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Supervisors:

Senior Research Fellow Rainis Haller, PhD
University of Tartu
Tartu, Estonia

Research Fellow Johann Langemets, PhD
University of Tartu
Tartu, Estonia

Associate Professor Vegard Lima, Dr Scient.
University of Agder
Kristiansand, Norway

Opponents:

Professor Vladimir Kadets, DSc
V. N. Karazin Kharkiv National University
Kharkiv, Ukraine

Assistant Professor Antonín Procházka, PhD
University of Franche-Comté
Besançon, France

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Contents

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Chapter 1

Introduction

1.1 Background

The geometry of the unit ball plays an important role in understanding the nature of Banach spaces. Over the years, mathematicians have proposed various properties and indices to classify Banach spaces. This thesis is focused on investigating properties which describe how large certain subsets of the unit ball can be. Typical examples of such subsets are slices of the unit ball.

One of the most well-known and widely studied properties of Banach spaces is the Radon–Nikodým property. It is known that if a Banach space has the Radon–Nikodým property, then its unit ball has slices of arbitrarily small diameter. Since the diameter of the unit ball of any non-trivial Banach space is 2, the maximal diameter of a slice can at most be 2 as well.

In recent times, the so-called diameter 2 properties have been extensively studied (see, e.g. [ALN1], [HLP1], [BLRo]–[BLR8]). A Banach space has the local diameter 2 property if every slice of the unit ball has diameter 2. If the same holds for all non-empty relatively weakly open subsets of the unit ball or convex combinations of slices of the unit ball, then the space has the diameter 2 property or strong diameter 2 property, respectively. Thus diameter 2 properties are diametrically opposite to the Radon–Nikodým property. These diameter 2 properties were named by T. A. Abrahamsen, V. Lima and O. Nygaard in [ALN1]. Using a lemma by J. Bourgain, it is not difficult to see that the strong diameter 2 property implies the diameter 2 property. A Banach space has the strong diameter 2 property if and only if the norm of the dual space is octahedral. The norm of a Banach space is octahedral if for any finite number elements from the unit sphere we can find another norm one element whose distance from them is arbitrarily close to 2.

Even further from the Radon–Nikodým property lies the Daugavet property. The study of the Daugavet property was initiated in the 1960s by I. Daugavet in [Dau] who showed that the compact operators on $C[0,1]$ are orthogonal to the identity operator. Around the year 2000, a geometric characterisation of the Daugavet property in terms of slices was found and since then the Daugavet property has received extensive attention (see [Wer2] for an overview of the Daugavet property). In a space with the Daugavet property, if a slice contains a point on the unit sphere, then this point is the endpoint of a line segment in the slice of length as close to 2 as we want. Also, if a space has the Daugavet property, then both the space itself and its dual are octahedral. In particular, these spaces enjoy the following strengthening of the strong diameter 2 property. A space has the diametral strong diameter 2 property if for any element in a convex combination of non-empty relatively weakly open subsets there exists an element in the convex combination such that their distance is arbitrarily close to one plus the norm of the original element.

In [Whi], in the 1960s, R. Whitley introduced the thickness index of a Banach space. The thickness index of a space is 2 if and only if its norm is octahedral. In [Rue], A. Rueda Zoca brought the idea of the thickness index to the study of spaces with the Daugavet property by introducing the Daugavet index of thickness. The new index quantifies how close a Banach space is to having the Daugavet property and is also related to Whitley's thickness index. The Daugavet index of a space is the infimum of radii of balls with centre on the sphere that can cover a non-empty relatively weakly open subset of the unit ball. The Daugavet index of a Banach space is 2 if and only it enjoys the Daugavet property and is zero if the space has the Radon–Nikodým property.

Another strengthening of the strong diameter 2 property was first implicitly noted in [ALN1] but was named in [ANP]. The symmetric strong diameter 2 property states that given a finite number of slices of the unit ball, there exists a direction such that all these slices contain a line segment of length almost 2 in the given direction. The prototype of a space with this property is c_0. Every known example of a space with the symmetric strong diameter 2 property contains a copy of c_0. It is also noteworthy that the symmetric strong diameter 2 property, just as the Daugavet property, implies the strong diameter 2 property but they are independent of each other. All spaces with the Daugavet property contain a copy of ℓ_1.

Roughness as a property was introduced in the 1970s by E. B. Leach and J. H. M. Whitfield in [LW]. Roughness describes how far a norm is from being Fréchet differentiable. In the 1980s, in [Dev], R. Deville expanded on the notion of roughness by introducing the notion of average roughness. A Banach space is 2-average rough if and only if the norm is octahedral. It is known that if every slice of the unit ball has diameter at least δ then its dual is δ-rough and if every convex combination of slices of the unit ball has diameter at least δ then its dual is δ-average rough. Therefore roughness, as dual property, fills the gap between the Radon–Nikodým property and the diameter 2 properties.

1.2 Summary of the thesis

The aim of the thesis is to systematically study diameter 2 properties and related notions, such as roughness of the norm and the Daugavet index of thickness. We give a complete description of how different strong diameter 2 properties, rough norms and the Daugavet index of thickness behave on absolute norms. We study what criteria must a metric space meet in order for the space of Lipschitz functions on the metric space to have the w^*-symmetric strong diameter 2 property. We quantitatively generalise known roughness results for spaces of bounded linear operators. From our study of the Daugavet index of thickness, we give a negative answer to a question posed by Y. Ivakhno in 2006 about the relationship between the diameter and radius of slices.

The thesis has been organised into four chapters. Here, in the first chapter we give an overview of both the thesis and diameter 2 properties in general and also introduce notions that we will be using throughout the thesis, such as absolute normalised norms.

In the second chapter we study various strong diameter 2 properties, with an emphasis on the symmetric strong diameter 2 property. We study the strong diameter 2 property in absolute sums and then use our results to show that the diametral strong diameter 2 property is not stable for general absolute sums. After that we turn our focus to the symmetric strong diameter 2 property by giving a complete characterisation of its behaviour on absolute sums. We then study the property in regards to subspaces and we end the chapter by studying the w^*-symmetric strong diameter 2 property in the space of Lipschitz functions on metric spaces. This chapter is mainly based on [HLLN] and [HLN].

The third chapter is on rough norms in Banach spaces which is a quantitative expansion on the notion of octahedrality. We study how roughness behaves on absolute sums and give a complete characterisation of how octahedrality behaves on absolute sums. The chapter ends with the study of roughness in ultrapowers of Banach spaces and in spaces of operators. This chapter is mostly based on [HLN] and [Nad].

Finally, the fourth chapter expands on the Daugavet index of thickness introduced by A. Rueda Zoca by formulating new indices and studying their various characteristics. We also connect the indices to diameter 2 properties and quantitatively advance the study of the Daugavet property by describing the indices on absolute sums and isomorphic properties. We use the Daugavet indices to solve an open problem on the r-big slice property stated by Y. Ivakhno in 2006. This chapter is mainly based on [HLLNR].

The main results of the thesis are contained in [HLN], [HLLN], [HLLNR] and [Nad].

1.3 Notation

The notation used in this thesis is standard (see, e.g. [FHHMZ]).

We shall consider only Banach spaces over the real field. Mostly, the Banach spaces under consideration will be infinite-dimensional.

Given a Banach space X, we denote by B_X the closed unit ball of X and by S_X the unit sphere and by $B(x,r)$ the closed ball with centre at x and radius $r > 0$.

For Banach spaces X and Y, by $\mathcal{L}(X,Y)$ we mean the space of all bounded linear operators from X to Y. For any $T \in \mathcal{L}(X,Y)$ its corresponding adjoint operator will be denoted T^*. An operator $T \in \mathcal{L}(X,Y)$ is of finite rank if its range is finite-dimensional.

By X^* we denote the topological dual of X which is the space of all bounded linear functionals from X to $\mathbb{R}$ and by X^{**} we denote the bidual. A Banach space X will be regarded as a subspace of its bidual X^{**} by means of the canonical embedding.

Given a subset $A \subset X$, we denote by $\operatorname{span}(A)$ and $\operatorname{conv}(A)$ the linear span and convex hull of the elements in A, respectively. We also denote by $\overline{A}$ the closure of A, by $\operatorname{diam}(A)$ the diameter of A and by $\operatorname{ext}(A)$ as the set of all extreme points of A.

We assume the reader is familiar with well-known notions and results of Banach spaces (such as the Hahn–Banach theorem, the Principle of Local Reflexivity, the Krein–Milman theorem etc.) and sometimes we shall use them without proper reference.

1.4 Preliminaries

We will now present some basic notions and properties that will be used throughout the thesis.

Definition 1.1. Let X be a Banach space and A a non-empty bounded subset of X. A *slice* of A is a set of the form

$$S(A, x^*, \alpha) := \{x \in A \mid x^*(x) > \sup_{y \in A} x^*(y) - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. Often, when it does not cause confusion, we will denote $S(x^*, \alpha) := S(B_X, x^*, \alpha)$ and refer to slices of the unit ball simply as slices.

If X is a dual space, then slices of A whose defining functional comes from (the canonical image of) the predual of X are called w^*-slices of A.

For a Banach space X, a *convex combination of slices* of B_X is a set of the form

$$\sum_{i=1}^{n} \lambda_i S(B_X, x_i^*, \alpha_i),$$

where $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n > 0$, with $\sum_{i=1}^{n} \lambda_i = 1$.

Note that every slice is a non-empty relatively weakly open subset of the unit ball. On the other hand, J. Bourgain observed that every non-empty relatively weakly open subset of the unit ball contains a convex combination of slices.

Lemma 1.2 (Bourgain's lemma, cf. [GGMS, Lemma II.1])**.** *Let X be a Banach space. If U is a non-empty relatively weakly open subset of B_X, then there exist $n \in \mathbb{N}$, slices $S_1, \ldots, S_n$ of B_X and $\lambda_1, \ldots, \lambda_n > 0$, with $\sum_{i=1}^{n} \lambda_i = 1$, such that $\sum_{i=1}^{n} \lambda_i S_i \subset U$.*

Remark 1.1. Bourgain's lemma actually holds for any convex subset C of a convex compact subset K in a Hausdorff locally convex topological vector space. Thus an analogue of Lemma 1.2 for the relative w^*-topology holds as well.

In [ALN1], the following properties were formulated.

Definition 1.3 (see [ALN1, Definition 1.1])**.** It is said that a Banach space has

(a) the *local diameter 2 property* (LD2P) if every slice of the unit ball has diameter two;

(b) the *diameter 2 property* (D2P) if every non-empty relatively weakly open subset of the unit ball has diameter two;

(c) the *strong diameter 2 property* (SD2P) if every convex combination of slices of the unit ball has diameter two.

By Lemma 1.2 and the fact that slices are relatively weakly open, it is clear that the following implications hold.

$$\text{SD2P} \Longrightarrow \text{D2P} \Longrightarrow \text{LD2P}$$

In general, these implications are not reversible. It is known that, for example, $c_0 \oplus_2 c_0$ has the D2P (see [ALN1, Theorem 3.2]) but fails the SD2P (see [ABP, Theorem 3.2] and [HL, Theorem 1]). Also, in [BLR2, Theorem 2.4], a renorming of c_0 was constructed that possesses the LD2P but fails the D2P.

Example 1.4. Classical examples of Banach spaces that enjoy the SD2P include c_0, ℓ_∞, $C[0,1]$, $L_1[0,1]$ and $L_\infty[0,1]$. More generally, non-reflexive M-embedded spaces (see [Lop, Proposition 2.3]), infinite-dimensional uniform algebras (see [ALN1, Theorem 4.2]) and spaces with the Daugavet property (see [ALN1, Theorem 4.4]) all have the SD2P.

It is natural to wonder whether diameter 2 properties of a dual space remain the same properties if, instead of all slices or relatively weakly open subsets, one considers only w^*-slices or relatively w^*-open subsets. The following example shows us that, in general, it is not the case.

Example 1.5 (see [HLP1, Example 1.1])**.** Every convex combination of w^*-slices of $B_{C[0,1]^*}$ has diameter 2 but there are slices with arbitrarily small diameter.

The example above motivates the definition of another set of diameter 2 properties.

Definition 1.6 (see [BLR3, page 2] and [HLP1, Definition 1.1])**.** Let X be a Banach space. It is said that X^* has

(a) the *w^*-local diameter 2 property* (w^*-LD2P) if every w^*-slice of the unit ball has diameter two;

(b) the *w^*-diameter 2 property* (w^*-D2P) if every non-empty relatively w^*-open subset of the unit ball has diameter two;

(c) the *w^*-strong diameter 2 property* (w^*-SD2P) if every convex combination of w^*-slices of the unit ball has diameter two.

As with and for similar reasons as the regular diameter 2 properties, we have the following chain of implications for the w^*-versions of diameter 2 properties.

$$w^*\text{-SD2P} \Longrightarrow w^*\text{-D2P} \Longrightarrow w^*\text{-LD2P}$$

Again, the first implication is due to the w^*-version of Bourgain's lemma (see Remark 1.1) and the second implication is due to the fact that w^*-slices are relatively w^*-open subsets.

Remark 1.2. Note that the diameter 2 properties and the corresponding w^*-versions of diameter 2 properties are connected in a natural way. Indeed, a Banach space X has the LD2P (resp., D2P, SD2P) if and only if its bidual X^{**} has the w^*-LD2P (resp., w^*-D2P, w^*-SD2P).

In a dual space one has the following implications. However, by Remark 1.2 and by Example 1.5, it is not difficult to see that, in a dual Banach space, they are not generally reversible.

$$\begin{array}{ccccc} \text{SD2P} & \Longrightarrow & \text{D2P} & \Longrightarrow & \text{LD2P} \\ \Downarrow & & \Downarrow & & \Downarrow \\ w^*\text{-SD2P} & \Longrightarrow & w^*\text{-D2P} & \Longrightarrow & w^*\text{-LD2P} \end{array}$$

We now turn our attention to the dual property of the SD2P. The notion of an octahedral norm was introduced by G. Godefroy in 1989 in order to characterise Banach spaces containing an isomorphic copy of ℓ_1.

Definition 1.7 (see [God, page 12])**.** Let X be a Banach space. It is said that the norm of X is *octahedral* if for every finite-dimensional subspace $E \subset X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|x + y\| > (1 - \varepsilon)(\|x\| + \|y\|) \quad \text{for every } x \in E.$$

Where it does not cause confusion, we shall refer to the space X being octahedral instead of its norm.

In [HLP1], R. Haller, J. Langemets and M. Põldvere gave the following useful equivalent formulation of octahedral norms.

Proposition 1.8 (see [HLP1, Proposition 2.2])**.** *Let X be a Banach space. Then X is octahedral if and only if whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x_i + y\| \geq 2 - \varepsilon \quad \textit{for every } i \in \{1, \dots, n\}.$$

With the previous equivalent formulation of octahedrality in mind, it is not difficult to check that the classical Banach spaces ℓ_1, $C[0,1]$, $L_1[0,1]$ and $L_\infty[0,1]$ are octahedral.

It is known (see [God, Remark II.5 2)] and [Dev, Remark (c), page 119]) that octahedrality and the SD2P are connected in a dual sense. A proof of this fact can be found in [HLP1, Theorem. 2.2] and [BLR1, Theorem 2.1].

Theorem 1.9 (see [God, Remark II.5 2)] and [Dev, Remark (c), page 119])**.** *Let X be a Banach space. Then X is octahedral if and only if X^* has the w^*-SD2P.*

An immediate consequence of Theorem 1.9 combined with Remark 1.2 is that a Banach space has the SD2P if and only if its dual is octahedral.

A remarkable class of Banach spaces with the SD2P and being octahedral at the same time is the class of spaces with the Daugavet property. The study of such spaces was initiated in 1963 by I. Daugavet in [Dau].

Definition 1.10 (see, e.g. [Wer2, Definition 2.1])**.** A Banach space X is said to have the *Daugavet property* if for every rank-1 operator $T\colon X \to X$ the following equation holds

$$\|I + T\| = 1 + \|T\|, \tag{1.1}$$

where I is the identity operator on X.

Equation (1.1) is referred to as the *Daugavet equation.* Note that in the definition of the Daugavet property, equation (1.1) is required to hold for only rank-1 operators from the space onto itself. It is known that if a space enjoys the Daugavet property then equation (1.1) holds for all weakly compact operators from the space onto itself (see [KSSW, Theorem 2.3]).

Classical examples of Banach spaces with the Daugavet property include $C[0,1]$, $L_1[0,1]$ and $L_\infty[0,1]$.

The Daugavet property has a very useful geometric description that allows us to easily connect it to the diameter 2 properties.

Proposition 1.11 (see [Shv, Lemmas 2 and 3])**.** *Let X be a Banach space. The following statements are equivalent.*

(i) *X has the Daugavet property.*

(ii) *For every $x \in S_X$, $\varepsilon > 0$ and slice S of B_X, there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.*

(iii) *For every $x \in S_X$, $\varepsilon > 0$ and non-empty relatively weakly open subset W of B_X, there exists $y \in W$ such that $\|x - y\| \geq 2 - \varepsilon$.*

(iv) *For every $x \in S_X$, $\varepsilon > 0$ and convex combination C of non-empty relatively weakly open subsets of B_X, there exists $y \in C$ such that $\|x - y\| > 2 - \varepsilon$.*

(v) *For every $x^* \in S_{X^*}$, $\varepsilon > 0$ and w^*-slice S of B_{X^*}, there exists $y^* \in S$ such that $\|x^* - y^*\| \geq 2 - \varepsilon$.*

(vi) *For every* $x^* \in S_{X^*}$, $\varepsilon > 0$ *and non-empty relatively* w^**-open subset* W *of* B_{X^*} *there exists* $y^* \in W$ *such that* $\|x^* - y^*\| \geq 2 - \varepsilon$.

(vii) *For every* $x^* \in S_{X^*}$, $\varepsilon > 0$ *and convex combination* C *of non-empty relatively* w^**-open subsets of* B_{X^*}, *there exists* $y^* \in C$ *such that* $\|x^* - y^*\| > 2 - \varepsilon$.

From (iv) and (vii) of Proposition 1.11, it follows that if X is a Banach space with the Daugavet property, then X has the SD2P and X^* has the w^*-SD2P, which, by Theorem 1.9, means that X is octahedral.

Let us sum up the relations between the properties that we have introduced so far in the diagram below. Note that the implications from top to bottom hold only for a dual Banach space.

$$\begin{array}{ccccccc}
\text{Daugavet property} & \Longrightarrow & \text{SD2P} & \Longrightarrow & \text{D2P} & \Longrightarrow & \text{LD2P} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & w^*\text{-SD2P} & \Longrightarrow & w^*\text{-D2P} & \Longrightarrow & w^*\text{-LD2P}
\end{array}$$

Throughout the thesis, we require the notion of absolute normalised norms, which are generalisations of ℓ_p-norms.

We recall that a norm N on $\mathbb{R}^2$ is *absolute* (see [BD]) if

$$N(a,b) = N(|a|,|b|) \qquad \text{for all } (a,b) \in \mathbb{R}^2$$

and *normalised* if

$$N(1,0) = N(0,1) = 1.$$

For $1 \leq p \leq \infty$, we denote the ℓ_p-norm on $\mathbb{R}^2$ by $\|\cdot\|_p$. Every norm $\|\cdot\|_p$ is absolute and normalised.

Example 1.12. Let $\lambda \in (0,1)$. The norm $\|\cdot\| = \max\{\|\cdot\|_\infty, \lambda\|\cdot\|_1\}$ is an absolute normalised norm which is not an ℓ_p-norm.

Moreover, if N is an absolute normalised norm on $\mathbb{R}^2$ (see [BD, Lemmas 21.1 and 21.2]), then

$$\|\cdot\|_\infty \leq N(\cdot) \leq \|\cdot\|_1$$

and if $(a,b),(c,d) \in \mathbb{R}^2$ such that $|a| \leq |c|$ and $|b| \leq |d|$, then

$$N(a,b) \leq N(c,d).$$

If N is an absolute normalised norm on $\mathbb{R}^2$, then its dual norm N^* defined by

$$N^*(c,d) = \max_{N(a,b)\leq 1} (|ac| + |bd|) \qquad \text{for all } (c,d) \in \mathbb{R}^2$$

is also an absolute normalised norm on $\mathbb{R}^2$.

If X and Y are Banach spaces and N is an absolute normalised norm on $\mathbb{R}^2$, then by $X \oplus_N Y$ we denote the product space $X \times Y$ with respect to the norm

$$\|(x,y)\|_N = N(\|x\|, \|y\|) \qquad \text{for all } x \in X \text{ and } y \in Y.$$

In the special case where N is the ℓ_p-norm, we write $X \oplus_p Y$.

Chapter 2

Strong diameter 2 properties

In this chapter, we first describe how the strong diameter 2 property and diametral strong diameter 2 property behave on absolute sums. We then show that the symmetric strong diameter two property is only preserved by ℓ_∞-sums. Working with w^*-slices, we show that $\mathrm{Lip}_0(M)$ has the w^*-version of the symmetric strong diameter 2 property for several classes of metric spaces M. This chapter is mainly based on [HLLN] and [HLN].

2.1 Introduction and definitions

In this section, we introduce the symmetric strong diameter 2 property which is a strengthening of the SD2P. The property implicitly appeared first in [ALN1] but was named in [ANP].

Definition 2.1. A Banach space X has the *symmetric strong diameter 2 property* (SSD2P) if for every $n \in \mathbb{N}$, finite family $S_1, \ldots, S_n$ of slices of B_X and $\varepsilon > 0$, there exist $x_1 \in S_1, \ldots, x_n \in S_n$ and $y \in B_X$ such that $x_i \pm y \in S_i$ for every $i \in \{1, \ldots, n\}$ and $\|y\| > 1 - \varepsilon$.

It is straightforward to verify that the SSD2P is formally stronger than the SD2P.

Lemma 2.2 (see [ALN1, Lemma 4.1])**.** *Let X be a Banach space. If X has the SSD2P, then it also has the SD2P.*

However, the converse of the lemma above does not hold. The Banach space $L_1[0,1]$ has the SD2P but fails the SSD2P (see Remark 2.3 below).

In [ALL], another strengthening of the SD2P was introduced by Abrahamsen, Langemets and Lima.

Definition 2.3 (see [ALL]). Let X be a Banach space. It is said that X is *almost square* (ASQ) if for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$ there exists a sequence $(y_k) \subset S_X$ such that $\|x_i \pm y_k\| \to 1$, for every $i \in \{1, \dots, n\}$, and $y_k \to 0$ weakly, as $k \to \infty$.

The prototype of an ASQ space is c_0 because every ASQ space contains c_0 isomorphically (see [ALL, Theorem 2.4]). However, observe that, by considering the constant 1 sequence, it is clear that ℓ_∞ is not ASQ.

From Theorem 2.5 (v) it is clear that ASQ implies the SSD2P. On the other hand, ℓ_∞ is an example of a space that enjoys the SSD2P but fails to be ASQ (see Proposition 2.14 below). Therefore we have the following chain of implications.

$$\text{ASQ} \Longrightarrow \text{SSD2P} \Longrightarrow \text{SD2P}$$

The following classes of spaces have the SSD2P:

(a) Lindenstrauss spaces (this follows by inspecting the proof of [ALN2, Proposition 4.6]);

(b) uniform algebras (see [ALN1, Theorem 4.2]);

(c) ASQ-spaces, in particular, Banach spaces which are M-ideals in their bidual (see [ALL]);

(d) Banach spaces with an infinite-dimensional centraliser (this follows by inspecting the proof of [ABP, Proposition 3.3]);

(e) somewhat regular linear subspaces of $C_0(L)$, whenever L is an infinite locally compact Hausdorff space (see [ANP]);

(f) Müntz spaces (this follows by inspecting the proof of [ALMN, Theorem 2.5]).

All of the spaces listed above contain an almost isometric copy of c_0, which leads us to a natural question.

Question 2.4. *Does every space with the SSD2P contain an isomorphic copy of c_0?*

On the other hand, every Banach space containing an isomorphic copy of c_0 can be equivalently renormed to have the SSD2P, in fact even to be ASQ (see [BLR6, Theorem 2.3]).

Let us now give equivalent criteria for the SSD2P. Throughout the chapter, let $\mathcal{O}(x)$ denote the set of all relatively weakly open neighbourhoods of x in B_X.

Theorem 2.5. *Let X be a Banach space. The following statements are equivalent.*

(i) *The space X has the SSD2P.*

(ii) *Whenever $n \in \mathbb{N}$, $U_1, \ldots, U_n$ are non-empty relatively weakly open subsets of B_X and $\varepsilon > 0$, there exist $x_1 \in U_1, \ldots, x_n \in U_n$ and $y \in B_X$ such that $x_i \pm y \in U_i$ for every $i \in \{1, \ldots, n\}$ and $\|y\| > 1 - \varepsilon$.*

(iii) *Whenever $n \in \mathbb{N}$, $C_1, \ldots, C_n$ are finite convex combinations of slices of B_X and $\varepsilon > 0$, there exist $x_1 \in C_1, \ldots, x_n \in C_n$ and $y \in B_X$ such that $x_i \pm y \in C_i$ for every $i \in \{1, \ldots, n\}$ and $\|y\| > 1 - \varepsilon$.*

(iv) *Whenever $n \in \mathbb{N}$, $C_1, \ldots, C_n$ are finite convex combinations of non-empty relatively weakly open subsets of B_X and $\varepsilon > 0$, there exist $x_1 \in C_1, \ldots, x_n \in C_n$ and $y \in B_X$ such that $x_i \pm y \in C_i$ for every $i \in \{1, \ldots, n\}$ and $\|y\| > 1 - \varepsilon$.*

(v) *Whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, there exist nets $(y_\alpha^1), \ldots, (y_\alpha^n) \subset S_X$ and $(z_\alpha) \subset S_X$ such that $y_\alpha^i \to x_i$ weakly, $z_\alpha \to 0$ weakly and $\|y_\alpha^i \pm z_\alpha\| \to 1$ for every $i \in \{1, \ldots, n\}$.*

(vi) *Whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, $U_i \in \mathcal{O}(x_i)$, $i \in \{1, \ldots, n\}$, $V \in \mathcal{O}(0)$ and $\varepsilon > 0$, there exist $y_1 \in U_i \cap S_X, \ldots, y_n \in U_n \cap S_X$ and $z \in V \cap S_X$ such that $\|y_i \pm z\| \leq 1 + \varepsilon$.*

Proof. (i) $\Rightarrow$ (ii). Let $n \in \mathbb{N}$ and assume that $U_1, \ldots, U_n$ are non-empty relatively weakly open subsets of B_X and that $\varepsilon > 0$. By Lemma 1.2, each U_i contains a convex combination of slices, say $U_i \supset \sum_{j=1}^{n_i} \lambda_i^j S_i^j$, with $\sum_{j=1}^{n_i} \lambda_i^j = 1$ and $\lambda_i^j > 0$, for each $i \in \{1, \ldots, n\}$. We apply the definition of the SSD2P to the family of all S_i^j to find $x_i^j \in S_i^j$ and $y \in B_X$ such that $x_i^j \pm y \in S_i^j$ and $\|y\| > 1 - \varepsilon$. Set $w_i := \sum_{j=1}^{n_i} \lambda_i^j x_i^j$. Then

$$w_i \in \sum_{j=1}^{n_i} \lambda_i^j S_i^j \subset U_i$$

and

$$w_i \pm y = \sum_{j=1}^{n_i} \lambda_i^j (x_i^j \pm y) \in \sum_{j=1}^{n_i} \lambda_i^j S_i^j \subset U_i.$$

This shows (i) $\Rightarrow$ (ii). The same proof also gives (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv), while (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) are trivial.

