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ERGE IDEON

Rational spline collocation for boundary value problems

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Rational spline collocation for boundary value problems

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Chapter 1

Introduction

Boundary value problems arise in several branches of scientific and engineering problems. For example problems involving the wave equation are often stated as boundary value problems. A large class of important boundary value problems are the Sturm−Liouville ones. There are many boundary value problems in mathematical physics corresponding to second order equations, e.g., in the studies of vibrations of a membrane one has to solve a homogeneous boundary value problem. Some model problems and most common general forms are presented in [8].

A boundary value problem consists of a differential equation on a given interval and an explicit condition (or conditions) that a solution must satisfy at one or several points. A simple and common form for a two point boundary value problem is

$$\begin{aligned} y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in (a, b), \\ y(a) = \alpha_1, \quad y(b) = \alpha_2 \end{aligned}$$

where α_1, α_2 and the endpoints a, b are known constants. Traditional methods for approximate solution of boundary value problems are finite difference method which only gives a discrete solution, and collocation method with polynomial splines. Many aspects of finite difference methods are introduced in [17, 20]. Quite a number of authors have studied collocation based on polynomial functions. Polynomial splines of order m (or degree $m-1$) are piecewise polynomials and usually defined in such a way that they are $m-2$ times continuously differentiable over the domain. Piecewise polynomials were used in approximation theory already in the early 1900s but the terminology $spline function$ was first introduced by Schoenberg in 1946 [49]. Schoenberg states that he started to use this terminology by the connection of piecewise polynomials with a certain mechanical device called a spline. This is a thin rod of some elastic material equipped with a groove and a set of weights with attached arms designed to fit into the groove. It appears that the use of piecewise polynomial functions offers significant advantages - it is simpler and more powerful (see, e.g., [14]). Piecewise polynomial functions are more adaptable to special problems too. Up until 1960 the theory of spline functions had a rather modest development and prior to the mid-1960s there were only a few papers which dealt with the problem of how well classes of smooth functions can be approximated by piecewise polynomials or splines.

For the collocation method quite general results about stability and convergence with polynomials for boundary value problems are obtained in [59, 60]. Piecewise polynomials are used as approximate solution for the same problem in [44]. Cubic spline approximation at the solution of the two point boundary value problem for second order linear differential equation is treated in [6, 10, 22] and also for fourth order boundary value problems in [23]. Collocation with some higher order splines for the second order boundary value problems is studied in [1]. Some surveys about other numerical methods for boundary value problems in ordinary differential equations could be found in [5, 28, 29]. Collocation procedures using piecewise polynomial functions with Gauss collocation points are applied to two quasilinear second order differential equations in [56]. Collocation for systems of boundary value problems with nonlinear differential equations are studied in [45] and linear systems of first order ordinary differential equations are considered with a regular singularity at one endpoint, under some assumptions on the smoothness of the coefficients that appear in the equations and in the boundary conditions in [13]. In [7] collocation by piecewise Hermite cubic polynomials is made for two point boundary value problem and uniform convergence together with some superconvergence for the derivatives is obtained. These problems are dealt with in several books about spline theory [12, 53] and numerical solution of differential equations. Several interpolatory approximations for collocation solutions to systems of two point boundary value problems are studied in [42]. Among others we refer the reader to [8] for thorough treatment and comprehensive bibliography and to works by Stechkin and Subbotin [57] and Schumaker [55] which give a comprehensive treatment of the theory of numerical analysis of polynomial spline functions. The paper [43] contains 93 references and is also useful as an introduction to the extensive literature on projection methods for the numerical solution of two point boundary value problems. For quadratic and cubic spline collocation method the convergence rate $O(h^2)$ is known [30, 39]. In some cases, the actual error is less for the quadratic splines and in other cases, the error is less for the cubic splines, see [39].

In the early 1900s there was also quite extensive development of interpolation using the piecewise polynomials. Interpolation is the most simple way to reconstruct a function according to discrete data. In a typical interpolation problem we suppose that we have n distinct datapoints $a = x_1 < \ldots < x_n = b$ and values y_i, $i = 1, \ldots, n$. We look for a continuous function f from a given class of functions in a way such that the graph of f passes through the given set of data points, that is, it satisfies $f(x_i) = y_i$, $i = 1, \ldots, n$, and the points x_i are called the interpolation nodes. The intensive development of the theory of interpolating splines began in the early 1960s. It led to achieving error bounds. Some of the early contributors are Ahlberg and Nilson [2], Birkhoff and de Boor [11], Ahlberg, Nilson, and Walsh [3, 4], Atkinson [9], Schoenberg [50, 51, 52].

Let us mention that $O(h^2)$ convergence rate of quadratic spline collocation for boundary value problems is based on superconvergence property of interpolating splines. This was discovered in [30] and developed extensively in [35, 39]. It was shown in [39] that the main part of error at quadratic spline collocation is actually several times less than was obtained in [30].

There is also the possibility of using piecewise rational functions. The first development of nonlinear spline spaces with the rational functions and generalizations of them was carried out by Schaback [46] and by Werner [61]. A very general space of rational splines was also defined by Schumaker [54]. A generalization for the results of [46] is presented in [16]. Since then various classes of rational splines have been studied. For example in [18] a class of rational C^2 quadratic/quadratic and in [19] a class of C^1 cubic/cubic splines for interpolation are considered. In [34] algorithms for interpolation by rational splines containing, as a special case, parabolic splines and piecewise-linear interpolation is discussed. A class of rational C^2 cubic/quadratic splines is studied in [24]. The accuracy $O(h^3)$ or $O(h^4)$ is achieved. These splines may have some advantages over rational linear/linear and quadratic/linear splines because of their possibly larger choice of coefficients but low degree rational splines are simpler and more convenient to use. They do not lose the accuracy too. For a smooth function y and interpolating linear/linear rational spline S it is known that $\|S - y\|_\infty = O(h^3)$, see, e.g., [25, 37]. For consistent data, the linear/linear rational spline interpolant of class C^1 always exists and is unique [37, 38]. In [25], the expansions on subintervals via the derivatives of the smooth function to interpolate could be found. They give the superconvergence of the spline values and its derivatives in certain points. In interpolation the linear/linear rational splines of class C^1 have the same accuracy as the classical quadratic splines and none of them have advantage in comparison of real errors [25, 32]. In such circumstances, it is natural to pose the question about relation between convergence rates of quadratic and linear/linear rational spline collocation for boundary value problems. This is one of the main topics of this thesis. In collocation with quadratic spline S it is known that $\|S - y\|_\infty = O(h^2)$, where y is the solution of the problem. For the proof see, e.g., [30, 39]. We will study such a problem in the case of linear/linear rational spline. We will also study a quadratic/linear rational spline collocation for boundary value problems and compare it to the cubic spline case. For the latter one the convergence rate $O(h^2)$ is known as well. Let us point out that quadratic/linear rational interpolating splines of class C^2 have the same accuracy as the classical cubic interpolating splines [37]. In some cases, the error is less for the cubic splines and in some cases, the error is less for the quadratic/linear rational splines. For a strictly convex (or strictly concave) smooth function y and interpolating quadratic/linear rational spline S it is known that $\|S-y\|_\infty = O(h^4)$, see, e.g., [36, 37]. A quadratic/linear rational spline interpolant of class C^2 exists and is unique and strictly convex for any strictly convex data [47]. It should be effective to use these splines in seeking the solutions with singularities of differential and integral equations. For the cubic splines, the expansions on subintervals via the derivatives of the smooth function to interpolate could be found, e.g., in [58]. They give the superconvergence of the spline values and its derivatives in certain points. We will study such a problem in the case of quadratic/linear rational spline interpolation.

While the interpolation problem is a linear one, the linear/linear rational spline interpolation as well as quadratic/linear rational spline interpolation is, in nature, a nonlinear method because it leads to a nonlinear system with respect to the spline parameters. Nevertheless, the complexity of these rational spline interpolation methods is the same as in polynomial spline case. It was shown in [15] that any strict convexity preserving interpolation method having certain regularity properties cannot be linear. Hopefully, similar result should also hold for strict monotonicity preservation. The problem of shape preserving interpolation has been considered by several authors [18, 19, 21, 24, 41, 48]. Firstly they have kept in mind monotonicity and convexity. A review with 164 references of shape preserving approximation methods and algorithms for approximating univariate functions or discrete data is given in [31]. Main ideas about the methods of solving nonlinear systems of equations could be found in [40].

We can say about the linear/linear rational spline collocation method that it is, in nature, a nonlinear method and it does not work in the case of nonmonotone solution. Fortunately, the method proposed in this thesis could be implemented so that it has the same complexity as collocation with quadratic splines, e.g., each step of Newton's method requires the solution of linear system with banded matrix and the number of steps is limited (relatively small). The practical solution process could be organized in such a way that on each step of Newton's iteration one has to solve a linear system having diagonal dominance in rows of its banded matrix. The solution of these systems, e.g., by Gaussian elimination, is known to be stable with respect to small errors in calculations and such kind of stability concerns also Newton's method itself. In opposite situation an adaptive strategy where in the region of strict monotonicity rational pieces and elsewhere polynomial pieces are used is a reasonable approach. Then once again the solution by adaptive method could give considerably better result at the same starting information (mainly the values of functions in differential equation). This procedure has the same complexity as quadratic spline collocation. However, to prove the existence of this combined spline as approximate solution in collocation method and to find an adequate error estimate is a challenge beyond this thesis. The theory of adaptive interpolation is developed, e.g., in [47] with cubic polynomial and quadratic/linear rational splines and in [38] for any data with quadratic polynomial and linear/linear rational splines. In [47] cubic polynomial and quadratic/linear rational pieces to retain strict convexity in the regions of strict convexity of data are used. The existence of such a coconvex spline interpolant is proved if data have weak alternation of second order divided differences on cubic sections. It is remarkable that the existence of those adaptive interpolating splines was quite complicated to prove. At spline collocation for boundary value problems this adaptive strategy needs to be investigated in future. It is natural that this research can be based on the results about rational spline collocation. Keeping this in mind, the main purpose of this thesis is to show the existence of linear/linear rational spline and quadratic/linear rational spline as approximate solution in collocation method for boundary value problems and also comparison of error estimates with respect to quadratic and cubic spline collocation respectively.

In the following we give a brief overview of the dissertation by chapters. The present work consists of eight chapters.

In Chapter 1 we already gave an overview of boundary value problems, examples of applications and a short review of main books and papers on spline collocation method with polynomial and rational splines.

In Chapter 2 we present the linear/linear rational splines and give three different representations: by spline values and first moments, by spline values and first derivatives in midpoints of subintervals and by spline values. There is also given the number of free parameters and some general remarks about linear/linear rational splines.

In Chapter 3 there are given two representations for quadratic/linear rational splines (by spline values and first moments and by spline values and second moments) as well as some general comments.

Chapter 4 is devoted to linear/linear rational spline interpolation problem. Firstly, the description of the method is given where linear/linear rational spline representation by spline values is used. In this chapter we show that for the interpolating linear/linear rational splines we obtain $\|S(x_i) - y(x_i)\|_\infty = O(h^4)$ on uniform mesh $x_i = a + ih$, $i = 0, \ldots, n$, and prove also the superconvergence of order h^3 for the first derivative and of order h^2 for the second derivative of S in certain points. In the end of the chapter, for comparison, the expansions of quadratic spline S and its derivatives are given. The results of Chapter 4 are published in [25].

In Chapter 5 we investigate the interpolation with quadratic/linear rational splines. As in previous chapter we start by giving the description of the method. In this chapter a representation by spline values and second moments is used. On a uniform mesh $x_i = a + ih$, $i = 0, \ldots, n$, in the case of sufficiently smooth function y the expansions of interpolating quadratic/linear rational spline S and its derivatives are obtained. They give the superconvergence of order h^4 for the first derivative, of order h^3 for the second derivative and of order h^2 for the third derivative of S in certain points. We compare the results with cubic spline interpolation method. The results of Chapter 5 are published in [26].

Chapter 6 is devoted to the study of the linear/linear rational spline collocation method. We use the spline representation by spline values and at first we represent the collocation method. Then the necessary transformations are established and the use of the Bohl-Brouwer fixed point theorem is shown. The convergence estimates using the special interpolation points are finally achieved and stated in Theorem 6.1. For the linear/linear rational spline S on uniform mesh it holds $\|S-y\|_\infty = O(h^2)$, where the solution y of the boundary value problem is a strictly monotone function. Established bound of error for the collocation method gives a dependence on the solution of the boundary value problem and its coefficients. We prove also convergence rates $\|S'-y'\|_\infty = O(h^2)$, $\|S''-y''\|_\infty = O(h)$ and the superconvergence of order h^2 for the second derivative of S in certain points. The results of this chapter have been submitted for publication in [27].

In Chapter 7 we investigate the quadratic/linear rational spline collocation method. As in previous chapter we start by introducing the collocation method with quadratic/linear rational splines. At this time the spline representation by spline values and first moments is used. For the quadratic/linear rational spline S on uniform mesh with the solution y of the boundary value problem which is a strictly convex (or strictly concave) function it holds $\|S-y\|_\infty = O(h^2)$. We prove also convergence rates $\|S'-y'\|_\infty = O(h^2)$ and $\|S''-y''\|_\infty = O(h^2)$.

In Chapter 8 there are given numerical tests which totally support the theoretical analysis.

Chapter 2

Representation of linear/linear rational splines

In this chapter we introduce three representations for linear/linear rational splines.

Consider a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b - a)/n$, $n \in \mathbb{N}$. We also need the points $\xi_i = x_{i-1} + h/2$, $i = 1, \ldots, n$.

Linear/linear rational spline is a function $S \in C^1[a, b]$ of the form

$$S(x) = a_i + \frac{c_i(x - \xi_i)}{1 + d_i(x - \xi_i)}, \quad x \in [x_{i-1}, x_i], \tag{2.1}$$

where $1 + d_i(x - \xi_i) > 0$. Let us point out that, in general, a linear/linear rational function on each particular interval $[x_{i-1}, x_i]$ may have the form

$$S(x) = \frac{\hat{a}_i + \hat{b}_i x}{\hat{c}_i + \hat{d}_i x},$$

where $\hat{c}_i + \hat{d}_i x < 0$ or $\hat{c}_i + \hat{d}_i x > 0$, but it could be transformed into (2.1).

Note that a straightforward reasoning shows that if there are two representations for function S on interval $[x_{i-1}, x_i]$, i.e.,

$$S(x) = a_i + \frac{c_i(x - \xi_i)}{1 + d_i(x - \xi_i)}$$

and

$$S(x) = \bar{a}_i + \frac{\bar{c}_i(x - \xi_i)}{1 + \bar{d}_i(x - \xi_i)}$$

with arbitrary $\xi_i \in [x_{i-1}, x_i]$, then $\bar{a}_i = a_i$ and $\bar{c}_i = c_i$. If, in addition, $\bar{c}_i = c_i \neq 0$, then $\bar{d}_i = d_i$.

According to the representation (2.1) we have $3n$ parameters to determine for constructing the spline. We require for C^1 continuity of S on $[a,b]$ which involves $2(n-1)$ conditions, namely that S and S' must be continuous at all interior knots $x_1,\dots,x_{n-1}$. That means the number of free parameters is $3n-2(n-1)=n+2$.

Observe at once that on $[x_{i-1},x_i]$ we have

$$S'(x)=\frac{c_i}{(1+d_i(x-\xi_i))^2}, \tag{2.2}$$

which means that S being in $C^1[a,b]$ is strictly increasing or strictly decreasing or constant on $[a,b]$.

2.1 Representation by spline values and first moments

Let $S(\xi_i)=S_i$, $i=1,\dots,n$, and $S'(x_i)=m_i$, $i=0,\dots,n$. We call by first moments the parameters m_i. From (2.1) and (2.2) we get

$$\begin{aligned}
S_i &= a_i,\\
m_{i-1} &= \frac{c_i}{\left(1-\dfrac{hd_i}{2}\right)^2},\\
m_i &= \frac{c_i}{\left(1+\dfrac{hd_i}{2}\right)^2}.
\end{aligned}$$

Consider the case $m_{i-1},m_i>0$ or $m_{i-1},m_i<0$. Otherwise we have $m_i=0$, $i=0,\dots,n$, and the spline S is constant.

From previous system we can uniquely express the parameters

$$\begin{aligned}
a_i &= S_i,\\
c_i &= \frac{4m_{i-1}}{\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}+1\right)^2},\\
d_i &= \frac{2\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}-1\right)}{h\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}+1\right)}.
\end{aligned}$$

Replacing them in (2.1) we obtain for $x\in[x_{i-1},x_i]$

$$S(x)=S_i+\frac{4hm_{i-1}(x-\xi_i)}{\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}+1\right)\left(h\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}+1\right)+2\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}-1\right)(x-\xi_i)\right)}.$$

Setting $x = x_{i-1} + \tau h$, $\tau \in [0, 1]$, the representation of linear/linear rational spline by spline values and first moments is

$$S(x) = S_i + \frac{h m_{i-1}(2\tau - 1)}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} + 1\right)\left(1 + \tau\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)}, \quad x \in [x_{i-1}, x_i]. \quad (2.3)$$

This also gives

$$S'(x) = \frac{m_{i-1}}{\left(1 + \tau\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)^2},$$

$$S''(x) = -\frac{2m_{i-1}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)}{h\left(1 + \tau\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)^3}.$$

The continuity of S' is guaranteed by the notation $S'(x_i) = m_i$, $i = 0, \ldots, n$. From (2.3) we get

$$S(x_i - 0) = S_i + \frac{h m_i}{\left(\frac{m_i}{m_{i-1}}\right)^{1/2} + 1}$$

and

$$S(x_i + 0) = S_{i+1} - \frac{h m_i}{\left(\frac{m_i}{m_{i+1}}\right)^{1/2} + 1},$$

so the continuity of S at the interior knots, i.e., $S(x_i - 0) = S(x_i + 0)$, $i = 1, \ldots, n-1$, leads to the internal equations

$$m_i\left(\frac{1}{\left(\frac{m_i}{m_{i-1}}\right)^{1/2} + 1} + \frac{1}{\left(\frac{m_i}{m_{i+1}}\right)^{1/2} + 1}\right) = \frac{S_{i+1} - S_i}{h}, \quad i = 1, \ldots, n-1.$$

2.2 Representation by spline values and first derivatives in midpoints of subintervals

Using the notations $S(x_i) = S_i$, $i = 0, \ldots, n$, $S'(\xi_i) = m_i$, $i = 1, \ldots, n$, we get from (2.1) and (2.2)

$$m_i = c_i,$$

$$S_{i-1} = a_i - \frac{hc_i}{2 - hd_i} = a_i - \frac{hm_i}{2 - hd_i}, \tag{2.4}$$

$$S_i = a_i + \frac{hc_i}{2 + hd_i} = a_i + \frac{hm_i}{2 + hd_i}. \tag{2.5}$$

Again, we assume that the spline is strictly increasing or strictly decreasing, otherwise it is constant and $m_i = 0$, $i = 1, \dots, n$.

Firstly, for expressing the parameters a_i, b_i and c_i, we find $S_i - S_{i-1}$. That is by (2.4) and (2.5)

$$S_i - S_{i-1} = a_i + \frac{hm_i}{2 + hd_i} - \left(a_i - \frac{hm_i}{2 - hd_i}\right) = \frac{hm_i}{2 + hd_i} + \frac{hm_i}{2 - hd_i}.$$

We obtain immediately

$$d_i = \pm \frac{2}{h}\left(1 - \frac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}.$$

Then from (2.5) we get

$$a_i = S_i - \frac{hm_i}{2\left(1 \pm \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}\right)}$$

and, in addition,

$$c_i = m_i.$$

Replacing a_i, c_i, d_i, $i = 1, \dots, n$, in (2.1) we have the representation by spline values and first derivatives in midpoints of subintervals

$$S(x) = S_i - \frac{hm_i}{2 \pm 2\left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}} + \frac{m_i(x - \xi_i)}{1 \pm \dfrac{2}{h}\left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}(x - \xi_i)},$$

$$x \in [x_{i-1}, x_i],$$

or setting $x = x_{i-1} + h\tau$, $\tau \in [0, 1]$,

$$S(x) = S_i - \frac{hm_i(1 - \tau)}{\left(1 \pm \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}\right)\left(1 \pm \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}(2\tau - 1)\right)}. \tag{2.6}$$

This also gives us

$$S'(x) = \frac{m_i}{\left(1 \pm \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2} (2\tau - 1)\right)^2}, \tag{2.7}$$

$$S''(x) = \frac{\mp 4m_i \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}}{h\left(1 \pm \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2} (2\tau - 1)\right)^3}.$$

The continuity conditions of S' can be written as $S'(x_i - 0) = S'(x_i + 0)$ or with the help of (2.7) in the form

$$\frac{m_i}{\left(1 \pm \left(1 - \dfrac{m_i}{\dfrac{S_i - S_{i-1}}{h}}\right)^{1/2}\right)^2} = \frac{m_{i+1}}{\left(1 \mp \left(1 - \dfrac{m_{i+1}}{\dfrac{S_{i+1} - S_i}{h}}\right)^{1/2}\right)^2}, \tag{2.8}$$

$$i = 1, \dots, n-1.$$

Note that in this representation we have two different splines with same parameters S_{i-1}, S_i and m_i, one with $+$ and the other with $-$ in the representation (2.6).

2.3 Representation by spline values

Let $S(x_i) = S_i$, $i = 0, \dots, n$, and $S(\xi_i) = \bar{S}_i$, $i = 1, \dots, n$. We assume that the spline is strictly increasing or strictly decreasing. We get from (2.1)

$$\bar{S}_i = a_i,$$

$$S_{i-1} = a_i - \frac{hc_i}{2 - hd_i},$$

$$S_i = a_i + \frac{hc_i}{2 + hd_i}.$$

This allows to express uniquely a_i, c_i and d_i via the spline values as

$$a_i = \bar{S}_i,$$

$$c_i = \frac{4(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})}{h(S_i - S_{i-1})},$$

$$d_i = \frac{2(2\bar{S}_i - S_{i-1} - S_i)}{h(S_i - S_{i-1})}.$$

After replacing the obtained parameters in (2.1) we have the representation by spline values

$$S(x) = \bar{S}_i + \frac{4(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})(x - \xi_i)}{h(S_i - S_{i-1}) + 2((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))(x - \xi_i)}, \ x \in [x_{i-1}, x_i], \tag{2.9}$$

or by setting $x = x_{i-1} + h\tau$, $\tau \in [0,1]$,

$$S(x) = \bar{S}_i + \frac{(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})(2\tau - 1)}{(2\tau - 1)\bar{S}_i - \tau S_{i-1} + (1-\tau)S_i}.$$

This also gives for $x \in [x_{i-1}, x_i]$

$$S'(x) = \frac{4h(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})(S_i - S_{i-1})}{(h(S_i - S_{i-1}) + 2((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))(x - \xi_i))^2}, \tag{2.10}$$

$$S''(x) = -\frac{16h(S_i - \bar{S}_i)(\bar{S}_i - S_{i-1})(S_i - S_{i-1})((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))}{(h(S_i - S_{i-1}) + 2((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))(x - \xi_i))^3}. \tag{2.11}$$

When the continuity of S is guaranteed by the representation (2.9), the continuity of S' i.e., $S'(x_i - 0) = S'(x_i + 0)$, $i = 1, \ldots, n-1$, at interior knots with the help of (2.10) leads to the internal equations

$$\frac{(S_i - \bar{S}_i)(S_i - S_{i-1})}{\bar{S}_i - S_{i-1}} = \frac{(\bar{S}_{i+1} - S_i)(S_{i+1} - S_i)}{S_{i+1} - \bar{S}_{i+1}}, \quad i = 1, \ldots, n-1. \tag{2.12}$$

Chapter 3

Representation of quadratic/linear rational splines

In this chapter we introduce two representations for quadratic/linear rational splines.

Consider a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \dots, n$, $h = (b - a)/n$, $n \in \mathbb{N}$.

Quadratic/linear rational spline is a function $S \in C^2[a, b]$ and on each particular subinterval has the form

$$S(x) = a_i + b_i(x - x_{i-1}) + \frac{c_i}{1 + d_i(x - x_{i-1})}, \quad x \in [x_{i-1}, x_i], \tag{3.1}$$

where $1 + d_i(x - x_{i-1}) > 0$. In general, a quadratic/linear rational function may have the form by five parameters

$$S(x) = \frac{\hat{a}_i + \hat{b}_i x + \hat{c}_i x^2}{\hat{d}_i + \hat{e}_i x}, \quad x \in [x_{i-1}, x_i],$$

where $\hat{d}_i + \hat{e}_i x < 0$ or $\hat{d}_i + \hat{e}_i x > 0$, but it could be transformed into (3.1).

From (3.1) we have for $x \in [x_{i-1}, x_i]$

$$S'(x) = b_i - \frac{c_i d_i}{(1 + d_i(x - x_{i-1}))^2} \tag{3.2}$$

and

$$S''(x) = \frac{2c_i d_i^2}{(1 + d_i(x - x_{i-1}))^3} \tag{3.3}$$

which means that S or $-S$ is strictly convex or first degree polynomial.

Let us note that if there are two different representations for function S on interval $[x_{i-1}, x_i]$, i.e.,

$$S(x) = a_i + b_i(x - x_{i-1}) + \frac{c_i}{1 + d_i(x - x_{i-1})}$$

and

$$S(x) = \bar{a}_i + \bar{b}_i(x - x_{i-1}) + \frac{\bar{c}_i}{1 + \bar{d}_i(x - x_{i-1})}$$

then in the assumption $\bar{c}_i = c_i \neq 0$ it follows $\bar{d}_i = d_i$. In the case $\bar{d}_i = d_i \neq 0$ we get $\bar{a}_i = a_i$, $\bar{b}_i = b_i$, $\bar{c}_i = c_i$ and if $\bar{d}_i = d_i = 0$, then $\bar{b}_i = b_i$, but a_i and c_i are not determined uniquely.

According to the representation (3.1) there are 4 parameters on each subinterval to determine, so altogether we have $4n$ parameters to determine for constructing the spline. From C^2 continuity of S on $[a, b]$, namely that S, S' and S'' must be continuous at all interior knots $x_1, \ldots, x_{n-1}$, we get $3(n-1)$ additional conditions. That means the number of free parameters is $4n - 3(n-1) = n + 3$.

3.1 Representation by spline values and first moments

Let us consider a subinterval $[x_{i-1}, x_i]$, $i = 1, \ldots, n$. Using

$$S(x_{i-1}) = S_{i-1}, \qquad S(x_i) = S_i, \qquad S'(x_{i-1}) = m_{i-1}, \qquad S'(x_i) = m_i$$

we get from (3.1) and (3.2) a system to determine the parameters a_i, b_i, c_i and d_i. Namely,

$$\begin{aligned} S_{i-1} &= a_i + c_i, \\ S_i &= a_i + b_i h + \frac{c_i}{1 + d_i h}, \\ m_{i-1} &= b_i - c_i d_i, \\ m_i &= b_i - \frac{c_i d_i}{(1 + d_i h)^2}, \end{aligned}$$

which gives

$$a_i = S_{i-1} - \frac{(S_i - S_{i-1} - h m_{i-1})^2 (h m_i - (S_i - S_{i-1}))}{(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))^2},$$

$$b_i = m_{i-1} + \frac{(S_i - S_{i-1} - h m_{i-1})^2}{h(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))},$$

$$c_i = \frac{(S_i - S_{i-1} - hm_{i-1})^2 (hm_i - (S_i - S_{i-1}))}{(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))^2},$$

$$d_i = \frac{2(S_i - S_{i-1}) - h(m_{i-1} + m_i)}{h(hm_i - (S_i - S_{i-1}))}.$$

Denote

$$e_i = \frac{(S_i - S_{i-1} - hm_{i-1})^2}{h(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))^2}$$

and replace a_i, b_i, c_i and d_i in (3.1). Then we obtain a representation by spline values and first moments

$$\begin{aligned} S(x) = {} & S_{i-1} - h(hm_i - (S_i - S_{i-1}))e_i \\ & + h(m_{i-1} + e_i)(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))(x - x_{i-1}) \\ & + \frac{h^2 (hm_i - (S_i - S_{i-1}))^2 e_i}{h(hm_i - (S_i - S_{i-1})) + (2(S_i - S_{i-1}) - h(m_{i-1} + m_i))(x - x_{i-1})}, \\ & \hfill x \in [x_{i-1}, x_i]. \end{aligned}$$

By changing the variable $x = x_{i-1} + \tau h$, $\tau \in [0, 1]$, we get

$$\begin{aligned} S(x) = {} & S_{i-1} + \tau h(m_{i-1} + e_i)h(2(S_i - S_{i-1}) - h(m_{i-1} + m_i)) \\ & - h(hm_i - (S_i - S_{i-1}))e_i \\ & + \frac{h(hm_i - (S_i - S_{i-1}))^2 e_i}{hm_i - (S_i - S_{i-1}) + \tau(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))}. \end{aligned} \tag{3.4}$$

From that we can find

$$\begin{aligned} S'(x) = {} & m_{i-1} + (2(S_i - S_{i-1}) - h(m_{i-1} + m_i))e_i \\ & - \frac{(hm_i - (S_i - S_{i-1}))^2 (2(S_i - S_{i-1}) - h(m_{i-1} + m_i))e_i}{(hm_i - (S_i - S_{i-1}) + \tau(2(S_i - S_{i-1}) - h(m_{i-1} + m_i)))^2} \end{aligned}$$

and

$$S''(x) = \frac{2(hm_i - (S_i - S_{i-1}))^2 (2(S_i - S_{i-1}) - h(m_{i-1} + m_i))^2 e_i}{h(hm_i - (S_i - S_{i-1}) + \tau(2(S_i - S_{i-1}) - h(m_{i-1} + m_i)))^3}. \tag{3.5}$$

The representation of S in terms of S_i and m_i ensures the continuity of S and S'. The continuity of S'' in the interior knots can be expressed by

$$S''(x_i - 0) = S''(x_i + 0), \quad i = 1, \ldots, n-1,$$

or with the help of (3.5) as

$$\frac{(hm_i - (S_i - S_{i-1}))^2}{S_i - S_{i-1} - hm_{i-1}} = \frac{(S_{i+1} - S_i - hm_i)^2}{hm_{i+1} - (S_{i+1} - S_i)}, \qquad i = 1, \ldots, n-1. \tag{3.6}$$

3.2 Representation by spline values and second moments

Let us use the notation $S(x_i) = S_i$ and $S''(x_i) = M_i$, $i = 0, \ldots, n$. We call the parameters M_i second moments. From (3.1) and (3.3) we get

$$\begin{aligned} S_{i-1} &= a_i + c_i, \\ S_i &= a_i + b_i h + \frac{c_i}{1 + d_i h}, \\ M_{i-1} &= 2c_i d_i^2, \\ M_i &= \frac{2c_i d_i^2}{(1 + d_i h)^3}. \end{aligned} \tag{3.7}$$

Consider at first the case $M_i \neq 0$. Then also $M_{i-1} \neq 0$ and $d_i \neq 0$. From (3.7) it follows immediately that

$$\begin{aligned} c_i &= \frac{M_{i-1}}{2d_i^2}, \\ a_i &= S_{i-1} - \frac{M_{i-1}}{2d_i^2}, \\ b_i &= \frac{1}{h}(S_i - S_{i-1}) + \frac{M_{i-1}}{2d_i(1 + d_i h)}. \end{aligned}$$

Now, by replacing the obtained parameters a_i, b_i, and c_i in (3.1) we can represent the quadratic/linear rational spline as

$$\begin{aligned} S(x) = S_{i-1} - \frac{M_{i-1}}{2d_i^2} + \left(\frac{S_i - S_{i-1}}{h} + \frac{M_{i-1}}{2d_i(1 + d_i h)} \right)(x - x_{i-1}) \\ + \frac{M_{i-1}}{2d_i^2(1 + d_i(x - x_{i-1}))}, \quad x \in [x_{i-1}, x_i]. \end{aligned} \tag{3.8}$$

This also gives for $x \in [x_{i-1}, x_i]$

$$S'(x) = \frac{S_i - S_{i-1}}{h} + \frac{M_{i-1}}{2d_i(1+d_ih)} - \frac{M_{i-1}}{2d_i(1+d_i(x-x_{i-1}))^2}, \tag{3.9}$$

$$S''(x) = \frac{M_{i-1}}{(1+d_i(x-x_{i-1}))^3} \tag{3.10}$$

and

$$S'''(x) = -\frac{3d_iM_{i-1}}{(1+d_i(x-x_{i-1}))^4}. \tag{3.11}$$

The continuity of S and S'' is guaranteed by the representation (3.8). The continuity of S', i.e., $S'(x_i - 0) = S'(x_i + 0)$, $i = 1, \ldots, n-1$, by using (3.9) leads to the equations

$$\frac{S_i - S_{i-1}}{h} + \frac{M_{i-1}h}{2(1+d_ih)^2} = \frac{S_{i+1} - S_i}{h} - \frac{M_ih}{2(1+d_{i+1}h)}.$$

From last two equations of (3.7) we get

$$1 + d_ih = \Big(\frac{M_{i-1}}{M_i}\Big)^{1/3}$$

and, thus, we have the internal equations

$$M_i^{2/3}\Big(M_{i-1}^{1/3} + M_{i+1}^{1/3}\Big) = \frac{2}{h^2}(S_{i-1} - 2S_i + S_{i+1}), \quad i = 1, \ldots, n-1. \tag{3.12}$$

These equations hold naturally in the case $M_i = 0$ (then $M_{i-1} = 0$ and $M_{i+1} = 0$) because then the spline is a linear function and (3.12) expresses the fact that its second order divided difference is equal to zero.

