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Compactness and null sequences defined by ℓ_p spaces

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Compactness and null sequences defined by ℓ_p spaces

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Chapter 1

Introduction

1.1 Background and summary of thesis

In 1955, A. Grothendieck [Gro55] described relatively compact subsets of a Banach space as sets contained in the closed convex hull of a norm null sequence. Nowadays this result is called the Grothendieck compactness principle.

Let $1 \leq p < \infty$. In 1980s, O. Reinov [Rei84] and J. Bourgain and O. Reinov [BR85] considered a stronger form of relative compactness: subsets of the closed convex hull of a norm p-summable sequence. In the current thesis these sets are called relatively p-compact in the sense of Bourgain–Reinov. In 2002, D. P. Sinha and A. K. Karn defined and studied a form of relative compactness, that lays "between" aforementioned types of relative compactness. They required the set to belong to the so called p-convex hull of a norm p-summable sequence. This type of sets are called in the present thesis relatively p-compact sets in the sense of Sinha–Karn.

A linear operator is said to be compact if it maps bounded sets to relatively compact ones. By using relatively p-compact subsets instead of relatively compact ones, the concepts of p-compact operators are obtained. In the recent years, the p-compact operators in the sense of Sinha–Karn have received great attention, whereas the p-compact operators in the sense of Bourgain–Reinov have had hardly any recognition. One aim of the thesis is to eliminate this shortcoming. For this we define a new kind of compactness that encompasses the above-mentioned compactnesses.

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, where p^* is the conjugate index of p. We say that a subset of a Banach space is relatively (p, r)-compact if it is a subset of the (p,r)-convex hull of a norm p-summable sequence (norm null sequence when $p=\infty$). The (p,r)-compact operator is defined in a natural way as a linear operator that maps bounded sets to relatively (p,r)-compact sets.

Our study includes new results for p-compactness in the sense of Bourgain–Reinov as it is exactly the $(p,1)$-compactness. The results in the thesis also include and clarify many results known from the literature for the operator ideal of p-compact operators in the sense of Sinha–Karn, as the (p,p^*)-compact operators coincide with the p-compact operators in the sense of Sinha–Karn.

In 2011, C. Piñeiro and J. M. Delgado [PD11] looked at relatively p-compact sets (in the sense of Sinha–Karn) in a Grothendieck-like manner. Lead by the relationship between norm null sequences and relatively compact sets in a Banach space, they defined p-null sequences in a Banach space and described the relatively p-compact sets as laying inside of the closed convex hull of a p-null sequence.

We extend the notion of p-null sequences to (p,r)-null sequences. In conjunction with [PD11], we also obtain a Grothendieck-like characterization of relatively (p,r)-compact sets as sets contained in the closed convex hull of a (p,r)-null sequence. This includes the Delgado–Piñeiro result because p-null sequences are exactly (p,p^*)-null sequences. Our proof is straightforward, which was not the case in [PD11].

The p-null sequences can be described as being exactly the norm null sequences that are relatively p-compact as subsets. This was discovered and proved by Piñeiro and Delgado [PD11] in the case of Banach spaces having a certain approximation property. For arbitrary Banach spaces, the characterization was proved by E. Oja [Oja12a], relying on the description of the space of p-null sequences as a Chevet–Saphar tensor product and using the Hahn–Banach theorem. Recently, the result was also proved by S. Lassalle and P. Turco [LT13] using a theory due to B. Carl and I. Stephani [CS84].

One of the key points of this thesis is the observation, that the Carl–Stephani theory can be used in a more efficient way than in [LT13] to prove the aforementioned description of p-null sequences. The method to do this does not rely on the Hahn–Banach theorem and therefore can and is used in a more general setting of quasi-Banach operator ideals. In this way, the (p,r)-null sequences as well as recent concepts of unconditionally (p,r)-null and weakly (p,r)-null sequences are characterized.

The objective of the thesis is to study the operator ideal of (p,r)-compact operators and (p,r)-null sequences in a Banach space in a direct and clear way. Among others, as particular cases, we give new straightforward proofs to previously known results concerning p-compactness and p-null sequences.

The thesis has been organized as follows.

Chapter 1 introduces the background of (p,r)-compactness and (p,r)-null sequences, provides a summary of the thesis, and describes basic notation used in the thesis.

Chapter 2 is an overview of notions and results needed in the following chapters. These include vector-valued sequence spaces, based on the book [DJT95] by Diestel, Jarchow, and Tonge; also operator ideals with some ways to construct new operator ideals as well as products of operator ideals. We refer to Pietsch's monographs [Pie80, Pie07] for the theory of operator ideals. Some preliminaries are also due to [ALO12].

In Chapter 3, we extend the notion of some well-known forms of relatively compact sets to relatively (p,r)-compact subsets of a Banach space. We go on to define the operator ideal of (p,r)-compact operators and describe its structure as an operator ideal. This chapter is mainly based on [ALO12].

In Chapter 4, we prove one of the main results. It is a description of (p,r)-compact operators as operators belonging to the operator ideal $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$. This allows us to equip the operator ideal $\mathcal{K}_{(p,r)}$ of (p,r)-compact operators with an s-norm from $\mathcal{N}_{(p,1,r^*)}$. We also show how to explicitly calculate the s-norm of an (p,r)-compact operator. This chapter is based on [ALO12].

In Chapter 5, results from the Chapter 4 are used to describe the surjective and injective hulls of $\mathcal{N}_{(p,1,r^*)}$. A description of $\mathcal{K}_{(p,r)}$ as product of operator ideals is given as well. This chapter is based on [ALO12].

In Chapter 6, we introduce a new term: the (p,r)-null sequence. The main aim of this chapter is to establish an omnibus theorem giving six equivalent properties for a sequence in a Banach space to be a (p,r)-null sequence. The method used to do this is self-contained and does not use the Hahn–Banach theorem as was mentioned above. This chapter relies on [AO12, AO15].

The notions from Chapters 3 and 6 involving the (p,r)-compactness are extended in Chapter 7 to the unconditional and weak (p,r)-compactnesses. Using the techniques developed previously, the unconditionally (p,r)-null and weakly (p,r)-null sequences are described. This chapter is mainly based on [AO15].

Main results of the thesis are contained in [ALO12, AO12, AO15].

1.2 Notation

Our notation is standard.

We consider Banach spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If X and Y are Banach spaces, then $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y. An operator $T \in \mathcal{L}(X, Y)$ is called *(weakly) compact*, if T maps bounded subsets of X to (weakly) relatively compact sets in Y. The space of all compact operators acting between X and Y is denoted by $\mathcal{K}(X, Y)$ and the space of weakly compact operators between X and Y is denoted by $\mathcal{W}(X, Y)$. An operator $T \in \mathcal{L}(X, Y)$ is of *finite rank* if its range is finite-dimensional. The space of all finite-rank operators acting between X and Y is denoted by $\mathcal{F}(X, Y)$. An operator $T \in \mathcal{L}(X, Y)$ is said to belong to $\overline{\mathcal{F}}(X, Y)$, the space of *approximable operators*, if it is a uniform limit of finite-rank operators.

The unit sphere of a Banach space X is denoted by S_X, the closed unit ball by B_X, the identity operator with I_X. By X^* we denote the dual space of X. The *natural surjection* $Q_X : \ell_1(B_X) \to X$ is defined as

$$Q_X(\lambda_x)_{x\in B_X} = \sum_{x\in B_X} \lambda_x x,\ (\lambda_x)_{x\in B_X} \in \ell_1(B_X),$$

and the *natural embedding* $J_X : X \to \ell_\infty(B_{X^*})$ is defined as

$$J_X x = (x^*(x))_{x^*\in B_{X^*}},\ x \in X,$$

where $\ell_1(B_X)$ and $\ell_\infty(B_{X^*})$ denote the Banach spaces of all absolutely summable scalar families (λ_x) where $x \in B_X$, and all bounded scalar families (λ_{x^*}) where $x^* \in B_{X^*}$, respectively.

For $1 \le p \le \infty$, let p^* denote the conjugate index of p (i.e., $1/p + 1/p^* = 1$ with the convention $1/\infty = 0$). We also use conventions $1\infty = \infty 1 = 1+\infty = \infty + 1 = \infty$ and $\infty/\infty = 1$.

We assume that the reader is familiar with well-known basic notions and theorems from the theory of Banach spaces and topological vector spaces (such as conjugate operator, reflexivity, the Hahn–Banach theorem, the Banach–Alaoglu theorem, ultrafilter, etc.).

Chapter 2

Preliminaries

This chapter is an overview of notions and results needed in the following chapters. These include vector-valued sequence spaces, based on the book [DJT95] by Diestel, Jarchow, and Tonge; also operator ideals with some ways to construct new operator ideals as well as products of operator ideals. We refer to Pietsch's monographs [Pie80, Pie07] for the theory of operator ideals. Some preliminaries are also due to [ALO12].

2.1 p-Summable sequences

Let $1 \leq p < \infty$ and let X be a Banach space.

A sequence $(x_n) \subset X$ belongs to $\ell_p(X)$, the space of *absolutely p-summable* sequences of X, if $(\|x_n\|) \in \ell_p$. The space $\ell_p(X)$ is a Banach space with the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}}.$$

The space of bounded sequences in X, denoted by $\ell_\infty(X)$, and the space of norm null sequences in X, denoted by $c_0(X)$, are Banach spaces with the norm

$$\|(x_n)\|_\infty = \sup_n \|x_n\|.$$

For a Banach space X, we denote by $\ell_p^w(X)$ the Banach space of *weakly* *p-summable* X-valued sequences with the norm given by

$$\|(x_n)\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty}|x^*(x_n)|^p\right)^{1/p} : x^* \in B_{X^*}\right\}.$$

The space of weakly bounded X-valued sequences $\ell_\infty^w(X)$ coincides with $\ell_\infty(X)$.

For a Banach space X, we denote by $c_0^w(X)$ the Banach space of *weakly null sequences* in X with the norm given by

$$\|(x_n)\|_\infty^w = \sup_{x^* \in B_{X^*}} \|x^*(x_n)\|_\infty = \|(x_n)\|_\infty .$$

It is well known that

$$c_0(X) \subset c_0^w(X) \subset \ell_\infty(X)$$

as closed subspaces, where equality $c_0(X) = c_0^w(X)$ holds only for Banach spaces X with Schur's property.

In [DJT95] the following description of spaces $\ell_p^w(X)$ is given.

Proposition 2.1 (see [DJT95, Proposition 2.2])**.** *The space $\mathcal{L}(\ell_{p^*}, X)$ is isometrically isomorphic to $\ell_p^w(X)$ for $1 < p < \infty$, and $\mathcal{L}(c_0, X)$ is isometrically isomorphic to $\ell_1^w(X)$ for $p = 1$ with the isometric isomorphism given by $A \mapsto (Ae_n)$, where (e_n) is the standard unit vector basis of ℓ_{p^*} (c_0 when $p = 1$).*

Example 2.2. Let $1 \le p \le \infty$. The unit vector basis (e_n) of $\ell_{p^*} \subset \ell_p^*$ (c_0 when $p = 1$) is weakly p-summable in ℓ_{p^*} (c_0 when $p = 1$) with weakly p-summable norm one, i.e., $(e_n) \in S_{\ell_p^w(\ell_{p^*})}$ ($(e_n) \in S_{\ell_1^w(c_0)}$ when $p = 1$).

Proof. Since $(e_n) \in \ell_\infty^w(\ell_1) = \ell_\infty(\ell_1)$ and $\|(e_n)\|_\infty = 1$, we may assume that $1 \le p < \infty$. According to Proposition 2.1 the isometric isomorphism from $\mathcal{L}(\ell_{p^*}, \ell_{p^*})$ onto $\ell_p^w(\ell_{p^*})$ gives us the correspondence $I_{\ell_{p^*}} \mapsto (e_n)$ when $1 < p < \infty$. Therefore $\|(e_n)\|_p^w = \|I_{\ell_{p^*}}\| = 1$ and $(e_n) \in S_{\ell_p^w(\ell_p^*)}$. For $p = 1$, the isometric isomorphism from $\mathcal{L}(c_0, c_0)$ onto $\ell_1^w(c_0)$ and the correspondence $I_{c_0} \mapsto (e_n)$ similarly yield $\|(e_n)\|_1^w = 1$. □

2.2 Operator ideals

Let X and Y be Banach spaces.

It is well known that $T \in \mathcal{F}(X,Y)$ if and only if there exist functionals $x_1^*, ..., x_n^* \in X^*$ and vectors $y_1, ..., y_n \in Y$ such that

$$Tx = \sum_{k=1}^{n} x_k^*(x) y_k, \ x \in X.$$

Following the standard tensor-product notation, for $x^* \in X^*$ and $y \in Y$ the operator $x \mapsto x^*(x)y$, $x \in X$, is denoted by $x^* \otimes y$. It is clear that $x^* \otimes y$ is a rank one operator if and only if $x^* \neq 0$ and $y \neq 0$. Therefore, $T \in \mathcal{F}(X,Y)$ if and only if T can be represented as a finite sum of rank one operators

$$T = \sum_{k=1}^{n} x_k^* \otimes y_k.$$

If for $T \in \mathcal{L}(X,Y)$ we write

$$T = \sum_{k=1}^{\infty} x_k^* \otimes y_k,$$

with $x_k^* \in X^*$ and $y_k \in Y$, then this means that

$$Tx = \sum_{k=1}^{\infty} x_k^*(x) y_k, \ x \in X.$$

The following definitions and results on operator ideals are mainly due to [Pie80]. Let $\mathcal{L}$ be the class of all bounded linear operators between arbitrary Banach spaces.

Definition 2.3 (see, e.g., [Pie07, 2.6.6.1])**.** An *operator ideal* $\mathcal{A}$ is a subclass of $\mathcal{L}$ such that the following holds:

1° the *components* $\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$ are linear subspaces of $\mathcal{L}(X,Y)$ for all Banach spaces X and Y,

2° $\mathcal{A}$ contains all rank one operators,

3° if X, Y, Z, W are Banach spaces and $A \in \mathcal{L}(X,Y)$, $T \in \mathcal{A}(Y,Z)$, $B \in \mathcal{L}(Z,W)$, then $BTA \in \mathcal{A}(X,W)$.

Example 2.4. The classes of all bounded linear operators $\mathcal{L}$, compact operators $\mathcal{K}$, weakly compact operators $\mathcal{W}$, approximable operators $\overline{\mathcal{F}}$, and finite-rank operators $\mathcal{F}$ are operator ideals.

Recall (see, e.g., [Pie07, 3.2.5.1]) that a map $\|\cdot\|$ from a linear space X to non-negative numbers is a *quasi-norm* if the following conditions are satisfied:

(Q1) $\|x\| = 0$ implies $x = 0$,

(Q2) there exists $\kappa \geq 0$ such that $\|x + y\| \leq \kappa(\|x\| + \|y\|)$ for $x, y \in X$,

(Q3) $\|\lambda x\| = |\lambda| \, \|x\|$ for $x \in X$, $\lambda \in \mathbb{K}$.

A quasi-norm is called an *s-norm*, $0 < s \leq 1$, if (Q2) is replaced with the *s-triangle inequality*

$$\|x + y\|^s \leq \|x\|^s + \|y\|^s , \; x, y \in X$$

(see, e.g., [Pie07, 3.2.5.2]). For $\kappa = 2^{1/s-1}$, (Q2) follows from the s-triangle inequality.

A 1-norm is just a norm, and an s-norm is also a t-norm if $0 < t < s \leq 1$.

An s-norm induces a metric topology on X that can be defined by $d(x, y) = \|x - y\|^s$. The space X is said to be *s-Banach* if it is complete for this metric (see, e.g., [Kal03])). In case of a quasi-norm, X is endowed with a metrizable topology where the sets $\{x \in X : \|x\| < \varepsilon\}$, $\varepsilon > 0$, form a base of neighbourhoods of 0.

Note that every quasi-norm is equivalent to an s-norm, where $\kappa = 2^{1/s-1}$ (see, e.g., [Pie07, 3.2.5.4]). It is well known that every s-norm is continuous in its topology whereas a quasi-norm might not be (see, e.g., [Pie80, 6.1.9]).

Definition 2.5 (see, e.g., [Pie07, 6.3.2.4])**.** Let $0 < s \leq 1$. An operator ideal $\mathcal{A}$ is called an *s-Banach operator ideal* and denoted by $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ if there exists a non-negative function $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}$, called an *s-norm*, such that

1° all components $\mathcal{A}(X, Y) = (\mathcal{A}(X, Y), \|\cdot\|_{\mathcal{A}})$ are s-Banach spaces,

2° $\|x^* \otimes y\|_{\mathcal{A}} = \|x^*\|\|y\|$ for all rank one operators $x^* \otimes y$,

3° if X, Y, Z, W are Banach spaces and $A \in \mathcal{L}(X, Y), T \in \mathcal{A}(Y, Z), B \in \mathcal{L}(Z, W)$, then $\|BTA\|_{\mathcal{A}} \leq \|B\|\|T\|_{\mathcal{A}}\|A\|$.

A 1-Banach operator ideal is called a *Banach operator ideal.*

An s-Banach operator ideal is a special case of quasi-Banach operator ideals (for quasi-Banach operator ideals, see [Pie80, 6.1.3 and 6.2.1]).

Example 2.6. The operator ideals $\mathcal{L}$, $\mathcal{K}$, $\overline{\mathcal{F}}$, and $\mathcal{W}$ are Banach operator ideals with respect to the operator norm. The operator ideal $\mathcal{F}$ can not be made even a quasi-Banach operator ideal (see, e.g., [Pie07, 6.3.2.4]).

Example 2.7 (see [Pie80, 18.1.2])**.** Let $0 < t \leq \infty$, $1 \leq u, v \leq \infty$, and $1/u + 1/v \leq 1 + 1/t$. An operator $T \in \mathcal{L}(X, Y)$ is called (t, u, v)-*nuclear* if

$$T = \sum_{n=1}^{\infty} \delta_n x_n^* \otimes y_n \tag{2.1}$$

with $(\delta_n) \in \ell_t$ ($(\delta_n) \in c_0$ when $t = \infty$), $(x_n^*) \in \ell_{v^*}^w(X^*)$, and $(y_n) \in \ell_{u^*}^w(Y)$. Denote

$$\|T\|_{\mathcal{N}_{(t,u,v)}} = \inf \|(\delta_n)\|_t \|(x_n^*)\|_{v^*}^w \|(y_n)\|_{u^*}^w,$$

where the infimum is taken over all (t, u, v)-*nuclear representations* (2.1) of T. Set

$$\frac{1}{s} = \frac{1}{t} + \frac{1}{u^*} + \frac{1}{v^*}.$$

Then $0 < s \leq 1$ and $(\mathcal{N}_{(t,u,v)}, \|\cdot\|_{\mathcal{N}_{(t,u,v)}})$ is an s-Banach operator ideal.

Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals. Recall that the *inclusion* $\mathcal{A} \subset \mathcal{B}$ means that $\mathcal{A}(X, Y) \subset \mathcal{B}(X, Y)$ for all Banach spaces X and Y. The *equality* of $\mathcal{A}$ and $\mathcal{B}$ is denoted as $\mathcal{A} = \mathcal{B}$ and means that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$.

Similar terminology is used for an s-Banach operator ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and a t-Banach operator ideal $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. The inclusion is denoted by $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \subset (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ or, shortly, $\mathcal{A} \subset \mathcal{B}$ if in addition $\|T\|_{\mathcal{A}} \geq \|T\|_{\mathcal{B}}$ for all Banach spaces X and Y, and for all $T \in \mathcal{A}(X, Y)$. Two u-Banach operator ideals $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ are *equal* if $\mathcal{A} = \mathcal{B}$ as operator ideals and also $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{B}}$ for all Banach spaces X and Y, and for all $T \in \mathcal{A}(X, Y)$.

Proposition 2.8 (see [Pie80, 18.1.5])**.** *Let $t_1 \leq t_2$, $u_1 \leq u_2$, and $v_1 \leq v_2$. If*

$$\frac{1}{u_1} + \frac{1}{v_1} - \frac{1}{t_1} \leq \frac{1}{u_2} + \frac{1}{v_2} - \frac{1}{t_2},$$

then $(\mathcal{N}_{(t_1,u_1,v_1)}, \|\cdot\|_{\mathcal{N}_{(t_1,u_1,v_1)}}) \subset (\mathcal{N}_{(t_2,u_2,v_2)}, \|\cdot\|_{\mathcal{N}_{(t_2,u_2,v_2)}})$.

2.3 Constructing new operator ideals

For an operator ideal $\mathcal{A}$ there are several ways to produce new operator ideals.

The components of $\mathcal{A}^{\text{dual}}$, the *dual operator ideal* of $\mathcal{A}$, are defined by

$$\mathcal{A}^{\text{dual}}(X,Y) = \{T \in \mathcal{L}(X,Y) : T^* \in \mathcal{A}(Y^*, X^*)\}.$$

If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an s-Banach operator ideal, then $\mathcal{A}^{\text{dual}}$ is also an s-Banach operator ideal with $\|T\|_{\mathcal{A}^{\text{dual}}} = \|T^*\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{\text{dual}}(X,Y)$ (see, e.g., [Pie07, 6.3.2.6]).

The components of $\mathcal{A}^{\text{inj}}$, the *injective hull* of $\mathcal{A}$, are defined by

$$\mathcal{A}^{\text{inj}}(X,Y) = \{T \in \mathcal{L}(X,Y) : J_Y T \in \mathcal{A}(X, \ell_\infty(B_{Y^*}))\}.$$

If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an s-Banach operator ideal, then $\mathcal{A}^{\text{inj}}$ is also an s-Banach operator ideal with $\|T\|_{\mathcal{A}^{\text{inj}}} = \|J_Y T\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{\text{inj}}(X,Y)$ (see, e.g., [Pie07, 6.3.2.7]).

The components of $\mathcal{A}^{\text{sur}}$, the *surjective hull* of $\mathcal{A}$ are defined by

$$\mathcal{A}^{\text{sur}}(X,Y) = \{T \in \mathcal{L}(X,Y) : TQ_X \in \mathcal{A}(\ell_1(B_X), Y)\}.$$

If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an s-Banach operator ideal, then $\mathcal{A}^{\text{sur}}$ is also an s-Banach operator ideal with $\|T\|_{\mathcal{A}^{\text{sur}}} = \|TQ_X\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{\text{sur}}(X,Y)$ (see, e.g., [Pie07, 6.3.2.7]).

Clearly $\mathcal{A} \subset \mathcal{A}^{\text{inj}}$ and $\mathcal{A} \subset \mathcal{A}^{\text{sur}}$ hold, and in the case of an s-Banach operator ideal $\mathcal{A}$, these inclusions are in the sense of s-Banach operator ideals (see, e.g., [Pie80, 8.4.3] and [Pie80, 8.5.3], respectively).

