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RAUNI LILLEMETS

Generating Systems of Sets and Sequences

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Generating Systems of Sets and Sequences

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Chapter 1

Introduction

1.1 Background and summary of the thesis

A. Pietsch created the theory of operator ideals in [Pi1], which has been widely adopted and permeates the contemporary field of Banach spaces. I. Stephani introduced the related notions of a generating system of sets and a generating system of sequences in [S]. Namely, given two generating systems of sets, one obtains an operator ideal by considering all of the operators that map the sets of the first system to the sets of the second system. Generating systems of sequences can be used to obtain generating systems of sets. In comparison to the theory of operator ideals, the notions of generating systems of sets and sequences have received considerably less attention.

One of the aims of this thesis is to study the classes of generating systems of sets and sequences and the relations between them; in particular, to show that there is a Galois connection between the former and a certain quotient class of the latter.

In [Pi1, 1.11.1], Pietsch remarks that "the collection of all operator ideals is something like a complete lattice with respect to the natural ordering". One of our objectives in this thesis is to study the lattice structure of various classes of operator ideals, generating systems of sets, and generating systems of sequences.

In [S], Stephani showed how one obtains operator ideals from generating systems of sets and vice versa, how one obtains generating systems of sets from operator ideals. From that study, the following notions emerged: the notion of a surjective operator ideal and of an ideal system of sets. We show that some of the results from [S] can also be seen through the lenses of a Galois connection between the classes of operator ideals and generating systems of sets.

A well-known example of a generating system of sets is the system of relatively compact sets, which is obtained from the system of convergent sequences in a certain way. Correspondingly, the operator ideal of compact operators consists of operators mapping bounded sets to relatively compacts sets. As sources of examples, we additionally consider several alternative notions of relative compactness. These notions have been inspired by a result proved by A. Grothendieck in his famous Memoir [G2]: a subset of a Banach space is relatively compact if and only if it is contained in the closed convex hull of a norm null sequence. Nowadays, this result is known as the Grothendieck compactness principle.

Let $1 \leq p < \infty$. If one replaces null sequences with p-summable sequences in the Grothendieck compactness principle, then one obtains a stronger form of relative compactness. This form of compactness was occasionally considered in the 1980s by O. Reinov [Re1] and J. Bourgain and O. Reinov [BR] in the study of approximation properties of order $s \leq 1$. In this thesis, such sets are said to be relatively p-compact in the sense of Bourgain–Reinov. In 2002, D. P. Sinha and A. K. Karn defined and studied in [SK1] another form of relative compactness, which lays "between" the aforementioned types of relative compactness. They required the set to belong to the so-called p-convex hull of a p-summable sequence. In the present thesis, sets of this type are said to be relatively p-compact in the sense of Sinha–Karn.

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, where p^* is the conjugate index of p. In this thesis, we study the notion of relatively (p, r)-compact sets, which encompasses the notions of relative p-compactness in the sense of Bourgain–Reinov (for $r = 1$) and in the sense of Sinha–Karn (for $r = p^*$). We observe that the system of relatively (p, r)-compact sets is a generating system of sets. By considering the operators which map bounded sets to relatively (p, r)-compact sets, the operator ideal of (p, r)-compact operators is obtained. Relatively p-compact operators in the sense of Bourgain–Reinov (for $r = 1$) and Sinha–Karn (for $r = p^*$) are special cases of this construction.

It was proven in [SK1] that the collection of all p-compact operators (in the sense of Sinha–Karn) is a Banach operator ideal. We describe the operator ideal $\mathcal{K}_{(p,r)}$ of all (p, r)-compact operators as the surjective hull of the operator ideal $\mathcal{N}_{(p,1,r^*)}$. This allows us to equip $\mathcal{K}_{(p,r)}$ with the corresponding s-norm of $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ and to prove that $\mathcal{K}_{(p,r)}$ is an s-Banach operator ideal.

In [SK1], the notion of relatively weakly p-compact sets was also studied. A more general notion of relatively weakly (p, r)-compact sets was introduced in [AO2], encompassing the weakly p-compact sets for $r = p^*$. Additionally, the notion of unconditionally weakly (p, r)-compact sets was introduced in [AO2], residing between the relatively (p, r)-compact sets and relatively weakly (p, r)-compact sets. The weakly p-compact, weakly (p,r)-compact operators, and unconditionally (p,r)-compact operators are defined in the obvious manner. Denote by $\mathcal{W}_{(p,r)}$ and $\mathcal{U}_{(p,r)}$ the operator ideals of all weakly (p,r)-compact operators and all unconditionally (p,r)-compact operators. It was proven in [SK1], that if $1 \leq p < \infty$, the operator ideal $\mathcal{W}_p = \mathcal{W}_{(p,p^*)}$ of all weakly p-compact operators is a Banach operator ideal. We prove that $\mathcal{W}_{(p,1)}$ and $\mathcal{U}_{(p,1)}$ are quasi-Banach operator ideals, where $1 \leq p < \infty$. We do so by proposing a general method for constructing generating systems of sets and operator ideals from a BK-space g and a normed system of sequences $\mathbf{h}$. We prove that the constructed operator ideal is always quasi-Banach provided that g and $\mathbf{h}$ satisfying certain assumptions.

