfolioOptim)
  library(zoo)
  k<-ncol(traindata)
  z<- nrow(traindata)
  distribution<- c(rep(0.6/floor(0.75*z),floor(0.75*z)),rep(0.2/floor(0.15*z)
,floor(0.15*z)),rep(0.2/(z-floor(0.75*z)-floor(0.15*z)-1),(z-floor(0.75*z)-fl
oor(0.15*z)-1))) # cretes the uniform stepwise distribution
  dat<-cbind( diff(traindata[,])/traindata[-1,] , distribution) #)matrix(1/(n
row(traindata)-1),(nrow(traindata)-1)))
  port_ret = 0.08# target portfolio return
  alpha_optim = 0.95
  a0 <- rep(1,k)
  Aconstr <- rbind(rep(1,k) , rep(-1,k)) # create the constraints for the wei
ghts
  bconstr <- c(1+1e-8, -1+1e-8)
  LB <- rep(0,k) # lower bound
  UB <- rep(0.6,k) # uper bound
  res <- BDportfolio_optim(dat, port_ret, "CVAR", alpha_optim,Aconstr, bconst
r, LB, UB, maxiter=150, tol=1e-8)
  return(t(res$thet))
}
```

Creating the platform to trade UPDATEIC and Capital Vector.

```
updateIC <- function(currentcapital,pastobservation, newobservation, weights)
{
  library(dplyr)
  investment<- currentcapital*weights # current proportion in each security
  numberofsecurites <- investment/pastobservation # number of securites NB: I
take fractions of that too
  newcapital<-numberofsecurites%*%newobservation # new capital
  return(newcapital)
}

Capitalvector<- function(traindata,testdata, algorithm, IC){
  # The algorithm should return the weights as a vector in the same order
  i<- nrow(testdata) # number of iterations over the algorithm and test data
```

```
  changeofcapital <- IC # Initial capital
  cumulativeweight<- rep(0,ncol(traindata))# first weights
  datevalues<- rownames(testdata) # get the date values form test data
  pastvalues <- traindata[nrow(traindata),] #last observation of the train da
ta
  for (k in 1:i) {
    currentweights<- algorithm (traindata) # train the mdoel

    cumulativeweight<- rbind(cumulativeweight,currentweights) # add them to t
he weight cumulative vector for plot

    newvalues  <- testdata[k,] # get the new values of tomorrow
    curentcapital <-updateIC(changeofcapital[k], pastvalues, newvalues, curre
ntweights) # updates the capital

    pastvalues <- newvalues # update the past values

    traindata <- rbind(traindata,newvalues)# add the observation to the train
data in the end
    rownames(traindata)[nrow(traindata)]<- datevalues[k] # add the date value
to that observation

    changeofcapital <- c(changeofcapital,curentcapital) # add the capital to
the cumulative capital vector fro plot in the future
  }
  new_list <- list(cumulativeweight,changeofcapital) # return the cumulative
weight and cumulative capital
  return(new_list)
}
```

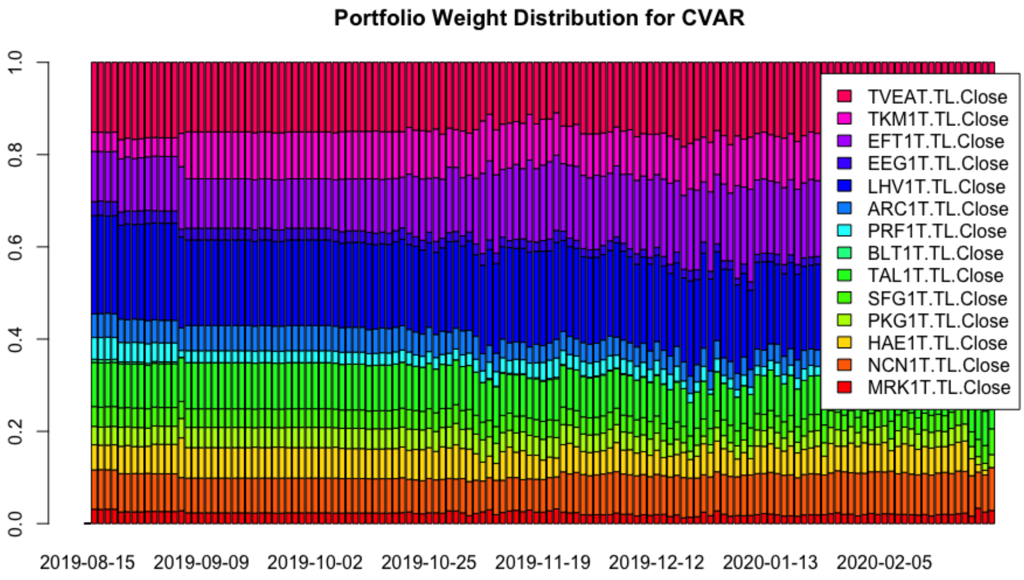
Evalutating the methods