(ii) $\Rightarrow$ (vi). Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $U_i \in \mathcal{O}(x_i)$, $i \in \{1, \dots, n\}$, $V \in \mathcal{O}(0)$, and $\varepsilon \in (0,1)$. By choosing $\delta \in (0, \varepsilon)$ small enough, there exist finite sets $A_i \subset S_{X^*}$ and $B \subset S_{X^*}$ such that

$$U_i \supset \bar{U}_i := \{x \in B_X \mid |x^*(x - x_i)| < \delta, x^* \in A_i\}$$

and

$$V \supset \bar{V} := \{x \in B_X \mid |x^*(x)| < \delta, x^* \in B\}.$$

Let

$$\bar{U}_i \supset \tilde{U}_i := \{x \in B_X \mid |x^*(x - x_i)| < \delta/2, x^* \in A_i\}$$

and

$$\bar{V} \supset \tilde{V} := \{x \in B_X \mid |x^*(x)| < \delta/2, x^* \in B\}.$$

For each $i \in \{1, \dots, n\}$, choose $x_i^* \in S_{X^*}$ such that $x_i^*(x_i) = 1$, and define $S_i := S(B_X, x_i^*, \delta/2)$. We apply (ii) to the relatively weakly open sets $W_i = S_i \cap \tilde{U}_i$ and $\tilde{V}$ and find $w_i \in W_i$ and $v \in \tilde{V}$ and $z \in B_X$ such that

$$w_i \pm z \in W_i, \quad v \pm z \in \tilde{V} \quad \text{and} \quad \|z\| > 1 - \frac{\delta}{2}.$$

Define $u_i := \frac{w_i}{\|w_i\|}$. Since $w_i \in S_i$, we get

$$\|w_i\| > 1 - \frac{\delta}{2} \quad \text{and} \quad \|u_i - w_i\| < \frac{\delta}{2}.$$

From this and $w_i \in \tilde{U}_i$ we have $u_i \in U_i$.

Next, note that $-(v \pm z) \in \tilde{V}$ hence $z = \frac{1}{2}(-v + z) + \frac{1}{2}(v + z) \in \tilde{V}$ by convexity. Since $\|z\| > 1 - \delta/2$, we get that $y := \frac{z}{\|z\|} \in V$.

Finally, note that

$$\|u_i \pm y\| \le \|w_i - u_i\| + \|w_i \pm z\| + \|z - y\| \le \frac{\delta}{2} + 1 + \frac{\delta}{2} < 1 + \varepsilon.$$

(v) $\Rightarrow$ (i). Let $n \in \mathbb{N}$, $S_1 := S(B_X, x_1^*, \alpha_1), \dots, S_n := S(B_X, x_n^*, \alpha_n)$ be slices of B_X and $\varepsilon \in (0,1)$. Find a $\delta > 0$ such that

$$\frac{1}{1+\delta} > 1 - \varepsilon \quad \text{and} \quad \frac{1 - 2\delta}{1 + \delta} > 1 - \alpha_i$$

for every $i \in \{1, \dots, n\}$. For every $i \in \{1, \dots, n\}$ choose an $x_i \in S_X \cap S(B_X, x_i^*, \delta)$. By (v) there are nets $(y_\alpha^i) \subset S_X$ and $(z_\alpha) \subset S_X$ such that $y_\alpha^i \to x_i$ weakly, $z_\alpha \to 0$ weakly and $\|y_\alpha^i \pm z_\alpha\| \to 1$ for every $i \in \{1, \dots, n\}$.

Find an index α_0 such that

$$y_{\alpha_0}^i \in S(B_X, x_i^*, \delta), \quad \|x_i^*(z_{\alpha_0})\| < \delta, \quad \text{and} \quad \max \|y_{\alpha_0}^i \pm z_{\alpha_0}\| \le 1 + \delta$$

for every $i \in \{1, \dots, n\}$. Finally, set $y_i := y_{\alpha_0}^i/(1+\delta)$ and $z := z_{\alpha_0}/(1+\delta)$. Then we have $y_i, y_i \pm z \in S_i$ and $z \in B_X$, with $\|z\| > 1 - \varepsilon$.

The implication (vi) $\Rightarrow$ (v) is straightforward. $\square$

2.2 Strong diameter 2 properties in absolute sums

In this section, we first characterise those absolute norms for which the absolute sum of two Banach spaces with the SD2P also has the SD2P. Recall that, by [ABP] and [HL], it is known that the only ℓ_p-norms which preserve the SD2P are the ℓ_1- and ℓ_∞-norms. However, there are many more absolute norms besides the ℓ_1- and ℓ_∞-norms that do preserve the SD2P. On the other hand, the diametral strong diameter 2 property is stable only for the ℓ_1- and ℓ_∞-norms, similar to the Daugavet property.

In order to characterise those absolute norms which preserve the SD2P, we introduce the following notion.

Definition 2.6. An element $(a,b) \in \mathbb{R}^2$ is called *positive* if $a \geq 0$ and $b \geq 0$. Let N be an absolute normalised norm on $\mathbb{R}^2$. We say that $\mathbb{R}^2$ has the *positive strong diameter 2 property* (positive SD2P) if whenever $n \in \mathbb{N}$, positive $f_1, \ldots, f_n \in S_{(\mathbb{R}^2, N^*)}$, $\alpha_1, \ldots, \alpha_n > 0$ and $\lambda_1, \ldots, \lambda_n \geq 0$, with $\sum_{i=1}^n \lambda_i = 1$, there are positive $(a_i, b_i) \in S(B_{(\mathbb{R}^2,N)}, f_i, \alpha_i)$, for every $i \in \{1, \ldots, n\}$, such that

$$N\Big(\sum_{i=1}^n \lambda_i (a_i, b_i)\Big) = 1.$$

Remark 2.1. Note that $(\mathbb{R}^2, N)$ has the positive SD2P if and only if there are $a, b \geq 0$ such that $N(a,1) = N(1,b) = 1$ and

$$N\Big(\frac{1}{2}(a,1) + \frac{1}{2}(1,b)\Big) = 1.$$

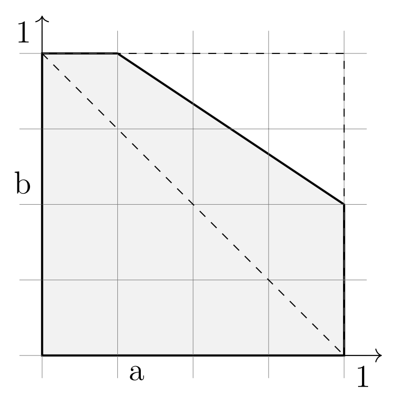

Figure 2.1: First quadrant of the unit ball of a $(\mathbb{R}^2, N)$ with the positive SD2P.

Theorem 2.7. *Let X and Y be Banach spaces with the SD2P and N an absolute normalised norm on $\mathbb{R}^2$. Then $X \oplus_N Y$ has the SD2P if and only if $(\mathbb{R}^2, N)$ has the positive SD2P.*

Proof. First, let us assume that $X \oplus_N Y$ has the SD2P. We will show that $(\mathbb{R}^2, N)$ has the positive SD2P. Let $n \in \mathbb{N}$, $(c_1, d_1), \ldots, (c_n, d_n)$ be positive elements in $S_{(\mathbb{R}^2, N^*)}$, $\alpha_1, \ldots, \alpha_n > 0$, $\lambda_1, \ldots, \lambda_n > 0$, with $\sum_{i=1}^n \lambda_i = 1$, and $\varepsilon > 0$. We will show that there are positive $(a_i, b_i) \in B_{(\mathbb{R}^2, N)}$ such that $c_i a_i + d_i b_i > 1 - \alpha_i$ and $N(\sum_{i=1}^n \lambda_i (a_i, b_i)) > 1 - \varepsilon$.

Let $(x_i^*, y_i^*) \in S_{X^* \oplus_{N^*} Y^*}$ be such that $\|x_i^*\| = c_i$ and $\|y_i^*\| = d_i$ for every $i \in \{1, \ldots, n\}$. Since $X \oplus_N Y$ has the SD2P, there are

$$(x_i, y_i) \in S(B_{X \oplus_N Y}, (x_i^*, y_i^*), \alpha_i)$$

such that $\| \sum_{i=1}^n \lambda_i (x_i, y_i) \|_N \geq 1 - \varepsilon$.

Take $(a_i, b_i) = (\|x_i\|, \|y_i\|)$. Then $c_i a_i + d_i b_i > 1 - \alpha_i$ because

$$c_i a_i + d_i b_i = \|x_i^*\| \|x_i\| + \|y_i^*\| \|y_i\| \geq x_i^*(x_i) + y_i^*(y_i) > 1 - \alpha_i$$

and

$$\begin{aligned} N\Big(\sum_{i=1}^n \lambda_i (a_i, b_i)\Big) &= N\Big(\sum_{i=1}^n \lambda_i \|x_i\|, \sum_{i=1}^n \lambda_i \|y_i\|\Big) \\ &\geq N\Big(\|\sum_{i=1}^n \lambda_i x_i\|, \|\sum_{i=1}^n \lambda_i y_i\|\Big) \\ &= \|\sum_{i=1}^n \lambda_i (x_i, y_i)\|_N \\ &\geq 1 - \varepsilon. \end{aligned}$$

Now assume that $(\mathbb{R}^2, N)$ has the positive SD2P. We use an idea from [HPP]. Let $n \in \mathbb{N}$, $S_1, \ldots, S_n$ be slices of $B_{X \oplus_N Y}$ defined by norm one functionals (x_i^*, y_i^*) and scalars $\alpha_i > 0$. Let $\lambda_i > 0$ be such that $\sum_{i=1}^n \lambda_i = 1$. We will show that the diameter of $\sum_{i=1}^n \lambda_i S_i$ is 2.

Let $\varepsilon > 0$. Consider slices $S_i^X = S(B_X, \frac{x_i^*}{\|x_i^*\|}, \frac{\alpha_i}{2})$ and $S_i^Y = S(B_Y, \frac{y_i^*}{\|y_i^*\|}, \frac{\alpha_i}{2})$. If $x_i^* = 0$, then $S_i^X = B_X$ and if $y_i^* = 0$, then $S_i^Y = B_Y$.

Since $(\mathbb{R}^2, N)$ has the positive SD2P, there are positive elements $(a_i, b_i) \in S(B_{(\mathbb{R}^2, N)}, (\|x_i^*\|, \|y_i^*\|), \delta)$ such that $N\Big(\sum_{i=1}^n \lambda_i (a_i, b_i)\Big) > 1 - \delta$, where $\delta > 0$ satisfies $(1 - \delta)(1 - \alpha_i/2) \geq 1 - \alpha_i$ for every $i \in \{1, \ldots, n\}$.

It turns out that $a_i S_i^X \times b_i S_i^Y \subset S_i$. Indeed, if $x \in S_i^X$ and $y \in S_i^Y$, then

$$\|(a_i x, b_i y)\|_N = N(a_i \|x\|, b_i \|y\|) \leq N(a_i, b_i) \leq 1$$

and
$$a_i x_i^*(x) + b_i y_i^*(y) > (1-\delta)(1-\frac{\alpha_i}{2}) \geq 1-\alpha_i.$$
Denote
$$a = \sum_{i=1}^n \lambda_i a_i \qquad \text{and} \qquad b = \sum_{i=1}^n \lambda_i b_i.$$
Suppose that $a \neq 0$ and $b \neq 0$. For every $i \in \{1, \dots, n\}$, denote
$$\mu_i = \frac{\lambda_i a_i}{a} \qquad \text{and} \qquad \nu_i = \frac{\lambda_i b_i}{b}.$$
As X and Y have the SD2P, then there are $\widehat{x}, \widehat{u} \in \sum_{i=1}^n \mu_i S_i^X$ and $\widehat{y}, \widehat{v} \in \sum_{i=1}^n \nu_i S_i^Y$ such that $\|\widehat{x} - \widehat{u}\| \geq 2-\varepsilon$ and $\|\widehat{y} - \widehat{v}\| \geq 2-\varepsilon$. Take $x = a\widehat{x}$, $y = b\widehat{y}$, $u = a\widehat{u}$ and $v = b\widehat{v}$. Then $(x,y),(u,v) \in \sum_{i=1}^n \lambda_i S_i$ because $x, u \in \sum_{i=1}^n \lambda_i a_i S_i^X$ and $y, v \in \sum_{i=1}^n \lambda_i b_i S_i^Y$. Finally,
$$\begin{aligned}\|(x,y)-(u,v)\|_N &= N(\|x-u\|, \|y-v\|) \\ &\geq (2-\varepsilon)N(a,b) \\ &> (2-\varepsilon)(1-\delta).\end{aligned}$$
Consider now the case where $a = 0$ or $b = 0$. Assume, without loss of generality, that $a = 0$. Since
$$\{0\} \times S_i^Y \subset S_i,$$
then
$$\{0\} \times \sum_{i=1}^n \lambda_i S_i^Y \subset \sum_{i=1}^n \lambda_i S_i.$$
As the diameter of $\sum_{i=1}^n \lambda_i S_i^Y$ is 2, there are $y, v \in \sum_{i=1}^n \lambda_i S_i^Y$ such that
$$\|y-v\| \geq 2-\varepsilon.$$
Thus $(0,y),(0,v) \in \sum_{i=1}^n \lambda_i S_i$. Now we have
$$\begin{aligned}\|(0,y)-(0,v)\|_N &= N(0, \|y-v\|) \\ &= \|y-v\| \\ &\geq 2-\varepsilon.\end{aligned}$$
□

In [BLR7], the diameter 2 properties were strengthened by requiring the abundance of diametral points, which does hold for Banach spaces with the Daugavet property. Here we will only consider the diametral strong diameter 2 property, which is the strengthening of the SD2P.

Definition 2.8 (see [BLR7, Definition 3.1])**.** Let X be a Banach space. We say that X has the *diametral strong diameter 2 property* (DSD2P) if given a convex combination of non-empty relatively weakly open subsets C of B_X, $x \in C$ and $\varepsilon > 0$, then there exists $y \in C$ such that $\|x - y\| > 1 + \|x\| - \varepsilon$.

Observe that the following chain of implications holds.

$$\text{Daugavet property} \Longrightarrow \text{DSD2P} \Longrightarrow \text{SD2P}$$

Indeed, by Lemma 1.2, it is not difficult to see that DSD2P implies the SD2P. However, the converse fails, as c_0 has the SD2P but fails to have the DSD2P. This is easily seen from [BLR7, Proposition 2.9] and the fact that $c_0^* = \ell_1$ has the RNP. Also, Banach spaces with the Daugavet property have the DSD2P (see [BLR7, Example 3.3]). Whether the converse holds seems to be unknown (see [BLR7, Question 4.1]).

Question 2.9. *Does there exist a Banach space X with the DSD2P that fails the Daugavet property?*

The only ℓ_p-norms, for $1 \leq p \leq \infty$, preserving the DSD2P are the ℓ_1- and ℓ_∞-norms (see [BLR7, Theorem 3.7] and [HPP, Theorem]). Since the SD2P is stable for absolute sums with the positive SD2P, one could wonder whether there are more absolute sums, which preserve the DSD2P and thus answer Question 2.9 negatively.

Since the DSD2P implies the SD2P and the latter is stable only for absolute norms with the positive SD2P (see Theorem 2.7 above), we can restrict our attention to them. Consider an absolute normalised norm N on $\mathbb{R}^2$, which differs from the ℓ_1-norm and ℓ_∞-norm, such that $(\mathbb{R}^2, N)$ has the positive SD2P. Thus for some $a, b \in [0, 1)$, with $a > 0$ or $b > 0$, N is defined by

$$N(c, d) = \max\left\{|c|, |d|, \frac{(1-b)|c| + (1-a)|d|}{1 - ab}\right\} \qquad \text{for all } (c, d) \in \mathbb{R}^2. \tag{2.1}$$

We will show that a direct sum with an absolute normalised norm N cannot have the DSD2P if N differs from the ℓ_1- or ℓ_∞-norms. To that end we require the following elementary lemma.

Lemma 2.10. *Let N be an absolute normalised norm defined by (2.1). Then there is a $\lambda \in (0, 1)$ such that*

$$N\Big(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b\Big) < 1 + N(\lambda, 1 - \lambda).$$

Proof. Assume that $\lambda \in (0,1)$. Denote

$$c = 2\lambda + (1-\lambda)a \qquad \text{and} \qquad d = 2(1-\lambda) + \lambda b.$$

It is straightforward to directly show that the condition

$$N(c,d) = \frac{(1-b)c + (1-a)d}{1-ab}$$

is equivalent to

$$\frac{a}{2+a-ab} \le \lambda \le \frac{2-ab}{2+b-ab},$$

and the condition

$$\frac{(1-b)c + (1-a)d}{1-ab} < 1 + N(\lambda, 1-\lambda)$$

is equivalent to

$$\lambda < \frac{a}{1+a} \qquad \text{or} \qquad \lambda > \frac{1}{1+b}.$$

Note that

$$\frac{a}{2+a-ab} \le \frac{a}{1+a} \le \frac{1}{1+b} \le \frac{2-ab}{2+b-ab},$$

where the first inequality is strict if and only if $a \neq 0$ and the last inequality is strict if and only if $b \neq 0$. □

Proposition 2.11. *Let X and Y be Banach spaces and N defined by (2.1). Then $X \oplus_N Y$ does not have the DSD2P.*

Proof. By Lemma 2.10, we choose $\lambda \in (0,1)$ such that

$$N\Big(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b\Big) < 1 + N(\lambda, (1-\lambda).$$

Denote

$$\delta = 1 + N(\lambda, 1-\lambda) - N\Big(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b\Big).$$

Choose any $\varepsilon \in (0, \delta/2)$. Let $\alpha > 0$ be such that if $(c_1, d_1), (c_2, d_2) \in \mathbb{R}^2$ satisfy the conditions $N(c_1, d_1), N(c_2, d_2) \le 1$, $|c_1| > 1-\alpha$ and $|d_2| > 1-\alpha$, then

$$N\Big(2\lambda + (1-\lambda)|c_2|, 2(1-\lambda) + \lambda|d_1|\Big) \le N\Big(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b\Big) + \varepsilon.$$

Fix any $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. Consider slices $S_1 = S(B_{X\oplus_N Y}, (x^*, 0), \alpha)$ and $S_2 = S(B_{X\oplus_N Y}, (0, y^*), \alpha)$. Choose $x \in S_X$ and $y \in S_Y$ such that $(x, 0) \in S_1$ and $(0, y) \in S_2$. Assuming that the Banach space $X \oplus_N Y$ has the DSD2P, there exist $(u_1, v_1) \in S_1$ and $(u_2, v_2) \in S_2$ such that

$$\begin{aligned}\tilde{N} := \|\lambda(x,0) + (1-\lambda)(0,y) - \lambda(u_1, v_1) - (1-\lambda)(u_2, v_2)\|_N \\ \geq \|\lambda(x,0) + (1-\lambda)(0,y)\|_N + 1 - \varepsilon.\end{aligned}$$

Since

$$\begin{aligned}\tilde{N} &= N\Big(\|\lambda x - \lambda u_1 - (1-\lambda)u_2\|, \|(1-\lambda)y - \lambda v_1 - (1-\lambda)v_2\|\Big) \leq \\ &\leq N\Big(2\lambda + (1-\lambda)\|u_2\|, 2(1-\lambda) + \lambda\|v_1\|\Big) \leq \\ &\leq N\Big(2\lambda + (1-\lambda)a, 2(1-\lambda) + \lambda b\Big) + \varepsilon = \\ &= 1 + N(\lambda, 1-\lambda) - \delta + \varepsilon,\end{aligned}$$

it follows that

$$\|\lambda(x,0) + (1-\lambda)(0,y)\|_N + 1 - \varepsilon \leq 1 + N(\lambda, 1-\lambda) - \delta + \varepsilon,$$

i.e. $\delta \leq 2\varepsilon$, which is a contradiction. □

Combining [BLR7, Theorem 3.8], [HPP, Theorem] and Proposition 2.11, we get the following corollary.

Corollary 2.12. *If $Z = X \oplus_N Y$ has the DSD2P, then either $Z = X \oplus_1 Y$ or $Z = X \oplus_\infty Y$.*

Due to Corollary 2.12, absolute sums can not be used to show that the Daugavet property and DSD2P are different properties and thus Question 2.9 remains open. On the other hand, Corollary 2.12 does easily show that the DSD2P and SD2P are indeed different.

2.3 SSD2P in absolute sums

We now turn our attention back to the SSD2P. We prove that, among absolute norms, the SSD2P is only preserved by the ℓ_∞-norm.

Theorem 2.13. *Let X and Y be Banach spaces.*

(a) *The Banach space $X \oplus_\infty Y$ has the SSD2P if and only if X or Y has the SSD2P.*

(b) *If N is an absolute normalised norm different from the ℓ_∞-norm, then $X \oplus_N Y$ does not have the SSD2P.*

Proof. (a). Assume first that X has the SSD2P and denote $Z := X \oplus_\infty Y$. Let $n \in \mathbb{N}$ and for every $i \in \{1, \dots, n\}$ let W_i be a non-empty relatively weakly open subset of B_Z containing an element (u_i, v_i) and $\varepsilon > 0$. Find non-empty relatively weakly open subsets $U_i \subset B_X$ and $V_i \subset B_Y$ such that

$$(u_i, v_i) \in U_i \times V_i \subset W_i.$$

Since X has the SSD2P, by Theorem 2.5 (ii), we can find $x_i \in U_i$ and $x \in B_X$ such that $x_i, x_i \pm x \in U_i$ for every $i \in \{1, \dots, n\}$ and $\|x\| > 1 - \varepsilon$. Set $z_i = (x_i, v_i)$ and $z = (x, 0)$. Then $z_i, z_i \pm z \in W_i$ and $\|z\| > 1 - \varepsilon$, which completes the proof.

Assume now that $X \oplus_\infty Y$ has the SSD2P. Suppose for contradiction that X and Y both fail to have the SSD2P.

Since X fails the SSD2P, there are $n \in \mathbb{N}$, non-empty relatively weakly open subsets $U_1, \dots, U_n$ of B_X and an $\varepsilon > 0$ such that for every $x_i \in U_i$ and for every $x \in B_X$, with $\|x\| > 1 - \varepsilon$, there is an index i_0 such that $x_{i_0} + x \notin U_{i_0}$ or $x_{i_0} - x \notin U_{i_0}$. Also, there are $m \in \mathbb{N}$, non-empty relatively weakly open subsets $V_1, \dots, V_m \in B_Y$ and $\delta > 0$ such that for every $y_j \in V_j$ and for every $y \in B_Y$, with $\|y\| > 1 - \delta$, there is an index j_0 such that $y_{j_0} + y \notin V_{j_0}$ or $y_{j_0} - y \notin V_{j_0}$.

Set $W_{ij} := U_i \times V_j$ for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Then each W_{ij} is a non-empty relatively open subset of $B_{X \oplus_\infty Y}$ and, by our assumption, there should be $(x_{i_0}, y_{j_0}) \in W_{i_0 j_0}$ and $(x, y) \in B_Z$ such that $(x_{i_0}, y_{j_0}) \pm (x, y) \in W_{i_0 j_0}$ and $\|(x, y)\| > 1 - \max\{\delta, \varepsilon\}$, which is impossible.

(b). Denote $Z := X \oplus_N Y$. Note that $N(1,1) > 1$ because N differs from the ℓ_∞-norm. Let $a \in (0,1)$ be such that $N(a, a) = 1$. Since $N(a, 1) > 1$ and $N(1, a) > 1$, there is a $\delta > 0$ such that if $N(u, v) \le 1$ and $u > 1 - \delta$, then $v < a - \delta$ or if $v > 1 - \delta$, then $u < a - \delta$. Fix an $\varepsilon > 0$ with $a - \delta \le (1 - \varepsilon)a$.

Consider slices $S_1 := S(B_Z, (x^*, 0), \delta)$ and $S_2 := S(B_Z, (0, y^*), \delta)$. Suppose for contradiction that Z has the SSD2P. Then there are $z_1 = (x_1, y_1) \in S_1$, $z_2 = (x_2, y_2) \in S_2$ and $w = (u, v) \in B_Z$ such that

$$z_1 \pm w \in S_1, \quad z_2 \pm w \in S_2 \quad \text{and} \quad \|w\| > 1 - \varepsilon.$$

Therefore $(x^*, 0)(z_1 \pm w) = x^*(x \pm u) > 1 - \delta$, which implies that $\|x_1 \pm u\| > 1 - \delta$. Similarly we have that $\|y_2 \pm v\| > 1 - \delta$. Hence $\|y_1 \pm v\| < a - \delta$ and $\|x_2 \pm u\| < a - \delta$. Now we see that

$$\|v\| \le \frac{1}{2}(\|y_1 + v\| + \|y_1 - v\|) < a - \delta.$$

Similarly, one has that $\|u\| < a - \delta$. Thus

$$\begin{aligned} 1 - \varepsilon < \|w\| &= N(\|u\|, \|v\|) \\ &\le N(a - \delta, a - \delta) \\ &\le N((1 - \varepsilon)a, (1 - \varepsilon)a) \\ &= (1 - \varepsilon)N(a, a) \\ &= 1 - \varepsilon, \end{aligned}$$

a contradiction. □

Remark 2.2. Note that Theorem 2.13 implies that ASQ is also preserved only by ℓ_∞-sums.

Remark 2.3. Since $L_1[0,1] = L_1[0, \frac{1}{2}] \oplus_1 L_1[\frac{1}{2}, 1]$, we immediately get, from Theorem 2.13, that $L_1[0,1]$ does not have the SSD2P.

If $(X_n)_{n=1}^\infty$ is a sequence of Banach spaces, then let us denote by $\ell_\infty(X_n)$ the Banach space of all bounded sequences $(x_n)_{n=1}^\infty$, where $x_n \in X_n$, with norm $\|(x_n)\| = \sup_n \|x_n\|$, and the c_0-sum by $c_0(X_n)$ with the same norm. Recall that $c_0(X_n)$ is always ASQ (see [ALL, Example 3.1]).

Proposition 2.14. *Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces. Then $\ell_\infty(X_n)$ has the SSD2P.*

Proof. Denote $Z := \ell_\infty(X_n)$ and $Z_0 := c_0(X_n)$. Let $P_k \colon Z \to Z_0$, $z = (x_n) \mapsto P_k(z) = (0, \ldots, 0, x_k, 0, \ldots)$. Observe that $(P_k(z))_{k=1}^\infty$ is a weakly null sequence in Z because it is a weakly null sequence in the subspace Z_0, where $Z_0^* = \ell_1(X_n^*)$.

Let $n \in \mathbb{N}$, $z_1, \ldots, z_m \in S_Z$. Choose $u = (u_n)_{n=1}^\infty \in S_Z$ such that $\|u_n\| = 1$ for every $n \in \mathbb{N}$. Define

$$y_k^i := z_i - P_k(z_i) \quad \text{and} \quad w_k := P_k(u).$$

Then $y_k^i \to z_i$ weakly and $w_k \to 0$ weakly, since both $(P_k(z_i))_{k=1}^\infty$ and $(P_k(u))_{k=1}^\infty$ are weakly null. By definition $\|y_k^i \pm w_k\| = 1$. From Theorem 2.5 (v), we see that Z has the SSD2P. □

2.4 SSD2P in ideals and subspaces

We start this section by proving that the SSD2P can be characterised in terms of separable ai-ideals. Then we show that the SSD2P can always be lifted from an M-ideal to the superspace. We end this section by showing that the SSD2P passes down to finite co-dimensional subspaces.