Chapter 4

Linear/linear rational spline interpolation

The interpolation problem with linear/linear rational splines is similar to that with quadratic splines. In the latter case the expansions on subintervals via the derivatives of the smooth function y to interpolate could be found, e.g., in [32, 33]. They give the superconvergence of the spline values and its derivatives in certain points. In this chapter we study such a problem in the case of linear/linear rational spline interpolant. First we give the description of the interpolation method, then we analyze the obtained nonlinear system and transform it to a more suitable form in order to get the expansions of the interpolant. Finally, this allows us to receive the superconvergence results.

4.1 Description of the method

Let $a = x_0 < x_1 < \ldots < x_n = b$ be a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b - a)/n$, $n \in \mathbb{N}$. We also need the points $\xi_i = x_{i-1} + h/2$, $i = 1, \ldots, n$.

In interpolation with linear/linear rational spline S, for a given data $\bar{y}_i$, $i = 1, \ldots, n$, let us require that the interpolation conditions

$$S(\xi_i) = \bar{y}_i, \quad i = 1, \ldots, n, \tag{4.1}$$

are satisfied. Since the number of free parameters (see Chapter 2) is $n + 2$ it is necessary to give two more conditions. So, in addition, we impose

$$S(a) = \alpha_1, \quad S(b) = \alpha_2 \tag{4.2}$$

or

$$S'(a) = \alpha_1, \quad S'(b) = \alpha_2. \tag{4.3}$$

A combination with one boundary condition from (4.2) and another from (4.3) at different endpoints is also allowed.

In this chapter we use the linear/linear rational spline representation by spline values (see Section 2.3). Replacing the values $\bar{S}_i$, $i = 1, \ldots, n$, from (4.1) in the internal equations (2.12) and considering them with two boundary conditions we obtain a nonlinear system with respect to the unknowns $S_0, \ldots, S_n$. The nonlinear system can successfully be solved by the ordinary iteration method, Gauss-Seidel method or Newton's method.

For a smooth function y and interpolating linear/linear rational spline S it is known that $\|S - y\|_\infty = O(h^3)$ (see, e.g., [37]). It is also known, that a linear/linear rational spline interpolant exists, is unique and preserves the monotonicity of the data only if y is strictly monotone or constant everywhere (see, e.g., [37, 38]).

4.2 Transformation of the system

First, we analyze the nonlinear system with respect to the unknowns $S_0, \ldots, S_n$.

Suppose that we have a function $y : [a, b] \to \mathbb{R}$ to interpolate and $\bar{y}_i = y(\xi_i)$, $i = 1, \ldots, n$. Denote $y_i = y(x_i)$, $i = 0, \ldots, n$, similar notation will be used in the case of derivatives.

Let us write equations (2.12) with replaced values $\bar{S}_i$ from (4.1) in the form

$$\begin{aligned} \varphi_i(S_{i-1}, S_i, S_{i+1}) = {} & (S_i - \bar{y}_i)(S_i - S_{i-1})(S_{i+1} - \bar{y}_{i+1}) \\ & - (\bar{y}_{i+1} - S_i)(S_{i+1} - S_i)(\bar{y}_i - S_{i-1}) = 0, \ i = 1, \ldots, n-1, \end{aligned} \tag{4.4}$$

introducing at the same time functions φ_i. In our further discussion we use the boundary conditions (4.2). We can write the system consisting of the boundary conditions and the internal equations (4.4) in the form

$$\begin{cases} h^2 (y_0')^2 (S_0 - \alpha_1) = 0, \\ \varphi_i(S_{i-1}, S_i, S_{i+1}) = (S_i - \bar{y}_i)(S_i - S_{i-1})(S_{i+1} - \bar{y}_{i+1}) \\ \qquad\qquad - (\bar{y}_{i+1} - S_i)(S_{i+1} - S_i)(\bar{y}_i - S_{i-1}) = 0, \\ \qquad\qquad\qquad i = 1, \ldots, n-1, \\ h^2 (y_n')^2 (S_n - \alpha_2) = 0. \end{cases} \tag{4.5}$$

Performing the Taylor expansion for (4.4) gives

$$
\begin{aligned}
\varphi_i(S_{i-1}, S_i, S_{i+1}) = \varphi_i(y_{i-1}, y_i, y_{i+1}) &+ \frac{\partial \varphi_i}{\partial S_{i-1}}(y_{i-1}, y_i, y_{i+1})(S_{i-1} - y_{i-1}) \\
&+ \frac{\partial \varphi_i}{\partial S_i}(y_{i-1}, y_i, y_{i+1})(S_i - y_i) \\
&+ \frac{\partial \varphi_i}{\partial S_{i+1}}(y_{i-1}, y_i, y_{i+1})(S_{i+1} - y_{i+1}) + \frac{\varphi_i''}{2!}(\xi_\lambda)\bar{h}^2 = 0
\end{aligned} \tag{4.6}
$$

with the difference vector $\bar{h} = (S_{i-1} - y_{i-1}, S_i - y_i, S_{i+1} - y_{i+1})$, some $\lambda \in (0, 1)$ and $\xi_\lambda = (y_{i-1}, y_i, y_{i+1}) + \lambda\bar{h}$. Our next aim is to find $\frac{\partial \varphi_i}{\partial S_{i-1}}(y_{i-1}, y_i, y_{i+1})$, $\frac{\partial \varphi_i}{\partial S_i}(y_{i-1}, y_i, y_{i+1})$, $\frac{\partial \varphi_i}{\partial S_{i+1}}(y_{i-1}, y_i, y_{i+1})$ and then $\varphi_i(y_{i-1}, y_i, y_{i+1})$. From (4.4) we get the partial derivatives for $i = 1, \ldots, n-1$, namely,

$$
\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}} &= -(S_i - \bar{y}_i)(S_{i+1} - \bar{y}_{i+1}) + (\bar{y}_{i+1} - S_i)(S_{i+1} - S_i), \\
\frac{\partial \varphi_i}{\partial S_i} &= (S_i - S_{i-1})(S_{i+1} - \bar{y}_{i+1}) + (S_i - \bar{y}_i)(S_{i+1} - \bar{y}_{i+1}) \\
&\quad + (S_{i+1} - S_i)(\bar{y}_i - S_{i-1}) + (\bar{y}_{i+1} - S_i)(\bar{y}_i - S_{i-1}), \\
\frac{\partial \varphi_i}{\partial S_{i+1}} &= (S_i - \bar{y}_i)(S_i - S_{i-1}) - (\bar{y}_{i+1} - S_i)(\bar{y}_i - S_{i-1}).
\end{aligned}
$$

Suppose in the following that $y \in C^4[a, b]$. Let us expand y_{i-1}, $\bar{y}_i$, $\bar{y}_{i+1}$ and y_{i+1} at the point x_i by Taylor formula up to the forth derivative as

$$
\begin{aligned}
y_{i-1} &= y_i - hy_i' + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4), \\
\bar{y}_i &= y_i + \frac{h}{2}y_i' - \frac{h^2}{8}y_i'' + \frac{h^3}{48}y_i''' - \frac{h^4}{384}y_i^{IV} + o(h^4), \\
\bar{y}_{i+1} &= y_i + \frac{h}{2}y_i' + \frac{h^2}{8}y_i'' + \frac{h^3}{48}y_i''' + \frac{h^4}{384}y_i^{IV} + o(h^4), \\
y_{i+1} &= y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4)
\end{aligned}
$$

with the rest terms $O(h^{4+\alpha})$ in the case $y^{IV} \in \text{Lip}\ \alpha$, $0 < \alpha \leq 1$. Then direct calculations yield

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}}(y_{i-1}, y_i, y_{i+1}) &= \frac{h^2}{4} y_i'^2 + \frac{h^3}{4} y_i' y_i'' + O(h^4), \\
\frac{\partial \varphi_i}{\partial S_i}(y_{i-1}, y_i, y_{i+1}) &= \frac{3}{2} h^2 y_i'^2 + O(h^4), \\
\frac{\partial \varphi_i}{\partial S_{i+1}}(y_{i-1}, y_i, y_{i+1}) &= \frac{h^2}{4} y_i'^2 - \frac{h^3}{4} y_i' y_i'' + O(h^4).
\end{aligned} \tag{4.7}$$

Similarly, from

$$\begin{aligned}
\varphi_i(y_{i-1}, y_i, y_{i+1}) &= (y_i - \bar{y}_i)(y_i - y_{i-1})(y_{i+1} - \bar{y}_{i+1}) \\
&\qquad - (\bar{y}_{i+1} - y_i)(y_{i+1} - y_i)(\bar{y}_i - y_{i-1}) = 0, \quad i = 1, \dots, n-1,
\end{aligned}$$

we get with the help of Taylor formula as shown above

$$\varphi_i(y_{i-1}, y_i, y_{i+1}) = \frac{h^6}{64}\Big((y_i')^2 y_i^{IV} - 4 y_i' y_i'' y_i''' + 3(y_i'')^3\Big) + o(h^6). \tag{4.8}$$

The entries in the matrix φ_i'' are consisting of the second order partial derivatives of φ_i as follows:

$$\begin{aligned}
&\frac{\partial^2 \varphi_i}{\partial S_{i-1}^2} = 0, \\
&\frac{\partial^2 \varphi_i}{\partial S_{i-1} \partial S_i} = \frac{\partial^2 \varphi_i}{\partial S_i \partial S_{i-1}} = -(S_{i+1} - \bar{y}_{i+1}) - (S_{i+1} - S_i), \\
&\frac{\partial^2 \varphi_i}{\partial S_{i-1} \partial S_{i+1}} = \frac{\partial^2 \varphi_i}{\partial S_{i+1} \partial S_{i-1}} = -(S_i - \bar{y}_i) + (\bar{y}_{i+1} - S_i), \\
&\frac{\partial^2 \varphi_i}{\partial S_i^2} = 2(S_{i+1} - \bar{y}_{i+1}) - 2(\bar{y}_i - S_{i-1}), \\
&\frac{\partial^2 \varphi_i}{\partial S_i \partial S_{i+1}} = \frac{\partial^2 \varphi_i}{\partial S_{i+1} \partial S_i} = (S_i - S_{i-1}) + (S_i - \bar{y}_i) + (\bar{y}_i - S_{i-1}), \\
&\frac{\partial^2 \varphi_i}{\partial S_{i+1}^2} = 0.
\end{aligned}$$

They are of order $O(h)$. Recall that $\|S - y\|_\infty = O(h^3)$. This means $\|\bar{h}\|_\infty = O(h^3)$ and altogether $\varphi_i''(\xi_\lambda)\bar{h}^2 = O(h^7)$. Taking now into account (4.7), (4.8) and the order of the rest term in (4.6), system (4.5) reduces to

$$
\begin{cases}
h^2(y_0')^2(S_0-\alpha_1)=0, \\
\dfrac{h^6}{64}\Big((y_i')^2y_i^{IV}-4y_i'y_i''y_i'''+3(y_i'')^3\Big) \\
\quad +\Big(\dfrac{h^2}{4}(y_i')^2+\dfrac{h^3}{4}y_i'y_i''+O(h^4)\Big)(S_{i-1}-y_{i-1}) \\
\quad +\Big(\dfrac{3}{2}h^2(y_i')^2+O(h^4)\Big)(S_i-y_i) \\
\quad +\Big(\dfrac{h^2}{4}(y_i')^2-\dfrac{h^3}{4}y_i'y_i''+O(h^4)\Big)(S_{i+1}-y_{i+1})+o(h^6)=0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,\dots,n-1, \\
h^2(y_n')^2(S_n-\alpha_2)=0.
\end{cases}
\tag{4.9}
$$

We assume that $y'(x)>0$ for all $x\in[a,b]$ or $y'(x)<0$ for all $x\in[a,b]$ which means that y is strictly monotone (recall that a linear/linear rational spline interpolant exists only if y is strictly monotone or constant everywhere). Consider now (4.9) as a linear system with respect to the unknowns S_i-y_i, $i=0,\dots,n$. Then its matrix has the diagonal dominance in rows for sufficiently small h. We look for the solution such that

$$
S_i=y_i+h^4[\psi(y)]_i+\beta_i,\quad i=0,\dots,n, \tag{4.10}
$$

where the continuous function $\psi(y)$ and numbers β_i will be specified later. Note, that the continuity of $\psi(y)$ gives $[\psi(y)]_{i-1}=[\psi(y)]_i+o(1)$, $[\psi(y)]_{i+1}=[\psi(y)]_i+o(1)$. Let us look more precisely the internal equations of (4.9). Replacing there S_i-y_i, $i=0,\dots,n$, with (4.10) we get

$$
\begin{aligned}
&\frac{h^6}{64}\Big((y_i')^2y_i^{IV}-4y_i'y_i''y_i'''+3(y_i'')^3\Big)\\
&\quad+\Big(\frac{h^2}{4}(y_i')^2+\frac{h^3}{4}y_i'y_i''+O(h^4)\Big)(h^4[\psi(y)]_i+\beta_{i-1})\\
&\quad+\Big(\frac{3}{2}h^2(y_i')^2+O(h^4)\Big)(h^4[\psi(y)]_i+\beta_i)\\
&\quad+\Big(\frac{h^2}{4}(y_i')^2-\frac{h^3}{4}y_i'y_i''+O(h^4)\Big)(h^4[\psi(y)]_i+\beta_{i+1})+o(h^6)=0.
\end{aligned}
$$

Determine now the function $\psi(y)$ so that the coefficient at h^6 is equal to 0, i.e.,

$$
\frac{1}{64}\Big((y_i')^2y_i^{IV}-4y_i'y_i''y_i'''+3(y_i'')^3\Big)+2(y_i')^2[\psi(y)]_i=0
$$

and it gives

$$[\psi(y)]_i = -\frac{1}{128}\Big(y_i^{IV} - 4\frac{y_i''y_i'''}{y_i'} + 3\frac{(y_i'')^3}{(y_i')^2}\Big), \quad i = 1, \ldots, n-1. \tag{4.11}$$

Extend (4.11) for $i = 0$ and $i = n$ as well, then choose $\beta_0 = o(h^4)$ and $\beta_n = o(h^4)$ (e.g., it may be $\beta_0 = \beta_n = 0$). This determines the values of α_1 and α_2. Thus, we pose the boundary conditions (4.2) in the form

$$\begin{aligned} S(a) &= y(a) - \frac{h^4}{128}\Big(y^{IV}(a) - 4\frac{y''(a)y'''(a)}{y'(a)} + 3\frac{(y''(a))^3}{(y'(a))^2}\Big) + o(h^4), \\ S(b) &= y(b) - \frac{h^4}{128}\Big(y^{IV}(b) - 4\frac{y''(b)y'''(b)}{y'(b)} + 3\frac{(y''(b))^3}{(y'(b))^2}\Big) + o(h^4). \end{aligned} \tag{4.12}$$

Therefore, we may write the system (4.9) as follows

$$\begin{cases} h^2(y_0')^2\beta_0 = o(h^6), \\ \Big(\frac{h^2}{4}(y_i')^2 + \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big)\beta_{i-1} \\ \quad +\Big(\frac{3}{2}h^2(y_i')^2 + O(h^4)\Big)\beta_i \\ \quad +\Big(\frac{h^2}{4}(y_i')^2 - \frac{h^3}{4}y_i'y_i'' + O(h^4)\Big)\beta_{i+1} + o(h^6) = 0, \quad i = 1, \ldots, n-1, \\ h^2(y_n')^2\beta_n = o(h^6). \end{cases}$$

This system has the matrix form $A\beta = g(\beta)$, where $\beta = (\beta_0, \ldots, \beta_n)$. The matrix A has a diagonal dominance in rows and the components of g are depending continuously on β. The equivalent system $\beta = A^{-1}g(\beta)$ has a solution by Bohl-Brouwer fixed point principle because $A^{-1}g$ maps a set $K = [-ch^4, ch^4]^{n+1}$ for some $c > 0$ into itself due to the fact that, for $\beta = O(h^4)$, we have $g(\beta) = o(h^6)$. Recall that the solution of the interpolation problem is unique and, consequently, β is uniquely determined. Thus, it holds $\beta_i = o(h^4)$, $i = 0, \ldots, n$, and in total, (4.10) is now

$$S_i = y_i - \frac{h^4}{128}\Big(y_i^{IV} - 4\frac{y_i''y_i'''}{y_i'} + 3\frac{(y_i'')^3}{(y_i')^2}\Big) + o(h^4), \quad i = 0, \ldots, n. \tag{4.13}$$

It could be also transformed into the form

$$S_i = y_i - \frac{h^4}{128}\Big(y_i'\Big(\frac{y'''}{y'}\Big)_i' - 3y_i''\Big(\frac{y''}{y'}\Big)_i'\Big) + o(h^4), \quad i = 0, \ldots, n.$$

4.3 Expansions of the interpolant

In this section we establish the expansions of interpolant S and its first and second derivatives on the whole particular interval.

We write the representation (2.9) with obvious notations A and B in the following form

$$S(x) = \bar{S}_i + \frac{A}{B} = y(x) + \frac{A - (y(x) - \bar{S}_i)B}{B}.$$

In the fractional term $\dfrac{A - (y(x) - \bar{S}_i)B}{B}$ we use $\bar{S}_i = \bar{y}_i$ and Taylor expansions at ξ_i, i.e.,

$$\begin{aligned}
S_i &= y_i - \frac{h^4}{128}\psi_i + o(h^4) \\
&= \bar{y}_i + \bar{y}_i'\frac{h}{2} + \frac{\bar{y}_i''}{2}\Big(\frac{h}{2}\Big)^2 + \frac{\bar{y}_i'''}{6}\Big(\frac{h}{2}\Big)^3 + \frac{\bar{y}_i^{IV}}{24}\Big(\frac{h}{2}\Big)^4 - \frac{h^4}{128}\bar{\psi}_i + o(h^4),
\end{aligned}$$

$$\begin{aligned}
S_{i-1} &= y_{i-1} - \frac{h^4}{128}\psi_{i-1} + o(h^4) \\
&= \bar{y}_i - \bar{y}_i'\frac{h}{2} + \frac{\bar{y}_i''}{2}\Big(\frac{h}{2}\Big)^2 - \frac{\bar{y}_i'''}{6}\Big(\frac{h}{2}\Big)^3 + \frac{\bar{y}_i^{IV}}{24}\Big(\frac{h}{2}\Big)^4 - \frac{h^4}{128}\bar{\psi}_i + o(h^4),
\end{aligned}$$

$$y(x) = \bar{y}_i + \bar{y}_i' th + \frac{\bar{y}_i''}{2}(th)^2 + \frac{\bar{y}_i'''}{6}(th)^3 + \frac{\bar{y}_i^{IV}}{24}(th)^4 + o(h^4).$$

This gives us for $x \in [x_{i-1}, x_i]$

$$\begin{aligned}
S(x) &= y(x) + \frac{t(1-4t^2)}{48}h^3\Big(2\bar{y}_i''' - 3\frac{(\bar{y}_i'')^2}{\bar{y}_i'}\Big) \\
&\quad + \frac{t^2}{48}h^4\Big(-(1+2t^2)\bar{y}_i^{IV} + 6\frac{\bar{y}_i''\bar{y}_i'''}{\bar{y}_i'} - 6(1-t^2)\frac{(\bar{y}_i'')^3}{(\bar{y}_i')^2}\Big) + o(h^4), \qquad (4.14)
\end{aligned}$$

with $x = \xi_i + th$, $t \in [-1/2, 1/2]$. Clearly, the expansion (4.14) at $x = x_i$ or $t = 1/2$ coincides with (4.13).

For the first derivative, proceeding similarly, i.e., using the form $S'(x) = y'(x) + \dfrac{A}{B}$ and same Taylor expansions at ξ_i we get from (2.10)

$$\begin{aligned}
S'(x) &= y'(x) + \frac{1-12t^2}{48}h^2\Big(2\bar{y}_i''' - 3\frac{(\bar{y}_i'')^2}{\bar{y}_i'}\Big) \\
&\quad - \frac{t}{24}h^3\Big((1+4t^2)\bar{y}_i^{IV} - 6\frac{\bar{y}_i''\bar{y}_i'''}{\bar{y}_i'} + 6(1-2t^2)\frac{(\bar{y}_i'')^3}{(\bar{y}_i')^2}\Big) + o(h^3). \qquad (4.15)
\end{aligned}$$

For the second derivative we obtain from (2.11)

$$S''(x) = y''(x) + th\Big(\bar{y}_i''' + \frac{3}{2}\frac{\bar{y}_i''^2}{\bar{y}_i'}\Big)$$

$$+ \frac{h^2}{24}\Big(- (1+12t^2)\bar{y}_i^{IV} + 6\frac{\bar{y}_i''\bar{y}_i'''}{\bar{y}_i'} - 6(1-6t^2)\frac{(\bar{y}_i'')^3}{(\bar{y}_i')^2}\Big) + o(h^2), \quad (4.16)$$

where, as before, $x = \xi_i + th$, $t \in [-1/2, 1/2]$.

Same technique allows to establish the expansions (4.13)−(4.16) in the case of boundary conditions (4.3) as

$$S'(a) = y'(a) - \frac{h^2}{12}\Big(y'''(a) - \frac{3}{2}\frac{(y''(a))^2}{y'(a)}\Big) + o(h^3),$$

$$S'(b) = y'(b) - \frac{h^2}{12}\Big(y'''(b) - \frac{3}{2}\frac{(y''(b))^2}{y'(b)}\Big) + o(h^3). \quad (4.17)$$

We have proved the following

Theorem 4.1. *Let y be a strictly monotone function and $y \in C^4[a,b]$. Then the linear/linear rational spline S of smoothness class C^1 satisfying interpolation conditions* (4.1) *and boundary conditions* (4.12) *or* (4.17) *expands like* (4.13)−(4.16)*.*

Remark 4.1. *If $y^{IV} \in \mathrm{Lip}\ \alpha$, $0 < \alpha \le 1$, then in previous formulae all the rest terms written as $o(h^k)$ for some k could be replaced by $O(h^{k+\alpha})$.*

4.4 Superconvergence rates

We derive our superconvergence rate results basing on the expansions of the interpolant (4.14)−(4.16) from the previous section.

It is clear that from (4.14) we get $S(x_i) = y(x_i) + O(h^4)$, $i = 0, \ldots, n$. Expansion (4.15) yields $S'(x) = y'(x) + O(h^3)$ in points $x = \xi_i + th$, corresponding to $t = \pm\sqrt{3}/6$, and lastly (4.16) gives $S''(\xi_i) = y''(\xi_i) + O(h^2)$, $i = 1, \ldots, n$.

The numerical experiments presented in Chapter 8 confirm the convergence rates predicted by the theory.

Similar results for quadratic splines were known earlier. They are given, e.g., in [32] in a slightly different form

$$S(x) = y(x) - \frac{t(1-t)(1-2t)}{12}h^3 y'''(x)$$

$$- \frac{(1-2t)^2(1+4t-4t^2)}{128}h^4 y^{IV}(x) + o(h^4),$$

$$S'(x) = y'(x) - \frac{1-6t-6t^2}{12} h^2 y'''(x) - \frac{t(1-t)(1-2t)}{6} h^3 y^{IV}(x) + o(h^3),$$

$$S''(x) = y''(x) + \frac{1-2t}{2} h y'''(x) - \frac{1-6t-6t^2}{6} h^2 y^{IV}(x) + o(h^2),$$

$$x \in [x_{i-1}, x_i], \quad x = x_{i-1} + th, \quad t \in [0,1].$$

We see that the superconvergence takes place in points x_i, $i = 0, \ldots, n$, for quadratic spline interpolant S, in $x = \xi_i \pm (\sqrt{3}/6)h$, $i = 1, \ldots, n$, for S' and in ξ_i, $i = 1, \ldots, n$, for S'', that means in the same points as for linear/linear rational spline interpolants.

Chapter 5

Quadratic/linear rational spline interpolation

The interpolation problem with quadratic/linear rational splines is similar to that with cubic splines, to which the expansions on subintervals via the derivatives of the smooth function to interpolate could be found, e.g., in [58]. We will study such a problem in the case of quadratic/linear rational spline interpolation. This needs expansions of a quadratic/linear rational spline interpolant with special boundary conditions and the establishment of them is the main purpose of this chapter. At first we give the description of the quadratic/linear rational spline interpolation method. Then we analyze the obtained nonlinear system and transform it to a more suitable form. In the end of this chapter we get the expansions of the interpolant and receive the superconvergence results.

5.1 Description of the method

Consider a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b-a)/n$, $n \in \mathbb{N}$. In this chapter we use the spline representation by spline values and second moments (Section 3.2).

Assume that there is given data y_i, $i = 0, \ldots, n$. In interpolation with quadratic/linear rational splines we look for a spline $S \in C^2[a, b]$ such that

$$S(x_i) = y_i, \quad i = 0, \ldots, n. \tag{5.1}$$

In addition, we set two boundary conditions

$$S'(a) = \alpha_1, \quad S'(b) = \alpha_2 \tag{5.2}$$

or

$$S''(a) = \alpha_1, \quad S''(b) = \alpha_2 \tag{5.3}$$

for given α_1 and α_2, which we will specify later. After replacing the values S_i, $i = 0, \dots, n$, from (5.1) in the internal equations (3.12) and considering them with two boundary conditions we obtain a nonlinear system with respect to the unknowns $M_0, \dots, M_n$.

For a strictly convex (or strictly concave) smooth function y and interpolating quadratic/linear rational spline S it is known that $\|S - y\|_\infty = O(h^4)$, see, e.g., [36, 37]. In [46] it is proven, that a quadratic/linear rational spline interpolant of class C^2 exists and is unique and strictly convex for any strictly convex data.

5.2 Transformation of the system and second moments of the interpolant

In this section we study the nonlinear system with respect to the unknowns $M_0, \dots, M_n$.

Suppose that we have a sufficiently smooth function $y : [a, b] \to \mathbb{R}$ to interpolate. Denote $y_i = y(x_i)$, $i = 0, \dots, n$, similar notation will be used in the case of derivatives.

