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Closed maximal regular one-sided ideals in topological algebras

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Chapter 1

Introduction

To study the structure of topological algebras, it is very useful to know the description of closed ideals in the topological algebra under the consideration. It is often very important to know which maximal ideals in a particular topological algebra are closed, because majority of topological algebras have both closed and not closed maximal ideals. It is well known that all maximal regular one-sided and two-sided ideals are closed in Q-algebras, but not only in Q-algebras. Necessary and sufficient conditions for a given topological algebra, in which all maximal one-sided ideals [15] or two-sided ideals [16, 14] are closed, are also known.

One of the simplest nontrivial topological algebras is the commutative Banach algebra $C(X)$ of continuous functions on a compact Hausdorff space X with uniform topology. Closed ideals in $C(X)$ were described in the first half of the previous century. Soon after that the description of closed ideals in the commutative Banach algebra $C_b(X)$ of bounded and continuous functions on a completely regular Hausdorff space X with uniform topology was found.

Next, in addition to other individual commutative Banach algebras, several mathematicians (first of them B. Yood in [46] and A. Hausner in [26]) started to consider the ideal problem in several Banach algebras of continuous A-valued functions on a completely regular Hausdorff space X (with uniform topology) in case of a unital commutative Banach algebra A (in this case the descriptions of maximal ideals with codimension one were mainly considered). Moreover, I. Kaplansky considered the ideal problem in [30] in the case of Banach algebra of bounded and continuous sections, defined by the family $\{(A_x, \tau_x) \colon x \in X\}$ of unital Banach algebras. Later on, many mathematicians (see, for example, W. E. Dietrich in [22]) described all closed maximal ideals with codimension one in the topological algebra $C(X, A)$ of continuous A-valued functions endowed with compact-open topology in case when A is a locally convex algebra with nontrivial continuous multiplicative linear functionals. W. J. Hery described in [27] all maximal ideals with codimension one in the algebra $C_b(X, A)$ of all bounded and continuous A-valued functions on a completely regular Hausdorff space X in case when A is a unital commutative topological (not necessarily locally convex) algebra.

In 1962, L. Waelbroeck showed in [45] that for the description of maximal ideals in unital noncommutative locally convex algebras it is possible to use the description of maximal ideals of the center of the given algebra. In 1968, G. R. Allan developed in [17] a method for describing all maximal ideals in unital noncommutative Banach algebras by using extendible ideals of the center of the given Banach algebra. As Mart Abel showed in [3], it was possible to generalize this method for the description of all closed maximal regular one-sided ideals in several Gelfand-Mazur algebras.

In 1996, J. W. Kitchen and D. A. Robbins gave in [32] (see also [28]) the description of all closed maximal ideals with codimension one in the locally convex algebra of bounded continuous sections, defined by a family $\{(A_x, \tau_x) \colon x \in X\}$ of unital commutative locally m-convex algebras.

The aim of the present Thesis was to obtain some more general results about the description of closed maximal regular ideals of topological algebras. Descriptions of closed maximal regular one- and two-sided ideals of topological algebras of (not necessarily bounded) continuous sections, defined by a family $\{(A_x, \tau_x) \colon x \in X\}$ of general (that is, not necessarily unital and not necessarily locally convex) topological algebras, are considered in the present Thesis.

Since the previous work in this field has been conducted for the locally convex case, using the technique of seminorms, new methods, involving neighbourhoods, had to be used in order to deal with the more general case.

Definitions of the main terms of the Thesis and several results, used later on, are presented in the second Chapter of the Thesis.

Descriptions of closed (in particular, maximal closed) one-sided ideals in the algebra of continuous sections defined by a family $\{(A_x, \tau_x) \colon x \in X\}$ of topological algebras (where the index set X is a completely regular Hausdorff space), are given in the third Chapter of the Thesis. Here the sections need not be bounded and the topology on the algebra of sections is not locally convex in general.

The topological algebra of continuous sections, defined by the family $\{(A/I, \tau_{A/I}) \colon I \in \mathcal{I}\}$ of quotient algebras of a given topological algebra A with respect to two-sided ideals $I \in \mathcal{I}$ in A, is considered in the fourth Chapter of the Thesis. Conditions, when A is densely embedded into that topological algebra of sections, and the description of a topological algebra Q such that A is densely embedded in the topological algebra $C_c(\mathcal{I}, Q)$ of continuous Q-valued functions on $\mathcal{I}$, endowed with the compact-open topology, have been given here.

The topological algebra of continuous sections, defined by the family $\{(A/P, \tau_{A/P})\colon P \in \pi(A)\}$ (here $\pi(A)$ denotes the set of all primitive ideals of a topological algebra A, defined by the closed maximal regular left (right) ideals) of quotient algebras of a topological algebra A with respect to the closed primitive ideals of A, are considered in the fifth Chapter. A description of all closed maximal regular one-sided ideals in A, using the algebra of sections described above, is also given here.

Some applications of the results, presented in the fifth Chapter, are given in the sixth Chapter. The descriptions of left and right topological Jacobson radicals are given, which are used to find the conditions when these radicals coincide, answering partly the question posed in 1964 by B. Yood ([47]).

The results, presented in this Thesis, have been published in [5, 6, 8] and the paper [7] has been accepted for publishing. The author of this Thesis has introduced his results at the Sixth Conference on Function Spaces (Edwardsville, 2010), at the 9th International Conference on Topological Algebras and their Applications (Holon, 2015), at the workshops "Algebra and its applications" (Vadsa, 2009 and Sokka, 2012) and several times at the seminar of topological algebras at the University of Tartu.

The author of this thesis is responsible for formulation of some statements, giving detailed proofs for the results and preparing all the manuscripts. The statements of the reasearch problems and many of the ideas of solutions belong to the supervisors.

Chapter 2

Preliminary definitions and results

Here we give some essential definitions and basic results that we use later on. Throughout the whole Thesis, let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers.

2.1 Algebra

An *assocative algebra A over* $\mathbb{K}$ (shortly, an *algebra*) is an associative ring which is also a linear space over $\mathbb{K}$ and satisfies

$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

for all $\lambda \in \mathbb{K}$ and $a, b \in A$. An algebra A is a *unital algebra* if the ring A is unital. Algebra B is a *subalgebra* of an algebra A if B is a subspace and a subring of A. A subalgebra B of A is called a *proper subalgebra of* A if $A \neq B$. For every linear space, ring or algebra A, we denote by θ_A the zero element of A. If an algebra or a ring A has the unit element, then we denote it by e_A.

2.2 Ideals

Let A be an algebra and I a proper subalgebra of A. We say that I is a *left* (*right* or *two-sided*) *ideal* of A if $AI \subset I$ ($IA \subset I$ or both, $AI \subset I$ and $IA \subset I$, respectively).

2.2.1 Regular ideal

A left (right or two-sided) ideal I of an algebra A is *regular* (sometimes also called *modular*) if there exists an element $u \in A$ such that $a - au \in I$

($a - ua \in I$ or $a - au, a - ua \in I$, respectively) for all $a \in A$. Such element u is called a *right* (*left* or *two-sided*, respectively) *regular unit for* I.

If u is a right (left or two-sided) regular unit for a left (right or two-sided, respectively) ideal I, then $u \notin I$ (see [37], Theorem 2.4.6(b), p. 235).

2.2.2 Maximal ideal

Let M be a left (right or two-sided) ideal of an algebra A. If for every left (right or two-sided, respectively) ideal I of A from the condition $M \subset I$ follows that $M = I$, then M is called a *maximal left* (*right* or *two-sided*, respectively) ideal in A.

Every regular left (right or two-sided) ideal of an algebra A is contained in a maximal regular left (right or two-sided, respectively) ideal of A (see [37], Theorem 2.4.6(d), p. 236).

2.2.3 Primitive ideal

Let M be a maximal regular left (right) ideal of an algebra A. Then the two-sided ideal $P = \{a \in A \colon aA \subset M\}$ ($P = \{a \in A \colon Aa \subset M\}$, respectively) is called a *primitive ideal* of A with respect to M.

Every primitive ideal is the intersection of all maximal regular left (right) ideals containing it (see [37], p. 447).

If M is a closed maximal regular left (right) ideal of a topological algebra A (see 2.13), then the primitive ideal with respect to M is closed (see [7]).

2.3 Quotient algebra

Let A be a linear space over $\mathbb{K}$, J a linear subspace of A and

$$\overline{a} = a + J = \{a + j \colon j \in J\}.$$

The set

$$A/J = \{\overline{a} \colon a \in A\},$$

with binary operations

a) $\overline{a} + \overline{b} = \overline{a + b}$ for all $a, b \in A$

and

b) $k\overline{b} = \overline{kb}$ for all $k \in \mathbb{K}$ and $b \in A$, is also a linear space over $\mathbb{K}$ and is called the *quotient space of A with respect to J*. The map $\kappa\colon A \to A/J$, defined by $\kappa(a) = \overline{a}$ for every $a \in A$, is called the *canonical homomorphism*. In this case $\ker \kappa = \{a \in A\colon \kappa(a) = \theta_{A/J}\} = J$.