If $\mathcal{A} = \mathcal{A}^{\text{sur}}$ (respectively, $\mathcal{A} = \mathcal{A}^{\text{inj}}$) as ($s$-Banach) operator ideals, then $\mathcal{A}$ is said to be a *surjective* (respectively, an *injective*) (s-Banach) operator ideal.

Example 2.9. The operator ideal $\mathcal{F}$ is injective, surjective, and $\mathcal{F}^{\text{dual}} = \mathcal{F}$ (see, e.g., [Pie80, 4.6.12], [Pie80, 4.7.12], and [Pie80, 4.4.7], respectively). Also $\mathcal{K}^{\text{dual}} = \mathcal{K}$, $\overline{\mathcal{F}}^{\text{dual}} = \overline{\mathcal{F}}$, and $\mathcal{W}^{\text{dual}} = \mathcal{W}$ as Banach operator ideals (see [DF93, 9.9]). The Banach operator ideals $\mathcal{K}$ and $\mathcal{W}$ are injective and surjective, but $\overline{\mathcal{F}}^{\text{inj}} = \mathcal{K} = \overline{\mathcal{F}}^{\text{sur}}$ (see [DF93, 9.7] and [DF93, 9.8]).

The following properties of $\mathcal{A}^{\text{dual}}$, $\mathcal{A}^{\text{inj}}$, and $\mathcal{A}^{\text{sur}}$ are from [Pie80, 8.2.3], [Pie80, 8.4.3], and [Pie80, 8.5.3], respectively.

Proposition 2.10. *Let $\mathcal{A}$ be an (s-Banach) operator ideal. Then $\mathcal{A}^{\text{inj}\,\text{inj}} = \mathcal{A}^{\text{inj}}$ and $\mathcal{A}^{\text{sur}\,\text{sur}} = \mathcal{A}^{\text{sur}}$ as (s-Banach) operator ideals.*

Proposition 2.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be (s-Banach) operator ideals. If $\mathcal{A} \subset B$, then $\mathcal{A}^{\text{dual}} \subset B^{\text{dual}}$, $\mathcal{A}^{\text{inj}} \subset B^{\text{inj}}$, and $\mathcal{A}^{\text{sur}} \subset B^{\text{sur}}$ as (s-Banach) operator ideals.*

It is well known, that every $T \in \mathcal{L}(X, Y)$ admits the canonical factorization

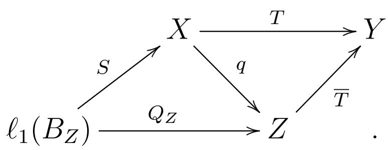
$$T = \overline{T}q,$$

where $q : X \to X/\ker T$ is the quotient map and $\overline{T} \in \mathcal{L}(X/\ker T, Y)$ is an injective operator that we shall call the *injective associate of* T.

Proposition 2.12. *Let* X *and* Y *be Banach spaces, and let* $\mathcal{A}$ *be an* s*-Banach operator ideal. If* $T \in \mathcal{A}(X, Y)$*, then* $\overline{T} \in \mathcal{A}^{\text{sur}}(X, Y)$ *and*

$$\|T\|_{\mathcal{A}^{\text{sur}}} \leq \left\|\overline{T}\right\|_{\mathcal{A}^{\text{sur}}} \leq \|T\|_{\mathcal{A}}.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Denote $Z = X/\ker T$. We can use the metric lifting property of $\ell_1(B_Z)$ for the surjection Q_Z (see, e.g., [Pie80, C.3.5 and C.3.6]). This property gives us an operator $S \in \mathcal{L}(\ell_1(B_Z), X)$ such that $Q_Z = qS$ and $\|S\| \leq (1+\varepsilon)\|Q_Z\| = 1+\varepsilon$. And we have the commutative diagram



Now we have

$$\overline{T}Q_Z = \overline{T}qS = TS \in \mathcal{A}(\ell_1(B_Z), Y).$$

This means that $\overline{T} \in \mathcal{A}^{\text{sur}}(X, Y)$. Moreover,

$$\|T\|_{\mathcal{A}^{\text{sur}}} = \left\|\overline{T}q\right\|_{\mathcal{A}^{\text{sur}}} \leq \|\overline{T}\|_{\mathcal{A}^{\text{sur}}} = \|\overline{T}Q_Z\|_{\mathcal{A}} = \|TS\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}.$$

Since this holds for every $\varepsilon > 0$, we have $\|T\|_{\mathcal{A}^{\text{sur}}} \leq \left\|\overline{T}\right\|_{\mathcal{A}^{\text{sur}}} \leq \|T\|_{\mathcal{A}}$, as desired. □

2.4 Products of operator ideals

Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals.

The *product* $\mathcal{A} \circ \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is an operator ideal that consists of all operators $T \in \mathcal{L}(X, Y)$ for which there are a Banach space Z and operators $A \in \mathcal{A}(X, Z)$ and $B \in \mathcal{B}(Z, Y)$ such that $T = BA$.

The product $(\mathcal{A} \circ \mathcal{B}, \|\cdot\|_{\mathcal{A} \circ \mathcal{B}})$ of an s-Banach operator ideal $\mathcal{A}$ and a t-Banach operator ideal $\mathcal{B}$ is a u-Banach operator ideal, where

$$u = \frac{st}{s+t},$$

whose components $\mathcal{A}\circ\mathcal{B}(X,Y)$ consist of all operators $T \in \mathcal{L}(X,Y)$ for which there are a Banach space Z and operators $A \in \mathcal{A}(X,Z)$ and $B \in \mathcal{B}(Z,Y)$ such that $T = BA$; the u-norm of T is given by

$$\|T\|_{\mathcal{B}\circ\mathcal{A}} = \inf \|B\|_{\mathcal{B}}\|A\|_{\mathcal{A}},$$

where the infimum is taken over all possible factorizations of T as above (see [Pie80, 7.1.2, the proof of the theorem]).

The product of two Banach operator ideals (i.e., the case when $s = t = 1$), in general, is not a Banach operator ideal but only a 1/2-Banach operator ideal (see [Oer03]), so that, in general, $u = st/(s+t)$ is the best "exponent". But it may happen that in some special cases $\|\cdot\|_{\mathcal{B}\circ\mathcal{A}}$ is actually a v-norm for some $v > u$. Oerter [Oer03, Proposition 3.1] has proved that if one of the operator ideals $\mathcal{A}$ or $\mathcal{B}$ is an s-Banach operator ideal and the other is a *closed* (or *classical*) Banach operator ideal, i.e., a Banach operator ideal with respect to the usual operator norm, then $\mathcal{A}\circ\mathcal{B}$ is an s-Banach operator ideal. For instance, $\overline{\mathcal{F}}\circ\mathcal{A}$, $\mathcal{A}\circ\overline{\mathcal{F}}$, $\mathcal{K}\circ\mathcal{A}$, and $\mathcal{A}\circ\mathcal{K}$ are s-Banach operator ideals whenever $\mathcal{A}$ is an s-Banach operator ideal.

To our best knowledge, the next result and the following Corollary 2.14 are of folklore and their proofs were unpublished.

Proposition 2.13 (cf. [ALO12, proof of Proposition 4.7])**.** *Let $\mathcal{A}$ be an s-Banach operator ideal and let $\mathcal{B}$ be a t-Banach operator ideal. Then $(\mathcal{B}\circ\mathcal{A})^{\mathrm{sur}} \subset \mathcal{B}^{\mathrm{sur}}\circ\mathcal{A}^{\mathrm{sur}}$ as u-Banach operator ideals, where $1/u = 1/s + 1/t$.*

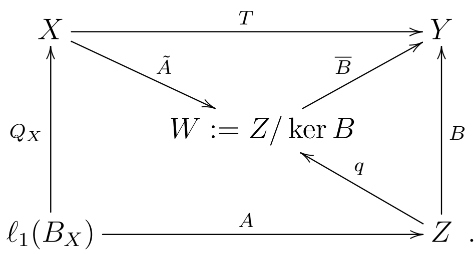
Proof. Let $1/u = 1/s + 1/t$. Then $\mathcal{B}\circ\mathcal{A}$ is a u-Banach operator ideal, and also $(\mathcal{B}\circ\mathcal{A})^{\mathrm{sur}}$ is. Similarly we get that $\mathcal{B}^{\mathrm{sur}}\circ\mathcal{A}^{\mathrm{sur}}$ is a u-Banach operator ideal.

Let an operator T belong to the component $(\mathcal{B}\circ\mathcal{A})^{\mathrm{sur}}(X,Y)$ where X and Y are Banach spaces. This means that $TQ_X \in \mathcal{A}(\ell_1(B_X),Y)$ and $\|T\|_{(\mathcal{B}\circ\mathcal{A})^{\mathrm{sur}}} = \|TQ_X\|_{\mathcal{B}\circ\mathcal{A}}$. Then for every $\varepsilon > 0$, there are a Banach space Z and operators $A \in \mathcal{A}(\ell_1(B_X),Z)$ and $B \in \mathcal{B}(Z,Y)$ such that $TQ_X = BA$ and

$$(1+\varepsilon)\|TQ_X\|_{\mathcal{B}\circ\mathcal{A}} \geq \|B\|_{\mathcal{B}}\|A\|_{\mathcal{A}}.$$

Denote $W := Z/\ker B$. Let $\overline{B} : W \to Y$ be the injective associate of B, then $B = \overline{B}q$, where $q : Z \to W$ is the quotient mapping. Define an operator $\tilde{A} : X \to W$ through the equation $\tilde{A}x = qA\lambda$, where $x \in X$ satisfies $x = Q_X\lambda$

for some $\lambda \in \ell_1(B_X)$. The operators are represented in the following diagram:



The definition of $\tilde{A}$ is correct. Indeed, for $x = Q_X\lambda = Q_X\mu$ we have

$$\overline{B}qA\lambda = BA\lambda = TQ_X\lambda = Tx = TQ_X\mu = BA\mu = \overline{B}qA\mu, \tag{2.2}$$

thus $qA\lambda = qA\mu$ due to the injectivity of $\overline{B}$. From the definition of $\tilde{A}$ it is clear that

$$\tilde{A}Q_X = qA. \tag{2.3}$$

The linearity of $\tilde{A}$ follows from the linearity of A, q and Q_X. For the boundedness of $\tilde{A}$ observe that for every $x_0 \in B_X$ we can choose $\lambda_0 \in S_{\ell_1(B_X)}$ as follows:

$$\lambda_0 = (\lambda_x)_{x\in B_X} = \begin{cases} 1 \text{ if } x = x_0, \\ 0 \text{ otherwise.} \end{cases}$$

Then $Q_X\lambda_0 = x_0$, hence $\tilde{A}x_0 = \tilde{A}Q_X\lambda_0 = qA\lambda_0$. Therefore $\|\tilde{A}x_0\| \le \|A\|$. Hence $\|\tilde{A}\| \le \|A\|$ and $\tilde{A} \in \mathcal{L}(X, W)$.

From (2.2) we also see that

$$T = \overline{B}\,\tilde{A}.$$

To show that $T \in \mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}}(X, Y)$ we have to prove that $\tilde{A} \in \mathcal{A}^{\text{sur}}(X, W)$ and $\overline{B} \in \mathcal{B}^{\text{sur}}(W, Y)$.

By Proposition 2.12, $\overline{B}$ belongs to $\mathcal{B}^{\text{sur}}(W, Y)$ and $\|\overline{B}\|_{\mathcal{B}^{\text{sur}}} \le \|B\|_{\mathcal{B}}$.

Secondly, from (2.3) we see that $\tilde{A}Q_X \in \mathcal{A}(\ell_1(B_X), W)$. Therefore $\tilde{A} \in \mathcal{A}^{\text{sur}}(X, W)$ and

$$\|\tilde{A}\|_{\mathcal{A}^{\text{sur}}} = \|qA\|_{\mathcal{A}} \le \|A\|_{\mathcal{A}}.$$

Thus, $T \in \mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}}(X, Y)$ as desired.

Finally, the above inequalities imply that

$$\begin{aligned} \|T\|_{\mathcal{B}^{\text{sur}}\circ\mathcal{A}^{\text{sur}}} &\le \|\overline{B}\|_{\mathcal{B}^{\text{sur}}}\|\tilde{A}\|_{\mathcal{A}^{\text{sur}}} \le \|B\|_{\mathcal{B}}\|A\|_{\mathcal{A}} \\ &\le (1+\varepsilon)\|TQ_X\|_{\mathcal{B}\circ\mathcal{A}} = (1+\varepsilon)\|T\|_{(\mathcal{B}\circ\mathcal{A})^{\text{sur}}} \end{aligned}$$

holds for every $\varepsilon > 0$. Therefore we have

$$\|T\|_{(\mathcal{B}\circ\mathcal{A})^{\text{sur}}} \geq \|T\|_{\mathcal{B}^{\text{sur}}\circ\mathcal{A}^{\text{sur}}}$$

as desired. □

Corollary 2.14. *Let $\mathcal{A}$ be an s-Banach operator ideal and let $\mathcal{B}$ be a t-Banach operator ideal. Then $(\mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}})^{\text{sur}} = \mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}}$ as u-Banach operator ideals, where $1/u = 1/s + 1/t$.*

Proof. Every v-Banach operator ideal $\mathcal{C}$ is contained in its surjective hull $\mathcal{C}^{\text{sur}}$ and $\mathcal{C}^{\text{sur}\,\text{sur}} = \mathcal{C}^{\text{sur}}$ as v-Banach operator ideals (see Section 2.3). Therefore $\mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}} \subset (\mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}})^{\text{sur}}$ and, by Proposition 2.13,

$$(\mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}})^{\text{sur}} \subset \mathcal{B}^{\text{sur}\,\text{sur}} \circ \mathcal{A}^{\text{sur}\,\text{sur}} = \mathcal{B}^{\text{sur}} \circ \mathcal{A}^{\text{sur}}$$

as u-Banach operator ideals. □

2.5 The operator $\Phi_{(x_n)}$

This section is based on [ALO12].

Let X be a Banach space. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$.

It is well known and easy to see that every $(x_n) \in \ell_p(X)$ defines an operator $\Phi_{(x_n)} \in \mathcal{L}(\ell_r, X)$ through the equality

$$\Phi_{(x_n)}(a_n) = \sum_{n=1}^{\infty} a_n x_n, \ (a_n) \in \ell_r.$$

Let (e_n) be the unit vector basis of $\ell_{r^*} \subset (\ell_r)^*$ (c_0 when $r = 1$), considered as coordinate functionals for ℓ_r. Then we clearly have

$$\Phi_{(x_n)} = \sum_{n=1}^{\infty} e_n \otimes x_n.$$

Proposition 2.15. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let X be a Banach space and let $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ when $p = \infty$). Then the operator $\Phi_{(x_n)} : \ell_r \to X$ is approximable, i.e., $\Phi_{(x_n)} \in \overline{\mathcal{F}}(\ell_r, X)$.*

Proof. Denoting $(x_n)_{n\leq m} := (x_1, ..., x_m, 0, 0, ...)$, we have

$$\Phi_{(x_n)_{n\leq m}} = \sum_{n=1}^{m} e_n \otimes x_n.$$

Now, since $1 \leq r \leq p^*$,

$$\|\Phi_{(x_n)} - \Phi_{(x_n)_{n\leq m}}\| = \sup_{(a_n)\in B_{\ell_r}} \left\| \sum_{n=m+1}^{\infty} a_n x_n \right\| \leq \sup_{(a_n)\in B_{\ell_r}} \sum_{n=m+1}^{\infty} |a_n| \|x_n\|$$

$$\leq \begin{cases} \sup_{(a_n)\in B_{\ell_r}} \sup_{n\geq m+1} |a_n| \sum_{n=m+1}^{\infty} \|x_n\| \text{ if } p = 1, \\ \sup_{(a_n)\in B_{\ell_r}} \left(\sum_{n=m+1}^{\infty} |a_n|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{n=m+1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} \text{ if } 1 < p < \infty, \\ \sup_{(a_n)\in B_{\ell_1}} \sup_{n\geq m+1} \|x_n\| \sum_{n=m+1}^{\infty} |a_n| \text{ if } p = \infty \end{cases}$$

$$\leq \begin{cases} \left(\sum_{n=m+1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty, \\ \sup_{n\geq m+1} \|x_n\| \text{ if } p = \infty \end{cases} \to_m 0.$$

Thus $\Phi_{(x_n)_{n\leq m}}$ converges to $\Phi_{(x_n)}$ in $\mathcal{L}(\ell_r, X)$ as $m \to \infty$. □

Since $\overline{\mathcal{F}} \subset \mathcal{K}$, we immediately get that $\Phi_{(x_n)} \in \mathcal{K}(\ell_r, X)$ for $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0$ when $p = \infty$). Even more is true.

Proposition 2.16 (see [ALO12, p. 149])**.** *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let X be a Banach space and let $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ when $p = \infty$). Then the operator $\Phi_{(x_n)} : \ell_r \to X$ is $(p, 1, r^*)$-nuclear, i.e., $\Phi_{(x_n)} \in \mathcal{N}_{(p,1,r^*)}$, and $\left\|\Phi_{(x_n)}\right\|_{\mathcal{N}_{(p,1,r^*)}} \leq \|x_n\|_p$.*

Proof. In Example 2.2 we showed that $(e_n) \in S_{\ell_r^w(\ell_{r^*})}$ ($(e_n) \in S_{\ell_1^w(c_0)}$ when $r = 1$). Therefore, from the representation

$$\Phi_{(x_n)} = \sum_{n=1}^{\infty} e_n \otimes x_n$$

and from

$$e_n \otimes x_n = \|x_n\| e_n \otimes (\|x_n\|^{-1} x_n),$$

it is clear that $(\|x_n\|) \in \ell_p$ (respectively, $(\|x_n\|) \in c_0$ when $p = \infty$), $(e_n) \in \ell_r^w(\ell_{r^*}) \subset \ell_r^w(\ell_r^*)$ (respectively, $(e_n) \in \ell_1^w(c_0) \subset \ell_1^w(\ell_\infty)$ when $r = 1$), and $(\|x_n\|^{-1} x_n) \in \ell_\infty(X) = \ell_\infty^w(X)$. Thus (cf. Example 2.7)

$$\Phi_{(x_n)} \in \mathcal{N}_{(p,1,r^*)}(\ell_r, X)$$

and

$$\|\Phi_{(x_n)}\|_{\mathcal{N}_{(p,1,r^*)}} \leq \|(x_n)\|_p,$$

as desired. □

The key observation for our approach in Chapter 4 is that $\overline{\Phi}_{(x_n)}$, the injective associate of $\Phi_{(x_n)}$, belongs to $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$. The following is immediate from Propositions 2.12 and 2.16.

Proposition 2.17. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let X be a Banach space and let $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ when $p = \infty$). Then $\overline{\Phi}_{(x_n)} \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(Z, X)$, where $Z = \ell_r / \ker \Phi_{(x_n)}$, and $\|\overline{\Phi}_{(x_n)}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \leq \|(x_n)\|$.*

Chapter 3

Compact operators defined by ℓ_p spaces: (p, r)-compact operators

In this chapter, we extend the notion of some well-known forms of relatively compact sets to relatively (p, r)-compact subsets of a Banach space. We go on to define the operator ideal of (p, r)-compact operators and describe its structure as an operator ideal. This chapter is mainly based on [ALO12].

3.1 Relatively (p, r)-compact subsets

Let X be a Banach space. Similarly to [DFL$^+$12] we call the next result the Grothendieck compactness principle (see [Gro55] or, e.g., [LT77, p. 30]).

Theorem 3.1 (Grothendieck). *A subset K of X is relatively compact if and only if there exists $(x_n) \in c_0(X)$ such that $K \subset \overline{\text{conv}}(x_n)$, the closed convex hull of the sequence (x_n).*

In the 1980s a stronger form of relative compactness was given. If one replaces $c_0(X)$ with the space $\ell_p(X)$, for some fixed real number $p \geq 1$, then one obtains a stronger form of relative compactness. This form of compactness was occasionally considered in the 1980s by Reinov [Rei84] and Bourgain and Reinov [BR85] in the study of approximation properties of order $s \leq 1$. Let us say, in this case, that K is *relatively p-compact in the sense of Bourgain–Reinov.*

Let $p \geq 1$ be a real number. The *p-convex hull* of a sequence $(x_n) \in \ell_p(X)$ is defined as

$$p\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_{p^*}} \right\}.$$

In 2002, another strong form of compactness (but weaker than the Bourgain–Reinov one) was introduced by Sinha and Karn [SK02] through the requirement that $K \subset p\text{-conv}(x_n)$ for some $(x_n) \in \ell_p(X)$. In this case, let us say that K is *relatively p-compact in the sense of Sinha–Karn.* Remark that the special case $p = 1$ was considered already in 1973 by Stephani [Ste73, Section 4] under the name of nuclearity (of sets) (see also Remark 6.8).

Definition 3.2. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We call

$$(p, r)\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r} \right\}$$

the (p, r)*-convex hull* of a sequence $(x_n) \in \ell_p(X)$.

It is a well-known folklore fact that, if $(x_n) \in \ell_\infty(X)$, then $(\infty, 1)\text{-conv}(x_n)$ is exactly $\overline{\text{absconv}}(x_n)$, the closed absolutely convex hull of (x_n). We include a proof for completeness (in the case when $(x_n) \in c_0(X)$, a proof is sketched in exercises in [FHH+01, pp. 22, 33]).

Proposition 3.3. *Let* $(x_n) \in \ell_\infty(X)$. *Then* $(\infty, 1)\text{-conv}(x_n) = \overline{\text{absconv}}(x_n)$.

Proof. Since

$$\text{absconv}(x_n) = \left\{ \sum_{n=1}^{m} a_n x_n : \sum_{n=1}^{m} |a_n| \leq 1, m \in \mathbb{N} \right\},$$

we clearly have

$$\text{absconv}(x_n) \subset (\infty, 1)\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : \sum_{n=1}^{\infty} |a_n| \leq 1 \right\} \subset \overline{\text{absconv}}(x_n).$$

Therefore, it remains to show that $(\infty, 1)\text{-conv}(x_n)$ is a closed subset. For this end, we prefer to apply an idea which will be also used in the proof of Theorem 6.5 below. Namely, observe that $(\infty, 1)\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_1})$ (the operator $\Phi_{(x_n)} \in \mathcal{L}(\ell_1, X)$ was defined in Section 2.5). The subset $\Phi_{(x_n)}(B_{\ell_1})$ is obviously convex. It is also weakly compact. Indeed, $B_{\ell_\infty} = B_{\ell_1^*}$ is weak* compact due to the well-known Banach–Alaoglu theorem and $\Phi_{(x_n)} \in \mathcal{L}(\ell_1^*, X)$ is weak*-to-weakly continuous (because $\Phi_{(x_n)}^*(X^*) \subset \ell_1$). Hence, as weakly closed convex sets are closed in the norm topology, $\Phi_{(x_n)}(B_{\ell_r})$ is closed. $\square$

We extend these previous forms of compactness as follows.