We show that $\mathcal{W}_\infty = \mathcal{W}_{(\infty,1)}$ is a Banach operator ideal. We prove that the operator ideal $\mathcal{V}$ of completely continuous operators can be expressed as a right-hand quotient $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. From this, the result [DFLORT, Theorem 1] follows: a weak version of the Grothendieck compactness principle holds only in Schur spaces.

Recall that a Banach space is said to have the approximation property if the identity operator can be approximated uniformly on compact sets by finite rank operators. In the spirit of this thesis, we study the approximation property by considering the system of all approximable sets, where an approximable set is defined as a bounded set on which the identity operator may be approximated uniformly. Similarly, we define an approximable sequence as a null sequence on which the identity operator may be approximated uniformly, and study the system of approximable sequences. Clearly, a Banach space has the approximation property if and only if the relatively compact sets are exactly the approximable sets. We prove a result reminiscent of the Grothendieck compactness principle: a subset of a Banach space is approximable if and only if it is contained in the closed convex hull of an approximable sequence. We also prove that there exists a non-approximable sequence which can be represented as a sum of three approximable sequences.

The thesis has been organized as follows.

Chapter 1 introduces the historical background of the relevant notions, provides a summary of the thesis and describes the notation used throughout the thesis.

In Chapter 2, we study the class OI of all operator ideals, the class GSet of all generating systems of sets, the class GSeq of all generating systems of sequences, and various relations between these classes. Stephani considered a domination relation on the class GSeq, which we denote as $\lesssim$. This relation is a preorder, which induces an equivalence relation $\sim$. Using a standard procedure, the class GSeq$/_\sim$ becomes an ordered class with the order induced from the preorder $\lesssim$. One of the main results of this thesis states that there is a Galois connection between the ordered classes GSet and GSeq$/_\sim$ (the order on the class GSet is defined in a natural way via inclusion). We say that a generating system of sets is sequentially generatable if it can be obtained from a generating systems of sequences. The Galois connection provides a useful characterization for sequentially generatable systems of sets. This chapter is mainly based on [Lil1].

In Chapter 3, we study the lattice structure on the class of operator ideals, the classes of generating systems of sets and sequences, and related classes. We then study the order properties of the various mappings between these classes. This chapter is mainly based on [Lil1].

In Chapter 4, we study the generating system $\mathbf{K}_{(p,r)}$ of all relatively (p,r)-compact sets and the operator ideal $\mathcal{K}_{(p,r)}$ of all (p,r)-compact operators. Relying on Chapter 2, we show that the system $\mathbf{K}_{(p,r)}$ is sequentially generatable if and only if $p=\infty$ and $r=1$, in which case $\mathbf{K}_{(p,r)}$ coincides with the system of all relatively compact sets $\mathbf{K}$. In turn, the system $\mathbf{K}_{(p,r)}$ provides answers and counterexamples to certain questions posed in the general context of Chapter 2. We prove that $\mathcal{K}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$. This allows us to equip $\mathcal{K}_{(p,r)}$ with the corresponding s-norm of $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ and to prove that $\mathcal{K}_{(p,r)}$ is an s-Banach operator ideal. This chapter is based on [ALO] and [Lil1].

In Chapter 5, we study sequentially generatable systems of sets $\mathbf{G}$, which satisfy $\mathbf{G} \leq \mathbf{K}$. We introduce the notion of a hereditarily almost autoapproximable sequence. Using this notion, we prove that the latter inequality $\mathbf{G} \leq \mathbf{K}$ is strict if and only if the system $\mathbf{G}$ is obtained from a generating system of sets $\mathbf{g}$ consisting entirely of hereditarily almost autoapproximable sequences. We also provide an example of such a system of sequences $\mathbf{g}$.

In Chapter 6, we study the generating system $\mathbf{W}_\infty$ of all relatively weakly ∞-compact sets and the operator ideal $\mathcal{W}_\infty$ of all weakly ∞-compact operators. We show that the operator ideal $\mathcal{W}_\infty$ is a Banach operator ideal. We prove that the equality $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$ holds (even in the context of Banach operator ideals). As a consequence, this provides an alternative proof for the following result from [DFLORT, Theorem 1]: the weak Grothendieck compactness principle holds only in Schur spaces. This chapter is based on [JLO] and [Lil2].