Let A be an algebra, J a two-sided ideal of A and $\overline{a}$, A/J, $\overline{a} + \overline{b}$, $k\overline{b}$ be defined as above for all $a, b \in A$ and $k \in \mathbb{K}$. It is well known that the quotient space A/J is an algebra with respect to the multiplication

c) $\overline{a}\,\overline{b} = \overline{ab}$ for all $a, b \in A$

(see [37], p. 5).

Assume that the quotient space A/J is an algebra with respect to the multiplication c) and J is a proper subspace of A, but not a left ideal. Then there exist $a \in A$ and $j \in J$ such that $aj \notin J$. Now

$$\overline{\theta_A} = \overline{a\theta_A} = \overline{a}\overline{\theta_A} = \overline{a}\overline{j} = \overline{aj},$$

but this is a contradiction since $\overline{\theta_A} \neq \overline{aj}$ and, therefore, J must be a left ideal in A or $J = A$. Similarily, J has to be a right ideal in A or $J = A$. So, A/J is an algebra if and only if J is a two-sided ideal in the algebra A or $J = A$.

Let A be an algebra and J a two-sided ideal in A. The *quotient algebra* A/J is the quotient space A/J which is also a quotient ring.

If J is a regular two-sided ideal in A, then A/J is a unital algebra.

2.4 A-module

Let A be an algebra and M a linear space over $\mathbb{K}$. The space M is called a *left* (*right*) *A-module* if the module multiplication $(a, m) \to am$ of $A \times M$ into M ($(m, a) \to ma$ of $M \times A$ into M, respectively) is a bilinear map which satisfies the condition $a_1(a_2m) = (a_1a_2)m$ ($(ma_1)a_2 = m(a_1a_2)$, respectively) for each $a_1, a_2 \in A$ and $m \in M$. If M' is a subspace of a left (right) A-module M and for any $m \in M'$ and $a \in A$ the product am (ma, respectively) is in M', then M' is called a *left* (*right*, respectively) *A-submodule* of M. A left (right) A-module M is called *nontrivial* if $AM \neq \{\theta_M\}$ ($MA \neq \{\theta_M\}$, respectively) and M is called *irreducible* if it is nontrivial and the only A-submodules of M are M and $\{\theta_M\}$.

2.5 Representation and antirepresentation

An *antihomomorphism* $\pi'\colon A \to B$ is a map from an algebra A into an algebra B that preserves addition and multiplication by scalars (as a homomorphism), but reverses the order of multiplication, that is, $\pi'(ab) = \pi'(b)\pi'(a)$ for all

$a, b \in A$. Let X be a linear space over $\mathbb{K}$. Any homomorphism π (antihomomorphism π') of an algebra A into $\mathcal{L}(X)$, the algebra of all linear maps from X to X, is called a *representation* (*antirepresentation*, respectively) of A on X. Every representation π (antirepresentation π') of A on X defines on X a left (right, respectively) module multiplication $\cdot_\pi$ ($\cdot_{\pi'}$, respectively), if we put $a \cdot_\pi x = \pi(a)(x)$ ($x \cdot_{\pi'} a = \pi'(a)(x)$, respectively) for each $a \in A$ and $x \in X$. In this case X becomes a left (right, respectively) A-module. We denote it by X_π ($X_{\pi'}$, respectively). A representation π (antirepresentation π', respectively) of A on X is *irreducible* if X_π ($X_{\pi'}$, respectively) is an irreducible left (right, respectively) A-module. Let $x_0 \in X$ and define $\varrho_{\pi,x_0} \colon A \to X_\pi$ by

$$\varrho_{\pi,x_0}(a) = a \cdot_\pi x_0,$$

and $\varrho_{\pi',x_0} \colon A \to X_{\pi'}$ by

$$\varrho_{\pi',x_0}(a) = x_0 \cdot_{\pi'} a,$$

for all $a \in A$. In case π is a representation of A on X and, thus, X_π is a left A-module, then

$$\ker \varrho_{\pi,x_0} = \{a \in A \colon a \cdot_\pi x_0 = \theta_A\}$$

is a left ideal of A. In case π' is an antirepresentation of A on X and, thus, $X_{\pi'}$ is a right A-module, then

$$\ker \varrho_{\pi',x_0} = \{a \in A \colon x_0 \cdot_{\pi'} a = \theta_A\}$$

is a right ideal of A. We say that x_0 is a *left* (*right*) *cyclic element* of X_π ($X_{\pi'}$, respectively), if $A \cdot_\pi x_0 = X_\pi$ ($x_0 \cdot_{\pi'} A = X_{\pi'}$, respectively). Moreover, denote by

$$\mathrm{id}_{\pi,r}(x_0) = \{e \in A \colon e \cdot_\pi x_0 = x_0\}$$

and

$$\mathrm{id}_{\pi',l}(x_0) = \{e \in A \colon x_0 \cdot_{\pi'} e = x_0\}$$

the sets of one-sided units for an element x_0.

2.6 Jacobson radical

The intersection $\mathrm{Rad}(A)$ of kernels of all irreducible representations of an algebra A on all linear spaces is called the *Jacobson radical of* A. If there are no irreducible representations of A at all, then A is called a *radical algebra* (in this case $\mathrm{Rad}(A) = A$) and if $\mathrm{Rad}(A) = \{\theta_A\}$, then A is called a *semi-simple algebra*.

The Jacobson radical $\mathrm{Rad}(A)$ can also be defined by kernels of all irreducible antirepresentations. Since those two definitions of a Jacobson radical coincide (see, for example, [35], p. 197), then the Jacobson radical is usually defined by the representations only.

If A is not a radical algebra, then the Jacobson radical $\mathrm{Rad}(A)$ is the intersection of all maximal regular left (or right) ideals in A (see [38], Theorem 2.3.2(ii), p. 55).

2.7 Topological space

A *topological space* is a pair (X, τ), where X is a set and τ is a collection of subsets of X, satisfying the following axioms:

a) the empty set $\emptyset$ and X itself belong to τ;

b) any (finite or infinite) union of elements of τ belongs to τ;

c) the intersection of any finite number of elements of τ belongs to τ.

The elements of τ are called *open sets* and the collection τ is called a *topology on* X. A set $F \subset X$ is said to be *closed in* X if $X \setminus F \in \tau$. A subset $\beta \subset \tau$ is called a *base* of τ if every element of τ can be written as a union of some elements of β. Let $S \subset X$. The intersection $\mathrm{cl}_X(S)$ of all closed sets of X, containing S, is called the closure of S in X. A set $V \subset X$ is called a *neighbourhood of* x *in* X if there exists $O \in \tau$ such that $x \in O \subset V$. A set $\mathcal{B}_x$ of neighbourhoods of $x \in X$ is a *base of neighbourhoods of* x if for every neighbourhood V of x, there exists $O \in \mathcal{B}_x$ such that $O \subset V$.

Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f\colon X \to Y$. It is said that the map f is *continuous* if $f^{-1}(U) \in \tau_X$ for every $U \in \tau_Y$.

Below we define some classes of topological spaces.

2.7.1 Hausdorff space

A topological space X is said to be a *Hausdorff* (or *separated*) *space* if for every x and y in X there exist neighbourhoods V_x of x and V_y of y in X satisfying

$$V_x \cap V_y = \emptyset.$$

2.7.2 Compact space

A subset $Y \subset X$ of a topological space X is said to be *compact* if for every open cover $\{U_i\colon i \in I\}$ of Y (that is, every U_i $(i \in I)$ is open in X and $Y \subset \bigcup_{i \in I} U_i$) there is a finite set $F \subset I$ such that $\{U_i\colon i \in F\}$ is still a cover of Y. A topological space X is called *compact* if X is compact subset in X. A subset $Z \subset X$ in a topological space X is called *relatively compact* if $\mathrm{cl}_X(Z)$ is compact in X.

2.7.3 Completely regular space

A topological space X is a *completely regular space* iff for every closed subset C of X and every point $x \in X \setminus C$, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(C) = \{1\}$.

2.8 Lebesgue covering dimension

Let X be a topological space and $\mathcal{U}$ an open cover of X. Consider how many elements of the cover $\mathcal{U}$ contain a given point in X. If this has a maximum over all the points of X, then this maximum is called the order of the cover $\mathcal{U}$. It is said that a topological space X has *Lebesgue covering dimension* m if for every open cover of that space, there is an open cover that refines it such that the refinement has order at most $m+1$. We denote by $\dim X$ the Lebesgue covering dimension of a completely regular space X (see, e.g., [25], p. 243). If a topological space X does not have Lebesgue covering dimension m for any $m \in \mathbb{N}$, the space X is said to have *infinite Lebesgue covering dimension.*

2.9 Product topology

Let $(Y_i, \tau_i)_{i\in I}$ be a family of topological spaces. The *product topology* τ on

$$Y = \prod_{i\in I} Y_i = \left\{ f\colon I \to \bigcup_{i\in I} Y_i, f(i) \in Y_i \right\}$$

is the weakest topology such that all the projections

$$\mathrm{pr}_j\colon Y \to Y_j, \quad (j \in I),$$

defined by $\mathrm{pr}_j(f) = f(j)$ for every $j \in I$ and $f \in Y$, are continuous.