```
set.seed(02011997)
IC=10000

Data <- Data2[,-ncol(Data2)]
traindata<- Data[1:(round(nrow(Data)*0.75)-1),]
testdata<- Data[round(nrow(Data)*0.75):nrow(Data),]

datevalues<- rownames(testdata)

capitalMK<- Capitalvector(traindata,testdata,MKportfolio,IC) # ccall for Var
portfolio
capital <- Capitalvector(traindata,testdata,CVARPortfolio,IC) # call for  CVa
r portfolio
capitalSDV <- Capitalvector(traindata,testdata,SDVportfolio,IC) # call for th
e Var portfolio
capitalLP <- Capitalvector(traindata,testdata,LPportfolio,IC) # call for LP p
ortfolio
```

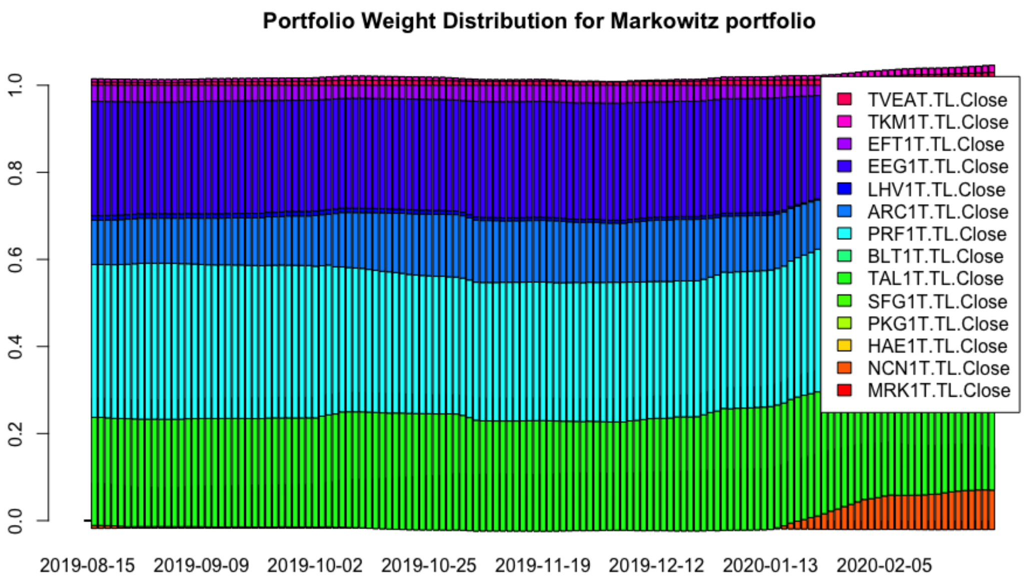
Plotting the results