Definition 3.4. We say that a subset K of X is *relatively* (p, r)*-compact* if $K \subset (p, r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p(X)$ (where $(x_n) \in c_0(X)$ if $p = \infty$).

As for the "extremal" cases, the $(p, 1)$-compactness is precisely the Bourgain–Reinov p-compactness, and the (p, p^*)-compactness is precisely the Sinha–Karn p-compactness with the $(1, \infty)$-compactness being the Stephani nuclearity. According to Grothendieck compactness principle, the $(\infty, 1)$-compactness coincides with the compactness (because $(\infty, 1)\text{-conv}(x_n)$ is precisely the closed absolutely convex hull of (x_n)).

It is convenient to look at (p, r)-convex hulls in the following way.

As every $(x_n) \in \ell_p(X)$ defines the operator $\Phi_{(x_n)} : \ell_r \to X$ (see Section 2.5), it is clear that

$$(p, r)\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_r}). \tag{3.1}$$

Remark 3.5. Concerning $1\text{-conv}(x_n) = (1, \infty)\text{-conv}(x_n)$ with $(x_n) \in \ell_1(X)$, we have $\Phi_{(x_n)} : \ell_\infty \to X$ and $1\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_\infty})$. This is the definition of the 1-convex hull of (x_n) as, e.g., in [AO12, CK10, DOPS09, PD11, Pie14]. But, e.g., in [ALO12, AO15, DPS10b, SK02, SK08], the 1-convex hull of (x_n) is defined as $\Phi_{(x_n)}(B_{c_0})$. Since $\Phi_{(x_n)}(B_{c_0}) \subset \Phi_{(x_n)}(B_{\ell_\infty}) \subset \Phi_{(\lambda_n x_n)}(B_{c_0})$ whenever $1 \le \lambda_n \to \infty$ is chosen such that $(\lambda_n x_n) \in \ell_1(X)$, both definitions of the 1-convex hulls yield the same notions of 1-compactness (see Definitions 3.4 and 3.7) and 1-null sequences (see Definitions 6.1 and 6.16). We prefer the definition of the 1-convex hull of (x_n) as $\Phi_{(x_n)}(B_{\ell_\infty})$, because this set is closed (even weakly compact) (see the proof of Theorem 6.5), and also for the notational purpose.

Definition 3.4 means by (3.1) that a subset K of X is relatively (p, r)-compact if and only if

$$K \subset \Phi_{(x_n)}(B_{\ell_r})$$

for some $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ when $p = \infty$); in particular, $\Phi_{(x_n)}(B_{\ell_r})$ itself is relatively (p, r)-compact.

Theorem 3.6. *Let X be a Banach space. Let $1 \le p \le q \le \infty$, $1 \le r \le p^*$, and $1 \le s \le q^*$. Let*

$$\frac{1}{q} + \frac{1}{s} \le \frac{1}{p} + \frac{1}{r}. \tag{3.2}$$

If a subset of X is relatively (p, r)-compact, then it is relatively (q, s)-compact.

Proof. First, let $1 \le p \le q = \infty$. Then $s = 1$. As already mentioned, the $(\infty, 1)$-compactness coincides with the usual compactness. Thus, we have to show that relatively (p, r)-compact sets are relatively compact. For this it is sufficient to show that $\Phi_{(x_n)}(B_{\ell_r})$ is relatively compact. But the latter immediately follows from the compactness of $\Phi_{(x_n)}$ (see Section 2.5).

If $1 \le p \le q < \infty$ and $r \le s$, then the assertion in clear as $\ell_p(X) \subset \ell_q(X)$ and $B_{\ell_r} \subset B_{\ell_s}$, hence

$$\Phi_{(x_n)}(B_{\ell_r}) \subset \Phi_{(x_n)}(B_{\ell_s})$$

for any $(x_n) \in \ell_p(X)$.

Finally, let $1 \le p \le q < \infty$ and $s < r$. If $(x_n) \in \ell_p(X)$, then

$$(y_n) := \left(C\|x_n\|^{\frac{p-q}{q}} x_n\right) \in \ell_q(X),$$

where $C := \|(x_n)\|_p^{\frac{q-p}{q}}$.

For $(a_n) \in B_{\ell_r}$, put

$$b_n := C^{-1}\|x_n\|^{\frac{q-p}{q}} a_n,$$

so that $b_n y_n = a_n x_n$, $n \in \mathbb{N}$. We shall show that $(b_n) \in B_{\ell_s}$.

Assume first that $r < \infty$. Then $\frac{r}{s} > 1$ and $\left(\frac{r}{s}\right)^* = \frac{r}{r-s}$, and we can apply Hölder's inequality to obtain

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} |b_n|^s\right)^{\frac{1}{s}} &= \left(\sum_{n=1}^{\infty} |C^{-1}\|x_n\|^{\frac{q-p}{q}} a_n|^s\right)^{\frac{1}{s}} \\
&\le \left(\left(\sum_{n=1}^{\infty} |a_n|^r\right)^{\frac{s}{r}} \left(\sum_{n=1}^{\infty} C^{\frac{sr}{s-r}}\|x_n\|^{\frac{q-p}{q}\frac{sr}{r-s}}\right)^{\frac{r-s}{r}}\right)^{\frac{1}{s}} \\
&= \left(\sum_{n=1}^{\infty} |a_n|^r\right)^{\frac{1}{r}} \left(\sum_{n=1}^{\infty} C^{\frac{sr}{s-r}}\|x_n\|^{\frac{q-p}{q}\frac{sr}{r-s}}\right)^{\frac{r-s}{sr}} \\
&\le C^{-1}\left(\|(x_n)\|_{\frac{q-p}{q}\frac{sr}{r-s}}\right)^{\frac{q-p}{q}} \\
&\le C^{-1}\|(x_n)\|_p^{\frac{q-p}{q}} = 1,
\end{aligned}$$

because, due to (3.2),

$$p \le \frac{q-p}{q}\frac{sr}{r-s} < \infty.$$

If $r = \infty$, then $p = 1$. Hence by (3.2), $s = q^*$ and

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |b_n|^s\right)^{\frac{1}{s}} &= \left(\sum_{n=1}^{\infty} |C^{-1}\|x_n\|^{\frac{q-1}{q}} a_n|^{q^*}\right)^{\frac{1}{q^*}} \\ &\leq C^{-1}\left(\sum_{n=1}^{\infty} \|x_n\|^{\frac{q-1}{q}q^*}\right)^{\frac{1}{q^*}} \\ &= C^{-1}\left(\sum_{n=1}^{\infty} \|x_n\|\right)^{\frac{q-1}{q}} = C^{-1}C = 1.\end{aligned}$$

Since

$$\Phi_{(x_n)}(a_n) = \sum_{n=1}^{\infty} a_n x_n = \sum_{n=1}^{\infty} b_n y_n,$$

and $(b_n) \in B_{\ell_s}$, we have

$$\Phi_{(x_n)}(B_{\ell_r}) \subset \Phi_{(y_n)}(B_{\ell_s}),$$

showing that the (p, r)-compactness implies the (q, s)-compactness also in this case. □

3.2 The operator ideal of (p, r)-compact operators

Let X and Y be Banach spaces. Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$.

Recall that a linear operator $T : X \to Y$ is *compact* if $T(B_X)$ is a relatively compact subset of Y. Using relatively p-compact subsets of Y (i.e., relatively (p, p^*)-compact subsets in our terminology) instead of relatively compact ones, Sinha and Karn [SK02] obtained the concept of *p-compact operators* (in the sense of Sinha–Karn). If one uses relatively p-compact subsets of Y in the sense of Bourgain–Reinov instead of relatively compact ones, then one obtains the notion of *p-compact operators in the sense of Bourgain–Reinov.*

Following [SK02], denote the class of all p-compact operators in the sense of Sinha–Karn by $\mathcal{K}_p$. Properties of $\mathcal{K}_p$ were studied in [SK02] and, e.g., in the recent papers [AMR10, CK10, DOPS09, DPS10a, DPS10b, GLT12, LT12, Oja12b, Pie14, SK08].

We define (p, r)-compact operators in an obvious way.

Definition 3.7. Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A linear operator $T : X \to Y$ is (p,r)*-compact* if $T(B_X)$ is a relatively (p,r)-compact subset of Y.

Let us denote the class of all (p,r)-compact operators acting between arbitrary Banach spaces by $\mathcal{K}_{(p,r)}$. For the "extremal" cases, it is clear that $\mathcal{K} = \mathcal{K}_{(\infty,1)}$, $\mathcal{K}_p = \mathcal{K}_{(p,p^*)}$, and the class of p-compact operators in the sense of Bourgain–Reinov is precisely $\mathcal{K}_{(p,1)}$.

Let $T : X \to Y$ be a linear operator. From (3.1) it is clear that $T \in \mathcal{K}_{(p,r)}(X,Y)$ if and only if $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$ for some $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0$ when $p = \infty$).

The class of compact operators $\mathcal{K}$ is a well-known operator ideal. It is also proved in [SK02, Theorem 4.2] that $\mathcal{K}_p$ is an operator ideal. We extend these results to the more general case of $\mathcal{K}_{(p,r)}$.

Proposition 3.8. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. The class of (p,r)-compact operators $\mathcal{K}_{(p,r)}$ is an operator ideal.*

Proof. Let $S, T \in \mathcal{K}_{(p,r)}(X,Y)$ be such that $S(B_X) \subset \Phi_{(x_n)}(B_{\ell_r})$ and $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$ for some $(x_n), (y_n) \in \ell_p(Y)$ ($(x_n), (y_n) \in c_0(X)$ when $p = \infty$), and let $a \in \mathbb{K}$. Put

$$z_n = \begin{cases} 2^{1/r} a x_{(n+1)/2} & \text{if } n \text{ is odd,} \\ 2^{1/r} y_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

It is easy and straightforward to verify that $(z_n) \in \ell_p(Y)$ ($(z_n) \in c_0(Y)$ when $p = \infty$) and for $(a_n), (b_n) \in B_{\ell_r}$ the sequence

$$c_n = \begin{cases} 2^{-1/r} a_{(n+1)/2} & \text{if } n \text{ is odd,} \\ 2^{-1/r} b_{n/2} & \text{if } n \text{ is even,} \end{cases}$$

is in B_{ℓ_r}. Now $(aS + T)(B_X) \subset \Phi_{(z_n)}(B_{\ell_r})$, meaning that $aS + T \in \mathcal{K}_{(p,r)}(X,Y)$. This shows that $\mathcal{K}_{(p,r)}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$.

The space $\mathcal{K}_{(p,r)}(X,Y)$ contains all rank one operators. Indeed, $x^* \otimes y \in \mathcal{K}_{(p,r)}(X,Y)$ because $(x^* \otimes y)(B_X) \subset \Phi_{(z_n)}(B_{\ell_r})$ for (z_n) with $z_1 = \|x^*\| y$, $z_2 = z_3 = \ldots = 0$.

Finally, let $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(Y,W)$. Then $BTA \in \mathcal{K}_{(p,r)}(Z,W)$ because $(BTA)(B_Z) \subset \Phi_{(w_n)}(B_{\ell_r})$ for $w_n = \|A\| B y_n$, $n \in \mathbb{N}$. □

Proposition 3.9. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. The operator ideal $\mathcal{K}_{(p,r)}$ is surjective, i.e., $\mathcal{K}_{(p,r)} = \mathcal{K}^{\mathrm{sur}}_{(p,r)}$.*

Proof. Let $T \in \mathcal{K}^{\text{sur}}_{(p,r)}(X, Y)$. Denote $Z = \ell_1(B_X)$. Then $TQ_X \in \mathcal{K}_{(p,r)}(Z, Y)$ and there exists $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0(Y)$ when $p = \infty$) such that $(TQ_X)(B_Z) \subset \Phi_{(y_n)}(B_{\ell_r})$. But $B_X \subset Q_X(B_Z)$ because for $x_0 \in B_X$ we have $x_0 = Q_X(\lambda_x)$, where

$$\lambda_x = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. □

Remark 3.10. Propositions 3.8 and 3.9 follow also from the theory of generating systems of sets introduced by Stephani in [Ste80]. Indeed, in [Lil13] it was proved, that the class of all relatively (p, r)-compact subsets is a generating system (see [Ste80] for the definition of the generating system of sets) and therefore it follows from [Ste73] and [Ste80] that $\mathcal{K}_{(p,r)}$ is a surjective operator ideal.

From Theorem 3.6 we immediately get the following inclusion result.

Corollary 3.11. *Let X be a Banach space. Let $1 \le p \le q \le \infty$, $1 \le r \le p^*$, and $1 \le s \le q^*$. Let*

$$\frac{1}{q} + \frac{1}{s} \le \frac{1}{p} + \frac{1}{r}.$$

Then (p, r)-compact operators are also (q, s)-compact, i.e., $\mathcal{K}_{(p,r)} \subset \mathcal{K}_{(q,s)}$ as operator ideals

Chapter 4

$\mathcal{K}_{(p,r)}$ as an s-Banach operator ideal and its s-norm

In this chapter, we prove one of the main results. It is a description of (p,r)-compact operators $\mathcal{K}_{(p,r)}$ as operators belonging to the operator ideal $\mathcal{N}^{\text{sur}}_{(p,1,r^*)}$. This allows us to equip the operator ideal $\mathcal{K}_{(p,r)}$ with an s-norm from $\mathcal{N}_{(p,1,r^*)}$. We also show how to explicitly calculate the s-norm of an (p,r)-compact operator. This chapter is based on [ALO12].

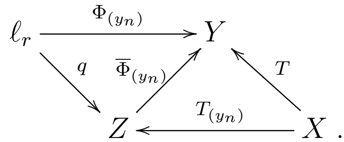
4.1 $\mathcal{K}_{(p,r)}$ as an s-Banach operator ideal

Let X and Y be Banach spaces, let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$, and let $T \in \mathcal{K}_{(p,r)}(X,Y)$. Similarly to the case of $\mathcal{K}_p = \mathcal{K}_{(p,p^*)}$ (see [SK02, pp. 20–21]), we have the *natural factorization* of T as follows.

Let $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0$ when $p = \infty$) such that $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. Denote $Z = \ell_r / \ker \Phi_{(y_n)}$. Then $\Phi_{(y_n)} = \overline{\Phi}_{(y_n)} q$, where $q : \ell_r \to Z$ is the quotient mapping and $\overline{\Phi}_{(y_n)} : Z \to Y$ is the injective associate of $\Phi_{(y_n)}$.

Let $x \in B_X$ and let $\alpha \in B_{\ell_r}$ satisfy $Tx = \Phi_{(y_n)}\alpha$. If $Tx = \Phi_{(y_n)}\beta$ for some (other) $\beta \in \ell_r$, then clearly $\alpha - \beta \in \ker \Phi_{(y_n)}$. Therefore each $x \in X$ determines an $\alpha \in \ell_r$, which is unique up to the elements of $\ker \Phi_{(y_n)}$. We also may assume that the α satisfies $\|\alpha\|_r \leq \|x\|$. (Indeed, for $x \in X \backslash \{0\}$, $\frac{x}{\|x\|} \in S_X$ determines a $\beta \in B_{\ell_r}$ such that $T(\frac{x}{\|x\|}) = q\beta$. Now $Tx = q\alpha$ where $\alpha := \|x\|\beta \in \ell_r$ and $\|\alpha\|_r = \|x\|\|\beta\|_r \leq \|x\|$.)

Hence one can define $T_{(y_n)} : X \to Z$ by $T_{(y_n)}x = q\alpha$, $x \in X$, where $\alpha \in \ell_r$ satisfies $\|\alpha\|_r \le \|x\|$ and $Tx = \Phi_{(y_n)}\alpha$. The operators are represented in the following diagram:



Since $\Phi_{(y_n)} = \overline{\Phi}_{(y_n)}q$, one immediately obtains the factorization

$$T = \overline{\Phi}_{(y_n)}T_{(y_n)}, \tag{4.1}$$

with $T_{(y_n)} \in \mathcal{L}(X, Z)$, $\|T_{(y_n)}\| \le 1$, and $\ker T_{(y_n)} = \ker T$. (Indeed, $Tx = \Phi_{(y_n)}\alpha = \overline{\Phi}_{(y_n)}q\alpha = \overline{\Phi}_{(y_n)}T_{(y_n)}x$ for every $x \in X$. The linearity of $T_{(y_n)}$ and the equality of the kernels follows from (4.1) because $\overline{\Phi}_{(y_n)}$ is linear and injective. Finally, if $\|x\| \le 1$, then $\left\|T_{(y_n)}x\right\| = \|q\alpha\|$, where $\|\alpha\|_r \le \|x\| \le 1$; hence $\left\|T_{(y_n)}x\right\| \le \|q\| = 1$.)

Theorem 4.1. *Let $1 \le p \le \infty$, $1 \le r \le p^*$. Then $\mathcal{K}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ as operator ideals.*

Proof. Let $T \in \mathcal{K}_{(p,r)}(X, Y)$. Since, by the natural factorization, $T = \overline{\Phi}_{(y_n)}T_{(y_n)}$ and $\overline{\Phi}_{(y_n)} \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(Z, Y)$, where $Z = \ell_r/\ker\Phi_{(y_n)}$ (see Proposition 2.17), we have $T \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(X, Y)$.

On the other hand, to see that $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)} \subset \mathcal{K}_{(p,r)}$ it suffices to prove that $\mathcal{N}_{(p,1,r^*)} \subset \mathcal{K}_{(p,r)}$ because $\mathcal{K}_{(p,r)}$ is surjective (see Proposition 3.9) and $\mathcal{A}^{\mathrm{sur}} \subset \mathcal{B}^{\mathrm{sur}}$ whenever $\mathcal{A} \subset \mathcal{B}$ (see Proposition 2.11).

Consider $T \in \mathcal{N}_{(p,1,r^*)}(X, Y)$. Then $T = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n$, where $(\sigma_n) \in \ell_p$ ($(\sigma_n) \in c_0$ for $p = \infty$), $(x_n^*) \in \ell_r^w(X^*)$, $\|y_n\| = O(1)$. We clearly may assume that $\|(x_n^*)\|_r^w = 1$. (Indeed, $\sigma_n x_n^* \otimes y_n = \sigma_n \frac{x_n^*}{\|(x_n^*)\|_r^w} \otimes \|(x_n^*)\|_r^w y_n$ and $(\|(x_n^*)\|_r^w y_n) \in \ell_\infty(Y)$.) Then $(\sigma_n y_n) \in \ell_p(Y)$ ($(\sigma_n y_n) \in c_0(Y)$ for $p = \infty$) and for every $x \in B_X$, we have

$$Tx = \sum_{n=1}^{\infty} x_n^*(x)\sigma_n y_n \in \Phi_{(\sigma_n y_n)}(B_{\ell_r}),$$

meaning that $T \in \mathcal{K}_{(p,r)}(X, Y)$. □

Recall that $\mathcal{N}_{(p,1,r^*)}$ is an s-Banach operator ideal, where

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{r} \tag{4.2}$$

(see Example 2.7). The equality $\mathcal{K}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ of operator ideals from Theorem 4.1 allows us to equip $\mathcal{K}_{(p,r)}$ with an s-norm in the following natural way.

Definition 4.2. Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Define

$$\|\cdot\|_{\mathcal{K}_{(p,r)}} := \|\cdot\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} .$$

Summing up we have the following.

Theorem 4.3. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Then $\mathcal{K}_{(p,r)} = (\mathcal{K}_{(p,r)}, \|\cdot\|_{\mathcal{K}_{(p,r)}})$ is an s-Banach operator ideal and $\mathcal{K}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ as s-Banach operator ideals, where s satisfies* (4.2).

Theorem 4.1 together with the inclusion theorem for (t,u,v)-nuclear operators [Pie80, 18.1.5] and Proposition 2.11 immediately yield the following result.

Corollary 4.4. *Let X be a Banach space. Let $1 \le p \le q \le \infty$, $1 \le r \le p^*$, and $1 \le s \le q^*$. Assume that $s \le r$ and*

$$\frac{1}{q} + \frac{1}{s} \le \frac{1}{p} + \frac{1}{r}.$$

Then

$$(\mathcal{K}_{(p,r)}, \|\cdot\|_{\mathcal{K}_{(p,r)}}) \subset (\mathcal{K}_{(q,s)}, \|\cdot\|_{\mathcal{K}_{(q,s)}}).$$

In particular, if $1 \le p \le q \le \infty$, then $(\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}) \subset (\mathcal{K}_q, \|\cdot\|_{\mathcal{K}_q})$ and $(\mathcal{K}_{(p,1)}, \|\cdot\|_{\mathcal{K}_{(p,1)}}) \subset (\mathcal{K}_{(q,1)}, \|\cdot\|_{\mathcal{K}_{(q,1)}})$.

Condition (4.2) defines s by the equality

$$s = \frac{pr}{p+r}.$$

Remark 4.5. The only case when

$$s = \frac{pr}{p+r} = 1$$

is precisely when $r = p^*$. This means that *from the all family of s-Banach operator ideals $\mathcal{K}_{(p,r)}$, only $\mathcal{K}_{(p,p^*)}$ is a Banach operator ideal.* As was already mentioned, $\mathcal{K}_{(p,p^*)} = \mathcal{K}_p$, the ideal of p-compact operators introduced in [SK02] by Sinha and Karn. In [SK02] also $\mathcal{K}_p$ was equipped with a Banach operator ideal norm, but in a different manner than in Definition 4.2. In the next section we will see, among others, that these definitions coincide (see Remark 4.8).

4.2 The s-norm of $\mathcal{K}_{(p,r)}$

Let $1 \le p \le \infty$, $1 \le r \le p^*$ and $s = pr/(p+r)$. It is important that we can explicitly calculate the s-norm $\|\cdot\|_{\mathcal{K}_{(p,r)}}$ (see Theorem 4.3) as in the following Theorem 4.6. Among others, this shows that the norm $\|\cdot\|_{\mathcal{K}_p}$ coincides with the Banach ideal norms introduced in [SK02] and [DPS10b] (see Remarks 4.8 and 4.9 below).