The sets

$$\prod_{i\in I} V_i \tag{2.1}$$

with $V_i \subset Y_i$ open for each $i \in I$ and $\{i \in I\colon V_i \neq Y_i\}$ finite, form a base of the product topology τ (see, e.g., [23], p. 44). So, the open sets in the product topology are unions of sets of the form (2.1).

If $s \in Y$ and U is a neighbourhood of s in Y, then there is a family $(U_i)_{i\in I}$ such that

$$\prod_{i\in I} U_i \subset U,$$

where U_i is a neighbourhood of $s(i)$ for all $i \in I$ and $\{i \in I\colon U_i \neq Y_i\}$ is finite (see [23], Theorem 7.5, p. 45).

2.10 Topological linear space

A *topological linear space* X over $\mathbb{K}$ (shortly, a *topological linear space* X) is defined as a linear space over $\mathbb{K}$ that is endowed with a topology such that the multiplication by scalars $\mathbb{K} \times X \to X$, defined by $(\lambda, x) \mapsto \lambda x$, and the addition $X \times X \to X$, defined by $(x, y) \mapsto x + y$, are continuous maps. Here $X \times X$ and $\mathbb{K} \times X$ are endowed with the product topologies.

Let O be a neighbourhood of x in a topological linear space X. If $x = \lambda y$, then there are neighbourhoods O_λ of λ in $\mathbb{K}$ and O_y of y in X such that $O_\lambda O_y \subset O$, and, if $x = y + z$, then there are neighbourhoods O_y of y and O_z of z in X such that $O_y + O_z \subset O$.

If V is a neighbourhood of zero (that is, a neighbourhood of θ_X) in X, then $x + V$ is a neighbourhood of x in X.

2.11 Quotient topology

Let X be a topological linear space, Y a linear subspace of X and $\kappa : X \to X/Y$ the canonical homomorphism. The *quotient topology on* X/Y is defined as the final topology with respect to κ, that is, the finest topology that makes κ continuous.

A subset $U \subset X/Y$ is open in the quotient topology if and only if $\kappa^{-1}(U)$ is open in X.

A subspace Y is closed in X if and only if X/Y is a Hausdorff space (see [42], result 2.3, p. 20).

2.12 Approximation property

It is said that a topological linear space X has the *approximation property* if for each (relatively) compact subset $K \subset X$ and each neighbourhood O of zero in X there is a linear continuous finite-dimensional operator $L\colon X \to X$ (i.e., $L(X)$ is a finite dimensional subspace in X) such that $L(x) - x \in O$ for each $x \in K$.

2.13 Topological algebra

Let A be an algebra endowed with a topology τ_A.

For every $a \in A$, let the maps $l_a\colon A \to A$ and $r_a\colon A \to A$ be defined by $l_a(x) = ax$ and $r_a(x) = xa$, respectively, for every $x \in A$. We say that the multiplication $A \times A \to A$, defined by $(a, b) \mapsto ab$, is *separately continuous*, if all the maps l_a and r_a $(a \in A)$ are continuous in τ_A.

The continuous multiplication, as a mapping $A \times A \to A$, is also called *jointly continuous* multiplication in contrast with separately continuous multiplication.

In general, separate continuity of multiplication need not imply joint continuity even in case of normed algebras (see [40], p. 247 and 272).

If the multiplication in A is jointly continuous, then it is separately continuous. If the topology of A is complete and metrizable, then separately continuous multiplication in A is jointly continuous (see, for example [48], Theorem 1.5, or [31], Theorem A.6.1).

A topological linear space A over $\mathbb{K}$ is called a *topological algebra over* $\mathbb{K}$ (shortly, a *topological algebra*) if there has been defined an associative multiplication in A such that

a) A is an algebra over $\mathbb{K}$;

b) the multiplication in A is separately continuous.

Let O be a neighbourhood of x in a topological algebra A. If $x = yz$, then there are neighbourhoods O_y of y and O_z of z in A such that $O_y z \subset O$ and $yO_z \subset O$. If the multiplication is jointly continuous, then, for every neighbourhood O of x, there are neighbourhoods O_y of y and O_z of z in A such that $O_y O_z \subset O$.

2.14 Quasi-invertibility

Let A be a topological algebra. An element r of A is called *left quasi-invertible* if there is an element $p \in A$ such that

$$p \circ r = p + r - pr = \theta_A,$$

that is, $A \circ r = A$. An element r of A is *topologically left quasi-invertible* if there is a net $(p_\lambda)_{\lambda\in\Lambda}$ in A such that the net $(p_\lambda \circ r)_{\lambda\in\Lambda}$ converges to θ_A, that is, $A \circ r$ is dense in A. We denote by $\operatorname{Qinv}_l A$ the set of all left quasi-invertible elements of A and by $\operatorname{Tqinv}_l A$ the set of all topologically left quasi-invertible elements of A. The sets $\operatorname{Qinv}_r A$ of all right quasi-invertible elements of A and

$\mathrm{Tqinv}_r\, A$ of all topologically right quasi-invertible elements of A are defined similarly.

It is said that an element $r \in A$ is (*topologically*) *quasi-invertible* if r is (topologically) left and (topologically) right quasi-invertible. We denote by ($\mathrm{Tqinv}\, A$) $\mathrm{Qinv}\, A$ the set of all (topologically) quasi-invertible elements of A.

2.15 Topological A-module

Let A be a topological algebra and M a left (right) A-module over the same field. If, in addition, M is a topological linear space (over the same field) in which the module multiplication is separately continuous, then M is called a *topological left* (*right*) *A-module.*

2.16 Some classes of topological algebras

In this section we give the definitions of some classes of topological algebras used in the present Thesis.

2.16.1 Banach algebra

Topological algebra A is called a *normed* algebra if the topology of A is defined by submultiplicative norm, that is $||ab|| \leq ||a||\,||b||$ for each $a, b \in A$.

A normed algebra which is complete in its norm is called a *Banach* algebra.

2.16.2 C^*-algebra

An involution on an algebra A is a map $^*\colon A \to A$ satisfying

a) $(a+b)^* = a^* + b^*$;

b) $(\lambda a)^* = \lambda^* a^*$;

c) $(ab)^* = b^* a^*$;

d) $(a^*)^* = a$

for all $a, b \in A$ and $\lambda \in \mathbb{K}$, where λ^* is the conjugate of λ. A Banach algebra in which the norm and involution are related by the condition $||a^*a|| = ||a||^2$ is called a *C^*-algebra.*

2.16.3 Locally pseudoconvex algebra and locally m-pseudoconvex algebra

A topological algebra A is called *locally pseudoconvex* if it has a base $\{U_i : i \in I\}$ of neighbourhoods of zero which consists of balanced (that is, $\mu a \in U_i$ if $a \in U_i$ and $|\mu| \leq 1$) and pseudoconvex (that is, $U_i + U_i \subset 2^{\frac{1}{k_i}} U_i$ for some $k_i \in (0, 1]$) sets U_i. It is well known (see [19], p. 189) that the topology on A can be given by a family $\{p_i : i \in I\}$ of k_i-homogeneus seminorms. Moreover, if $k_i = 1$ for every $i \in I$, then the algebra A is called a *locally convex* algebra. In particular, if all seminorms p_i of a locally pseudoconvex algebra A satisfy the condition $p_i(ab) \leq p_i(a)p_i(b)$ for all $a, b \in A$, then A is called a *locally multiplicatively pseudoconvex* or a *locally m-pseudoconvex* algebra. Similarly, if $k_i = 1$ for every $i \in I$, then the algebra A is called a *locally m-convex* algebra.

2.16.4 Q-algebra

A topological algebra A is called a *Q-algebra*, if the set $\operatorname{Qinv} A$ of all quasi-invertible elements of A is open in A. In case when A is a unital algebra, then A is a Q-algebra, if the set $\operatorname{Inv} A$ of all invertible elements of A is open in A.

All Banach algebras and many others are Q-algebras.

2.16.5 Advertive algebra

We say that a topological algebra A is *left advertive*, if $\operatorname{Tqinv}_l A = \operatorname{Qinv}_l A$, *right advertive*, if $\operatorname{Tqinv}_r A = \operatorname{Qinv}_r A$, and *advertive*, if it is both left and right advertive.

It is known that every Q-algebra is an advertive algebra (see, for example, [11], Proposition 2).

2.16.6 Simplicial algebra

In topological algebras, every closed regular left (right) ideal is contained in at least one maximal regular left (right, respectively) ideal which is not necessarily closed. A topological algebra A is called *left* (*right*) *simplicial*, if every closed regular left (right, respectively) ideal of A is contained in a closed maximal regular left (right, respectively) ideal of A, and *simplicial* (or *normal* in the sense of Michael, see [34]), if it is both left and right simplicial. It is known (see [13], Corollary 5) that every commutative locally m-pseudoconvex Hausdorff unital algebra is simplicial.