Recall that a linear operator $\varphi\colon Y^* \to X^*$ is called a *Hahn–Banach extension operator* if $\varphi(y^*)(y) = y^*(y)$ and $\|\varphi(y^*)\| = \|y^*\|$ for every $y \in Y$ and $y^* \in Y^*$.

Let us introduce the notion of ai-ideals of Banach spaces. Recall that the Principle of Local Reflexivity says that finite-dimensional subspaces of X^{**} are locally complemented in X by almost isometries (see, e.g. [FHHMZ, Theorem 6.3]). Almost isometric ideals extend this to arbitrary subspaces. We will use the characterisation of ai-ideals from [ALN2, Theorem 1.4] as our definition.

Definition 2.15 (see [ALN2, Theorem 1.4])**.** Let X be a Banach space. A subspace Y of X is called an *almost isometric ideal* (ai-ideal) in X if there exists a Hahn–Banach operator $\varphi\colon Y^* \to X^*$ such that for every $\varepsilon > 0$, finite-dimensional subspace $E \subset X$ and finite-dimensional subspace $F \subset Y^*$ there exists $T\colon E \to Y$ which satisfies

(a) $Te = e$ for every $e \in Y \cap E$,

(b) $(1+\varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1+\varepsilon)\|e\|$ for every $e \in E$,

(c) $\varphi f^*(e) = f^*(Te)$ for every $e \in E$ and $f^* \in F$.

Notice that the Principle of Local Reflexivity states the every Banach space X is an ai-ideal in its bidual X^{**}.

It is known that the SD2P can be characterised by its separable ai-ideals (see [Abr, Proposition 3.2]). Let us now show that the SSD2P can also be described in such a manner.

Proposition 2.16. *Let X be Banach space and Y be its closed subspace. If X has the SSD2P and Y is an ai-ideal in X, then Y has the SSD2P.*

Proof. Suppose that X has the SSD2P and let $\varphi\colon Y^* \to X^*$ be a Hahn–Banach extension operator from Definition 2.15. Let $n \in \mathbb{N}$, $y_1, \dots, y_n \in S_Y$, $U_i \in \mathcal{O}(y_i)$ and $V \in \mathcal{O}(0)$ in Y and $\varepsilon > 0$.

Let $\delta > 0$ be such that $(1+\delta)^2 + \delta < 1 + \varepsilon$. By choosing δ even smaller if necessary there exist finite sets $A_i \subset S_{Y^*}$ and $B \subset S_{Y^*}$ such that

$$U_i \supset \bar{U}_i := \{y \in B_Y \mid |y^*(y - y_i)| < \delta, y^* \in A_i\}$$

and

$$V \supset \bar{V} := \{y \in B_Y \mid |y^*(y)| < \delta, y^* \in B\}.$$

Define corresponding neighbourhoods in X by

$$\tilde{U}_i := \{x \in B_X \mid |\varphi(y^*)(x - y_i)| < \frac{\delta}{2}, y^* \in A_i\}$$

and
$$\tilde{V} := \{x \in B_X \mid |\varphi(y^*)(x)| < \frac{\delta}{2}, y^* \in B\}.$$

By Theorem 2.5 (vi), there exist $x_i \in \tilde{U}_i \cap S_X$ and $z \in \tilde{V} \cap S_X$ such that $\|x_i \pm z\| \leq 1 + \delta$.

Define $E := \operatorname{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\} \subset X$ and $F := \operatorname{span}(A_1 \cup \ldots \cup A_n \cup B) \subset Y^*$. Both E and F are finite-dimensional. Since Y is an ai-ideal in X, there exists a bounded linear operator $T\colon E \to Y$ such that

(a) $Te = e$ for every $e \in E \cap Y$,

(b) $(1 - \frac{\delta}{2})\|e\| \leq \|Te\| \leq (1 + \frac{\delta}{2})\|e\|$ for every $e \in E$,

(c) $\varphi(f)(e) = f(Te)$ for every $e \in E$ and $f \in F$.

Define $u_i := Tx_i/\|Tx_i\|$ and $v := Tz/\|Tz\|$. Then $\|u_i - Tx_i\| \leq \delta/2$ and $\|v - Tz\| \leq \delta/2$ hence

$$\begin{aligned} |y^*(u_i - y_i)| &\leq |y^*(Tx_i - y_i)| + \frac{\delta}{2} = |y^*(T(x_i - y_i))| + \frac{\delta}{2} \\ &= |\varphi(y^*)(x_i - y_i)| + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

and similarly $|y^*(v)| < \delta$. This means that $u_i \in U_i \cap S_Y$, for every $i \in \{1, \ldots, n\}$, and $v \in V \cap S_Y$. Finally,

$$\begin{aligned} \|u_i \pm v\| &= \|Tx_i - u_i\| + \|T(x_i \pm z)\| + \|Tz - w\| \\ &\leq \frac{\delta}{2} + (1 + \frac{\delta}{2})\|x_i \pm z\| + \frac{\delta}{2} \leq (1 + \delta)^2 + \delta \leq 1 + \varepsilon. \end{aligned}$$

From Theorem 2.5 (vi) we get that Y has the SSD2P. □

Corollary 2.17. *Let X be a Banach space. If X^{**} has the SSD2P, then X also has the SSD2P.*

We now turn our attention to the converse of Proposition 2.16.

Proposition 2.18. *Let X be a Banach space. If every infinite-dimensional separable ai-ideal of X has the SSD2P, then X has the SSD2P.*

Proof. Suppose that every infinite-dimensional separable ai-ideal of X has the SSD2P.

Let $n \in \mathbb{N}$, $S_1, \ldots, S_n$ be slices of B_X, where $S_i = S(x_i^*, \alpha_i)$ for each $i \in \{1, \ldots, n\}$ and $\varepsilon > 0$. By [Abr, Theorem 1.5] find a separable ai-ideal Y in X such that $\operatorname{span}(x_1^*, \ldots, x_n^*) \subset \varphi(Y^*)$, where $\varphi\colon Y^* \to X^*$ is the almost isometric Hahn–Banach extension operator. Now, for each $i \in \{1, \dots, n\}$ find $y_i^* \in X^*$ such that $y_i^* = \varphi(x_i^*)$. Define $\tilde{S}_i = S(B_Y, y_i^*, \alpha_i)$. Choose $y \in \tilde{S}_i$ and note that due to $x_i^*(y) = \varphi(y_i^*)(y) = y_i^*(y) > 1 - \alpha_i$ we have $\tilde{S}_i \subset S_i$.

Since Y has the SSD2P, there exist $x_i \in \tilde{S}_i \subset S_i$ and $y \in B_Y \subset B_X$, such that $x_i \pm y \in \tilde{S}_i \subset S_i$ for every $i \in \{1, \dots, n\}$ and $\|y\| > 1 - \varepsilon$. We can thus conclude that X has the SSD2P. □

Combining Propositions 2.16 and 2.18, we immediately have the following characterisation of the SSD2P.

Proposition 2.19. *Let X be a Banach space. Then X has the SSD2P if and only if every infinite-dimensional separable ai-ideal of X has the SSD2P.*

Let us turn to M-ideals and let us show that the SSD2P can be lifted from an M-ideal to the superspace.

Definition 2.20 (see [HWW, Definition 1.1])**.** Let X be a Banach space and $Y \subset X$ a closed subspace. It is said that the subspace Y is an *M-ideal in X* if there exists a norm one projection $P\colon X^* \to X^*$ with $\ker P = Y^\perp = \{x^* \in X^* \mid x^*|_Y = 0\}$ and

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \quad \text{for every } x^* \in X^*.$$

Example 2.21 (see, e.g. [HWW, page 4])**.** The sequence space c_0 is an M-ideal in its bidual ℓ_∞.

In [HL], it was shown that if Y is an M-ideal in a Banach space X and Y has the SD2P, then X has the SD2P. A similar statement holds for the SSD2P as well.

Proposition 2.22. *Let Y be a proper closed subspace of a Banach space X. If Y is an M-ideal in X and Y has the SSD2P, then X has the SSD2P.*

Proof. The proof is modelled on the proof of [HL, Proposition 3]. Let $n \in \mathbb{N}$, $S_i := S(B_X, x_i^*, \alpha_i)$, $i \in \{1, \dots, n\}$, be slices and let $\varepsilon > 0$.

Let $P\colon X^* \to X^*$, with $\ker P = Y^\perp$, be the M-ideal projection. Define

$$y_i^* := \frac{Px_i^*}{\|Px_i^*\|} \quad \text{and} \quad \beta_i := \frac{\varepsilon(1 - \|Px_i^*\|) + \varepsilon^2}{\|Px_i^*\|} > 0.$$

Since Y has the SSD2P, there exist $u_i \in S(B_Y, y_i^*, \beta_i)$ and $v \in B_Y$ with $u_i \pm v \in S(B_Y, y_i^*, \beta_i)$ and $\|v\| > 1 - \varepsilon$. Note that we then have $|y_i^*(v)| < \beta_i$. The choice of β_i means that

$$Px_i^*(u_i) > (\|Px_i^*\| - \varepsilon)(1 + \varepsilon).$$

If we happen to have $Px_i^* = 0$, we set $u_i = 0$ and use the v we get from the rest of the slices. If $Px_i^* = 0$ for every $i \in \{1, \dots, n\}$, then use any $v \in S_Y$.

Find $x_1, \dots, x_n \in X$ such that

$$(x_i^* - Px_i^*)(x_i) > (\|x_i^* - Px_i^*\| - \varepsilon)(1 + \varepsilon).$$

By [Wer1, Proposition 2.3], for each $i \in \{1, \dots, n\}$ there is a net $z_{\alpha,i}$ in Y such that $z_{\alpha,i} \to x_i$ in the $\sigma(X, Y^*)$-topology and

$$\limsup \|y + (x_i - z_{\alpha,i})\| \leq 1$$

for every $y \in B_Y$. Hence we may choose $z_i \in Y$ such that

$$\begin{gathered}
\|u_i + x_i - z_i\| < 1 + \varepsilon \\
\|u_i \pm v + x_i - z_i\| < 1 + \varepsilon \\
|P(x_i^*)(x_i - z_i)| < \varepsilon.
\end{gathered}$$

Define

$$y_i := \frac{u_i + x_i - z_i}{1 + \varepsilon} \quad \text{and} \quad w := \frac{v}{1 + \varepsilon}.$$

Then

$$\begin{aligned}
x_i^*(y_i) &= \frac{x_i^*(u_i + x_i - z_i)}{1 + \varepsilon} \\
&= \frac{Px_i^*(u_i) + (x_i^* - Px_i^*)(x_i) + Px_i^*(x_i - z_i)}{1 + \varepsilon} \\
&> \frac{(\|Px_i^*\| - \varepsilon)(1 + \varepsilon) + (\|x_i^* - Px_i^*\| - \varepsilon)(1 + \varepsilon) - \varepsilon}{1 + \varepsilon} \\
&> \|x_i^*\| - 3\varepsilon = 1 - 3\varepsilon.
\end{aligned}$$

Also, $1 \geq \|w\| \geq \frac{1-\varepsilon}{1+\varepsilon}$ and

$$\begin{aligned}
x_i^*(y_i \pm w) &> 1 - 3\varepsilon \pm \frac{\|Px_i^*\|}{1 + \varepsilon} y_i^*(v) > 1 - 3\varepsilon - \frac{\|Px_i^*\|}{1 + \varepsilon} \beta_i \\
&= 1 - 3\varepsilon - \frac{\varepsilon - \varepsilon\|Px_i^*\| + \varepsilon^2}{1 + \varepsilon} > 1 - 4\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can choose it as small as we like so that $y_i \in S_i$, $y_i \pm w \in S_i$ and $\|w\|$ is as close to 1 as we like. □

The SD2P-version of the following result is [ALN1, Theorem 4.10], however its proof in [ALN1] actually proves the SSD2P-version.

Theorem 2.23. *Let X be a Banach space and Y its proper closed subspace. If Y is an M-ideal in X, that is $X^* = Z \oplus_1 Y^\perp$ for some non-empty subspace Z of X^*, and moreover, if Z is 1-norming for X, then both X and Y have the SSD2P.*

In particular, if X is non-reflexive and an M-ideal in X^{**}, then both X and X^{**} have the SSD2P.

Remark 2.4. Similar results to Proposition 2.22 and Theorem 2.23 *cannot* hold for ASQ spaces, because c_0 is an M-ideal in $\ell_\infty = (c_0)^{**}$ and c_0 is ASQ while ℓ_∞ is not.

Recall that if C is a closed convex bounded subset of a Banach space X, then it is said that C is *strongly regular* if for every non-empty subset $D \subset C$ and $\varepsilon > 0$, there exists a convex combination of slices $S \subset D$ such that $\operatorname{diam} S < \varepsilon$ (see [GGMS, III A. δ)]). In turn, a Banach space X is said to be strongly regular if every closed, convex and bounded subset of X has convex combination of slices with an arbitrarily small diameter.

In [BLR5, Theorem 2.2], it was shown that the SD2P of X passes down to a subspace Y of X whenever the quotient X/Y is strongly regular. We will show that a similar statement holds for the SSD2P as well.

Proposition 2.24. *Let X be a Banach space and Y a closed subspace. If X has the SSD2P and X/Y is strongly regular, then Y has the SSD2P as well. In particular, SSD2P passes down to finite co-dimensional subspaces.*

Proof. The proof uses the ideas of the proof of [BLR5, Theorem 2.2]. Let X be a Banach space with the SSD2P and X/Y be strongly regular. Fix $n \in \mathbb{N}$, $U_1, \dots, U_n$ non-empty relatively weakly open subsets of B_Y and $\varepsilon > 0$. Denote by $\pi\colon X \to X/Y$ the quotient map and $A_i := \pi(U_i)$ for each $i \in \{1, \dots, n\}$. Notice that every A_i is a convex subset of $B_{X/Y}$ containing zero. By [GGMS, Proposition III.6] the closure of each A_i is the closure of the strongly regular points of A_i. Let $\delta > 0$ and for each index i choose a strongly regular point $a_i \in A_i$ such that

$$\|a_i\| < \delta.$$

For each $i \in \{1, \dots, n\}$ find $n_i \in \mathbb{N}$, $\mu^i_1, \dots, \mu^i_{n_i} > 0$ such that $\sum_{j=1}^{n_i} \mu^i_j = 1$ and slices $S(B_{X/Y}, (a^i_j)^*, \alpha^i_j)$ such that

$$a_i \in \sum_{j=1}^{n_i} \mu^i_j S(B_{X/Y}, (a^i_j)^*, \alpha^i_j) \cap \overline{A}_i \tag{2.2}$$

and

$$\operatorname{diam}\left(\sum_{j=1}^{n_i} \mu^i_j S(B_{X/Y}, (a^i_j)^*, \alpha^i_j) \cap \overline{A}_i \right) < \delta.$$

Note that due to equation (2.2) we have

$$V_i := \sum_{j=1}^{n_i} \mu_j^i S(B_X, \pi^*(a_j^i)^*, \alpha_j^i) \cap U_i \neq \emptyset \quad \text{for each } i \in \{1, \dots, n\}.$$

According to our assumption we have that for each $i \in \{1, \dots, n\}$ there exist $x_i \in V_i$ and $y \in B_X$ such that $x_i \pm y \in V_i$ and $\|y\| > 1-\delta$. Fix $i \in \{1, \dots, n\}$. Denote $x_i := \sum_{j=1}^{n_i} \mu_j^i x_j^i$. Since

$$\pi(x_i) \in \sum_{j=1}^{n_i} \mu_j^i S(B_{X/Y}, (a_j^i)^*, \alpha_j^i) \cap A_i,$$

we have $\|\pi(x_i)\| \leq \|\pi(x_i) - a_i\| + \|a_i\| \leq 2\delta$. Then we also have $\|\pi(y)\| \leq \|\pi(x_i - y)\| + \|\pi(x_i)\| \leq 4\delta$.

Choose $u_i \in B_Y$ such that $\|u_i - x_i\| < 3\delta$ and $v \in B_Y$ such that $\|v - y\| < 5\delta$. By choosing δ small enough we get that

$$u_i \in U_i \quad \text{and} \quad u_i \pm \frac{v}{\|v\|} \in U_i.$$

Therefore Y has the SSD2P.

□

2.5 w^*-SSD2P in spaces of Lipschitz functions

We now turn our attention to the space of Lipschitz functions on metric spaces and we show that $\mathrm{Lip}_0(M)$ has the w^*-version of the SSD2P for many classes of metric spaces M.

Definition 2.25. A dual Banach space X^* has the *w^*-symmetric strong diameter 2 property* (w^*-SSD2P) if for $n \in \mathbb{N}$, $S_1, \dots, S_n$ w^*-slices of B_{X^*} and $\varepsilon > 0$ there exist $x_1^* \in S_1, \dots, x_n^* \in S_n$ and $y^* \in B_{X^*}$, such that $x_i^* \pm y^* \in S_i$ for every $i \in \{1, \dots, n\}$ and $\|y^*\| > 1 - \varepsilon$.

Observe that in a dual Banach space the SSD2P implies the w^*-SSD2P because every w^*-slice is also a slice. However, the converse is open.

Question 2.26. *Does there exist a dual Banach space with the w^*-SSD2P but without the SSD2P?*

The w^*-SSD2P is stronger than the w^*-SD2P. Indeed, by [ALN1], the space $\ell_\infty \oplus_1 \ell_\infty$ has the SD2P (hence also the w^*-SD2P) but ℓ_1-sums never have the w^*-SSD2P (the proof is analogous to the proof of Theorem 2.13). We also note that a Banach space X has the SSD2P if and only if X^{**} has the w^*-SSD2P because, by Goldstine's theorem, B_X is w^*-dense in $B_{X^{**}}$ and the norm on X^{**} is w^*-lower semicontinuous.

Let us now give a brief introduction to the space of Lipschitz functions. Let M be a pointed metric space with a metric d and the origin denoted by 0. The space $\mathrm{Lip}_0(M)$ of all Lipschitz functions $f\colon M \to \mathbb{R}$, with $f(0) = 0$, is a Banach space with the norm

$$\|f\| = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in M, x \neq y\right\}.$$

It is known that $\mathrm{Lip}_0(M)$ is a dual space, whose canonical predual is the Lipschitz-free space, also known as the Arens–Eells space, $\mathcal{F}(M)$, which is the norm closed linear subspace of $\mathrm{Lip}_0(M)^*$ spanned by the evaluation functionals δ_x with $x \in M$. If $\mu = \sum_{i=1}^{n} a_i\delta_{x_i}$ is an element in $\mathcal{F}(M)$, with $x_i \in M \setminus \{0\}$ and $a_i \neq 0$, for every $i \in \{1, \dots, n\}$, then we will denote the *support* of μ by $\mathrm{supp}(\mu) := \{x_1, \dots, x_n\}$.

To read more on the topic of Lipschitz and Lipschitz-free spaces, we encourage the reader to consult [Wea].

We now turn our attention to the long trapezoid property introduced in [PR] in order to characterise the w^*-SD2P in the space of Lipschitz functions.

Combining [PR, Theorem 3.1] with the dual description of octahedral spaces from Theorem 1.9, one has the following result.

Theorem 2.27 (see [PR, Theorem 3.1])**.** *Let M be a metric space. The following statements are equivalent.*

(i) *The Banach space $Lip_0(M)$ has the w^*-SD2P.*

(ii) *The metric space M has the LTP, that is, if for every finite subset $N \subset M$ and $\varepsilon > 0$, there exist $u, v \in M$, with $u \neq v$, such that*

$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leq d(x, u) + d(y, v)$$

holds for every $x, y \in N$.

Example 2.28 (see [PR, Example 3.4])**.** Let M be a metric space. If M is unbounded or $\inf\{d(x, y) \mid x, y \in M, x \neq y\} = 0$, then M has the LTP.

Let us start the study of w^*-SSD2P in Lipschitz spaces by giving conditions on the underlying metric space M which enable the Lipschitz space $\mathrm{Lip}_0(M)$ to have the w^*-SSD2P.

Proposition 2.29. *If M is an unbounded metric space, then $\mathrm{Lip}_0(M)$ has the w^*-SSD2P.*

Proof. Let $n \in \mathbb{N}$, $S_i := S(B_{\mathrm{Lip}_0(M)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, n\}$, be w^*-slices of $B_{\mathrm{Lip}_0(M)}$, where $\mu_i \in \operatorname{span}\{\delta_x \mid x \in M\}$ and $\varepsilon > 0$. We want to show that there exist $f_i \in S_i$ and $\varphi \in B_{\mathrm{Lip}_0(M)}$ such that

$$f_i \pm \varphi \in S_i \qquad \text{and} \qquad \|\varphi\| > 1 - \varepsilon.$$

Choose $g_i \in S_i$ with $g_i(\mu_i) = 1$ for $i \in \{1, \dots, n\}$. Denote $N := \{0\} \cup \bigcup_{i=1}^n \operatorname{supp}(\mu_i)$. The main idea of the proof is to find norm preserving extensions f_i of $g_i|_N$ such that $f_i|_{M \setminus B(0,s)} = 0$ and $\varphi|_{B(0,t)} = 0$ for suitable $0 < s < t$. Since N is a finite subset of M, there is an $r > 0$ such that $N \subset B(0, r)$. Let $s := 2r$. Then for every $x \in B(0, r)$ and $y \in M \setminus B(0, s)$ we have $d(x, 0) \leq d(x, y)$. Since M is unbounded, there exists $u \in M \setminus B(0, s)$.

Let $\delta > 0$ be such that $(1 - \delta)^2 > \max\{1 - \varepsilon, 1 - \alpha_i\}$. Find a $t > 0$ such that for every $x \in B(0, s)$ and $y \in M \setminus B(0, t)$ one has

$$d(x, y) \geq (1 - \delta)(d(x, u) + d(u, y)).$$

For example, any t with $\delta t \geq 2(s + d(0, u))$ suffices.

Since M is unbounded, there exists $v \in M \setminus B(0, t)$ such that

$$\frac{d(v, 0) - t}{d(v, 0)} > 1 - \delta, \text{ that is } \delta \cdot d(v, 0) > t.$$

Define $\tilde{\varphi} \colon B(0, t) \cup \{v\} \to \mathbb{R}$ by $\tilde{\varphi}|_{B(0,t)} = 0$ and $\tilde{\varphi}(v) = d(v, 0) - t$. Then $\|\tilde{\varphi}\| \leq 1$ because for any $x \in B(0, t)$ we have

$$\frac{|\tilde{\varphi}(v) - \tilde{\varphi}(x)|}{d(v, x)} = \frac{|\tilde{\varphi}(v) - 0|}{d(v, x)} \leq \frac{d(v, 0) - t}{d(v, 0) - d(0, x)} \leq 1.$$

Also, $\|\tilde{\varphi}\| > 1 - \delta$ because

$$\|\tilde{\varphi}\| \geq \frac{\tilde{\varphi}(v) - \tilde{\varphi}(0)}{d(v, 0)} = \frac{d(v, 0) - t}{d(v, 0)} > 1 - \delta.$$

For every $i \in \{1, \dots, n\}$ define $\tilde{f}_i \colon N \cup (M \setminus B(0, s)) \to \mathbb{R}$ by $\tilde{f}_i|_N = g_i$ and $\tilde{f}_i|_{M \setminus B(0,s)} = 0$. Then $\|\tilde{f}_i\| \leq 1$ because for any $x \in N$ and $y \in M \setminus B(0, s)$ we have

$$|\tilde{f}_i(x) - \tilde{f}_i(y)| = |\tilde{f}_i(x) - \tilde{f}_i(0)| \leq d(x, 0) \leq d(x, y).$$

Consider $f_i := (1-\delta)\tilde{f}_i$ and $\varphi := (1-\delta)\tilde{\varphi}$ and extend them to M while preserving the norm. Observe that $\|f_i \pm \varphi\| \leq 1$, because for any $x \in B(0,s)$ and $y \in M \setminus B(0,t)$ we have

$$\begin{aligned}|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(y)| &= |f_i(x) \pm \varphi(y)| \\ &\leq |f_i(x)| + |\varphi(y)| = |f_i(x) - f_i(u)| + |\varphi(y) - \varphi(u)| \\ &\leq (1-\delta)d(x,u) + (1-\delta)d(u,y) \leq d(x,y).\end{aligned}$$

Finally, note that $\|\varphi\| = (1-\delta)\|\tilde{\varphi}\| > (1-\delta)^2 > 1-\varepsilon$ and

$$\begin{aligned}(f_i \pm \varphi)(\mu_i) &= f_i(\mu_i) = (1-\delta)\tilde{f}_i(\mu_i) \\ &= (1-\delta)g_i(\mu_i) = 1-\delta \\ &> (1-\delta)^2 > 1-\alpha_i.\end{aligned}$$

□

We continue by showing that Lipschitz spaces over infinite discrete metric spaces also enjoy the w^*-SSD2P.

Proposition 2.30. *If M is an infinite discrete metric space, then $\mathrm{Lip}_0(M)$ has the w^*-SSD2P.*

Proof. Let $n \in \mathbb{N}$, $S_i := S(B_{\mathrm{Lip}_0(M)}, \mu_i, \alpha_i)$, $i \in \{1,\dots,n\}$, be w^*-slices of $B_{\mathrm{Lip}_0(M)}$, where $\mu_i \in \operatorname{span}\{\delta_x \mid x \in M\}$.

Let $N = \{0\} \cup \bigcup_{i=1}^n \operatorname{supp}(\mu_i)$. For every $i \in \{1,\dots,n\}$ choose $g_i \in S_{\mathrm{Lip}_0(N)}$ such that $g_i(\mu_i) = 1$ and let $x_i, y_i \in N$ be such that

$$g_i(x_i) - g_i(y_i) = d(x_i, y_i) = 1.$$

Fix any two different elements $u, v \in M \setminus N$. Define $f_i \in S_i$ and $\varphi \in S_{\mathrm{Lip}_0(M)}$ by setting

$$f_i(x) := \begin{cases} g_i(x), & \text{if } x \in N, \\ \dfrac{g_i(x_i) + g_i(y_i)}{2}, & \text{if } x = u \text{ or } x = v, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi(x) := \begin{cases} \dfrac{g_i(x_i) - g_i(y_i)}{2} = \dfrac{1}{2}, & \text{if } x = u, \\ \dfrac{g_i(y_i) - g_i(x_i)}{2} = -\dfrac{1}{2}, & \text{if } x = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_i, f_i \pm \varphi \in S_i$ and $\|\varphi\| = 1$.

□

For each $n \in \mathbb{N}$, denote

$$K_n := \{x = (x_k) \in \ell_\infty \mid x_k \in \{1, \dots, n\}\}$$

with the metric inherited from ℓ_∞. In order to describe the space K_n we shall require the following lemma.

Lemma 2.31. *Let $n \in \mathbb{N}$ and let N be a finite subset of K_n. Then there are $u, v \in K_n \setminus N$ satisfying*

(a) *$d(u, v) = 1$;*

(b) *for every $x \in N$ one has $d(x, u) = d(x, v)$;*

(c) *for every $x, y \in N$ one has $d(x, y) \leq d(x, u)$ or $d(x, y) \leq d(y, u)$.*

In particular, K_n has the LTP.