Theorem 4.6. *Let X and Y be Banach spaces. Assume that $1 \le p \le \infty$ and $1 \le r \le p^*$. Let $T \in \mathcal{K}_{(p,r)}(X,Y)$. Then*

$$\|T\|_{\mathcal{K}_{(p,r)}} = \inf \|T_{(y_n)}\| \|(y_n)\| = \inf \|(y_n)\|,$$

where the both infimums are taken over all sequences $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0$ when $p = \infty$) such that

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} a_n y_n : (a_n) \in B_{\ell_r} \right\}.$$

Proof. Let $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0$ when $p = \infty$) be such that $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. We know that $T = \overline{\Phi}_{(y_n)} T_{(y_n)}$, $\|T_{(y_n)}\| \le 1$, and $\|\overline{\Phi}_{(y_n)}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \le \|(y_n)\|$ (see (4.1) and Proposition 2.17). Hence,

$$\begin{aligned} \|T\|_{\mathcal{K}_{(p,r)}} &= \|T\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \le \|\overline{\Phi}_{(y_n)}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \|T_{(y_n)}\| \\ &\le \|T_{(y_n)}\| \|(y_n)\| \le \|(y_n)\|. \end{aligned}$$

Consequently,

$$\|T\|_{\mathcal{K}_{(p,r)}} \le \inf \|T_{(y_n)}\| \|(y_n)\| \le \inf \|(y_n)\|.$$

On the other hand, from the factorization theorem of (t,u,v)-nuclear operators (see [Pie80, 18.1.3]), we know that the $(p,1,r^*)$-nuclear operator TQ_X factorizes as follows:

$$\begin{array}{ccccc} Z := \ell_1(B_X) & \xrightarrow{Q_X} & X & \xrightarrow{T} & Y \\ \Big\downarrow{\scriptstyle A} & & & & \Big\uparrow{\scriptstyle B} \\ \ell_r & & \xrightarrow{\quad \Delta \quad} & & \ell_1 \end{array}$$

where $\Delta \in \mathcal{L}(\ell_r, \ell_1)$ is a diagonal operator of the form $\Delta(a_n) = (\sigma_n a_n)$ with $(\sigma_n) \in \ell_p$ ($(\sigma_n) \in c_0$ when $p = \infty$), $A \in \mathcal{L}(Z, \ell_r)$, $B \in \mathcal{L}(\ell_1, Y)$, and

$$\|TQ_X\|_{\mathcal{N}_{(p,1,r^*)}} = \inf \|B\| \|(\sigma_n)\| \|A\|,$$

where the infimum is taken over the all possible factorizations.

Let $\varepsilon > 0$. Choose A, (σ_n), and B as above so that

$$\varepsilon + \|T\|_{\mathcal{K}_{(p,r)}} = \varepsilon + \|TQ_X\|_{\mathcal{N}_{(p,1,r^*)}} \geq \|B\|\|(\sigma_n)\|\|A\| = \|(\sigma_n)\|,$$

because we clearly may assume that $\|A\| = \|B\| = 1$. Since $B_X \subset Q_X(B_Z)$, we have

$$T(B_X) \subset (B\Delta A)(B_Z) \subset (B\Delta)(B_{\ell_r}) = \left\{ \sum_{n=1}^{\infty} a_n \sigma_n B e_n : (a_n) \in B_{\ell_r} \right\},$$

where (e_n) is the unit vector basis of ℓ_r (c_0 when $r = \infty$). Put $y_n = \sigma_n B e_n$. Then $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0$ when $p = \infty$), $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$, and $\|(y_n)\| \leq \|(\sigma_n)\|$. Therefore,

$$\|T\|_{\mathcal{K}_{(p,r)}} \geq \inf \|(y_n)\|.$$

This concludes the proof. □

Let us spell out the immediate special case of Theorems 4.3 and 4.6 which characterizes the p-compact operators $\mathcal{K}_{(p,1)}$ in the Bourgain–Reinov sense.

Corollary 4.7. *Let X and Y be Banach spaces. Let $1 \leq p \leq \infty$. The operator ideal $\mathcal{K}_{(p,1)} = \mathcal{N}^{\text{sur}}_{(p,1,\infty)}$ is a $\frac{p}{p+1}$-Banach operator ideal. The $\frac{p}{p+1}$-norm of $T \in \mathcal{K}_{(p,1)}(X,Y)$ is calculated as follows:*

$$\|T\|_{\mathcal{K}_{(p,1)}} = \inf \|T_{(y_n)}\|\|(y_n)\| = \inf \|(y_n)\|,$$

where the both infimums are taken over all sequences $(y_n) \in \ell_p(Y)$ such that

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} a_n y_n : (a_n) \in B_{\ell_1} \right\}.$$

Remark 4.8. In [SK02, Theorem 4.2], the Banach operator ideal norm was introduced in the operator ideal $\mathcal{K}_p$ through the formula

$$\|T\|_{\kappa_p} := \inf \|T_{(y_n)}\|\|y_n\|,$$

which is the special case with $r = p^*$ of the first equality in Theorem 4.6. Thus Corollary 4.4 extends the inclusion result [SK02, Proposition 4.3] for Banach operator ideals $\mathcal{K}_p$.

Remark 4.9. Recently, Delgado, Piñero, and Serrano [DPS10b] made a thorough study of the operator ideal $\mathcal{K}_p$, but they defined the Banach operator ideal norm in $\mathcal{K}_p$ through the formula

$$\|T\|_{k_p} := \inf \|y_n\|,$$

which is the special case with $r = p^*$ of the second equality in Theorem 4.6. They proved (see [DPS10b, Proposition 3.15]) that the Banach operator ideal norms from [SK02] and [DPS10b] coincide: $\|T\|_{k_p} = \|T\|_{\kappa_p}$; this equality is also contained in Theorem 4.6.

Remark 4.10. One of the main results of [DPS10b] (see [DPS10b, Proposition 3.11]) is that $(\mathcal{K}_p, \|\cdot\|_{k_p}) = (\mathcal{N}^p, \|\cdot\|_{\mathcal{N}^p})^{\mathrm{sur}}$, the Banach operator ideal of *right p-nuclear operators*. Since, by definition, $(\mathcal{N}^p, \|\cdot\|_{\mathcal{N}^p}) = (\mathcal{N}_{(p,1,p)}, \|\cdot\|_{\mathcal{N}_{(p,1,p)}})$ (cf. [Pie80, 18.1.1] and, e.g., [Rya02, p. 140]), this result is contained as the special case with $r = p^*$ in Theorems 4.3 and 4.6. In [DPS10b], to prove this result, the authors used a roundabout approach, first describing $\mathcal{K}_p^{\mathrm{dual}}$, and relied on Reinov's recent study [Rei00] on operators with p-nuclear adjoints.

Chapter 5

Applications to some related operator ideals

In this chapter, results from the previous chapter are used to describe the surjective and injective hulls of $\mathcal{N}_{(p,1,r^*)}$. A description of $\mathcal{K}_{(p,r)}$ as product of operator ideals is given as well. This chapter is based on [ALO12].

5.1 Surjective and injective hulls of $\mathcal{N}_{(p,1,r^*)}$

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We start with an alternative description of the $\frac{pr}{p+r}$-Banach operator ideal $(\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}, \|\cdot\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}})$ in the same vain as the description of the surjective hull of the Banach operator ideal $(\mathcal{N}, \|\cdot\|_{\mathcal{N}}) = (\mathcal{N}_{(1,1,1)}, \|\cdot\|_{\mathcal{N}_{(1,1,1)}})$ of *nuclear operators* in [Pie80, 8.5.5]. This is the following immediate consequence of Theorems 4.3 and 4.6; it contains the description of $\mathcal{N}^{\mathrm{sur}}$ from [Pie80, 8.5.5].

Theorem 5.1. *Let X and Y be Banach spaces. Assume that $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A linear operator $T : X \to Y$ belongs to $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(X, Y)$ if and only if there exists $(y_n) \in \ell_p(Y)$ ($(y_n) \in c_0(X)$ when $p = \infty$) such that*

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} a_n y_n : (a_n) \in B_{\ell_r} \right\}.$$

In this case, the $\frac{pr}{p+r}$-norm of T is given by

$$\|T\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} = \inf \|(y_n)\|,$$

where the infimum is taken over all admissible sequences (y_n).

The injective hull $\mathcal{N}^{\text{inj}}$ of the Banach operator ideal $\mathcal{N} = \mathcal{N}_{(1,1,1)}$ is described, e.g., in [Pie80, 8.4.5]. In Theorem 5.3 below, we have the following description of the $\frac{pr}{p+r}$-Banach operator ideal $\mathcal{N}^{\text{inj}}_{(p,r^*,1)}$ as the dual operator ideal of $\mathcal{K}_{(p,r)}$.

Lemma 5.2 (see [Pie80, 8.4.4])**.** *Let* X, Y *and* Z *be Banach spaces, and let* $\mathcal{A}$ *be an* s*-Banach operator ideal. If* $\|Sx\| \le \|Tx\|$, $x \in X$, *for* $S \in \mathcal{L}(X, Y)$ *and* $T \in \mathcal{A}(X, Z)$, *then* $S \in \mathcal{A}^{\text{inj}}(X, Y)$ *with* $\|S\|_{\mathcal{A}^{\text{inj}}} \le \|T\|_{\mathcal{A}}$.

Theorem 5.3. *Let* X *and* Y *be Banach spaces. Assume that* $1 \le p \le \infty$ *and* $1 \le r \le p^*$. *Let* $T : X \to Y$ *be a linear operator. Then the following statements are equivalent:*

(a) $T \in \mathcal{N}^{\text{inj}}_{(p,r^*,1)}(X, Y)$,

(b) $T \in \mathcal{K}^{\text{dual}}_{(p,r)}(X, Y)$,

(c) *there exists* $(x_n^*) \in \ell_p(X^*)$ $((x_n^*) \in c_0(X^*)$ *when* $p = \infty)$ *such that*

$$\|Tx\| \le \|(x_n^*(x))\|_{r^*} \;\forall x \in X. \tag{5.1}$$

Moreover, in this case, the $\frac{pr}{p+r}$*-norm of* T *is given by*

$$\|T\|_{\mathcal{N}^{\text{inj}}_{(p,r^*,1)}} = \|T\|_{\mathcal{K}^{\text{dual}}_{(p,r)}} = \inf \|(x_n^*)\|_p,$$

where the infimum is taken over all $(x_n^*) \in \ell_p(X^*)$ $((x_n^*) \in c_0(X^*)$ *when* $p = \infty)$ *satisfying* (5.1).

Proof. To simplify notation, we omit below the corresponding s-norms.

(a)⇒(b). From [Pie80, 8.3.3] we know that $\mathcal{N}_{(p,r^*,1)} \subset \mathcal{N}^{\text{reg}}_{(p,r^*,1)}$, the regular hull of $\mathcal{N}_{(p,r^*,1)}$, and $\mathcal{N}^{\text{reg}}_{(p,r^*,1)} = \mathcal{N}^{\text{dual}}_{(p,1,r^*)}$ (see [Pie80, 18.1.6]). Hence, recalling that injectivity and surjectivity are dual properties, in particular, $\mathcal{A}^{\text{dual inj}} \subset \mathcal{A}^{\text{sur dual}}$ for any s-Banach operator ideal $\mathcal{A}$ (see [Pie80, 8.5.9]), we have

$$\mathcal{N}^{\text{inj}}_{(p,r^*,1)} \subset \mathcal{N}^{\text{dual inj}}_{(p,1,r^*)} \subset \mathcal{N}^{\text{sur dual}}_{(p,1,r^*)} = \mathcal{K}^{\text{dual}}_{(p,r)}$$

(see Theorem 4.3). The above inclusion also means that

$$\|T\|_{\mathcal{K}^{\text{dual}}_{(p,r)}} \le \|T\|_{\mathcal{N}^{\text{inj}}_{(p,r^*,1)}} \;\forall T \in \mathcal{N}^{\text{inj}}_{(p,r^*,1)}.$$

(b)$\Rightarrow$(c). Let $T \in \mathcal{K}^{\text{dual}}_{(p,r)}(X,Y)$. Then $T^* \in \mathcal{K}_{(p,r)}(Y^*, X^*)$. Hence (see Theorem 4.6), for every $\varepsilon > 0$, there exists $(x_n^*) \in \ell_p(X^*)$ ($(x_n^*) \in c_0(X^*)$ when $p = \infty$) such that

$$T^*(B_{Y^*}) \subset \left\{ \sum_{n=1}^{\infty} a_n x_n^* : (a_n) \in B_{\ell_r} \right\}$$

and

$$\|(x_n^*)\|_p \le \|T\|_{\mathcal{K}^{\text{dual}}_{(p,r)}} + \varepsilon.$$

For every $x \in X$, we clearly have

$$\|Tx\| = \sup_{\|y^*\|\le 1} |(T^*y^*)(x)| \le \sup \left\{ \sum_{n=1}^{\infty} |a_n x_n^*(x)| : (a_n) \in B_{\ell_r} \right\} = \|(x_n^*(x))\|_{r^*}.$$

It follows that

$$\inf \|(x_n^*)\|_p \le \|T\|_{\mathcal{K}^{\text{dual}}_{(p,r)}},$$

whenever the infimum is taken over all $(x_n^*) \in \ell_p(X^*)$ ($(x_n^*) \in c_0(X)$ when $p = \infty$) satisfying (5.1).

(c)$\Rightarrow$(a). Following an idea from the proof of [Pie80, 8.4.5], we define an operator $A \in \mathcal{L}(X, \ell_{r^*})$ by $Ax = (x_n^*(x))$, $x \in X$. Clearly

$$Ax = \sum_{n=1}^{\infty} x_n^*(x) e_n, x \in X,$$

where (e_n) is the unit vector basis for ℓ_{r^*} (c_0 when $r = 1$). From the representation

$$A = \sum_{n=1}^{\infty} x_n^* \otimes e_n$$

and from

$$x_n^* \otimes e_n = \|x_n^*\| \frac{x_n^*}{\|x_n^*\|} \otimes e_n,$$

we have that $(\|x_n^*\|) \in \ell_p$ (respectively, $(\|x_n^*\|) \in c_0$ when $p = \infty$), $(\|x_n^*\|^{-1} x_n^*) \in \ell_\infty(X^*) = \ell_\infty^w(X^*)$, and $(e_n) \in \ell_r^w(\ell_{r^*}) \subset \ell_r^w(\ell_r^*)$ (respectively, $(e_n) \in \ell_1^w(c_0) \subset \ell_1^w(\ell_\infty)$ when $r = 1$) (cf. Proposition 2.16). Hence (see Example 2.7) $A \in \mathcal{N}_{(p,r^*,1)}(X, \ell_{r^*})$ and $\|A\|_{\mathcal{N}_{(p,r^*,1)}} \le \|(x_n^*)\|_p$.

Since $\|Tx\| \le \|Ax\|$ for all $x \in X$, by Lemma 5.2, we immediately get that $T \in \mathcal{N}^{\text{inj}}_{(p,r^*,1)}(X,Y)$ and

$$\|T\|_{\mathcal{N}^{\text{inj}}_{(p,r^*,1)}} \le \|A\|_{\mathcal{N}_{(p,r^*,1)}} \le \|(x_n^*)\|_p.$$

This proves the theorem. $\square$

Remark 5.4. Recall that the Banach operator ideal $\mathcal{N}_p$ of *p-nuclear operators* is defined by $(\mathcal{N}_p, \|\cdot\|_{\mathcal{N}_p}) = (\mathcal{N}_{(p,p,1)}, \|\cdot\|_{\mathcal{N}_{(p,p,1)}})$ (see [Pie80, 18.2.1]). In this classical case, the description (a)⇔(c) of $\mathcal{N}_p^{\mathrm{inj}}$ as the Banach operator ideal $\mathcal{QN}_p$ of quasi-nuclear operators is due to Persson and Pietsch [PP69, Theorem 39]. In [DPS10b, Corollary 3.4], it is proved that $\mathcal{K}_p^{\mathrm{dual}} = \mathcal{QN}_p$ as operator ideals. The special case of (b)⇔(c) with $r = p^*$, i.e., $r^* = p$, tells us that $\mathcal{K}_p^{\mathrm{dual}} = \mathcal{QN}_p$ as *Banach* operator ideals.

We know that the $\frac{2tu}{2(u-t)+3tu}$-Banach operator ideals $\mathcal{N}_{(t,u,2)}$ and $\mathcal{N}_{(t,2,u)}$ (see Example 2.7) are, respectively, surjective and injective (see [Pie80, 18.1.8]). Therefore, we have the following immediate consequence of Theorems 4.3 and 5.3.

Corollary 5.5. *Let $1 \le p \le 2$. Then $\mathcal{K}_{(p,2)} = \mathcal{N}_{(p,1,2)}$ and $\mathcal{K}_{(p,2)}^{\mathrm{dual}} = \mathcal{N}_{(p,2,1)}$ as $\frac{2p}{2+p}$-Banach operator ideals.*

Remark 5.6. The special case with $p = 2$ tells us that $\mathcal{K}_2 = \mathcal{N}^2$ (for $\mathcal{N}^2$, see Remark 4.10) and $\mathcal{K}_2^{\mathrm{dual}} = \mathcal{N}_2$ as Banach operator ideals. The latter equality was established in [SK08, Corollary 3.8].

Let us recall the notion of an ultrastable s-Banach operator ideal. Let $\mathcal{U}$ be an ultrafilter on an arbitrary index set I. If $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are families of Banach spaces and $T_i \in \mathcal{L}(X_i, Y_i)$, then the linear operator

$$(T_i)_{\mathcal{U}} : (\prod_{i\in I} X_i)_{\mathcal{U}} \to (\prod_{i\in I} Y_i)_{\mathcal{U}}$$

(where $(\prod_{i\in I} X_i)_{\mathcal{U}}$ denotes the ultraproduct of the family $(X_i)_{i\in I}$ (see, e.g., [Pie80, 8.8.3])), defined by

$$(T_i)_{\mathcal{U}}(x_i)_{\mathcal{U}} = (T_i x_i)_{\mathcal{U}},\ (x_i)_{\mathcal{U}} \in (\prod_{i\in I} X_i)_{\mathcal{U}},$$

is called the *ultraproduct of the operators* T_i. An s-Banach operator ideal $\mathcal{A}$ is *ultrastable*, if

$$(T_i)_{\mathcal{U}} \in \mathcal{A}((\prod_{i\in I} X_i)_{\mathcal{U}}, (\prod_{i\in I} Y_i)_{\mathcal{U}}) \text{ and } \|(T_i)_{\mathcal{U}}\|_{\mathcal{A}} \le \lim_{\mathcal{U}} \|T_i\|_{\mathcal{A}}$$

for every ultrafilter $\mathcal{U}$ on I and family of operators $T_i \in \mathcal{A}(X_i, Y_i)$ with $\sup_{i\in I} \|T_i\|_{\mathcal{A}} < \infty$ (see, e.g., [Pie80, 8.8.5]).

If $0 < t < \infty$ and $1 + 1/t > 1/u + 1/v$, then the s-Banach operator ideal $\mathcal{N}_{(t,u,v)}$, where s satisfies $1/s = 1/t + 1/u^* + 1/v^*$, is ultrastable (see [Pie80, 18.1.9]), and this property is preserved under surjective and injective hulls (see [Pie80, 8.8.9 and 8.8.8]). Therefore, we have the following immediate consequence of Theorems 4.3 and 5.3, showing that $\mathcal{K}_{(p,r)}$ and $\mathcal{K}_{(p,r)}^{\text{dual}}$ are "almost always" ultrastable.

Corollary 5.7. *Let $1 \leq p < \infty$ and $1 \leq r < p^*$. Then the $\frac{pr}{p+r}$-Banach operator ideals $\mathcal{K}_{(p,r)}$ and $\mathcal{K}_{(p,r)}^{\text{dual}}$ are ultrastable.*

5.2 Description of $\mathcal{K}_{(p,r)}$ via products

Let X and Y be Banach spaces and $1 \leq p < \infty$.

Recall that an operator $T \in \mathcal{L}(X, Y)$ belongs to the Banach operator ideal $\mathcal{P}_p$ of *absolutely p-summing* operators (*p-summing* operators in [DJT95]) if it takes weakly p-summing sequences (x_n) of X to absolutely p-summing sequences (Tx_n) of Y, and one defines $\|T\|_{\mathcal{P}_p} := \inf c$, where $c \geq 0$ satisfies

$$\|(Tx_n)_{n=1}^m\|_p \leq c \,\|(x_n)_{n=1}^m\|_p^w$$

for every finite family $x_1, ..., x_m \in X$ (see, e.g., [Pie07, 6.3.6.2]). If $p = \infty$ then $\mathcal{P}_\infty = \mathcal{L}$ (see [Pie80, 17.3.1]). The operators from $\mathcal{P}_p^{\text{dual}}(X, Y)$ are also called *Cohen strongly p^*-summing operators* (described in [Coh73]).

In [SK02, Proposition 5.3], it was shown that $\mathcal{K}_p \subset \mathcal{P}_p^{\text{dual}}$ as Banach operator ideals. On the other hand, in [DPS10b, Proposition 3.13], it was proved that $\mathcal{P}_p^{\text{dual}} \circ \mathcal{K} \subset \mathcal{K}_p$ as Banach operator ideals. We shall improve these results by showing that, in fact, $\mathcal{K}_p = \mathcal{P}_p^{\text{dual}} \circ \mathcal{K}$ (see Corollary 5.10 below). This multiplication formula is a special case of our multiplication formula for the $\frac{pr}{p+r}$-Banach operator ideal $\mathcal{K}_{(p,r)}$ (see Theorem 5.8 below).

Let $0 < t \leq \infty$, $1 \leq u, v \leq \infty$, and $1/u + 1/v \leq 1 + 1/t$. Recall that the s-Banach operator ideal $\mathcal{I}_{(t,u,v)}$ of *(t, u, v)-integral operators*, where s satisfies $1/s = 1/t + 1/u^* + 1/v^*$, is defined as the maximal hull of (t, u, v)-nuclear operators:

$$(\mathcal{I}_{(t,u,v)}, \|\cdot\|_{\mathcal{I}_{(t,u,v)}}) := (\mathcal{N}_{(t,u,v)}^{\max}, \|\cdot\|_{\mathcal{N}_{(t,u,v)}^{\max}})$$

(see [Pie80, 19.1.1]).

In turn, recall that the *maximal hull* $(\mathcal{A}^{\max}, \|\cdot\|_{\mathcal{A}^{\max}})$ of an s-Banach operator ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ may be defined as follows (see [Pie80, 8.7.4]). An operator $T \in \mathcal{L}(X, Y)$ belongs to $\mathcal{A}^{\max}(X, Y)$ if there exists $c \geq 0$ such that

$$\|BTA\|_{\mathcal{A}} \leq c \,\|B\| \,\|A\|$$

for all $A \in \mathcal{F}(Z, X)$ and $B \in \mathcal{F}(Y, W)$, where Z and W are arbitrary Banach spaces; in this case, one defines $\|T\|_{\mathcal{A}^{\max}} := \inf c$.

Theorem 5.8. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Then*

$$\mathcal{K}_{(p,r)} = \mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)} \circ \mathcal{K}$$

as $\frac{pr}{p+r}$-Banach operator ideals.