Every Q-algebra is simplicial algebra.

2.16.7 Gelfand-Mazur algebra

A topological algebra A is called a *Gelfand-Mazur algebra*, if the quotient algebra A/M is topologically isomorphic to $\mathbb{K}$ for each closed regular two-sided ideal M of A which is maximal as left or right ideal. Thus, Gelfand-Mazur algebras are exactly those topological algebras in which the Gelfand theory (that is well known in case of commutative Banach algebras) holds. Most of the topological algebras, that are applicable in practice, are Gelfand-Mazur algebras. Interested reader can find main classes of these algebras in [2] or [10].

2.16.8 Waelbroeck algebra

A topological algebra A is called a *Waelbroeck algebra* (see [33]) or an *algebra with continuous inverse* (see [45]), if A is a Q-algebra in which the quasi-inversion (if A has the unit element, then inversion) is continuous.

Banach algebras and many other topological algebras are Waelbroeck algebras.

2.16.9 Algebra $\mathcal{L}_\tau(X)$

Let X be a topological linear space and $\mathcal{L}_c(X)$ the set of all continuous linear maps from X to X. If we define the addition of elements and the multiplication over $\mathbb{K}$ in $\mathcal{L}_c(X)$ point-wise and the multiplication of elements in $\mathcal{L}_c(X)$ by the composition, then $\mathcal{L}_c(X)$ is an algebra (non-commutative in general).

We endow $\mathcal{L}_c(X)$ with the topology τ of simple convergence. A base of neighbourhoods of $f \in \mathcal{L}_c(X)$ in τ consists of sets

$$T_f(S, O) = \{L \in \mathcal{L}_c(X)\colon (L - f)(S) \subset O\},$$

where S is a finite subset of X and O is a neighbourhood of zero in X (see [42], Chapter 3).

The algebra $\mathcal{L}_\tau(X)$ ($\mathcal{L}_c(X)$ endowed with the topology τ of simple convergence) is a topological algebra. To prove this, we need to show that the multiplication by scalars and the addition in $\mathcal{L}_\tau(X)$ are continuous and the multiplication in $\mathcal{L}_\tau(X)$ is separately continuous.

Let $f \in \mathcal{L}_\tau(X)$ and O_f be a neighbourhood of f in τ. In order to prove the continuity of a map, it is sufficient to consider only the base neighbourhoods, so let $O_f = T_f(S, O)$, for some finite set $S \subset X$ and some neighbourhood O of zero in X.

Let $\lambda \in \mathbb{K}$ and $g \in \mathcal{L}_\tau(X)$ be such that $f = \lambda g$. Since X is a topological linear space, there are neighbourhoods $V_{\lambda,x} \subset \mathbb{K}$ of λ and $U_{g(x)} \subset X$ of $g(x)$ such that $V_{\lambda,x}U_{g(x)} \subset f(x) + O$ for every $x \in S$.

Let now

$$V_\lambda = \bigcap_{x \in S} V_{\lambda,x}$$

and

$$U = \bigcap_{x \in S} (U_{g(x)} - g(x)).$$

Then U is a neighbourhood of zero in X and $T_g(S,U)$ is a neighbourhood of g in $\mathcal{L}_\tau(X)$. If $f' \in V_\lambda T_g(S,U)$, then there are $\lambda' \in V_\lambda$ and $g' \in T_g(S,U)$ such that $f' = \lambda' g'$. Now

$$\begin{aligned}
(f' - f)(x) &= (\lambda' g' - f)(x) \\
&= \lambda'[(g' - g)(x) + g(x)] - f(x) \\
&\in V_\lambda[U + g(x)] - f(x) \\
&\subset V_{\lambda,x}[U_{g(x)} - g(x) + g(x)] - f(x) \\
&= V_{\lambda,x}U_{g(x)} - f(x) \\
&\subset f(x) + O - f(x) = O
\end{aligned}$$

for every $x \in S$, which means that $f' \in T_f(S,O)$. Therefore, $V_\lambda T_g(S,U) \subset O_f$ and the multiplication by scalars is continuous in $\mathcal{L}_\tau(X)$.

Let $f = g + h$ for some $g, h \in \mathcal{L}_\tau(X)$. Since X is a linear topological space, there is a neighbourhood V of zero in X such that $V + V \subset O$. So, $T_g(S,V)$ is a neighbourhood of g and $T_h(S,V)$ is a neighbourhood of h in $\mathcal{L}_\tau(X)$. If $f' \in T_g(S,V) + T_h(S,V)$, then there are $g' \in T_g(S,V)$ and $h' \in T_h(S,V)$ such that $f' = g' + h'$. Now

$$\begin{aligned}
(f' - f)(x) &= (g' + h' - g - h)(x) \\
&= (g' - g)(x) + (h' - h)(x) \\
&\in V + V \subset O
\end{aligned}$$

for every $x \in S$, which means that $f' \in T_f(S,O)$. Therefore,

$$T_g(S,V) + T_h(S,V) \subset O_f$$

and the addition is continuous in $\mathcal{L}_\tau(X)$.

Let now $g, h \in \mathcal{L}_\tau(X)$ and $f = g \circ h$ (the composition of maps h and g). Then $h(S)$ is a finite set and $g^{-1}(O)$ is a neighbourhood of zero in X. Hence,

$T_g(h(S), O)$ is a neighbourhood of g and $T_h(S, g^{-1}(O))$ is a neighbourhood of h in $\mathcal{L}_\tau(X)$. If $g' \in T_g(h(S), O)$, then

$$(g' \circ h - f)(x) = (g' \circ h - g \circ h)(x) = (g' - g)(h(x)) \in O$$

for every $x \in S$, which means that $g' \circ h \in O_f$. Similarily, if $h' \in T_h(S, g^{-1}(O))$, then

$$(g \circ h' - f)(x) = (g \circ h' - g \circ h)(x) = g((h' - h)(x)) \in g(g^{-1}(O)) \subset O$$

for every $x \in S$, which means that $g \circ h' \in O_f$. Therefore, $T_g(h(S), O) \circ h \subset O_f$ and $g \circ T_h(S, g^{-1}(O)) \subset O_f$. Hence, the multiplication is separately continuous in $\mathcal{L}_\tau(X)$. Consequently, the algebra $\mathcal{L}_\tau(X)$ is a topological algebra.

If X is a Hausdorff linear space, then $\mathcal{L}_\tau(X)$ is a Hausdorff algebra. To show this, let $f, g \in \mathcal{L}_\tau(X)$. If $f \neq g$, then there is $x \in X$ such that $f(x) \neq g(x)$. Since X is a Hausdorff space, then there are neighbourhoods O_1 of $f(x)$ and O_2 of $g(x)$ in X such that $O_1 \cap O_2 = \emptyset$. Now, neighbourhoods $T_g(\{x\}, O_1)$ of f and $T_g(\{x\}, O_2)$ of g in $\mathcal{L}_\tau(X)$ are disjoint.

2.17 Topological radicals

The *left* (*right*) *topological radical* of a topological algebra A is defined as the intersection $\mathrm{rad}_l(A)$ ($\mathrm{rad}_r(A)$, respectively) of kernels of all continuous irreducible representations (antirepresentations, respectively) of A on all Hausdorff linear spaces. If A has no continuous irreducible representations (antirepresentations) at all, then we say that A is a *topologically left* (*right*, respectively) *radical algebra*. In this case $\mathrm{rad}_l(A) = A$ ($\mathrm{rad}_r(A) = A$, respectively). In case $\mathrm{rad}_l(A) \neq A$ ($\mathrm{rad}_r(A) \neq A$), we say that A is a *topologically left* (*right*, respectively) *nonradical algebra* and if $\mathrm{rad}_l(A) = \{\theta_A\}$ ($\mathrm{rad}_r(A) = \{\theta_A\}$), we say that A is a *topologically left* (*right*, respectively) *semi-simple algebra*.

In general, $\mathrm{rad}_l(A) \neq \mathrm{rad}_r(A)$. A class of topological algebras, in which $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$, is described in [4].

Chapter 3

Closed ideals in algebras of sections

Let A and X be a topological spaces and $\pi\colon A \to X$ a continuous surjection such that $A_x = \pi^{-1}(x)$ is a topological algebra for evry $x \in X$. A map $s\colon X \to A$ is a *section of* π if $\pi(s(x)) = x$ for all $x \in X$. So, $s(x) \in A_x = \pi^{-1}(x)$ for every $x \in X$. We denote by τ_x the subset topology on A_x, defined by the topology of A. Now we have a family $(A_x, \tau_x)_{x\in X}$ of topological algebras. Consider the family

$$\mathcal{F} \subset \left\{ s\colon X \to \bigcup_{x\in X} A_x\colon\ s(x) \in A_x \text{ for each } x \in X \right\} = \prod_{x\in X} A_x$$

of continuous sections. Let $\mathrm{ev}_x\colon \mathcal{F} \to A_x$ be an *evaluation map* for each $x \in X$, that is, $\mathrm{ev}_x(s) = s(x)$ for every $s \in \mathcal{F}$, and therefore,

$$\mathrm{ev}_x(\mathcal{F}) = \{s(x)\colon s \in \mathcal{F}\} \subset A_x.$$

Notice, that $\mathrm{ev}_x = \mathrm{pr}_x|_{\mathcal{F}}$ for each $x \in X$, where $\mathrm{pr}_x\colon \prod_{y\in X} A_y \to A_x$ is the projection for every $x \in X$.