Proof. Choose an $u \in K_n \setminus N$ such that $u(i) \in \{0, n\}$ for every $i \in \mathbb{N}$. For such an element u, the condition (c) holds. We will construct a suitable $v \in K_n \setminus N$, such that it differs from u in only one coordinate i_0. Let $I \subset \mathbb{N}$ be a finite subset such that for every $x \in N$ there is an $i \in I$ such that $d(x, u) = |x(i) - u(i)|$. Fix an $i_0 \in \mathbb{N} \setminus I$. Let $v(i_0)$ be such that $|u(i_0) - v(i_0)| = 1$. Hence condition (a) holds and we will check condition (b). Let $x \in N$. Clearly, $d(x, v) \geq d(x, u)$ because $u(i) = v(i)$ for every $i \in I$. For the reverse inequality, observe that

$$|x(i_0) - v(i_0)| = \begin{cases} 1, & \text{if } x(i_0) = u(i_0), \\ |x(i_0) - u(i_0)| - 1, & \text{if } x(i_0) \neq u(i_0). \end{cases}$$

Hence $d(x, u) \geq d(x, v)$ and condition (b) holds. □

From the previous lemma and Theorem 2.27 we see that $\mathrm{Lip}_0(K_n)$ has the w^*-SD2P. Y. Ivakhno proved that K_n, $n \in \{1, 2\}$, has the LD2P, leaving the question open for $n \geq 3$ (see [Iva]). We shall now show that K_n, $n \in \mathbb{N}$, has the w^*-SSD2P. Note that Proposition 2.30 shows that K_1 has the w^*-SSD2P. As for $n \geq 2$, we start with K_2 because it slightly differs from the general case.

Proposition 2.32. *The Banach space $\mathrm{Lip}_0(K_2)$ has the w^*-SSD2P.*

Proof. Let $n \in \mathbb{N}$, $S_i := S(B_{\mathrm{Lip}_0(K_2)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, n\}$, be w^*-slices of $B_{\mathrm{Lip}_0(K_2)}$, where $\mu_i \in \mathrm{span}\{\delta_x \mid x \in M\}$. We show that there exist $f_i \in S_i$ and $\varphi \in B_{\mathrm{Lip}_0(K_2)}$ such that

$$f_i \pm \varphi \in S_i \qquad \text{and} \qquad \|\varphi\| = 1.$$

Let $N = \{0\} \cup \bigcup_{i=1}^{n} \operatorname{supp}(\mu_i)$ and let $u, v \in K_2$ be as in Lemma 2.31 for N. For every μ_i choose $g_i \in S_{\mathrm{Lip}_0(K_2)}$ such that $g_i(\mu_i) = 1$. The main idea of the proof is to define φ such that $\varphi = 0$ outside $\{u, v\}$ and f_i are norm preserving extensions of $g_i|_N$ satisfying $f_i(u) = f_i(v)$ and $|f_i(x) - f_i(y)| \leq 1$ for every $x, y \notin N$. For $j \in \{1, 2\}$ set

$$N_j := \{x \in N \mid d(x, u) = d(x, v) = j\}.$$

For every g_i define its norm preserving extension g_i^+ from N to $N \cup \{u, v\}$ by taking

$$\begin{aligned} g_i^+(u) &:= \min\{g_i(x) + d(x, u) \mid x \in N\}, \\ g_i^+(v) &:= \max\{g_i^+(x) - d(x, v) \mid x \in N \cup \{u\}\}. \end{aligned}$$

This means that g_i^+ is the maximal extension from N to u and then minimal extension to v preserving the Lipschitz constant (see [McS] or [Wea, page 18]). Note that $g_i^+(v) + 1 = g_i^+(u)$. Indeed, for every $k, l \in \{1, 2\}$,

$$(\min_{x \in N_k} g_i(x) + k) - (\max_{x \in N_l} g_i(x) - l) \geq 1,$$

that is,

$$\max_{x \in N_l} g_i(x) - \min_{x \in N_k} g_i(x) \leq k + l - 1,$$

because for $x \in N_l$ and $y \in N_k$

$$g_i(x) - g_i(y) \leq d(x, y) \leq \begin{cases} d(x, u) = l \leq k + l - 1 \\ \text{or} \\ d(y, u) = k \leq k + l - 1, \end{cases}$$

by Lemma 2.31.

If there is an element $x \in K_2 \setminus N$ such that

$$N_2^1(x) := \{y \in N_2 \mid d(x, y) = 1\} \neq \emptyset,$$

then choose arbitrarily a_i^x from the set

$$[\max_{y \in N_2^1(x)} g_i(y) - 1, \min_{y \in N_2^1(x)} g_i(y) + 1] \cap [g_i^+(u) - 1, g_i^+(u)].$$

Note that the latter intersection is non-empty because

$$\max_{y \in N_2^1(x)} g_i(y) - 1 \leq g_i^+(u) = g_i^+(v) + 1$$

and

$$\min_{y\in N_2^1(x)} g_i(y) + 1 \geq g_i^+(u) - 1.$$

Define

$$\varphi(x) := \begin{cases} \frac{1}{2}, & \text{if } x = u, \\ -\frac{1}{2}, & \text{if } x = v, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_i(x) := \begin{cases} g_i(x), & \text{if } \ x \in N, \\ a_i^x, & \text{if } \ x \in K_2 \setminus N \text{ and } d(x, N_2) = 1, \\ g_i^+(u) - \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then $f_i \in S_i$, $(f_i \pm \varphi)(\mu_i) = f_i(\mu_i) > 1 - \alpha_i$ and $\|\varphi\| = 1$. To check that $\|f_i \pm \varphi\| \leq 1$, we argue by cases.

Case 1. If $x \in N$ and $y = u$, then

$$|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(u)| = \left| g_i(x) - \left(g_i^+(u) - \frac{1}{2} \pm \frac{1}{2} \right) \right| =$$
$$= \begin{cases} |g_i(x) - g_i^+(u)| \leq d(x, u) \\ \text{or} \\ |g_i(x) - g_i^+(u) + 1| = |g_i(x) - g_i^+(v)| \leq d(x, v) = d(x, u). \end{cases}$$

Case 2. If $x \in K_2 \setminus (N \cup \{u, v\})$ and $y = u$, then

$$|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(u)| =$$
$$= \begin{cases} |a_i^x - (g_i^+(u) - \frac{1}{2} \pm \frac{1}{2})| \leq 1 \leq d(x, u) \\ \text{or} \\ |(g_i^+(u) - \frac{1}{2}) - (g_i^+(u) - \frac{1}{2} \pm \frac{1}{2})| = \frac{1}{2} \leq d(x, u), \end{cases}$$

because $a_i^x \in [g_i^+(u) - 1, g_i^+(u)]$.

Case 3. If $x \in K_2 \setminus (N \cup \{u, v\})$ and $y \in N$, then

$$|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(y)| = |f_i(x) - f_i(y)| \leq d(x, y),$$

because $\|f_i\| \leq 1$.

The other cases are trivial or similar to the ones above. Hence $\|f_i \pm \varphi\| \leq 1$, which completes the proof. □

Let us now show that in the general case for K_n, where $n \geq 3$, K_n has the w^*-SSD2P.

Proposition 2.33. *The Banach space* $\mathrm{Lip}_0(K_n)$, $n \geq 3$, *has the* w^**-SSD2P.*

Proof. Let $n \geq 3$, $k \in \mathbb{N}$, $S_i := S(B_{\mathrm{Lip}_0(K_n)}, \mu_i, \alpha_i)$, $i \in \{1, \dots, k\}$, be w^*-slices of $B_{\mathrm{Lip}_0(K_n)}$, where $\mu_i \in \operatorname{span}\{\delta_x \mid x \in M\}$. We show that there exist $f_i \in S_i$ and $\varphi \in B_{\mathrm{Lip}_0(K_n)}$ such that

$$f_i \pm \varphi \in S_i \qquad \text{and} \qquad \|\varphi\| = 1.$$

Set $N = \{0\} \cup \bigcup_{i=1}^{k} \operatorname{supp}(\mu_i)$. Choose $u \in K_n$ such that $d(u, N) \geq 2$ and for every j one has $u(j) = 0$ or $u(j) = n$. (The geometrical idea behind choosing u is the following. There are uncountably many elements $u \in K_n$ with coordinates 0 or n. Closed balls of K_n with such centres u and with radius 1 do not intersect each other. Since N is finite, such balls can be chosen without elements from N (to visualise K_n imagine an infinite-dimensional Rubik's cube).

Define $\varphi(u) = 1$ and $\varphi = 0$ otherwise. Now we deal with the f_i's. For every μ_i choose $g_i \in S_{\mathrm{Lip}_0(K_n)}$ such that $g_i(\mu_i) = 1$. For every g_i let g_i^+ be its norm preserving extension from N to $K_n \setminus B(u, 1)$, where $B(u, 1)$ is the closed ball with centre u and radius 1. Let

$$a_i = \frac{1}{2}\Big(\max_{x \in S(u,2)} g_i^+(x) + \min_{x \in S(u,2)} g_i^+(x) \Big).$$

Note that since $\mathrm{Lip}(g_i^+) = 1$, the values of the function g_i^+ on the sphere $S(u, 2)$ differ by no more than 2. Moreover,

$$\min_{x \in S(u,2)} g_i^+(x) \leq n - 2$$

and

$$\max_{x \in S(u,2)} g_i^+(x) \geq -n + 2.$$

This is because $S(u, 2)$ contains a point of distance $n - 2$ from the origin. Therefore $a_i \in [-n + 1, n - 1]$.

Define $f_i \colon K_n \to \mathbb{R}$ by

$$f_i(x) = \begin{cases} a_i, & \text{if } d(x, u) \leq 1, \\ g_i^+(x) & \text{otherwise.} \end{cases}$$

Then $f_i \in S_i$ and $f_i \pm \varphi \in S_i$. Let us verify that $\|f_i \pm \varphi\| \leq 1$. Fix $x \in B(u, 1)$ and $y \in K_n \setminus B(u, 1)$.

If $x \neq u$, then $\varphi(x) = \varphi(y) = 0$ and therefore

$$|(f_i \pm \varphi)(x) - (f_i \pm \varphi)(y)| = |f_i(x) - f_i(y)| \leq d(x, y).$$

For $x = u$, fix $z \in S(u,1)$ such that $d(u,y) = d(u,z) + d(z,y)$. Then

$$\begin{aligned} |(f_i \pm \varphi)(u) - (f_i \pm \varphi)(y)| &= |f_i(z) \pm \varphi(u) - f_i(y)| \\ &\leq |f_i(z) - f_i(y)| + 1 \\ &\leq d(z,y) + d(u,z) \\ &= d(u,y). \end{aligned}$$

□

We now collect the known examples of metric spaces M such that $\mathrm{Lip}_0(M)$ has the w^*-SSD2P into a single result.

Theorem 2.34. *If M is an infinite metric space satisfying at least one of the following conditions:*

(a) $\sup\{d(x,y) \mid x,y \in M, x \neq y\} = \infty$;

(b) $\inf\{d(x,y) \mid x,y \in M, x \neq y\} = 0$;

(c) *M is a discrete metric space;*

(d) *$M = K_n$, where $n \in \mathbb{N}$,*

then $\mathrm{Lip}_0(M)$ *has the w^*-SSD2P.*

Proof. (a), (c) and (d) are Propositions 2.29, 2.30, 2.32 and 2.33, respectively. An inspection of the proof of [BLR8, Theorem 2.4] shows (b). □

Quite recently, A. Ostrak described, in [Ost], which property the underlying metric space needs to possess in order for the Lipschitz space to enjoy the w^*-SSD2P.

Theorem 2.35 (see [Ost, Theorem 2.1] and [Ost, Definition 1.3])**.** *Let M be a pointed metric space. The following statements are equivalent.*

(i) *The Banach space* $\mathrm{Lip}_0(M)$ *has the w^*-SSD2P.*

(ii) *The metric space M has the SLTP, that is, in addition to having the LTP (see Theorem 2.27), for every finite subset $N \subset M$ and $\varepsilon > 0$, there exist $u, v \in M, u \neq v$, such that for any $x, y \in N$ and for any $x, y, z, w \in N$,*

$$\begin{aligned} &(1-\varepsilon)(2d(u,v) + d(x,y) + d(z,w)) \\ &\qquad \leq d(x,u) + d(y,u) + d(z,v) + d(w,v). \end{aligned}$$

Also, Ostrak gave an example (see [Ost, Example 3.1]) of a metric space M that has the LTP but fails the SLTP. In turn, it shows that there exists a metric space M such that $\mathrm{Lip}_0(M)$ has w^*-SD2P that fails the w^*-SSD2P.

Remark 2.5. Very recently, J. Langemets and A. Rueda Zoca showed in [LaR] that $\mathrm{Lip}_0(M)$ even has the SSD2P for all of the metric spaces M, except for $M = K_2$, from Theorem 2.34.

Question 2.36. *Does the space* $\mathrm{Lip}_0(K_2)$ *have the SSD2P?*

Chapter 3

Rough norms in Banach spaces

In this chapter, we investigate roughness of norms in Banach spaces. It is known that a Banach space has the LD2P (resp., SD2P) if and only if its dual is 2-rough (resp., 2-average rough). We study roughness and fully describe the behaviour of octahedral norms in absolute sums. We end the chapter by studying roughness in ultrapowers of Banach spaces and in spaces of bounded linear operators. This chapter is mainly based on [HLN] and [Nad].

3.1 Introduction and definitions

The central notion of this chapter – roughness of the norm – was introduced by E. B. Leach and J. H. M. Whitfield, in [LW], in order to study when a Banach space does not have an equivalent Fréchet differentiable norm. A strengthening of roughness, called average roughness, was introduced by R. Deville in [Dev]. The dual characterisation of roughness and average roughness is well known (see Theorems 3.3 and 3.5 below).

Definition 3.1. Let $\delta > 0$. We say that a Banach space X is

(a) *δ-rough* if for every $x \in X$ we have

$$\limsup_{\|y\|\to 0} \frac{\|x+y\| + \|x-y\| - 2\|x\|}{\|y\|} \geq \delta;$$

(b) *δ-average rough* if for $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ we have

$$\limsup_{\|y\|\to 0} \frac{1}{n} \sum_{i=1}^{n} \frac{\|x_i+y\| + \|x_i-y\| - 2\|x_i\|}{\|y\|} \geq \delta.$$

From the definition above it is clear that δ-average roughness implies δ-roughness, however the converse does not hold in general. For example, the space $\ell_1 \oplus_2 \ell_1$ is 2-rough but $\sqrt{2}$-average rough (see Example 3.23 below). Also, observe that the notion of δ-roughness makes sense only when $\delta \in (0, 2]$ because no space can be δ-rough for $\delta > 2$.

It is known that 2-average rough norms are exactly the norms which are octahedral.

Proposition 3.2 (see [Dev, Proposition 3] and [God, page 12])**.** *Let X be a Banach space. Then X is octahedral if and only if it is 2-average rough.*

It is well known that the classical Banach spaces $\ell_1, C[0,1], L_1[0,1]$ and $L_\infty[0,1]$ are octahedral, i.e. 2-average rough.

In 1978, K. John and V. Zizler proved that rough norms can be dually characterised in terms of diameters of w^*-slices.

Theorem 3.3 (see [JZ, Proposition 1])**.** *Let X be a Banach space and $\delta > 0$. The following statements are equivalent.*

(i) *The norm of X is δ-rough.*

(ii) *The diameter of every w^*-slice of B_{X^*} is at least δ.*

Since a Banach space X has the LD2P if and only if its bidual X^{**} has the w^*-LD2P, we immediately have the following.

Corollary 3.4. *Let X be a Banach space. Then X has the LD2P if and only if X^* is 2-rough.*

In 1989, Deville proved that a similar dual characterisation as for rough norms also holds for average rough norms.

Theorem 3.5 (see [Dev, Theorem 1])**.** *Let X be a Banach space and $\delta > 0$. The following statements are equivalent.*

(i) *The norm of X is δ-average rough.*

(ii) *The diameter of every convex combination of w^*-slices of B_{X^*} is at least δ.*

Again, from the equivalence that a Banach space X has the SD2P if and only if its bidual X^{**} has the w^*-SD2P, we immediately have the following.

Corollary 3.6. *Let X be a Banach space. Then X has the SD2P if and only if X^* is 2-average rough.*

Remark 3.1. In [HLP2], the authors introduced another variant of roughness that lies between roughness and average roughness and is also dually connected to the diameter properties. Namely, a Banach space is *weakly δ-average rough* if every non-empty relatively weak* open subset of the dual unit ball has diameter greater than or equal to δ.

The following proposition gives us various ways to check whether a space is average rough, which we will make use of in the remainder of the chapter.

Proposition 3.7. *Let X be a Banach space and $\delta > 0$. The following statements are equivalent.*

(i) *The Banach space X is δ-average rough.*

(ii) *Whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ there is a $y \in X$ such that $\|y\| \leq \varepsilon$ and*

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y\| + \|x_i - y\|\Big) > (\delta - \varepsilon)\|y\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\|.$$

(iii) *Whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$ there is a $y \in X$ such that $\|y\| \leq \varepsilon$ and*

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y\| + \|x_i - y\|\Big) > (\delta - \varepsilon)\|y\| + 2.$$

Proof. (i)$\Leftrightarrow$(ii). Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$. Denote for each $y \in X$

$$f(y) = \frac{1}{n}\sum_{i=1}^{n}\frac{\|x_i + y\| + \|x_i - y\| - 2\|x_i\|}{\|y\|}.$$

It is sufficient to show that $\limsup_{\|y\|\to 0} f(y) \geq \delta$ if and only if for each $\varepsilon > 0$ there exists $y \in X$ such that $\|y\| \leq \varepsilon$ and $f(y) > \delta - \varepsilon$. According to the definition of the limit supremum we have

$$\limsup_{\|y\|\to 0} f(y) = \inf_{\varepsilon > 0}\sup\{f(y) \mid \|y\| \leq \varepsilon\}.$$

Therefore $\lim_{\|y\|\to 0} f(y) \geq \delta$ if and only if $\inf_{\varepsilon>0}\sup\{f(y) \mid \|y\| \leq \varepsilon\} \geq \delta$. The last inequality means that for $\varepsilon > 0$ we have $\sup\{f(y) \mid \|y\| \leq \varepsilon\} \geq \delta$, i.e. for each $\varepsilon > 0$ there exists $y \in X$ such that $\|y\| \leq \varepsilon$ and $f(y) > \delta - \varepsilon$.

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii). Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$. Suppose first that $x_1, \dots, x_n \in B_X$. Let $I \subset \{1, \dots, n\}$ be the set of indices for which $i \in I$ if and only if $x_i \neq 0$. If $I = \emptyset$, then we can obviously take any $y \in X$ such that $\|y\| = \varepsilon$ because in that case

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y\| + \|x_i - y\|\Big) &= 2\|y\| \geq (\delta - \varepsilon)\|y\| \\ &= (\delta - \varepsilon)\|y\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\|.\end{aligned}$$

Let us assume that $I \neq \emptyset$. From (iii) we get a $y \in X$ such that $\|y\| \leq \varepsilon$ and

$$\frac{1}{|I|}\sum_{i \in I}\left(\left\|\frac{x_i}{\|x_i\|} + y\right\| + \left\|\frac{x_i}{\|x_i\|} - y\right\|\right) > (\delta - \varepsilon)\|y\| + 2.$$

Therefore

$$\begin{aligned}&\frac{1}{|I|}\sum_{i \in I}\Big(\|x_i + y\| + \|x_i - y\|\Big) \\ &\quad\geq \frac{1}{|I|}\sum_{i \in I}\left(\left\|\frac{x_i}{\|x_i\|} + y\right\| + \left\|\frac{x_i}{\|x_i\|} - y\right\| - 2\left\|x_i - \frac{x_i}{\|x_i\|}\right\|\right) \\ &\quad> \Big((\delta - \varepsilon)\|y\| + 2\Big) - \frac{2}{|I|}\sum_{i \in I}\Big(1 - \|x_i\|\Big) \\ &\quad= (\delta - \varepsilon)\|y\| + \frac{2}{|I|}\sum_{i \in I}\|x_i\|.\end{aligned}$$

Which means that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y\| + \|x_i - y\|\Big) &= \frac{1}{n}\left(\sum_{i \in I}\Big(\|x_i + y\| + \|x_i - y\|\Big) + \sum_{i \notin I} 2\|y\|\right) \\ &> \frac{1}{n}\left(|I|\,(\delta - \varepsilon)\,\|y\| + 2\sum_{i \in I}\|x_i\| + \sum_{i \notin I}\delta\|y\|\right) \\ &= \left(\delta - \frac{|I|}{n}\varepsilon\right)\|y\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\| \\ &\geq (\delta - \varepsilon) + \frac{2}{n}\sum_{i=1}^{n}\|x_i\|.\end{aligned}$$

Suppose now that $x_1, \dots, x_n \in X$ such that $c := \max_{1\le i\le n} \|x_i\| > 1$. As $x_1/c, \dots, x_n/c \in B_X$, from the first part of the proof of this case we get a $z \in X$ such that $\|z\| \le \varepsilon/c$ and

$$\frac{1}{n}\sum_{i=1}^{n}\left(\left\|\frac{x_i}{c}+z\right\|+\left\|\frac{x_i}{c}-z\right\|\right) > \left(\delta-\frac{\varepsilon}{c}\right)\|z\| + \frac{2}{n}\sum_{i=1}^{n}\frac{\|x_i\|}{c}.$$

Multiplying by c, we get

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i+cz\|+\|x_i-cz\|\Big) > \left(\delta-\frac{\varepsilon}{c}\right)\|cz\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\|.$$

If we define $y = cz$, we have $\|y\| \le \varepsilon$ and

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i+y\|+\|x_i-y\|\Big) &> \left(\delta-\frac{\varepsilon}{c}\right)\|cz\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\| \\ &\ge (\delta-\varepsilon)\|y\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\|.\end{aligned}$$

Therefore (ii) holds. □

Remark 3.2. The equivalences in Proposition 3.7 remain true if either of the following holds:

(a) one replaces $\|y\| \le \varepsilon$ with $\|y\| = \varepsilon$;

(b) one replaces $\frac{1}{n}\sum_{i=1}^n$ with $\sum_{i=1}^n \lambda_i$, where $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$. This can be derived from both (ii) and (iii) by using (a) and the fact that for every $\lambda_1, \dots, \lambda_n > 0$, with $\sum_{i=1}^n \lambda_i = 1$, and for every $\varepsilon > 0$, there are $k_1, \dots, k_n \in \mathbb{N}$ such that $\sum_{i=1}^n |\lambda_i - \frac{k_i}{m}| < \varepsilon$, where $m = \sum_{i=1}^n k_i$.

Proposition 3.8. *Let X be a Banach space and $\delta > 0$. The following statements are equivalent.*

(i) *The Banach space X is δ-rough.*

(ii) *Whenever $x \in X$, and $\varepsilon > 0$ there is a $y \in X$ such that $\|y\| \le \varepsilon$ and*

$$\|x+y\|+\|x-y\| > (\delta-\varepsilon)\|y\| + 2\|x\|.$$

(iii) *Whenever $x \in S_X$ and $\varepsilon > 0$ there is a $y \in X$ such that $\|y\| \le \varepsilon$ and*

$$\|x+y\|+\|x-y\| > (\delta-\varepsilon)\|y\| + 2.$$

Proof. The proof of Proposition 3.8 is essentially the proof of Proposition 3.7 with $n = 1$. □

Remark 3.3. The equivalences in Proposition 3.8 remain true if one replaces $\|y\| \le \varepsilon$ with $\|y\| = \varepsilon$.

3.2 Roughness in absolute sums

In this section, we study how average roughness behaves on absolute sums. We characterise how average roughness lifts from components to the ℓ_1-sum. Then we show that given an arbitrary absolute normalised norm N, the N-sum of two Banach spaces that are δ-average rough is $\delta/N(1,1)$-average rough. We will then prove that our results are sharp in the general case and conclude that for every $\delta \in (1,2]$ there exists a Banach space which is exactly δ-average rough. We also describe how average roughness passes down from absolute sums to component spaces. We end the section by describing how δ-roughness behaves on absolute sums.

For ℓ_1-sums it is sufficient that only one of the components to be δ-average rough in order for the ℓ_1-sum to also be δ-average rough.

Proposition 3.9. *Let X and Y be Banach spaces. If X or Y is δ-average rough for some $\delta > 0$, then $X \oplus_1 Y$ is also δ-average rough.*

Proof. We consider only the case where X is δ-average rough. The case where Y is δ-average rough is analogous. We will prove that $Z := X \oplus_1 Y$ is δ-average rough. Let $n \in \mathbb{N}$, $z_1 := (x_1, y_1), \ldots, z_n := (x_n, y_n) \in S_Z$ and $\varepsilon > 0$. By Proposition 3.7, it suffices to show that there exists $z = (x, y) \in Z$ such that $\|z\|_1 = \varepsilon$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|z_i + z\|_1 + \|z_i - z\|_1\Big) \geq (\delta - \varepsilon)\|z\|_1 + 2.$$

Since X is δ-average rough, there is an $x \in X$ such that $\|x\| = \varepsilon$ and

$$\sum_{i=1}^{n}\frac{1}{n}\Big(\|x_i + x\| + \|x_i - x\|\Big) \geq (\delta - \varepsilon)\|x\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\|.$$

It follows that, for $z = (x, 0)$ we have $\|z\|_1 = \|x\| = \varepsilon$ and

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\Big(\|z_i + z\|_1 + \|z_i - z\|_1\Big)\\
&= \frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + x\| + \|y_i\| + \|x_i - x\| + \|y_i\|\Big)\\
&\geq (\delta - \varepsilon)\|x\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i\| + \frac{2}{n}\sum_{i=1}^{n}\|y_i\|\\
&= (\delta - \varepsilon)\|z\|_1 + 2.
\end{aligned}$$

□

Using the fact that 2-average roughness is equivalent to having an octahedral norm, we immediately get the following result.

Corollary 3.10 (see [HLP1, Proposition 3.12]). *If X or Y is octahedral, then $X \oplus_1 Y$ is also octahedral.*

However, when $1 < p < \infty$, the ℓ_p-sum of Banach spaces cannot be octahedral.

Proposition 3.11 (see [HLP1, Proposition 4.7]). *Let X and Y be Banach spaces. If $1 < p < \infty$, then $X \oplus_p Y$ is not octahedral.*

Let $\delta > 0$. Let us now give the main result of this section, which quantifies Proposition 3.11.