Proof. Denote $\mathcal{B} = \mathcal{I}_{(p,1,r^*)}$. We know (see [Pie80, 19.1.10]) that

$$\mathcal{N}_{(p,1,r^*)} = \mathcal{B} \circ \overline{\mathcal{F}}$$

as $\frac{pr}{p+r}$-Banach operator ideals. Passing to surjective hulls (see Theorem 4.3) and using Proposition 2.13, we get that

$$\mathcal{K}_{(p,r)} = (\mathcal{B} \circ \overline{\mathcal{F}})^{\mathrm{sur}} \subset \mathcal{B}^{\mathrm{sur}} \circ \overline{\mathcal{F}}^{\mathrm{sur}} = \mathcal{B}^{\mathrm{sur}} \circ \mathcal{K}$$

as u-Banach operator ideals (where $u = \frac{pr}{p+r+pr}$), due to $\overline{\mathcal{F}}^{\mathrm{sur}} = \mathcal{K}$ (see Example 2.9).

On the other hand, let $T \in (\mathcal{B}^{\mathrm{sur}} \circ \mathcal{K})(X, Y)$. Then for every $\varepsilon > 0$, there are a Banach space Z and operators $A \in \mathcal{K}(X, Z)$ and $B \in \mathcal{B}^{\mathrm{sur}}(Z, Y)$ such that $T = BA$ and

$$(1 + \varepsilon)\|T\|_{\mathcal{B}^{\mathrm{sur}} \circ \mathcal{K}} \ge \|B\|_{\mathcal{B}^{\mathrm{sur}}}\|A\| = \|BQ_Z\|_{\mathcal{B}}\|A\|.$$

Since $AQ_X \in \mathcal{K}(\ell_1(B_X), Z)$, we can use the almost norm-preserving lifting property for compact operators with an $L_1(\mu)$-space as the domain space due to Grothendieck [Gro56, Corollary 5 on p. 24] (see, e.g., [Lin64]). This property gives us an operator $C \in \mathcal{K}(\ell_1(B_X), \ell_1(B_Z))$ such that $Q_Z C = AQ_X$ and

$$\|C\| \le (1 + \varepsilon)\|AQ_X\| \le (1 + \varepsilon)\|A\|.$$

The operators are represented in the following diagram:

$$\begin{array}{ccccc} \ell_1(B_X) & \xrightarrow{Q_X} & X & \xrightarrow{T} & Y \\ {\scriptstyle C}\big\downarrow & & {\scriptstyle A}\big\downarrow & \nearrow {\scriptstyle B} & \\ \ell_1(B_Z) & \xrightarrow{Q_Z} & Z & & \end{array}.$$

Since $\ell_1(B_Z)$ has the approximation property (i.e., its identity operator can be approximated, uniformly on relatively compact sets, by finite-rank operators), $\mathcal{K}(\ell_1(B_X), \ell_1(B_Z)) = \overline{\mathcal{F}}(\ell_1(B_X), \ell_1(B_Z))$.

Now we have $TQ_X = BQ_ZC$, where $C \in \overline{\mathcal{F}}(\ell_1(B_X), \ell_1(B_Z))$ and $BQ_Z \in \mathcal{B}(\ell_1(B_Z), Y)$, and

$$\begin{aligned}(1+\varepsilon)^2\|T\|_{\mathcal{B}^{\mathrm{sur}}\circ\mathcal{K}} &\geq (1+\varepsilon)\|BQ_Z\|_{\mathcal{B}}\|A\| \geq \|BQ_Z\|_{\mathcal{B}}\|C\| \\ &\geq \|TQ_X\|_{\mathcal{B}\circ\overline{\mathcal{F}}} = \|T\|_{(\mathcal{B}\circ\overline{\mathcal{F}})^{\mathrm{sur}}}.\end{aligned}$$

This means that $\mathcal{B}^{\mathrm{sur}} \circ \mathcal{K} \subset (\mathcal{B} \circ \overline{\mathcal{F}})^{\mathrm{sur}}$ as u-Banach operator ideals.

Thus, we have established that $\mathcal{K}_{(p,r)} = \mathcal{B}^{\mathrm{sur}} \circ \mathcal{K}$ as u-Banach operator ideals. Since $\mathcal{K}_{(p,r)}$ is a $\frac{pr}{p+r}$-Banach operator ideal, also $\mathcal{B}^{\mathrm{sur}} \circ \mathcal{K}$ is, and the equality holds in the sense of $\frac{pr}{p+r}$-Banach operator ideals. □

Remark 5.9. One has $\mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)} = \mathcal{I}^{\mathrm{inj\,dual}}_{(p,r^*,1)}$ as s-Banach operator ideals. Indeed, by [Pie80, 19.1.4],

$$\mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)} = \mathcal{I}^{\mathrm{dual\,sur}}_{(p,r^*,1)} = \mathcal{I}^{\mathrm{inj\,dual}}_{(p,r^*,1)},$$

because $\mathcal{A}^{\mathrm{dual\,sur}} = \mathcal{A}^{\mathrm{inj\,dual}}$ for any quasi-Banach operator ideal $\mathcal{A}$ (see [Pie80, 8.5.9]).

Since $\mathcal{P}_p = \mathcal{I}^{\mathrm{inj}}_{(p,p,1)}$ for $1 \leq p < \infty$ and $\mathcal{P}_\infty = \mathcal{L}$ as Banach operator ideals (see [Pie80, 19.2.7, 17.3.1] respectively), recalling that $\mathcal{K}_p = \mathcal{K}_{(p,p^*)}$, Theorem 5.8 and Remark 5.9 immediately yield the promised result.

Corollary 5.10. *Let* $1 \leq p \leq \infty$. *Then*

$$\mathcal{K}_p = \mathcal{P}_p^{\mathrm{dual}} \circ \mathcal{K}$$

as Banach operator ideals.

Chapter 6

The (p, r)-null sequences

We introduce a new term: the (p, r)-null sequence. The main aim of this chapter is to establish an omnibus theorem giving six equivalent properties for a sequence in a Banach space to be a (p, r)-null sequence. The method used to do this relies on the theory of Carl–Stephani [CS84], which is also introduced. This chapter is based on [AO12, AO15].

6.1 Elementary observations on (p, r)-null sequences

Let X be a Banach space.

Let $1 \leq p < \infty$. In [PD11] Delgado and Piñeiro called a sequence (x_n) in X *p-null* if for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$ with $\|(z_k)\|_p \leq \varepsilon$ such that $x_n \in p\text{-conv}(z_k)$ for all $n \geq N$ (see [PD11]).

We extend the notion of p-null sequences due to Delgado and Piñeiro [PD11] as follows.

Definition 6.1. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We call a sequence (x_n) in X *(p, r)-null* if for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$ ($(z_k) \in c_0(X)$ when $p = \infty$) with $\|(z_k)\|_p \leq \varepsilon$ such that $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$.

As $p\text{-conv}(z_k)$ coincides with the (p, p^*)-convex hull of $(z_k) \in \ell_p(X)$, p-null sequences in X are precisely the (p, p^*)-null sequences.

Let us make a couple of elementary observations (see Propositions 6.2 and 6.3).

Proposition 6.2. *A sequence (x_n) in a Banach space X is $(\infty, 1)$-null if and only if $x_n \to 0$.*

Proof. Suppose that $(x_n) \subset X$ is $(\infty, 1)$-null. Then for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in c_0(X)$, $\|(z_k)\|_\infty \le \varepsilon$, such that $x_n = \sum_{k=1}^\infty a_k^n z_k$, where $(a_k^n)_{k=1}^\infty \in B_{\ell_1}$, for all $n \ge N$.

We have, for all $n \ge N$,

$$\|x_n\| \le \sum_{k=1}^\infty \|a_k^n z_k\| \le \varepsilon \sum_{k=1}^\infty |a_k^n| \le \varepsilon,$$

and therefore $x_n \to 0$.

Let now $x_n \to 0$. This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n\| \le \varepsilon$ for all $n \ge N$. Now the sequence

$$z_k = \begin{cases} 0 & \text{if } k < N, \\ x_k & \text{if } k \ge N, \end{cases}$$

is in $c_0(X)$ with $\|(z_k)\|_\infty \le \varepsilon$ and $x_n \in (\infty, 1)$-conv(z_k) for all $n \ge N$. □

Proposition 6.3. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. If a sequence (x_n) in a Banach space X is (p, r)-null, then $x_n \to 0$ and (x_n) is relatively (p, r)-compact.*

Proof. Assume $1 \le p < \infty$ as for $p = \infty$ the assertion is clear from Proposition 6.2. Since (x_n) is (p, r)-null, for every $\varepsilon > 0$ there are $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$, $\|(z_k)\|_p \le \varepsilon$, such that $x_n = \sum_{k=1}^\infty a_k^n z_k$, where $(a_k^n)_{k=1}^\infty \in B_{\ell_r}$, for all $n \ge N$. Hence, for all $n \ge N$,

$$\|x_n\| \le \sum_{k=1}^\infty \|a_k^n z_k\| \le \|(a_k^n)_k\|_{p^*} \|(z_k)\|_p \le \|(a_k^n)_k\|_r \|(z_k)\|_p \le \varepsilon,$$

and therefore $x_n \to 0$.

Since $\{x_N, x_{N+1}, ...\} \subset (p, r)$-conv$(z_k)$ and $(z_k) \in \ell_p(X)$, the sequence

$$y_k = \begin{cases} x_k & \text{if } k < N, \\ z_{k-N+1} & \text{if } k \ge N, \end{cases}$$

is in $\ell_p(X)$ and $x_n \in (p, r)$-conv(y_k) for all $n \in \mathbb{N}$. This means that (x_n) is relatively (p, r)-compact. □

We shall see in Section 6.5 below that the converse of Proposition 6.3 holds as well.

6.2 Relatively (p, r)-compact sets are generated by (p, r)-null sequences

Let X be a Banach space. According to the Grothendieck compactness principle (see Theorem 3.1), a subset K of X is relatively compact if and only if it is contained in the closed convex hull of a null sequence.

Delgado and Piñeiro obtained the following Grothendieck-like result (see [PD11, Theorem 2.5]).

Theorem 6.4 (Delgado–Piñeiro)**.** *Let $1 \leq p < \infty$. A subset K of a Banach space X is relatively p-compact if and only if K is contained in the closed convex hull of a p-null sequence.*

The proof of Theorem 6.4 in [PD11] is not self-contained. It relies on some theory of p-compact operators developed by Delgado, Piñeiro, and Serrano in [DPS10b] (see [DPS10b, Corrollary 3.4, Propositions 3.5 and 3.8, and Theorem 3.13]) and uses a characterization of operators having absolutely p-summing adjoints (see [PD11, Proposition 2.4]).

In the case when $r = p^*$, the following theorem reduces to Theorem 6.4, also giving it an easy and elementary direct proof.

Theorem 6.5. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. A subset K of a Banach space X is relatively (p, r)-compact if and only if K is contained in the closed convex hull of a (p, r)-null sequence.*

Remark that the special case $p = \infty$ of Theorem 6.5 coincides with the Grothendieck compactness principle, because relatively $(\infty, 1)$-sets are precisely relatively compact sets and $(\infty, 1)$-null sequences are precisely null sequences.

Proof of Theorem 6.5. For the "if " part, let $(x_n) \subset X$ be a (p, r)-null sequence. By Proposition 6.3, (x_n) is relatively (p, r)-compact. Thus $(x_n) \subset \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p(X)$. The set $\Phi_{(z_k)}(B_{\ell_r})$ is clearly absolutely convex. It is also weakly compact. Indeed, if $1 < r < \infty$, then ℓ_r is reflexive and therefore B_{ℓ_r} is weakly compact. If $r = 1$ (or $r = \infty$), then the proof is similar to the proof of Proposition 3.3. Indeed, then $B_{\ell_1} = B_{c_0^*}$ (or

$B_{\ell_\infty} = B_{\ell_1^*}$) is weak* compact due to the well-known Banach–Alaoglu theorem and $\Phi_{(z_k)} \in \mathcal{L}(c_0^*, X)$ (or $\Phi_{(z_k)} \in \mathcal{L}(\ell_1^*, X)$) is weak*-to-weakly continuous (because $\Phi_{(z_k)}^*(X^*) \subset c_0$ (or $\Phi_{(z_k)}^*(X^*) \subset \ell_1$)). Hence, as weakly closed convex sets are closed in the norm topology, $\Phi_{(z_k)}(B_{\ell_r})$ is a closed absolutely convex subset of X containing (x_n). Therefore $\overline{\operatorname{conv}}(x_n) \subset \Phi_{(z_k)}(B_{\ell_r})$ meaning that $\overline{\operatorname{conv}}(x_n)$ is relatively (p,r)-compact.

For the "only if" part, let us assume that $K \subset X$ is relatively (p,r)-compact. We clearly may assume that $K = \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p(X)$. We are going to construct a (p,r)-null sequence (x_n) such that $K \subset \overline{\operatorname{conv}}(x_n)$.

Similarly to the very beginning of the proof of [DOPS09, Theorem 2.1, (c)⇒(a)] (or of [PD11, Theorem 2.5]), we choose $\alpha_k \searrow 0$ such that $(\alpha_k^{-1} z_k) \in \ell_p(X)$, and we consider the compact diagonal operator $D : \ell_r \to \ell_r$, defined by $D(\beta_k) = (\alpha_k \beta_k)$, $(\beta_k) \in \ell_r$, and $\Phi := \Phi_{(\alpha_k^{-1} z_k)} : \ell_r \to X$. Then, clearly, $\Phi_{(z_k)} = \Phi D$.

Since $D(B_{\ell_r})$ is a relatively compact subset of ℓ_r, by the Grothendieck compactness principle, there exists a sequence $(\Gamma_n) \subset \ell_r$ such that $\Gamma_n \to 0$ and $D(B_{\ell_r}) \subset \overline{\operatorname{conv}}(\Gamma_n)$. Denote $x_n = \Phi\Gamma_n$. Then $K \subset \overline{\operatorname{conv}}(x_n)$, and it remains to show that (x_n) is a (p,r)-null sequence.

Let $\Gamma_n = (\gamma_k^n)_{k=1}^\infty$. Then $\sum_{k=1}^\infty |\gamma_k^n|^r = \|\Gamma_n\|_r^r \underset{n}{\to} 0$ if $r < \infty$ or $\sup_k |\gamma_k^n| \underset{n}{\to} 0$ if $r = \infty$. Let us only consider the former case, the latter case being similar.

Let $\varepsilon > 0$ be fixed. Choose $\delta > 0$ satisfying $\delta^r \le 1 - 2^{-r}$ and $\delta^p(2^p\nu^p + 1) \le \varepsilon^p$, where $\nu := \|(\alpha_k^{-1} z_k)\|_p$. Then there exists $N \in \mathbb{N}$ such that

$$\sum_{k=1}^\infty |\gamma_k^n|^r < \delta^r$$

if $n \ge N$, and also

$$\sum_{k>N} \|\alpha_k^{-1} z_k\|^p < \delta^p.$$

Now

$$\begin{aligned} x_n &= \sum_{k=1}^\infty \gamma_k^n \alpha_k^{-1} z_k = \sum_{k=1}^N \gamma_k^n \alpha_k^{-1} z_k + \sum_{k>N} \gamma_k^n \alpha_k^{-1} z_k \\ &= \sum_{k=1}^N \frac{\gamma_k^n}{2\delta} 2\delta \alpha_k^{-1} z_k + \sum_{k>N} \gamma_k^n \alpha_k^{-1} z_k = \sum_{k=1}^\infty \delta_k^n y_k, \end{aligned}$$

where

$$(\delta_k^n)_{k=1}^\infty := \Big(\frac{\gamma_1^n}{2\delta}, \ldots, \frac{\gamma_N^n}{2\delta}, \gamma_{N+1}^n, \gamma_{N+2}^n, \ldots\Big) \in \ell_r$$

and

$$(y_k)_{k=1}^{\infty} := (2\delta\alpha_1^{-1}z_1, ..., 2\delta\alpha_N^{-1}z_N, \alpha_{N+1}^{-1}z_{N+1}, \alpha_{N+2}^{-1}z_{N+2}, ...) \subset X.$$

Observe that

$$\begin{aligned} \|(y_k)\|_p^p &= 2^p\delta^p \sum_{k=1}^{N} \|\alpha_k^{-1}z_k\|^p + \sum_{k>N} \|\alpha_k^{-1}z_k\|^p \\ &< 2^p\delta^p\nu^p + \delta^p = \delta^p(2^p\nu^p + 1) \le \varepsilon^p, \end{aligned}$$

i.e., $(y_k) \in \ell_p(X)$ and $\|(y_k)\|_p \le \varepsilon$. Observe also that, for every $n \ge N$,

$$\|(\delta_k^n)_k\|_r^r = \sum_{k=1}^{N} \frac{|\gamma_k^n|^r}{2^r\delta^r} + \sum_{k>N} |\gamma_k^n|^r < \frac{\delta^r}{2^r\delta^r} + \delta^r = \frac{1}{2^r} + \delta^r \le 1,$$

i.e., $(\delta_k^n)_k \in B_{\ell_r}$ if $n \ge N$. Hence, for every $n \ge N$, we have

$$x_n = \sum_{k=1}^{\infty} \delta_k^n y_k \in (p, r)\text{-conv}(y_k),$$

as desired. □

6.3 Some classes of bounded sets

Let us introduce some useful notation which is inspired by [Ste80], but seems to be more suggestive than the notation in [Ste80].

Let $\boldsymbol{b}$ denote the class of all bounded subsets of all Banach spaces, and let $\boldsymbol{g}$ be a subclass of $\boldsymbol{b}$. Let X be a Banach space. Following [Ste80, Definition 1.1], we denote by $\boldsymbol{g}(X)$ the family of subsets of X which are of type $\boldsymbol{g}$. For instance, $\boldsymbol{b}(X)$ is the family of all bounded subsets of X.

We denote by $\boldsymbol{w}$ and $\boldsymbol{k}$, respectively, the classes of all relatively weakly compact and relatively compact subsets of all Banach spaces. It is convenient to denote by $\boldsymbol{k}_{(p,r)}$ the class of all relatively (p,r)-compact sets in all Banach spaces. In particular, as stated in Section 3.1, $\boldsymbol{k} = \boldsymbol{k}_{(\infty,1)}$ and $\boldsymbol{k}_p := \boldsymbol{k}_{(p,p^*)}$, the class of all relatively p-compact sets.

Let $\mathcal{A}$ be an operator ideal. Denote by $\mathcal{A}(\boldsymbol{g})$ the subclass of $\boldsymbol{b}$, which is given as

$$\mathcal{A}(\boldsymbol{g})(X) = \{E \subset X : E \subset T(F) \text{ for some } F \in \boldsymbol{g}(Y) \text{ and } T \in \mathcal{A}(Y, X)\}$$

where X is an arbitrary Banach space (in [Ste80], the notation $\mathcal{A} \circ \boldsymbol{g}$ is used).

In this notation, the Grothendieck compactness principle (see Theorem 3.1) reads as follows.

Proposition 6.6 (Grothendieck). *One has* $\boldsymbol{k} = \overline{\mathcal{F}}(\boldsymbol{b}) = \mathcal{K}(\boldsymbol{b})$.

Proof. Let X be a Banach space and let $K \in \boldsymbol{k}(X)$. Grothendieck's criterion gives us a sequence $(x_n) \in c_0(X)$ such that $K \subset \Phi_{(x_n)}(B_{\ell_1})$. Since $\Phi_{(x_n)} \in \overline{\mathcal{F}}(\ell_1, X)$, it is clear that K is of type $\overline{\mathcal{F}}(\boldsymbol{b})$. But $\overline{\mathcal{F}}(\boldsymbol{b}) \subset \mathcal{K}(\boldsymbol{b})$ because $\overline{\mathcal{F}} \subset \mathcal{K}$. Finally, if K is of type $\mathcal{K}(\boldsymbol{b})$, then it is relatively compact. □

Proposition 6.6 says, in particular, that $\boldsymbol{k}_{(\infty,1)} = \mathcal{K}_{(\infty,1)}(\boldsymbol{b})$. Using the definitions of $\boldsymbol{k}_{(p,r)}$ and $\mathcal{K}_{(p,r)}$ together with the observation (see Proposition 2.16) that $\Phi_{(x_n)}$ belongs to the operator ideal $\mathcal{N}_{(p,1,r^*)}$, the above proof yields also the general case.

Proposition 6.7. *Let* $1 \le p \le \infty$ *and* $1 \le r \le p^*$. *Then* $\boldsymbol{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\boldsymbol{b}) = \mathcal{K}_{(p,r)}(\boldsymbol{b})$.

Remark 6.8. Using the notion of ideal system of sets (see [Ste73]), the equalities $\boldsymbol{k} = \mathcal{K}(\boldsymbol{b})$ and $\boldsymbol{w} = \mathcal{W}(\boldsymbol{b})$ were observed in [Ste80]. In the special case $p = 1$, $r = \infty$, the left-hand equality $\boldsymbol{k}_1 = \boldsymbol{k}_{(1,\infty)} = \mathcal{N}(\boldsymbol{b})$ of Proposition 6.7 was proved in [Ste73].

Recall (see [FS81, Theorem 2.5] or, e.g., [Pie80, 18.3.2]) that $\mathcal{N}_{(\infty,1,\infty)}$ coincides with the operator ideal K_1 of *classical* 1-compact operators. Following Fourie and Swart [FS79] or Pietsch [Pie80, 18.3.1 and 18.3.2], a linear operator $T : Y \to X$ is called 1-*compact*, i.e., $T \in \mathrm{K}_1(Y, X)$, if there exist $A \in \mathcal{K}(Y, \ell_1)$ and $B \in \mathcal{K}(\ell_1, X)$ such that $T = BA$. This means that T admits a compact factorization through ℓ_1. By an important theorem due to Johnson (see [Joh71]) $\overline{\mathcal{F}}(X, Y) = \mathrm{K}_1(X, Y)$ *for all Banach spaces* X *if and only if* Y *is an* $\mathcal{L}_1$-space. Recall that a Banach space X is an $\mathcal{L}_p$-*space*, $1 \le p \le \infty$, if there exists $\lambda > 1$ such that for each finite-dimensional subspace E of X there are a finite-dimensional subspace F of X with $E \subset F$ and an isomorphism $T : F \to \ell_p^{\dim F}$ satisfying $\|T\| \, \|T^{-1}\| \le \lambda$ (see, e.g., [JL01, pp. 57–60] for the definition and properties of $\mathcal{L}_p$-spaces). Since there are Banach spaces that are not $\mathcal{L}_1$-spaces (for instance, infinite-dimensional Hilbert spaces (see, e.g., [DF93, 23.3])), K_1 is strictly contained in $\overline{\mathcal{F}}$. Hence, the following special case of Proposition 6.7 slightly improves Proposition 6.6.

Corollary 6.9. *One has* $\boldsymbol{k} = \mathrm{K}_1(\boldsymbol{b})$.

6.4 $\mathcal{A}$-null sequences and $\mathcal{A}$-compact sets

Let us now describe some relevant notions from the Carl–Stephani theory [CS84], which is based on earlier work by Stephani [Ste72, Ste73, Ste80].