We shall call $\mathcal{F}$ an *algebra of continuous sections for the family* $(A_x, \tau_x)_{x\in X}$ *of topological algebras* (in short, *an algebra of sections*) if it fulfills the following conditions:

(F1) $\mathcal{F}$ is an algebra under pointwise defined operations;

(F2) $A_x = \mathrm{ev}_x(\mathcal{F})$ for any $x \in X$;

(F3) $\mathcal{F}$ is a two-sided $\mathcal{C}(X)$-module.

By (F1) and (F3), the algebraic operations in $\mathcal{F}$ are defined by

1) $(s+t)(x) = s(x) + t(x)$ for $s, t \in \mathcal{F}$ and $x \in X$;

2) $(s\lambda)(x) = (\lambda s)(x) = \lambda s(x)$ for $\lambda \in \mathbb{K}$, $s \in \mathcal{F}$ and $x \in X$;

3) $(st)(x) = s(x)t(x)$ for $s, t \in \mathcal{F}$ and $x \in X$;

4) $(fs)(x) = f(x)s(x) = (sf)(x)$ for $f \in \mathcal{C}(X)$, $s \in \mathcal{F}$ and $x \in X$.

The zero element $\theta_{\mathcal{F}}$ in $\mathcal{F}$ is defined by $\theta_{\mathcal{F}}(x) = \theta_{A_x}$ for all $x \in X$ and the topology we consider on $\mathcal{F}$ is induced by the product topology of

$$\prod_{x \in X} A_x.$$

Note that, from the algebraic point of view, $\mathcal{F}$ is a subdirect product of the algebras A_x $(x \in X)$.

If H is a subalgebra of $\mathcal{F}$, then $\mathrm{ev}_x(H)$ is a subalgebra of A_x. If I is a left (right or two-sided) ideal in $\mathcal{F}$, then $\mathrm{ev}_x(I)$ is a left (right or two-sided, respectively) ideal in A_x or $\mathrm{ev}_x(I) = A_x$.

For $x \in X$ and $H_x \subset A_x$, let

$$\mathcal{F}(x, H_x) = \{s \in \mathcal{F} \colon s(x) \in H_x\}.$$

Now one can write $\mathcal{F} = \mathcal{F}(x, A_x)$ for any $x \in X$. If I_x is a left (right or two-sided) ideal in A_x, then $\mathcal{F}(x, I_x)$ is a left (right or two-sided, respectively) ideal in $\mathcal{F}$. We also mention that

$$\begin{aligned}\bigcap_{x \in X} \mathcal{F}(x, H_x) &= \bigcap_{x \in X} \{s \in \mathcal{F} \colon s(x) \in H_x\} \\ &= \{s \in \mathcal{F} \colon s(x) \in H_x,\ x \in X\} = \mathcal{F} \cap \prod_{x \in X} H_x,\end{aligned}$$

where $H_x \subset A_x$ for all $x \in X$.

Let A be a topological algebra. A net $(e_\lambda)_{\lambda \in \Lambda}$ in A is called a *left approximate identity* (*right approximate identity*) *of* A, abbreviated l.a.i. (r.a.i., respectively), if

$$\lim_\lambda e_\lambda a = a \quad (\lim_\lambda a e_\lambda = a, \text{ respectively})$$

for every $a \in A$. An *approximate identity* (a.i. in short) is a l.a.i. which is also a r.a.i.

The structure of algebras of continuous and bounded sections, defined by Banach algebras or locally convex algebras, and ideals of such algebras has been investigated by Kaplansky ([30]), Naimark ([35]), Arzumanyan and

Grigoryan ([18]), Roch and Silbermann ([39]), Kitchen and Robbins ([32]), Scărlătescu-Murea ([41]) and others. In those papers, algebras of sections for a family of Banach algebras ([30, 35, 39]), C^*-algebras ([18]), commutative locally multiplicatively convex algebras ([32]) and locally convex *-algebras ([41]), were studied. The sections here are continuous but not bounded (or bounded on elements of a cover of X) in general and therefore the topology is not necessarily locally convex as is the case in e. g. [29].

Our aim in the paper [5] was to give the description of ideals of algebras of sections in case when $(A_x, \tau_x)_{x \in X}$ are topological algebras with an a. i. and the space X is a completely regular *Hausdorff* space, also known as a *Tychonoff* space.

Proposition 1. *Let $\mathcal{F}$ be an algebra of sections for a family $(A_x, \tau_x)_{x \in X}$ of topological algebras. If $\mathcal{F}$ has a l.a.i (r.a.i.) then every closed left (right, respectively) ideal in $\mathcal{F}$ is a $\mathcal{C}(X)$-submodule.*

The next result is similar to Theorem 3.1 in [30].

Theorem 2. *Let $\mathcal{F}$ be an algebra of sections for a family $(A_x, \tau_x)_{x \in X}$ of topological algebras indexed by a set X. Let $K \subset X$ be a non-empty subset and I_x a closed left (right or two-sided) ideal in A_x for each $x \in K$. Then*

$$I = \bigcap_{x \in K} \mathcal{F}(x, I_x)$$

is a closed left (right or two-sided, respectively) ideal in $\mathcal{F}$.

Similarly to the description of some closed ideals of $\mathcal{F}$ in the previous theorem, we can describe some dense subsets in $\mathcal{F}$.

Theorem 3. *Let $\mathcal{F}$ be an algebra of sections for a family $(A_x, \tau_x)_{x \in X}$ of topological algebras indexed by a completely regular Hausdorff space X. If $D_x \subset A_x$ is a dense subset for each x in a finite subset $P \subset X$, then*

$$D = \bigcap_{x \in P} \mathcal{F}(x, D_x)$$

is dense in $\mathcal{F}$.

The next corollary describes some closed ideals of $\mathcal{F}$ in the form of an intersection of closed ideals with known structure.

Corollary 4. *Let $(A_x, \tau_x)_{x \in X}$ be a family of topological algebras indexed by a completely regular Hausdorff space X. Let $\mathcal{F}$ be an algebra of sections,* *as above, and $I \subset \mathcal{F}$ a closed left (right or two-sided) ideal which is also a $\mathcal{C}(X)$-submodule. Then there is a subset $K \subset X$ such that $\operatorname{ev}_x(I)$ is a left (right or two-sided, respectively) ideal in A_x for all $x \in K$ and*

$$I = \bigcap_{x \in K} \mathcal{F}\left(x, \overline{\operatorname{ev}_x(I)}\right) = \bigcap_{x \in X} \mathcal{F}\left(x, \overline{\operatorname{ev}_x(I)}\right).$$

Now we give the description of all maximal closed ideals in $\mathcal{F}$.

Theorem 5. *Let $\mathcal{F}$ be an algebra of sections for a family $(A_x, \tau_x)_{x \in X}$ of topological algebras indexed by a completely regular Hausdorff space X. If every closed left (right or two-sided) ideal in $\mathcal{F}$ is a $\mathcal{C}(X)$-submodule and I is a maximal closed left (right or two-sided) ideal in $\mathcal{F}$, then there is a unique $x_0 \in X$ and a unique maximal closed left (right or two-sided, respectively) ideal $I_{x_0} \subset A_{x_0}$ such that*

$$I = \mathcal{F}(x_0, I_{x_0}).$$

The next theorem gives a simple description for some quotient spaces of algebras of sections.

Theorem 6. *Let $\mathcal{F}$ be an algebra of sections for a family $(A_x, \tau_x)_{x \in X}$ of topological algebras indexed by a completely regular Hausdorff space X. If I_{x_0} is a two-sided ideal in A_{x_0} for some $x_0 \in X$, then A_{x_0}/I_{x_0} and $\mathcal{F}/\mathcal{F}(x_0, I_{x_0})$ are topologically isomorphic.*

In the following (see also [9]), we present one special case when the set K is closed.

Theorem 7. *Let $\mathcal{F}$ be an algebra of sections for a family $(A_x, \tau_x)_{x \in X}$ of unital topological algebras indexed by a completely regular Hausdorff space X. Assume that the set*

$$s^{-1}\left(\bigcup_{x \in X} \operatorname{Inv} A_x\right) \tag{3.1}$$

is open in X for each $s \in \mathcal{F}$. Then for every closed left (right or two-sided) ideal $I \subset \mathcal{F}$, which is a $\mathcal{C}(X)$-submodule of $\mathcal{F}$, the set $K = \{x \in X \colon \operatorname{ev}_x(I) \neq A_x\}$ is closed in X.