```
plot_data<- capital[[1]]
Rainbow_colours = rainbow(ncol(as.zoo(plot_data)))
rownames(plot_data)<-c(rownames(traindata)[nrow(traindata)],rownames(testdata
```

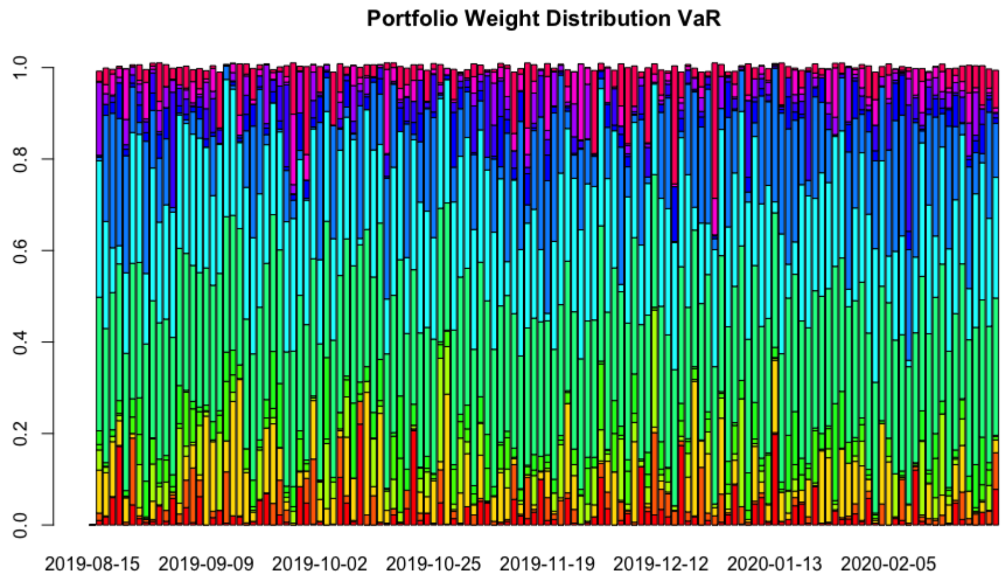
```
))
plot_data_xts <- xts(plot_data, as.Date(rownames(plot_data)))
barplot(plot_data_xts, main= "Portfolio Weight Distribution for CVAR", legend
= T,col=Rainbow_colours)
```



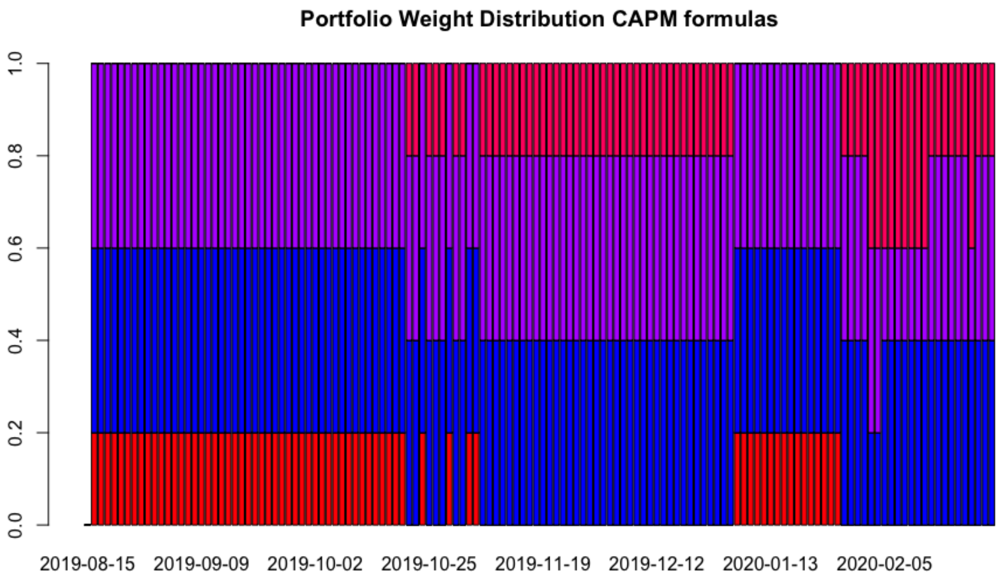