Firstly we write the equations (3.12) with replaced values S_i from (5.1) in the following form

$$\varphi_i(M_{i-1}, M_i, M_{i+1}) = M_i^{2/3}\left(M_{i-1}^{1/3} + M_{i+1}^{1/3}\right) - \frac{2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 0, \quad i = 1, \dots, n-1, \tag{5.4}$$

introducing at the same time functions φ_i. Now we consider the boundary conditions (5.3) and internal equations (5.4) in the form

$$\begin{cases} M_0 - \alpha_1 = 0, \\ \varphi_i(M_{i-1}, M_i, M_{i+1}) = M_i^{2/3}\left(M_{i-1}^{1/3} + M_{i+1}^{1/3}\right) - \dfrac{2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, n-1, \\ M_n - \alpha_2 = 0. \end{cases}$$

This system can be transformed with the help of Taylor expansion at (5.4) into

$$
\begin{cases}
M_0 - \alpha_1 = 0, \\
\varphi_i(y''_{i-1}, y''_i, y''_{i+1}) + \dfrac{\partial \varphi_i}{\partial M_{i-1}}(y''_{i-1}, y''_i, y''_{i+1})(M_{i-1} - y''_{i-1}) \\
\qquad + \dfrac{\partial \varphi_i}{\partial M_i}(y''_{i-1}, y''_i, y''_{i+1})(M_i - y''_i) \\
\qquad + \dfrac{\partial \varphi_i}{\partial M_{i+1}}(y''_{i-1}, y''_i, y''_{i+1})(M_{i+1} - y''_{i+1}) + \dfrac{\varphi''_i}{2!}(\xi_\lambda)\bar{h}^2 = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n-1, \\
M_n - \alpha_2 = 0
\end{cases}
\tag{5.5}
$$

with the difference vector $\bar{h} = (M_{i-1} - y''_{i-1}, M_i - y''_i, M_{i+1} - y''_{i+1})$, some number $\lambda \in (0,1)$ and $\xi_\lambda = (y''_{i-1}, y''_i, y''_{i+1}) + \lambda \bar{h}$. Now let us investigate $\varphi_i(y''_{i-1}, y''_i, y''_{i+1})$ and the partial derivatives as well as the rest term in (5.5). We calculate from (5.4) for $i = 1, \ldots, n-1$

$$
\begin{aligned}
\frac{\partial \varphi_i}{\partial M_{i-1}}(M_{i-1}, M_i, M_{i+1}) &= \frac{1}{3}\Big(\frac{M_i}{M_{i-1}}\Big)^{2/3}, \\
\frac{\partial \varphi_i}{\partial M_i}(M_{i-1}, M_i, M_{i+1}) &= \frac{2}{3}\Big(\Big(\frac{M_{i-1}}{M_i}\Big)^{1/3} + \Big(\frac{M_{i+1}}{M_i}\Big)^{1/3}\Big), \\
\frac{\partial \varphi_i}{\partial M_{i+1}}(M_{i-1}, M_i, M_{i+1}) &= \frac{1}{3}\Big(\frac{M_i}{M_{i+1}}\Big)^{2/3}.
\end{aligned}
\tag{5.6}
$$

Suppose in the following that $y \in C^4[a,b]$. We assume that $y''(x) > 0$ for all $x \in [a,b]$ or $y''(x) < 0$ for all $x \in [a,b]$ which means that y or $-y$ is strictly convex. Let us expand y_{i-1}, y_{i+1}, y''_{i-1} and y''_{i+1} at the point x_i by Taylor formula up to the forth derivative as

$$
\begin{aligned}
y_{i-1} &= y_i - hy'_i + \frac{h^2}{2}y''_i - \frac{h^3}{6}y'''_i + \frac{h^4}{24}y_i^{IV} + o(h^4), \\
y_{i+1} &= y_i + hy'_i + \frac{h^2}{2}y''_i + \frac{h^3}{6}y'''_i + \frac{h^4}{24}y_i^{IV} + o(h^4), \\
y''_{i-1} &= y''_i - hy'''_i + \frac{h^2}{2}y_i^{IV} + o(h^2), \\
y''_{i+1} &= y''_i + hy'''_i + \frac{h^2}{2}y_i^{IV} + o(h^2).
\end{aligned}
$$

First two expansions give for $\varphi_i(y''_{i-1}, y''_i, y''_{i+1})$

$$
\frac{2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 2y''_i + \frac{1}{6}h^2 y_i^{IV} + o(h^2).
$$

Next, y_i'' together with y_{i-1}'' gives

$$\frac{y_i''}{y_{i-1}''} = 1 + h\frac{y_i'''}{y_i''} - \frac{h^2}{2}\frac{y_i^{IV}}{y_i''} + h^2\frac{(y_i''')^2}{(y_i'')^2} + o(h^2)$$

and y_i'' with y_{i+1}''

$$\frac{y_i''}{y_{i+1}''} = 1 - h\frac{y_i'''}{y_i''} - \frac{h^2}{2}\frac{y_i^{IV}}{y_i''} + h^2\frac{(y_i''')^2}{(y_i'')^2} + o(h^2).$$

Then by (5.6) with the help of expansions $(1+x)^{2/3} = 1 + \frac{2}{3}x - \frac{1}{9}x^2 + o(x^2)$ and $(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 + o(x^2)$ direct calculations yield

$$\frac{\partial\varphi_i}{\partial M_{i-1}}(y_{i-1}'', y_i'', y_{i+1}'') = \frac{1}{3} + \frac{2}{9}h\frac{y_i'''}{y_i''} - \frac{1}{9}h^2\frac{y_i^{IV}}{y_i''} + \frac{5}{27}h^2\Big(\frac{y_i'''}{y_i''}\Big)^2 + o(h^2),$$

$$\frac{\partial\varphi_i}{\partial M_i}(y_{i-1}'', y_i'', y_{i+1}'') = \frac{4}{3} + \frac{2}{9}h^2\frac{y_i^{IV}}{y_i''} - \frac{4}{27}h^2\Big(\frac{y_i'''}{y_i''}\Big)^2 + o(h^2),$$

$$\frac{\partial\varphi_i}{\partial M_{i+1}}(y_{i-1}'', y_i'', y_{i+1}'') = \frac{1}{3} - \frac{2}{9}h\frac{y_i'''}{y_i''} - \frac{1}{9}h^2\frac{y_i^{IV}}{y_i''} + \frac{5}{27}h^2\Big(\frac{y_i'''}{y_i''}\Big)^2 + o(h^2).$$

From (5.4) we get

$$\varphi_i(y_{i-1}'', y_i'', y_{i+1}'') = \frac{1}{6}h^2 y_i^{IV} - \frac{2}{9}h^2\frac{(y_i''')^2}{y_i''} + o(h^2)$$

which we replace in (5.5).

We look for the solution of the obtained system such that

$$M_i = y_i'' + h^2[\psi(y)]_i + \beta_i, \quad i = 0, \ldots, n, \tag{5.7}$$

where we suppose the function $\psi(y)$ to be continuous. Then $[\psi(y)]_{i-1} = [\psi(y)]_i + o(1)$ and $[\psi(y)]_{i+1} = [\psi(y)]_i + o(1)$.

The entries in the matrix φ_i'' as second order partial derivatives of φ_i could be calculated from (5.6):

$$\frac{\partial^2\varphi_i}{\partial M_{i-1}^2} = -\frac{2}{9}\Big(\frac{M_i}{M_{i-1}}\Big)^{2/3}\frac{1}{M_{i-1}},$$

$$\frac{\partial^2\varphi_i}{\partial M_{i-1}\partial M_i} = \frac{\partial^2\varphi_i}{\partial M_i\partial M_{i-1}} = \frac{2}{9}\Big(\frac{M_i}{M_{i-1}}\Big)^{2/3}\frac{1}{M_i},$$

$$\frac{\partial^2 \varphi_i}{\partial M_{i-1}\partial M_{i+1}} = \frac{\partial^2 \varphi_i}{\partial M_{i+1}\partial M_{i-1}} = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial M_i^2} = -\frac{2}{9}\Big(\Big(\frac{M_{i-1}}{M_i}\Big)^{1/3} + \Big(\frac{M_{i+1}}{M_i}\Big)^{1/3}\Big)\frac{1}{M_i}$$

$$\frac{\partial^2 \varphi_i}{\partial M_i\partial M_{i+1}} = \frac{\partial^2 \varphi_i}{\partial M_{i+1}\partial M_i} = \frac{2}{9}\Big(\frac{M_{i+1}}{M_i}\Big)^{1/3}\frac{1}{M_{i+1}},$$

$$\frac{\partial^2 \varphi_i}{\partial M_{i+1}^2} = -\frac{2}{9}\Big(\frac{M_i}{M_{i+1}}\Big)^{2/3}\frac{1}{M_{i+1}}.$$

We see, that they contain a multiplier M_j^{-1}, $j = i-1, i, i+1$, of the expressions in (5.6) and are of order $O(1)$ provided we suppose, e.g., that $\beta_i = O(h)$. Then, in the case $\beta_i = O(h^2)$, due to the three-diagonality of the matrix φ_i'', we have $\varphi_i''(\xi_\lambda)\bar{h}^2 = O(h^4)$ and the system (5.5) could be written as

$$\begin{cases} y_0'' + h^2[\psi(y)]_0 + \beta_0 - \alpha_1 = 0, \\[2mm] \dfrac{1}{6}h^2 y_i^{IV} - \dfrac{2}{9}h^2\dfrac{(y_i''')^2}{y_i''} \\[2mm] \quad +\Big(\dfrac{1}{3} + \dfrac{2}{9}h\dfrac{y_i'''}{y_i''} - \dfrac{1}{9}h^2\dfrac{y_i^{IV}}{y_i''} + \dfrac{5}{27}h^2\Big(\dfrac{y_i'''}{y_i''}\Big)^2\Big)(h^2[\psi(y)]_i + \beta_{i-1}) \\[2mm] \quad +\Big(\dfrac{4}{3} + \dfrac{2}{9}h^2\dfrac{y_i^{IV}}{y_i''} - \dfrac{4}{27}h^2\Big(\dfrac{y_i'''}{y_i''}\Big)^2\Big)(h^2[\psi(y)]_i + \beta_i) \\[2mm] \quad +\Big(\dfrac{1}{3} - \dfrac{2}{9}h\dfrac{y_i'''}{y_i''} - \dfrac{1}{9}h^2\dfrac{y_i^{IV}}{y_i''} + \dfrac{5}{27}h^2\Big(\dfrac{y_i'''}{y_i''}\Big)^2\Big)(h^2[\psi(y)]_i + \beta_{i+1}) + o(h^2) = 0, \\[2mm] \hfill i = 1, \dots, n-1, \\[2mm] y_n'' + h^2[\psi(y)]_n + \beta_n - \alpha_2 = 0. \end{cases} \tag{5.8}$$

Determine the function $\psi(y)$ so that the coefficient at h^2 in interior equations is equal to 0. This gives

$$\psi(y) = -\frac{1}{12}\Big(y^{IV} - \frac{4}{3}\frac{(y''')^2}{y''}\Big). \tag{5.9}$$

Let us choose α_1 and α_2 so that $\beta_0 = o(h^2)$ and $\beta_n = o(h^2)$ (e.g., it may be

$\beta_0 = \beta_n = 0$), thus, we pose the boundary conditions (5.3) in the form

$$S''(a) = y''(a) - \frac{h^2}{12}\Big(y^{IV}(a) - \frac{4}{3}\frac{(y'''(a))^2}{y''(a)}\Big) + o(h^2),$$
$$S''(b) = y''(b) - \frac{h^2}{12}\Big(y^{IV}(b) - \frac{4}{3}\frac{(y'''(b))^2}{y''(b)}\Big) + o(h^2). \tag{5.10}$$

Finally, we get from (5.8) a system of the form $A\beta = \Phi(\beta)$ with respect to the unknowns $\beta = (\beta_0, \ldots, \beta_n)$ having the matrix A with diagonal dominance in rows and the components of Φ depending continuously on β. The equivalent system $\beta = A^{-1}\Phi(\beta)$ has a solution by Bohl-Brouwer fixed point principle because $A^{-1}\Phi$ maps a set $K = [-ch^2, ch^2]^{n+1}$ for some $c > 0$ into itself due to the fact that, for $\beta = O(h^2)$, we have $\Phi(\beta) = o(h^2)$. Recall that the solution of the interpolation problem is unique and, consequently, β is uniquely determined. Thus, it holds $\beta_i = o(h^2)$, $i = 0, \ldots, n$, and we arrive at the estimate

$$M_i = y_i'' - \frac{h^2}{12}\Big(y_i^{IV} - \frac{4}{3}\frac{(y_i''')^2}{y_i''}\Big) + o(h^2), \quad i = 0, \ldots, n. \tag{5.11}$$

Note that in the case $y^{IV} \in \text{Lip}\,\alpha$, $0 < \alpha \le 1$, we have the error terms $O(h^{2+\alpha})$ instead of $o(h^2)$ in all earlier expansions and estimates.

5.3 Expansions of the interpolant

In this section the expansions of interpolants on the whole particular interval will be established.

Setting $x = x_{i-1} + th$, $t \in [0,1]$ and replacing S_{i-1} and S_i in (3.8), (3.9), (3.10) as well as (3.11) by y_{i-1} and y_i, respectively, we write them in the form

$$S(x) = y_{i-1} - \frac{t(1-t)h^2 M_{i-1}}{2(1+d_i h)(1+d_i th)}, \tag{5.12}$$

$$S'(x) = \frac{y_i - y_{i-1}}{h} + \frac{(t - 1 + t(1+d_i th))hM_{i-1}}{2(1+d_i h)(1+d_i th)^2}, \tag{5.13}$$

$$S''(x) = \frac{M_{i-1}}{(1+d_i th)^3} \tag{5.14}$$

and

$$S''(x) = \frac{-3M_{i-1}d_i}{(1+d_i th)^4}. \tag{5.15}$$

Using $1+d_ih = (M_{i-1}/M_i)^{1/3}$ and (5.11) we establish with the help of Taylor formula the expansion

$$1+d_ith = 1+t\Big(-\frac{h}{3}\frac{y_i'''}{y_i''}+h^2\Big(\frac{1}{6}\frac{y_i^{IV}}{y_i''}-\frac{1}{9}\Big(\frac{y_i'''}{y_i''}\Big)^2\Big)\Big)+o(h^2).$$

Similarly we can express $(1+d_ith)^2$, $(1+d_ith)^3$, $(1+d_ith)^4$ and d_i needed in (5.12), (5.13), (5.14), (5.15). Finally, by direct calculations, the Taylor expansion in $x\in[x_{i-1},x_i]$ gives

$$S(x) = y(x)-\frac{t^2(1-t)^2}{24}h^4\Big(y^{IV}(x)-\frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big)+o(h^4), \tag{5.16}$$

$$S'(x) = y'(x)-\frac{t(1-t)(1-2t)}{12}h^3\Big(y^{IV}(x)-\frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big)+o(h^3), \tag{5.17}$$

$$S''(x) = y''(x)-\frac{1-6t(1-t)}{12}h^2\Big(y^{IV}(x)-\frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big)+o(h^2), \tag{5.18}$$

$$S'''(x) = y'''(x)+\frac{1-2t}{2}h\Big(y^{IV}(x)-\frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big)+o(h). \tag{5.19}$$

Note that (5.18) at $x=x_i$ coincides with (5.11).

The boundary conditions (5.2) have to be used in the form

$$S'(a)=y'(a)+o(h^3),\quad S'(b)=y'(b)+o(h^3). \tag{5.20}$$

Recall that we assumed $y\in C^4[a,b]$. Suppose now that $y\in C^5[a,b]$. The reasoning of Section 5.2 gives then (5.11) with the rest term $o(h^3)$ instead of $o(h^2)$. Doing once more the calculations, we obtain

$$\begin{aligned}1+d_ith = 1+t\Big(&-\frac{h}{3}\frac{y_i'''}{y_i''}+h^2\Big(\frac{1}{6}\frac{y_i^{IV}}{y_i''}-\frac{1}{9}\Big(\frac{y_i'''}{y_i''}\Big)^2\Big)\\ &+h^3\Big(-\frac{1}{36}\frac{y_i^{V}}{y_i''}+\frac{1}{108}\frac{y_i'''y_i^{IV}}{(y_i'')^2}+\frac{1}{81}\Big(\frac{y_i'''}{y_i''}\Big)^3\Big)\Big)+o(h^3)\end{aligned}$$

and then for $x\in[x_{i-1},x_i]$ we get the expansions

$$\begin{aligned}S(x) = y(x)&-\frac{t^2(1-t)^2}{24}h^4\Big(y^{IV}(x)-\frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big)\\ &-\frac{t(1-t)(1-2t)(1+3t(1-t))}{180}h^5\Big(y^{V}(x)-\frac{10}{3}\frac{y'''(x)y^{IV}(x)}{y''(x)}+\frac{20}{9}\frac{(y'''(x))^3}{(y''(x))^2}\Big)\\ &+o(h^5),\end{aligned} \tag{5.21}$$

$$S'(x) = y'(x) - \frac{t(1-t)(1-2t)}{12}h^3\Big(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big)$$
$$- \frac{2-45t^2(1-t)^2}{360}h^4 y^V(x) + \frac{1-24t^2(1-t)^2}{54}h^4\frac{y'''(x)y^{IV}(x)}{y''(x)} \tag{5.22}$$
$$- \frac{2-51t^2(1-t)^2}{162}h^4\frac{(y'''(x))^3}{(y''(x))^2} + o(h^4),$$

$$S''(x) = y''(x) - \frac{1-6t(1-t)}{12}h^2\Big(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big) \tag{5.23}$$
$$+ \frac{t(1-t)(1-2t)}{6}h^3\Big(y^V(x) - 4\frac{y'''(x)y^{IV}(x)}{y''(x)} + \frac{28}{9}\frac{(y'''(x))^3}{(y''(x))^2}\Big) + o(h^3),$$

$$S'''(x) = y'''(x) + \frac{1-2t}{2}h\Big(y^{IV}(x) - \frac{4}{3}\frac{(y'''(x))^2}{y''(x)}\Big) \tag{5.24}$$
$$+ \frac{1-6t(1-t)}{12}h^2\Big(y^V(x) - \frac{16}{3}\frac{y'''(x)y^{IV}(x)}{y''(x)} + \frac{44}{9}\frac{(y'''(x))^3}{(y''(x))^2}\Big) + o(h^2).$$

The boundary conditions (5.2) have to be specified now as

$$\begin{aligned} S'(a) &= y'(a) - h^4\Big(\frac{1}{180}y^V(a) - \frac{1}{54}\frac{y'''(a)y^{IV}(a)}{y''(a)} + \frac{1}{81}\frac{(y'''(a))^3}{(y''(a))^2}\Big) + o(h^4), \\ S'(b) &= y'(b) - h^4\Big(\frac{1}{180}y^V(b) - \frac{1}{54}\frac{y'''(b)y^{IV}(b)}{y''(b)} + \frac{1}{81}\frac{(y'''(b))^3}{(y''(b))^2}\Big) + o(h^4). \end{aligned} \tag{5.25}$$

We have proved the following

Theorem 5.1. *Let y (or $-y$) be a strictly convex function. In the case $y \in C^4[a,b]$ the quadratic/linear rational spline S of smoothness class C^2 satisfying interpolation conditions* (5.1) *and boundary conditions* (5.10) *or* (5.20) *expands as shown in* (5.16)$-$(5.19). *If $y \in C^5[a,b]$ the expansions* (5.21)$-$(5.24) *hold provided the boundary conditions* (5.10) *with the rest terms $o(h^3)$ instead of $o(h^2)$ or* (5.25) *are used.*

Remark 5.1. *If $y^{IV} \in \mathrm{Lip}\ \alpha$ or $y^V \in \mathrm{Lip}\ \alpha$, $0 < \alpha \le 1$, then in previous formulae all the rest terms written as $o(h^k)$ for some k could be replaced by $O(h^{k+\alpha})$.*

5.4 Superconvergence rates

Basing on expansions (5.17)−(5.19) it is now immediate to obtain superconvergence assertions.

From (5.17) we get $S'(x) = y'(x) + O(h^4)$ in points $x = x_i$ and $x = (x_{i-1}+x_i)/2$, (5.18) yields $S''(x) = y''(x) + O(h^3)$ in points $x = x_i + th$, corresponding to $t = (3 \pm \sqrt{3})/6$ and (5.19) gives $S'''(x) = y'''(x) + O(h^2)$ in points $(x_{i-1} + x_i)/2$.

In Chapter 8 there are numerical experiments which confirm the theoretical convergence rates.

The expansions for cubic spline interpolants, which were known earlier could be found, e.g., in [58]. In the case $y \in C^5[a,b]$, for $x \in [x_{i-1}, x_i]$

$$S(x) = y(x) - \frac{t^2(1-t)^2}{24} h^4 y^{IV}(x)$$

$$- \frac{t(1-t)(1-2t)(1+3t(1-t))}{180} h^5 y^V(x) + o(h^5),$$

$$S'(x) = y'(x) - \frac{t(1-t)(1-2t)}{12} h^3 y^{IV}(x) - \frac{2-45t^2(1-t)^2}{360} h^4 y^V(x) + o(h^4),$$

$$S''(x) = y''(x) - \frac{1-6t(1-t)}{12} h^2 y^{IV}(x) + \frac{t(1-t)(1-2t)}{6} h^3 y^V(x) + o(h^3),$$

$$S'''(x) = y'''(x) + \frac{1-2t}{2} h y^{IV}(x) + \frac{1-6t(1-t)}{12} h^2 y^V(x) + o(h^2).$$

We see that the superconvergence takes place in the same points as well for quadratic/linear rational and cubic spline interpolants.

Chapter 6

Linear/linear rational spline collocation

In this chapter we will study the linear/linear rational spline collocation method for a boundary value problem of second order ordinary linear differential equation. Before, let us describe the problem and assumptions about it with corresponding consequences.

We consider the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in (a, b), \tag{6.1}$$

with boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta. \tag{6.2}$$

Suppose that the problem has a solution $y \in C^2[a, b]$. Let p, q, f be continuous and $q(x) \leq q < 0$, $x \in (a, b)$. Then the solution of problem (6.1), (6.2) is unique. Let us prove that.

We show that the homogeneous problem corresponding to (6.1), (6.2), i.e.,

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad x \in (a, b), \tag{6.3}$$

$$y(a) = 0, \quad y(b) = 0 \tag{6.4}$$

has only the trivial solution.

Firstly, let $p \in C^1[a, b]$. We use the change of variable

$$y(x) = e^{-\frac{1}{2}\int p(x)dx} z(x).$$

Then

$$y'(x) = e^{-\frac{1}{2}\int p(x)dx}(-\frac{1}{2}p(x)z(x) + z'(x)),$$

$$y''(x) = e^{-\frac{1}{2}\int p(x)dx}(\frac{1}{4}p^2(x)z(x) - p(x)z'(x) - \frac{1}{2}p'(x)z(x) + z''(x)).$$

After replacing the obtained derivatives in (6.3), we have

$$
\begin{aligned}
&z''(x) + \big(-\frac{1}{2}p'(x) - \frac{1}{4}p^2(x) + q(x)\big) z(x) = 0, \quad x \in (a,b), \qquad (6.5)\\
&z(a) = 0, \;\; z(b) = 0.
\end{aligned}
$$

Let z be the solution to (6.5). Now multiply the equation (6.5) by z and integrate. Then

$$\int_a^b z''(x)z(x)dx + \int_a^b (-\frac{1}{2}p'(x) - \frac{1}{4}p^2(x) + q(x))z^2(x)dx = 0$$

which gives us with the help of integration by parts

$$-\int_a^b (z'(x))^2 dx + \int_a^b p(x)z(x)z'(x)dx - \frac{1}{4}\int_a^b p^2(x)z^2(x)dx + \int_a^b q(x)z^2(x)dx = 0.$$

Note that

$$
\begin{aligned}
\int_a^b (z'(x))^2 dx - \int_a^b p(x)z(x)z'(x)dx \;\; + \frac{1}{4}\int_a^b p^2(x)z^2(x)dx \\
= \int_a^b (\frac{1}{2}p(x)z(x) - z'(x))^2 dx \geq 0
\end{aligned}
$$

and $\int_a^b q(x)z^2(x)dx \leq 0$. So we get $\int_a^b q(x)z^2(x)dx = 0$ which implies $z = 0$ a.e. in $[a,b]$ and it follows $z = 0$. This, in turn, gives $y = 0$ for the solution of (6.3), (6.4).

Secondly, let $p \notin C^1[a,b]$, but $p \in C[a,b]$. We can take $p_n \in C^1[a,b]$ so that $\|p_n - p\|_C \to 0$. Now let

$$
\begin{aligned}
y(x) &= e^{-\frac{1}{2}\int_a^x p_n(x)dx} z_n(x),\\
y(x) &= e^{-\frac{1}{2}\int_a^x p(x)dx} z(x)
\end{aligned}
$$

which gives

$$
\begin{aligned}
z_n(x) &= e^{\frac{1}{2}\int_a^x p_n(x)dx} y(x),\\
z(x) &= e^{\frac{1}{2}\int_a^x p(x)dx} y(x)
\end{aligned}
$$

and $\|z_n - z\|_C \to 0$. Then z_n is the solution of the counterpart of (6.5) with p_n and we get by the same discussion as in first part of the proof that $z_n = 0$. Thus, $z = 0$ and $y = 0$.

Note that in this reasoning it is sufficient to suppose that $q(x) < 0$ a.e. in $[a,b]$.

6.1 Description of the spline collocation method

A mesh $a = x_0 < x_1 < \ldots < x_n = b$ will be used representing the spline knots $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b-a)/n$, $n \in \mathbb{N}$. We define the collocation points $\xi_i = x_{i-1} + h/2$, $i = 1, \ldots, n$.

In our discussion we use again the linear/linear rational spline representation by spline values (Section 2.3). We remind that in this representation we have the notations $S(x_i) = S_i$, $i = 0, \ldots, n$, and $S(\xi_i) = \bar{S}_i$, $i = 1, \ldots, n$.

In collocation we require the spline S to satisfy the differential equation at points ξ_i and the boundary conditions:

$$S''(\xi_i) + p(\xi_i)S'(\xi_i) + q(\xi_i)S(\xi_i) = f(\xi_i), \quad i = 1, \ldots, n, \tag{6.6}$$

$$S(a) = \alpha, \quad S(b) = \beta. \tag{6.7}$$

Internal equations (2.12) together with equations (6.6) and (6.7) form a nonlinear system with respect to the unknowns $\bar{S}_1, \ldots, \bar{S}_n$, $S_0, \ldots, S_n$.

Note, that the collocation problem is similar for quadratic spline case. The convergence rate $O(h^2)$ for quadratic spline collocation method is known [30, 39]. Let us mention that $O(h^2)$ convergence rate is based on superconvergence property of interpolating splines. This was discovered in [30] and developed extensively in [35, 39]. It was shown in [39] that the main part of error at quadratic spline collocation is actually several times less than was obtained in [30]. We will study such a problem in the case of linear/linear rational splines.

6.2 Transformation of the system and main properties of the derivatives

Let us start our investigation by transforming the system (2.12), (6.6), (6.7) to a suitable form in order to establish some estimates which will be used in the proof of existence of solution.

Likewise to (4.4) in linear/linear rational spline interpolation discussion let us introduce functions φ_i and write equations (2.12) in the form

$$\begin{aligned}\varphi_i(S_{i-1}, \bar{S}_i, S_i, \bar{S}_{i+1}, S_{i+1}) &= (S_i - \bar{S}_i)(S_i - S_{i-1})(S_{i+1} - \bar{S}_{i+1}) \\ &- (\bar{S}_{i+1} - S_i)(S_{i+1} - S_i)(\bar{S}_i - S_{i-1}) = 0, \quad i = 1, \ldots, n-1.\end{aligned} \tag{6.8}$$

Observe that if y is the solution of (6.1) we may write for $i = 1, \ldots, n$

$$y''(\xi_i) + p(\xi_i)y'(\xi_i) + q(\xi_i)y(\xi_i) = f(\xi_i). \tag{6.9}$$

Using (2.10) and (2.11) at point ξ_i in (6.6) we get for $i = 1, \dots, n$

$$\frac{-16(\bar{S}_i - S_{i-1})(S_i - \bar{S}_i)(2\bar{S}_i - S_{i-1} - S_i)}{h^2(S_i - S_{i-1})^2} \qquad (6.10)$$
$$+ p(\xi_i)\frac{4(\bar{S}_i - S_{i-1})(S_i - \bar{S}_i)}{h(S_i - S_{i-1})} + q(\xi_i)\bar{S}_i = f(\xi_i).$$

Now subtract equation (6.10) from (6.9) and multiply the result by $h^2(S_i - S_{i-1})$. This gives the following equations:

$$\begin{aligned} &\psi_i(S_{i-1}, \bar{S}_i, S_i) \\ &\quad = 16(\bar{S}_i - S_{i-1})(S_i - \bar{S}_i)(2\bar{S}_i - S_{i-1} - S_i) + h^2(S_i - S_{i-1})^2 y''(\xi_i) \\ &\qquad - p(\xi_i)(4h(\bar{S}_i - S_{i-1})(S_i - S_{i-1})(S_i - \bar{S}_i) + h^2(S_i - S_{i-1})^2 y'(\xi_i)) \\ &\qquad - h^2 q(\xi_i)(S_i - S_{i-1})^2(\bar{S}_i - y(\xi_i)) = 0, \qquad i = 1, \dots, n, \end{aligned} \qquad (6.11)$$

where we also introduce functions ψ_i.

We write the boundary conditions (with the help of notations φ_0 and φ_n for the left side of equations) as

$$\varphi_0(S_0) = h^2(S_0 - \alpha) = 0, \quad \varphi_n(S_n) = h^2(S_n - \beta) = 0. \qquad (6.12)$$

Let $S = (S_0, \bar{S}_1, S_1, \dots, \bar{S}_n, S_n)$. We point out, that although we denote by S a spline and the vector of its knot values, from the context it is clear which object we deal with. Now the internal equations (6.8), collocation equations (6.11) and boundary equations (6.12) together form a nonlinear system $F(S) = 0$ with $F : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ and $F(S) = (\varphi_0(S), \psi_1(S), \varphi_1(S), \dots, \psi_n(S), \varphi_n(S))$ (here and in the sequel we allow the whole vector of spline knot values S in $\varphi_i(S)$ and $\psi_i(S)$ instead of some particular components as in (6.8), (6.11) and (6.12)). The system $F(S) = 0$ will be considered as basic system and it contains all the requirements on spline values. Our aim is to show that this system has an appropriate solution and then obtain the convergence results.

In order to establish our goals we need to use the linear/linear rational spline interpolant S satisfying (4.1) and (4.12) for the solution y of problem (6.1), (6.2). Denote this special interpolant by S_* and define its vector of values by $S^* = (S_*(x_0), S_*(\xi_1), S_*(x_1), \dots, S_*(\xi_n), S_*(x_n)) \in \mathbb{R}^{2n+1}$. It occurs that around the vector S^* we succeed in finding a set with sufficiently small size containing the vector of values of approximate solution.

Let us assume now and in the sequel that the solution y of (6.1), (6.2) is such that $y \in C^4[a, b]$ and $y'(x) > 0$ for all $x \in [a, b]$ or $y'(x) < 0$ for all $x \in [a, b]$. In addition, let $p', q \in C[a, b]$ and $q(x) \le q < 0$, $x \in [a, b]$. Our aim is to estimate

$F(S^*)$ and $F'(S^*)$. Let $y_i = y(x_i)$, $i = 0, \ldots, n$, and $\bar{y}_i = y(\xi_i)$, $i = 1, \ldots, n$, similar notation will be used in the case of derivatives and other functions. Based on (4.13) and using for the special interpolant S_* the Taylor expansions up to the fourth derivative in the rest

$$S^*_{i-1} = \bar{y}_i - \frac{h}{2}\bar{y}'_i + \frac{h^2}{8}\bar{y}''_i - \frac{h^3}{48}\bar{y}'''_i + \frac{h^4}{384}\bar{y}^{IV}_i - \frac{h^4}{128}\bar{\rho}_i + o(h^4),$$

$$S^*_i = \bar{y}_i + \frac{h}{2}\bar{y}'_i + \frac{h^2}{8}\bar{y}''_i + \frac{h^3}{48}\bar{y}'''_i + \frac{h^4}{384}\bar{y}^{IV}_i - \frac{h^4}{128}\bar{\rho}_i + o(h^4),$$

where

$$\rho = y^{IV} - 4\frac{y''y'''}{y'} + 3\frac{(y'')^3}{(y')^2},$$

straightforward calculations give

$$\psi_i(S^*) = h^6\Big(-\frac{1}{24}(\bar{y}'_i)^2\bar{y}^{IV}_i + \frac{1}{4}\bar{y}'_i\bar{y}''_i\bar{y}'''_i - \frac{1}{4}(\bar{y}''_i)^3 + p(\xi_i)\Big(\frac{1}{24}(\bar{y}'_i)^2\bar{y}'''_i - \frac{1}{16}\bar{y}'_i(\bar{y}''_i)^2\Big)\Big) + o(h^6). \qquad (6.13)$$

Due to (2.12) or (6.8) we have $\varphi_i(S^*) = 0$, $i = 1, \ldots, n-1$. In addition, special boundary conditions (4.12) for S_* assure $\varphi_0(S^*) = o(h^6)$ and $\varphi_n(S^*) = o(h^6)$. Thus, we can conclude that $\|F(S^*)\|_\infty = O(h^6)$.