Let $\mathcal{A}$ be an operator ideal.

Following [CS84, Lemma 1.2], a sequence (x_n) in a Banach space X is called $\mathcal{A}$-*null* if there exist a Banach space Y, a null sequence (y_n) in Y, and $T \in \mathcal{A}(Y, X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$.

Using the notation of Section 6.3 and following [CS84, Theorem 1.2], we say (as in [LT13]) that a subset K of a Banach space X is $\mathcal{A}$-*compact* if K is of type $\mathcal{A}(\boldsymbol{k})$, i.e. $K \in \mathcal{A}(\boldsymbol{k})(X)$.

Using Propositions 6.6 and 6.7, we shall see now that the relatively (p, r)-compact sets, $\mathcal{N}_{(p,1,r^*)}$-compact sets, and $\mathcal{K}_{(p,r)}$-compact sets are all the same.

Proposition 6.10. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then $\boldsymbol{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\boldsymbol{k}) = \mathcal{K}_{(p,r)}(\boldsymbol{k})$.*

Proof. We know that $\mathcal{N}_{(p,1,r^*)}$ is a minimal operator ideal (see [Pie80, 18.1.4]). This means that $\mathcal{N}_{(p,1,r^*)} = \overline{\mathcal{F}} \circ \mathcal{N}_{(p,1,r^*)} \circ \overline{\mathcal{F}}$ (see [Pie80, 4.8.6]). Hence, using Propositions 6.7 and 6.6, we have

$$\begin{aligned} \mathcal{K}_{(p,r)}(\boldsymbol{k}) &\subset \mathcal{K}_{(p,r)}(\boldsymbol{b}) = \boldsymbol{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\boldsymbol{b}) = (\overline{\mathcal{F}} \circ \mathcal{N}_{(p,1,r^*)})(\overline{\mathcal{F}}(\boldsymbol{b})) \\ &= \overline{\mathcal{F}} \circ \mathcal{N}_{(p,1,r^*)}(\boldsymbol{k}) \subset \mathcal{N}_{(p,1,r^*)}(\boldsymbol{k}) \subset \mathcal{K}_{(p,r)}(\boldsymbol{k}). \end{aligned}$$

This shows that $\boldsymbol{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\boldsymbol{k}) = \mathcal{K}_{(p,r)}(\boldsymbol{k})$. □

Remark 6.11. The second equality in Proposition 6.10 also follows from the general Carl–Stephani theory. Indeed, for any operator ideal $\mathcal{A}$, it is known (see [CS84, p. 79]) that a subset is $\mathcal{A}$-compact if and only if it is $\mathcal{A}^{\mathrm{sur}}$-compact. And (see Theorem 4.1) $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)} = \mathcal{K}_{(p,r)}$.

The following special case of Proposition 6.10 (when $p = \infty$) improves Corollary 6.9 and complements Proposition 6.6.

Corollary 6.12. *One has $\boldsymbol{k} = \mathrm{K}_1(\boldsymbol{k}) = \overline{\mathcal{F}}(\boldsymbol{k}) = \mathcal{K}(\boldsymbol{k})$.*

6.5 Characterizations of (p, r)-null sequences

The p-null sequences can be characterized via p-compactness as follows.

Theorem 6.13 (Delgado–Piñeiro–Oja). *Let $1 \le p < \infty$. A sequence (x_n) in a Banach space X is p-null if and only if (x_n) is null and relatively p-compact.*

Theorem 6.13 was discovered in [PD11, Proposition 2.6] and proved in the case of Banach spaces enjoying a version of the approximation property depending on p (by [Oja12b], this version of the approximation property coincides with the classical one for closed subspaces of $L_p(\mu)$-spaces). For arbitrary Banach spaces, Theorem 6.13 was proved in [Oja12a]. The proof of Theorem 6.13 in [Oja12a] relies on the description of the space of p-null sequences as a Chevet–Saphar tensor product $c_0 \hat{\otimes}_{d_p} X$, also established in [Oja12a].

Remark 6.14. Theorem 6.13 is the "limit" case $r = p^*$ of the equivalence (a)⇔(b) in Theorem 6.17 below. This is, in fact, the only special case when Theorem 6.17 could be proved by the method in [Oja12a]. The reason is simple: the method in [Oja12a] uses the Hahn–Banach theorem. But the (p,r)-context provides a suitable norm only if $r = p^*$, and in all other cases merely quasi-norms are available (cf. Remark 4.5). But, as is well known, quasi-normed spaces do not enjoy the Hahn–Banach theorem.

Very recently, an alternative natural proof of Theorem 6.13 was found by Lassalle and Turco [LT13] who rediscovered and applied a powerful theory due to Carl and Stephani [CS84] from 1984. Lassalle–Turco's proof in [LT13] relies on the following operator ideal version of Theorem 6.13, deduced from the Carl–Stephani theory in [LT13, Proposition 1.4].

Theorem 6.15 (Lassalle–Turco). *Let $\mathcal{A}$ be an operator ideal. A sequence (x_n) in a Banach space X is $\mathcal{A}$-null if and only if (x_n) is null and $\mathcal{A}$-compact.*

A starting point for the present section was the observation that, in the proof of Theorem 6.13, Theorem 6.15 could be used in a more efficient way than in [LT13]. In particular, the technical result [LT13, Proposition 1.5] would not be needed in the proof. Even more, it is obtained for "free" as a by-product (see Remark 6.18).

Theorem 6.17 below is an omnibus theorem, which provides six equivalent properties for a sequence in a Banach space to be a (p,r)-null sequence. One of these properties is to be a uniformly (p,r)-null sequence, which is a natural (formal) strengthening of a (p,r)-null sequence.

Definition 6.16. Let $1 \le p \le \infty$ and $1 \le r \le p^*$. We call a sequence (x_n) in a Banach space X *uniformly (p,r)-null* if there exists $(z_k) \in B_{\ell_p(X)}$ ($(z_k) \in c_0(X)$ when $p = \infty$) with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in \varepsilon\,(p,r)\text{-conv}(z_k)$ for all $n \ge N$.

We say that (x_n) is *uniformly p-null* if it is uniformly (p,p^*)-null. The latter property was implicitly used in a result by Lassalle and Turco asserting (in the above terminology) that the p-null sequences are always uniformly p-null (concerning the proof, see Remark 6.18).

Theorem 6.17. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(a) *(x_n) is (p,r)-null,*

(b) *(x_n) is null and relatively (p,r)-compact,*

(c) *(x_n) is null and $\mathcal{N}_{(p,1,r^*)}$-compact,*

(d) *(x_n) is null and $\mathcal{K}_{(p,r)}$-compact,*

(e) *(x_n) is $\mathcal{N}_{(p,1,r^*)}$-null,*

(f) *(x_n) is $\mathcal{K}_{(p,r)}$-null,*

(g) *(x_n) is uniformly (p,r)-null.*

Proof. The implication (a)⇒(b) is exactly Proposition 6.3.

Implications (b)⇔(c)⇔(d) are immediate from Proposition 6.10.

Implications (c)⇔(e) and (d)⇔(f) are immediate from Theorem 6.15.

To prove that (f)⇒(g), let (x_n) be a $\mathcal{K}_{(p,r)}$-null sequence. Then there are a null sequence (y_n) in a Banach space Y and an operator $T \in \mathcal{K}_{(p,r)}(Y,X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$. The (p,r)-compactness of T gives us a sequence $(w_k) \in \ell_p(X)$ such that $T(B_Y) \subset (p,r)\text{-conv}(w_k)$. Now $(z_k) := \left(\frac{w_k}{\|(w_k)\|_p}\right) \in B_{\ell_p(X)}$, and let $\varepsilon > 0$. As (y_n) is null in Y, for $\varepsilon_0 := \frac{\varepsilon}{\|(w_k)\|_p}$ there exists $N \in \mathbb{N}$ such that $Ty_n \in \varepsilon_0 T(B_Y)$ for all $n \ge N$. Hence,

$$x_n \in \varepsilon_0 (p,r)\text{-conv}(w_k) = \varepsilon_0 \left\|(w_k)\right\|_p (p,r)\text{-conv}(z_k) = \varepsilon (p,r)\text{-conv}(z_k)$$

for all $n \ge N$, as desired.

The implication (g)⇒(a) is clear from the definitions, because if $(z_k) \in B_{\ell_p(X)}$ ($(z_k) \in B_{c_0(X)}$ when $p = \infty$), then $(\varepsilon z_k) \in \varepsilon B_{\ell_p(X)}$ ($(\varepsilon z_k) \in \varepsilon B_{c_0(X)}$ when $p = \infty$) and $(p,r)\text{-conv}(\varepsilon z_k) = \varepsilon (p,r)\text{-conv}(z_k)$. □

Remark 6.18. The technical Lassalle–Turco result [LT13, Proposition 1.5] (inspired by [AMR10, Theorem 1]) to prove Theorem 6.13 is not needed. Even more, this technical result appears as a simple by-product of our proof:

it is precisely the special case of the implication (a)⇒(g) when $r = p^*$, $p \neq \infty$. For $p = \infty$, the implication (a)⇒(g) of Theorem 6.17 together with Proposition 6.2 gives us that null sequences in a Banach space are uniformly $(\infty, 1)$-null. This coinciding is not immediate from the definitions.

Remark 6.19. For completeness, let us show how Theorem 6.17, (b)⇒(a), could be used in the proof of the "only if" part of Theorem 6.5. Indeed, by the beginning of this proof, we have $x_n = \Phi\Gamma_n$, where $(\Gamma_n) \subset \ell_r$ is a null sequence, and we need to show that (x_n) is a (p,r)-null sequence. Observe that (x_n) is a null sequence, and since $(x_n) \subset M\Phi(B_{\ell_r})$ for some $M > 0$, where $\Phi = \Phi_{(w_k)}$ with $(w_k) \in \ell_p(X)$, the sequence (x_n) is also relatively (p,r)-compact. Hence, by Theorem 6.17, (b)⇒(a), (x_n) is a (p,r)-null sequence. The idea of the above proof essentially belongs to Kim [Kim14, proof of Corollary 1.3], where it was applied in the context of unconditionally p-null sequences (see Remark 7.10).

Let $\mathcal{A}$ be an operator ideal. Let K be an $\mathcal{A}$-compact set and let (x_n) be an $\mathcal{A}$-null sequence. If $\mathcal{B}$ is a larger operator ideal than $\mathcal{A}$, i.e. $\mathcal{A} \subset \mathcal{B}$, then, by definitions, clearly, K is also $\mathcal{B}$-compact and (x_n) is $\mathcal{B}$-null. The equality

$$\mathcal{K}_{(p,r)} = \mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)} \circ \mathcal{K},$$

from Theorem 5.8 enables us to extend characterizations (d) and (f) of (p,r)-null sequences of Theorem 6.17 to even more larger operator ideal than $\mathcal{K}_{(p,r)}$, namely to $\mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)}$.

Proposition 6.20. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(a) *(x_n) is (p,r)-null,*

(b) *(x_n) is null and $\mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)}$-compact,*

(c) *(x_n) is $\mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)}$-null.*

Proof. As was mentioned, $\mathcal{K}_{(p,r)} = \mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)} \circ \mathcal{K}$. Hence, using Propositions 6.7 and 6.6, we have

$$\boldsymbol{k}_{(p,r)} = \mathcal{K}_{(p,r)}(\boldsymbol{b}) = \mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)}(\mathcal{K}(\boldsymbol{b})) = \mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)}(\boldsymbol{k}).$$

This shows that relatively (p,r)-compact sets are exactly $\mathcal{I}^{\mathrm{sur}}_{(p,1,r^*)}$-compact sets. The claim now follows from Theorems 6.17 and 6.15. □

Since $\mathcal{P}_p = \mathcal{I}^{\text{inj}}_{(p,p,1)}$ as Banach operator ideals (see [Pie80, 19.2.7]), Remark 5.9 gives $\mathcal{P}_p^{\text{dual}} = \mathcal{I}^{\text{sur}}_{(p,1,p)}$ when $1 \leq p < \infty$. Therefore we can spell out, from Theorem 6.17 and Proposition 6.20, the following omnibus characterization of p-null sequences.

Corollary 6.21. *Let $1 \leq p < \infty$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(a) *(x_n) is p-null,*

(b) *(x_n) is null and relatively p-compact,*

(c) *(x_n) is null and $\mathcal{N}^p$-compact,*

(d) *(x_n) is null and $\mathcal{K}_p$-compact,*

(e) *(x_n) is null and $\mathcal{P}_p^{\text{dual}}$-compact,*

(f) *(x_n) is $\mathcal{N}^p$-null,*

(g) *(x_n) is $\mathcal{K}_p$-null,*

(h) *(x_n) is $\mathcal{P}_p^{\text{dual}}$-null,*

(i) *(x_n) is uniformly p-null.*

Remark 6.22. The equivalence (a)$\Leftrightarrow$(b) is precisely Theorem 6.13 we started Section 6.5 with.

Chapter 7

Unconditionally and weakly (p, r)-null sequences

The notions form Chapters 3 and 6 involving the (p, r)-compactness are extended to the unconditional and weak (p, r)-compactnesses. Using the techniques developed previously, the unconditionally (p, r)-null and weakly (p, r)-null sequences are described. This chapter is mainly based on [AO15].

7.1 Unconditional and weak (p, r)-compactnesses

Let X be a Banach space, and let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$.

The (uniformly) (p, r)-null sequences and (p, r)-compactness in X are defined in terms of (p, r)-convex hulls using the space $\ell_p(X)$ of absolutely p-summable sequences in X. In general, (p, r)-convex hulls can be defined using the space $\ell_p^w(X)$ of weakly p-summable sequences in X. This is a pretty old idea, going back at least to the paper [CS93, p. 51] by Castillo and Sanchez in 1993. In [CS93], the (p, p^*)-convex hull of $(x_n) \in \ell_p^w(X)$ was considered under the name of p^*-convex hull of (x_n). In 2002, Sinha and Karn [SK02] developed some of their theory of p-compactness in a more general context of weak p-compactness. In [SK02], also the (p, p^*)-convex hull of $(x_n) \in \ell_p^w(X)$ was used but under the name of p-convex hull of $(x_n) \in \ell_p^w(X)$.

"Between" absolutely p-summable sequences $\ell_p(X)$ and weakly p-summable sequences $\ell_p^w(X)$, there is the Banach space $\ell_p^u(X)$ of *unconditionally p-summable sequences* (see, e.g., [DF93, 8.2, 8.3]; we follow [BCFP12] in our terminology). The space $\ell_p^u(X)$ was introduced and thoroughly studied by Fourie and Swart [FS79] in 1979. The space $\ell_p^u(X)$ is defined as the (closed) subspace of $\ell_p^w(X)$, formed by the $(x_n) \in \ell_p^w(X)$ satisfying $(x_n) = \lim_{N\to\infty}(x_1, ..., x_N, 0, 0, ...)$ in $\ell_p^w(X)$. Remark that $\ell_\infty^u(X) = c_0(X)$ as Banach spaces (see [FS79, p. 351] or, e.g., [DF93, 8.2]) and recall that $\ell_\infty^w(X) = \ell_\infty(X)$.

The unconditionally p-summable sequences of X are characterized in [FS79, Lemma 1.2] as follows.

Lemma 7.1 (Fourie–Swart). *A sequence (x_n) in a Banach space X is an element of $\ell_p^u(X)$ if and only if there exist $(\delta_n) \in c_0$ and $(y_n) \in \ell_p^w(X)$ such that $x_n = \delta_n y_n$ for all $n \in \mathbb{N}$.*

Corollary 7.2. *If $(x_n) \in \ell_p^u(X)$, then there exists $1 \le \lambda_n \to \infty$ such that $\lambda_n x_n \in \ell_p^u(X)$.*

Proof. For $(x_n) \in \ell_p^u(X)$ there exist $(\delta_n) \in c_0$ and $(y_n) \in \ell_p^w(X)$ such that $x_n = \delta_n y_n$, $n \in \mathbb{N}$. Without loss of generality we may assume that $\delta_n \le 1$ for all n. Now by Lemma 7.1 $(\lambda_n x_n) \in \ell_p^u(X)$, where $\lambda_n := \delta_n^{-1/2}$. We also have that $1 \le \lambda_n \to \infty$. □

In the present Chapter 7, we shall assume that the definition of the (p,r)-convex hull (p,r)-conv(x_n) (see Definition 3.2) is extended to $(x_n) \in \ell_p^w(X)$ as follows (cf. Remark 3.5).

Definition 7.3. Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Let $(x_n) \in \ell_p^w(X)$. If $r \ne \infty$, then call

$$(p,r)\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r} \right\}$$

the (p,r)-*convex hull* of (x_n). If $r = \infty$ and $(x_n) \in \ell_1^u(X)$, then we define

$$1\text{-conv}(x_n) := (1,\infty)\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_\infty} \right\}.$$

If $r = \infty$ and $(x_n) \in \ell_1^w(X)$, then we define

$$1\text{-co}(x_n) := (1,\infty)\text{-co}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0} \right\}.$$

Now, for $(x_n) \in \ell_p^w(X)$, the operator $\Phi_{(x_n)} : \ell_r \to X$ ($\Phi_{(x_n)} : c_0 \to X$ when $r = \infty$) is also well defined and

$$(p,r)\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_r})$$

if $r \neq \infty$, and

$$1\text{-co}(x_n) = (1,\infty)\text{-co}(x_n) = \Phi_{(x_n)}(B_{c_0}).$$

But $\Phi_{(x_n)}$ need not be a compact operator any more (see, e.g., Section 7.3). It follows from [FS79, Theorem 1.4] that $\Phi_{(x_n)}$ is compact whenever $(x_n) \in \ell_p^u(X)$. In fact, $\Phi_{(x_n)} : \ell_{p^*} \to X$ ($\Phi_{(x_n)} : c_0 \to X$ when $p = 1$) *is compact if and only if* $(x_n) \in \ell_p^u(X)$ (see [FS79, Theorem 1.4] or, e.g., [DF93, 8.2]).

We define relatively unconditionally (respectively, weakly) (p,r)-compact sets in X by replacing $\ell_p(X)$ with $\ell_p^u(X)$ (respectively, with $\ell_p^w(X)$) in the definition of relatively (p,r)-compact sets. Recall that $c_0^w(X)$ is the closed subspace of $\ell_\infty^w(X) = \ell_\infty(X)$ consisting of all weakly null sequences in X (see, e.g., [DJT95, p. 33]).

Definition 7.4. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We say that a subset K of X is *relatively unconditionally* (p,r)*-compact* if $K \subset (p,r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p^u(X)$. We say that K is *relatively weakly* (p,r)*-compact* if $K \subset (p,r)\text{-conv}(x_n)$ ($K \subset 1\text{-co}(x_n)$ when $r = \infty$) for some $(x_n) \in \ell_p^w(X)$ ($(x_n) \in c_0^w(X)$ when $p = \infty$).

The classes of relatively unconditionally (p,r)-compact sets and relatively weakly (p,r)-compact sets in all Banach spaces are denoted, respectively, by $\boldsymbol{u}_{(p,r)}$ and $\boldsymbol{w}_{(p,r)}$.

According to Remark 3.5, if $(x_n) \in \ell_1(X)$, then

$$(1,\infty)\text{-co}(x_n) \subset (1,\infty)\text{-conv}(x_n) \subset (1,\infty)\text{-co}(\lambda_n x_n) \tag{7.1}$$

whenever $1 \leq \lambda_n \to \infty$ is chosen such that $(\lambda_n x_n) \in \ell_1(X)$. If $(x_n) \in \ell_1^u(X)$, then there also exists (λ_n), $1 \leq \lambda_n \to \infty$, such that $(\lambda_n x_n) \in \ell_p^u(X)$ (see Corollary 7.2) and therefore (7.1) holds. Hence

$$\boldsymbol{k}_{(p,r)} \subset \boldsymbol{u}_{(p,r)} \subset \boldsymbol{w}_{(p,r)}.$$

We clearly also have that

$$\boldsymbol{u}_{(p,r)} \subset \boldsymbol{u}_{(\infty,1)} = \boldsymbol{k}_{(\infty,1)} = \boldsymbol{k}.$$

Definition 7.5. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A linear operator T from a Banach space Y to X is *unconditionally* (respectively, *weakly*) (p,r)*-compact* if $T(B_Y)$ is a relatively unconditionally (respectively, weakly) (p,r)-compact subset of X.

Let $\mathcal{U}_{(p,r)}$ and $\mathcal{W}_{(p,r)}$ denote the classes of all unconditionally and weakly (p,r)-compact operators acting between arbitrary Banach spaces, so that

$$\mathcal{K}_{(p,r)} \subset \mathcal{U}_{(p,r)} \subset \mathcal{W}_{(p,r)} \text{ and } \mathcal{U}_{(p,r)} \subset \mathcal{K}.$$

It is clear from the definitions that

$$\boldsymbol{u}_{(p,r)} = \mathcal{U}_{(p,r)}(\boldsymbol{b}) \text{ and } \boldsymbol{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\boldsymbol{b}).$$

An easy straightforward verification, as in the case of $\mathcal{K}_{(p,r)}$ in Propositions 3.8 and 3.9, shows that $\mathcal{U}_{(p,r)}$ *and* $\mathcal{W}_{(p,r)}$ *are surjective operator ideals.*

Note that $\mathcal{W}_{(p,p^*)} = \mathcal{W}_p$, the class of *weakly* p*-compact operators*, studied in [SK02]. Similarly, in all cases, we shall write "p-" instead of "(p,p^*)-", and speak, for instance, about the operator ideal $\mathcal{U}_p$ of *unconditionally* p*-compact operators.*

7.2 Unconditionally (p,r)-null sequences

Let X be a Banach space, and let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. It is rather easy to see that our approach in Chapter 6 goes through if $\ell_p(X)$ is replaced with the larger space $\ell_p^u(X)$. Let us start by fixing the relevant terminology.

We define unconditionally (p,r)-null sequences in a Banach space X by replacing $\ell_p(X)$ with $\ell_p^u(X)$ in the corresponding definition of (p,r)-null sequences (see Definition 6.1).

Definition 7.6. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We call a sequence (x_n) in X *unconditionally* (p,r)*-null* if for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p^u(X)$ with $\|(z_k)\|_p^w \leq \varepsilon$ such that $x_n \in (p,r)\text{-conv}(z_k)$ for all $n \geq N$.

The p-conv(z_k) coincides with the (p,p^*)-convex hull of $(z_k) \in \ell_p^u(X)$ and therefore we call the unconditionally (p,p^*)-null sequences *unconditionally* p*-null.* These sequences have been studied also in [Kim14].

Similarly to Section 6.1, let us make some elementary observations. Recalling that $\ell_\infty^u(X) = c_0(X)$, we immediately have by Proposition 6.2 the following.