Theorem 3.12. *Let X and Y be Banach spaces, N an absolute normalised norm on $\mathbb{R}^2$ and $\gamma > 0$ be such that $\|\cdot\|_\infty \geq \gamma N(\cdot)$. If X and Y are δ-average rough for some $\delta > 0$, then $X \oplus_N Y$ is $\gamma\delta$-average rough.*

Proof. Assume that X and Y are δ-average rough. We will prove that $Z = X \oplus_N Y$ is $\gamma\delta$-average rough. Let $n \in \mathbb{N}$, $z_1 = (x_1, y_1), \dots, z_n = (x_n, y_n) \in S_Z$ and $\varepsilon > 0$. By Proposition 3.7, it suffices to show that there exists $z = (x, y) \in Z$ such that $\|z\|_N = \varepsilon N(1,1)$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|z_i + z\|_N + \|z_i - z\|_N\Big) \geq (\delta - \varepsilon)\gamma\|z\|_N + 2.$$

Choose $c_i, d_i \geq 0$ such that $N^*(c_i, d_i) = 1$ and $c_i\|x_i\| + d_i\|y_i\| = 1$. Denote

$$c = \frac{1}{n}\sum_{i=1}^{n} c_i \qquad \text{and} \qquad d = \frac{1}{n}\sum_{i=1}^{n} d_i.$$

Note that $c + d \geq 1$ because $c_i + d_i \geq N^*(c_i, d_i) = 1$. Consider first the case where $c \neq 0$ and $d \neq 0$. Denote

$$\mu_i = \frac{1}{n}\frac{c_i}{c} \qquad \text{and} \qquad \nu_i = \frac{1}{n}\frac{d_i}{d}.$$

Observe that $\mu_1 + \cdots + \mu_n = \nu_1 + \cdots + \nu_n = 1$. Since X and Y are δ-average rough, by Proposition 3.7, there are $x \in X$ and $y \in Y$ such that $\|x\| = \|y\| = \varepsilon$ and

$$\sum_{i=1}^{n}\mu_i\Big(\|x_i + x\| + \|x_i - x\|\Big) \geq (\delta - \varepsilon)\|x\| + 2\sum_{i=1}^{n}\mu_i\|x_i\|$$

and

$$\sum_{i=1}^{n} \nu_i\Big(\|y_i + y\| + \|y_i - y\|\Big) \geq (\delta - \varepsilon)\|y\| + 2\sum_{i=1}^{n} \nu_i\|y_i\|.$$

It follows that, for $z = (x, y)$ we have $\|z\|_N = \varepsilon N(1, 1)$ and

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} \Big(\|z_i + z\|_N + \|z_i - z\|_N\Big) \\
&\qquad \geq \frac{1}{n}\sum_{i=1}^{n} N\big(\|x_i + x\| + \|x_i - x\|, \|y_i + y\| + \|y_i - y\|\big) \\
&\qquad \geq \frac{1}{n}\sum_{i=1}^{n} \Big(c_i\big(\|x_i + x\| + \|x_i - x\|\big) + d_i\big(\|y_i + y\| + \|y_i - y\|\big)\Big) \\
&\qquad = c\sum_{i=1}^{n} \mu_i\big(\|x_i + x\| + \|x_i - x\|\big) + d\sum_{i=1}^{n} \nu_i\big(\|y_i + y\| + \|y_i - y\|\big) \\
&\qquad \geq c\Big((\delta - \varepsilon)\|x\| + 2\sum_{i=1}^{n} \mu_i\|x_i\|\Big) + d\Big((\delta - \varepsilon)\|y\| + 2\sum_{i=1}^{n} \nu_i\|y_i\|\Big) \\
&\qquad = (\delta - \varepsilon)(c\|x\| + d\|y\|) + \frac{2}{n}\sum_{i=1}^{n}(c_i\|x_i\| + d_i\|y_i\|) \\
&\qquad = (\delta - \varepsilon)(c + d)\max\{\|x\|, \|y\|\} + 2 \\
&\qquad \geq (\delta - \varepsilon)\gamma N(\|x\|, \|y\|) + 2 \\
&\qquad = (\delta - \varepsilon)\gamma\|z\|_N + 2.
\end{aligned}$$

Consider now the case where $c = 0$, which means that $c_i = 0$ and $d_i = 1$ for every $i \in \{1, \ldots, n\}$. This implies that $\|y_i\| = 1$ for every $i \in \{1, \ldots, n\}$. Since Y is δ-average rough, by Proposition 3.7, there exists a $y \in Y$ such that $\|y\| = \varepsilon N(1, 1)$ and

$$\sum_{i=1}^{n} \frac{1}{n}\Big(\|y_i + y\| + \|y_i - y\|\Big) \geq (\delta - \varepsilon)\|y\| + 2.$$

Therefore for $z = (0, y)$ we have $\|z\|_N = \|y\| = \varepsilon N(1, 1)$ and

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} \Big(\|z_i + z\|_N + \|z_i - z\|_N\Big) \\
&\qquad \geq \frac{1}{n}\sum_{i=1}^{n} \Big(\|y_i + y\| + \|y_i - y\|\Big) \\
&\qquad \geq (\delta - \varepsilon)\|y\| + 2 \\
&\qquad \geq (\delta - \varepsilon)\gamma\|z\|_N + 2.
\end{aligned}$$

The case where $d = 0$ is similar to the case $c = 0$. We have thus proved that $X \oplus_N Y$ is $\gamma\delta$-average rough. □

Applying Theorem 3.12 to ℓ_p-norms, we get the following corollary.

Corollary 3.13. *Let X and Y be Banach spaces which are δ-average rough for some $\delta > 0$.*

(a) *The Banach space $X \oplus_\infty Y$ is δ-average rough.*

(b) *The Banach space $X \oplus_p Y$ is $2^{-1/p}\delta$-average rough for $1 < p < \infty$.*

Again, using the equivalence between 2-average roughness and octahedrality, we get the following result.

Corollary 3.14. *If Banach spaces X and Y are octahedral and $1 < p < \infty$, then $X \oplus_p Y$ is $2^{1-1/p}$-average rough.*

Let us now show that for ℓ_p-sums, in general, the average roughness value $2^{1-1/p}$ from Corollary 3.14 is sharp.

Proposition 3.15. *Let X and Y be Banach spaces and $1 < p < \infty$. Then $X \oplus_p Y$ is not δ-average rough for any $\delta > 2^{1-1/p}$.*

Proof. We will prove that $Z = X \oplus_p Y$ is not δ-average rough for any $\delta > 2^{1-1/p}$. Consider the elements $z_1 = (x_0, 0)$ and $z_2 = (0, y_0)$ in Z, where $x_0 \in S_X$ and $y_0 \in S_Y$. It suffices to show that there is a function $f : (0, \infty) \to \mathbb{R}$ such that $f(\varepsilon) \to 0$, when $\varepsilon \to 0$, and that for every $\varepsilon > 0$ and $z \in Z$, where $\|z\| = \varepsilon$,

$$\frac{1}{2}\Big(\|z_1 + z\|_p + \|z_1 - z\|_p + \|z_2 + z\|_p + \|z_2 - z\|_p\Big) \leq \Big(2^{1-1/p} + f(\varepsilon)\Big)\|z\|_p + 2.$$

Let $\varepsilon \in (0, 1)$. Let $z = (x, y) \in Z$ be such that $\|z\|_p = \varepsilon$. By Maclaurin's formula,

$$(1 + \|x\|)^p = 1 + p\|x\| + \frac{p(p-1)}{2}(1 + \xi)^{p-2}\|x\|^2,$$

for some $\xi \in (0, \|x\|)$. Observe that

$$\begin{aligned} \|z_1 \pm z\|_p^p &= \|x_0 \pm x\|^p + \|y\|^p \\ &\leq (1 + \|x\|)^p + \|y\|^p \\ &= 1 + p\|x\| + \frac{p(p-1)(1 + \xi)^{p-2}}{2}\|x\|^2 + \|y\|^p. \end{aligned} \tag{3.1}$$

We continue by considering the cases $1 < p \leq 2$ and $p > 2$ separately. In both cases we will use the generalised Bernoulli's inequality, which says that for any $t \geq 0$ we have $(1+t)^{1/p} \leq 1 + t/p$.

Case 1. Assume that $1 < p \leq 2$. Since $\xi \in (0, \|x\|)$, we have

$$(1+\xi)^{p-2} \leq (1+0)^{p-2} = 1.$$

Combining the estimate (3.1) with Bernoulli's inequality, we get

$$\begin{aligned}\|z_1 \pm z\|_p &\leq \left(1 + p\|x\| + \frac{p(p-1)}{2}\|x\|^2 + \|y\|^p\right)^{1/p}\\ &\leq 1 + \|x\| + \frac{p-1}{2}\|x\|^2 + \frac{\|y\|^p}{p}.\end{aligned}$$

Similarly, we obtain

$$\|z_2 \pm z\|_p \leq 1 + \frac{\|x\|^p}{p} + \frac{p-1}{2}\|y\|^2 + \|y\|.$$

Therefore

$$\begin{aligned}&\frac{1}{2}\Big(\|z_1 + z\|_p + \|z_1 - z\|_p + \|z_2 + z\|_p + \|z_2 - z\|_p\Big)\\ &\quad\leq \left(1 + \|x\| + \frac{p-1}{2}\|x\|^2 + \frac{\|y\|^p}{p}\right) + \left(1 + \frac{\|x\|^p}{p} + \frac{p-1}{2}\|y\|^2 + \|y\|\right)\\ &\quad= 2 + \|x\| + \|y\| + \frac{p-1}{2}(\|x\|^2 + \|y\|^2) + \frac{1}{p}(\|x\|^p + \|y\|^p)\\ &\quad\leq 2 + 2^{1-1/p}\|(x,y)\|_p + \frac{p-1}{2}\varepsilon^2 + \frac{\varepsilon^p}{p}\\ &\quad= 2 + \left(2^{1-1/p} + \frac{p-1}{2}\varepsilon + \frac{\varepsilon^{p-1}}{p}\right)\|z\|_p.\end{aligned}$$

Thus, for $1 < p \leq 2$, we can take

$$f(\varepsilon) = \frac{p-1}{2}\varepsilon + \frac{\varepsilon^{p-1}}{p}.$$

Case 2. Assume that $p > 2$. Since $\xi \in (0, \|x\|)$ and $\|x\| \leq \varepsilon < 1$, we have

$$(1+\xi)^{p-2} \leq (1+\|x\|)^{p-2} \leq (1+\varepsilon)^{p-2} < 2^{p-2}.$$

Combining this estimate with (3.1) and Bernoulli's inequality, we get

$$\begin{aligned}\|z_1 \pm z\|_p &\leq \big(1 + p\|x\| + p(p-1)2^{p-3}\|x\|^2 + \|y\|^p\big)^{1/p}\\ &\leq 1 + \|x\| + (p-1)2^{p-3}\|x\|^2 + \frac{\|y\|^p}{p}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|z_2 \pm z\|_p &\le (\|x\|^p + 1 + p\|y\| + p(p-1)2^{p-3}\varepsilon^2)^{1/p} \\ &\le 1 + \frac{\|x\|^p}{p} + (p-1)2^{p-3}\|y\|^2 + \|y\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\Big(\|z_1 + z\|_p + \|z_1 - z\|_p + \|z_2 + z\|_p + \|z_2 - z\|_p\Big) \\ &\le \left(1 + \|x\| + (p-1)2^{p-3}\varepsilon^2 + \frac{\|y\|^p}{p}\right) \\ &\qquad + \left(1 + \frac{\|x\|^p}{p} + (p-1)2^{p-3}\varepsilon^2 + \|y\|\right) \\ &= 2 + (\|x\| + \|y\|) + (p-1)2^{p-3}(\|x\|^2 + \|y\|^2) + \frac{1}{p}(\|x\|^p + \|y\|^p) \\ &\le 2 + 2^{1-1/p}\|(x,y)\|_p + (p-1)2^{p-3}2^{1-2/p}\varepsilon^2 + \frac{\varepsilon^p}{p} \\ &= \left(2^{1-1/p} + (p-1)2^{p-2/p-2}\varepsilon + \frac{\varepsilon^{p-1}}{p}\right)\|z\| + 2.\end{aligned}$$

Thus, for $p > 2$, we can take

$$f(\varepsilon) = (p-1)2^{p-2/p-2}\varepsilon + \frac{\varepsilon^{p-1}}{p}.$$

Hence $X \oplus_p Y$ is not δ-average rough for any $\delta > 2^{1-1/p}$. □

Combining the previous two results we can show that there is a space that is exactly δ-average rough for every δ greater than 1.

Theorem 3.16. *For any $\delta \in (1,2]$ there is a dual Banach space, which is δ-average rough and is not γ-average rough for any $\gamma > \delta$.*

Proof. If $\delta = 2$, we can take ℓ_1. If $\delta \in (1,2)$, then there is a $q \in (1,\infty)$ such that $\delta = 2^{1/q}$. Let $p \in (1,\infty)$ be such that $1/p + 1/q = 1$. Since ℓ_1 is octahedral, then, by Corollary 3.14 and Proposition 3.15, the Banach space $\ell_1 \oplus_p \ell_1$ is δ-average rough and is not γ-average rough for any $\gamma > \delta$. □

When taking into account the duality of δ-average roughness and the diameter of convex combinations of slices (see Theorem 3.5), we get the following corollary.

Corollary 3.17. *For any $\delta \in (1,2]$ there is a Banach space in which the minimal diameter of convex combination of slices is exactly δ.*

In Chapter 4, we will see that the results of Theorem 3.16 and Corollary 3.17 can be extended to any $\delta \in (0,2]$ (see Proposition 4.11 and Corollary 4.12).

In order to investigate, when average roughness passes down from absolute sums, we will make use of the following elementary lemma.

Lemma 3.18. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $(1,0)$ is an extreme point of the unit ball $B_{(\mathbb{R}^2,N)}$. Then $(1,0)$ is a strongly exposed point of $B_{(\mathbb{R}^2,N)}$, which is strongly exposed by the functional $(1,0) \in B_{(\mathbb{R}^2,N^*)}$. In particular, for every $\varepsilon > 0$ there is a $\gamma > 0$ such that, whenever $(a,b) \in B_{(\mathbb{R}^2,N)}$ and $a > 1-\gamma$, then $|b| < \varepsilon$.*

Proof. Let N be such a norm on $\mathbb{R}^2$ that the point $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$. Because the notions of an exposed and strongly exposed point coincide in a finite-dimensional space (see [AB, Lemma 7.84]), let us show that $(1,0)$ is an exposed point. Consider the set

$$E = \{(a,b) \in B_{(\mathbb{R}^2,N)} \mid (1,0)(a,b) = 1\} = \{(1,b) \in B_{(\mathbb{R}^2,N)}\}.$$

Let us show that $E = \{(1,0)\}$. If it weren't the case, then there would exist $b > 0$ such that $(1,b) \in E$. Because N is an absolute norm, we also get that $(1,-b) \in E$. However, notice that $(1,0) = \frac{1}{2}(1,b) + \frac{1}{2}(1,-b)$, which is a contradiction. Therefore (1,0) is an exposed point.

Because (1,0) is also a strongly exposed point exposed by the functional (1,0), we see that $\operatorname{diam} S((1,0),\alpha) \to 0$ if $\alpha \to 0$. Thus for each $\varepsilon > 0$ we get a $\gamma > 0$ such that $\operatorname{diam} S((1,0),\gamma) < \varepsilon$, i.e. given $a > 1-\gamma$, we have $|b| = \|(a,b)-(a,0)\| < \varepsilon$. □

Using the lemma above, we can present the final result on average roughness of this section.

Proposition 3.19. *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$ such that $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N^*)}$. If $X \oplus_N Y$ is δ-average rough for some $\delta > 0$, then X is δ-average rough.*

Proof. Assume that $Z = X \oplus_N Y$ is δ-average rough. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$ and $\varepsilon \in (0,\delta)$. We will show that there is a $u \in X$ such that $\|u\| \leq \varepsilon$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i+u\| + \|x_i-u\|\Big) > \Big(\delta-\varepsilon\Big)\|u\| + 2.$$

By Lemma 3.18, there is a $\gamma \in (0, \frac{2\varepsilon}{3})$ such that, whenever $N^*(a,b) \leq 1$ and $a > 1-\gamma$, then $|b| < \frac{\varepsilon}{3}$.

Consider $(x_i, 0) \in S_Z$. Since Z is δ-average rough, there is a $z = (u,v) \in Z$ such that $\|z\|_N = \frac{\gamma}{2n}$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|(x_i,0)+(u,v)\|_N + \|(x_i,0)-(u,v)\|_N\Big) > (\delta-\gamma/2)\|z\|_N + 2.$$

Choose $a_i, b_i, c_i, d_i \geq 0$, with $N^*(a_i,b_i) = N^*(c_i,d_i) = 1$, such that

$$a_i\|x_i+u\| + b_i\|v\| = N(\|x_i+u\|, \|v\|)$$

and

$$c_i\|x_i-u\| + d_i\|v\| = N(\|x_i-u\|, \|v\|).$$

Then we have

$$\frac{1}{n}\sum_{i=1}^{n}\Big(a_i(\|x_i\|+\|u\|)+b_i\|v\|+c_i(\|x_i\|+\|u\|)+d_i\|v\|\Big) > (\delta-\gamma/2)\|z\|_N+2,$$

which implies that

$$\frac{a_i}{n} + \frac{n-1}{n} + 1 + 2\|z\|_N > (\delta-\gamma/2)\|z\|_N + 2.$$

It follows that $a_i > 1-\gamma$ and hence $b_i < \varepsilon/3$ for every $i \in \{1,\dots,n\}$. Similarly, one obtains that $c_i > 1-\gamma$ and $d_i < \varepsilon/3$ for every $i \in \{1,\dots,n\}$.

Therefore

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i+u\| + \|x_i-u\|\Big)\\
&\qquad\geq \frac{1}{n}\sum_{i=1}^{n}\Big(a_i\|x_i+u\| \pm b_i\|v\| + c_i\|x_i-u\| \pm d_i\|v\|\Big)\\
&\qquad> (\delta-\gamma/2)\|z\|_N + 2 - 2\frac{\varepsilon}{3}\|z\|_N\\
&\qquad= (\delta-\gamma/2-2\frac{\varepsilon}{3})\|z\|_N + 2\\
&\qquad> (\delta-\varepsilon)\|u\| + 2.
\end{aligned}$$

□

Remark 3.4. One can prove similarly to Proposition 3.19 that if N is an absolute normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2,N^*)}$ and $X \oplus_N Y$ is δ-average rough for some $\delta > 0$, then Y is δ-average rough.

Clearly $(1,0)$ and $(0,1)$ are extreme points of $\mathbb{R}^2$ equipped with the ℓ_p-norm for every $1 \leq p < \infty$. Hence the following is immediate.

Corollary 3.20. *Let X and Y be Banach spaces and $1 < p \leq \infty$. If $X \oplus_p Y$ is δ-average rough, then X and Y are δ-average rough.*

We end this section by studying roughness in absolute sums. There is a notable difference between roughness and average roughness, namely roughness is stable with respect to all absolute normalised norms.

Theorem 3.21. *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$. If X and Y are δ-rough for some $\delta > 0$, then $X \oplus_N Y$ is also δ-rough.*

Proof. Suppose that X and Y are δ-rough. Let $(x,y) \in S_{X\oplus_N Y}$ and $\varepsilon > 0$. Consider two cases.

Case 1. First, assume that $x \neq 0$ and $y \neq 0$. Because X is δ-rough, there exists $x_0 \in X$ such that $\|x_0\| = \varepsilon$ and

$$\left\|\frac{x}{\|x\|} + x_0\right\| + \left\|\frac{x}{\|x\|} - x_0\right\| > (\delta - \varepsilon)\|x_0\| + 2.$$

Analogously, because Y is δ-rough, there exists $y_0 \in Y$ such that $\|y_0\| = \varepsilon$ and

$$\left\|\frac{y}{\|y\|} + y_0\right\| + \left\|\frac{y}{\|y\|} - y_0\right\| > (\delta - \varepsilon)\|y_0\| + 2.$$

Denote $(u,v) := (\|x\|x_0, \|y\|y_0)$. It is obvious that $\|(u,v)\|_N = \varepsilon$. Notice that

$$\begin{aligned}
&\|(x,y) + (u,v)\|_N + \|(x,y) - (u,v)\|_N \\
&\qquad \geq N(\|x+u\| + \|x-u\|, \|y+v\| + \|y-v\|) \\
&\qquad = N\Bigg(\|x\|\left(\left\|\frac{x}{\|x\|} + x_0\right\| + \left\|\frac{x}{\|x\|} - x_0\right\|\right), \\
&\qquad\qquad \|y\|\left(\left\|\frac{y}{\|y\|} + y_0\right\| + \left\|\frac{y}{\|y\|} - y_0\right\|\right)\Bigg) \\
&\qquad > N\Big(\|x\|\big((\delta-\varepsilon)\varepsilon + 2\big), \|y\|\big((\delta-\varepsilon)\varepsilon + 2\big)\Big) \\
&\qquad \geq (\delta - \varepsilon)\|(u,v)\|_N + 2.
\end{aligned}$$

Case 2. Assume now that $x = 0$ or $y = 0$. Without loss of generality assume that $x = 0$, which means that $\|y\| = 1$. Because Y is δ-rough, there exists $v \in Y$ such that $\|v\| = \varepsilon$ and

$$\|y + v\| + \|y - v\| > (\delta - \varepsilon)\|v\| + 2.$$

Notice that

$$\begin{aligned}\|(x,y)+(0,v)\|_N+\|(x,y)-(0,v)\|_N&\\ &=\|y+v\|+\|y-v\|\\ &>(\delta-\varepsilon)\|v\|+2\\ &=(\delta-\varepsilon)\|(0,v)\|_N+2.\end{aligned}$$

Therefore $X\oplus_N Y$ is δ-rough. □

Remark 3.5. Given two Banach spaces, only one of the spaces needs to be δ-rough for their ℓ_1-sum to be δ-rough. The proof is essentially the proof of Proposition 3.9 with $n=1$.

Applying Theorem 3.21 to ℓ_p-norms, we get the following result.

Corollary 3.22. *Let X and Y be Banach spaces. If X and Y are δ-rough, then $X\oplus_p Y$ is also δ-rough.*

We can now give the following example which shows that roughness and average roughness are indeed distinct properties.

Example 3.23. The space $\ell_1\oplus_2\ell_1$ is 2-rough (by Theorem 3.21) and $\sqrt{2}$-average rough (by Theorem 3.12) but it is not δ-average rough for any $\delta>\sqrt{2}$ (by Proposition 3.15).

Proposition 3.24. *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$ such that $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N^*)}$. If $X\oplus_N Y$ is δ-rough for some $\delta>0$, then X is δ-rough.*

Proof. The proof of Proposition 3.24 is essentially the proof of Proposition 3.19 with $n=1$. □

Corollary 3.25. *Let X and Y be Banach spaces and $1<p\leq\infty$. If $X\oplus_p Y$ is δ-rough, then X and Y are δ-rough.*

3.3 Octahedral norms in absolute sums

In this section, we characterise those absolute norms which preserve octahedrality from component spaces – there are many such norms besides the ℓ_1- and ℓ_∞-norm.

Drawing inspiration from Proposition 1.8, a new form of octahedrality for absolute norms was defined in [HLN].

Definition 3.26. An element $(a,b) \in \mathbb{R}^2$ is called *positive* if $a \geq 0$ and $b \geq 0$. Let N be an absolute normalised norm on $\mathbb{R}^2$. We say that $(\mathbb{R}^2, N)$ is *positively octahedral* if whenever $n \in \mathbb{N}$ and positive $(a_1,b_1),\ldots,(a_n,b_n) \in S_{(\mathbb{R}^2,N)}$ there is a positive $(c,d) \in S_{(\mathbb{R}^2,N)}$ such that

$$N\big((a_i,b_i)+(c,d)\big) = 2 \quad \text{for every } i \in \{1,\ldots,n\}.$$

Remark 3.6. Note that $(\mathbb{R}^2, N)$ is positively octahedral if and only if there is a $(c,d) \in S_{(\mathbb{R}^2,N)}$ such that

$$N\big((1,0)+(c,d)\big) = 2 \qquad \text{and} \qquad N\big((0,1)+(c,d)\big) = 2.$$

Since the unit ball of $\mathbb{R}^2$ with an absolute norm is completely characterised by the first quadrant, we may illustrate the positively octahedral spaces as shown in Figure 3.1.

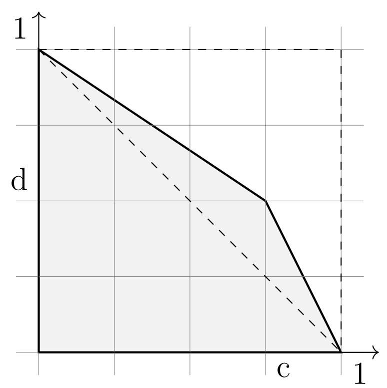

Figure 3.1: First quadrant of the unit ball of a positively octahedral $(\mathbb{R}^2, N)$.

Analogous to Theorem 2.7, the next theorem allows us to describe octahedrality of an absolute sum only by describing the norm on $\mathbb{R}^2$.

Theorem 3.27. *Let X and Y be octahedral Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$. Then $X \oplus_N Y$ is octahedral if and only if $(\mathbb{R}^2, N)$ is positively octahedral.*

Proof. First, assume that $X \oplus_N Y$ is octahedral. Let $n \in \mathbb{N}$, positive elements $(a_1,b_1),\ldots,(a_n,b_n) \in S_{(\mathbb{R}^2,N)}$ and $\varepsilon > 0$. We will show that there is a positive $(c,d) \in S_{(\mathbb{R}^2,N)}$ such that

$$N\Big((a_i,b_i)+(c,d)\Big) > 2-\varepsilon \quad \text{for every } i \in \{1,\ldots,n\}.$$

Let $x_i \in X$ and $y_i \in Y$ be such that $\|x_i\| = a_i$ and $\|y_i\| = b_i$. Since $X \oplus_N Y$ is octahedral, there exists a $(u, v) \in S_{X\oplus_N Y}$ such that $\|(u, v)\|_N = 1$ and

$$\|(x_i, y_i) + (u, v)\|_N > 2 - \varepsilon \quad \text{for every } i \in \{1, \dots, n\}.$$

Take $c = \|u\|$ and $d = \|v\|$. Then for every $i \in \{1, \dots, n\}$

$$\begin{aligned} N\Big((a_i, b_i) + (c, d)\Big) &= N(a_i + c, b_i + d) \\ &= N(\|x_i\| + \|u\|, \|y_i\| + \|v\|) \\ &\geq N(\|x_i + u\|, \|y_i + v\|) \\ &> 2 - \varepsilon. \end{aligned}$$

Assume now that $(\mathbb{R}^2, N)$ is positively octahedral. Let $(x_1, y_1), \dots, (x_n, y_n) \in X \oplus_N Y$ be with norm one and $\varepsilon > 0$. We will show that there is a $(u, v) \in X \oplus_N Y$ with norm one such that

$$\|(x_i, y_i) + (u, v)\|_N \geq (1 - \varepsilon)(2 - \varepsilon) \quad \text{for every } i \in \{1, \dots, n\}.$$

Since $(\mathbb{R}^2, N)$ is positively octahedral, there is a positive $(c, d) \in S_{(\mathbb{R}^2,N)}$ such that

$$N(\|x_i\| + c, \|y_i\| + d) \geq 2 - \varepsilon \quad \text{for every } i \in \{1, \dots, n\}.$$

Since X and Y are octahedral, there are $x \in S_X$ and $y \in S_Y$ such that

$$\|x_i + tx\| \geq (1 - \varepsilon)(\|x_i\| + t) \quad \text{for every } t \geq 0$$

and

$$\|y_i + ty\| \geq (1 - \varepsilon)(\|x_i\| + t) \quad \text{for every } t \geq 0.$$

Take $u = cx$ and $v = dy$. It follows that $\|(u, v)\|_N = 1$ and

$$\begin{aligned} \|(x_i, y_i) + (u, v)\|_N &= N(\|x_i + cx\|, \|y_i + dy\|) \\ &\geq (1 - \varepsilon)N\Big(\|x_i\| + c, \|y_i\| + d\Big) \\ &\geq (1 - \varepsilon)(2 - \varepsilon). \end{aligned}$$

□

As is the case with the SD2P and octahedrality (see Theorem 1.9), the positive SD2P (see Definition 2.6) and positive octahedrality are also dually connected.