Next, we are interested in the matrix $F'(S^*)$. In the rows corresponding to equations (6.11) there are

$$\begin{aligned}\frac{\partial\psi_i}{\partial S_{i-1}} &= 16(S_i - \bar{S}_i)((S_i - \bar{S}_i) - 2(\bar{S}_i - S_{i-1})) - 2h^2(S_i - S_{i-1})\bar{y}''_i \\ &\quad + p(\xi_i)(4h(S_i - \bar{S}_i)((\bar{S}_i - S_{i-1}) - (S_i - S_{i-1})) - 2h^2(S_i - S_{i-1})\bar{y}'_i) \\ &\quad + 2h^2q(\xi_i)(\bar{S}_i - \bar{y}_i)(S_i - S_{i-1}),\end{aligned}$$

$$\begin{aligned}\frac{\partial\psi_i}{\partial \bar{S}_i} &= -16((\bar{S}_i - S_{i-1}) - (S_i - \bar{S}_i))^2 + 32(\bar{S}_i - S_{i-1})(S_i - \bar{S}_i) \\ &\quad - 4hp(\xi_i)((S_i - S_{i-1})((S_i - \bar{S}_i) - (\bar{S}_i - S_{i-1}))) - h^2q(\xi_i)(S_i - S_{i-1})^2,\end{aligned}$$

$$\begin{aligned}\frac{\partial\psi_i}{\partial S_i} &= 16(\bar{S}_i - S_{i-1})((\bar{S}_i - S_{i-1}) - 2(S_i - \bar{S}_i)) + 2h^2(S_i - S_{i-1})\bar{y}''_i \\ &\quad - p(\xi_i)(4h(\bar{S}_i - S_{1-1})((S_i - S_{i-1}) + (S_i - \bar{S}_i)) + 2h^2(S_i - S_{i-1})\bar{y}'_i) \\ &\quad - 2h^2q(\xi_i)(\bar{S}_i - \bar{y}_i)(S_i - S_{i-1}).\end{aligned}$$

They give

$$\frac{\partial \psi_i}{\partial S_{i-1}} + \frac{\partial \psi_i}{\partial \bar{S}_i} + \frac{\partial \psi_i}{\partial S_i} = -h^2 q(\xi_i)(S_i - S_{i-1})^2. \tag{6.14}$$

Using again the Taylor formula at the point ξ_i we may write

$$-h^2 q(\xi_i)(S_i^* - S_{i-1}^*)^2 = -h^4 q(\xi_i)(\bar{y}_i')^2 + O(h^5)$$

and, in addition, have the partial derivatives for $i = 1, \ldots, n$ in the form

$$\begin{aligned}
\frac{\partial \psi_i}{\partial S_{i-1}}(S^*) = & -4h^2(\bar{y}_i')^2 + h^3 p(\xi_i)(\bar{y}_i')^2 \\
& - h^4\Big(\frac{1}{3}\bar{y}_i'\bar{y}_i''' - \frac{3}{4}(\bar{y}_i'')^2 - \frac{1}{2}p(\xi_i)\bar{y}_i'\bar{y}_i''\Big) + O(h^5),
\end{aligned}$$

$$\frac{\partial \psi_i}{\partial \bar{S}_i}(S^*) = 8h^2(\bar{y}_i')^2 + h^4\Big(\frac{2}{3}\bar{y}_i'\bar{y}_i''' - \frac{3}{2}(\bar{y}_i'')^2 - p(\xi_i)\bar{y}_i'\bar{y}_i'' - q(\xi_i)(\bar{y}_i')^2\Big) + O(h^5), \tag{6.15}$$

$$\begin{aligned}
\frac{\partial \psi_i}{\partial S_i}(S^*) = & -4h^2(\bar{y}_i')^2 - h^3 p(\xi_i)(\bar{y}_i')^2 \\
& - h^4\Big(\frac{1}{3}\bar{y}_i'\bar{y}_i''' - \frac{3}{4}(\bar{y}_i'')^2 - \frac{1}{2}p(\xi_i)\bar{y}_i'\bar{y}_i''\Big) + O(h^5).
\end{aligned}$$

It is now immediate to verify that, for small h, we get the diagonal dominance in rows corresponding to the collocation equations. Namely (recall the assumption $q(x) \le q < 0,\ x \in (a, b)$)

$$\Big|\frac{\partial \psi_i}{\partial \bar{S}_i}(S^*)\Big| - \Big|\frac{\partial \psi_i}{\partial S_{i-1}}(S^*)\Big| - \Big|\frac{\partial \psi_i}{\partial S_i}(S^*)\Big| = -h^4 q(\xi_i)(\bar{y}_i')^2 + O(h^5).$$

In the rows of matrix $F'(S^*)$ corresponding to equations (6.8) we calculate the derivatives

$$\frac{\partial \varphi_i}{\partial S_{i-1}} = -(S_i - \bar{S}_i)(S_{i+1} - \bar{S}_{i+1}) + (\bar{S}_{i+1} - S_i)(S_{i+1} - S_i),$$

$$\frac{\partial \varphi_i}{\partial \bar{S}_i} = -(S_i - S_{i-1})(S_{i+1} - \bar{S}_{i+1}) - (\bar{S}_{i+1} - S_i)(S_{i+1} - S_i),$$

$$\frac{\partial \varphi_i}{\partial S_i} = (S_i - S_{i-1} + S_i - \bar{S}_i)(S_{i+1} - \bar{S}_{i+1}) + (\bar{S}_i - S_{i-1})(\bar{S}_{i+1} - S_i + S_{i+1} - S_i),$$

$$\frac{\partial \varphi_i}{\partial \bar{S}_{i+1}} = -(S_i - S_{i-1})(S_i - \bar{S}_i) - (\bar{S}_i - S_{i-1})(S_{i+1} - S_i),$$

$$\frac{\partial \varphi_i}{\partial S_{i+1}} = (S_i - S_{i-1})(S_i - \bar{S}_i) - (\bar{S}_i - S_{i-1})(\bar{S}_{i+1} - S_i).$$

Obviously, then

$$\frac{\partial \varphi_i}{\partial S_{i-1}} + \frac{\partial \varphi_i}{\partial \bar{S}_i} + \frac{\partial \varphi_i}{\partial S_i} + \frac{\partial \varphi_i}{\partial \bar{S}_{i+1}} + \frac{\partial \varphi_i}{\partial S_{i+1}} = 0. \tag{6.16}$$

Observe also that, for $z = (S_0, \bar{S}_1, S_1, \ldots, \bar{S}_n, S_n)$, (6.16) yields

$$\sum_j \frac{\partial^2 \varphi_i}{\partial z_j \partial z_k} = 0 \tag{6.17}$$

and

$$\sum_j \frac{\partial^3 \varphi_i}{\partial z_j \partial z_k \partial z_l} = 0$$

for all i, k, l.

Using $S_i^* = y_i - \frac{h^4}{128}\rho_i + o(h^4)$ and $\bar{S}_i^* = \bar{y}_i$ we perform the following Taylor expansions

$$\begin{aligned}
S_{i-1}^* &= y_i - hy_i' + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} - \frac{h^4}{128}\rho_i + o(h^4),\\
\bar{S}_i^* &= y_i - \frac{h}{2}y_i' + \frac{h^2}{8}y_i'' - \frac{h^3}{48}y_i''' + \frac{h^4}{384}y_i^{IV} + o(h^4),\\
\bar{S}_{i+1}^* &= y_i + \frac{h}{2}y_i' + \frac{h^2}{8}y_i'' + \frac{h^3}{48}y_i''' + \frac{h^4}{384}y_i^{IV} + o(h^4),\\
S_{i+1}^* &= y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} - \frac{h^4}{128}\rho_i + o(h^4)
\end{aligned} \tag{6.18}$$

which in all give

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}}(S^*) &= \frac{1}{4}h^2(y_i')^2 + \frac{1}{4}h^3 y_i' y_i'' + h^4(\frac{1}{48}y_i' y_i''' + \frac{7}{64}(y_i'')^2) + O(h^5),\\
\frac{\partial \varphi_i}{\partial \bar{S}_i}(S^*) &= -h^2(y_i')^2 - \frac{1}{2}h^3 y_i' y_i'' - h^4(\frac{1}{3}y_i' y_i''' - \frac{1}{8}(y_i'')^2) + O(h^5),\\
\frac{\partial \varphi_i}{\partial S_i}(S^*) &= \frac{3}{2}h^2(y_i')^2 + h^4(\frac{5}{8}y_i' y_i''' - \frac{15}{32}(y_i'')^2) + O(h^5),\\
\frac{\partial \varphi_i}{\partial \bar{S}_{i+1}}(S^*) &= -h^2(y_i')^2 + \frac{1}{2}h^3 y_i' y_i'' - h^4(\frac{1}{3}y_i' y_i''' - \frac{1}{8}(y_i'')^2) + O(h^5),\\
\frac{\partial \varphi_i}{\partial S_{i+1}}(S^*) &= \frac{1}{4}h^2(y_i')^2 - \frac{1}{4}h^3 y_i' y_i'' + h^4(\frac{1}{48}y_i' y_i''' + \frac{7}{64}(y_i'')^2) + O(h^5).
\end{aligned} \tag{6.19}$$

Clearly, by (6.19), there is no diagonal dominance in rows of $F'(S^*)$ corresponding to the internal equations. Let us mention that, instead of (6.18) and (6.19), we could use shorter expansions to detect this nondominance but we need them in such a form later.

Because of the absence of diagonal dominance in rows of $F'(S^*)$ corresponding to the internal equations we replace the equations (6.8) in $F(S) = 0$ by equations

$$\tilde{\varphi}_i(S) = -\frac{1}{2}\psi_i(S) + \frac{16}{3}\varphi_i(S) - \frac{1}{2}\psi_{i+1}(S) = 0 \tag{6.20}$$

and, thereby, transform the basic system into the form $\tilde{F}(S) = MF(S) = 0$, where the entries of M are $m_{i,i-1} = m_{i,i+1} = -\frac{1}{2}$, $m_{ii} = \frac{16}{3}$ in rows corresponding to the internal equations, $m_{ii} = 1$ in other rows and $m_{ij} = 0$ elsewhere. This assures $\|M\|_{\infty\longrightarrow\infty} \leq$ const and $\|M^{-1}\|_{\infty\longrightarrow\infty} \leq$ const which we will use later.

Basing on (6.15) and (6.19) we calculate

$$\Big|\frac{\partial\tilde{\varphi}_i}{\partial S_i}(S^*)\Big| - \Big|\frac{\partial\tilde{\varphi}_i}{\partial S_{i-1}}(S^*)\Big| - \Big|\frac{\partial\tilde{\varphi}_i}{\partial \bar{S}_i}(S^*)\Big| - \Big|\frac{\partial\tilde{\varphi}_i}{\partial \bar{S}_{i+1}}(S^*)\Big| - \Big|\frac{\partial\tilde{\varphi}_i}{\partial S_{i+1}}(S^*)\Big|$$
$$= -h^4 q(x_i)(y_i')^2 + O(h^5).$$

Thus, for small h, the matrix $\tilde{F}'(S^*)$ has the diagonal dominance in rows. This implies the invertibility of $\tilde{F}'(S^*)$ and hence, of $F'(S^*)$. All in all, we achieve $\|(\tilde{F}'(S^*))^{-1}\|_{\infty\longrightarrow\infty} = O(h^{-4})$ and $\|(F'(S^*))^{-1}\|_{\infty\longrightarrow\infty} = O(h^{-4})$.

6.3 Application of fixed point principle

In this section we use the Bohl-Brouwer fixed point theorem to show the existence of solution of the basic system (6.8), (6.11), (6.12) which we denoted in previous section by $F(S) = 0$. Firstly, we transform the basic system to an equivalent form and then carry out an analysis with the help of Taylor expansion.

Consider the equation $S = G(S)$ where $G(S) = S - (F'(S^*))^{-1}F(S)$. Then G is continuous. The equation $F(S) = 0$ is equivalent to $S = G(S)$. Let us introduce the sets $K = \prod_{i=0}^{2n} K_i,\;\; K_i = [S_i^* - ch^2, S_i^* + ch^2]$ and

$$\begin{aligned}\tilde{K} = \{S \in K \mid\; &|\,(\bar{S}_i - S_*(\xi_i)) - (S_{i-1} - S_*(x_{i-1}))\,| \leq \tilde{c}h^3,\\ &|\,(S_i - S_*(x_i)) - (\bar{S}_i - S_*(\xi_i))\,| \leq \tilde{c}h^3,\; i = 1, \ldots, n\}\end{aligned}$$

with numbers $c > 0$ and $\tilde{c} > 0$ independent of h and which will be specified later. It is clear that the set $\tilde{K}$ is convex and compact. Our main purpose is now to show that $G : \tilde{K} \to \tilde{K}$. This allows us to use Bohl-Brouwer fixed point theorem.

To begin with we show that $G : \tilde{K} \to K$. The Taylor expansion of $F(S)$ at point S^* gives

$$G(S) - S^* = -\,(F'(S^*))^{-1}F(S^*) - \frac{1}{2}(F'(S^*))^{-1}F''(S^*)(S-S^*)^2 - \frac{1}{6}(F'(S^*))^{-1}F'''(S^*)(S-S^*)^3. \tag{6.21}$$

From the previous section we already know that $\|(F'(S^*))^{-1}\| = O(h^{-4})$ and $\|F(S^*)\| = O(h^6)$. It follows $\|(F'(S^*))^{-1}F(S^*)\| \le c_0h^2$ for some $c_0 > 0$.

In order to estimate the terms with second and third derivatives in (6.21) we state the following technical result.

Lemma 6.1. *Suppose $A = (a_{ij})$ is a $n \times n$ matrix with $\sum_{j=1}^n a_{ij} = 0$, $i = 1, \dots, n$, and $\sum_{i=1}^n a_{ij} = 0$, $j = 1, \dots, n$. Then $(Ax, x) = (ABz, Bz) = (B^TABz, z)$ for all $x \in \mathbb{R}^n$ with $z_1 = 0$, $z_i = x_i - x_{i-1}$, $i = 2, \dots, n$, and $B = (b_{ij})$ with $b_{ij} = 1$ for $i \ge j$ and $b_{ij} = 0$ for $i < j$.*

Proof. We have

$$\begin{aligned}(Ax,x) &= \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}x_j\Big)x_i = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x_j - x_1)\Big)x_i = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(\sum_{k=1}^j z_k)\Big)x_i \\ &= \sum_{i=1}^n\sum_{j=1}^n a_{ij}(Bz)_jx_i = (ABz, x) = (Bz, A^Tx) = (Bz, A^TBz) = (ABz, Bz).\end{aligned}$$

□

In order to estimate $(F'(S^*))^{-1}F''(S^*)(S-S^*)^2$ we point out, that the components of $F''(S^*)(S-S^*)^2$ are $\varphi_i''(S^*)(S-S^*)^2$ or $\psi_i''(S^*)(S-S^*)^2$. Any symmetric matrix $\varphi_i''(S^*)$, $i = 1, \dots, n-1$, has only one 5x5 nonzero diagonal block with the property (6.17). Let us use now Lemma 6.1, where we take $A = \varphi_i''(S^*)$. Then B^TAB has only one 4x4 nonzero diagonal block with entries of order $O(h)$. By the inclusion $S \in \tilde{K}$ we can conclude $\varphi_i''(S^*)(S-S^*)^2 = O(h^7)$, $i = 1, \dots, n-1$. Clearly, $\varphi_0'' = \varphi_n'' = 0$.

For the $\psi_i''(S^*)(S-S^*)^2$ we calculate from (6.14)

$$\frac{\partial^2\psi_i}{\partial S_{i-1}^2} + \frac{\partial^2\psi_i}{\partial \bar{S}_i\partial S_{i-1}} + \frac{\partial^2\psi_i}{\partial S_i\partial S_{i-1}} = 2h^2q(\xi_i)(S_i - S_{i-1}),$$

$$\frac{\partial^2\psi_i}{\partial S_{i-1}\partial \bar{S}_i} + \frac{\partial^2\psi_i}{\partial \bar{S}_i^2} + \frac{\partial^2\psi_i}{\partial S_i\partial \bar{S}_i} = 0,$$

$$\frac{\partial^2\psi_i}{\partial S_{i-1}\partial S_i} + \frac{\partial^2\psi_i}{\partial \bar{S}_i\partial S_i} + \frac{\partial^2\psi_i}{\partial S_i^2} = -2h^2q(\xi_i)(S_i - S_{i-1}).$$

Let us split $\psi_i''(S^*)$ as A_0+A_1 where A_0 has only a 3x3 nonzero diagonal block of second order derivatives of ψ_i except $\dfrac{\partial^2\psi_i}{\partial S_{i-1}^2}$ replaced by $\dfrac{\partial^2\psi_i}{\partial S_{i-1}^2}-2h^2q(\xi_i)(S_i-S_{i-1})$ and $\dfrac{\partial^2\psi_i}{\partial S_i^2}$ replaced by $\dfrac{\partial^2\psi_i}{\partial S_i^2}+2h^2q(\xi_i)(S_i-S_{i-1})$. Then A_0 has the property of zero sum of entries by rows and columns needed in Lemma 6.1. In this case B^TA_0B has a 2x2 nonzero diagonal block with entries of order $O(h)$ and hence $A_0(S-S^*)^2=O(h^7)$ due to $S\in\tilde{K}$. The matrix A_1 has only two nonzero entries of order $O(h^3)$ and $A_1(S-S^*)^2=O(h^7)$ by $S\in K$. Thus, $\psi_i''(S^*)(S-S^*)^2=O(h^7)$. Again, due to $\|(F'(S^*))^{-1}\|=O(h^{-4})$ we establish the estimate

$$\|(F'(S^*))^{-1}F''(S^*)(S-S^*)^2\|\le(c_2+\tilde{c}_2c^2)h^3$$

for some $c_2,\tilde{c}_2>0$ and all $S\in\tilde{K}$ (c^2 appears from the estimate of $A_1(S-S^*)^2$).

Similar reasoning allows to show $\|(F'(S^*))^{-1}F'''(S^*)(S-S^*)^3\|\le(c_3+\tilde{c}_3c^3)h^4$ for some $c_3,\tilde{c}_3>0$ and all $S\in\tilde{K}$.

All estimates together yield that it holds $G(S)\in K$ for all $S\in\tilde{K}$ if

$$c_0h^2+\frac{1}{2}(c_2+\tilde{c}_2c^2)h^3+\frac{1}{6}(c_3+\tilde{c}_3c^3)h^4\le ch^2 \tag{6.22}$$

which takes place if we choose, e.g., $c=2c_0$ and then h sufficiently small. All in all, $G:\tilde{K}\to K$.

Our next goal is to show that $G(S)\in\tilde{K}$ for $S\in\tilde{K}$. We know already that $G(S)\in K$. Additional conditions in $\tilde{K}$ concern adjacent components. Consider now (6.21) as $G(S)-S^*=Ax$ where $A=(F'(S^*))^{-1}=(a_{ij})$ and $x\in\mathbb{R}^{2n+1}$, $\|x\|=O(h^6)$. Then we have

$$[Ax]_i-[Ax]_{i+1}=\sum_j(a_{ij}-a_{i+1,j})x_j$$

and in the case of $a_{ij}-a_{i+1,j}=O(h^{-2})$ we get

$$|\,[Ax]_i-[Ax]_{i+1}\,|\le\sum_j|\,a_{ij}-a_{i+1,j}\,|\,|\,x_j\,|=O(h^3)$$

as needed for $G(S)\in\tilde{K}$. Actually, here we will transform the basic system $F(S)=0$ into the form $\tilde{F}(S)=MF(S)=0$ with some regular suitable matrix M having the property that $\|M\|$ and $\|M^{-1}\|$ are bounded uniformly in n (therefore in h). It turns out that we get the suitable matrix M if we replace the equations $\varphi_i(S)=0$, $i=1,\dots,n-1$, by the equations

$$\tilde{\varphi}_i(S)=\frac{1}{2}\psi_i(S)+8\varphi_i(S)+\frac{1}{2}\psi_{i+1}(S)=0.$$

Then we see the entries of $\tilde{F}'(S^*)$ from (6.15) and (6.19) as

$$\begin{aligned}
\frac{\partial\tilde{\varphi}_i}{\partial S_{i-1}}(S^*) &= h^3(4y_i'y_i'' + \frac{1}{2}p_i(y_i')^2) \\
&\quad + h^4(-\frac{1}{2}y_i'y_i''' + \frac{3}{4}(y_i'')^2 - \frac{1}{4}p_iy_i'y_i'' - \frac{1}{4}p_i'(y_i')^2) + O(h^5), \\
\frac{\partial\tilde{\varphi}_i}{\partial \bar{S}_i}(S^*) &= -4h^2(y_i')^2 - 8h^3y_i'y_i'' \\
&\quad + h^4(-\frac{4}{3}y_i'y_i''' + \frac{5}{4}(y_i'')^2 - \frac{1}{2}p_iy_i'y_i'' - \frac{1}{2}q_i(y_i')^2) + O(h^5), \\
\frac{\partial\tilde{\varphi}_i}{\partial S_i}(S^*) &= 8h^2(y_i')^2 + h^4(\frac{11}{3}y_i'y_i''' - 4(y_i'')^2 + \frac{3}{2}p_iy_i'y_i'' + \frac{1}{2}p_i'(y_i')^2) + O(h^5),
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
\frac{\partial\tilde{\varphi}_i}{\partial \bar{S}_{i+1}}(S^*) &= -4h^2(y_i')^2 + 8h^3y_i'y_i'' \\
&\quad + h^4(-\frac{4}{3}y_i'y_i''' + \frac{5}{4}(y_i'')^2 - \frac{1}{2}p_iy_i'y_i'' - \frac{1}{2}q_i(y_i')^2) + O(h^5), \\
\frac{\partial\tilde{\varphi}_i}{\partial S_{i+1}}(S^*) &= h^3(-4y_i'y_i'' - \frac{1}{2}p_i(y_i')^2) \\
&\quad + h^4(-\frac{1}{2}y_i'y_i''' + \frac{3}{4}(y_i'')^2 - \frac{1}{4}p_iy_i'y_i'' - \frac{1}{4}p_i'(y_i')^2) + O(h^5)
\end{aligned}$$

together with those obtained from (6.12). For (6.23), we observe the diagonal dominance in rows with the difference $-h^4q_i(y_i')^2 + O(h^5)$.

It follows now that $G(S) - S^* = (F'(S^*))^{-1}x = (\tilde{F}'(S^*))^{-1}Mx$ with $\|Mx\| = O(h^6)$ and we can use the Bohl-Brouwer fixed point theorem, by which the set $\tilde{K}$ contains a fixed point of the function G which is also a solution of the basic system.

To complete this section we need to prove the following lemma for the matrix $A = (\tilde{F}'(S^*))^{-1} = (a_{ij})$

Lemma 6.2. *It holds* $\mid a_{ij} - a_{i+1,j} \mid = O(h^{-2})$.

Proof. We give the proof in the case of $p = 0$ in equation (6.1) which simplifies considerably the writings.

Dividing the equations $\psi_i(S) = 0$ by $4h^2(\bar{y}_i')^2$ we transform the derivatives in

(6.15) into the form

$$
\begin{aligned}
&-(1+h^2 r_i)+O(h^3),\\
&2(1+h^2(r_i-s_i))+O(h^3),\\
&-(1+h^2 r_i)+O(h^3),
\end{aligned}
\tag{6.24}
$$

where $r_i = \dfrac{1}{12}\dfrac{\bar{y}_i'''}{\bar{y}_i'} - \dfrac{1}{4}\Big(\dfrac{\bar{y}_i''}{\bar{y}_i'}\Big)^2$, $s_i = \dfrac{1}{8}\bar{q}_i$.

We will eliminate in the equations of $\tilde{F}'(S^*)A = I$ (I being identity matrix) containing the derivatives from (6.23) the first and the last ones. For that, let us expand (6.15) and also (6.15) for $i+1$ at the point x_i. The standard elimination and division by $4h^2(y_i')^2$ gives the equations with coefficients of the form (6.24) with $r_i = \dfrac{7}{12}\dfrac{y_i'''}{y_i'} - \dfrac{11}{16}\Big(\dfrac{y_i''}{y_i'}\Big)^2 + \dfrac{1}{8}q_i$ and $s_i = \dfrac{1}{8}q_i$. Dividing also equations (6.12) by h^2 we will study the equation $\tilde{F}'(S^*)A = I$ with the entries of $\tilde{F}'(S^*)$ of the form (6.24) in interior rows and show that here, for $A=(a_{ij})$, it holds $\mid a_{ij} - a_{i+1,j} \mid = O(1)$.

For simplicity in writings we suppose that the rows and columns of A have the indices from 1 to n. We omit in (6.24) and in calculations the terms of $O(h^3)$ (see the remark at the end of the proof).

Let us start with the first column of $\tilde{F}'(S^*)A$. Clearly, $a_{11}=1$. From

$$-(1+h^2r_2)a_{11} + 2(1+h^2(r_2-s_2))a_{21} - (1+h^2r_2)a_{31} = 0$$

we obtain

$$a_{21} - \frac{1}{2}(1+h^2 s_2)a_{31} = \frac{1}{2}(1+h^2 s_2) \tag{6.25}$$

and

$$a_{21} - a_{31} = \frac{1}{2}(-1+h^2 s_2)a_{31} + \frac{1}{2}(1+h^2 s_2).$$

Straightforward calculations lead to the equalities

$$a_{i1} - \frac{i-1}{i}(1+h^2(b_{i2}s_2+\ldots+b_{ii}s_i))a_{i+1,1} = \frac{1}{i}(1+h^2(c_{i2}s_2+\ldots+c_{ii}s_i)) \tag{6.26}$$

and

$$
\begin{aligned}
a_{i1} - a_{i+1,1} = {}& \frac{1}{i}(-1+h^2((i-1)b_{i2}s_2+\ldots+(i-1)b_{ii}s_i))a_{i+1,1}\\
&+\frac{1}{i}(1+h^2(c_{i2}s_2+\ldots+c_{ii}s_i)), \quad i=2,\ldots,n-1.
\end{aligned}
\tag{6.27}
$$

In (6.25) we have $b_{22}=c_{22}=1$. Successive increasing of index i allows to establish the relations

$$
\begin{aligned}
&b_{ij} = \frac{i-2}{i}b_{i-1,j}, \quad j=2,\ldots,i-1, \quad b_{ii} = \frac{2(i-1)}{i},\\
&c_{ij} = c_{i-1,j} + b_{ij}, \quad j=2,\ldots,i-1, \quad c_{ii} = b_{ii}.
\end{aligned}
$$

Then, for $i \geq 2$,

$$(i-1)\sum_{j=2}^{i} b_{ij} = (i-1)\Big(\sum_{j=2}^{i-1} b_{ij} + b_{ii}\Big) = (i-1)\Big(\sum_{j=2}^{i-1} \frac{i-2}{i} b_{i-1,j} + \frac{2(i-1)}{i}\Big)$$

$$= 2\frac{i-1}{i}(i-1) + \frac{i-1}{i}\Big(\frac{2(i-2)}{i-1}(i-2) + \frac{i-2}{i-1}(i-3)\sum_{j=2}^{i-2} b_{i-2,j}\Big)$$

$$< 2((i-1)+(i-2)+\ldots+b_{22}) = i(i-1).$$

Thus, $\sum_{j=2}^{i} b_{ij} \leq i \leq n$ and, consequently,

$$h^2 \sum_{j=2}^{i} b_{ij} s_j = O(h) \quad \text{with} \quad \sum_{j=2}^{i} b_{ij} s_j < 0. \tag{6.28}$$

Similarly, we establish

$$\sum_{j=2}^{i} c_{ij} \leq i(i-1) \quad \text{and} \quad h^2 \sum_{j=2}^{i} c_{ij} s_j = O(1). \tag{6.29}$$

Write (6.26) in the form

$$\frac{a_{i1}}{i-1} = (1+b_i)\frac{a_{i+1,1}}{i} + \frac{1}{(i-1)i}(1+c_i) \tag{6.30}$$

where $b_i = h^2 \sum_{j=2}^{i} b_{ij} s_j$ and $c_i = h^2 \sum_{j=2}^{i} c_{ij} s_j$. Denote now $z_i = \dfrac{a_{i1}}{i-1}$, $i = 2, \ldots, n$. Evidently, $a_{n1} = 0$ and $z_n = 0$. By (6.29), from (6.30) we get $|\, z_{n-1} \,| \leq \dfrac{C_1}{(n-2)(n-1)}$ with some constant C_1. Then, successively using (6.30), (6.28) which implies $1 + b_i \leq 1$, and also the estimate of z_{n-1}, we have

$$|\, z_{n-i} \,| \leq C_2\Big(\frac{1}{(n-i-1)(n-i)} + \ldots + \frac{1}{(n-2)(n-1)}\Big) \leq C,$$

C_2, C being positive constants. We have shown that (recall that $a_{11} = 1$ and $a_{n1} = 0$)

$$\frac{a_{i1}}{i-1} = O(1), \; i = n-1, \ldots, 2. \tag{6.31}$$

Now (6.27) with the help of (6.28), (6.29) and (6.31) yields

$$|\, a_{i1} - a_{i+1,1} \,| = O(1), \; i = 1, \ldots, n-1. \tag{6.32}$$

The calculations in the second column of $\tilde{F}'(S^*)A$ are almost identical. E.g., $a_{12} = 0$ and formula (6.26) is replaced by

$$a_{i2} - \frac{i-1}{i}(1 + h^2(b_{i2}s_2 + \ldots + b_{ii}s_i))a_{i+1,2} = \frac{1}{i}(1 + h^2(c_{i2}s_2 + \ldots + c_{ii}s_i - r_2))$$

with corresponding difference in (6.27). The counterpart of (6.32) holds.