Proposition 7.7. *For a sequence* (x_n) *in a Banach space* X *the following statements are equivalent:*

(a) (x_n) *is unconditionally* $(\infty, 1)$*-null,*

(b) (x_n) *is* $(\infty, 1)$*-null,*

(c) $x_n \to 0$.

Proposition 7.8. *Let* $1 \le p \le \infty$ *and* $1 \le r \le p^*$. *If a sequence* (x_n) *in a Banach space* X *is unconditionally* (p,r)*-null, then* $x_n \to 0$ *and* (x_n) *is relatively unconditionally* (p,r)*-compact.*

Proof. Let (x_n) be unconditionally (p,r)-null. Then, as in the proof of Proposition 6.3, for every $\varepsilon > 0$ there are $N \in \mathbb{N}$ and $(z_k) \in \ell_p^u(X)$, $\|(z_k)\|_p^w \le \varepsilon$, such that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_r}$, for all $n \ge N$. Hence,

$$\|x_n\| = \sup_{x^* \in B_{X^*}} |x^*(x_n)| \le \sup_{x^* \in B_{X^*}} \sum_{k=1}^{\infty} |a_k^n x^*(z_k)| \le \|(a_k^n)_k\|_r \, \|(z_k)\|_p^w \le \varepsilon,$$

for all $n \ge N$, and therefore $x_n \to 0$.

Since $\{x_N, x_{N+1}, ...\} \subset (p,r)\text{-conv}(z_k)$ and $(z_k) \in \ell_p^u(X)$, the sequence

$$y_k = \begin{cases} x_k & \text{if } k < N, \\ z_{k-N+1} & \text{if } k \ge N, \end{cases}$$

is in $\ell_p^u(X)$ and $x_n \in (p,r)\text{-conv}(y_k)$ for all $n \in \mathbb{N}$. This means that (x_n) is relatively unconditionally (p,r)-compact. □

Recall that the Grothendieck compactness principle (Theorem 3.1) was extended to relatively p-compact sets by Delgado and Piñeiro (Theorem 6.4). The Delgado–Piñeiro theorem in turn was very recently extended to relatively unconditionally p-compact sets by [Kim14, Corollary 1.3] as follows.

Theorem 7.9 (Kim)**.** *Let* $1 \le p < \infty$. *A subset* K *of a Banach space* X *is relatively unconditionally* p*-compact if and only if* K *is contained in the closed convex hull of an unconditionally* p*-null sequence.*

Remark 7.10. The proof of Theorem 7.9 in [Kim14] is not self-contained. It relies on [Kim14, Theorem 1.2] which is the special case $r = p^*$ of Theorem 7.13, (b)$\Leftrightarrow$(a), below (see also the remark after Corollary 7.15 concerning the proof of [Kim14, Theorem 1.2]).

The proof of Theorem 6.5 can be easily adopted to give an easy and direct proof of the following theorem, which contains Theorem 7.9 as the special case $r = p^*$.

Theorem 7.11. *Let $1 \le p < \infty$ and $1 \le r \le p^*$. A subset K of a Banach space X is relatively unconditionally (p,r)-compact if and only if K is contained in the closed convex hull of an unconditionally (p,r)-null sequence.*

As in the case of relatively (p,r)-compactness, let us remark that the special case $p = \infty$ of Theorem 7.11 coincides with the Grothendieck compactness principle, because relatively unconditionally $(\infty,1)$-compact sets are exactly relatively compact sets and unconditionally $(\infty,1)$-null sequences are exactly null sequences.

Proof of Theorem 7.11. It is very similar to the proof of Theorem 6.5.

For the "if " part, let $(x_n) \subset X$ be an unconditionally (p,r)-null sequence. By Proposition 7.8, (x_n) is relatively unconditionally (p,r)-compact. Thus $(x_n) \subset \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p^u(X)$. According to the proof of Theorem 6.5, the set $\Phi_{(z_k)}(B_{\ell_r})$ is a closed absolutely convex subset of X containing (x_n). Therefore $\overline{\text{conv}}(x_n) \subset \Phi_{(z_k)}(B_{\ell_r})$ meaning that $\overline{\text{conv}}(x_n)$ is relatively unconditionally (p,r)-compact.

For the "only if" part, let us assume that $K \subset X$ is relatively unconditionally (p,r)-compact. We clearly may assume that $K = \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p^u(X)$. We are going to construct an unconditionally (p,r)-null sequence (x_n) such that $K \subset \overline{\text{conv}}(x_n)$.

We can choose $\lambda_k \to \infty$ such that $(\lambda_k z_k) \in \ell_p^u(X)$ (see Corollary 7.2), and we consider the compact diagonal operator $D : \ell_r \to \ell_r$, defined by $D(\beta_k) = (\lambda_k^{-1}\beta_k)$, $(\beta_k) \in \ell_r$, and $\Phi := \Phi_{(\lambda_k z_k)} : \ell_r \to X$. Then, clearly, $\Phi_{(z_k)} = \Phi D$.

Since $D(B_{\ell_r})$ is a relatively compact subset of ℓ_r, by the Grothendieck compactness principle, there exists a sequence $(\Gamma_n) \subset \ell_r$ such that $\Gamma_n \to 0$ and $D(B_{\ell_r}) \subset \overline{\text{conv}}(\Gamma_n)$. Denote $x_n = \Phi\Gamma_n$. Then $K \subset \overline{\text{conv}}(x_n)$, and it remains to show that (x_n) is an unconditionally (p,r)-null sequence.

Let $\Gamma_n = (\gamma_k^n)_{k=1}^\infty$. Then $\sum_{k=1}^\infty |\gamma_k^n|^r = \|\Gamma_n\|_r^r \underset{n}{\to} 0$ if $r < \infty$ or $\sup_k |\gamma_k^n| \underset{n}{\to} 0$ if $r = \infty$. Let us only consider the former case, the latter case being similar.

Let $\varepsilon > 0$ be fixed. Choose $\delta > 0$ satisfying $\delta^r \le 1 - 2^{-r}$ and $\delta^p(2^p\nu^p + 1) \le \varepsilon^p$, where

$$\nu := \|(\lambda_k z_k)\|_p^w = \sup_{x^* \in B_{X^*}} (\sum_{k=1}^\infty |x^*(\lambda_k z_k)|^p)^{1/p}.$$

Then there exists $N \in \mathbb{N}$ such that

$$\sum_{k=1}^\infty |\gamma_k^n|^r < \delta^r$$

if $n \geq N$. Moreover, $(\lambda_k z_k)$ being an element of $\ell_p^u(X)$ means that

$$\lim_{m\to\infty} \|(\lambda_k z_k)_{k=m}^{\infty}\|_p^w = \lim_{m\to\infty} \sup_{x^*\in B_{X^*}} \left(\sum_{k=m}^{\infty} |x^*(\lambda_k z_k)|^p\right)^{\frac{1}{p}} = 0.$$

Therefore one can also assume that

$$\sum_{k>N} |x^*(\lambda_k z_k)|^p < \delta^p,\ x^* \in B_{X^*}.$$

Now

$$\begin{aligned} x_n &= \sum_{k=1}^{\infty} \gamma_k^n \lambda_k z_k = \sum_{k=1}^{N} \gamma_k^n \lambda_k z_k + \sum_{k>N} \gamma_k^n \lambda_k z_k \\ &= \sum_{k=1}^{N} \frac{\gamma_k^n}{2\delta} 2\delta \lambda_k z_k + \sum_{k>N} \gamma_k^n \lambda_k z_k = \sum_{k=1}^{\infty} \delta_k^n y_k, \end{aligned}$$

where

$$(\delta_k^n)_{k=1}^{\infty} := (\frac{\gamma_1^n}{2\delta}, ..., \frac{\gamma_N^n}{2\delta}, \gamma_{N+1}^n, \gamma_{N+2}^n, ...) \in \ell_r$$

and

$$(y_k)_{k=1}^{\infty} := (2\delta\lambda_1 z_1, ..., 2\delta\lambda_N z_N, \lambda_{N+1} z_{N+1}, \lambda_{N+2} z_{N+2}, ...) \subset X.$$

Observe that

$$\begin{aligned} (\|(y_k)\|_p^w)^p &= \sup_{x^*\in B_{X^*}} \sum_{k=1}^{\infty} |x^*(y_k)|^p \leq \sum_{k=1}^{N} |x^*(2\delta\lambda_k z_k)|^p + \sum_{k>N} |x^*(\lambda_k z_k)|^p \\ &< 2^p \delta^p \nu^p + \delta^p = \delta^p(2^p\nu^p + 1) \leq \varepsilon^p, \end{aligned}$$

i.e., $(y_k) \in \ell_p^w(X)$ and $\|(y_k)\|_p^w \leq \varepsilon$. The sequence (y_k) is chosen such that $(y_k)_{k>N} = (\lambda_k z_k)_{k>N}$, thus for $n > N$, we have

$$\|(y_k)_{k=n}^{\infty}\|_p^w = \|(\lambda_k z_k)_{k=n}^{\infty}\|_p^w \to_n 0,$$

meaning that $(y_k) \in \ell_p^u(X)$.

Observe also that, for every $n \geq N$,

$$\|(\delta_k^n)_k\|_r^r = \sum_{k=1}^{N} \frac{|\gamma_k^n|^r}{2^r\delta^r} + \sum_{k>N} |\gamma_k^n|^r < \frac{\delta^r}{2^r\delta^r} + \delta^r = \frac{1}{2^r} + \delta^r \leq 1,$$

i.e., $(\delta_k^n)_k \in B_{\ell_r}$ if $n \geq N$. Hence, for every $n \geq N$, we have

$$x_n = \sum_{k=1}^{\infty} \delta_k^n y_k \in (p,r)\text{-conv}(y_k),$$

as desired. □

We define uniformly unconditionally (p,r)-null sequences in a Banach space X by replacing $\ell_p(X)$ with $\ell_p^u(X)$ in the corresponding definition of uniformly (p,r)-null sequences (see Definition 6.16).

Definition 7.12. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We call a sequence (x_n) in a Banach space X *uniformly unconditionally* (p,r)*-null* if there exists $(z_k) \in B_{\ell_p^u(X)}$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in \varepsilon\,(p,r)\text{-conv}(z_k)$ for all $n \geq N$.

Let $(x_n) \in \ell_p^u(X)$. Then, by Lemma 7.1, $x_n = \delta_n y_n$ for some $(\delta_n) \in c_0$ and $(y_n) \in \ell_p^w(X)$. Since, clearly,

$$\Phi_{(x_n)} = \sum_{n=1}^{\infty} e_n \otimes x_n = \sum_{n=1}^{\infty} \delta_n e_n \otimes y_n$$

we have analogously to Proposition 2.16

$$\Phi_{(x_n)} \in \mathcal{N}_{(\infty,p^*,r^*)}(\ell_r, X).$$

Similarly, as in Section 4.1, we get that

$$\mathcal{U}_{(p,r)} = \mathcal{N}^{\text{sur}}_{(\infty,p^*,r^*)}.$$

This implies that

$$\mathcal{U}_{(p,r)} = \mathcal{K} \circ \mathcal{U}_{(p,r)} \circ \mathcal{K}.$$

Indeed, as in the proof of Proposition 6.10, $\mathcal{N}_{(\infty,p^*,r^*)} = \overline{\mathcal{F}} \circ \mathcal{N}_{(\infty,p^*,r^*)} \circ \overline{\mathcal{F}}$, and therefore

$$\mathcal{U}_{(p,r)} = (\overline{\mathcal{F}} \circ \mathcal{N}_{(\infty,p^*,r^*)} \circ \overline{\mathcal{F}})^{\text{sur}} \subset \overline{\mathcal{F}}^{\text{sur}} \circ \mathcal{N}^{\text{sur}}_{(\infty,p^*,r^*)} \circ \overline{\mathcal{F}}^{\text{sur}} = \mathcal{K} \circ \mathcal{U}_{(p,r)} \circ \mathcal{K},$$

because $\overline{\mathcal{F}}^{\text{sur}} = \mathcal{K}$ (see Example 2.9).

Further, similarly to Proposition 6.7, we have

$$\boldsymbol{u}_{(p,r)} = \mathcal{N}_{(\infty,p^*,r^*)}(\boldsymbol{b}) = \mathcal{U}_{(p,r)}(\boldsymbol{b}),$$

which implies (cf. Proposition 6.10 and its proof) that

$$\boldsymbol{u}_{(p,r)} = \mathcal{N}_{(\infty,p^*,r^*)}(\boldsymbol{k}) = \mathcal{U}_{(p,r)}(\boldsymbol{k}). \tag{7.2}$$

Using the above facts, we come to the omnibus characterization of unconditionally (p,r)-null sequences.

Theorem 7.13. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(a) *(x_n) is unconditionally (p,r)-null,*

(b) *(x_n) is null and relatively unconditionally (p,r)-compact,*

(c) *(x_n) is null and $\mathcal{N}_{(\infty,p^*,r^*)}$-compact,*

(d) *(x_n) is null and $\mathcal{U}_{(p,r)}$-compact,*

(e) *(x_n) is $\mathcal{N}_{(\infty,p^*,r^*)}$-null,*

(f) *(x_n) is $\mathcal{U}_{(p,r)}$-null,*

(g) *(x_n) is uniformly unconditionally (p,r)-null.*

Proof. The implication (a)⇒(b) is exactly Proposition 7.8.

Implications (b)⇔(c)⇔(d) are clear from (7.2).

Implications (c)⇔(e) and (d)⇔(f) are immediate from Theorem 6.15.

To prove that (f)⇒(g), let (x_n) be a $\mathcal{U}_{(p,r)}$-null sequence. Then there are a null sequence (y_n) in a Banach space Y and an operator $T \in \mathcal{U}_{(p,r)}(Y,X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$. As T is an unconditionally (p,r)-compact operator, we have a sequence $(w_k) \in \ell_p^u(X)$ such that $T(B_Y) \subset (p,r)\text{-conv}(w_k)$.

Now $(z_k) := \left(\frac{w_k}{\|(w_k)\|_p^w}\right) \in B_{\ell_p^u(X)}$, and let $\varepsilon > 0$. As (y_n) is null in Y, for $\varepsilon_0 := \frac{\varepsilon}{\|(w_k)\|_p^u}$ there exists $N \in \mathbb{N}$ such that $Ty_n \in \varepsilon_0 T(B_Y)$ for all $n \ge N$. Hence,

$$x_n \in \varepsilon_0 (p,r)\text{-conv}(w_k) = \varepsilon_0 \|(w_k)\|_p^w (p,r)\text{-conv}(z_k) = \varepsilon (p,r)\text{-conv}(z_k)$$

for all $n \ge N$, as desired.

The implication (g)⇒(a) is clear from the definitions, because if $(z_k) \in B_{\ell_p^u(X)}$, then $(\varepsilon z_k) \in \varepsilon B_{\ell_p^u(X)}$ and $(p,r)\text{-conv}(\varepsilon z_k) = \varepsilon (p,r)\text{-conv}(z_k)$. □

Remark 7.14. Theorem 7.13 together with Proposition 7.7 tells us that the uniformly unconditionally $(\infty,1)$-null sequences coincide with null sequences.

Recall (see [FS81, Theorem 2.5] or, e.g., [Pie80, 18.3.2]) that $\mathcal{N}_{(\infty,p,p^*)}$ coincides with the operator ideal K_p of *classical* p-compact operators. Following Fourie and Swart [FS79] or Pietsch [Pie80, 18.3.1 and 18.3.2], a linear operator $T : Y \to X$ is called *p-compact*, i.e., $T \in \mathrm{K}_p(Y,X)$, if there exist

$A \in \mathcal{K}(Y, \ell_p)$ and $B \in \mathcal{K}(\ell_p, X)$ such that $T = BA$. Remark (see [Oja12b] and [Pie14]) that $\mathcal{K}_p$ and K_p are notably different as operator ideals.

Since $\mathcal{U}_{p^*} = \mathcal{U}_{(p^*,p)} = \mathcal{N}^{\mathrm{sur}}_{(\infty,p,p^*)}$, one has $\mathrm{K}_p^{\mathrm{sur}} = \mathcal{U}_{p^*}$ as a *description of the surjective hull of* K_p.

Let us spell out, from Theorem 7.13, an omnibus characterization of unconditionally p-null (i.e., (p,p^*)-null) sequences.

Corollary 7.15. *Let* $1 \le p \le \infty$. *For a sequence* (x_n) *in a Banach space* X *the following statements are equivalent:*

(a) (x_n) *is unconditionally* p*-null,*

(b) (x_n) *is null and relatively unconditionally* p*-compact,*

(c) (x_n) *is null and* K_{p^*}*-compact,*

(d) (x_n) *is null and* $\mathcal{U}_p$*-compact,*

(e) (x_n) *is* K_{p^*}*-null,*

(f) (x_n) *is* $\mathcal{U}_p$*-null,*

(g) (x_n) *is uniformly unconditionally* p*-null.*

Let us remark, that the equivalence (a)⇔(b) for $p < \infty$ in Corollary 7.15 is also proved by Kim (see [Kim14, Theorem 1.2]). Kim's proof in [Kim14] is modelled after the proof of Theorem 1.1 in [Oja12a], relying on the description of the space of unconditionally p-null sequences as a Fourie–Swart tensor product, also established in [Kim14, Theorem 1.1]. Therefore, since Kim's method is the same as in [Oja12a] and, in particular, uses the Hahn–Banach theorem, it would not work for obtaining the more general equivalence (a)⇔(b) in Theorem 7.13 (cf. Remark 6.14 concerning the p-null sequences).

7.3 Weakly (p,r)-null sequences and weakly $\mathcal{A}$-null sequences

Let $1 \le p < \infty$, $1 \le r \le p^*$, and let X be a Banach space.

What about the weakly (p,r)-null sequences? It would be natural to expect that they would form a subclass of weakly null sequences, but not a subclass of null sequences as in the case of (p,r)-null sequences (which might be called also absolutely (p,r)-null sequences) or unconditionally (p,r)-null sequences. This means that we cannot employ the "verbatim" definition: replacing $\ell_p(X)$ with $\ell_p^w(X)$.

Indeed (see the proof of Proposition 7.8), such a "weakly" (p,r)-null sequence would always be a null sequence. And, for instance, looking at $X = \ell_{p^*}$ ($X = c_0$ when $p = 1$), every null sequence (x_n) in X would be uniformly "weakly" (p,p^*)-null, because the unit vector basis (e_k) of X belongs to $B_{\ell_p^w(X)}$ and, since $\Phi_{(e_k)} = I_X$, we have $x_n = \Phi_{(e_k)} x_n \in \|x_n\|\, p\text{-conv}(e_k)$.

To motivate a definition for weakly (p,r)-null sequences, let us make the following observation from Theorem 6.17, yielding two more characterizations of (p,r)-null sequences.

Proposition 7.16. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(i) *(x_n) is (p,r)-null,*

(ii) *for every $\varepsilon > 0$ there exist $(z_k) \in \ell_p(X)$ ($(z_k) \in c_0(X)$ when $p = \infty$) and $N \in \mathbb{N}$ such that $\|x_n\| \le \varepsilon$ and $x_n \in (p,r)\text{-conv}(z_k)$ for all $n \ge N$,*

(iii) *there exists $(z_k) \in \ell_p(X)$ ($(z_k) \in c_0(X)$ when $p = \infty$) with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n\| \le \varepsilon$ and $x_n \in (p,r)\text{-conv}(z_k)$ for all $n \ge N$.*

Proof. The implication (i)⇒(ii) is clear from the proof of Theorem 6.17, the first part of (a)⇒(b).

From (ii), it is clear that $x_n \to 0$, and also (fixing, e.g., $\varepsilon = 1$ and looking at the proof of Theorem 6.17, the second part of (a)⇒(b)) that (x_n) is relatively (p,r)-compact. By Theorem 6.17, (b)⇒(a), (x_n) is (p,r)-null, meaning that (ii)⇒(i). By Theorem 6.17, (b)⇒(g), (x_n) is uniformly (p,r)-null. Hence, assuming that $\varepsilon \le 1$, condition (iii) holds (similarly to the implication (i)⇒(ii) above).

Finally, (iii)⇒(ii) is more than obvious, and we saw above that (ii)⇔(i). □

Looking at Proposition 7.16, and accordingly with Definition 7.4, it seems to be natural to make the following definitions.

Definition 7.17. Let $1 \le p \le \infty$, $1 \le r \le p^*$. Let (x_n) be a sequence in a Banach space X. We call (x_n) *weakly (p,r)-null* if for every $x^* \in X^*$

and every $\varepsilon > 0$ there exist $(z_k) \in \ell_p^w(X)$ ($(z_k) \in c_0^w(X)$ when $p = \infty$) and $N \in \mathbb{N}$ such that $|x^*(x_n)| \le \varepsilon$ and $x_n \in (p,r)\text{-conv}(z_k)$ ($(x_n) \in 1\text{-co}(z_k)$ when $r = \infty$) for all $n \ge N$.

Definition 7.18. Let $1 \le p \le \infty$, $1 \le r \le p^*$. Let (x_n) be a sequence in a Banach space X. We call (x_n) *uniformly weakly (p,r)-null* if there exists $(z_k) \in \ell_p^w(X)$ ($(z_k) \in c_0^w(X)$ when $p = \infty$) with the following property: for every $x^* \in X^*$ and every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x^*(x_n)| \le \varepsilon$ and $x_n \in (p,r)\text{-conv}(z_k)$ ($(x_n) \in 1\text{-co}(z_k)$ when $r = \infty$) for all $n \ge N$.

Recall that the (uniformly) (unconditionally) $(\infty,1)$-null sequences and the null sequences coincide (see Propositions 6.2, 7.7, and Remark 7.14).

For weakly $(\infty,1)$-null sequences we get a similar result very easily.

Proposition 7.19. *For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(a) (x_n) *is uniformly weakly $(\infty,1)$-null,*

(b) (x_n) *is weakly $(\infty,1)$-null,*

(c) (x_n) *is weakly null.*

Proof. The implications (a)⇒(b)⇒(c) are clear from the definitions.

To prove (c)⇒(a), let (x_n) be a weakly null sequence, i.e., $(x_n) \in c_0^w(X)$. This means that for every $x^* \in X^*$ and every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x^*(x_n)| \le \varepsilon$ for all $n \ge N$. Since $x_n \in (\infty,1)\text{-conv}(x_n)$ for all $n \in \mathbb{N}$, we may take $(z_k) = (x_k)$. □

Let $\mathcal{A}$ be an operator ideal. In the present context, it would be natural to complement the Carl–Stephani theory with the concepts of weakly $\mathcal{A}$-null sequences and weakly $\mathcal{A}$-compact sets as follows.

Definition 7.20. We call a sequence (x_n) in a Banach space X *weakly $\mathcal{A}$-null* if there exist a Banach space Y, a weakly null sequence (y_n) in Y, and $T \in \mathcal{A}(Y,X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$.

Recall that $\boldsymbol{w}$ denotes the class of all relatively weakly compact sets.