Remark 8. *If A_x is a Q-algebra for every $x \in X$, then the set, described by (3.1), is open in X for each $s \in \mathcal{F}$.*

Chapter 4

Representations of topological algebras

Let A be a topological algebra and $\mathcal{I}$ a nonempty set of two-sided ideals of A. We consider the product

$$\Gamma = \prod_{I\in\mathcal{I}} A/I \tag{4.1}$$

of the quotient algebras. Here the topology of each quotient algebra is the quotient topology (see section 2.11) and the topology of the product space is the product topology (see section 2.9). For each $I \in \mathcal{I}$, let $\kappa_I\colon A \to A/I$ be the canonical homomorphism. For each $a \in A$ and $I \in \mathcal{I}$, let $a\hat{}(I) = \kappa_I(a)$. Then $a\hat{} \in \Gamma$ for each $a \in A$. We define $\Upsilon\colon A \to \Gamma$ with $\Upsilon(a) = a\hat{}$ for each $a \in A$ and study the properties of the map Υ. This is done in [2] for the case when A is a unital topological algebra over $\mathbb{C}$ and the set $\mathcal{I}$ of ideals is generated by the maximal ideals of the center of the algebra A.

Several mathematicians have studied representations of topological algebras as subalgebras of some topological algebras of vector-valued functions.

In [41], the set $\mathcal{I}$ of ideals is generated by a unital commutative locally C^*-subalgebra (see [24]) of a unital hereditarily complete[1] locally C^*-algebra A.

We study the case when there is a nonempty collection $\mathcal{I}$ of two-sided ideals of a topological algebra A and a topological algebra Q which is topologically isomorphic to all of the quotient algebras A/I, where $I \in \mathcal{I}$.

Example 9. If A is a Gelfand-Mazur algebra and $\mathcal{I} \subset m(A)$ (the set of all closed regular two-sided ideals of A which are maximal as left or right ideals), then $Q = \mathbb{K}$.

[1] Topological algebra A is *hereditarily complete*, if it is complete and for every closed two-sided ideal I of A, the quotient algebra A/I is also complete.

Example 10. Let X be a compact Hausdorff space, A be a Banach algebra, $\mathcal{A} = C(X, A)$ (the algebra of all continuous maps from X to A) endowed with the uniform topology, $\varepsilon_x \colon \mathcal{A} \to A$ be an onto homomorphism, defined by $\varepsilon_x(f) = f(x)$ for each $f \in C(X, A)$ and $x \in X$. Here every $\ker \varepsilon_x$ is a closed two-sided ideal in $\mathcal{A}$ because all ε_x $(x \in X)$ are continuous. Then $\pi_x \colon \mathcal{A}/\ker \varepsilon_x \to \varepsilon_x(\mathcal{A}) = A$, defined by $\pi_x(f + \ker \varepsilon_x) = f(x)$ for each $f \in \mathcal{A}$ and $x \in X$, is a continuous isomorphism. Since $\mathcal{A}$ and A are Banach algebras, then π_x is an open map for each $x \in X$ by the open-mapping theorem. Hence, $\mathcal{A}/\ker \varepsilon_x$ and A are topologically isomorphic for each $x \in X$. It means that, in this case, $Q = A$.

For a Banach algebra A, a similar construction is used in Proposition 2.2.31 in [21] (p. 180), where the quotient algebras are matrix algebras.

In [43] and [44], a locally convex algebra E is represented as a subalgebra of $C(\mathcal{M}(E, F), F)$, where F is a locally convex algebra, $\mathcal{M}(E, F)$ is the set of all nonzero continuous algebra homomorphisms of E into F, equipped with the topology of simple convergence on E, and $C(\mathcal{M}(E, F), F)$ is the algebra of all continuous F-valued maps on $\mathcal{M}(E, F)$.

Our aim is to represent A as a subalgebra of $C_c(\mathcal{I}, Q)$ of all continuous functions from $\mathcal{I}$ to Q in compact-open topology.

The next proposition asserts that Γ, defined by (4.1) for topological algebras, is a topological algebra.

Proposition 11. *Let X be a nonempty set and $(A_x, \tau_x)_{x \in X}$ a family of topological algebras. Then Γ, defined by (4.1), is, under pointwise defined algebraic operations and the product topology, a topological algebra.*

Let X and X_λ $(\lambda \in \Lambda)$ be topological spaces and $f_\lambda \colon X \to X_\lambda$ for every $\lambda \in \Lambda$. It is said that the collection $\{f_\lambda \colon \lambda \in \Lambda\}$ of maps *separates points from closed sets in* X iff whenever K is a closed subset of X and $x \notin K$, then $f_\lambda(x) \notin \mathrm{cl}_{X_\lambda}(f_\lambda(K))$ for some $\lambda \in \Lambda$.

Next, we need the following result.

Lemma 12. *Let X be a topological space, X_λ a completely regular space and $f_\lambda \colon X \to X_\lambda$ a continuous map for each $\lambda \in \Lambda$. If the collection $\{f_\lambda \colon \lambda \in \Lambda\}$ of maps separates points from closed sets in X, then X is a completely regular space.*

Using Lemma 12, we have the next result.

Proposition 13. *Let A be a topological algebra and $\mathcal{I}$ a collection of two-sided ideals of A. The following statements are equivalent:*

a) *for any finite subset* $\mathcal{I}_0 \subset \mathcal{I}$ *and nonempty sets* $V_I \in \tau_{A/I}$ $(I \in \mathcal{I}_0)$ *the intersection*

$$\bigcap_{I\in\mathcal{I}_0} \kappa_I^{-1}(V_I)$$

is nonempty;

b) $\Upsilon(A)$ *is dense in* Γ.

In case when there is a nonempty collection $\mathcal{I}$ of two-sided ideals of A and a topological algebra (Q, τ_Q) which is topologically isomorphic to all of the quotient algebras A/I, where $I \in \mathcal{I}$, let $\varrho_I\colon A/I \to Q$ be the topological isomorphisms for all $I \in \mathcal{I}$. Now, for each $a \in A$, let $a^{\sim}\colon \mathcal{I} \to Q$ be defined by $a^{\sim}(I) = \varrho_I(\kappa_I(a))$ for all $I \in \mathcal{I}$.

We endow the set $\mathcal{I}$ with the initial topology $\tau_{\mathcal{I}}$ defined by the family $\{a^{\sim}\colon a \in A\}$, that is, with the smallest topology on $\mathcal{I}$ such that all the maps $a^{\sim}$ $(a \in A)$ are continuous.

Next we have the following result that gives conditions under which $\Upsilon(A)$ is dense in $C_c(\mathcal{I}, Q)$.

Theorem 14. *Let* $\mathcal{I}$ *be a completely regular Hausdorff space and* Q *a Hausdorff algebra with unit* e_Q. *Let* A *be a* $C_c(\mathcal{I}, \mathbb{K})$*-module and a* Q*-module with unit* e_A *such that*

$$(\alpha e_A)^{\sim}(I) = \alpha(I)e_A{}^{\sim}(I) \tag{4.2}$$

for all $\alpha \in C_c(\mathcal{I}, \mathbb{K})$ *and* $I \in \mathcal{I}$. *Moreover, let one of the following conditions be true:*

a) Q *is locally convex;*

b) Q *has the approximation property;*

c) $\dim Q$ *is finite.*

Then[2] $\mathfrak{L}(C_c(\mathcal{I}, \mathbb{K})Q) \subset \Upsilon(A)$ *and* $\overline{\Upsilon(A)} = C_c(\mathcal{I}, Q)$.

Example 15. We show that the algebra $A = C_c(X, \mathbb{K})$ of continuous functions on a completely regular Hausdorff space X in compact-open topology satisfies the condition (4.2). For each $x \in X$ and $f \in A$ define $\delta_x\colon A \to \mathbb{K}$ by $\delta_x(f) = f(x)$. Then $\mathcal{I} = \{\ker \delta_x\colon x \in X\}$ is the set of all closed maximal ideals of A and the map $\nu\colon X \to \{\delta_x\colon x \in X\}$, defined by $\nu(x) = \delta_x$ for each $x \in X$ is a homeomorphism (see [22], Theorem 1). Moreover, the map $\sigma\colon \{\delta_x\colon x \in X\} \to \mathcal{I}$, defined by $\sigma(\delta_x) = \ker \delta_x$ for every $x \in X$, is an

[2]Here $\mathfrak{L}(B)$ denotes the linear span of B.

injection by Lemma 7.2 in [33] and obviously an onto map. Therefore, we can define the topology on $\mathcal{I}$ such that $\mathcal{I}$ is a completely regular Hausdorff space. Since A is a commutative Gelfand-Mazur algebra (see [10]) and every closed maximal regular ideal of A has the form $\ker \delta_x$, then $\varrho_x\colon A/\ker \delta_x \to \mathbb{K}$, defined by $\varrho_x(f + \ker \delta_x) = f(x)$ for each $f \in A$, is a topological isomorphism for every $x \in X$. Let $\kappa_x\colon A \to A/\ker \delta_x$ be the canonical map, then $\delta_x = \varrho_x \circ \kappa_x$ for every $x \in X$. We define the module multiplication in A over $C_c(\mathcal{I}, \mathbb{K})$ by

$$(\alpha f)(x) = \alpha[\sigma \circ \nu(x)]f(x) = \alpha(\ker \delta_x)f(x)$$

for each $\alpha \in C_c(\mathcal{I}, \mathbb{K})$, $f \in C_c(X, \mathbb{K})$ and $x \in X$. Then A is a $C_c(\mathcal{I}, \mathbb{K})$-module and $Q = \mathbb{K}$. Now,

$$(\alpha e_A)^{\sim}(\ker \delta_x) = (\varrho_x \circ \kappa_x)(\alpha e_A) = \delta_x(\alpha e_A) = (\alpha e_A)(x)$$

$$= \alpha[\sigma \circ \nu(x)]1 = \alpha(\ker \delta_x)\tilde{e}_A(\ker \delta_x),$$

that is, (4.2) holds for all $\alpha \in C_c(\mathcal{I}, \mathbb{K})$ and $I \in \mathcal{I}$.