Let us discuss more generally columns with indices $k = 2, \ldots, \dfrac{n+1}{2}$ (observe that in our reasoning n is actually odd number and due to the symmetrical structure of $F'(S^*)$ there is no need to consider $k > \dfrac{n+1}{2}$ because we could start from the right bottom of A with $a_{nn} = 1$). We have $a_{1k} = 0$ and get

$$a_{ik} - \frac{i-1}{i}(1 + h^2(b_{i2}s_2 + \ldots + b_{ii}s_i))a_{i+1,k} = 0, \; i < k, \tag{6.33}$$

which implies $\left|\dfrac{a_{ik}}{i-1}\right| \leq \left|\dfrac{a_{i+1,k}}{i}\right|$, $i = 1, \ldots, k-1$. From (6.33) it follows

$$a_{ik} - a_{i+1,k} = \frac{1}{i}(-1 + h^2((i-1)b_{i2}s_2 + \ldots + (i-1)b_{ii}s_i))a_{i+1,k}, \; i < k. \tag{6.34}$$

Again, direct calculations give

$$\begin{aligned} a_{ik} - \frac{i-1}{i}&(1 + h^2(b_{i2}s_2 + \ldots + b_{ii}s_i))a_{i+1,k} \\ &= \frac{k-1}{i}(1 + h^2(c_{i2}s_2 + \ldots + c_{ii}s_i - r_k)), \quad i = k, \ldots, n-1, \end{aligned} \tag{6.35}$$

and this implies

$$\begin{aligned} a_{ik} - a_{i+1,k} = &\frac{1}{i}(-1 + h^2(b_{i2}s_2 + \ldots + b_{ii}s_i))a_{i+1,k} \\ &+ \frac{k-1}{i}(1 + h^2(c_{i2}s_2 + \ldots + c_{ii}s_i - r_k)), \quad i = k, \ldots, n-1. \end{aligned} \tag{6.36}$$

Denoting here $z_i = \dfrac{a_{ik}}{i-1}$, $i = 2, \ldots, n$, we have $z_n = 0$. The recursion (6.35) has the form

$$z_i = \frac{k-1}{(i-1)i}(1 + c_i) + (1 + b_i)z_{i+1}, \; i = k, \ldots, n-1. \tag{6.37}$$

Basing again on (6.29), it holds $\mid 1 + c_i \mid \leq C_1 \mid$ and $\mid z_{n-1} \mid \leq \dfrac{k-1}{(n-1)(n-2)}C_1$. Successive use of (6.37) gives in this case for $i = 2, \ldots, n-k$

$$\mid z_{n-i} \mid \leq C_2(k-1)\Big(\frac{1}{(k-1)k} + \ldots + \frac{2}{(n-2)(n-1)}\Big).$$

But

$$\sum_{i=k}^{n-1} \frac{1}{(i-1)i} \leq \sum_{i=k-1}^{\infty} \frac{1}{i^2} \leq \int_{k-1}^{\infty} \frac{dx}{x^2} = \frac{1}{k-1}$$

and hence, we have shown that

$$\left| \frac{a_{ik}}{i-1} \right| = O(1), \; i = 2, \ldots, n-1.$$

This, in turn, by (6.34) and (6.36) gives

$$|\; a_{ik} - a_{i+1,k} \;| = O(1), \; i = 1, \ldots, n-1,$$

which was the assertion of Lemma 6.2.

Remark. We can observe that the rest terms in (6.15) and (6.23) and those occurring in calculations do not spoil the reasoning. Namely, if $\tilde{F}'(S^*) = A + B$, $\|B\| = O(h^5)$ with $\|A^{-1}\| = O(h^{-4})$ and $|\; [A^{-1}x]_i - [A^{-1}x]_{i+1} \;| = O(h^3)$ for $\|x\| = O(h^6)$, then $(\tilde{F}'(S^*))^{-1}x = A^{-1}(I - (BA^{-1})(I + BA^{-1})^{-1})x$ and we have also $|\; [(\tilde{F}'(S^*))^{-1}x]_i - [(\tilde{F}'(S^*))^{-1}x]_{i+1} \;| = O(h^3)$. The described elimination means that we replace $F(S) = 0$ by $(I + E)F(S) = 0$, E being a diagonal matrix with entries of order $O(h)$. Thus, the matrix $M = I + E$ is suitable.

This completes the proof of Lemma 6.2. □

6.4 Convergence estimates

In this section we establish in uniform norm the convergence rates of collocation method (6.6), (6.7) with linear/linear rational splines for boundary value problem (6.1), (6.2).

Let us remind, that in previous section we proved that the set $\tilde{K}$ contains a fixed point of the function G which is also a solution of the basic system (6.8), (6.11), (6.12). Denote by S here the linear/linear rational spline having as knot values this vector in $\tilde{K}$. Recall that S_* is a special linear/linear rational spline interpolant to the solution y of (6.1), (6.2). From [25] we know (see also Chapter 4), that

$$\| \; S_* - y \; \|_\infty = O(h^3) \tag{6.38}$$

and

$$\| \; S'_* - y' \; \|_\infty = O(h^2). \tag{6.39}$$

Firstly, we determine c_0 for the convergence estimate. The constant c_0 is obtained from the estimate of $(\tilde{F}'(S^*))^{-1}\tilde{F}(S^*)$ where $\tilde{F}(S) = MF(S)$ with suitable matrix M. Suppose we divide (6.11) by $-h^4 q(\xi_i)(\bar{y}'_i)^2$ and (6.20) by $-h^4 q(x_i)(y'_i)^2$ which we include into the matrix M. Then the matrix $\tilde{F}'(S^*)$ has the diagonal dominance in rows with the difference 1 and, consequently, $\| \; (\tilde{F}'(S^*))^{-1} \; \| \leq 1$. Then, on the other hand, as $\varphi_i(S^*) = 0$, by (6.20) the estimate of $\tilde{\psi}_i(S^*)$ reduces to the estimate of $\psi_i(S^*)$ which, by (6.13), gives that

$$c_0 = \frac{1}{24}\max_{1\le i\le n}\left|\frac{y^{IV} - py''' - 6\dfrac{y'''y''}{y'} + 6\dfrac{(y'')^3}{(y')^2} + \dfrac{3}{2}p\dfrac{(y'')^2}{y'}}{q}(\xi_i)\right|. \tag{6.40}$$

Note that, instead of points ξ_i, $i = 1,\dots,n$, we could use in determination of c_0 knots x_i, $i = 1,\dots,n-1$.

Before achieving in uniform norm the convergence rates, we need some auxiliary results. If we take $c = c_0 + c_1 h$ with sufficiently large $c_1 > 0$ then (6.22) holds. This means that

$$|S(x_i) - S_*(x_i)| \le c_0 h^2 + o(h^2),\ i = 1,\dots,n-1, \tag{6.41}$$

and

$$|S(\xi_i) - S_*(\xi_i)| \le c_0 h^2 + o(h^2),\ i = 1,\dots,n. \tag{6.42}$$

Now from the estimate $|S(\xi_i) - y(\xi_i)| \le |S(\xi_i) - S_*(\xi_i)| + |S_*(\xi_i) - y(\xi_i)|$ we get by (6.42) and (6.38)

$$|S(\xi_i) - y(\xi_i)| \le c_0 h^2 + o(h^2). \tag{6.43}$$

Proposition 6.1. *For the solution S of the problem* (6.6), (6.7) *having the vector of values in $\tilde K$ it holds $\|S^{(k)}\|_\infty = \max_{1\le i\le n}\max_{x_{i-1}\le x\le x_i}|S^{(k)}(x)| \le C_k$, $k \ge 1$.*

Proof. From (2.1) we obtain

$$S^{(k)}(x) = \frac{(-1)^{k-1}k!c_i d_i^{k-1}}{(1 + d_i(x-\xi_i))^{k+1}},\ x \in [x_{i-1}, x_i].$$

We indicated in Section 2.3 that c_i and d_i could be expressed via the spline values at x_i and ξ_i. Namely,

$$c_i = \frac{4(S_i - \bar S_i)(\bar S_i - S_{i-1})}{h(S_i - S_{i-1})}, \quad d_i = \frac{2(2\bar S_i - S_{i-1} - S_i)}{h(S_i - S_{i-1})}.$$

Then, by (6.41), (6.42) and (6.38), we find that $c_i = 1 + O(h)$ and $d_i = O(1)$ which implies the assertion. □

Next, we show the superconvergence of order h^2 for S'' in certain points.

From the equalities

$$(S - S_*)(x_i) - (S - S_*)(\xi_i) = O(h^3)$$

and

$$(S - S_*)(x_{i-1}) - (S - S_*)(\xi_i) = O(h^3),$$

we obtain

$$(S' - S'_*)(\xi_i) = \frac{(S - S_*)(x_i) - (S - S_*)(x_{i-1})}{h} + O(h^2) = O(h^2).$$

Therefore, due to (6.39) we have

$$S'(\xi_i) - y'(\xi_i) = O(h^2). \tag{6.44}$$

By (6.1) and (6.6)

$$S''(\xi_i) - y''(\xi_i) + p(\xi_i)(S'(\xi_i) - y'(\xi_i)) + q(\xi_i)(S(\xi_i) - y(\xi_i)) = 0$$

which, with the help of (6.43) and (6.44) gives

$$S''(\xi_i) - y''(\xi_i) = O(h^2). \tag{6.45}$$

The Taylor expansion for $x \in [x_{i-1}, x_i]$

$$S'(x) - y'(x) = S'(\xi_i) - y'(\xi_i) + (S''(\xi_i) - y''(\xi_i))(x - \xi_i) + \frac{1}{2}(S''' - y''')(\eta_i)(x - \xi_i)^2$$

due to (6.44), (6.45) and Proposition 6.1 implies

$$\max_{a \le x \le b} |S'(x) - y'(x)| = O(h^2). \tag{6.46}$$

The expansion

$$S''(x) - y''(x) = S''(\xi_i) - y''(\xi_i) + (S''' - y''')(\bar{\eta}_i)(x - \xi_i), \; x \in [x_{i-1}, x_i],$$

(6.45) and Proposition 6.1 gives

$$\max_{a \le x \le b} |S''(x) - y''(x)| = O(h). \tag{6.47}$$

Finally, the expansion for $x \in [x_{i-1}, x_i]$

$$S(x) - y(x) = S(\xi_i) - y(\xi_i) + (S' - y')(\zeta_i)(x - \xi_i)$$

and (6.46) thanks to (6.43) yields

$$\max_{a \le x \le b} |S(x) - y(x)| \le c_0 h^2 + o(h^2). \tag{6.48}$$

We have proved as our main result in this chapter the following

Theorem 6.1. *Let the solution $y \in C^4[a, b]$ of boundary value problem* (6.1), (6.2) *be strictly monotone. Then, for sufficiently small h, the collocation problem* (6.6), (6.7) *has a linear/linear rational spline S as solution with convergence estimates* (6.48) (*c_0 is determined in* (6.40)), (6.46), (6.47) *and* (6.45). *The rest term $o(h^2)$ in* (6.48) *is actually $O(h^{2+\alpha})$ in the case $y^{IV} \in \mathrm{Lip}\ \alpha$, $0 < \alpha \le 1$.*

In collocation with quadratic splines, see, e.g., [39], it is known the estimate

$$\|S - y\|_\infty \leq \frac{h^2}{24} \max_{1 \leq i \leq n} \left| \frac{y^{IV} - py'''}{q}(\xi_i) \right| + o(h^2).$$

Let us point out, that in comparison to (6.48), the additional members in c_0 could make the main part of the estimate for rational splines smaller or greater than in case of quadratic splines. In Chapter 8 there is a numerical example in which the rational splines have really considerably smaller error.

Chapter 7

Quadratic/linear rational spline collocation

Chapter 7 is dedicated to quadratic/linear rational spline collocation method. At first we introduce the method.

As in previous chapter we consider the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in (a, b), \tag{7.1}$$

with boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta. \tag{7.2}$$

Suppose that the problem has a solution $y \in C^4[a, b]$. Let p, q, f be continuous and $q(x) \leq q < 0$, $x \in (a, b)$. Then the solution of problem (7.1), (7.2) is unique as we showed in Chapter 6.

7.1 Description of the spline collocation method

Let $a = x_0 < x_1 < \ldots < x_n = b$ be a uniform partition of the interval $[a, b]$ with knots $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b - a)/n$, $n \in \mathbb{N}$.

We use the quadratic/linear rational spline representation by spline values and first moments (Section 3.1). In this representation we have the notations $S(x_i) = S_i$ and $S'(x_i) = m_i$, $i = 0, \ldots, n$.

In collocation we require the spline S to satisfy the differential equation at points x_i and the boundary conditions:

$$S''(x_i) + p(x_i)S'(x_i) + q(x_i)S(x_i) = f(x_i), \quad i = 0, \ldots, n, \tag{7.3}$$

$$S(a) = \alpha, \quad S(b) = \beta. \tag{7.4}$$

Internal equations (3.6) together with equations (7.3) and (7.4) form a nonlinear system with respect to the unknowns $S_0, \ldots, S_n$, $m_0, \ldots, m_n$.

Let us mention, that this collocation problem is similar to cubic spline case. The convergence rate $O(h^2)$ with adequate error estimate for cubic spline collocation method is known [39]. We will study such a problem in the case of quadratic/linear rational splines.

7.2 Transformation of the system

We start by transforming the system (3.6), (7.3), (7.4) to a more appropriate form.

Calculating $S''(x_i)$ with the help of (3.5) at point x_i on subinterval $[x_{i-1}, x_i]$ we have

$$\frac{2(hm_i - (S_i - S_{i-1}))^2}{h^2(S_i - S_{i-1} - hm_{i-1})} + p(x_i)m_i + q(x_i)S_i = f(x_i), \; i = 1, \ldots, n. \tag{7.5}$$

Similarly, $S''(x_i)$ calculated on subinterval $[x_i, x_{i+1}]$ gives

$$\frac{2(S_{i+1} - S_i - hm_i)^2}{h^2(hm_{i+1} - (S_{i+1} - S_i))} + p(x_i)m_i + q(x_i)S_i = f(x_i), \; i = 0, \ldots, n-1. \tag{7.6}$$

Note that if y is the solution of (7.1) we may write for $i = 0, \ldots, n$

$$y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = f(x_i). \tag{7.7}$$

Let us now subtract (7.7) from (7.5) and from (7.6), then multiply the results by $h^2(S_i - S_{i-1} - hm_{i-1})$ and by $h^2(hm_{i+1} - (S_{i+1} - S_i))$, respectively. This gives

$$\begin{aligned} &2(hm_i - (S_i - S_{i-1}))^2 - h^2(S_i - S_{i-1} - hm_{i-1})y''(x_i) \\ &\qquad + h^2 p(x_i)(m_i - y'(x_i))(S_i - S_{i-1} - hm_{i-1}) \\ &\qquad + h^2 q(x_i)(S_i - y(x_i))(S_i - S_{i-1} - hm_{i-1}) = 0, \quad i = 1, \ldots, n, \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} &2(S_{i+1} - S_i - hm_i)^2 - h^2(hm_{i+1} - (S_{i+1} - S_i))y''(x_i) \\ &\qquad + h^2 p(x_i)(m_i - y'(x_i))(hm_{i+1} - (S_{i+1} - S_i)) \\ &\qquad + h^2 q(x_i)(S_i - y(x_i))(hm_{i+1} - (S_{i+1} - S_i)) = 0, \quad i = 0, \ldots, n-1. \end{aligned} \tag{7.9}$$

Adding the last two equations for $i = 1, \ldots, n-1$, we have

$$\begin{aligned}
\psi_i(S_{i-1},&m_{i-1}, S_i, m_i, S_{i+1}, m_{i+1}) \\
&= 2((hm_i - (S_i - S_{i-1}))^2 + (S_{i+1} - S_i + hm_i)^2) \\
&\quad - h^2(h(m_{i+1} - m_{i-1}) - (S_{i+1} - 2S_i + S_{i-1}))y''(x_i) \\
&\quad + h^2 p(x_i)(m_i - y'(x_i))(h(m_{i+1} - m_{i-1}) - (S_{i+1} - 2S_i + S_{i-1})) \\
&\quad + h^2 q(x_i)(S_i - y(x_i))(h(m_{i+1} - m_{i-1}) - (S_{i+1} - 2S_i + S_{i-1})) = 0,
\end{aligned} \tag{7.10}$$

where we introduce the functions ψ_i, $i = 1, \ldots, n-1$.

We also introduce the functions φ_i, $i = 1, \ldots, n-1$, and write the equations (3.6) in the form

$$\begin{aligned}
\varphi_i(S_{i-1}, m_{i-1}, S_i, m_i, S_{i+1}, m_{i+1}) = &\ (hm_i - (S_i - S_{i-1}))^2(hm_{i+1} - (S_{i+1} - S_i)) \\
&- (S_{i+1} - S_i - hm_i)^2(S_i - S_{i-1} - hm_{i-1}) = 0.
\end{aligned} \tag{7.11}$$

We write the boundary conditions (7.4) (with the help of notations ψ_0 and ψ_n for the left side of equations) as

$$\psi_0(S_0) = h^2(S_0 - \alpha) = 0, \quad \psi_n(S_n) = h^2(S_n - \beta) = 0. \tag{7.12}$$

Let us define the function $\tilde{\varphi}_0$ by (7.9) in the case of $i = 0$ and $\tilde{\varphi}_n$ by (7.8) in the case of $i = n$. More precisely,

$$\begin{aligned}
\tilde{\varphi}_0(S_0, m_0, S_1, m_1) = &\ 2(S_1 - S_0 - hm_0)^2 - h^2(hm_1 - (S_1 - S_0))y''(x_0) \\
&+ h^2 p(x_0)(m_0 - y'(x_0))(hm_1 - (S_1 - S_0)) \\
&+ h^2 q(x_0)(S_0 - y(x_0))(hm_1 - (S_1 - S_0)) = 0
\end{aligned} \tag{7.13}$$

and

$$\begin{aligned}
\tilde{\varphi}_n(S_{n-1}, m_{n-1}, S_n, m_n) = &\ 2(hm_n - (S_n - S_{n-1}))^2 - h^2(S_n - S_{n-1} - hm_{n-1})y''(x_n) \\
&+ h^2 p(x_n)(m_n - y'(x_n))(S_n - S_{n-1} - hm_{n-1}) \\
&+ h^2 q(x_n)(S_n - y(x_n))(S_n - S_{n-1} - hm_{n-1}) = 0.
\end{aligned} \tag{7.14}$$

Define also $\tilde{\varphi}_1$ by (7.8) in the case $i = 1$ and similarly $\tilde{\varphi}_{n-1}$ by (7.9) in the case of $i = n-1$, i.e.,

$$\begin{aligned}
\tilde{\varphi}_1(S_0, m_0, S_1, m_1) = &\ 2(hm_1 - (S_1 - S_0))^2 - h^2(S_1 - S_0 - hm_0)y''(x_1) \\
&+ h^2 p(x_1)(m_1 - y'(x_1))(S_1 - S_0 - hm_0) \\
&+ h^2 q(x_1)(S_1 - y_1)(S_1 - S_0 - hm_0) = 0
\end{aligned} \tag{7.15}$$

and

$$\tilde{\varphi}_{n-1}(S_{n-1}, m_{n-1}, S_n, m_n) =$$

$$\begin{aligned} & 2(S_n - S_{n-1} - hm_{n-1})^2 - h^2(hm_n - (S_n - S_{n-1}))y''(x_{n-1}) \\ & + h^2 p(x_{n-1})(m_{n-1} - y'(x_{n-1}))(hm_n - (S_n - S_{n-1})) \\ & + h^2 q(x_{n-1})(S_{n-1} - y(x_{n-1}))(hm_n - (S_n - S_{n-1})) = 0. \end{aligned} \tag{7.16}$$

Then define $\varphi_0 = \tilde{\varphi}_0 - \tilde{\varphi}_1$ and $\varphi_n = \tilde{\varphi}_n - \tilde{\varphi}_{n-1}$.

Denote $S = (S_0, \ldots, S_n)$ and $m = (m_0, \ldots, m_n)$. Let us point out, that although we denote by S a spline and the vector of its knot values, from the context it is clear which object we deal with. Now the collocation equations (7.10), internal equations (7.11), boundary equations (7.12), equations $\varphi_0(S, m) = 0$ and $\varphi_n(S, m) = 0$ together form a system $\Phi(S, m) = 0$ and $\Psi(S, m) = 0$ with functions $\Phi : \mathbb{R}^{2n+2} \to \mathbb{R}^{n+1}$ and $\Psi : \mathbb{R}^{2n+2} \to \mathbb{R}^{n+1}$, where $\Phi(S, m) = (\varphi_0(S, m), \ldots, \varphi_n(S, m))$ and $\Psi(S, m) = (\psi_0(S, m), \ldots, \psi_n(S, m))$ (here and in the sequel we allow the whole vector of spline knot values S and derivative values m in $\varphi_i(S, m)$ and $\psi_i(S, m)$ instead of some particular components as in (7.10)−(7.16)). The system $\Phi(S, m) = 0$, $\Psi(S, m) = 0$ will be considered as basic system and it contains all the requirements on spline values and its derivatives. Our aim is to show that this system has an appropriate solution and obtain the convergence results.

In order to establish our results we have to use the quadratic/linear rational spline interpolant S satisfying (5.1) and (5.25) for the solution y of problem (7.1), (7.2). Denote this special interpolant by S_*. Define its vector of values by $S^* = (S_*(x_0), \ldots, S_*(x_n)) \in \mathbb{R}^{n+1}$ and let $m^* = (S'_*(x_0), \ldots, S'_*(x_n)) \in \mathbb{R}^{n+1}$. It occurs that around the vector (S^*, m^*) we succeed in finding a set with sufficiently small size containing the vector of values and derivatives of approximate solution. We know that $S^*_i = y_i$ and $m^*_i - y'_i = o(h^3)$ ([26] or see Chapter 5). If $y \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$, then

$$m^*_i - y'_i = O(h^{3+\alpha}). \tag{7.17}$$

We transform the basic system to an equivalent form and then carry out an analysis.

7.3 Estimate of the error of approximate solution

In this section we estimate the error in the case of the solution of basic system.

The Taylor expansion of $\Psi(S, m)$ at point (S^*, m) gives

$$\Psi(S, m) = \Psi(S^*, m) + \Psi_S(S^*, m)(S - S^*) + \frac{1}{2}\Psi_{SS}(S - S^*)^2 \tag{7.18}$$

where we use the notation $\Psi_S(S,m) = (\psi_{0S}(S,m),\ldots,\psi_{nS}(S,m))$, and, in turn, with the components $\psi_{iS}(S,m) = (\dfrac{\partial\psi_i}{\partial S_0}(S,m),\ldots,\dfrac{\partial\psi_i}{\partial S_n}(S,m))$, but any component ψ_{iSS} of Ψ_{SS} is a matrix consisting of second order partial derivatives by variables S_i, $i=0,\ldots,n$. If the matrix $\Psi_S(S^*,m)$ is invertible then due to (7.18) the equation $\Psi(S,m)=0$ is equivalent to

$$S - S^* = -\Psi_S^{-1}(S^*,m)(\Psi(S^*,m) + \frac{1}{2}\Psi_{SS}(S-S^*)^2). \tag{7.19}$$

Suppose that

$$S_i - S_i^* = O(h^2),\ i=1,\ldots,n-1, \tag{7.20}$$

$$(S_{i-1}-S_{i-1}^*) - 2(S_i - S_i^*) + (S_{i+1}-S_{i+1}^*) = O(h^4),\ i=1,\ldots,n-1, \tag{7.21}$$

$$2h(m_i - m_i^*) - ((S_{i+1}-S_{i+1}^*) - (S_{i-1}-S_{i-1}^*)) = O(h^5),\ i=1,\ldots,n-1, \tag{7.22}$$

$$h(m_0 - m_0^*) - (S_1 - S_1^*) = O(h^4), \tag{7.23}$$

$$h(m_n - m_n^*) - (S_{n-1} - S_{n-1}^*) = O(h^4). \tag{7.24}$$

In those assumptions, in the case of sufficiently small h, the basic system has a solution which we will show later. Thus, in estimation of the error we may use (7.20)–(7.24).

Let us add that in those assumptions also, for $i=1,\ldots,n-1$,

$$h(m_i - m_i^*) - ((S_{i+1}-S_{i+1}^*) - (S_i - S_i^*)) = O(h^4), \tag{7.25}$$

$$h(m_i - m_i^*) - ((S_i - S_i^*) - (S_{i-1}-S_{i-1}^*)) = O(h^4) \tag{7.26}$$

where (7.25) is obtained by subtracting (7.21) from (7.22) and (7.26) comes by adding (7.21) and (7.22).

Let us assume without loss of generality that $y''(x) > 0$, $x \in [a,b]$ (see Introduction) (in the case $y''(x) < 0$, $x\in[a,b]$, the changes in the following text are obvious).

The main problem in (7.19) is the study of interior equations with indices $i=1,\ldots,n-1$. Let us start with the study of $\Psi_S(S^*,m)$. As before, let $y_i = y(x_i)$, $i=0,\ldots,n$, similar notation will be used in the case of derivatives and other functions.

From (7.10) we get

$$\frac{\partial \psi_i}{\partial S_{i-1}} = 4(hm_i - (S_i - S_{i-1})) + h^2 y_i'' - h^2 p_i(m_i - y_i') - h^2 q_i(S_i - y_i),$$

$$\begin{aligned} \frac{\partial \psi_i}{\partial S_i} = & -4(S_{i+1} - 2S_i + S_{i-1}) - 2h^2 y_i'' + 2h^2 p_i(m_i - y_i') + 2h^2 q_i(S_i - y_i) \\ & + h^2 q_i(h(m_{i+1} - m_{i-1}) - (S_{i+1} - 2S_i + S_{i-1})), \end{aligned} \tag{7.27}$$

$$\frac{\partial \psi_i}{\partial S_{i+1}} = 4(S_{i+1} - S_i - hm_i) + h^2 y_i'' - h^2 p_i(m_i - y_i') - h^2 q_i(S_i - y_i).$$

With the help of the following Taylor expansions

$$\begin{aligned} S_{i-1}^* &= y_i - hy_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{IV} + o(h^4), \\ S_{i+1}^* &= y_i + hy_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{IV} + o(h^4), \\ m_i^* &= y_i' - hy_i'' + \frac{h}{2} y_i''' - \frac{h^2}{6} y_i^{IV} + o(h^3), \\ m_{i+1}^* &= y_i' + hy_i'' + \frac{h}{2} y_i''' + \frac{h^2}{6} y_i^{IV} + o(h^3) \end{aligned} \tag{7.28}$$

we can calculate

$$\begin{aligned} hm_i^* - (S_i^* - S_{i-1}^*) &= \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{IV} + o(h^4), \\ hm_{i+1}^* - (S_{i+1}^* - S_i^*) &= \frac{h^2}{2} y_i'' + \frac{h^3}{3} y_i''' + \frac{h^4}{8} y_i^{IV} + o(h^4), \\ S_i^* - S_{i-1}^* - hm_{i-1}^* &= \frac{h^2}{2} y_i'' - \frac{h^3}{3} y_i''' + \frac{h^4}{8} y_i^{IV} + o(h^4), \\ S_{i+1}^* - S_i^* - hm_i^* &= \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{IV} + o(h^4) \end{aligned} \tag{7.29}$$

and then

$$\begin{aligned} & h(m_{i+1} - m_{i-1}) - (S_{i+1}^* - 2S_i^* + S_{i-1}^*) \\ &= h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*) + h(m_{i+1}^* - m_{i-1}^*) - (S_{i+1}^* - 2S_i^* + S_{i-1}^*) \\ &= h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*) + hm_{i+1}^* - (S_{i+1}^* - S_i^*) + (S_i^* - S_{i-1}^* - hm_{i-1}^*) \\ &= h^2 y_i'' + \frac{1}{4} h^4 y_i^{IV} + h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*) + o(h^4). \end{aligned} \tag{7.30}$$

We get from (7.27) due to (7.29) and (7.30)

$$\begin{aligned}
\frac{\partial \psi_i}{\partial S_{i-1}}(S^*,m) =&\ 4(hm_i^* - (S_i^* - S_{i-1}^*)) + 4h(m_i - m_i^*) + h^2 y_i''\\
&- h^2 p_i(m_i - y_i') - h^2 q_i(S_i^* - y_i)\\
=&\ 3h^2 y_i'' - \frac{2}{3}h^3 y_i''' + \frac{h^4}{6}y_i^{IV} + 4h(m_i - m_i^*) - h^2 p_i(m_i - y_i') + o(h^4),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi_i}{\partial S_i}(S^*,m) =&- 4(S_{i+1}^* - 2S_i^* + S_{i-1}^*) - 2h^2 y_i'' + 2h^2 p_i(m_i - y_i')\\
&+ 2h^2 q_i(S_i^* - y_i) + h^2 q_i(h(m_{i+1}^* - m_{i-1}^*) - (S_{i+1}^* - 2S_i^* + S_{i-1}^*))\\
&+ h^2 q_i(h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*))\\
=&\ - 6h^2 y_i'' - \frac{h^4}{3}y_i^{IV} + h^4 q_i y_i'' + 2h^2 p_i(m_i - y_i')\\
&+ h^2 q_i(h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*)) + o(h^4),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi_i}{\partial S_{i+1}}(S^*,m) =&\ 4(S_{i+1}^* - S_i^* - hm_i^*) - 4h(m_i - m_i^*) + h^2 y_i''\\
&- h^2 p_i(m_i - y_i') - h^2 q_i(S_i^* - y_i)\\
=&\ 3h^2 y_i'' + \frac{2}{3}h^3 y_i''' + \frac{h^4}{6}y_i^{IV} - 4h(m_i - m_i^*) - h^2 p_i(m_i - y_i') + o(h^4).
\end{aligned}$$

Note that in the case of $m_i - m_i^* = O(h^2)$ and due to (7.17) we have for small h the diagonal dominance in rows, namely

$$\Big|\frac{\partial \psi_i}{\partial S_i}(S^*,m)\Big| - \Big|\frac{\partial \psi_i}{\partial S_{i-1}}(S^*,m)\Big| - \Big|\frac{\partial \psi_i}{\partial S_{i+1}}(S^*,m)\Big| = -h^4 q_i y_i'' + o(h^4).$$

In addition, for ψ_0 and ψ_n we have the diagonal dominance in rows as well. This implies the invertibility of $\Psi_S(S^*,m)$.