```
plot_data<- capitalMK[[1]]
Rainbow_colours = rainbow(ncol(as.zoo(plot_data)))
colnames(plot_data) <- tickers[-length(tickers)]
rownames(plot_data)<-c(rownames(traindata)[nrow(traindata)],rownames(testdata
))
plot_data_xts <- xts(plot_data, as.Date(rownames(plot_data)))
barplot(plot_data_xts, main= "Portfolio Weight Distribution for Markowitz por
tfolio", col=Rainbow_colours,legend=T)
```



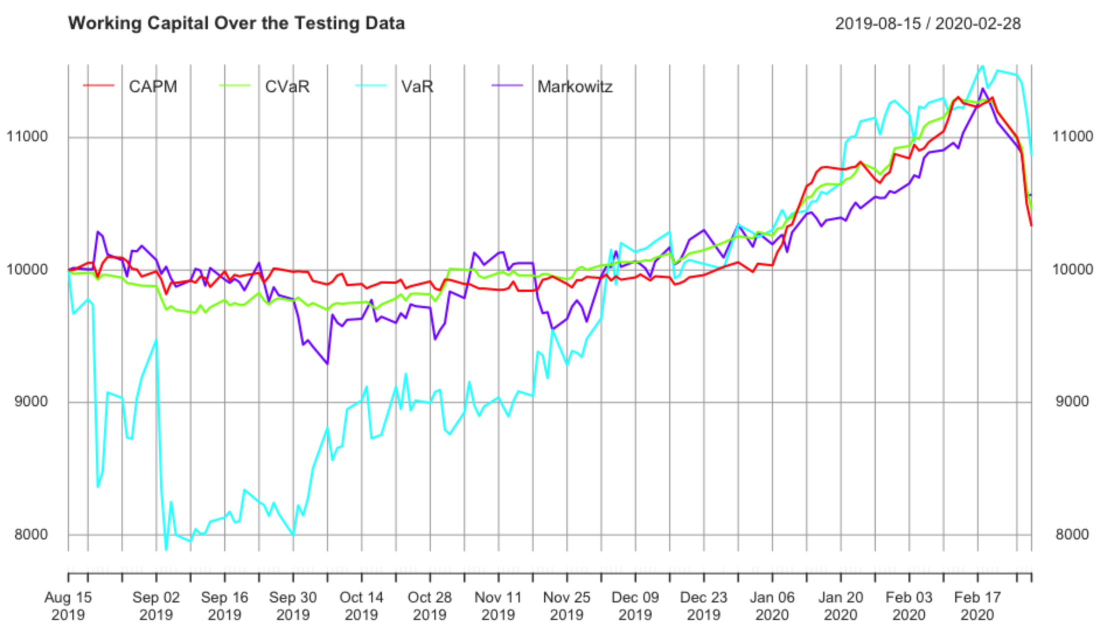
```
plot_data<- capitalSDV[[1]]
Rainbow_colours = rainbow(ncol(as.zoo(plot_data)))
colnames(plot_data) <- tickers[-length(tickers)]
rownames(plot_data)<-c(rownames(traindata)[nrow(traindata)],rownames(testdata
))
plot_data_xts <- xts(plot_data, as.Date(rownames(plot_data)))
barplot(plot_data_xts, main= "Portfolio Weight Distribution VaR", col=Rainbow
_colours)
```




```
plot_data<- capitalLP[[1]]
Rainbow_colours = rainbow(ncol(as.zoo(plot_data)))
colnames(plot_data) <- tickers[-length(tickers)]
rownames(plot_data)<-c(rownames(traindata)[nrow(traindata)],rownames(testdata
))
plot_data_xts <- xts(plot_data, as.Date(rownames(plot_data)))
barplot(plot_data_xts, main= "Portfolio Weight Distribution CAPM formulas", c
ol=Rainbow_colours)
```