Example 16. It is possible to show in a similar way that the algebra $C(X, A)$ of Example 10 satisfies the condition (4.2). In this case we take $Q = A$ and $\mathcal{I} = \{\ker \varepsilon_x\colon x \in X\}$, where ε_x is defined as in Example 10, define the module multiplication over $C_c(\mathcal{I}, A)$ by

$$(\alpha f)(x) = \alpha(\ker \varepsilon_x)f(x)$$

for each $\alpha \in C_c(\mathcal{I}, A)$, $f \in C(X, A)$ and $x \in X$ and the module multiplication over A by

$$(af)(x) = af(x)$$

for each $a \in A$, $f \in C(X, A)$ and $x \in X$. Then $\mathcal{I}$ is a nonempty set of closed two-sided ideals in A because ε_x is a continuous homomorphism from $C(X, A)$ to A for each $x \in X$.

Chapter 5

Closed maximal one-sided ideals

All closed maximal regular one-sided ideals in some classes of Gelfand-Mazur algebras are described in [1, 2, 3] and [36]. All closed maximal regular one-sided ideals in topological algebras are described in [7] by primitive ideals.

The next lemma describes the properties of the quotient map which is also a group homomorpism.

Lemma 17. *Let A be an Abelian group, I, J, M and N subgroups of A, P a subset of A, Q a nonempty subset of A, and R a subset of A/I. Let $\kappa_I\colon A \to A/I$ be the quotient map. Then*

a) $\kappa_I^{-1}(\kappa_I(P)) = I + P$,

b) $\kappa_I(P) = A/I$ *if and only if* $I + P = A$,

c) $\kappa_I^{-1}(\kappa_I(M)) = M$ *if and only if* $I \subset M$,

d) *if* $I \subset M$ *and* $I \subset N$, *then* $\kappa_I(M) = \kappa_I(N)$ *if and only if* $M = N$,

e) *if* $\kappa_I(P) = \kappa_J(Q)$, *then* $I = J$,

f) $\kappa_I(\kappa_I^{-1}(R)) = R$.

Now we turn our attention to the properties of the preimage of a left (right or two-sided) ideal. Parts a) and b) of Lemma 18 are similar to parts (i) and (ii) of Lemma 1.2.17 in [19], respectively.

Lemma 18. *Let A and B be algebras and $f\colon A \to B$ a homomorphism.*

a) *For any left (right or two-sided) ideal J of B, $f^{-1}(J)$ is a left (respectively, right or two-sided) ideal of A or $f^{-1}(J) = A$.*

b) *If J is a regular left (right or two-sided) ideal of B with a right (respectively, left or two-sided) regular unit $u \in f(A)$, then $f^{-1}(J)$ is a regular left (respectively, right or two-sided) ideal of A with any $v \in f^{-1}(u)$ as a right (respectively, left or two-sided) regular unit of $f^{-1}(J)$.*

c) *If $I \subset A$ is a left (right or two-sided) ideal and $\ker f \subset I$, then $f(I) \neq B$.*

The next result remains true also in case of groups, subgroups and group homomorphisms. In case A and B are algebras, by an epimorphism $f\colon A \to B$ we mean a surjective homomorphism.

Lemma 19. *Let A and B be algebras, $M \subset A$ a maximal left (right or two-sided) ideal of A, and $f\colon A \to B$ an epimorphism. Then the following statements are equivalent:*

a) $\ker f \subset M$;

b) $\ker f + M \neq A$;

c) $f(M) \neq B$;

d) *$f(M)$ is a maximal left (respectively, right or two-sided) ideal of B.*

Now we enrich the previous results with topology.

Lemma 20. a) *Let A and B be topological algebras, $I \subset A$ a closed left (right or two-sided) ideal of A, $f\colon A \to B$ an epimorphism and $\mathrm{cl}_B\, f(I) \neq B$. Then $\mathrm{cl}_B\, f(I)$ is a closed left (respectively, right or two-sided) ideal of B.*

b) *Let $f\colon A \to B$ be a continuous open epimorphism, I a closed regular left (right or two-sided) ideal of A and $\ker f \subset I$. Then $\mathrm{cl}_B\, f(I) \neq B$.*

c) *Let $f\colon A \to B$ be a continuous open epimorphism, M a closed maximal regular left (right or two-sided) ideal of A and $\ker f \subset M$. Then $f(M)$ is a closed maximal regular left (respectively, right or two-sided) ideal of B.*

We denote by $M_l(A)$ ($M_r(A)$) the set of all maximal left (respectively, right) ideals of A and by $m_l(A)$ ($m_r(A)$ or $m_t(A)$) the set of all closed maximal regular left (respectively, right or two-sided) ideals of A. In addition, let $\pi_l(A)$ ($\pi_r(A)$) be the set of all closed primitive ideals of A, defined by closed maximal regular left (respectively, right) ideals. For any closed ideal $I \subset A$, let

$$h_k^A(I) = \{M \in m_k(A)\colon I \subset M\},$$

where $k = l$, $k = r$ or $k = t$.

Using the previous lemmas, we get the following results. First, we describe the relationship between closed maximal regular left (right or two-sided) ideals in topological algebras A and B, provided that there is a continuous open epimorphism $f\colon A \to B$.

Theorem 21. *Let A and B be topological algebras and $f\colon A \to B$ a continuous open epimorphism. Then the map $\varphi\colon h_k^A(\ker f) \to m_k(B)$, where $k = l$, $k = r$ or $k = t$, defined by $\varphi(M) = f(M)$ for every $M \in h_k^A(\ker f)$, is a bijection.*

The next theorem gives a description of all closed maximal regular one-sided ideals of a topological algebra using primitive ideals.

Theorem 22. *Let A be a topological algebra and $M \in m_l(A)$ ($M \in m_r(A)$). Then*

$$M = \kappa_P^{-1}(\mathcal{M})$$

for some $P \in \pi_l(A)$ ($P \in \pi_r(A)$, respectively) and $\mathcal{M} \in m_l(A/P)$ ($\mathcal{M} \in m_r(A/P)$, respectively), where $\kappa_P\colon A \to A/P$ is the quotient map.

Note that the sets $m_k(A/P)$ ($P \in \pi_k(A)$, $k = l$ or $k = r$) in the next theorem are disjoint. Indeed, let $P_1, P_2 \in \pi_k(A)$ and suppose that there is

$$\mathcal{M} \in m_k(A/P_1) \cap m_k(A/P_2).$$

Then $\mathcal{M} \subset (A/P_1) \cap (A/P_2)$ and there are $a_1, a_2 \in A$ such that

$$a_1 + P_1 = a_2 + P_2 \in \mathcal{M}.$$

Therefore, $(a_1 - a_2) + P_1 = P_2$. Since $\theta_A \in P_2$, we get that $-(a_1 - a_2) \in P_1$ and thus, $(a_1 - a_2) + P_1 = P_1$. It follows that $P_1 = P_2$.

Theorem 23. *Let A be a topological algebra and the map*

$$\Lambda\colon \bigcup_{P \in \pi_k(A)} m_k(A/P) \to m_k(A)$$

(here $k = l$ or $k = r$) be defined by

$$\Lambda(\mathcal{M}) = \kappa_P^{-1}(\mathcal{M})$$

for every $\mathcal{M} \in m_k(A/P)$ and $P \in \pi_k(A)$, where $\kappa_P\colon A \to A/P$ is the quotient map. Then Λ is a surjection and for every fixed P an injection. Hence, the restriction $\Lambda|_{m_k(A/P)}$ is a bijection for every fixed P.