Now, let us proceed with the study of $\Psi(S^*,m)$. With the help of the first equality of (7.29) we get

$$hm_i - (S_i^* - S_{i-1}^*) = h(m_i - m_i^*) + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4) \tag{7.31}$$

and the last one from (7.29) gives

$$S_{i+1}^* - S_i^* - hm_i = -h(m_i - m_i^*) + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4). \tag{7.32}$$

Then (7.28) implies

$$S_{i+1}^* - 2S_i^* + S_{i-1}^* = h^2 y_i'' + \frac{h^4}{12} y_i^{IV} + o(h^4). \tag{7.33}$$

Due to (7.30), (7.31) and (7.32) by direct calculations we have (7.10) as

$$\begin{aligned}\psi_i(S^*, m) = \frac{1}{9} h^6 (y_i''')^2 - \frac{h^6}{12} y_i'' y_i^{IV} + 4h^2 (m_i - m_i^*)^2 \\ - \frac{4}{3} h^4 y_i''' (m_i - m_i^*) - h^2 y_i'' (h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*)) \\ + h^4 p_i y_i'' (m_i - m_i^*) + o(h^6).\end{aligned} \tag{7.34}$$

Next, we are interested in the components of Ψ_{SS}. From (7.27) we get the second derivatives

$$\frac{\partial^2 \psi_i}{\partial S_{i-1}^2} = 4, \qquad \frac{\partial^2 \psi_i}{\partial S_{i-1} \partial S_i} = -4 - h^2 q_i, \qquad \frac{\partial^2 \psi_i}{\partial S_{i-1} \partial S_{i+1}} = 0,$$

$$\frac{\partial^2 \psi_i}{\partial S_i \partial S_{i-1}} = -4 - h^2 q_i, \qquad \frac{\partial^2 \psi_i}{\partial S_i^2} = 8 + 4h^2 q_i, \qquad \frac{\partial^2 \psi_i}{\partial S_i \partial S_{i+1}} = -4 - h^2 q_i,$$

$$\frac{\partial^2 \psi_i}{\partial S_{i+1} \partial S_{i-1}} = 0, \qquad \frac{\partial^2 \psi_i}{\partial S_{i+1} \partial S_i} = -4 - h^2 q_i, \qquad \frac{\partial^2 \psi_i}{\partial S_{i+1}^2} = 4.$$

Therefore

$$\begin{aligned}\frac{1}{2} \psi_{iSS} (S - S^*)^2 = 2((S_{i-1} - S_{i-1}^*) - (S_i - S_i^*))^2 + 2((S_i - S_i^*) - (S_{i+1} - S_{i+1}^*))^2 \\ - h^2 q_i (S_i - S_i^*)((S_{i-1} - S_{i-1}^*) - 2(S_i - S_i^*) + (S_{i+1} - S_{i+1}^*))\end{aligned}$$

and taking into account (7.20) and (7.21) then

$$\begin{aligned}\frac{1}{2} \psi_{iSS} (S - S^*)^2 = 2((S_{i-1} - S_{i-1}^*) - (S_i - S_i^*))^2 \\ + 2((S_i - S_i^*) - (S_{i+1} - S_{i+1}^*))^2 + O(h^8).\end{aligned} \tag{7.35}$$

Let us now study $\Psi(S^*, m) + \frac{1}{2} \Psi_{SS} (S - S^*)^2$. Observe that basing on (7.25) and (7.26) we can write

$$4h^2(m_i - m_i^*)^2 + 2((S_i - S_i^*) - (S_{i-1} - S_{i-1}^*))^2 + 2((S_{i+1} - S_{i+1}^*) - (S_i - S_i^*))^2$$

$$- 4h(m_i - m_i^*)((S_{i+1} - S_{i+1}^*) - (S_i - S_i^*) + (S_i - S_i^*) - (S_{i-1} - S_{i-1}^*))$$

$$+ 4h(m_i - m_i^*)(S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*))$$

$$= 2(h(m_i - m_i^*) - ((S_{i+1} - S_{i+1}^*) - (S_i - S_i^*)))^2$$

$$+ 2(h(m_i - m_i^*) - ((S_i - S_i^*) - (S_{i-1} - S_{i-1}^*)))^2$$

$$4h(m_i - m_i^*)(S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*))$$

$$= 4h(m_i - m_i^*)(S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*)) + O(h^8).$$

This with (7.34) and (7.35) leads to

$$\psi_i(S^*, m) + \frac{1}{2}\psi_{iSS}(S - S^*)^2 = \frac{h^6}{12}\Big(\frac{4}{3}(y_i''')^2 - y_i'' y_i^{IV}\Big) \tag{7.36}$$

$$- h^2 y_i''(h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*)) - \frac{4}{3}h^4 y_i'''(m_i - m_i^*)$$

$$+ h^4 p_i y_i''(m_i - m_i^*) + 4h(m_i - m_i^*)(S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*)) + o(h^6).$$

Replacing $2h(m_i - m_i^*)$ in (7.36) by $S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*) + O(h^5)$ and due to (7.22) we see that (7.36) reduces to

$$\psi_i(S^*, m) + \frac{1}{2}\psi_{iSS}(S - S^*)^2$$

$$= \frac{h^6}{12}\Big(\frac{4}{3}(y_i''')^2 - y_i'' y_i^{IV}\Big) - h^2 y_i''(h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*))$$

$$+ \Big(-\frac{2}{3}h^3 y_i''' + \frac{h^3}{2}p_i y_i'' + 2(S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*))\Big)(S_{i+1} - S_{i+1}^* - (S_{i-1} - S_{i-1}^*))$$

$$+ o(h^6). \tag{7.37}$$

Consider now (7.19) in the form

$$\Psi_S(S^*, m)(S - S^*) = -(\Psi(S^*, m) + \frac{1}{2}\Psi_{SS}(S - S^*)^2). \tag{7.38}$$

For the definiteness, let us assume that y^{IV}, p, $q \in \mathrm{Lip}\ 1$ (if only y^{IV}, p, $q \in C[a, b]$, the obvious appropriate changes should be made in the text). We have already studied $\Psi_S(S^*,m)$ and $\Psi(S^*,m)+\dfrac{1}{2}\Psi_{SS}(S-S^*)^2$. Now (7.38) has the matrix form $A(S-S^*)=b$, where we take the components

$$\Big(-\frac{2}{3}h^3y_i'''+\frac{h^3}{2}p_iy_i''+2(S_{i+1}-S^*_{i+1}-(S_{i-1}-S^*_{i-1}))\Big)(S_{i+1}-S^*_{i+1}-(S_{i-1}-S^*_{i-1}))$$

(coming from (7.37)) to the left hand side. Then

$$\begin{aligned} a_{i,i-1}&=3h^2y_i''+\frac{1}{6}h^4y_i^{IV}-h^2p_i(m_i-y_i')+O(h^5),\\ a_{i,i}&=-6h^2y_i''-\frac{1}{3}h^4y_i^{IV}+2h^2p_i(m_i-y_i')+h^4q_iy_i''+O(h^5),\\ a_{i,i+1}&=3h^2y_i''+\frac{1}{6}h^4y_i^{IV}-h^2p_i(m_i-y_i')+O(h^5)\end{aligned}\tag{7.39}$$

and

$$b_i=\frac{h^6}{12}\Big(y_i''y_i^{IV}-\frac{4}{3}(y_i''')^2\Big)+h^2y_i''(h(m_{i+1}-m^*_{i+1})-h(m_{i-1}-m^*_{i-1}))+O(h^7).\tag{7.40}$$

In the matrix A, in interior rows for $1,\dots,n-1$, we observe the diagonal dominance in rows with the difference $-h^4q_iy_i''+O(h^5)$.

Denote for brevity $u_i=S_i-S_i^*$, $v_i=2h(m_i-m_i^*)$, $i=0,\dots,n$, and $w_i=2h(m_i-m_i^*)-((S_{i+1}-S^*_{i+1})-(S_{i-1}-S^*_{i-1}))$, $i=1,\dots,n-1$. Then $w_i=v_i-(u_{i+1}-u_{i-1})$ and $v_i=w_i+(u_{i+1}-u_{i-1})$, $i=1,\dots,n-1$. Later we need also $w_0=v_0-2u_1$ and $w_n=v_n-2u_{n-1}$. Now we can write the interior equations of $A(S-S^*)=b$ as

$$\begin{aligned}&(3h^2y_i''+\alpha_i)u_{i-1}-(6h^2y_i''+2\alpha_i-h^4q_iy_i'')u_i+(3h^2y_i''+\alpha_i)u_{i+1}\\ &=\frac{1}{2}h^2y_i''(u_{i+2}-2u_{i+1}+u_i)+h^2y_i''(u_{i+1}-2u_i+u_{i-1})\\ &\quad+\frac{1}{2}h^2y_i''(u_i-2u_{i-1}+u_{i-2})+c_ih^6+\frac{1}{2}h^2y_i''(w_{i+1}-w_{i-1})+O(h^7)\end{aligned}\tag{7.41}$$

where $\alpha_i=\dfrac{1}{6}h^4y_i^{IV}-h^2p_i(m_i-y_i')$ and $c_i=\dfrac{1}{12}\Big(y_i''y_i^{IV}-\dfrac{4}{3}(y_i''')^2\Big)$.

Taking $h^2y_i''(u_{i+1}-2u_i+u_{i-1})$ to the left hand side of the equation, we get therein

$$(2h^2y_i''+\alpha_i)u_{i-1}-(4h^2y_i''+2\alpha_i-h^4q_iy_i'')u_i+(2h^2y_i''+\alpha_i)u_{i+1}$$

or

$$(2h^2y_i''+\alpha_i)(u_{i-1}-(2-h^2q_i)u_i+u_{i+1}).$$

Now we can write (7.41) for $i = 2, \ldots, n-2$ as

$$(2h^2 y_i'' + \alpha_i)(u_{i-1} - (2 - h^2 q_i)u_i + u_{i+1})$$

$$= \frac{1}{2} h^2 y_i''((u_i - (2 - h^2 q_{i+1})u_{i+1} + u_{i+2}) + (u_{i-2} - (2 - h^2 q_{i-1})u_{i-1} + u_i))$$

$$- \frac{1}{2} h^2 y_i''(h^2 q_{i-1} u_{i-1} - 2h^2 q_i u_i + h^2 q_{i+1} u_{i+1}) + c_i h^6 + \frac{1}{2} h^2 y_i''(w_{i+1} - w_{i-1}) + O(h^7).$$

Dividing the last equation by $h^2 q_i y_i''$ and denoting $z_i = u_{i-1} - (2 - h^2 q_i)u_i + u_{i+1}$, $i = 1, \ldots, n-1$, we have

$$-\frac{1}{2} \frac{z_{i-1}}{q_i} + (2 + \tilde{\alpha}_i)\frac{z_i}{q_i} - \frac{1}{2} \frac{z_{i+1}}{q_i} = \tilde{c}_i h^4 + O(h^5)$$

with $\tilde{\alpha}_i = O(h)$ and

$$\tilde{c}_i = \frac{1}{12} \frac{y_i^{IV} - \frac{4}{3} \frac{(y_i''')^2}{y_i''}}{q_i}. \tag{7.42}$$

Note that, e.g., $\frac{z_{i-1}}{q_i} = \frac{q_{i-1}}{q_i} \frac{z_{i-1}}{q_{i-1}} = \left(1 + \frac{q_{i-1} - q_i}{q_i}\right) \frac{z_{i-1}}{q_{i-1}} = (1 + O(h)) \frac{z_{i-1}}{q_{i-1}}$ which gives for $i = 2, \ldots, n-2$

$$-\frac{1}{2}(1 + O(h))\frac{z_{i-1}}{q_{i-1}} + (2 + \tilde{\alpha}_i)\frac{z_i}{q_i} - \frac{1}{2}(1 + O(h))\frac{z_{i+1}}{q_{i+1}} = \tilde{c}_i h^4 + O(h^5). \tag{7.43}$$

Remark. Dividing the interior equations for $i = 1, \ldots, n-1$ of the system $A(S - S^*) = b$ by $-3h^2 y_i''$, first and last equations by h^2, we obtain a system $\tilde{A}(S - S^*) = \tilde{b}$ with the matrix $\tilde{A}$ where its entries are as in (6.24) for $i = 1, \ldots, n-1$. Basing on the proof of Lemma 6.2, we have for $\tilde{A}^{-1} = (a_{ij})$ that $\mid a_{ij} - a_{i+1,j} \mid = O(1)$. This implies for the solution (S, m) of the basic system

$$(S_i - S_i^*) - (S_{i-1} - S_{i-1}^*) = O(h^3), \quad i = 1, \ldots, n. \tag{7.44}$$

Next, we will focus our attention to the study of $\tilde{\varphi}_0(S, m)$. Using the Taylor expansion at point (S, m^*) we have

$$\tilde{\varphi}_0(S, m) = \tilde{\varphi}_0(S, m^*) + \tilde{\varphi}_{0m}(S, m^*)(m - m^*) + \frac{1}{2}\tilde{\varphi}_{0mm}(m - m^*)^2$$

and therefore, by the following Taylor expansions at (S^*, m^*)

$$\tilde{\varphi}_0(S, m^*) = \tilde{\varphi}_0(S^*, m^*) + \tilde{\varphi}_{0S}(S^*, m^*)(S - S^*) + \frac{1}{2}\tilde{\varphi}_{0SS}(S - S^*)^2,$$

$$\tilde{\varphi}_{0m}(S, m^*)(m - m^*) = \tilde{\varphi}_{0m}(S^*, m^*)(m - m^*) + \tilde{\varphi}_{0mS}(m - m^*)(S - S^*),$$

the equation $\tilde{\varphi}_0(S, m) = 0$ is equivalent to

$$
\begin{aligned}
& -\tilde{\varphi}_{0m}(S^*, m^*)(m - m^*) - \tilde{\varphi}_{0S}(S^*, m^*)(S - S^*) \qquad (7.45) \\
& = \tilde{\varphi}_0(S^*, m^*) + \frac{1}{2}\tilde{\varphi}_{0SS}(S - S^*)^2 + \tilde{\varphi}_{0mS}(m - m^*)(S - S^*) + \frac{1}{2}\tilde{\varphi}_{0mm}(m - m^*)^2.
\end{aligned}
$$

From (7.13) we calculate

$$\frac{\partial \tilde{\varphi}_0}{\partial S_0} = -4(S_1 - S_0 - hm_0) - h^2 y_0'' + h^2 p_0(m_0 - y_0'),$$

$$\frac{\partial \tilde{\varphi}_0}{\partial S_1} = 4(S_1 - S_0 - hm_0) + h^2 y_0'' - h^2 p_0(m_0 - y_0'),$$

$$\frac{\partial \tilde{\varphi}_0}{\partial m_0} = -4h(S_1 - S_0 - hm_0) + h^2 p_0(hm_1 - (S_1 - S_0)),$$

$$\frac{\partial \tilde{\varphi}_0}{\partial m_1} = -h^3 y_0'' + h^3 p_0(m_0 - y_0')$$

and then

$$\frac{\partial^2 \tilde{\varphi}_0}{\partial S_0^2} = 4, \qquad \frac{\partial^2 \tilde{\varphi}_0}{\partial S_0 \partial S_1} = -4,$$

$$\frac{\partial^2 \tilde{\varphi}_0}{\partial S_1 \partial S_0} = -4, \qquad \frac{\partial^2 \tilde{\varphi}_0}{\partial S_1^2} = 4,$$

$$\frac{\partial^2 \tilde{\varphi}_0}{\partial m_0 \partial S_0} = 4h + h^2 p_0, \qquad \frac{\partial^2 \tilde{\varphi}_0}{\partial m_0 \partial S_1} = -4h - h^2 p_0,$$

$$\frac{\partial^2 \tilde{\varphi}_0}{\partial m_1 \partial S_0} = 0, \qquad \frac{\partial^2 \tilde{\varphi}_0}{\partial m_1 \partial S_1} = 0,$$

$$\frac{\partial^2 \tilde{\varphi}_0}{\partial m_0^2} = 4h^2, \qquad \frac{\partial^2 \tilde{\varphi}_0}{\partial m_0 \partial m_1} = h^3 p_0,$$

$$\frac{\partial^2 \tilde{\varphi}_0}{\partial m_1 \partial m_0} = h^3 p_0, \qquad \frac{\partial^2 \tilde{\varphi}_0}{\partial m_1^2} = 0.$$

Thus, using obtained partial derivatives, (7.13) and (7.28) by direct calculations we have (7.45) as

$$
\begin{aligned}
&(2h^3y_0'' + \frac{2}{3}h^4y_0''' - \frac{h^4}{2}p_0y_0'' + O(h^5))(m_0 - m_0^*) + (h^3y_0'' + o(h^5))(m_1 - m_1^*) \\
&\qquad - (3h^2y_0'' + \frac{2}{3}h^3y_0''' + \frac{h^4}{6}y_0^{IV} + O(h^5))(S_1 - S_1^*) \qquad (7.46) \\
&\quad = -\frac{h^6}{24}\Big(y_0''y_0^{IV} - \frac{4}{3}(y_0''')^2\Big) + 2(S_1 - S_1^*)^2 + 2h^2(m_0 - m_0^*)^2 + O(h^7).
\end{aligned}
$$

Recall that $u_1 = S_1 - S_1^*$, $v_0 = 2h(m_0 - m_0^*)$, $w_0 = v_0 - 2u_1$, $v_1 = 2h(m_1 - m_1^*)$ and $w_1 = v_1 - 2u_2$ with the property $u_2 - 2u_1 = O(h^4)$. Then we have $\frac{h^3}{3}y_0'''(v_0 - 2u_1) = \frac{h^3}{3}y_0'''w_0 = O(h^7)$ and we can write (7.46) as

$$
\begin{aligned}
&(h^2y_0'' - \frac{h^3}{4}p_0y_0'' + O(h^5))v_0 + \frac{1}{2}(h^2y_0'' + o(h^5))v_1 \\
&\qquad - (3h^2y_0'' + O(h^5))u_1 = -C_0 + 2u_1^2 + \frac{1}{2}v_0^2 + O(h^7), \qquad (7.47)
\end{aligned}
$$

where $C_0 = \frac{h^6}{24}\Big(y_0''y_0^{IV} - \frac{4}{3}(y_0''')^2\Big)$.

Note that it holds $\frac{1}{2}v_0^2 = 2u_1^2 + O(h^7)$. Thus, altogether

$$
\begin{aligned}
&(h^2y_0'' - \frac{h^3}{4}p_0y_0'' + O(h^3))v_0 + \frac{1}{2}\Big(h^2y_0'' + O(h^3)\Big)v_1 - (3h^2y_0'' + O(h^3))u_1 \\
&\qquad = -C_0 + O(h^7). \qquad (7.48)
\end{aligned}
$$

Next, we are interested in $\tilde{\varphi}_1(S, m)$. From (7.15) we get

$$
\begin{aligned}
\frac{\partial\tilde{\varphi}_1}{\partial S_0} &= 4(hm_1 - (S_1 - S_0)) + h^2y_1'' - h^2p_1(m_1 - y_1') - h^2q_1(S_1 - y_1), \\
\frac{\partial\tilde{\varphi}_1}{\partial S_1} &= -4(hm_1 - (S_1 - S_0)) - h^2y_1'' + h^2p_1(m_1 - y_1') + h^2q_1(S_1 - y_1) \\
&\quad + h^2q_1(S_1 - S_0 - hm_0).
\end{aligned}
$$

Then, by (7.17) and (7.29) we have

$$
\begin{aligned}
\frac{\partial\tilde{\varphi}_1}{\partial S_0}(S^*, m^*) &= 3h^2y_0'' + \frac{7}{3}h^3y_0''' + h^4y_0^{IV} + o(h^4), \\
\frac{\partial\tilde{\varphi}_1}{\partial S_1}(S^*, m^*) &= -3h^2y_0'' - \frac{7}{3}h^3y_0''' - h^4y_0^{IV} + \frac{h^4}{2}q_0y_0'' + o(h^4).
\end{aligned}
$$

Now we can calculate the second derivatives

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial S_0^2} = 4, \qquad \frac{\partial^2 \tilde{\varphi}_1}{\partial S_0 \partial S_1} = -4 - h^2 q_1,$$

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial S_1 \partial S_0} = -4 - h^2 q_1, \qquad \frac{\partial^2 \tilde{\varphi}_1}{\partial S_1^2} = 4 + 2h^2 q_1$$

and by direct calculations

$$\tilde{\varphi}_{1SS}(S - S^*)^2 = 4u_1^2 + O(h^8). \tag{7.49}$$

From (7.15) we find also

$$\begin{aligned} \frac{\partial \tilde{\varphi}_1}{\partial m_0} &= h^3 y_1'' - h^3 p_1(m_1 - y_1') - h^3 q_1(S_1 - y_1), \\ \frac{\partial \tilde{\varphi}_1}{\partial m_1} &= 4h(hm_1 - (S_1 - S_0)) + h^2 p_1(S_1 - S_0 - hm_0) \end{aligned} \tag{7.50}$$

and then by using (7.17) and (7.29)

$$\frac{\partial \tilde{\varphi}_1}{\partial m_0}(S^*, m^*) = h^3 y_0'' + h^4 y_0''' + O(h^5),$$

$$\frac{\partial \tilde{\varphi}_1}{\partial m_1}(S^*, m^*) = 2h^2 y_0'' + \frac{4}{3}h^4 y_0''' + \frac{h^4}{2} p_0 y_0'' + O(h^5).$$

Partial derivatives (7.50) give

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial m_0^2} = 0, \qquad \frac{\partial^2 \tilde{\varphi}_1}{\partial m_0 \partial m_1} = -h^3 p_1,$$

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial m_1 \partial m_0} = -h^3 p_1, \qquad \frac{\partial^2 \tilde{\varphi}_1}{\partial m_1^2} = 4h^2.$$

Now we can calculate

$$\tilde{\varphi}_{1mm}(m - m^*)^2 = 4h^2(m_1 - m_1^*)^2 + O(h^7). \tag{7.51}$$

Similarly we get from (7.50)

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial m_0 \partial S_0} = 0, \qquad \frac{\partial^2 \tilde{\varphi}_1}{\partial m_0 \partial S_1} = -h^3 q_1,$$

$$\frac{\partial^2 \tilde{\varphi}_1}{\partial m_1 \partial S_0} = 4h - h^2 p_1, \qquad \frac{\partial^2 \tilde{\varphi}_1}{\partial m_1 \partial S_1} = -4h + h^2 p_1$$

and, thus,

$$\tilde{\varphi}_{1mS}(m - m^*)(S - S^*) = O(h^7). \tag{7.52}$$

Also we calculate from (7.15) with the help of (7.29)

$$\tilde{\varphi}_1(S^*, m^*) = -\frac{h^6}{24}\Big(y_0'' y_0^{IV} - \frac{4}{3}(y_0''')^2\Big) + O(h^7). \tag{7.53}$$

Due to Taylor expansion the equation $\tilde{\varphi}_1(S, m) = 0$ is equivalent to

$$\tilde{\varphi}_1(S, m^*) + \tilde{\varphi}_{1m}(S, m^*)(m - m^*) + \frac{1}{2}\tilde{\varphi}_{1mm}(m - m^*)^2 = 0$$

which, with the help of expansions

$$\tilde{\varphi}_1(S, m^*) = \tilde{\varphi}_1(S^*, m^*) + \tilde{\varphi}_{1S}(S^*, m^*)(S - S^*) + \frac{1}{2}\tilde{\varphi}_{1SS}(S - S^*)^2,$$

$$\tilde{\varphi}_{1m}(S, m^*)(m - m^*) = \tilde{\varphi}_{1m}(S^*, m^*)(m - m^*) + \tilde{\varphi}_{1mS}(m - m^*)(S - S^*),$$

gives

$$\begin{aligned}\tilde{\varphi}_{1m}(S^*, m^*) + \tilde{\varphi}_{1S}(S^*, m^*)(S - S^*) = {}& -\tilde{\varphi}_1(S^*, m^*) - \frac{1}{2}\tilde{\varphi}_{1SS}(S - S^*)^2 \\ & - \tilde{\varphi}_{1mS}(m - m^*)(S - S^*) - \frac{1}{2}\tilde{\varphi}_{1mm}(m - m^*)^2.\end{aligned}$$

This can now be written as

$$\begin{aligned}&(h^3 y_0'' + h^4 y_0''' + O(h^5))(m_0 - m_0^*) \\ &\qquad + (2h^2 y_0'' + \frac{4}{3}h^4 y_0'' + \frac{h^4}{2}p_0 y_0'' + O(h^5))(m_1 - m_1^*) \\ &\qquad - (3h^2 y_0'' + \frac{7}{3}h^3 y_0''' + h^4 y_0^{IV} + \frac{h^4}{2}q_0 y_0'' + o(h^4))(S_1 - S_1^*) \\ &\quad = \frac{h^6}{24}(y_0'' y_0^{IV} - \frac{4}{3}(y_0''')^2) - 2u_1^2 - 2h^2(m_1 - m_1^*)^2 + O(h^7)\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{2}(h^2 y_0'' + h^3 y_0''' + O(h^4))v_0 + (h^2 y_0'' + \frac{2}{3}h^4 y_0'' + \frac{h^3}{4}p_0 y_0'' + O(h^4))v_1 \\ &\quad - (3h^2 y_0'' + \frac{7}{3}h^3 y_0''' + h^4 y_0^{IV} + \frac{h^4}{2}q_0 y_0'' + O(h^5))u_1 \\ &= C_0 - 2u_1^2 - 2u_1^2 + O(h^7)\end{aligned} \tag{7.54}$$

where C_0 is defined in (7.47).

Note that $\frac{1}{2}v_0 + \frac{2}{3}v_1 - \frac{7}{3}u_1 = O(h^4)$, $-\frac{h^3}{4}p_0 y_0'' v_0 = -\frac{h^4}{4}p_0 y_0'' v_1 + O(h^7)$ and $-\frac{h^2}{2}q_0 y_0'' u_1 = -\frac{h^4}{4}q_0 y_0'' v_0 + O(h^8)$. Therefore, if we add (7.47) and (7.54) we get

$$\Big(\frac{3}{2}h^2 y_0'' - \frac{h^4}{4}q_0 y_0''\Big)v_0 + \frac{3}{2}h^2 y_0'' v_1 - 6h^2 y_0'' u_1 = O(h^7). \tag{7.55}$$

Observe here the diagonal dominance in rows with the difference $-\frac{h^4}{4}q_0 y_0''$. Let us point out that we use this equation later as equation $[B(m-m^*)]_0 = d_0(S,m)$.

Also, for $i=1$, equation (7.41) is

$$\begin{aligned}&(3h^2 y_1'' + \alpha_1)u_0 - (6h^2 y_1'' + 2\alpha_1 - h^4 q_1 y_1'')u_1 + (3h^2 y_1'' + \alpha_1)u_2\\ &= c_1 h^6 + h^2 y_1''(h(m_2 - m_2^*) - h(m_0 - m_0^*)) + O(h^7)\end{aligned} \tag{7.56}$$

where $\alpha_1 = \frac{1}{6}h^4 y_1^{IV} - h^2 p_1(m_1 - y_1')$ and $c_1 = \frac{1}{12}\Big(y_1'' y_1^{IV} - \frac{4}{3}(y_1''')^2\Big)$.

Basing on (7.48) and using again the notations w_i, u_i, $i = 0,\dots,3$, we transform the right hand side of (7.56) as

$$\begin{aligned}&c_1 h^6 + \frac{1}{2}h^2 y_1''\tilde{c}_0 + \frac{1}{2}h^2 y_1''(w_2 + \frac{1}{2}w_1 + u_3 - u_1 + \frac{1}{2}u_2 - 3u_1) + O(h^7)\\ &= \frac{5}{4}c_1 h^6 + \frac{1}{2}h^2 y_1''(w_2 + \frac{1}{2}w_1) + \frac{1}{2}h^2 y_1''(u_3 - 2u_2 + u_1) + h^2 y_1''(u_2 - 2u_1 + u_0)\\ &\quad + \frac{1}{4}h^2 y_1''(u_2 - 2u_1) + O(h^7)\end{aligned}$$

where $\tilde{c}_0 = \frac{C_0}{h^2 y_0''}$. Next we move $h^2 y_1''(u_2 - 2u_1 + u_0)$ to the left hand side. Then we are able to write (7.56) (taking into account that $u_0 = 0$) as

$$\begin{aligned}&-(4h^2 y_1'' + 2\alpha_1 - h^4 q_1 y_1'')u_1 + (2h^2 y_1'' + \alpha_1)u_2\\ &\qquad = \frac{5}{4}c_1 h^6 + \frac{1}{2}h^2(u_3 - 2u_2 + u_1) + \frac{1}{4}h^2 y_1''(u_2 - 2u_1) + O(h^7)\end{aligned}$$

or

$$\begin{aligned}&(2h^2 y_1'' + \alpha_1)(u_0 - (2 - h^2 q_1)u_1 + u_2) = \frac{1}{2}h^2 y_1''(u_1 - (2 - h^2 q_2)u_2 + u_3)\\ &\quad + \frac{1}{4}h^2 y_1''(u_2 - (2 - h^2 q_1)u_1) + \frac{5}{4}c_1 h^6 + h^4 y_1''(q_1 u_1 - \frac{1}{2}q_2 u_2) + O(h^7),\end{aligned}$$

which with the help of notation $z_i = u_{i-1} - (2 - q_i h^2)u_i + u_{i+1}$, $i = 1, 2$, gives

$$\Big(\frac{7}{4}h^2 y_1'' + \alpha_1\Big)z_1 - \frac{1}{2}h^2 y_1'' z_2 = \frac{5}{4}c_1 h^6.$$

Divide the last equation by $\frac{5}{4}h^2 q_1 y_1''$, then, because of $\frac{z_2}{q_1} = (1 + O(h))\frac{z_2}{q_2}$, we get

$$\Big(\frac{7}{5} + O(h)\Big)\frac{z_1}{q_1} - \frac{2}{5}\Big(1 + O(h)\Big)\frac{z_2}{q_2} = \tilde{c}_1 h^4 \tag{7.57}$$

where $\tilde{c}_1$ has the same meaning as was for $i = 2, \ldots, n-2$.

Due to the diagonal dominance in rows with the difference $1 + O(h)$ in the system consisting in (7.43), (7.57) and the counterpart of (7.57) for $i = n-1$, it holds

$$\Big|\frac{z_i}{q_i}\Big| \leq \max_{1 \leq i \leq n-1} \mid \tilde{c}_i h^4 \mid + O(h^5).$$

Now

$$u_{i-1} - (2 - h^2 q_i)u_i + u_{i+1} = z_i$$

or

$$\frac{1}{q_i}u_{i-1} - (\frac{2}{q_i} - h^2)u_i + \frac{1}{q_i}u_{i+1} = \frac{z_i}{q_i}, \quad i = 2, \ldots, n-2. \tag{7.58}$$

Here we can calculate the difference of domination as

$$\Big|\frac{2}{q_i} - h^2\Big| - \Big|\frac{1}{q_i}\Big| - \Big|\frac{1}{q_i}\Big| = h^2.$$

If $i = 0$, then $u_i = 0$ $(S_0 - S_0^* = 0)$.