```
plot_data<- cbind(capitalLP[[2]],capital[[2]],capitalSDV[[2]],capitalMK[[2]])
colnames(plot_data) <- c("CAPM","CVaR","VaR","Markowitz")
rownames(plot_data)<-c(rownames(traindata)[nrow(traindata)],rownames(testdata
))
Rainbow_colours = rainbow(ncol(as.zoo(plot_data)))
plot_data_xts <- xts(plot_data, as.Date(rownames(plot_data)))
invisible(plot.xts(plot_data_xts,main= "Working Capital Over the Testing Data
", ylab = "Capital", col=Rainbow_colours ))
addLegend("topleft",
          legend.names=colnames(plot_data_xts),
          lty=rep(1,4),
          cex=1,
          ncol = 4)
```



The return of the portfolios with respect to a buy and hold strategy.

```
CVaR_buyhold <- updateIC(IC, traindata[nrow(traindata),],testdata[nrow(testda
ta),],capital[[1]][2,])/IC-1
CVaR_end <- capital[[2]][136]/IC-1

MK_buyhold <- updateIC(IC, traindata[nrow(traindata),],testdata[nrow(testdata
),],capitalMK[[1]][2,])/IC-1
MK_end <- capitalMK[[2]][136]/IC-1

LP_buyhold <- updateIC(IC, traindata[nrow(traindata),],testdata[nrow(testdata
),],capitalLP[[1]][2,])/IC-1
LP_end <- capitalLP[[2]][136]/IC-1

VaR_buyhold <- updateIC(IC, traindata[nrow(traindata),],testdata[nrow(testdat
a),],capitalSDV[[1]][2,])/IC-1
VaR_end <- capitalSDV[[2]][136]/IC-1

print(paste0("The Markowitz portfolio return is:",MK_end*100,"%. The return f
rom a buy and hold strategy are: ",MK_buyhold*100,"%. The % ratio of how bett
er the dayly update is: ", ((MK_end-MK_buyhold)/MK_end)*100,"%" ))

## [1] "The Markowitz portfolio return is:5.70673221682472%. The return from
a buy and hold strategy are: 3.81950647403242%. The % ratio of how better the
dayly update is: 33.0701646947501%"

print(paste0("The VaR portfolio return is:",VaR_end*100,"%. The return from a
buy and hold strategy are: ",VaR_buyhold*100,"%. The % ratio of how better th
e dayly update is: ", ((VaR_end-VaR_buyhold)/VaR_end)*100,"%" ))

## [1] "The VaR portfolio return is:8.63883515202519%. The return from a buy
and hold strategy are: -20.342147483274%. The % ratio of how better the dayly
update is: 335.473268389722%"

print(paste0("The CVaR portfolio return is:",CVaR_end*100,"%. The return from
a buy and hold strategy are: ",MK_buyhold*100,"%. The % ratio of how better t
he dayly update is: ", ((CVaR_end-CVaR_buyhold)/CVaR_end)*100,"%" ))

## [1] "The CVaR portfolio return is:4.60993399185723%. The return from a buy
and hold strategy are: 3.81950647403242%. The % ratio of how better the dayly
update is: 36.761348940207%"

print(paste0("The CAPM portfolio return is:",LP_end*100,"%. The return from a
buy and hold strategy are: ",LP_buyhold*100,"%. The % ratio of how better the
dayly update is: ", ((LP_end-LP_buyhold)/MK_end)*100,"%" ))

## [1] "The CAPM portfolio return is:3.30871044294387%. The return from a buy
and hold strategy are: 4.49188841278705%. The % ratio of how better the dayly
update is: -20.7330206655729%"
```

NB: Please note that this is an R-markdown output that can be obtained from the .Rmd attached to this document.

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I, Omar Setihe

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Omar Setihe 18/08/2020