Proposition 3.28. *Let N be an absolute normalised norm on $\mathbb{R}^2$. The space $(\mathbb{R}^2, N)$ has the positive SD2P if and only if $(\mathbb{R}^2, N^*)$ is positively octahedral.*

Proof. First, assume that $(\mathbb{R}^2, N)$ has the positive SD2P. So there are $a, b \geq 0$ such that $N(a, 1) = N(1, b) = 1$ and

$$N\Big(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\Big) = 1.$$

Let $c, d \geq 0$ be such that $N^*(c, d) = 1$ and

$$(c, d)\Big(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\Big) = 1.$$

It implies that $(c, d)(a, 1) = (c, d)(1, b) = 1$. Hence

$$N^*((1, 0) + (c, d)) = ((1, 0) + (c, d))(1, b) = 2$$

and

$$N^*((0, 1) + (c, d)) = ((0, 1) + (c, d))(a, 1) = 2.$$

Therefore $(\mathbb{R}^2, N^*)$ is positively octahedral.

Now let us assume that $(\mathbb{R}^2, N^*)$ is positively octahedral. So there exist $c, d \geq 0$ such that $N^*(c, d) = 1$ and

$$N^*((1, 0) + (c, d)) = 2 \qquad \text{and} \qquad N^*((0, 1) + (c, d)) = 2.$$

Let $a, b, x, y \geq 0$ be such that $N(a, y) = 1$, $N(x, b) = 1$,

$$((1, 0) + (c, d))(x, b) = 2,$$

and

$$((0, 1) + (c, d))(a, y) = 2.$$

It follows that $(1, 0)(x, b) = 1$ and $(0, 1)(a, y) = 1$, which means that $x = y = 1$. Hence

$$N\Big(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\Big) = (c, d)\Big(\frac{1}{2}(a, 1) + \frac{1}{2}(1, b)\Big) = \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore $(\mathbb{R}^2, N)$ has the positive SD2P. □

The duality between the SD2P and octahedrality, Theorem 3.27 and Proposition 3.28 together yield Proposition 2.7 thus allowing for a less technical proof.

3.4 Roughness of ultrapowers

In this section, we will study the average roughness of ultrapowers of Banach spaces. We will present results on how octahedrality, average roughness and the Daugavet property behave on ultrapowers.

For that we require the notion of an ultrapower in Banach spaces. Fix a Banach space X and $\mathcal{U}$ an ultrafilter on an index set $\mathcal{I}$. Given $(x_\alpha)_{\alpha\in\mathcal{I}} \in \mathbb{R}$, it is said that $\lim_{\mathcal{U}} x_\alpha = x$ if for each $\varepsilon > 0$ we have $\{\alpha \in \mathcal{I} \mid |x_\alpha - x| < \varepsilon\} \in \mathcal{U}$. Denote by $\ell_\infty(\mathcal{I}, X)$ the set of all bounded nets in X indexed by $\mathcal{I}$. Let $N_{\mathcal{U}} \subset \ell_\infty(\mathcal{I}, X)$ be such that $(x_\alpha)_{\alpha\in\mathcal{I}} \in N_{\mathcal{U}}$ if and only if $\lim_{\mathcal{U}} \|x_\alpha\| = 0$. Notice that $N_{\mathcal{U}}$ is closed. The *ultrapower* of X, denoted by $X^{\mathcal{U}}$, is the quotient space $\ell_\infty(\mathcal{I}, X)/N_{\mathcal{U}}$ equipped with the quotient norm $\|[(x_\alpha)]\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_\alpha\|$.

Recall that Banach spaces with the Daugavet property are octahedral, i.e. 2-average rough. In [KW], V. Kadets and D. Werner proved that the Daugavet property does not necessarily pass to ultrapowers of Banach spaces.

Theorem 3.29 (see [KW, Theorem 3.3])**.** *There exists a Banach space X with the Daugavet property such that its ultrapower $X^{\mathcal{U}}$ does not have the Daugavet property.*

On the other hand, J. D. Hardtke showed, in [Har], that a Banach space and its ultrapower are octahedral at the same time, contrasting the instability of the Daugavet property.

Proposition 3.30 (see [Har, Proposition 4.1])**.** *Let X be a Banach space and $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. Then X is octahedral if and only if its ultrapower $X^{\mathcal{U}}$ is octahedral.*

In [Nad], the work of Hardtke was extended to average roughness. We will include the proof of the following proposition for completeness.

Proposition 3.31. *Let X be a Banach space, $\mathcal{U}$ an ultrafilter on an index set $\mathcal{I}$ and $\delta > 0$. Then X is δ-average rough if and only if its ultrapower $X^{\mathcal{U}}$ is δ-average rough.*

Proof. First, suppose that X is δ-average rough. Let $n \in \mathbb{N}$, $z_1, \dots, z_n \in S_{X^{\mathcal{U}}}$, where for each $i \in \{1, \dots, n\}$, (x_i^α) is a representative of z_i. From our assumption we get that for every $\alpha \in \mathcal{I}$ there exists $x_\alpha \in X$ such that $\|x_\alpha\| = \varepsilon$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i^\alpha + x_\alpha\| + \|x_i^\alpha - x_\alpha\|\Big) > (\delta - \varepsilon)\|x_\alpha\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i^\alpha\|.$$

Let $z = [(x_\alpha)]$. Notice that $\|z\|_{\mathcal{U}} = \varepsilon$. Because $\lim_{\mathcal{U}} \|x_i^\alpha\| = \|z_i\|_{\mathcal{U}} = 1$, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(\|z_i + z\|_{\mathcal{U}} + \|z_i - z\|_{\mathcal{U}}\Big) &= \lim_{\mathcal{U}}\left(\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i^\alpha + x_\alpha\| + \|x_i^\alpha - x_\alpha\|\Big)\right)\\ &\geq \lim_{\mathcal{U}}\Big((\delta-\varepsilon)\|x_\alpha\| + \frac{2}{n}\sum_{i=1}^{n}\|x_i^\alpha\|\Big)\\ &= (\delta-\varepsilon)\|z\|_{\mathcal{U}} + 2.\end{aligned}$$

Therefore $X^{\mathcal{U}}$ is δ-average rough.

Now suppose that $X^{\mathcal{U}}$ is δ-average rough. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$ and $\varepsilon \in (0,1)$. It is sufficient to find $y \in X$ such that $\|y\| = \varepsilon$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y\| + \|x_i - y\|\Big) > (\delta - 4\varepsilon)\|y\| + 2.$$

By the canonical embedding, we can view X as a subspace of $X^{\mathcal{U}}$. Notice that $x_1, \ldots, x_n \in S_{X^{\mathcal{U}}}$. Because $X^{\mathcal{U}}$ is δ-average rough there exists $y = [(y_\alpha)] \in X^{\mathcal{U}}$ such that $\|y\|_{\mathcal{U}} = \varepsilon$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y\|_{\mathcal{U}} + \|x_i - y\|_{\mathcal{U}}\Big) > (\delta - \varepsilon)\|y\| + 2.$$

By properties of the ultralimit we get that

$$K := \left\{\alpha \in \mathcal{I} \;\middle|\; \frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y_\alpha\| + \|x_i - y_\alpha\|\Big) > (\delta-\varepsilon)\|y\| + 2 - \varepsilon^2\right\} \in \mathcal{U}.$$

Because $\lim_{\mathcal{U}} \|y_\alpha\| = \|y\|_{\mathcal{U}} = \varepsilon$ we have $L := \{\alpha \in \mathcal{I} \mid \big|\|y_\alpha\| - \varepsilon\big| < \varepsilon^2\} \in \mathcal{U}$. Choose $\beta \in K \cap L$. Then $\big|\|y_\beta\| - \varepsilon\big| < \varepsilon^2$ and

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i + y_\beta\| + \|x_i - y_\beta\|\Big) &> (\delta-\varepsilon)\varepsilon + 2 - \varepsilon^2\\ &= (\delta - 2\varepsilon)\varepsilon + 2.\end{aligned}$$

Let $y = \varepsilon\frac{y_\beta}{\|y_\beta\|}$. Because $\beta \in L$ we have $\|y - y_\beta\| = \big|\|y_\beta\| - \varepsilon\big| < \varepsilon^2$. Therefore

$$\|x_i \pm y\| \geq \|x_i \pm y_\beta\| - \|y - y_\beta\| > \|x_i \pm y_\beta\| - \varepsilon^2.$$

Now it is not difficult to see that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i+y\|+\|x_i-y\|\Big) &> \frac{1}{n}\sum_{i=1}^{n}\Big(\|x_i+y_\beta\|+\|x_i-y_\beta\|-2\varepsilon^2\Big)\\ &> (\delta-2\varepsilon)\varepsilon+2-2\varepsilon^2\\ &= (\delta-4\varepsilon)\|y\|+2.\end{aligned}$$

Therefore X is δ-average rough. □

We also have the same behaviour for roughness, with the proof being analogous to the proof of Proposition 3.31 by taking $n = 1$.

Proposition 3.32. *Let X be a Banach space, $\mathcal{U}$ an ultrafilter on an index set $\mathcal{I}$ and $\delta > 0$. Then X is δ-rough if and only if its ultrapower $X^{\mathcal{U}}$ is δ-rough.*

3.5 Roughness of spaces of operators

In this section, we will investigate necessary and sufficient conditions for the space of bounded linear operators between two Banach spaces to be rough or average rough.

We start by extending the notion of octahedrality.

Definition 3.33 (see [HLP2, Definition 2.1])**.** Let X be a Banach space. It is said that X is *alternatively octahedral* if for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\max\{\|x_i+y\|, \|x_i-y\|\} > 2-\varepsilon \quad \text{for every } i \in \{1,\dots,n\}.$$

Using Proposition 1.8, it is clear that every octahedral space is alternatively octahedral. However, one can easily verify that the spaces c_0, ℓ_1^n and ℓ_∞^n, where $n \in \mathbb{N}$, are alternatively octahedral but fail to be octahedral.

In [HLP2], sufficient conditions for the space of bounded linear operators to be rough or average rough were given.

Theorem 3.34 (see [HLP2, Theorem 2.1] and [HLP2, Theorem 2.2])**.** *Let X and Y be Banach spaces, $\delta > 0$ and $H \subset \mathcal{L}(X, Y)$ such a closed subspace that includes the finite rank operators.*

(a) *If X^* is δ-average rough (resp., δ-rough) and Y is alternatively octahedral, then H is δ-average rough (resp., δ-rough).*

(b) *If Y is δ-average rough (resp., δ-rough) and X^* is alternatively octahedral, then H is δ-average rough (resp., δ-rough).*

Let us now give necessary conditions in order for the space of bounded linear operators to be average rough.

Theorem 3.35. *Let X and Y be Banach spaces, $\delta_1 > 2\delta_2 > 0$ and $H \subset \mathcal{L}(X, Y)$ such a closed subspace that includes the finite rank operators. Let H be δ_1-average rough.*

(a) *If X^* is not δ_2-rough, then Y is $(\delta_1 - 2\delta_2)$-average rough.*

(b) *If Y is not δ_2-rough, then X^* is $(\delta_1 - 2\delta_2)$-average rough.*

Proof. (a). Suppose that X^* is not δ_2-rough. Let $n \in \mathbb{N}$, $y_1, \ldots, y_n \in S_Y$ and $\varepsilon \in (0, 1/3)$. Due to our assumption there is a w^*-slice $S(B_{X^{**}}, x^*, \alpha)$ such that $\operatorname{diam}\big(S(x^*, \alpha)\big) \leq \delta_2$. For each $i \in \{1, \ldots, n\}$ define $S_i = x^* \otimes y_i$ and notice that $\|S_i\| = 1$. Because H is δ_1-average rough, there exists $T \in H$ such that $\|T\| < \min\{\varepsilon, \alpha/3\}$ and

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\|S_i + T\| + \|S_i - T\|\Big) > (\delta_1 - \varepsilon)\|T\| + 2.$$

For $j \in \{1, 2\}$ let $y^*_{j,i} \in S_{Y^*}$ and $x_{j,i} \in S_X$ be such that

$$y^*_{j,i}\big((S_i + (-1)^{j+1}T)x_{j,i}\big) \geq \|S_i + (-1)^{j+1}T\| - \varepsilon\|T\|$$

or equivalently

$$x^*(x_{j,i})y^*_i(y_i) + (-1)^{j+1}y^*_{j,i}(Tx_{j,i}) \geq \|S_i + (-1)^{j+1}T\| - \varepsilon\|T\|.$$

Without loss of generality we may assume that $x^*(x_{j,i}), y^*_{j,i}(y) > 0$. Now we see that

$$\begin{aligned}
x^*(x_{j,i}) &\geq x^*(x_{j,i})y^*_{j,i}(y_i) \\
&\geq \|S_i + (-1)^{j+1}T\| - \varepsilon\|T\| - |y^*_{j,i}(Tx_{j,i})| \\
&\geq \|S_i\| - (2 + \varepsilon)\|T\| \\
&> 1 - 3\|T\| \geq 1 - \alpha.
\end{aligned}$$

Which means that $x_{1,i}, x_{2,i} \in S(x^*, \alpha)$. Denote $v = Tx_{1,1}$. Then $\|v\| \leq$

$\|T\| \le \varepsilon$ and

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\Big(\|y_i+v\|+\|y_i-v\|\Big)\\
&\qquad = \frac{1}{n}\sum_{i=1}^{n}\Big(\|y_i+Tx_{1,1}\|+\|y_i-Tx_{1,1}\|\Big)\\
&\qquad \ge \frac{1}{n}\sum_{i=1}^{n}\Big(\|y_i+Tx_{1,i}\|+\|y_i-Tx_{2,i}\|\\
&\qquad\qquad -\|Tx_{1,i}-Tx_{1,1}\|-\|Tx_{2,i}-Tx_{1,1}\|\Big)\\
&\qquad \ge \frac{1}{n}\sum_{i=1}^{n}\Big(y_1^*(y_i+Tx_1)+y_2^*(y_i-Tx_2)\\
&\qquad\qquad -\|T\|\left(\|x_{1,i}-x_{1,1}\|+\|x_{2,i}-x_{1,1}\|\right)\Big)\\
&\qquad \ge \frac{1}{n}\sum_{i=1}^{n}\Big(x^*(x_{1,i})y_{1,i}^*(y_i)+y_{1,i}^*(Tx_{1,i})\\
&\qquad\qquad +x^*(x_{2,i})y_{2,i}^*(y_i)-y_{2,i}^*(Tx_{2,i})\Big)-2\delta_2\|T\|\\
&\qquad \ge \frac{1}{n}\sum_{i=1}^{n}\Big(\|S_i+T\|+\|S_i-T\|\Big)-2\varepsilon\|T\|-2\delta_2\|T\|\\
&\qquad > (\delta_1-\varepsilon)\|T\|+2-(2\delta_2+2\varepsilon)\|T\|\\
&\qquad \ge (\delta_1-2\delta_2-3\varepsilon)\|v\|+2.
\end{aligned}$$

Therefore Y is $\delta_1-2\delta_2$-average rough.

The proof of (b) is analogous to the proof of (a). □

It is said that a Banach space is *non-rough* if it is not δ-rough for any $\delta > 0$. Recall that due to the dual characterisation of roughness (see Theorem 3.3) it is obvious that a Banach space X is non-rough whenever B_{X^*} possesses w^*-slices of arbitrarily small diameter.

The necessary conditions obtained in [HLP2] are a direct consequence of the previous theorem.

Corollary 3.36 (see [HLP2, Theorem 3.1])**.** *Let X and Y be Banach spaces, $\delta > 0$ and $H \subset \mathcal{L}(X,Y)$ such a closed subspace that includes the finite rank operators. Let H be δ-average rough.*

(a) *If X^* is non-rough, then Y is δ-average rough.*

(b) *If Y is non-rough, then X^* is δ-average rough.*

A similar result to Theorem 3.35 holds for roughness as well. Actually, the result is slightly sharper.

Theorem 3.37. *Let X and Y be Banach spaces, $\delta_1 > \delta_2 > 0$ and $H \subset \mathcal{L}(X, Y)$ such a closed subspace that includes the finite rank operators. Let H be δ_1-rough.*

(a) *If X^* is not δ_2-rough, then Y is $(\delta_1 - \delta_2)$-rough.*

(b) *If Y is not δ_2-rough, then X^* is $(\delta_1 - \delta_2)$-rough.*

Proof. The proof is analogous to the proof of Theorem 3.35. □

Again, we recover the corresponding necessary conditions from [HLP2].

Corollary 3.38 (see [HLP2, Theorem 3.1])**.** *Let X and Y be Banach spaces, $\delta > 0$ and $H \subset \mathcal{L}(X, Y)$ such a closed subspace that includes the finite rank operators. Let H be δ-rough.*

(a) *If X^* is non-rough, then Y is δ-rough.*

(b) *If Y is non-rough, then X^* is δ-rough.*

Chapter 4

Daugavet indices of thickness

In [Rue], A. Rueda Zoca introduced the Daugavet index of thickness in order to quantitatively measure how distant a Banach space is from having the Daugavet property. In this chapter, we continue the investigation of the behaviour of this index and also consider two new versions of the Daugavet index of thickness. We then study the behaviour of the Daugavet indices on absolute sums of Banach spaces. Finally, we connect these indices to diameter 2 properties and prove that an open question posed by Y. Ivakhno in 2006 about the relation between the radius and diameter of slices has a negative answer. This chapter is mainly based on [HLLNR].

4.1 Introduction and definitions

In [Whi], R. Whitley introduced the thickness index of Banach spaces, which is essentially the inner measure of non-compactness of the unit sphere. In [CPS], it was shown that if X is an infinite-dimensional Banach space, then Whitley's original index is equal to the following index

$$T(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in S_X \\ \text{with } B_X \subset \bigcup_{i=1}^{n} B(x_i, r) \end{array} \right\}.$$

In [God, Remark II.5 2)], Godefroy observed that Whitley's thickness index is closely related to octahedrality. Namely, for a Banach space X we have $T(X) = 2$ if and only if X is octahedral (see [God]).

Motivated by all of the above and the fact that Banach spaces with the Daugavet property are octahedral, Rueda Zoca introduced, in [Rue], the Daugavet index of thickness.

$$\mathcal{T}(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } n \in \mathbb{N},\, x_1, \dots, x_n \in S_X \text{ and relatively} \\ w\text{-open } W \text{ in } B_X \text{ such that } \emptyset \neq W \subset \bigcup_{i=1}^{n} B(x_i, r) \end{array} \right\}.$$

Rueda Zoca also noted (see [Rue, page 10]) that $\mathcal{T}(X) \leq T(X)$, for any Banach space X, and that the inequality can be strict, for example, $0 = \mathcal{T}(\ell_1) < T(\ell_1) = 2$.

As was pointed out in [Rue, Lemma 4.2], one can consider $n = 1$ in the definition of $\mathcal{T}(\cdot)$ without changing the index. From here on we will consider only $n = 1$.

Notice that $0 \leq \mathcal{T}(X) \leq 2$ for any Banach space X and for example, $\mathcal{T}(\ell_1) = 0$, $\mathcal{T}(c_0) = \mathcal{T}(\ell_\infty) = 1$ and $\mathcal{T}(C[0,1]) = 2$. In fact, $\mathcal{T}(X) = 2$ if and only if X has the Daugavet property (see Proposition 1.11). Clearly, if X has the Radon–Nikodým property, we have $\mathcal{T}(X) = 0$. The converse, however, does not hold in general (see Remark 4.2 below).

The following new Daugavet indices of thickness were defined in [HLLNR]:

$$\mathcal{T}^s(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a slice } S \text{ of } B_X \\ \text{such that } S \subset B(x, r) \end{array} \right\}$$

and

$$\mathcal{T}^{cc}(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and } C \text{ a convex combination} \\ \text{of slices of } B_X \text{ such that } C \subset B(x, r) \end{array} \right\}.$$

Remark 4.1. Notice that, due to Lemma 1.2, we can consider weakly open sets instead of slices in the definition of $\mathcal{T}^{cc}(\cdot)$ without changing the index.

Observe that for every Banach space X we have

$$0 \leq \mathcal{T}^{cc}(X) \leq \mathcal{T}(X) \leq \mathcal{T}^s(X) \leq 2$$

and we will prove that every value between 0 and 2 can be attained (see Theorem 4.13 below).

Remark 4.2. The inequalities above can be strict. Let X be the renorming of c_0 from [BLR2, Theorem 2.4] which has the LD2P but B_X contains relatively weakly open subsets of arbitrarily small diameter. Due to Lemma 4.17 we have $\mathcal{T}^s(X) \geq 1$ but at the same time $\mathcal{T}(X) = 0$. Analogously, if we take Y to be the renorming of c_0 from [BLR3, Theorem 2.5], we get that $\mathcal{T}(Y) \geq 1$ but $\mathcal{T}^{cc}(Y) = 0$.

Rueda Zoca noted that when X is a dual space it is worthwhile to also consider the index

$$\mathcal{T}_{w^*}(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a relatively } w^*\text{-open} \\ W \text{ in } B_X \text{ such that } \emptyset \neq W \subset B(x, r) \end{array} \right\}$$

and we will additionally consider the following w^*-indices

$$\mathcal{T}^s_{w^*}(X) = \inf\left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a } w^*\text{-slice} \\ S \text{ of } B_X \text{ such that } S \subset B(x,r) \end{array} \right\}$$

and

$$\mathcal{T}^{cc}_{w^*}(X) = \inf\left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exist } x \in S_X \text{ and a convex} \\ \text{combination } C \text{ of relatively } w^*\text{-open} \\ \text{subsets of } B_X \text{ such that } C \subset B(x,r) \end{array} \right\}.$$

As with the regular index, for any Banach space X, we have

$$0 \leq \mathcal{T}^{cc}_{w^*}(X^*) \leq \mathcal{T}_{w^*}(X^*) \leq \mathcal{T}^s_{w^*}(X^*) \leq 2. \tag{4.1}$$

In [Rue], it is shown that $\mathcal{T}(X) = \mathcal{T}_{w^*}(X^*) = 2$ if and only if X has the Daugavet property. It is not difficult to see that $\mathcal{T}^s(X) = \mathcal{T}^s_{w^*}(X^*) = 2$ and $\mathcal{T}^{cc}(X) = \mathcal{T}^{cc}_{w^*}(X^*) = 2$ if and only if X has the Daugavet property as well (see Proposition 1.11).

Also, observe that

$$\mathcal{T}^s(X^{**}) \leq \mathcal{T}^s_{w^*}(X^{**}) \leq \mathcal{T}^s(X) \tag{4.2}$$

and

$$\mathcal{T}(X^{**}) \leq \mathcal{T}_{w^*}(X^{**}) \leq \mathcal{T}(X) \tag{4.3}$$

and

$$\mathcal{T}^{cc}(X^{**}) \leq \mathcal{T}^{cc}_{w^*}(X^{**}) \leq \mathcal{T}^{cc}(X). \tag{4.4}$$

Remark 4.3. Let us make some observations about the above indices.

(a) By considering the biduals of the Banach spaces which give us the strict inequalities between the regular indices and taking into account equations (4.2)–(4.4), one has that the inequalities in equation (4.1) can, in general, be strict. Indeed, let X be the Banach space from Remark 4.2 such that $\mathcal{T}^s(X) \geq 1$ and $\mathcal{T}(X) = 0$. Since X has the LD2P, we have that X^{**} has the w^*-LD2P and thus $T^s_{w^*}(X^{**}) \geq 1$ (cf. Lemma 4.17). On the other hand, $\mathcal{T}_{w^*}(X^{**}) = 0$ because of equation (4.3) and $\mathcal{T}(X) = 0$. Analogously, the Banach space Y from Remark 4.2 gives us that $\mathcal{T}^{cc}_{w^*}(Y^{**}) < \mathcal{T}_{w^*}(Y^{**})$.

(b) Given a dual Banach space X^*, the inequality $\mathcal{T}(X^*) \leq \mathcal{T}_{w^*}(X^*)$ may be strict. Indeed, if $X = C[0,1]$, then $\mathcal{T}^{cc}_{w^*}(X^*) = 2$ since X has the Daugavet property. However, $\mathcal{T}^s(X^*) = 0$ since B_{X^*} contains slices of arbitrarily small diameter (see Example 1.5). This shows that all of the first inequalities of equations (4.2)–(4.4) can be strict.

(c) Again take $X = C[0,1]$. Then $\mathcal{T}^{cc}(X) = 2$ since X has the Daugavet property. However, $\mathcal{T}^{s}_{w^*}(X^{**}) < 2$ since X^* fails the Daugavet property. This shows that all of the second inequalities of equations (4.2)–(4.4) can be strict.

Rueda Zoca showed that the Daugavet index has a connection to the Daugavet equation even if $\mathcal{T}(\cdot) < 2$.

Proposition 4.1 (see [Rue, Proposition 4.4])**.** *Let X be a Banach space. For every norm one and weakly compact operator $T\colon X \to X$, it follows that*

$$\|I + T\| \geq \mathcal{T}(X)$$

and

$$\|I + T\| \geq \mathcal{T}_{w^*}(X^*).$$

In light of Proposition 4.1, it was asked in [Rue, Problem 5.3] whether the equality

$$\inf\left\{ \|I + T\| \;\middle|\; \begin{array}{l} T \in \mathcal{L}(X),\ \|T\| = 1 \text{ and} \\ T \text{ is weakly compact} \end{array} \right\} = \max\{\mathcal{T}(X), \mathcal{T}_{w^*}(X^*)\} \quad (4.5)$$

holds for every Banach space X. We will now show that equality (4.5) does not hold in general. We begin by observing that the proof of [Rue, Proposition 4.4] actually shows the following.

Proposition 4.2. *Let X be a Banach space. Then, for every norm one and weakly compact operator $T\colon X \to X$, it follows that*

$$\|I + T\| \geq \max\{\mathcal{T}^s(X), \mathcal{T}^s_{w^*}(X^*)\}.$$

Remark 4.4. By [BLR2, Theorem 2.4], there exists an equivalent renorming Z of c_0 such that all slices of B_Z have diameter two and there are relatively weakly open subsets of B_Z with arbitrarily small diameter. Thus, by Lemma 4.17 below, we have $\mathcal{T}^s(Z) \geq 1$ but $\mathcal{T}(Z) = 0 = \mathcal{T}_{w^*}(Z^*)$ (notice that Z^* has the Radon–Nikodým property because it is isomorphic to ℓ_1 and the result follows from [FHHMZ, Theorem 11.8]).

Therefore equality (4.5) fails for this Banach space Z and it is natural to ask the following question.

Question 4.3. *Does the equality*

$$\inf\left\{ \|I + T\| \;\middle|\; \begin{array}{l} T \in \mathcal{L}(X),\ \|T\| = 1, \\ \textit{and } T \textit{ is weakly compact} \end{array} \right\} = \max\{\mathcal{T}^s(X), \mathcal{T}^s_{w^*}(X^*)\}$$

hold for every Banach space X?