If $i = 1$, we can take $a_{i,i-1} = 0$ in (7.39) and divide the equation (7.41) by $3y_1''$. This gives

$$(-2h^2 + O(h^4))u_1 + (h^2 + O(h^4))u_2 = O(h^6). \tag{7.59}$$

Observe here the diagonal dominance in rows with the difference $h^2 + O(h^4)$.

The system (7.58), (7.59) and its counterpart for $i = n-1$ allows to get the estimate

$$\mid u_i \mid \leq \frac{1}{h^2} \max_{1 \leq i \leq n-1} \Big|\frac{z_i}{q_i}\Big| \leq \max_{1 \leq i \leq n-1} \tilde{c}_i h^2 + \ O(h^3), \ i = 1, \ldots, n-1.$$

All in all,

$$\|S - S^*\|_\infty \leq c_0 h^2 + O(h^3), \tag{7.60}$$

where

$$c_0 = \frac{1}{12} \max_{1 \leq i \leq n-1} \left| \frac{y_i^{IV} - \frac{4}{3}\frac{(y_i''')^2}{y_i''}}{q_i} \right|. \tag{7.61}$$

7.4 Transformation of internal equations

In this section we study the internal equations $\varphi_i(S,m)=0$, $i=1,\ldots,n-1$.

The Taylor expansion of $\Phi(S,m)$ at point (S^*,m) gives

$$\Phi(S,m)=\Phi(S^*,m)+\Phi_S(S^*,m)(S-S^*)+\frac{1}{2}\Phi_{SS}(S^*,m)(S-S^*)^2+\frac{1}{6}\Phi_{SSS}(S-S^*)^3. \tag{7.62}$$

This with the help of the following Taylor expansions at (S^*,m^*)

$$\Phi(S^*,m)=\Phi(S^*,m^*)+\Phi_m(S^*,m^*)(m-m^*)+\frac{1}{2}\Phi_{mm}(S^*,m^*)(m-m^*)^2 + \frac{1}{6}\Phi_{mmm}(m-m^*)^3,$$

$$\Phi_S(S^*,m)=\Phi_S(S^*,m^*)+\Phi_{Sm}(S^*,m^*)(m-m^*)+\frac{1}{2}\Phi_{Smm}(m-m^*)^2,$$

$$\Phi_{SS}(S^*,m)=\frac{1}{2}\Phi_{SS}(S^*,m^*)+\frac{1}{2}\Phi_{SSm}(m-m^*)$$

gives

$$\begin{aligned}\Phi_m(S^*,m^*)(m-m^*) &= -(\Phi(S^*,m^*)+\frac{1}{2}\Phi_{mm}(S^*,m^*)(m-m^*)^2\\ &+\frac{1}{6}\Phi_{mmm}(m-m^*)^3+\Phi_S(S^*,m^*)(S-S^*)+\Phi_{Sm}(S^*,m^*)(S-S^*)(m-m^*)\\ &+\frac{1}{2}\Phi_{Smm}(S-S^*)(m-m^*)^2+\frac{1}{2}\Phi_{SS}(S^*,m^*)(S-S^*)^2\\ &+\frac{1}{2}\Phi_{SSm}(S-S^*)^2(m-m^*)+\frac{1}{6}\Phi_{SSS}(S-S^*)^3)\end{aligned} \tag{7.63}$$

which we begin to study mainly for the components with indices $i=1,\ldots,n-1$.

Note that in the interior components of $\Phi(S^*,m^*)$ it holds $\varphi_i(S^*,m^*)=0$, $i=1,\ldots,n-1$, as S_* is a quadratic/linear rational spline. If $\Phi_m(S^*,m^*)$ is invertible then by writing (7.63) as $\varphi_{im}(S^*,m^*)(m-m^*)=d_i(S,m)$ we get

$$m-m^*=\Phi_m^{-1}(S^*,m^*)d(S,m). \tag{7.64}$$

Also, we use later $\varphi_{im}(S^*,m^*)(m-m^*)=d_i(S,m)$ as equations in system $B(m-m^*)=d$.

We calculate from (7.11) the components of $\varphi_{im}(S,m)$ as

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial m_{i-1}} &= h(S_{i+1} - S_i - hm_i)^2, \\
\frac{\partial \varphi_i}{\partial m_i} &= 2h(hm_i - (S_i - S_{i-1}))(hm_{i+1} - (S_{i+1} - S_i)) \\
&\quad + 2h(S_{i+1} - S_i - hm_i)(S_i - S_{i-1} - hm_{i-1}), \\
\frac{\partial \varphi_i}{\partial m_{i+1}} &= h(hm_i - (S_i - S_{i-1}))^2.
\end{aligned} \tag{7.65}$$

With the help of (7.28) we get

$$\frac{\partial \varphi_i}{\partial m_{i-1}}(S^*,m^*) = \frac{h^5}{4}(y_i'')^2 + \frac{h^6}{6}y_i''y_i''' + \frac{h^7}{24}y_i''y_i^{IV} + \frac{h^7}{36}(y_i''')^2 + o(h^7),$$

$$\frac{\partial \varphi_i}{\partial m_i}(S^*,m^*) = h^5(y_i'')^2 + \frac{h^7}{3}y_i''y_i^{IV} - \frac{2}{9}h^7(y_i''')^2 + o(h^7),$$

$$\frac{\partial \varphi_i}{\partial m_{i+1}}(S^*,m^*) = \frac{h^5}{4}(y_i'')^2 - \frac{h^6}{6}y_i''y_i''' + \frac{h^7}{24}y_i''y_i^{IV} + \frac{h^7}{36}(y_i''')^2 + o(h^7).$$

Clearly, there is diagonal dominance in rows and this implies the invertibility of $\Phi_m(S^*,m^*)$ if we have also appropriate transformations of equations $\varphi_0(S,m)=0$ and $\varphi_n(S,m)=0$.

Similarly we calculate from (7.11) the components of $\varphi_{iS}(S,m)$ as

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}} &= 2(hm_i - (S_i - S_{i-1}))(hm_{i+1} - (S_{i+1} - S_i)) + (S_{i+1} - S_i - hm_i)^2, \\
\frac{\partial \varphi_i}{\partial S_i} &= -2(hm_i - (S_i - S_{i-1}))(hm_{i+1} - (S_{i+1} - S_i)) - (S_{i+1} - S_i - hm_i)^2 \\
&\quad + (hm_i - (S_i - S_{i-1}))^2 + 2(S_i - S_{i-1} - hm_{i-1})(S_{i+1} - S_i - hm_i), \\
\frac{\partial \varphi_i}{\partial S_{i+1}} &= -2(S_i - S_{i-1} - hm_{i-1})(S_{i+1} - S_i - hm_i) - (hm_i - (S_i - S_{i-1}))^2.
\end{aligned} \tag{7.66}$$

Note that $\dfrac{\partial \varphi_i}{\partial S_{i-1}} + \dfrac{\partial \varphi_i}{\partial S_i} + \dfrac{\partial \varphi_i}{\partial S_{i+1}} = 0.$

Using again the Taylor expansions (7.28) we obtain

$$\frac{\partial \varphi_i}{\partial S_{i-1}}(S^*,m^*) = \frac{3}{4}h^4(y_i'')^2 + \frac{h^5}{3}y_i''y_i''' + \frac{5}{24}h^6y_i''y_i^{IV} - \frac{h^6}{12}(y_i''')^2 + o(h^6),$$

$$\frac{\partial \varphi_i}{\partial S_i}(S^*, m^*) = -\frac{2}{3}h^5 y_i'' y_i''' + o(h^6),$$

$$\frac{\partial \varphi_i}{\partial S_{i+1}}(S^*, m^*) = -\frac{3}{4}h^4 (y_i'')^2 + \frac{h^5}{3} y_i'' y_i''' - \frac{5}{24}h^6 y_i'' y_i^{IV} + \frac{h^6}{12}(y_i''')^2 + o(h^6).$$

We calculate for $\varphi_{iS}(S^*, m)$

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i-1}}(S^*, m) = {} & \frac{\partial \varphi_i}{\partial S_{i-1}}(S^*, m^*) + 2h(m_i - m_i^*)(hm_{i+1}^* - (S_{i+1}^* - S_i^*)) \\
& + 2h(m_{i+1} - m_{i+1}^*)(hm_i^* - (S_i^* - S_{i-1}^*)) - 2h(m_i - m_i^*)(S_{i+1}^* - S_i^* + hm_i^*) \\
& + 2h^2(m_i - m_i^*)(m_{i+1} - m_{i+1}^*) + h^2(m_i - m_i^*)^2 + O(h^5),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_i}(S^*, m) = {} & \frac{\partial \varphi_i}{\partial S_i}(S^*, m^*) - (2h(m_i - m_i^*)(hm_{i+1}^* - (S_{i+1}^* - S_i^*)) \\
& + 2h(m_{i+1} - m_{i+1}^*)(hm_i^* - (S_i^* - S_{i-1}^*)) + 2h^2(m_i - m_i^*)(m_{i+1} - m_{i+1}^*) \\
& - 2h(m_i - m_i^*)(S_{i+1}^* - S_i^* + hm_i^*) - 2h(m_i - m_i^*)(hm_i^* - (S_i^* - S_{i-1}^*)) \qquad (7.67) \\
& + 2h(m_{i-1} - m_{i-1}^*)(S_{i+1}^* - S_i^* - hm_i^*) + 2h(m_i - m_i^*)(S_i^* - S_{i-1}^* - hm_{i-1}^*) \\
& - 2h^2(m_{i-1} - m_{i-1}^*)(m_i - m_i^*)) + O(h^5),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial S_{i+1}}(S^*, m) = {} & \frac{\partial \varphi_i}{\partial S_{i+1}}(S^*, m^*) - 2h(m_i - m_i^*)(hm_i^* - (S_i^* - S_{i-1}^*)) \\
& + 2h(m_{i-1} - m_{i-1}^*)(S_{i+1}^* - S_i^* - hm_i^*) + 2h(m_i - m_i^*)(S_i^* - S_{i-1}^* - hm_{i-1}^*) \\
& - 2h^2(m_{i-1} - m_{i-1}^*)(m_i - m_i^*) - h^2(m_i - m_i^*)^2 + O(h^5).
\end{aligned}$$

Second order partial derivatives are calculated from (7.65) and are

$$\frac{\partial^2 \varphi_i}{\partial m_{i-1}^2} = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i-1} \partial m_i} = -2h^2(S_{i+1} - S_i - hm_i),$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i-1} \partial m_{i+1}} = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial m_i \partial m_{i-1}} = -2h^2(S_{i+1} - S_i - hm_i),$$

$$\frac{\partial^2 \varphi_i}{\partial m_i^2} = 2h^2(hm_{i+1} - (S_{i+1} - S_i)) - 2h^2(S_i - S_{i-1} - hm_{i-1}),$$

$$\frac{\partial^2 \varphi_i}{\partial m_i \partial m_{i+1}} = 2h^2(hm_i - (S_i - S_{i-1})),$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i+1} \partial m_{i-1}} = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i+1} \partial m_i} = 2h^2(hm_i - (S_i - S_{i-1})),$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i+1}^2} = 0.$$

They give

$$\frac{\partial^2 \varphi_i}{\partial m_{i-1}^2}(S^*, m^*) = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i-1} \partial m_i}(S^*, m^*) = -h^4 y_i'' + O(h^5),$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i-1} \partial m_{i+1}}(S^*, m^*) = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial m_i \partial m_{i-1}}(S^*, m^*) = -h^4 y_i'' + O(h^5),$$

$$\frac{\partial^2 \varphi_i}{\partial m_i^2}(S^*, m^*) = O(h^5),$$

$$\frac{\partial^2 \varphi_i}{\partial m_i \partial m_{i+1}}(S^*, m^*) = h^4 y_i'' + O(h^5),$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i+1} \partial m_{i-1}}(S^*, m^*) = 0,$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i+1} \partial m_i}(S^*, m^*) = h^4 y_i'' + O(h^5),$$

$$\frac{\partial^2 \varphi_i}{\partial m_{i+1}^2}(S^*, m^*) = 0.$$

From (7.66) we find also with the help of Taylor expansions

$$\frac{\partial^2 \varphi_i}{\partial S_{i-1}^2}(S^*, m^*) = h^2 y_i'' + \frac{2}{3} h^3 y_i''' + \frac{1}{4} h^4 y_i^{IV} + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_{i-1} \partial S_i}(S^*, m^*) = -h^2 y_i'' - \frac{4}{3} h^3 y_i''' - \frac{1}{4} h^4 y_i^{IV} + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_{i-1} \partial S_{i+1}}(S^*, m^*) = \frac{2}{3} h^3 y_i''' + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_i \partial S_{i-1}}(S^*, m^*) = -h^2 y_i'' - \frac{4}{3} h^3 y_i''' - \frac{1}{4} h^4 y_i^{IV} + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_i^2}(S^*, m^*) = \frac{8}{3} h^3 y_i''' + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_i \partial S_{i+1}}(S^*, m^*) = h^2 y_i'' - \frac{4}{3} h^3 y_i''' + \frac{1}{4} h^4 y_i^{IV} + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_{i+1} \partial S_{i-1}}(S^*, m^*) = \frac{2}{3} h^3 y_i''' + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_{i+1} \partial S_i}(S^*, m^*) = h^2 y_i'' - \frac{4}{3} h^3 y_i''' + \frac{1}{4} h^4 y_i^{IV} + o(h^4),$$

$$\frac{\partial^2 \varphi_i}{\partial S_{i+1}^2}(S^*, m^*) = -h^2 y_i'' + \frac{2}{3} h^3 y_i''' - \frac{1}{4} h^4 y_i^{IV} + o(h^4)$$

and for every j, k

$$\sum_j \frac{\partial^2 \varphi_i}{\partial S_j \partial S_k} = 0, \quad \sum_k \frac{\partial^2 \varphi_i}{\partial S_j \partial S_k} = 0.$$

Similarly we get for every j, k, l

$$\frac{\partial^3 \varphi_i}{\partial m_j \partial m_k \partial m_l} = O(h^3). \tag{7.68}$$

Let us start with the estimate of

$$\Phi_S(S^*,m)(S-S^*) = \Phi_S(S^*,m^*)(S-S^*) + \Phi_{Sm}(S^*,m^*)(S-S^*)(m-m^*) + \frac{1}{2}\Phi_{Smm}(S-S^*)(m-m^*)^2.$$

Using (7.29) and

$$2h(m_i - m_i^*) = v_i = w_i + (u_{i+1} - u_{i-1})$$

we can write due to (7.67)

$$\begin{aligned}
\frac{\partial\varphi_i}{\partial S_{i-1}}(S^*,m) = {}& \frac{\partial\varphi_i}{\partial S_{i-1}}(S^*,m^*) \\
&+ (w_i + (u_{i+1} - u_{i-1}))(\frac{h^3}{6}y_i''' + \frac{h^4}{12}y_i^{IV} + o(h^4)) \\
&+ (w_{i+1} + (u_{i+2} - u_i))(\frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4)) \\
&+ h(m_i - m_i^*)(w_{i+1} + (u_{i+2} - u_i)) + h^2(m_i - m_i^*)^2 + O(h^5),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\varphi_i}{\partial S_i}(S^*,m) = {}& \frac{\partial\varphi_i}{\partial S_i}(S^*,m^*) \\
&+ 2h^2(m_i - m_i^*)((m_{i-1} - m_{i-1}^*) - (m_{i+1} - m_{i+1}^*)) \\
&- (w_i + (u_{i+1} - u_{i-1}))(\frac{h^4}{6}y_i^{IV} + o(h^4)) \\
&- (w_{i-1} + (u_i - u_{i-2}))(\frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4)) \\
&- (w_{i+1} + (u_{i+2} - u_i))(\frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4)) + O(h^5),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\varphi_i}{\partial S_{i+1}}(S^*,m) = {}& \frac{\partial\varphi_i}{\partial S_{i+1}}(S^*,m^*) \\
&+ (w_i + (u_{i+1} - u_{i-1}))(-\frac{h^3}{6}y_i''' + \frac{h^4}{12}y_i^{IV} + o(h^4)) \\
&+ (w_{i-1} + (u_i - u_{i-2}))(\frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y_i^{IV} + o(h^4)) \\
&- h(m_i - m_i^*)(w_{i-1} + (u_i - u_{i-2})) - h^2(m_i - m_i^*)^2 + O(h^5).
\end{aligned}$$

These partial derivatives allow to form the expression $\varphi_{iS}(S^*,m)(S-S^*)$. In that the terms in $\varphi_{iS}(S^*,m)(S-S^*)$ with $\frac{h^2}{2}y_i''w_i$ give

$$\begin{aligned}
&\frac{h^2}{2}y_i''(w_iu_{i-1}+w_{i+1}u_{i-1}-w_iu_{i-1}-w_iu_{i+1}+w_{i-1}u_{i+1}+w_iu_{i+1}-(w_{i+1}+w_{i-1})u_i)\\
&= \frac{h^2}{2}y_i''(w_{i+1}(u_{i-1}-u_i)+w_{i-1}(u_{i+1}-u_i))
\end{aligned}$$

which we can estimate by cc_3h^{10} (here and in the sequel, in similar reasonings, c denotes a positive constant depending only on the boundary value problem).

We estimate other terms with w_i by $cc_3h^{10}+cc_3h^{11}$.

Members with $\frac{h^2}{2}y_i''(u_{i+1}-u_{i-1})$ are

$$\begin{aligned}
&\frac{h^2}{2}y_i''((u_{i+2}-u_i)u_{i-1}+(u_i-u_{i-2})u_{i+1}-(u_{i+2}-u_{i-2})u_i)\\
&=\frac{h^2}{2}y_i''((u_{i+2}-u_i)(u_{i-1}-u_i)+(u_i-u_{i-2})(u_{i+1}-u_i))\\
&=\frac{h^2}{2}y_i''((-u_{i+2}+2u_{i+1}-u_i)(u_i-u_{i-1})-2(u_{i+1}-u_i)(u_i-u_{i-1})\\
&\quad+(-u_{i-2}+2u_{i-1}-u_i)(u_{i+1}-u_i)+2(u_i-u_{i-1})(u_{i+1}-u_i))
\end{aligned}$$

which give cc_2h^9.

Similarly we can estimate other members in $\varphi_{iS}(S^*,m)(S-S^*)$ by the term $ch^9+cc_0h^9+cc_0^2h^9+cc_3h^{10}$.

All in all,

$$|\ \varphi_{iS}(S^*,m)(S-S^*)\ | \le cc_2h^9+ch^9+cc_0h^9+cc_0^2h^9+cc_3h^{10}+cc_3h^{11}.$$

In $\frac{1}{2}\varphi_{imm}(S^*,m^*)(m-m^*)^2$ members with $\frac{1}{2}h^4y_i''$ are

$$\begin{aligned}
&h^4y_i''(m_i-m_i^*)\frac{1}{2h}(w_{i+1}-w_{i-1}+(u_{i+2}-2u_{i+1}+u_i)\\
&\qquad\qquad+2(u_{i+1}-2u_i+u_{i-1})+(u_i-2u_{i-1}+u_{i-2}))
\end{aligned}$$

and can be estimated by $cc_0c_2h^9+cc_3c_0h^{10}$.

Other terms in $\frac{1}{2}\varphi_{imm}(S^*,m^*)(m-m^*)^2$ give $cc_0^2h^9$, thus,

$$\left|\frac{1}{2}\varphi_{imm}(S^*,m^*)(m-m^*)^2\right| \le cc_0c_2h^9+cc_0^2h^9+cc_3c_0h^{10}.$$

Because of (7.68) we have

$$\left|\frac{1}{6}\varphi_{immm}(m-m^*)^3\right| \leq cc_0^3h^9.$$

For $\dfrac{1}{2}\varphi_{iSS}(S^*,m^*)(S-S^*)^2$ we calculate

$$\frac{1}{2}h^2y_i''(-u_{i-1}+2u_i-u_{i+1})(u_{i+1}-u_{i-1})$$

$$+\frac{1}{3}h^3y_i'''(u_{i-1}-2u_i+u_{i+1})^2+\frac{1}{8}h^4y_i^{IV}(-u_{i-1}+2u_i-u_{i+1})(u_{i+1}-u_{i-1})$$

in which the terms in the sum can be estimated by $cc_2h^9+cc_2h^{11}+cc_2h^{11}$, respectively, therefore,

$$\left|\frac{1}{2}\varphi_{iSS}(S^*,m^*)(S-S^*)^2\right| \leq cc_2h^9+cc_2h^{11}. \tag{7.69}$$

Due to Lemma 6.1 the members in $\dfrac{1}{2}\varphi_{iSSm}(S-S^*)^2(m-m^*)$ can be estimated by cc_0h^9.

Similar reasoning allows to show that

$$\left|\frac{1}{6}\varphi_{iSSS}(S-S^*)^3\right| = |(u_{i+1}-u_i)(u_i-u_{i-1})(u_{i+1}-u_{i-1})| \leq ch^9.$$

7.5 Existence of the solution for collocation problem

In this section we use the Bohl-Brouwer fixed point theorem to show the existence of solution of the basic system $\Phi(S,m)=0$, $\Psi(S,m)=0$.

Consider the equations $S=G(S,m)$, where

$$G(S,m)=S^*-\Psi_S^{-1}(S^*,m)(\Psi(S^*,m)+\frac{1}{2}\Psi_{SS}(S-S^*)^2)$$

and $m=H(S,m)$ where

$$H(S,m)= m^*+\Phi_m^{-1}(S^*,m^*)(-\Phi(S^*,m^*)-\frac{1}{2}\Phi_{mm}(S^*,m^*)(m-m^*)^2$$

$$-\frac{1}{6}\Phi_{mmm}(m-m^*)^3-\Phi_S(S^*,m^*)(S-S^*)-\Phi_{Sm}(S^*,m^*)(S-S^*)(m-m^*)$$

$$-\frac{1}{2}\Phi_{Smm}(S-S^*)(m-m^*)^2-\frac{1}{2}\Phi_{SS}(S^*,m^*)(S-S^*)^2$$

$$-\frac{1}{2}\Phi_{SSm}(S-S^*)^2(m-m^*)-\frac{1}{6}\Phi_{SSS}(S-S^*)^3)$$

and which are equivalent to $\Psi(S,m)=0$ and $\Phi(S,m)=0$, respectively.

Let $F(S,m)=(G(S,m),H(G(S,m),m))$ with $F:\mathbb{R}^{2n+2}\to\mathbb{R}^{2n+2}$ and let us introduce the set

$$\begin{aligned}
K=\{(S,m)\in\mathbb{R}^{2n+2}\mid\ & S_0=\alpha,\ S_n=\beta,\\
& \mid S_i-S_i^*\mid\ \le c_1h^2,\ i=1,\dots,n-1,\\
& \mid (S_{i-1}-S_{i-1}^*))-2(S_i-S_i^*)+(S_{i+1}-S_{i+1}^*)\mid\ \le c_2h^4,\ i=1,\dots,n-1,\\
& \mid 2h(m_i-m_i^*)-((S_{i+1}-S_{i+1}^*)-(S_{i-1}-S_{i-1}^*))\mid\le c_3h^5,\ i=1,\dots,n-1,\\
& \mid 2h(m_0-m_0^*)-2(S_1-S_1^*)\mid\ \le\tilde{c}_3h^4,\\
& \mid 2h(m_n-m_n^*)-2(S_{n-1}-S_{n-1}^*)\mid\ \le\tilde{c}_3h^4\}
\end{aligned}$$

with numbers $c_1>0$, $c_2>0$, $c_3>0$, $\tilde{c}_3>0$ which will be specified later. The set K is convex and compact.

Our main purpose is now to show that $F:K\to K$. This allows us to use Bohl-Brouwer fixed point theorem. Let $(S,m)\in K$ and $\bar{S}=G(S,m)$, $\bar{m}=H(\bar{S},m)=H(G(S,m),m)$. We show that then $(\bar{S},\bar{m})\in K$.

Firstly, from the estimate (7.60) we obtain

$$\|\bar{S}-S^*\|_\infty\le c_0h^2+O(h^3). \tag{7.70}$$

If we choose e.g., $c_1=2c_0$, then $c_0h^2+O(h^3)\le c_1h^2$ holds with sufficiently small value of h.

In Section 7.3 we showed that for $i=1,\dots,n-1$

$$\left|\frac{\bar{z}_i}{q_i}\right|\le h^4\max_{1\le i\le n-1}\mid\tilde{c}_i\mid+O(h^5)$$

or

$$|\bar{z}_i|\le h^4\mid q_i\mid\max_{1\le i\le n-1}\mid\tilde{c}_i\mid+O(h^5),$$

where

$$\bar{z}_i=\bar{u}_{i-1}-(2-h^2q_i)\bar{u}_i+\bar{u}_{i+1}, \tag{7.71}$$

$\bar{u}_i=\bar{S}_i-S_i^*$ and where $\tilde{c}_i$ is given in (7.42). From (7.71) we get

$$\bar{u}_{i-1}-2\bar{u}_i+\bar{u}_{i+1}=\bar{z}_i-h^2q_i\bar{u}_i$$

which due to (7.70) implies

$$\mid\bar{u}_{i-1}-2\bar{u}_i+\bar{u}_{i+1}\mid\ \le\ \mid\bar{z}_i\mid -h^2\mid q_i\mid\mid\bar{u}_i\mid\ \le c_2h^4$$

with suitable choice of the constant c_2.

In Section 7.3 we constructed a system $A(S-S^*)=b$ with equations corresponding to (7.39), (7.40) and having indices $i=2,\dots,n-2$. Divide those equations by $h^2 y_i''$ and move $h(m_{i+1} - m_{i+1}^*) - h(m_{i-1} - m_{i-1}^*)$ to the left hand side. Then the higher order term of the new system $\tilde{A}(S - S^*) = \tilde{b}$ in the left hand side is

$$3u_{i-1} - 6u_i + 3u_{i+1} - \frac{1}{2}(v_{i+1} - v_{i-1}).$$

Similarly, after dividing the components with indices $i = 1, \dots, n-1$ in system $B(m - m^*) = d$ by $\dfrac{8}{h^4 (y_i'')^2}$ the main term in left hand side is $v_{i-1} + 4v_i + v_{i+1} - 6(u_{i+1} - u_{i-1})$ and we denote the obtained system by $\tilde{B}(m - m^*) = \tilde{d}$.

Let us form now the equation

$$\begin{aligned} &[\tilde{A}(S - S^*)]_{i-1} - [\tilde{A}(S - S^*)]_{i+1} \\ &\qquad - \frac{1}{2}[\tilde{B}(m - m^*)]_{i-1} + 2[\tilde{B}(m - m^*)]_i - \frac{1}{2}[\tilde{B}(m - m^*)]_{i+1} \\ &\qquad = \tilde{b}_{i-1} - \tilde{b}_{i+1} - \frac{1}{2}\tilde{d}_{i-1} + 2\tilde{d}_i - \frac{1}{2}\tilde{d}_{i+1}. \end{aligned} \tag{7.72}$$

Then the main terms in the left hand side of the obtained equations give

$$\begin{aligned} &(3u_{i-2} - 6u_{i-1} + 3u_i - \frac{1}{2}(v_i - v_{i-2})) - (3u_i - 6u_{i+1} + 3u_{i+2} - \frac{1}{2}(v_{i+2} - v_i)) \\ &- \frac{1}{2}((v_{i-2} + 4v_{i-1} + v_i) - 6(u_i - u_{i-2})) + 2((v_{i-1} + 4v_i + v_{i+1}) - 6(u_{i+1} - u_{i-1})) \\ &- \frac{1}{2}((v_i + 4v_{i+1} + v_{i+2}) - 6(u_{i+2} - u_i)) = 6(v_i - (u_{i+1} - u_{i-1})) = 6w_i. \end{aligned}$$

Actually, we transformed and managed with the systems $A(\bar{S} - S^*) = d(S, m)$ and $B(\bar{m} - m^*) = d(\bar{S}, m)$, thus, we get $6\bar{w}_i$ in (7.72). Observe that the terms together besides $6\bar{w}_i$ on the left hand side of (7.72) could be estimated by $O(h^5)$ and we have already estimated $\tilde{b}_i$ and $\tilde{d}_i$ by $O(h^5)$ (See Sections 7.3 and 7.4). This gives $| \bar{w}_i | \le c_3 h^5$ for suitable choice of c_3.

Next, the equation (7.55) allows to determine the equation $[B(m - m^*)]_0 = d_0$ participating in the formation of $m = H(S, m)$. From

$$(\frac{3}{2}hy_0'' - \frac{h^2}{4}q_0 y_0'')h\bar{w}_0 + \frac{3}{2}h^2 y_0'' \bar{w}_1 = d_0(\bar{S}, m)$$

provided we have $\bar{w}_1 = O(h^5)$ we get $\bar{w}_0 = O(h^4)$ because $d_0(\bar{S}, m) = O(h^6)$. To prove $\bar{w}_1 = O(h^5)$ we use the ideas of the case $i = 2, \dots, n-2$ and form the equation

$$\begin{aligned} &-[\tilde{A}(S - S^*)]_2 - \frac{1}{2}[\tilde{B}(m - m^*)]_2 + [\tilde{A}(S - S^*)]_1 + \frac{5}{2}[\tilde{B}(m - m^*)]_1 - 3[\tilde{B}(m - m^*)]_0 \\ &\qquad = -\tilde{b}_2 - \frac{1}{2}\tilde{d}_2 + \tilde{b}_1 + \frac{5}{2}\tilde{d}_1 - 3\tilde{d}_0. \end{aligned} \tag{7.73}$$

E.g., the main terms in the left hand side of (7.73) give $6\bar{w}_1$.

All in all, $F : K \to K$ and we can use the Bohl-Brouwer fixed point theorem, by which the set K contains a fixed point of the function F which is also a solution of the basic system.

7.6 Convergence estimates

In this section we establish in uniform norm the convergence rates of collocation method described in Section 7.1 with quadratic/linear rational splines for boundary value problem (7.1), (7.2).