For every non-empty subset S of a topological algebra A, let

$$I(S) = \mathrm{cl}_A\left\{\sum_{k=1}^{n} a_k s_k + s_k b_k + c_k s_k d_k + \lambda_k s_k \colon n \in \mathbb{N}, a_k, b_k, c_k, d_k \in A, s_k \in S, \lambda_k \in \mathbb{K} \text{ for all } k\right\}.$$

Then $I(S)$ is a minimal closed two-sided ideal of A which contains S. Using this, we give another description of all closed maximal regular left or right ideals of a topological algebra.

Theorem 24. *Let A be a topological algebra, $M \in m_k(A)$ ($k = l$ or $k = r$) and E a subalgebra of A such that $I(M \cap E) \subset M$. Then*

$$M = \kappa^{-1}_{I(M\cap E)}(\mathcal{M})$$

for some $\mathcal{M} \in m_k(A/I(M \cap E))$, where $\kappa_{I(M\cap E)} \colon A \to A/I(M \cap E)$ is the quotient map.

Remark 25. *In case of two-sided ideals, Theorem* 21 *is proved in* [38]*, Theorem* 2.6.6*, and in case when the subalgebra E of A is a subalgebra of the centre of A, Theorem* 24 *is proved in* [1, 2, 3] *and* [36].

Chapter 6

Topological Jacobson radicals

In the paper [8], some applications for the results obtained in [7] are given.

The left-sided case of the following lemma is Proposition 4 in [20] p. 120. Here we present also the right-sided case. For the denotions, see 2.5.

Lemma 26. *Let X be a left (right) A-module, π (π', respectively) be a representation (antirepresentation, respectively) of A on X and $x_0 \in X \setminus \{\theta_X\}$.*

a) *Each element of* $\mathrm{id}_r(x_0)$ ($\mathrm{id}_l(x_0)$, *respectively) is a right (left, respectively) regular unit for the left ideal* $\ker \varrho_{\pi,x_0}$ (*right ideal* $\ker \varrho_{\pi',x_0}$, *respectively).*

b) *If x_0 is a left (right, respectively) cyclic element, then* $\mathrm{id}_r(x_0)$ ($\mathrm{id}_l(x_0)$, *respectively) is non-void and* $\ker \varrho_{\pi,x_0}$ *is a regular left* ($\ker \varrho_{\pi',x_0}$ *is a regular right, respectively) ideal of A.*

c) *If X is irreducible, then x_0 is a left (right, respectively) cyclic element and* $\ker \varrho_{\pi,x_0}$ ($\ker \varrho_{\pi',x_0}$, *respectively) is a maximal regular left (right, respectively) ideal.*

Theorem 1 in [12] considers the case when the A-module X is commutative, that is, $ax = xa$ for all $x \in X$ and $a \in A$. Here we examine the case when the multiplication over A is not necessarily commutative.

Theorem 27. *Let A be a right (left) topologically nonradical algebra. Then* $\mathrm{rad}_r(A)$ ($\mathrm{rad}_l(A)$, *respectively) is equal to*

a) *the intersection of all closed maximal regular right (left, respectively) ideals of A;*

b) *the intersection of all primitive ideals of A which are defined by closed maximal regular right (left, respectively) ideals of A.*

That is,

$$\mathrm{rad}_k(A) = \bigcap_{P \in \pi_k(A)} P = \bigcap_{M \in m_k(A)} M$$

where $k = r$ $(k = l$*, respectively*$)$.

Hence, $\mathrm{rad}_r(A)$ $(\mathrm{rad}_l(A)$*, respectively*$)$ *is closed in* A.

The description of the class of unital topological algebras in which all maximal one-sided ideals are closed, is given in [15] and the class in which all maximal two-sided ideals are closed is described in [16] (for the nonunital case, see [14]). In these cases, $\mathrm{rad}_l(A) = \mathrm{rad}_r(A) = \mathrm{Rad}(A)$, by Theorem 27. This is the case when A is, e.g., a Q-algebra. Here we consider the case when maximal regular ideals are not necessarily closed.

Theorem 28. *Let* A *be a topological algebra and* $\kappa_P \colon A \to A/P$ *be the quotient map for every* $P \in \pi(A)$. *Then* $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$ *if and only if*

$$\bigcap_{P \in \pi_l(A)} \kappa_P^{-1}(\mathrm{rad}_l(A/P)) = \bigcap_{Q \in \pi_r(A)} \kappa_Q^{-1}(\mathrm{rad}_r(A/Q)). \tag{6.1}$$

Replacing the condition (6.1) with stronger assumptions, gives us the following two corollaries.

Corollary 29. *Let* A *be a topological algebra. If* $\pi_l(A) = \pi_r(A)$ *and* $\mathrm{rad}_l(A/P) = \mathrm{rad}_r(A/P)$ *for every* $P \in \pi_l(A)$, *then* $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$.

Corollary 30. *Let* A *be a topological algebra. If* $\pi_l(A) = \pi_r(A)$ *and* A/P *is simplicial and advertive for every* $P \in \pi_l(A)$, *then* $\mathrm{rad}_l(A) = \mathrm{rad}_r(A)$.

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Kokkuvõte

Kinnised maksimaalsed regulaarsed ühepoolsed ideaalid topoloogilistes algebrates

Topoloogilise algebra uurimisel on kasulik teada selle kinniste ideaalide kirjeldust. Sealjuures on eriti kasulik teada kõikide kinniste maksimaalsete ideaalide kirjeldust. On teada, et kõik maksimaalsed regulaarsed ühe- ja kahepoolsed ideaalid on kinnised Q-algebrates. Lisaks on teada tarvilikud ja piisavad tingimused topoloogilise algebra jaoks, mille korral selle kõik maksimaalsed ühepoolsed ([15]) või kahepoolsed ideaalid ([16, 14]) on kinnised.

Käesoleva väitekirja eesmärgiks oli saada üldisemaid tulemusi kinniste maksimaalsete regulaarsete ideaalide kohta topoloogilistes algebrates. Kirjeldatud on kõik kinnised maksimaalsed regulaarsed ühe- ja kahepoolsed ideaalid üldiste topoloogiliste algebrate pere $\{(A_x, \tau_x) \colon x \in X\}$ poolt defineeritud pidevate lõigete topoloogilises algebras.

Väitekiri koosneb kuuest põhiosast, kirjanduse loetelust, originaalartiklite koopiatest ja autori elulookirjeldusest.

Esimeses põhiosas, tehakse ajalooline sissejuhatus käesolevasse töösse.

Teises osas esitatakse topoloogiliste algebrate ja nende ideaalide põhilised definitsioonid ja tulemused, mis on vajalikud järgnevate osade mõistmiseks.

Kolmas kuni kuues osa käsitlevad töö aluseks olevat nelja publikatsiooni, vastavalt [5], [6], [7] ja [8].

Kolmas peatükk annab ülevaate kinnistest, sealhulgas maksimaalsetest kinnistest, ühepoolsetest ideaalidest topoloogiliste algebrate pere $\{(A_x, \tau_x) \colon x \in X\}$ poolt defineeritud pidevate lõigete topoloogilises algebras. Indeksite hulgaks X on siin täielikult regulaarne Hausdorffi ruum.

Topoloogilise algebra A faktoralgebrate pere $\{(A/I, \tau_{A/I}) \colon I \in \mathcal{I}\}$ poolt defineeritud pidevate lõigete algebrat käsitleb neljas peatükk. Siin moodustavad indeksite hulga $\mathcal{I}$ topoloogilise algebra A kõik kahepoolsed ideaalid. Antakse tingimused, mille korral A esitus pidevate lõigete algebras on kõikjal tihe. Leitakse kirjeldus topoloogilisele algebrale Q, mille korral topoloogilise algebra A saab esitada pidevate funktsioonide algebra $C_c(\mathcal{I}, Q)$ kõikjal tiheda alamalgebrana, kus $\mathcal{I}$ on varustatud kompaktse-lahtise topoloogiaga.

Viiendas peatükis käsitletakse topoloogilise algebra A faktoralgebrate pere $\{(A/P, \tau_{A/P})\colon P \in \pi(A)\}$ poolt defineeritud pidevate lõigete topoloogilist algebrat. Indeksite hulga $\pi(A)$ moodustavad siin A kõigi kinniste maksimaalsete regulaarsete vasakpoolsete (parempoolsete) ideaalide poolt defineeritud primitiivsed ideaalid. Selle lõigete algebra abil saadakse topoloogilise algebra A kõigi kinniste maksimaalsete regulaarsete ühepoolsete ideaalide kirjeldus.

Kuuendas peatükis rakendatakse viienda peatüki tulemusi. Vasakpoolse ja parempoolse topoloogilise Jacobsoni radikaali kirjelduse abil leitakse tingimused, mille korral need radikaalid langevad kokku. See annab järjekordse osalise vastuse B. Yoodi 1964. aastal püstitatud küsimusele ([47]).

Töö autor on sõnastanud osa tõestatud väidetest, esitanud detailsed tõestused tulemustele ning toimetanud kõikide artiklite käsikirju. Juhendajad on püstitanud uurimisprobleemid ja andnud suure osa lahendusideedest.

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