Definition 7.21. We say that a subset K of a Banach space X is *weakly $\mathcal{A}$-compact* if K is of type $\mathcal{A}(\boldsymbol{w})$, i.e., $K \in \mathcal{A}(\boldsymbol{w})(X)$.

Two basic facts in the Carl–Stephani theory [CS84] are that the classes of $\mathcal{A}$-null and $\mathcal{A}^{\mathrm{sur}}$-null sequences coincide, and so also do $\mathcal{A}$-compact and $\mathcal{A}^{\mathrm{sur}}$-compact sets. The "weak" versions of these results do not hold.

Indeed, let $\mathcal{V}$ denote the operator ideal of *completely continuous* operators, i.e., of operators that take weakly null sequences to null sequences. Then $\mathcal{V}^{\mathrm{sur}} = \mathcal{L}$ (see, e.g., [Pie80, 4.7.13]). Consequently, the weakly $\mathcal{V}$-null sequences are (precisely, because null sequences are $\mathcal{K}$-null, hence $\mathcal{V}$-null) the null sequences, but the weakly $\mathcal{V}^{\mathrm{sur}}$-null sequences are precisely the weakly null sequences. Similarly, the weakly $\mathcal{V}$-compact sets are precisely relatively compact:

$$\mathcal{V}(\boldsymbol{w}) = \mathcal{V}(\mathcal{W}(\boldsymbol{b})) = (\mathcal{V}\circ\mathcal{W})(\boldsymbol{b}) = \mathcal{K}(\boldsymbol{b}) = \boldsymbol{k}$$

(see Remark 6.8 for the equality $\boldsymbol{w} = \mathcal{W}(\boldsymbol{b})$ and, e.g., [Pie80, 3.1.3] for the equality $\mathcal{V}\circ\mathcal{W} = \mathcal{K}$). But $\mathcal{V}^{\mathrm{sur}}(\boldsymbol{w}) = \boldsymbol{w}$.

However, for our purposes, the following analogue of the Lassalle–Turco Theorem 6.15, characterizing *weakly* $\mathcal{A}$-null sequences, will be sufficient.

Proposition 7.22. *Let $\mathcal{A}$ be an operator ideal and let (x_n) be a sequence in a Banach space X.*

(a) *If (x_n) is weakly $\mathcal{A}$-null, then (x_n) is weakly null and weakly $\mathcal{A}$-compact.*

(b) *If (x_n) is weakly null and weakly $\mathcal{A}$-compact, then (x_n) is weakly $\mathcal{A}^{\mathrm{sur}}$-null.*

In particular, if $\mathcal{A}$ is surjective, then (x_n) is weakly $\mathcal{A}$-null if and only if (x_n) is weakly null and weakly $\mathcal{A}$-compact.

Proof. (a) We have $x_n = Ty_n$ for some $T \in \mathcal{A}(Y,X)$ and weakly null sequence (y_n) in Y. Hence (x_n) is weakly null. Since (y_n) is relatively weakly compact in Y, (x_n) is weakly $\mathcal{A}$-compact.

(b) We know that $(x_n) \subset T(K)$ for some $T \in \mathcal{A}(Y,X)$ and weakly compact subset K of Y. We may and shall assume that $0 \in K$. Denote by $\overline{T}$ the injective associate of T. Then $T = \overline{T}q$, where $q : Y \to Z := Y/\ker T$ is the quotient mapping, and $\overline{T} \in \mathcal{A}^{\mathrm{sur}}(Z,X)$ (see Proposition 2.12).

If $q(K)$ and $\overline{T}(q(K)) = T(K)$ are endowed with their weak topologies from Z and X, respectively, then $\overline{T} : q(K) \to T(K)$ is a continuous bijection, hence a homeomorphism. Let $x_n = Tk_n = \overline{T}qk_n$ for some $k_n \in K$ and let $z_n = qk_n$. Then $z_n = \overline{T}^{-1}x_n \to \overline{T}^{-1}(0) = 0$ weakly (recall that $0 \in K$ and (x_n) is weakly null by the assumption). Since $x_n = \overline{T}z_n$ for all $n \in \mathbb{N}$, (x_n) is weakly $\mathcal{A}^{\mathrm{sur}}$-null. $\square$

We saw (in Sections 6.3, 6.4, 7.1, 7.2) that $\boldsymbol{k}_{(p,r)} = \mathcal{K}_{(p,r)}(\boldsymbol{b}) = \mathcal{K}_{(p,r)}(\boldsymbol{k})$ and, similarly, $\boldsymbol{u}_{(p,r)} = \mathcal{U}_{(p,r)}(\boldsymbol{b}) = \mathcal{U}_{(p,r)}(\boldsymbol{k})$. Also $\boldsymbol{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\boldsymbol{b})$ (see Section 7.1). In general, $\mathcal{W}_{(p,r)}(\boldsymbol{b}) \neq \mathcal{W}_{(p,r)}(\boldsymbol{k})$. Indeed, as was mentioned in the beginning of Section 7.3, for $X = \ell_{p^*}$ ($X = c_0$ when $p = 1$), one has $\Phi_{(e_k)} = I_X$. Hence, $\mathcal{W}_p(X,X) = \mathcal{L}(X,X)$ and therefore $\mathcal{W}_p(\boldsymbol{b})(X) = \boldsymbol{b}(X)$, but $\mathcal{W}_p(\boldsymbol{k})(X) = \boldsymbol{k}(X)$. We shall need the fact that in many cases $\mathcal{W}_{(p,r)}(\boldsymbol{b}) = \mathcal{W}_{(p,r)}(\boldsymbol{w})$.

Proposition 7.23. *Let $1 \leq p < \infty$ and $1 < r \leq p^*$ with $r < \infty$ if $p = 1$. Then*

$$\mathcal{W}_{(p,r)} = \mathcal{W}_{(p,r)} \circ \mathcal{W} \text{ and } \boldsymbol{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\boldsymbol{w}).$$

Proof. Let X and Y be Banach spaces and $T \in \mathcal{W}_{(p,r)}(Y,X)$. As in the case of $\mathcal{W}_p$ in [SK02, pp. 20–21] and of $\mathcal{K}_{(p,r)}$ (see Section 4.1), we get a natural factorization $T = \overline{\Phi}_{(x_n)}S$ with $(x_n) \in \ell_p^w(X)$, where $\overline{\Phi}_{(x_n)}$ is the injective associate of $\Phi_{(x_n)}$ and $S \in \mathcal{L}(Y,Z)$, with $Z := \ell_r/\ker\Phi_{(x_n)}$. Since $\Phi_{(x_n)} \in \mathcal{W}_{(p,r)}(\ell_r, X)$, we have $\overline{\Phi}_{(x_n)} \in \mathcal{W}_{(p,r)}^{\mathrm{sur}}(Z,X) = \mathcal{W}_{(p,r)}(Z,X)$, because $\mathcal{W}_{(p,r)}$ is surjective. Since ℓ_r is reflexive, also Z is, and therefore $S \in \mathcal{W}(Y,Z)$. This proves that $\mathcal{W}_{(p,r)} = \mathcal{W}_{(p,r)} \circ \mathcal{W}$. Now, using this, we have

$$\boldsymbol{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\boldsymbol{b}) = (\mathcal{W}_{(p,r)} \circ \mathcal{W})(\boldsymbol{b}) = \mathcal{W}_{(p,r)}(\mathcal{W}(\boldsymbol{b})) = \mathcal{W}_{(p,r)}(\boldsymbol{w}). \qquad \square$$

Remark 7.24. We do not know whether Proposition 7.23 holds in the "limit" case $r = 1$, i.e., for $\mathcal{W}_{(p,1)}$ ($\mathcal{W}_{(\infty,1)}$ included). It does not hold in the other "limit" case $p = 1$, $r = \infty$, i.e., for $\mathcal{W}_1 = \mathcal{W}_{(1,\infty)}$. Indeed, as we saw above, $\mathcal{W}_1(c_0,c_0) = \mathcal{L}(c_0,c_0)$, and hence

$$\boldsymbol{w}_1(c_0) = \mathcal{W}_1(\boldsymbol{b})(c_0) = \boldsymbol{b}(c_0) \neq \boldsymbol{w}(c_0) = \mathcal{W}_1(\boldsymbol{w}).$$

In particular, $\mathcal{W}_{(1,\infty)} \not\subset \mathcal{W}$. In all other cases $\mathcal{W}_{(p,r)} \subset \mathcal{W}$. For $r \neq 1$, this is clear from Proposition 7.23. But $\mathcal{W}_{(p,1)} \subset \mathcal{W}_{(p,r)}$ (by the definition of $\mathcal{W}_{(p,\cdot)}$, because $B_{\ell_1} \subset B_{\ell_r}$). Also $\mathcal{W}_{(\infty,1)} \subset \mathcal{W}$. Indeed, if $(x_n) \in c_0^w(X)$, i.e., (x_n) is weakly null, then $\Phi_{(x_n)}(B_{\ell_1}) = \overline{\operatorname{absconv}}(x_n)$ (recall that $c_0^w(X) \subset \ell_\infty(X)$ and see Proposition 3.3) is a weakly compact subset of X (thanks to Krein's theorem (see, e.g., [FHH+01, Theorem 3.58]) asserting that the closed convex hull of a weakly compact subset of a Banach space is also weakly compact).

Remark 7.25. In the case $p = 1$, $1 \leq r \leq p^*$, including also the case $p = 1$, $r = \infty$ (cf. Remark 7.24), Proposition 7.23 holds in a strong form for a large class of Banach spaces X. Namely, for X that does not contain c_0 isomorphically. In this case (and only in this case), $\ell_1^w(X) = \ell_1^u(X)$, by the classical Bessaga–Pełczyński theorem [BP58, Theorem 5] (see, e.g., [DF93, 8.3]). Therefore (see Section 7.2),

$$\mathcal{W}_{(1,r)}(Y,X) = \mathcal{U}_{(1,r)}(Y,X) = (\mathcal{K} \circ \mathcal{U}_{(1,r)} \circ \mathcal{K})(Y,X)$$

for all Banach spaces Y, and

$$\boldsymbol{w}_{(1,r)}(X) = \boldsymbol{u}_{(1,r)}(X) = \mathcal{U}_{(1,r)}(\boldsymbol{k})(X) = \mathcal{N}_{(\infty,\infty,r^*)}(\boldsymbol{k})(X).$$

Keeping in mind that the operator ideal $\mathcal{W}_{(p,r)}$ is surjective (see Section 7.1) we come to an omnibus characterization of weakly (p,r)-null sequences.

Theorem 7.26. *Let $1 \le p < \infty$ and $1 < r \le p^*$ with $r < \infty$ if $p = 1$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

(a) *(x_n) is weakly (p,r)-null,*

(b) *(x_n) is weakly null and relatively weakly (p,r)-compact,*

(c) *(x_n) is weakly null and weakly $\mathcal{W}_{(p,r)}$-compact,*

(d) *(x_n) is weakly $\mathcal{W}_{(p,r)}$-null,*

(e) *(x_n) is uniformly weakly (p,r)-null.*

Proof. (a)⇒(b) It is clear from the definition that $x_n \to 0$ weakly. Also, by the definition, we have (fixing, e.g., $\varepsilon = 1$) $N \in \mathbb{N}$ and $(z_k) \in \ell_p^w(X)$ such that $\{x_N, x_{N+1}, ...\} \subset (p,r)$-conv$(z_k)$. Continuing verbatim to the proof of Proposition 7.8, we see that (x_n) is relatively weakly (p,r)-compact.

Implications (b)⇔(c) and (c)⇔(d) are immediate from Propositions 7.23 and 7.22, respectively.

To prove that (d)⇒(e), let (x_n) be a weakly $\mathcal{W}_{(p,r)}$-null sequence. Then there are a weakly null sequence (y_n) in a Banach space Y and an operator $T \in \mathcal{W}_{(p,r)}(Y,X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$. The weak (p,r)-compactness of T gives us a sequence $(w_k) \in \ell_p^w(X)$ such that $T(B_Y) \subset (p,r)$-conv(w_k). We also have an $M > 0$ such that $\|y_n\| \le M$ for all $n \in \mathbb{N}$. Now $(z_k) := (Mw_k) \in \ell_p^w(X)$ and $x_n \in (p,r)$-conv(z_k) for all $n \in \mathbb{N}$. As (x_n) is weakly null in X, for every $x^* \in X^*$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x^*(x_n)| \le \varepsilon$ for all $n \ge N$. Hence, (x_n) is uniformly weakly (p,r)-null.

The implication (e)⇒(a) is clear from the definitions. □

Remark 7.27. As we saw, all implications of Theorem 7.26, except (b)$\Rightarrow$(c), also hold in the "limit" cases $r=1$ and $p=1$, $r=\infty$. In the proof, we used that the implication (b)$\Rightarrow$(c) is immediate from Proposition 7.23 (see also Remark 7.24). We do not know whether Theorem 7.26 holds in these cases. If $p=1$ and $1\le r\le p^*$, Theorem 7.26 holds in a stronger form for those Banach spaces X that do not contain c_0 isomorphically. Indeed, by Remark 7.25, in condition (b), "weakly $(1,r)$-compact" is the same as "unconditionally $(1,r)$-compact" and in condition (c) "weakly $\mathcal{W}_{(1,r)}$-compact" is the same as "$\mathcal{U}_{(1,r)}$-compact" and also the same as "$\mathcal{N}_{(\infty,\infty,r^*)}$-compact". In condition (d), "weakly $\mathcal{W}_{(1,r)}$-null" is the same as "weakly $\mathcal{U}_{(1,r)}\circ\mathcal{K}$-null", which is the same as "$\mathcal{U}_{(1,r)}$-null", since compact operators take weakly null sequences to null sequences, i.e., $\mathcal{K}\subset\mathcal{V}$ (see, e.g., [Pie80, 1.11.4]). This shows that in the special case when $p=1$, $1\le r\le p^*$, and X does not contain c_0 isomorphically, all conditions of Theorem 7.13 are equivalent to the conditions of Theorem 7.26.

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Ruumide ℓ_p poolt defineeritud kompaktsus ja nulljadad

Kokkuvõte

Funktsionaalanalüüsi rakendustes mängib olulist rolli hulkade ja operaatorite kompaktsus. Aastast 1955 pärineva Grothendiecki kompaktsuse kriteeriumi kohaselt on Banachi ruumi alamhulk suhteliselt kompaktne parajasti siis, kui ta sisaldub mingi nulliks koonduva jada kinnises kumeras kattes. Sellest kriteeriumist lähtudes on järgnevatel aastatel toodud sisse mitmeid uusi tugevamaid kompaktsuse versioone, mis omakorda defineerivad erinevaid kompaktsete operaatorite klasse.

Käesoleva väitekirja põhieesmärk on välja töötada kompaktsuse vormide ning vastavate operaatorite klasside ühtne teooria. Selleks on sisse toodud (p,r)-kompaktsuse mõiste nii hulkade kui ka operaatorite jaoks, mis erijuhul $p=\infty$ ühtib klassikalise kompaktsusega ning juhul $r=1$ osutub Bourgain–Reinovi p-kompaktsuseks. Erijuhul $r=p^*$, kus $p^*=p/(p-1)$, on tegemist viimasel ajal intensiivselt uuritud Sinha–Karni p-kompaktsusega. Väitekirjas on loodud tunduvalt lihtsam alternatiivne teooria muuhulgas ka Sinha–Karni p-kompaktsuse käsitlemiseks ning saadud tulemused parendavad mitmeid varasemaid.

Väitekirja esimene peatükk sisaldab ülevaadet varasematest kompaktsuse vormidest ning nende seosest (p,r)-kompaktsusega, väitekirja kokkuvõtet ning väitekirjas kasutatavate tähistuste kirjeldust.

Väitekirja teises peatükis tutvustatakse jadaruume, mille elementideks on jadad Banachi ruumis, (kvaasi-Banachi) operaatorideaali, selle sürjektiivset ning injektiivset katet, tehteid (kvaasi-Banachi) operaatorideaalidega, operaatori injektiivset kaaslast ning defineeritakse operaator $\Phi_{(x_n)}$. Välja on toodud mõisted ja tulemused, mis on järgnevate osade mõistmiseks olulised. Võtmetähtsusega on tähelepanek (vt.[ALO12]), et operaatori $\Phi_{(x_n)}$ injektiivne kaaslane kuulub teatavat tüüpi tuumaoperaatorite ideaali sürjektiivsesse kattesse.

Lause 2.17. *Olgu $1 \leq p \leq \infty$ ja $1 \leq r \leq p^*$. Kui X on Banach ruum ja $(x_n) \in \ell_p(X)$ (vastavalt $(x_n) \in c_0(X)$, kui $p=\infty$), siis $\overline{\Phi}_{(x_n)} \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(Z,X)$, kus $Z=\ell_r/\ker\Phi_{(x_n)}$, ja $\|\overline{\Phi}_{(x_n)}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \leq \|(x_n)\|$.*

Väitekirja kolmandas peatükis üldistatakse varasemaid kompaktsuse vorme, tuues sisse (p,r)-kompaktsuse järgmisel viisil.

Definitsioon. Olgu X ja Y Banachi ruumid. Kehtigu $1 \le p \le \infty$ ja $1 \le r \le p^*$. Öeldakse, et ruumi Y osahulk K on *suhteliselt (p,r)-kompaktne*, kui leidub jada $(y_n) \in \ell_p(Y)$ (vastavalt $(y_n) \in c_0(Y)$, kui $p = \infty$) nii, et $K \subset \{\sum_{n=1}^{\infty} a_n y_n : (a_n) \in B_{\ell_p^*}\}$. Pidevat lineaarset operaatorit $T : X \to Y$ nimetatakse *(p,r)-kompaktseks*, kui $T(B_X)$ on ruumi Y suhteliselt (p,r)-kompaktne alamhulk.

Osutub, et (p,r)-kompaktsete operaatorite klass $\mathcal{K}_{(p,r)}$ moodustab operaatorideaali. Veelgi enam, (p,r)-kompaktsete operaatorite ideaal on sürjektiivne.

Lause 3.9. *Olgu $1 \le p \le \infty$ ja $1 \le r \le p^*$. Operaatorideaal $\mathcal{K}_{(p,r)}$ on sürjektiivne ehk $\mathcal{K}_{(p,r)} = \mathcal{K}_{(p,r)}^{\mathrm{sur}}$.*

Selle peatüki tulemused tuginevad peamiselt artiklile [ALO12].

Väitekirja neljandas peatükis tõestatakse väitekirja ühe põhitulemusena, et (p,r)-kompaktsete operaatorite klass $\mathcal{K}_{(p,r)}$ on samastatav tuumaoperaatorite ideaali $\mathcal{N}_{(p,1,r^*)}$ sürjektiivse kattega.

Teoreem 4.1. *Olgu $1 \le p \le \infty$ ja $1 \le r \le p^*$. Siis $\mathcal{K}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}^{\mathrm{sur}}$, kusjuures see võrdus kehtib operaatorideaalide võrdusena.*

Teoreem 4.1 võimaldab (p,r)-kompaktsete operaatorite ideaali varustada $(p,1,r^*)$-tuumaoperaatorite kvaasinormiga. Selliselt defineeritud kvaasinorm on erijuhul $r = p^*$ kooskõlas kirjanduses avaldatud Sinha–Karni p-kompaktsete operaatorite normidega. Neljas peatükk põhineb artiklil [ALO12].

Viiendas peatükis rakendatakse neljanda peatüki põhitulemusi $(p,1,r^*)$-tuumaoperaatorite ideaali sürjektiivse ning injektiivse katte kirjeldamiseks. Tuletatakse ka (p,r)-kompaktsete operaatorite esitus teatavate operaatorite korrutisena.

Teoreem 5.8. *Olgu $1 \le p \le \infty$ ja $1 \le r \le p^*$. Siis*

$$\mathcal{K}_{(p,r)} = \mathcal{I}_{(p,1,r^*)}^{\mathrm{sur}} \circ \mathcal{K},$$

kusjuures see võrdus kehtib kvaasi-Banachi operaatorideaalide võrdusena.

Erijuhul $r = p^*$ parendatakse teoreemiga 5.8 varasemaid Sinha–Karni p-kompaktsete operaatorite kohta käivaid tulemusi. See peatükk põhineb artiklil [ALO12].

Kuuendas peatükis tuuakse kirjanduses uuritud p-nulljada üldistusena sisse (p,r)-nulljada mõiste.

Definitsioon 6.1. Olgu $1 \leq p \leq \infty$ ja $1 \leq r \leq p^*$. Öeldakse, et jada (x_n) ruumis X on (p,r)-*nulljada* kui iga arvu $\varepsilon > 0$ jaoks leiduvad $N \in \mathbb{N}$ ja $(z_k) \in \ell_p(X)$ (vastavalt $(z_k) \in c_0(X)$, kui $p = \infty$) nii, et $\|(z_k)\|_p \leq \varepsilon$ ja $x_n \in (p,r)\text{-conv}(z_k)$ iga $n \geq N$ korral.

Kasutades Carl–Stephani teooriat, mis käsitleb operaatorideaalide poolt tekitatud kompaktsust, tõestatakse väitekirja ühe põhitulemusena järgmine teoreem.

Teoreem 6.17. *Kehtigu $1 \leq p \leq \infty$ ja $1 \leq r \leq p^*$. Olgu (x_n) jada Banachi ruumis X. Siis järgmised väited on samaväärsed:*

(a) *(x_n) on (p,r)-nulljada,*

(b) *(x_n) on suhteliselt (p,r)-kompaktne nulljada,*

(c) *(x_n) on $\mathcal{N}_{(p,1,r^*)}$-kompaktne nulljada,*

(d) *(x_n) on $\mathcal{K}_{(p,r)}$-kompaktne nulljada,*

(e) *(x_n) on $\mathcal{N}_{(p,1,r^*)}$-nulljada,*

(f) *(x_n) on $\mathcal{K}_{(p,r)}$-nulljada,*

(g) *(x_n) on ühtlaselt (p,r)-nulljada.*

Teoreem 6.17 iseloomustab asjaolu, et (x_n) on (p,r)-nulljada, kuuel erineval moel. Erijuhul $r = p^*$ sisaldab teoreem 6.17 varasemat p-nulljada Grothendiecki tüüpi kirjeldust. Kuues peatükk põhineb artiklitel [AO12] ja [AO15].

Seitsmendas peatükis tuuakse (p,r)-nulljada üldistustena sisse tingimatu (p,r)-nulljada ning nõrga (p,r)-nulljada mõisted. Kuuendas peatükis välja töötatud meetodi universaalsus võimaldab seitsmendas peatükis tõestada teoreemiga 6.17 sarnaseid tulemusi ka tingimatu (p,r)-nulljada ja nõrga (p,r)-nulljada jaoks. See peatükk põhineb artiklil [AO15].

Väitekirja olulisemad tulemused on ilmunud/ilmumas artiklites [ALO12, AO12, AO15].

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