We remind that in previous section we proved that the set K contains a fixed point of the function F which is also a solution of the basic system $\Phi(S, m) = 0$, $\Psi(S, m) = 0$. Denote now by S the quadratic/linear rational spline having as knot values this vector in K and we also remind that S_* is a special quadratic/linear rational spline interpolant to the solution y of (7.1), (7.2). It is known, that

$$\| S_* - y \|_\infty = O(h^4) \tag{7.74}$$

and

$$\| S_*' - y' \|_\infty = O(h^3). \tag{7.75}$$

Proposition 7.1. *For the solution S of the problem* (7.3), (7.4) *having the vector of values in K it holds $\|S^{(k)}\|_\infty = \max_{1\le i\le n} \max_{x_{i-1}\le x\le x_i} |S^{(k)}(x)| \le C_k$, $k \ge 2$.*

Proof. From (3.1) we obtain (if $k \ge 2$)

$$S^{(k)}(x) = \frac{(-1)^k k! c_i d_i^k}{(1 + d_i(x - x_{i-1}))^{k+1}}, \ x \in [x_{i-1}, x_i].$$

We indicated in Section 3.1 that c_i and d_i could be expressed via the first moments m_i and spline values S_i. Namely,

$$c_i = \frac{(S_i - S_{i-1} - hm_{i-1})^2 (hm_i - (S_i - S_{i-1}))}{(2(S_i - S_{i-1}) - h(m_{i-1} + m_i))^2},$$

$$d_i = \frac{2(S_i - S_{i-1}) - h(m_{i-1} + m_i)}{h(hm_i - (S_i - S_{i-1}))}.$$

With the help of (7.25), (7.26) and (7.29) we find

$$d_i = \frac{-\dfrac{1}{6}h^3 y_i''' + O(h^4)}{h\Big(\dfrac{h^2}{2} y_i'' + O(h^3)\Big)} = -\frac{y_i'''}{3y_i''} + O(h)$$

and then

$$c_i d_i^2 = \frac{\frac{h^4}{4}(y_i'')^2 + O(h^5)}{h^2\Big(\frac{h^2}{2}y_i'' + O(h^3)\Big)} = \frac{1}{2}(y_i'' + O(h))$$

which implies the assertion. $\square$

From (7.22) with the help of (7.44) we get

$$S'(x_i) - S_*'(x_i) = \frac{(S_{i+1} - S_{i+1}^*) - (S_{i-1} - S_{i-1}^*)}{2h} + O(h^4) = O(h^2).$$

Thus, we obtain from

$$|S'(x_i) - y'(x_i)| \leq |S'(x_i) - S_*'(x_i)| + |S_*'(x_i) - y'(x_i)|$$

by (7.75) that it holds

$$S'(x_i) - y'(x_i) = O(h^2). \tag{7.76}$$

From the estimate $|S(x_i) - y(x_i)| \leq |S(x_i) - S_*(x_i)| + |S_*(x_i) - y(x_i)|$ we get by (7.70) and (7.74)

$$|S(x_i) - y(x_i)| \leq c_0 h^2 + O(h^3) \tag{7.77}$$

where c_0 is determined in (7.61).

Next, by (7.1) and (7.3) we may write

$$S''(x_i) - y''(x_i) + p(x_i)(S'(x_i) - y'(x_i)) + q(x_i)(S(x_i) - y(x_i)) = 0$$

which, with the help of (7.77) and (7.76) gives

$$S''(x_i) - y''(x_i) = O(h^2). \tag{7.78}$$

The Taylor expansion for $x \in [x_{i-1}, x_i]$

$$S'(x) - y'(x) = S'(x_i) - y'(x_i) + (S''(x_i) - y''(x_i))(x - x_i) + \frac{1}{2}(S''' - y''')(\xi_i)(x - x_i)^2$$

thanks to (7.76), (7.78) and Proposition 7.1 gives

$$\max_{a \leq x \leq b} |S'(x) - y'(x)| = O(h^2). \tag{7.79}$$

Now our aim is to show $\|S'' - y''\|_\infty = O(h^2)$. Consider the expansion

$$y''(x) = y''(x_i) + y'''(x_i)(x - x_i) + O(h^2),\ x \in [x_{i-1}, x_i],$$

which, by setting $x = x_{i-1} + \tau h$, $\tau \in [0,1]$, is

$$y''(x) = y''(x_i) + (\tau - 1)hy'''(x_i) + O(h^2), \ \tau \in [0,1]. \tag{7.80}$$

In (3.3) we find by using the representations for c_i and d_i from Section 3.1 and (7.29) that

$$c_i d_i^2 = \frac{(S - S_{i-1} - hm_{i-1})^2}{h^2(hm_i - (S_i - S_{i-1}))} = \frac{\frac{h^4}{4}(y_i'')^2 - \frac{h^5}{3}y_i''y_i''' + O(h^6)}{\frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + O(h^4)} = \frac{h^2}{2}\Big(y_i'' - hy_i''' + O(h^2)\Big)$$

and by $d_i = -\dfrac{y_i'''}{3y_i''}$ which we already obtained in proof of Proposition 7.1 we get

$$(1 + d_i(x - x_{i-1}))^3 = 1 + 3d_i(x - x_{i-1}) + O(h^2) = 1 - \frac{y_i'''}{y_i''}(x - x_{i-1}) + O(h^2).$$

Thus, we have for $x \in [x_{i-1}, x_i]$

$$S''(x) = \frac{h^2 y_i'' - h^3 y_i''' + O(h^4)}{h^2(1 - \frac{y_i'''}{y_i''}(x - x_{i-1}) + O(h^2))} = y_i'' - hy_i''' + y_i'''(x - x_{i-1}) + O(h^2)$$

or by changing the variable $x = x_{i-1} + \tau h$, $\tau \in [0,1]$

$$S''(x) = y_i'' + (\tau - 1)hy_i''' + O(h^2).$$

This with expansion (7.80) implies

$$\max_{a \le x \le b} |S''(x) - y''(x)| = O(h^2). \tag{7.81}$$

But the expansion for $x \in [x_{i-1}, x_i]$

$$S(x) - y(x) = S(x_i) - y(x_i) + (S' - y')(\zeta_i)(x - x_i)$$

and (7.79) due to (7.77) yields (for $y \in C^4[a,b]$ as general case)

$$\max_{a \le x \le b} |S(x) - y(x)| \le c_0 h^2 + o(h^2). \tag{7.82}$$

We have proved as our main result in this chapter the following

Theorem 7.1. *Let the solution $y \in C^4[a,b]$ of boundary value problem* (7.1), (7.2) *be strictly convex. Then, for sufficiently small h, the collocation problem* (7.3), (7.4) *has a quadratic/linear rational spline S as solution with convergence estimates* (7.82) *(c_0 is determined in* (7.61)*),* (7.79) *and* (7.81)*. The rest term $o(h^2)$ in* (7.82) *is actually $O(h^{2+\alpha})$ in the case $y^{IV} \in \mathrm{Lip}\ \alpha$, $0 < \alpha \le 1$.*

In collocation with cubic splines, see, e.g., [39], it is known the estimate

$$\|S - y\|_\infty \leq \frac{h^2}{12} \max_{1 \leq i \leq n-1} \left| \frac{y^{IV}}{q}(x_i) \right| + o(h^2).$$

In comparison to (7.82), the additional members in c_0 could make the main part of the estimate for quadratic/linear rational splines smaller or greater than in case of cubic splines. In Chapter 8 there is a numerical example in which the rational splines have really considerably smaller error.

Chapter 8

Numerical examples

In this chapter we present the numerical examples.

In the thesis we deal with quadratic, cubic, linear/linear and quadratic/linear rational splines. We point out, that it would be easy to construct an example of boundary value or interpolation problem whose exact solution is one of the mentioned splines but is not any other of them. For example let a quadratic spline be the exact solution of the test problem but not any other spline written above. In comparison, this gives an advantage to quadratic splines because the error in interpolation or collocation is automatically zero for them and different from zero for all other types of splines. Our aim in numerical examples is to take a simple and ordinary function and show that in that case the rational splines have better results.

We interpolated the function

$$y(x) = \frac{1}{x^2}$$

on the interval $[-2, -\frac{1}{5}]$.

The boundary value problem which we solved by collocation method is

$$y'' - xy' - 8x^4 y = \frac{6}{x^4} + \frac{2}{x^2} - 8x^2, \quad x \in (-2, -\frac{1}{5}),$$

$$y(-2) = \frac{1}{4}, \quad y(-\frac{1}{5}) = 25$$

with the exact solution $y(x) = \dfrac{1}{x^2}$.

The results could be found in the following sections and tables.

8.1 Linear/linear rational spline interpolation

We interpolated the test function by linear/linear rational spline S as described in Section 4.1. The boundary conditions (4.2) with

$$\alpha_1 = y_0 + \frac{3}{64}h^4\frac{1}{x_0^6}, \quad \alpha_2 = y_n + \frac{3}{64}h^4\frac{1}{x_n^6}$$

were used. The "three-diagonal" nonlinear system to determine the values of S_i was solved by Newton's method. The iterations were stopped at $\|S^k - S^{k-1}\|_\infty \leq 10^{-10}$, S^k being the sequence of approximations to the vector $S = (S_0, \ldots, S_n)$. The errors $\varepsilon_n = S(z_i) - y(z_i)$ and $\varepsilon''_n = S''(z_i) - y''(z_i)$ were calculated in certain superconvergence points z_i. Results of numerical tests in Tables 8.1–8.2 support the theory.

Table 8.1: Numerical results for $\varepsilon_n = S(z_i) - y(z_i)$, $i = 1, 2, 3$.

	$z_1 = -1.55$		$z_2 = -1.1$		$z_3 = -0.65$	
n	ε_n	$\varepsilon_{\frac{n}{2}}/\varepsilon_n$	ε_n	$\varepsilon_{\frac{n}{2}}/\varepsilon_n$	ε_n	$\varepsilon_{\frac{n}{2}}/\varepsilon_n$
16	$5.381 \cdot 10^{-7}$		$4.189 \cdot 10^{-6}$		$9.697 \cdot 10^{-5}$	
32	$3.379 \cdot 10^{-8}$	15.931	$2.641 \cdot 10^{-7}$	15.861	$6.170 \cdot 10^{-6}$	15.716
64	$2.114 \cdot 10^{-9}$	15.984	$1.654 \cdot 10^{-8}$	15.967	$3.881 \cdot 10^{-7}$	15.984
128	$1.322 \cdot 10^{-10}$	15.991	$1.035 \cdot 10^{-9}$	15.981	$2.429 \cdot 10^{-8}$	15.978
256	$8.262 \cdot 10^{-12}$	16.001	$6.467 \cdot 10^{-11}$	16.004	$1.519 \cdot 10^{-9}$	15.991

Table 8.2: Numerical results for $\varepsilon''_n = S''(z_i) - y''(z_i)$, $i = 1, 2$.

	$z_1 = \frac{a+b}{2} - \frac{h}{2}$		$z_2 = \frac{a+b}{2} + \frac{h}{2}$	
n	ε''_n	$\varepsilon''_{\frac{n}{2}}/\varepsilon''_n$	ε''_n	$\varepsilon''_{\frac{n}{2}}/\varepsilon''_n$
16	$-2.602 \cdot 10^{-3}$		$-4.789 \cdot 10^{-3}$	
32	$-7.639 \cdot 10^{-4}$	3.406	$-4.037 \cdot 10^{-3}$	4.618
64	$-1.066 \cdot 10^{-4}$	3.697	$-2.408 \cdot 10^{-4}$	4.306
128	$-5.370 \cdot 10^{-5}$	3.847	$-5.798 \cdot 10^{-5}$	4.153
256	$-1.369 \cdot 10^{-5}$	3.923	$-1.422 \cdot 10^{-5}$	4.077
512	$-3.522 \cdot 10^{-6}$	3.964	$-3.455 \cdot 10^{-6}$	4.116

8.2 Quadratic/linear rational spline interpolation

We also interpolated the test function by quadratic/linear rational spline S as described in Section 5.1. The boundary conditions (5.3) with

$$\alpha_1 = y''_0 + \frac{2}{3}h^2\frac{1}{x_0^6}, \quad \alpha_2 = y''_n + \frac{2}{3}h^2\frac{1}{x_n^6}$$

were used. Likewise to linear/linear rational spline case the "three-diagonal" nonlinear system to determine the values M_i was solved by Newton's method and the iterations were stopped at $\|M^k - M^{k-1}\|_\infty \leq 10^{-10}$, M^k being the sequence of approximations to the vector $M = (M_0, \dots, M_n)$. The errors $\varepsilon'_n = S'(z_i) - y'(z_i)$ and $\varepsilon'''_n = S'''(z_i) - y'''(z_i)$ were calculated in certain superconvergence points z_i. Again, the results confirm the convergence rates predicted by the theory.

Table 8.3: Numerical results for $\varepsilon'_n = S'(-1.1) - y'(-1.1)$

n	16	32	64	128	256	512
ε'_n	$1.18 \cdot 10^{-5}$	$7.55 \cdot 10^{-7}$	$4.75 \cdot 10^{-8}$	$2.97 \cdot 10^{-9}$	$1.86 \cdot 10^{-10}$	$1.16 \cdot 10^{-11}$
$\varepsilon'_{\frac{n}{2}}/\varepsilon'_n$		15.6055	15.9101	15.9774	15.9938	16.0075

Table 8.4: Numerical results for $\varepsilon'''_n = S'''(z_i) - y'''(z_i)$, $i = 1, 2$.

	$z_1 = \frac{a+b}{2} - \frac{h}{2}$		$z_2 = \frac{a+b}{2} + \frac{h}{2}$	
n	ε'''_n	$\varepsilon'''_{\frac{n}{2}}/\varepsilon'''_n$	ε'''_n	$\varepsilon'''_{\frac{n}{2}}/\varepsilon'''_n$
16	$-6.9813 \cdot 10^{-3}$		$-1.4140 \cdot 10^{-2}$	
32	$-2.1037 \cdot 10^{-3}$	3.3186	$-3.0075 \cdot 10^{-3}$	4.7017
64	$-5.7679 \cdot 10^{-4}$	3.6473	$-6.8979 \cdot 10^{-4}$	4.3600
128	$-1.5091 \cdot 10^{-4}$	3.8221	$-1.6504 \cdot 10^{-4}$	4.1796
256	$-3.8583 \cdot 10^{-5}$	3.9113	$-4.0362 \cdot 10^{-5}$	4.0889
512	$-9.9922 \cdot 10^{-6}$	3.8613	$-9.7364 \cdot 10^{-6}$	4.1454

8.3 Linear/linear rational spline collocation

We solved the test problem with linear/linear rational splines S as described in the beginning of Chapter 6. The nonlinear system to determine the values of S_i and $\bar{S}_i$ was solved by Newton's method and the iterations were stopped at $\|S^k - S^{k-1}\|_\infty \leq 10^{-10}$, S^k being the sequence of approximations to the vector $S = (S_0, \bar{S}_1, S_1, \dots, S_n, \bar{S}_n)$. In addition, we present the results for the test problem by quadratic spline collocation method, see, e.g., [39]. In [39], for implementation, B-splines are used. In following tests, we used the first moments for quadratic spline S_2. Given numbers in Table 8.5 in the upper row are the errors calculated approximately on tenfold refined grid as

$$\delta_n = \max_{1 \leq i \leq n} \max_{0 \leq k \leq 10} |(S_2 - y)(x_{i-1} + kh/10)|. \tag{8.1}$$

Similarly to (8.1) we calculated ε_n as being the approximate values of $\|S - y\|_\infty$ where S is the linear/linear rational spline. They are presented in the middle row of Table 8.5. The results show that the error is approximately ten times less for linear/linear rational spline. Numbers in third row of Table 8.5 confirm the convergence rate $O(h^2)$ for linear/linear rational spline collocation established in Chapter 6.

In our example we have the functions

$$g_1(x) = \left| \frac{y^{IV} - py''' - 6\dfrac{y'''y''}{y'} + 6\dfrac{(y'')^3}{(y')^2} + \dfrac{3}{2}p\dfrac{(y'')^2}{y'}}{q}(x) \right| = \frac{3(x^2+4)}{8x^{10}},$$

$$g_2(x) = \left| \frac{y^{IV} - py'''}{q}(x) \right| = \frac{3(5-x^2)}{x^{10}}$$

and they are increasing if $x < 0$. Then $g_2\Big(-\dfrac{1}{5}\Big)/g_1\Big(-\dfrac{1}{5}\Big) = \dfrac{992}{101}$ which is consistent with numerical results presented in Table 8.5.

Table 8.5: Numerical results for δ_n and ε_n

n	16	32	64	128	256	512
δ_n	$8.60 \cdot 10^{-1}$	$3.13 \cdot 10^{-1}$	$8.70 \cdot 10^{-2}$	$2.20 \cdot 10^{-2}$	$5.60 \cdot 10^{-3}$	$1.40 \cdot 10^{-3}$
ε_n	$2.10 \cdot 10^{-1}$	$4.17 \cdot 10^{-2}$	$9.67 \cdot 10^{-3}$	$2.37 \cdot 10^{-3}$	$5.88 \cdot 10^{-4}$	$1.47 \cdot 10^{-4}$
$\varepsilon_{\frac{n}{2}}/\varepsilon_n$		5.048	4.308	4.089	4.027	4.009

We also calculated approximately the errors $\|S' - y'\|_\infty$ and $\|S'' - y''\|_\infty$ on tenfold refined grid as in (8.1) which we denote by ε'_n and ε''_n, respectively. Results of numerical tests presented in Table 8.6 support the established theoretical ones.

The errors $\delta''_n = S''(z_i) - y''(z_i)$ were calculated in certain superconvergence points z_i. Results of numerical tests are presented in Table 8.7 and they confirm convergence rate $O(h^2)$ predicted by theory.

Table 8.6: Numerical results for ε'_n and ε''_n

n	16	32	64	128	256	512
ε'_n	14.21	3.45	$8.56 \cdot 10^{-1}$	$2.14 \cdot 10^{-1}$	$5.34 \cdot 10^{-2}$	$1.34 \cdot 10^{-2}$
$\varepsilon'_{\frac{n}{2}}/\varepsilon'_n$		4.122	4.026	4.006	4.002	4.0004
ε''_n	888.117	344.384	151.128	70.645	34.129	16.771
$\varepsilon''_{\frac{n}{2}}/\varepsilon''_n$		2.579	2.279	2.139	2.070	2.035

Table 8.7: Numerical results for $\delta''_n = S''(z_i) - y''(z_i)$, $i = 1, 2$.

	$z_1 = \frac{a+b}{2} - \frac{h}{2}$		$z_2 = \frac{a+b}{2} + \frac{h}{2}$	
n	δ''_n	$\delta''_{\frac{n}{2}}/\delta''_n$	δ''_n	$\delta''_{\frac{n}{2}}/\delta''_n$
16	$-3.675 \cdot 10^{-1}$		$-3.404 \cdot 10^{-1}$	
32	$-7.400 \cdot 10^{-2}$	4.966	$-7.124 \cdot 10^{-2}$	4.778
64	$-1.722 \cdot 10^{-2}$	4.297	$-1.690 \cdot 10^{-2}$	4.216
128	$-4.214 \cdot 10^{-3}$	4.087	$-4.174 \cdot 10^{-3}$	4.049
256	$-1.046 \cdot 10^{-3}$	4.028	$-1.041 \cdot 10^{-3}$	4.009
512	$-2.607 \cdot 10^{-4}$	4.012	$-2.601 \cdot 10^{-4}$	4.002

8.4 Quadratic/linear rational spline collocation

Finally, we solved the test problem with quadratic/linear rational splines S (see Chapter 7). Again, we solved the arising nonlinear system by Newton's method and the iterations were stopped at $\|m^k - m^{k-1}\|_\infty \le 10^{-10}$ and $\|S^k - S^{k-1}\|_\infty \le 10^{-10}$, m^k being the sequence of approximations to the vector $m = (m_0, \ldots, m_n)$ and S^k being the sequence of approximations to the vector $S = (S_0, \ldots, S_n)$. For comparison, we have solved the test problem with cubic spline collocation method, where we used the representation by second moments [39].

The errors δ_n for cubic splines and ε_n for quadratic/linear rational splines were calculated as in (8.1). The results in Table 8.8 show that the error is approximately fifteen times less for quadratic/linear rational spline. Numbers in third row of Table 8.8 confirm the theoretical convergence rate for quadratic/linear rational spline collocation method.

Here we have functions

$$g_1(x) = \left| \frac{y^{IV} - \frac{4}{3}\frac{(y''')^2}{y''}}{q}(x) \right| = \frac{1}{x^{10}}, \qquad g_2(x) = \left| \frac{y^{IV}}{q}(x) \right| = \frac{15}{x^{10}}$$

and they are increasing if $x < 0$. Then $g_2\Big(-\frac{1}{5}\Big)/g_1\Big(-\frac{1}{5}\Big) = 15$ and this is consistent with numerical results presented in Table 8.8.

Table 8.8: Numerical results for δ_n and ε_n

n	16	32	64	128	256	512
δ_n	$36.18 \cdot 10^{-1}$	$7.88 \cdot 10^{-1}$	$1.87 \cdot 10^{-1}$	$4.60 \cdot 10^{-2}$	$1.14 \cdot 10^{-2}$	$2.86 \cdot 10^{-3}$
ε_n	$2.04 \cdot 10^{-1}$	$4.98 \cdot 10^{-2}$	$1.23 \cdot 10^{-2}$	$3.06 \cdot 10^{-3}$	$7.63 \cdot 10^{-4}$	$1.91 \cdot 10^{-4}$
$\varepsilon_{\frac{n}{2}}/\varepsilon_n$		4.087	4.057	4.019	4.005	4.001

We also calculated approximately the errors $\|S' - y'\|_\infty$ and $\|S'' - y''\|_\infty$ on tenfold refined grid as in (8.1) which we denote here by ε'_n and ε''_n, respectively. Results of numerical tests presented in Table 8.9 support the established theoretical ones.

Table 8.9: Numerical results for ε'_n and ε''_n

n	16	32	64	128	256	512
ε'_n	4.18	1.18	$3.06 \cdot 10^{-1}$	$7.73 \cdot 10^{-2}$	$1.94 \cdot 10^{-2}$	$4.85 \cdot 10^{-3}$
$\varepsilon'_{\frac{n}{2}}/\varepsilon'_n$		3.541	3.852	3.960	3.990	3.997
ε''_n	61.40	24.69	8.43	2.50	$6.92 \cdot 10^{-1}$	$1.82 \cdot 10^{-1}$
$\varepsilon''_{\frac{n}{2}}/\varepsilon''_n$		2.487	2.928	3.377	3.608	3.797

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Kokkuvõte

Rajaülesannete lahendamine ratsionaalsplainidega kollokatsioonimeetodil

Paljud matemaatika, füüsika ja teiste teadusalade probleemid on formuleeritavad rajaülesannete kujul. Traditsioonilised meetodid rajaülesannete lahendamiseks on võrgumeetod, mis annab ainult diskreetse lahendi, ja polünomiaalsete splainidega kollokatsioonimeetod.

Käesolevas töös vaadeldakse hariliku teist järku diferentsiaalvõrrandi rajaülesande

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in (a, b),$$

$$y(a) = \alpha, \quad y(b) = \beta$$

lahendamist kollokatsioonimeetodil kasutades lineaar/lineaar ja ruut/lineaar ratsionaalsplaine.

Eeldatakse, et ülesandel on olemas lahend, mis on piisava siledusega ja p, q, f on pidevad ning $q(x) \leq q < 0$, $x \in (a, b)$. Sellisel juhul on lahend ühene, mis on ka peatükis 6 tõestatud.

Vaatleme ühtlase jaotusega lõiku $[a, b]$ punktidega $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b - a)/n$, $n \in \mathbb{N}$. Rajaülesande lahendamist ruut-ja kuupsplainidega kollokatsioonimeetodil on uurinud mitmed autorid. On teada ka, et ruutsplainide korral põhineb koonduvuskiiruse $O(h^2)$ tõestus interpoleerimise superkoonduvusel [30, 35, 39]. Kuupsplainidega interpolatsiooniülesandes on antud sõlmedes x_i, $i = 0, \ldots, n$, väärtused y_i, $i = 0, \ldots, n$, ja otsitakse splaini S, mis rahuldaks tingimusi $S(x_i) = y_i$, $i = 0, \ldots, n$. Ruutsplainidega interpolatsiooniülesandes on antud väärtused $\bar{y}_i$, $i = 1, \ldots, n$, sõlmedes $\xi_i = (x_{i-1} + x_i)/2$, $i = 1, \ldots, n$, ja nõutakse, et $S(\xi_i) = \bar{y}_i$, $i = 1, \ldots, n$. Lineaar/lineaar ratsionaalsplainidega interpolatsiooniülesande formuleering on sama nagu ruutsplainidega ja ruut/lineaar ratsionaalsplainidega interpolatsiooniülesandel nagu kuupsplainidega. Interpoleerimise korral on teada, et ratsionaalsplainid lähendavad mõningaid funktsioone paremini kui polünomiaalsed splainid. Seetõttu võivad ratsionaalsplainid anda mõnedes rajaülesannetes paremaid tulemusi. Käesoleva töö põhiprobleemiks ongi uurida rajaülesannete lahendamist lineaar/lineaar ja ruut/lineaar ratsionaalsplainidega kollokatsioonimeetodil. Tulemusi võrreldakse vastavalt ruut-ja kuupsplain-kollokatsioonimeetodite tulemustega.

Kuna lineaar/lineaar ratsionaalsplain on monotoonne ja ruut/lineaar ratsionaalsplain S (või $-S$) on kumer lõigus $[a, b]$, siis on mõtet rajaülesannet selliste splainidega ligikaudselt lahendada vaid siis, kui on teada, et rajaülesande täpne lahend on sama omadusega.

Kui vaadeldav ruut-ja kuupsplainidega rajaülesanne on lineaarne, siis ratsionaalsplainidega ülesanne on mittelineaarne, sest see viib mittelineaarse võrrandisüsteemi lahendamiseni.

Käesolev doktoritöö koosneb 8 peatükist. Esimeses peatükis antakse lühike ülevaade tööst, rajaülesande ja ratsionaalsplainide ajaloost ning tutvustatakse teiste autorite artikleid antud valdkonna kohta.

Teises peatükis tutvustatakse lineaar/lineaar ratsionaalsplaine, kirjeldatakse nende vabade parameetrite arvu ja üldisi omadusi. Töös antakse 3 erinevat esitust: kõigepealt punktides x_i leitud esimeste momentide ja splaini väärtuste kaudu punktides ξ_i, teisena osalõikude keskpunktis arvutatud tuletise ja splaini väärtuste kaudu punktides x_i ning viimaseks splaini väärtuste abil punktides x_i ja ξ_i.

Kaks esitust ruut/lineaar ratsionaalsplainide jaoks leiab töö kolmandast peatükist. Nimelt on toodud esitus esimeste momentide ja splaini väärtuste abil sõlmedes x_i ja samuti teiste momentide ja splaini sõlmväärtuste kaudu. Samast peatükist leiab ka üldisi tulemusi ruut/lineaar ratsionaalsplainide kohta.

Töö neljas peatükk käsitleb lineaar/lineaar ratsionaalsplainidega interpoleerimist. Alustatakse interpolatsiooniülesande tutvustamisega ja kasutatakse splaini esitust väärtuste kaudu. Näidatakse, et interpoleerivate lineaar/lineaar ratsionaalsplainide S ja piisavalt sileda funktsiooni y jaoks kehtib $\|S(x_i)-y(x_i)\|_\infty = O(h^4)$ ühtlase jaotusega lõigus punktidega $x_i = a + ih$, $i = 0, \ldots, n$. Tõestatakse ka h^3 järku superkoonduvus splaini S esimeste tuletiste ja h^2 järku teiste tuletiste jaoks teatud punktides. Sarnased tulemused ruutsplainide kohta on teada varasemast, võrdlus saadud lineaar/lineaar ratsionaalsplainide tulemustega on toodud peatüki lõpus.

Viies peatükk on pühendatud interpoleerimisele ruut/lineaar ratsionaalsplainidega. Siin kasutatakse splaini esitust teiste momentide ja väärtuste kaudu. Tõestatakse interpoleerivate ruut/lineaar ratsionaalsplainide S ja piisavalt sileda funktsiooni y jaoks ühtlase jaotusega lõigus h^4 järku superkoonduvus splaini esimeste tuletiste, h^3 järku teiste tuletiste ja h^2 järku kolmandate tuletiste jaoks teatud punktides. Saadud tulemusi võrreldakse kuupsplainide tulemustega, mis on varasemast teada.

Kuuendas peatükis on uuritud kollokatsioonimeetodit lineaar/lineaar ratsionaalsplainidega. Peatüki algusest leiab meetodi kirjelduse ja võrdluse ruutsplain-kollokatsioonimeetodiga. Edasi on tõestatud $O(h^2)$ koonduvuskiirus ühtlase jaotusega lõigus lineaar/lineaar ratsionaalsplainidega kollokatsioonimeetodi jaoks. Tõestus põhineb splainidega interpoleerimise superkoonduvuse tulemustele. Samuti on saadud koonduvuskiirused $\|S' - y'\|_\infty = O(h^2)$ ja $\|S'' - y''\|_\infty = O(h)$ lineaar/lineaar ratsionaalsplaini S ja rangelt monotoonse rajaülesande lahendi y korral. Osutub, et kollokatsioonipunktides toimub splaini teiste tuletiste korral h^2 järku superkoondumine.

Seitsmendas peatükis vaadeldakse ruut/lineaar ratsionaalsplainidega kollokatsioonimeetodit. Nagu eelmises peatükis, alustatakse ka siin meetodi kirjeldamisega ning võrreldakse seda kuupsplain-kollokatsioonimeetodiga. Uurimisel kasutatakse splaini esitust esimeste momentide ja väärtuste kaudu ning tõestatakse, et ruut/lineaar ratsionaalsplainide S jaoks ühtlase jaotusega lõigus rajaülesande lahendi y, mis on rangelt kumer (või rangelt nõgus), korral kehtib $\|S-y\|_\infty = O(h^2)$. Samuti näidatakse $\|S' - y'\|_\infty = O(h^2)$ ja $\|S'' - y''\|_\infty = O(h^2)$ kehtivust.

Viimasest peatükist leiab arvuliste katsete tulemused nii interpolatsiooni- kui rajaülesande jaoks. On võetud tavaline lihtne funktsioon ning näidatud, et selle korral annavad ratsionaalsplainid võrreldes ruut-ja kuupsplainidega paremaid tulemusi. Saadud tulemused on ka täielikus kooskõlas töös toodud teoreetilistega.

Peatüki 4 tulemused on avaldatud artiklis [25] ja peatüki 5 tulemused artiklis [26]. Publitseerimisele on suunatud peatüki 6 tulemused [27] ning dissertatsioonis esitatut on tutvustatud viiel rahvusvahelisel teaduskonverentsil.

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"The 16th International Conference on Mathematical Modelling and Analysis" Siguldas Lätis (2011),

"International Conference on Optimization and Analysis of Structures (OAS 2011)" Tartus Eestis (2011),

"The 17th International Conference on Mathematical Modelling and Analysis" Tallinnas Eestis (2012).

List of publications

1. E. Ideon and P. Oja. Linear/linear rational spline interpolation. Math. Model. Anal., 15 (2010), 447-455.
2. E. Ideon and P. Oja. Quadratic/linear rational spline interpolation. Math. Model. Anal., 18 (2013), 250-259.
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