Let us now show that the Daugavet index $\mathcal{T}^s(\cdot)$ behaves well with respect to the Banach–Mazur distance. Recall that it is a distance between two isomorphic Banach spaces X and Y, which is defined by

$$d(X,Y) := \inf\{\|L\|\|L^{-1}\| \mid L\colon X \to Y \text{ is an isomorphism}\}.$$

Proposition 4.4. *Let X be a Banach space and $r \in [0,2]$. If for every $\delta > 0$, there exists a Banach space Y, which is isomorphic to X, such that $d(X,Y) < 1+\delta$, with $\mathcal{T}^s(Y) = r$, then $\mathcal{T}^s(X) \geq r$.*

Proof. Let $S(B_X, x^*, \alpha)$ be a slice of B_X, $x \in S_X$ and $\varepsilon > 0$. Let $\delta \in (0, \min\{\alpha, \varepsilon\})$ and let us find a Banach space Y with $\mathcal{T}^s(Y) = r$ such that $d(X,Y) < 1+\delta$. Then there exist an isomorphism $L : X \to Y$ and such that $\|L\| = 1$ and $\|L^{-1}\| < 1+\delta$.

Consider

$$y^* := \frac{(L^{-1})^* x^*}{\|(L^{-1})^* x^*\|} \in S_{Y^*} \quad \text{and} \quad y := \frac{Lx}{\|Lx\|} \in S_Y.$$

Since $\mathcal{T}^s(Y) = r$, we can find a $v \in S(B_Y, y^*, \delta^2)$ such that $\|v - y\| \geq r - \varepsilon$. Denote $u := \frac{L^{-1}v}{(1+\delta)}$ and observe that $u \in S(B_X, x^*, \delta) \subset S(B_X, x^*, \alpha)$. Our aim now is to show that $\|u - x\| \geq r - 3\varepsilon$. Indeed,

$$\begin{aligned}
r - \varepsilon \leq \|v - \frac{Lx}{\|Lx\|}\| &\leq \|L^{-1}v - \frac{x}{\|Lx\|}\| \\
&\leq \|u - x\| + \|(1+\delta)u - u\| + \|x - \frac{x}{\|Lx\|}\| \\
&\leq \|u - x\| + \delta + \delta \\
&< \|u - x\| + 2\varepsilon.
\end{aligned}$$

Hence $\|u - x\| \geq \mathcal{T}^s(Y) - 3\varepsilon$ and from the arbitrariness of ε, we have that $\mathcal{T}^s(X) \geq \mathcal{T}^s(Y)$. □

Note that we do not know whether the previous proposition holds for the other Daugavet indices.

Question 4.5. *Does the analogue of Proposition 4.4 hold for $\mathcal{T}(\cdot)$ and $\mathcal{T}^{cc}(\cdot)$?*

An application of Proposition 4.4 together with the fact that $\mathcal{T}^s(X) = 2$ means that X has the Daugavet property immediately gives us that the class of Banach spaces with the Daugavet property is closed with respect to the Banach–Mazur distance.

Corollary 4.6. *Let X be a Banach space. If for every $\delta > 0$, there exists a Banach space Y, which is isomorphic to X, such that $d(X,Y) < 1+\delta$ and Y has the Daugavet property, then X also has the Daugavet property.*

4.2 Daugavet indices in absolute sums

In this section, we describe how the Daugavet indices behave on absolute normalised norms, in particular on ℓ_p-norms. As a consequence of our results, we prove that for each $r \in [0,2]$ there is a Banach space X such that $\mathcal{T}^s(X) = \mathcal{T}(X) = \mathcal{T}^{cc} = r$.

We start by recalling a result about the behaviour of $\mathcal{T}(\cdot)$ on ℓ_p-sums from [Rue].

Proposition 4.7 (see [Rue, Proposition 4.5])**.** *Let X and Y be Banach spaces. Then*

(a) $\mathcal{T}(X \oplus_1 Y) \leq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$*;*

(b) $\mathcal{T}(X \oplus_p Y) \leq (\frac{(2^{1/p}+1)^p}{2})^{1/p}$ *for every* $1 < p < \infty$*;*

(c) $\mathcal{T}(X \oplus_\infty Y) \geq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$*, where equality holds if* $\mathcal{T}(X \oplus_\infty Y) > 1$*.*

Next, we establish a lower estimate for all of the Daugavet indices of thickness in absolute sums.

Proposition 4.8. *Let X and Y be Banach spaces, N be an absolute normalised norm on $\mathbb{R}^2$ and $\gamma > 0$ is such that $N(\cdot) \geq \gamma \|\cdot\|_1$. Then*

(a) $\mathcal{T}^s(X \oplus_N Y) \geq 2\gamma\big(\min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\} - 1\big)$*;*

(b) $\mathcal{T}(X \oplus_N Y) \geq 2\gamma\big(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1\big)$*;*

(c) $\mathcal{T}^{cc}(X \oplus_N Y) \geq 2\gamma\big(\min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\} - 1\big)$*.*

In particular, $\mathcal{T}(X \oplus_p Y) \geq 2^{1/p}\big(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1\big)$ *whenever* $1 < p < \infty$*.*

Proof. We will only prove (b) because the proofs of (a) and (c) are very similar.

(b). Without loss of generality let us assume that $\min\{\mathcal{T}(X), \mathcal{T}(Y)\} > 1$, otherwise the lower bound we are trying to prove trivially holds. Let $\varepsilon > 0$ be such that $\min\{\mathcal{T}(X), \mathcal{T}(Y)\} > 1 + \varepsilon$. Denote $Z := X \oplus_N Y$. Let $(\tilde{x}, \tilde{y}) \in S_Z$ and let W be a non-empty relatively weakly open subset of B_Z.

Without loss of generality we may assume that

$$W = \{z \in B_Z \mid |z_i^*(z) - z_i^*(z_0)| < 1,\ i \in \{1, \dots, n\}\},$$

for some $z_i^* = (x_i^*, y_i^*) \in Z^*$ and $z_0 = (x_0, y_0) \in S_Z$.

Define now

$$U := \begin{cases} \{x \in B_X \mid |x_i^*(x) - x_i^*(\frac{x_0}{\|x_0\|})| < \frac{1}{2\|x_0\|},\ i \in \{1, \dots, n\}\}, & \text{if } x_0 \neq 0, \\ \{x \in B_X \mid |x_i^*(x)| < \frac{1}{2},\ i \in \{1, \dots, n\}\}, & \text{if } x_0 = 0, \end{cases}$$

and

$$V := \begin{cases} \{y \in B_Y \mid |y_i^*(y) - y_i^*(\frac{y_0}{\|y_0\|})| < \frac{1}{2\|y_0\|},\ i \in \{1, \dots, n\}\}, & \text{if } y_0 \neq 0, \\ \{y \in B_Y \mid |y_i^*(y)| < \frac{1}{2},\ i \in \{1, \dots, n\}\}, & \text{if } y_0 = 0. \end{cases}$$

From now on we will distinguish two cases.

Case 1. Assume first that $\tilde{x} \neq 0$ and $\tilde{y} \neq 0$. Due to the definition of the Daugavet index we can find $u \in U$ and $v \in V$ such that $\|\frac{\tilde{x}}{\|\tilde{x}\|} - u\| \geq \mathcal{T}(X) - \varepsilon/2$ and $\|\frac{\tilde{y}}{\|\tilde{y}\|} - v\| \geq \mathcal{T}(Y) - \varepsilon/2$.

Claim. If two elements e and $\tilde{e}$ in the unit ball of a Banach space E satisfy that $\|e + \tilde{e}\| \geq 1 + \alpha$ for some $\alpha \in [0, 1]$, then $\|\lambda e + \mu \tilde{e}\| \geq (\lambda + \mu)\alpha$ for every $\lambda, \mu \geq 0$. Indeed, the claim follows from the inequalities

$$(\lambda + \mu)(1 + \alpha) \leq (\lambda + \mu)\|e + \tilde{e}\| \leq \|\lambda e + \mu \tilde{e}\| + \lambda + \mu.$$

Since

$$\mathcal{T}(X) - \varepsilon/2 = 1 + (\mathcal{T}(X) - 1 - \varepsilon/2) > 1 + \varepsilon/2,$$

we can apply the Claim above and get that

$$\|\tilde{x} - \|x_0\|u\| \geq (\|\tilde{x}\| + \|x_0\|)(\mathcal{T}(X) - 1 - \varepsilon/2).$$

Similarly, one obtains $\|\tilde{y} - \|y_0\|v\| \geq (\|\tilde{y}\| + \|y_0\|)(\mathcal{T}(Y) - 1 - \varepsilon/2)$.

Observe that $(\|x_0\|u, \|y_0\|v) \in W$ and

$$\begin{aligned} \|(\tilde{x}, \tilde{y}) - (\|x_0\|u, \|y_0\|v)\|_N &= N\Big(\|\tilde{x} - \|x_0\|u\|, \|\tilde{y} - \|y_0\|v\|\Big) \\ &\geq N\Big((\|\tilde{x}\| + \|x_0\|)(\mathcal{T}(X) - 1 - \varepsilon/2), \\ &\qquad (\|\tilde{y}\| + \|y_0\|)(\mathcal{T}(Y) - 1 - \varepsilon/2)\Big) \\ &\geq N\Big(\|\tilde{x}\| + \|x_0\|, \|\tilde{y}\| + \|y_0\|\Big)(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1 - \varepsilon/2) \\ &\geq \gamma(\|\tilde{x}\| + \|\tilde{y}\| + \|x_0\| + \|y_0\|)(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1 - \varepsilon/2) \\ &\geq 2\gamma(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1 - \varepsilon/2). \end{aligned}$$

Hence, by the arbitrariness of our choice of ε, we have that $\mathcal{T}(X \oplus_N Y) \geq 2\gamma(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1)$.

Case 2. Assume now that $\tilde{y} = 0$, hence $\|\tilde{x}\| = 1$ (the case when $\tilde{x} = 0$ follows similarly). Now we can find $u \in U$ such that $\|\tilde{x} - u\| \geq \mathcal{T}(X) - \varepsilon/2$ and let $v \in V \cap S_Y$. Again, note that $(\|x_0\|u, \|y_0\|v) \in W$ and

$$\begin{aligned}\|(\tilde{x}, 0) - (\|x_0\|u, \|y_0\|v)\|_N &= N\Big(\|\tilde{x} - \|x_0\|u\|, \|y_0\|\Big)\\ &\geq N\Big((1 + \|x_0\|)(\mathcal{T}(X) - 1 - \varepsilon/2), \|y_0\|\Big)\\ &\geq N\Big(1 + \|x_0\|, \frac{\|y_0\|}{(\mathcal{T}(X) - 1 - \varepsilon/2)}\Big)(\mathcal{T}(X) - 1 - \varepsilon/2)\\ &\geq \gamma(1 + \|x_0\| + \|y_0\|)(\mathcal{T}(X) - 1 - \varepsilon/2)\\ &\geq 2\gamma(\mathcal{T}(X) - 1 - \varepsilon/2),\end{aligned}$$

Again, by the arbitrariness of our choice of ε, we conclude that $\mathcal{T}(X \oplus_N Y) \geq 2\gamma(\min\{\mathcal{T}(X), \mathcal{T}(Y)\} - 1)$. □

Remark 4.5. With an almost identical proof one can generalise Proposition 4.8 to a finite direct sum of Banach spaces equipped with a correspondingly defined absolute normalised norm on said direct sum and the estimates will remain the same.

We now turn our attention to the upper estimate of these Daugavet indices of thickness.

Proposition 4.9. *Let X and Y be Banach spaces, N be an absolute normalised norm on $\mathbb{R}^2$ and $\Gamma > 0$ is such that $N(\cdot) \leq \Gamma\|\cdot\|_\infty$. If $(1, 0)$ or $(0, 1)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$, then $\mathcal{T}^s(X \oplus_N Y) \leq \Gamma$. In particular, $\mathcal{T}^s(X \oplus_p Y) \leq 2^{1/p}$ whenever $1 < p < \infty$.*

Proof. Denote $Z := X \oplus_N Y$ and let $\varepsilon > 0$, $x \in S_X$ and $y \in S_Y$. Assume that $e = (0, 1)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$ (the proof for the other case is similar). Then, by Lemma 3.18, the element e is actually a strongly exposed point, which allows us to fix a $\delta > 0$ such that, whenever $(a, b) \in B_{(\mathbb{R}^2,N)}$ and $b > 1 - \delta$, then $|a| < \varepsilon$. Find $y^* \in S_{Y^*}$ with $y^*(y) = 1$. If $(u, v) \in S(B_Z, (0, y^*)), \delta)$, then $\|v\| \geq y^*(v) > 1 - \delta$. By our assumption $\|u\| < \varepsilon$. Therefore

$$\begin{aligned}\|(u, v) - (x, 0)\|_N &= N\Big(\|u - x\|, \|v\|\Big)\\ &\leq N\Big(1 + \|u\|, \|v\|\Big)\\ &\leq (1 + \|u\|)N(1, 1)\\ &\leq \Gamma(1 + \varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{T}^s(Z) \leq \Gamma$. □

Remark 4.6. As was the case with Proposition 4.8, one can also generalise Proposition 4.9 to a finite direct sum of Banach spaces equipped with a correspondingly defined absolute normalised norm on said direct sum and the estimates will remain the same.

Since $\mathcal{T}(\cdot) \leq \mathcal{T}^s(\cdot)$, then the obtained upper bound from Proposition 4.9 improves the previously known estimate from Proposition 4.7. Moreover, from Proposition 4.8 we know that this estimate is sharp. We summarise this in the following result.

Theorem 4.10. *Let X and Y be Banach spaces and $1 < p < \infty$. If X and Y both have the Daugavet property, then $\mathcal{T}^s(X \oplus_p Y) = \mathcal{T}(X \oplus_p Y) = \mathcal{T}^{cc}(X \oplus_p Y) = 2^{1/p}$.*

Proof. Since X and Y have the Daugavet property, then $\mathcal{T}^{cc}(X) = \mathcal{T}^{cc}(Y) = 2$ by Proposition 1.11. From Proposition 4.8 we get that $\mathcal{T}^{cc}(X \oplus_p Y) \geq 2^{1/p}$ and, by Proposition 4.9, we have that $\mathcal{T}^s(X \oplus_p Y) \leq 2^{1/p}$. Therefore

$$2^{1/p} \leq \mathcal{T}^{cc}(X \oplus_p Y) \leq \mathcal{T}(X \oplus_p Y) \leq \mathcal{T}^s(X \oplus_p Y) \leq 2^{1/p},$$

which completes the proof. □

Let us now show that there are spaces whose Daugavet index of thickness can be less than one.

Proposition 4.11. *For every $r > 0$ there exists a Banach space X such that*

$$\mathcal{T}^{cc}(X) = \mathcal{T}(X) = \mathcal{T}^s(X) = \frac{r}{1+r}.$$

Proof. The proof is inspired by the example exhibited in [LoR, Theorem 2.1]. Fix $r > 0$ and define

$$U^* := \operatorname{conv}\left(B_{\ell_1 \oplus_\infty \mathbb{R}} \cup \{(0, 1+r), (0, -1-r)\}\right),$$

which clearly is a w^*-compact set in $(c_0 \oplus_1 \mathbb{R})^*$. Consequently, there is a norm $|||\cdot|||$ on $c_0 \oplus \mathbb{R}$ whose unit ball is

$$U := \{(x, \beta) \in c_0 \oplus \mathbb{R} \mid \phi(x, \beta) \leq 1 \text{ for every } \phi \in U^*\}.$$

Consider $X := (c_0 \oplus \mathbb{R}, |||\cdot|||)$ and let us prove that X satisfies the desired requirements. It is clear that $U^* = B_{X^*}$. Also, it is clear that

$$\operatorname{ext}(U^*) = \{(0, \pm(1+r))\} \cup \{(\xi e_n, \psi 1) \mid n \in \mathbb{N} \text{ and } \xi, \psi \in \{-1, 1\}\}. \quad (4.6)$$

Therefore $\operatorname{ext}(U^*)' = \{(0, \pm 1)\} \subset \frac{1}{1+r}U^*$.

Also note that $B_{\ell_1 \oplus_\infty \mathbb{R}} \subset U^* \subset (1+r)B_{\ell_1 \oplus_\infty \mathbb{R}}$, so

$$\frac{1}{1+r}B_{c_0 \oplus_1 \mathbb{R}} \subset U \subset B_{c_0 \oplus_1 \mathbb{R}}.$$

Consequently, for each pair $(x, \beta) \in X$, it follows that

$$\|(x,\beta)\|_1 \leq |||(x,\beta)||| \leq (1+r)\|(x,\beta)\|_1. \tag{4.7}$$

Define $S_\delta := S(B_X, (0, 1+r), \delta)$. Fix an element $(x, \beta) \in S_\delta$ and let us estimate $|||(x,\beta) - (0, \frac{1}{1+r})|||$. To this end, notice that

$$1 \geq (1+r)\beta = (0, 1+r)(x, \beta) > 1 - \delta,$$

which gives us

$$\frac{1}{1+r} \geq \beta \geq \frac{1-\delta}{1+r}.$$

Let us show that $\|x\|_\infty \leq \frac{r+\delta}{1+r}$. Assume for contradiction that there exists $n \in \mathbb{N}$ such that $|x_n| > \frac{r+\delta}{1+r}$. Choose $\xi = \operatorname{sign}(x_n)$ and define $x^* := (\xi e_n, 1) \in U^* \subset B_{X^*}$. Then

$$1 \geq x^*((x,\beta)) = |x_n| + \beta > \frac{r+\delta}{1+r} + \frac{1-\delta}{1+r} = \frac{r+\delta+1-\delta}{1+r} = 1,$$

a contradiction. So $\|x\|_\infty \leq \frac{r+\delta}{1+r}$. Also, notice that, by equation (4.7), we have $(0, \frac{1}{1+r}) \in S_\delta$ (note that $\|(0, \frac{1}{1+r})||| \leq (1+r)\|(0, \frac{1}{1+r})\|_1 = 1$).

Let us estimate $\|(x,\beta) - (0, \frac{1}{1+r})\|$. By the Krein–Milman theorem

$$\|(x,\beta) - (0, \frac{1}{1+r})\| = \sup_{x^* \in \operatorname{ext}(U^*)} |x^*((x,\beta) - (0, \frac{1}{1+r}))|.$$

Given $x^* \in \operatorname{ext}(U^*)$, we have $x^* = (y^*, \lambda)$, where $y^* \in \ell_1$ and $\lambda \in \mathbb{R}$. We will distinguish two cases.

Case 1. First, assume that $y^* = 0$. This implies, according to equation (4.6), that $|\lambda| = 1 + r$. Since $\frac{1}{1+r} \geq \beta \geq \frac{1-\delta}{1+r}$, we get that

$$|x^*((x,\beta) - (0, \frac{1}{1+r}))| = |\lambda||\beta - \frac{1}{1+r}| \leq (1+r)\frac{\delta}{1+r} = \delta.$$

Case 2. Now assume that $y^* \neq 0$. Then, by equation (4.6), $|\lambda| = 1$ and $y^* = \pm e_k$ for suitable $k \in \mathbb{N}$. Hence

$$\begin{aligned} |x^*((x,\beta) - (0, \frac{1}{1+r}))| &= |y^*(x) + \lambda(\beta - \frac{1}{1+r})| \\ &\leq \|x\|_\infty + |\lambda||\beta - \frac{1}{1+r}| \\ &\leq \frac{r+\delta}{1+r} + \frac{\delta}{1+r}. \end{aligned}$$

Taking the above inequalities into account, we get that

$$\|(x,\beta)-(0,\frac{1}{1+r})\| \leq \frac{r+\delta}{1+r}+\frac{\delta}{1+r}.$$

This means that $S_\delta \subset B((0,\frac{1}{1+r}),\frac{r+\delta}{1+r}+\frac{\delta}{1+r})$ and therefore $\mathcal{T}^s(X) \leq \frac{r+\delta}{1+r}+\frac{\delta}{1+r}$. Since $\delta>0$ was arbitrary, we get that $\mathcal{T}^s(X) \leq \frac{r}{1+r}$.

For the second part of the proof, let us prove that $\mathcal{T}^{cc}(X) \geq \frac{r}{1+r}$, for which we will prove that every convex combination of slices of B_X has diameter at least $\frac{2r}{1+r}$. The proof is motivated by [LoR, Proposition 2.2]. Take a convex combination of slices $C=\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha_i)$, an element $\sum_{i=1}^n \lambda_i x_i \in C$ and $\varepsilon \in (0,\frac{r}{1+r})$. Given $i \in \{1,\ldots,n\}$, define

$$A_i := \{f \in \operatorname{ext}(B_{X^*}) \mid |f(x_i)| > \frac{1}{1+r}+\varepsilon\}.$$

Since $\operatorname{ext}(B_{X^*})' \subset \frac{1}{1+r}B_{X^*}$, a compactness argument implies that each A_i is finite. Consequently, we can take

$$y \in \left(\bigcap_{i=1}^n \bigcap_{f\in A_i} \ker(f)\cap\ker(x_i^*)\right)\cap S_X.$$

Fix $i \in \{1,\ldots,n\}$. Let us show that

$$x_i \pm (\frac{r}{1+r}-\varepsilon)y \in S(B_{X^*}, x_i^*, \alpha_i).$$

First, notice that

$$x_i^*(x_i \pm (\frac{r}{1+r}-\varepsilon)y) = x_i^*(x_i) > 1-\alpha$$

since $y \in \ker(x_i^*)$. On the other hand let us prove that $\|x_i \pm (\frac{r}{1+r}-\varepsilon)y\| \leq 1$. To this end, notice that

$$\|x_i \pm (\frac{r}{1+r}-\varepsilon)y\| = \sup_{f\in\operatorname{ext}(B_{X^*})} |f(x_i \pm (\frac{r}{1+r}-\varepsilon)y)|.$$

Given $f \in \operatorname{ext}(B_{X^*})$, we have two cases to consider.

Case 1. If $f \in A_i$, we get that $f(y)=0$ and so

$$|f(x_i \pm (\frac{r}{1+r}-\varepsilon)y)| = |f(x_i)| \leq 1.$$

Case 2. If $f \notin A_i$, then $|f(x_i)| \leq \frac{1}{1+r}+\varepsilon$ and so

$$|f(x_i \pm (\frac{r}{1+r}-\varepsilon)y)| \leq |f(x_i)| + (\frac{r}{1+r}-\varepsilon)|f(y)| \leq \frac{1}{1+r}+\frac{r}{1+r} = 1.$$

This implies that $\sum_{i=1}^n \lambda_i(x_i \pm (\frac{r}{1+r} - \varepsilon)y) \in C$, so

$$\begin{aligned}\operatorname{diam}(C) &\geq \left\| \sum_{i=1}^n \lambda_i \left(x_i + (\frac{r}{1+r} - \varepsilon)y \right) - \sum_{i=1}^n \lambda_i \left(x_i - (\frac{r}{1+r} - \varepsilon)y \right) \right\| \\ &= 2\frac{r}{1+r} - 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\operatorname{diam}(C) \geq \frac{2r}{1+r}$. Consequently, if there exists a convex combination of slices C such that $C \subset B(0,\rho)$, then

$$\frac{2r}{1+r} \leq \operatorname{diam}(C) \leq \operatorname{diam}(B(0,\rho)) = 2\rho.$$

Therefore $\rho \geq \frac{r}{1+r}$, which implies that $\mathcal{T}^{cc}(X) \geq \frac{r}{1+r}$.

Hence we have proved that

$$\frac{r}{1+r} \leq \mathcal{T}^{cc}(X) \leq \mathcal{T}(X) \leq \mathcal{T}^s(X) \leq \frac{r}{1+r},$$

as desired. □

Observe that the proof of Proposition 4.11 even gives the following result.

Corollary 4.12. *For any $\delta \in (0,2]$ there is a Banach space in which the minimal diameter of slices (resp., non-empty relatively weakly open sets, convex combination of slices) is exactly δ.*

Proof. If $\delta = 2$, then take $X = c_0$. Fix $\delta \in (0,2)$ and find an $r > 0$ such that $\delta = \frac{2r}{r+1}$. Consider the space X from the proof of Proposition 4.11. We know that every convex combination of slices of B_X has diameter at least δ and that for each $\varepsilon > 0$ there exists a slice which has a diameter less than $\delta + \varepsilon$. Therefore the minimal diameter of slices, non-empty relatively weakly open sets and convex combination of slices is exactly δ. □

Note that Corollary 4.12 extends Corollary 3.17 from $\delta \in (1,2]$ to $\delta \in (0,2]$, in the case of convex combinations of slices.

As a consequence of Theorem 4.10 and Proposition 4.11, we get the following result.

Theorem 4.13. *For every $r \in [0,2]$ there exists a Banach space X such that $\mathcal{T}^s(X) = \mathcal{T}(X) = \mathcal{T}^{cc}(X) = r$.*

Proof. First observe that, if r equals $0, 1$ or 2, then take X to be ℓ_1, c_0 or $C[0,1]$, respectively. If $r \in (0,1)$, then there exists a $s \in (0,\infty)$ such that $r = \frac{s}{s+1}$ and apply Proposition 4.11 to s. If $r \in (1,2)$, then there exists a $p \in (1,\infty)$ such that $r = 2^{1/p}$ and apply Theorem 4.10 to $X = C[0,1] \oplus_p C[0,1]$. □

In Proposition 4.7 (a), it seems to be unknown whether the inequality can be strict (see Question 4.16). However, for the index $\mathcal{T}^s(\cdot)$, we always have equality.

Proposition 4.14. *Let X and Y be Banach spaces. Then*

(a) $\mathcal{T}^s(X \oplus_1 Y) = \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$*;*

(b) $\mathcal{T}^s(X \oplus_p Y) \leq 2^{1/p}$ *for every* $1 < p < \infty$*;*

(c) $\mathcal{T}^s(X \oplus_\infty Y) \geq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$ *and equality holds if* $\mathcal{T}^s(X \oplus_\infty Y) > 1$.

Proof. (a). Let us first show that $\mathcal{T}^s(X \oplus_1 Y) \geq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. Set $Z := X \oplus_1 Y$ and let $S(B_Z, (x^*, y^*), \alpha)$ be a slice of B_Z, $(x, y) \in S_Z$ and $\varepsilon > 0$.

Without loss of generality suppose that $\|x^*\| = 1$. Thus we have two cases, either $x = 0$ or $x \neq 0$.

Case 1. Assume first that $x = 0$, hence $\|y\| = 1$. Now find an element $u \in S_X$ such that $x^*(u) > 1 - \alpha$. Observe that $(u, 0) \in S(B_Z, (x^*, y^*), \alpha)$ and

$$\|(u, 0) - (0, y)\| = 2 \geq \mathcal{T}^s(X).$$

Case 2. Assume now that $x \neq 0$. Consider the slice $S(B_X, x^*, \alpha)$ and $\frac{x}{\|x\|} \in S_X$. Find an $u \in S(B_X, x^*, \alpha)$ such that $\|\frac{x}{\|x\|} - u\| \geq \mathcal{T}^s(X) - \varepsilon$. Now $(u, 0) \in S(B_Z, (x^*, y^*), \alpha)$ and

$$\begin{aligned}
\|(x, y) - (u, 0)\| &= \|x - u\| + \|y\| \\
&\geq \|\frac{x}{\|x\|} - u\| - \|\frac{x}{\|x\|} - x\| + \|y\| \\
&\geq \mathcal{T}^s(X) - \varepsilon - (1 - \|x\|) + \|y\| \\
&= \mathcal{T}^s(X) - \varepsilon.
\end{aligned}$$

Therefore $\mathcal{T}^s(X \oplus_1 Y) \geq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$.

The proof of $\mathcal{T}^s(X \oplus_1 Y) \leq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$ is the same as the proof of [Rue, Proposition 4.5 (2)], except with $m = 1$.

(b). This follows immediately from Proposition 4.9.

(c). Let us first show that $\mathcal{T}^s(X \oplus_\infty Y) \geq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. Denote $Z := X \oplus_\infty Y$. Let $S(B_Z, (x^*, y^*), \alpha)$ be a slice of B_Z, $(x, y) \in S_Z$ and $\varepsilon > 0$.

Define

$$S^X := \begin{cases} S(B_X, \dfrac{x^*}{\|x^*\|}, \alpha), & \text{if } x^* \neq 0, \\ B_X, & \text{if } x^* = 0, \end{cases}$$

and

$$S^Y := \begin{cases} S(B_Y, \frac{y^*}{\|y^*\|}, \alpha), & \text{if } y^* \neq 0, \\ B_Y, & \text{if } y^* = 0. \end{cases}$$

Observe that $S^X \times S^Y \subset S(B_Z, (x^*, y^*), \alpha)$. Since $\max\{\|x\|, \|y\|\} = 1$, we will suppose from now on that $\|x\| = 1$. Hence there exists an $x_0 \in S^X$ such that $\|x_0 - x\| > \mathcal{T}^s(X) - \varepsilon$. Let $y_0 \in S^Y$ be arbitrary. We have that

$$\|(x_0, y_0) - (x, y)\| = \max\{\|x_0 - x\|, \|y_0 - y\|\} \geq \mathcal{T}^s(X) - \varepsilon.$$

The case when $\|y\| = 1$ is similar. Therefore, by the arbitrariness of ε, we see that $\mathcal{T}^s(X \oplus_\infty Y) \geq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$.

Assume now that $\mathcal{T}^s(X \oplus_\infty Y) > 1$ and let us show that then $\mathcal{T}^s(X \oplus_\infty Y) \leq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. Fix an $\varepsilon > 0$ such that $\mathcal{T}^s(X \oplus_\infty Y) - \varepsilon > 1$. Let $x \in S_X$ and $S(B_X, x^*, \alpha)$ be a slice of B_X. Observe that $S((x^*, 0), \alpha)$ is a slice of B_Z and $(x, 0) \in S_Z$. Thus there is an element $(u, v) \in S(B_Z, (x^*, 0), \alpha)$ such that

$$1 < \mathcal{T}^s(X \oplus_\infty Y) - \varepsilon \leq \|(x, 0) - (u, v)\| = \max\{\|x - u\|, \|v\|\}.$$

Since $\|v\| \leq 1$, then we must have that $\|x - u\| \geq \mathcal{T}^s(X \oplus_\infty Y) - \varepsilon$. Notice that $u \in S(B_X, x^*, \alpha)$, hence $\mathcal{T}^s(X) \geq \mathcal{T}^s(X \oplus_\infty Y) - \varepsilon$. Finally, by the arbitrariness of ε, we conclude that $\mathcal{T}^s(X \oplus_\infty Y) \leq \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$. □

Remark 4.7. The inequality in Proposition 4.14 (c) can be strict if we remove the assumption on $\mathcal{T}^s(X \oplus_\infty Y)$. Indeed, let $X = c_0$ and $Y = \mathbb{R}$, then $X \oplus_\infty Y$ is isometrically isomorphic to c_0 and

$$T^s(X \oplus_\infty Y) = 1 > 0 = \mathcal{T}^s(\mathbb{R}) = \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}.$$

We end this section by studying the index $\mathcal{T}^{cc}(\cdot)$ in ℓ_p-sums.

Proposition 4.15. *Let X and Y be Banach spaces. Then*

(a) $\mathcal{T}^{cc}(X \oplus_1 Y) \leq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$;

(b) $\mathcal{T}^{cc}(X \oplus_p Y) \leq 2^{1/p}$ *for every* $1 < p < \infty$;

(c) $\mathcal{T}^{cc}(X \oplus_\infty Y) \geq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$ *and equality holds if* $\mathcal{T}^{cc}(X \oplus_\infty Y) > 1$.

Proof. (a). Let $\varepsilon > 0$ and assume without loss of generality that $\mathcal{T}^{cc}(X) = \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$. Fix $n \in \mathbb{N}$, $x \in S_X$ and a convex combination of slices $\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha)$ such that

$$\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha) \subset B(x, \mathcal{T}^{cc}(X) + \varepsilon).$$

Let $\delta \in (0, \alpha)$ and set $Z := X \oplus_1 Y$. Observe that

$$S(B_Z, (x_i^*, 0), \delta) \subset S(B_X, x_i^*, \alpha) \times \delta B_Y$$

for every $i \in \{1, \dots, n\}$. Therefore

$$\begin{aligned}\sum_{i=1}^n \lambda_i S(B_Z, (x_i^*, 0), \delta) &\subset \sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha) \times \delta B_Y \\ &\subset B(x, \mathcal{T}^{cc}(X) + \varepsilon) \times \delta B_Y \\ &\subset B((x, 0), \mathcal{T}^{cc}(X) + \varepsilon + \delta).\end{aligned}$$

Since ε and δ can be chosen to be arbitrarily small, we have that $\mathcal{T}^{cc}(X \oplus_1 Y) \le \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$.

(b). This follows immediately from the inequality $\mathcal{T}^{cc}(\cdot) \le \mathcal{T}^{s}(\cdot)$ and Proposition 4.9.

(c). Let us first show that $\mathcal{T}^{cc}(X \oplus_\infty Y) \ge \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$. Denote $Z := X \oplus_\infty Y$. Let $n \in \mathbb{N}$, for every $i \in \{1, \dots, n\}$ let $S(B_Z, (x_i^*, y_i^*), \alpha)$ be slices of B_Z, $\lambda_i > 0$, with $\sum_{i=1}^n \lambda_i = 1$, $(x, y) \in S_Z$ and $\varepsilon > 0$. Denote $S := \sum_{i=1}^n \lambda_i S(B_Z, (x_i^*, y_i^*), \alpha)$.

Define

$$S_i^X := \begin{cases} S(B_X, \dfrac{x_i^*}{\|x_i^*\|}, \alpha), & \text{if } x_i^* \neq 0, \\ B_X, & \text{if } x_i^* = 0, \end{cases}$$

and

$$S_i^Y := \begin{cases} S(B_Y, \dfrac{y_i^*}{\|y_i^*\|}, \alpha), & \text{if } y_i^* \neq 0, \\ B_Y, & \text{if } y_i^* = 0. \end{cases}$$

Denote $S^X := \sum_{i=1}^n \lambda_i S_i^X$ and $S^Y := \sum_{i=1}^n \lambda_i S_i^Y$. Notice that $S_i^X \times S_i^Y \subset S(B_Z, (x_i^*, y_i^*), \alpha)$, which gives us $S^X \times S^Y \subset S$. Since $\max\{\|x\|, \|y\|\} = 1$, we will suppose from now on that $\|x\| = 1$. Hence there exists an $x_0 \in S^X$ such that $\|x_0 - x\| > \mathcal{T}^{cc}(X) - \varepsilon$. Let $y_0 \in S^Y$ be arbitrary. We have that

$$\|(x_0, y_0) - (x, y)\| = \max\{\|x_0 - x\|, \|y_0 - y\|\} \ge \mathcal{T}^{cc}(X) - \varepsilon.$$

The case when $\|y\| = 1$ is similar. Therefore, by the arbitrariness of ε, we see that $\mathcal{T}^{cc}(X \oplus_\infty Y) \geq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$.

Assume now that $\mathcal{T}^{cc}(X \oplus_\infty Y) > 1$ and let us show that then $\mathcal{T}^{cc}(X \oplus_\infty Y) \leq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$. Fix an $\varepsilon > 0$ such that $\mathcal{T}^{cc}(X \oplus_\infty Y) - \varepsilon > 1$. Let $x \in S_X$, $S(B_X, x_i^*, \alpha)$ be slices of B_X and $\lambda_i > 0$, such that $\sum_{i=1}^n \lambda_i = 1$. Observe that for each i we have that $S(B_Z, (x_i^*, 0), \alpha)$ is a slice of B_Z and $(x, 0) \in S_Z$. Thus there is an element $(u, v) \in \sum_{i=1}^n \lambda_i S(B_Z, (x_i^*, 0), \alpha)$ such that

$$1 < \mathcal{T}^{cc}(X \oplus_\infty Y) - \varepsilon \leq \|(x, 0) - (u, v)\| = \max\{\|x - u\|, \|v\|\}.$$

Since $\|v\| \leq 1$, then we must have that $\|x - u\| \geq \mathcal{T}^{cc}(X \oplus_\infty Y) - \varepsilon$. Notice that $u \in \sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha)$, hence $\mathcal{T}^{cc}(X) \geq \mathcal{T}^{cc}(X \oplus_\infty Y) - \varepsilon$. Finally, from the arbitrariness of ε, we conclude that $\mathcal{T}^{cc}(X \oplus_\infty Y) \leq \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$. □

Remark 4.8. The same example as in Remark 4.7 shows that the inequality in Proposition 4.15 (c) can be strict if we remove the assumption on $\mathcal{T}^{cc}(X \oplus_\infty Y)$.

From Proposition 4.14 (a), we know that we have the equality $\mathcal{T}^s(X \oplus_1 Y) = \min\{\mathcal{T}^s(X), \mathcal{T}^s(Y)\}$ holds for all Banach spaces X and Y. However, we do not know whether the corresponding equalities hold for the indices $\mathcal{T}(\cdot)$ and $\mathcal{T}^{cc}(\cdot)$ as well.

Question 4.16. *Let X and Y be Banach spaces.*

(a) $\mathcal{T}(X \oplus_1 Y) = \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$*?*

(b) $\mathcal{T}^{cc}(X \oplus_1 Y) = \min\{\mathcal{T}^{cc}(X), \mathcal{T}^{cc}(Y)\}$*?*

4.3 Connections with diameter 2 properties

In this final section, we connect the Daugavet indices of thickness to diameter two properties and the r-big slice property, introduced by Ivakhno in [Iva].

We start with an elementary lemma connecting diameter 2 properties to Daugavet indices.

Lemma 4.17. *Let X be a Banach space.*

(a) *If X has the LD2P, then* $\mathcal{T}^s(X) \geq 1$.

(b) *If X has the D2P, then* $\mathcal{T}(X) \geq 1$.

(c) *If X has the SD2P, then $\mathcal{T}^{cc}(X) \geq 1$.*

Proof. We will only prove (b) because the others are analogous. Let W be a non-empty relatively weakly open subset of B_X and $x \in S_X$. Let $\varepsilon > 0$. Since X has the D2P, we can find $u, v \in W$ with $\|u - v\| \geq 2 - \varepsilon$. Thus

$$2 - \varepsilon \leq \|u - v\| \leq \|x - u\| + \|x - v\|$$

so the radius of a ball covering W is at least $1 - \varepsilon/2$ for any $\varepsilon > 0$. Therefore $\mathcal{T}(X) \geq 1$. □

Recall that for a bounded set C of a Banach space X the *radius of C* is defined as

$$r(C) := \inf\{r > 0 \mid C \subset B(x, r) \text{ for some } x \in X\}.$$

A Banach space X is said to have the *r-big slice property* if every slice of B_X is of radius one (see [Iva]). Observe that the r-big slice property of a Banach space X implies that $\mathcal{T}^s(X) \geq 1$. Moreover, if X has the LD2P, then X has the r-big slice property. Ivakhno asked if the converse is true (see [Iva, page 96]).

In view of Ivakhno's question (note Lemma 4.17), given a Banach space X, the following questions make sense.

(a) If $\mathcal{T}^s(X) \geq 1$, does X have the LD2P?

(b) If $\mathcal{T}(X) \geq 1$, does X have the D2P?

(c) If $\mathcal{T}^{cc}(X) \geq 1$, does X have the SD2P?

A negative answer to (c) is not difficult to see, as the following remark shows.

Remark 4.9. Let $1 < p < \infty$ and Y be a Banach space with the Daugavet property. If we take $X := Y \oplus_p Y$, then $\mathcal{T}^{cc}(X) = 2^{1/p} > 1$ (see Theorem 4.10) but, from both [ABP, Theorem 3.2] and [HL, Theorem 1], we know that there is a non-empty convex combination of slices of B_X with diameter strictly less than 2, i.e it fails the SD2P.

We end this section by proving that the answer to Ivakhno's question (i.e. the answer to (a)) is negative. Indeed, we have the following result.

Theorem 4.18. *There exists a Banach space X with the r-big slice property (hence $\mathcal{T}^s(X) \geq 1$) and there exists $\varphi \in S_{X^*}$ such that*

$$\inf_{\alpha > 0} \operatorname{diam}(S(B_X, \varphi, \alpha)) \leq \sqrt{2}.$$

In order to prove the theorem above, let us introduce some notions. Let $m, n \in \mathbb{N}$ and let us define

$$T := \{(\alpha_1, \dots, \alpha_n) \mid \alpha_1, \dots, \alpha_n \in \mathbb{N}\} \cup \{\emptyset\}.$$

Given $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_m) \in T \setminus \{\emptyset\}$, we say that

$$(\alpha_1, \dots, \alpha_n) \leq (\beta_1, \dots, \beta_m) \Leftrightarrow \begin{cases} |(\alpha_1, \dots, \alpha_n)| \leq |(\beta_1, \dots, \beta_m)| \\ \alpha_i = \beta_i \text{ for every } i \in \{1, \dots, n\}, \end{cases}$$

where $|(\alpha_1, \dots, \alpha_n)| := n$ and $|\emptyset| := 0$ and we say that $\min T = \emptyset$. This binary relation defines a partial order on T.

Let us define a *segment* in T as a totally ordered and finite subset $S \subset T$. Given $x : T \to \mathbb{R}$, let us consider

$$\|x\| = \sup \left(\sum_{i=1}^{n} \left(\sum_{t \in S_i} x(t) \right)^2 \right)^{\frac{1}{2}}, \tag{4.8}$$

where the supremum is taken over all families of disjoint segments $S_1, \dots, S_n$ of T. For $x \colon T \to \mathbb{R}$ and $S \subset T$ a segment in T, let us define

$$f_S(x) := \sum_{t \in S} x(t).$$

The space JT_∞ is defined as the completion of the space of finitely non-zero functions defined on T, i.e. functions $x \colon T \to \mathbb{R}$ such that $\{t \in T \mid x(t) \neq 0\}$ is finite, with the norm given by equation 4.8. Given $\alpha \in T$, let us define

$$e_\alpha(\beta) := \begin{cases} 1, & \text{if } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is known that $(e_\alpha)_{\alpha \in T}$ is a Schauder basis for JT_∞ and that JT_∞ is a dual space. We denote by $(e_\alpha^*)_{\alpha \in T}$ the biorthogonal sequence of $(e_\alpha)_{\alpha \in T}$. Then $B_\infty := \overline{\operatorname{span}}\{e_\alpha^* \mid \alpha \in T\}$, where the closure is taken in JT_∞^*, is a complete predual of JT_∞.

The space JT_∞ was introduced in [GM], where it was shown that B_∞ fails the Radon–Nikodým property. Moreover, every infinite-dimensional subspace of JT_∞ contains an isomorphic copy of ℓ_2 and so JT_∞ does not contain isomorphic copies of ℓ_1.

Let us start with the following lemma about weakly null sequences in JT_∞, which will help us prove Theorem 4.18.

Lemma 4.19. *Let $\{t_n \mid n \in \mathbb{N}\}$ be the set of successors of a given element $t \in T$. Then $(e_{t_n})_{n\in\mathbb{N}}$ is weakly null.*

Proof. Let $m \in \mathbb{N}$, $x = \sum_{j=1}^m \alpha_j e_{t_j}$, where $\alpha_j \in \mathbb{R}$, and let us prove that $\|x\| \leq \left(\sum_{j=1}^m \alpha_j^2\right)^{\frac{1}{2}}$. To this end, fix a family of disjoint segments $S_1, \ldots, S_k$. Since $\{t_n \mid n \in \mathbb{N}\}$ are incomparable, notice that for every $i \in \{1, \ldots, k\}$ we have that $S_i \cap \{t_1, \ldots, t_m\}$ has, at most, one element. Define A to be the set of such indices $i \in \{1, \ldots, k\}$ such that $S_i \cap \{t_1, \ldots, t_m\} =: \{t_{k_i}\}$. Notice that, since the segments S_i are disjoint, then $t_{k_i} \neq t_{k_j}$ if $i \neq j$, with $i, j \in A$. Now

$$\sum_{i=1}^m \left(\left(\sum_{t\in S_i} x(t)\right)^2\right)^{\frac{1}{2}} = \left(\sum_{i\in A} \alpha_{k_i}^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^m \alpha_i^2\right)^{\frac{1}{2}}.$$

Taking the supremum over the family of disjoint segments and we get

$$\|x\| \leq \left(\sum_{i=1}^m \alpha_i^2\right)^{\frac{1}{2}}.$$

The previous estimate implies, by the arbitrariness of $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, that the linear operator $\Phi\colon \ell_2 \to JT_\infty$ given by

$$\Phi(e_n) = e_{t_n}$$

is continuous. The w–w-continuity of Φ and the fact that $(e_n) \to 0$ weakly in ℓ_2 concludes the lemma. □

We are now ready to prove Theorem 4.18.

Proof of Theorem 4.18. Let $X = B_\infty$, the predual of JT_∞ described above. The existence of $\varphi \in S_{X^*}$ satisfying our requirements follows from [BLR4, Theorem 2.2]. Observe that in order to show that X has the r-big slice property, it suffices to show that for every w^*-slice S of $B_{X^{**}} = B_{JT_\infty^*}$, we have $r(S) \geq 1$. To this end, fix a w^*-slice $S := S(B_{JT_\infty^*}, x, \alpha)$, for a suitable finitely-supported function $x\colon T \to \mathbb{R}$ of norm one, $x^* \in JT_\infty^*$ and $\varepsilon > 0$. We shall find an element $y^* \in S$ such that $\|x^* - y^*\| \geq 1 - \varepsilon$. To this end, by [BLR4, Lemma 2.1], we can find an element $g \in S$ of the form $g := \sum_{i=1}^n \lambda_i f_{S_i}$, where $\lambda_1, \ldots, \lambda_n > 0$, with $\sum_{i=1}^n \lambda_i^2 = 1$, and $S_1, \ldots, S_n$ is a family of disjoint segments.

Fix $i \in \{1, \ldots, n\}$. Since the set of successors of a given element is infinite and the fact that the support of x is finite, we can assume (by adding elements which do not belong to $\mathrm{supp}(x)$ into S_i keeping the disjointness condition on the segments $S_1, \ldots, S_n$ if needed) that, if t_i is the maximum element of S_i, then for every $z \geq t_i$ one has $z \notin \operatorname{supp}(x)$ and that $\{t_1, \ldots, t_n\}$ are at the same level. For every $i \in \{1, \ldots, n\}$ fix $\{t_n^i \mid n \in \mathbb{N}\}$ the set of successors of t_i. By Lemma 4.19, it follows that $(e_{t_n^i})$ is weakly null, which means that $x^*(e_{t_n^i}) \to 0$. Consequently, we can find $n \in \mathbb{N}$ large enough so that $x^*(e_{t_n^i}) < \lambda\varepsilon$, for $\lambda := \min_{1\leq i\leq n} \lambda_i$, holds for every $i \in \{1, \ldots, n\}$.

Define $R_i := S_i \cup \{t_n^i \mid n \in \mathbb{N}\}$. Notice that $y^* := \sum_{i=1}^n \lambda_i f_{R_i}$ is a norm-one element (because $\{R_1, \ldots, R_n\}$ is still a family of disjoint segments) and that $y^* \in S$. Indeed, notice that $y^*(x) = g(x)$ because $t_n^i \notin \operatorname{supp}(x)$ for any $i \in \{1, \ldots, n\}$. Define $z\colon T \to \mathbb{R}$ by $z = \sum_{i=1}^n \lambda_i e_{t_n^i}$. Notice that $y^*(z) = \sum_{i=1}^n \lambda_i^2 = 1$ by our assumptions. Moreover, similar estimates to the ones in the proof of Lemma 4.19 give us that $\|z\| \leq 1$ in JT_∞. Also,

$$x^*(z) = \sum_{i=1}^n \lambda_i x^*(e_{t_n^i}) < \varepsilon,$$

which gives us

$$\|y^* - x^*\| \geq (y^* - x^*)(z) > 1 - \varepsilon$$

as desired. □

Remark 4.10. Notice that $\mathcal{T}(B_\infty) = 0$ since the unit ball of B_∞ contains non-empty relatively weakly open subsets of arbitrarily small diameter (in fact, B_∞ has the *convex point of continuity property*, see [GMS, Theorem 2.2]). Consequently, question (b) remains open.

Question 4.20. *If $\mathcal{T}(X) \geq 1$, does X have the D2P?*

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Summary in Estonian

Banachi ruumi ühikkera suurtest viiludest

Banachi ruumide geomeetrias mängib ühikkera struktuur tähtsat rolli. Aastate jooksul on matemaatikud välja pakkunud mitmeid omadusi ja indekseid selleks, et paremini eristada Banachi ruume üksteisest. Käesolevas töös uuritakse omadusi, mis kirjeldavad, kui suured saavad olla ühikkera teatud alamhulgad, näiteks viilud. Kuna Banachi ruumi ühikkera diameeter on kaks, siis mistahes viilu diameeter saab olla ülimalt kaks.

Viimastel aastatel on palju tähelepanu saanud diameeter-2 omadused, mis kirjeldavad selliseid Banachi ruume, kus mistahes viilu (mittetühja suhteliselt nõrgalt lahtise alamhulga või viilude kumera kombinatsiooni) diameeter on suurima võimaliku väärtusega ehk kaks. Selle suuna süstemaatilise uurimise käivitasid 2013. aastal T. A. Abrahamsen, V. Lima ja O. Nygaard. Kui Banachi ruumi mistahes viilude kumera kombinatsiooni diameeter on kaks, siis öeldakse, et ruumil on tugev diameeter-2 omadus.

Käesoleva väitekirja põhieesmärk on süsteemselt uurida diameeter-2 omadusi ja nendega seotud mõisteid, nagu näiteks normi karedus ning Daugaveti tihkuse indeks. Töös kirjeldatakse täielikult ära erinevate tugevate diameeter-2 omaduste, karedate normide ja Dauagaveti tihkuse indeksite stabiilsustulemused absoluutse normiga summaruumide ja komponentruumide vahel. Uuritakse tingimusi, mida meetriline ruum peab rahuldama, et vastaval Lipschitzi ruumil oleks $*$-nõrk sümmeetriline tugev diameeter-2 omadus. Töös üldistatakse ka kvantitatiivselt varasemalt teadaolevaid operaatorruumide kareduse tulemusi. Daugaveti tihkuse indeksi uurimise tulemusena vastatakse Y. Ivakhno poolt 2006. aastal püstitatud küsimus viilude diameetri ja raadiuse vahekorra kohta.

Väitekiri koosneb neljast peatükist. Esimene peatükk sisaldab ülevaadet töö põhilistest mõistetest ja nende tekkimise ajaloost. Samuti kirjeldatakse väitekirja ülesehitust ja töös kasutatavaid tähistusi.

Teises peatükis tuuakse sisse sümmeetrilise tugeva diameeter-2 omaduse mõiste. Uuritakse (sümmeetrilise, diametraalse) tugeva diameeter-2 omaduse stabiilsustulemusi absoluutsetel summadel. Seejärel näidatakse, et sümmeetrilise tugeva diameeter-2 omadus kandub ülemruumilt alamruumidele ja vastupidi sarnaselt tavalisele tugevale diameeter-2 omadusele. Peatüki lõpus uuritakse $*$-nõrka sümmeetrilise tugeva diameeter-2 omadust Lipschitzi ruumides. See peatükk tugineb artiklitele [HLLN] ja [HLN].

Kolmandas peatükis uuritakse Banachi ruumi kareduse mõistet. Esmalt vaadeldakse komponentruumide ja absoluutse normiga varustatud summaruumi kareduse vahelist seost. Seejärel kirjeldatakse Banachi ruumi ultraastme kareduse seost lähteruumi karedusega ning antakse tarvilikud ja piisavad tingimused pidevate lineaarsete operaatorite ruumi kareduseks. See peatükk põhineb peamiselt artiklil [HLN] ja magistritööl [Nad].

Viimases ehk neljandas peatükis uuritakse Daugaveti tihkuse indeksit. Esmalt kirjeldatakse Daugaveti tihkuse indeksi käitumist absoluutsetel summadel. Seejärel seotakse Daugaveti tihkuse indeksid diameeter-2 omadustega, mille abil vastatakse eitavalt Y. Ivakhno poolt 2006. aastal püstitatud lahtisele küsimusele lokaalse diameeter-2 omaduse ja r-suurte viilude omaduse samaväärsuse kohta. See peatükk tugineb põhiliselt artiklile [HLLNR].

Index

Curriculum vitae

Name: Rihhard Nadel
Date and place of birth: 03/08/1991, Tallinn, Estonia
Citizenship: Estonian
Address: University of Tartu, Institute of Mathematics, Narva mnt 18, 51009 Tartu, Estonia
Phone: +372 5836 5577
E-mail: rihhard.nadel@ut.ee

Education:

1999–2011	Tallinn German Gymnasium
2011–2014	University of Tartu, bachelor's studies in mathematics, Bachelor of Science
2014–2016	University of Tartu, master's studies in mathematics, Master of Science (Mathematics)
2016–	University of Tartu, doctoral studies in mathematics
2017–2018	Cranfield University, master's studies in operational research, Master of Science (Military Operational Research)

Languages: Estonian, English, German

Employment:

2014–2015	conscription in Estonian Defence Forces
2016–	Estonian Defence Forces, analyst

Fields of scientific interest:
Functional analysis – geometry of Banach spaces and spaces of operators

Curriculum vitae

NIMI: Rihhard Nadel
SÜNNIAEG JA -KOHT: 03.08.1991, Tallinn, Eesti
KODAKONDSUS: Eesti
AADRESS: Tartu Ülikooli matemaatika ja statistika instituut, Narva mnt 18, 51009 Tartu, Eesti
TELEFON: +372 5836 5577
E-MAIL: rihhard.nadel@ut.ee

HARIDUS:

1999–2011	Tallinna Saksa Gümnaasium
2011–2014	Tartu Ülikool, bakalaureuseõpingud matemaatikas, Loodusteaduste bakalaureus (Matemaatika)
2014–2016	Tartu Ülikool, magistriõpingud matemaatikas, Loodusteaduste magister (Matemaatika)
2016–	Tartu Ülikool, doktorantuur matemaatikas
2017–2018	Cranfield University, magistriõpingud operatsioonianalüüsis, Loodusteaduste magister (Military Operational Research)

KEELTEOSKUS: eesti, inglise, saksa

TEENISTUSKÄIK:

2014–2015	ajateenistus Kaitseväes
2016–	Kaitsevägi, analüütik

TEADUSLIKUD HUVID:
Funktsionaalanalüüs – Banachi ruumide ja operaatorite ruumide geomeetria

List of original publications

1. R. Haller, J. Langemets and R. Nadel, *Stability of average roughness, octahedrality, and strong diameter 2 properties of Banach spaces with respect to absolute sums*, Banach J. Math. Anal. **12** (2018), no. 1, 222–239.

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