In Chapter 7, we propose a general method for constructing generating systems of sets and quasi-Banach operator ideals. This method is inspired by the construction of generating systems of sets $\mathbf{K}_{(p,r)}$ and $\mathbf{W}_\infty$, and the corresponding operator ideals $\mathcal{K}_{(p,r)}$ and $\mathcal{W}_\infty$. This construction produces a quasi-Banach operator ideal from a BK-space g and a normed system of sequences $\mathbf{h}$, provided that g and $\mathbf{h}$ satisfy certain criteria. Among other examples, we prove that the operator ideals $\mathcal{W}_{(p,1)}$ and $\mathcal{U}_{(p,1)}$ are quasi-Banach operator ideals (for $1 \leq p < \infty$).

Chapter 8 begins with an overview about some of the known results concerning the approximation property. After this, we give the definitions of an approximable set and an approximable sequence. We prove a Grothendieck-like criterion for describing the approximable sets in a Banach space via the approximable sequences in this space. We also prove that there exists a non-approximable sequence, which can be represented as a sum of three approximable sequences.

1.2 Notation

We always consider X and Y to be Banach spaces over the same field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators acting from X to Y; if $X = Y$, we use the notation $\mathcal{L}(X)$ instead. An operator $T \in \mathcal{L}(X,Y)$ is of *finite rank* if its range is finite-dimensional. Denote by $\mathcal{F}(X,Y)$ the space of finite rank operators acting from X to Y; if $X = Y$, we use the notation $\mathcal{F}(X)$ instead. For an operator $T\colon X \to Y$, we denote by $\ker T$ and $\operatorname{ran} T$ its kernel and range, respectively.

The closed unit ball and the unit sphere of X are denoted by B_X and S_X, respectively. The identity operator of X is denoted by I_X and the dual space of X is denoted by X^*. The closure of a set K is denoted by $\overline{K}$, its linear span by $\operatorname{span} K$, and its absolutely convex hull by $\operatorname{absconv} K$. The norm closures of the two latter sets are denoted by $\overline{\operatorname{span}} K$ and $\overline{\operatorname{absconv}} K$, respectively. A closure of K with respect to a topology τ is denoted by $\overline{K}^{\tau}$.

By using the term "sequence", we implicitly assume that we are dealing with an infinite sequence. In the rare cases when we need to consider finite sequences, we state this explicitly. Denote by e_k the unit sequence (δ_{jk}), where δ_{jk} is Kronecker's symbol. We use the shorthand $\{x_k\} = \{x_k \mid k \in \mathbb{N}\}$ for any sequence (x_k). For brevity, we also put $\overline{x} = (x_k), \overline{y} = (y_k)$, etc., and $\overline{\alpha} = (\alpha_k), \overline{\beta} = (\beta_k)$, etc. We use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $1 \le p \le \infty$, let p^* denote the conjugate index of p (i.e., $1/p + 1/p^* = 1$ with the convention $1/\infty = 0$).

Let I be any set. A family of numbers $(\lambda_x)_{x\in I}$ (defined on the set I) is said to be *summable* if the directed system of all finite partial sums converges (see [Pi1, A.4.6]). In that case, the limit is denoted by $\sum_{x\in I} \lambda_x$. A family of numbers $(\lambda_x)_{x\in I}$ is said to be *absolutely summable* if the family $(|\lambda_x|)_{x\in I}$ is summable (see [Pi1, A.4.7]). Let us denote by $\ell_1[I]$ the Banach space of all absolutely summable families of numbers $(\lambda_x)_{x\in I}$ with the norm

$$\|(\lambda_x)_{x\in I}\| = \sum_{x\in I} |\lambda_x|.$$

Note that in [Pi1], notation $\ell_1(I)$ is used instead of $\ell_1[I]$ (see [Pi1, C.1.3]). The *natural surjection* $Q_X\colon \ell_1[B_X] \to X$ is defined by

$$Q_X\big((\lambda_x)_{x\in B_X}\big) = \sum_{x\in B_X} \lambda_x x.$$

A functional $f \in X^*$, where X is a topological vector space, is said to *vanish* on a set A if $f(x) = 0$ for all $x \in A$. A functional f is said to vanish on a sequence (x_k) if $f(x_k) = 0$ for all $k \in \mathbb{N}$.

If τ_1 and τ_2 are topologies on a set X so that $\tau_1 \subset \tau_2$, then τ_2 is said to be *stronger* than τ_1; likewise, τ_1 is said to be *weaker* than τ_2.

We consider the following set-theoretical structures, in addition to the sets themselves:

(i) 1st order classes, which consist of sets (e.g., a generating system of sequences);

(ii) 2nd order classes, which consist of 1st order classes (e.g., the class GSeq);

(iii) 3rd order classes, which consist of 2nd order classes (e.g., the quotient $\mathrm{GSeq}/_\sim$).

In the following we refer to all of them simply as classes.

A relation "$\le$" (on a set or a class) is said to be an order if it is reflexive, antisymmetric, and transitive. If $(A, \le)$ is an ordered class, then A^∂ denotes the class ordered by the reverse order $\ge$.

We assume that the reader is familiar with well-known basic notions and results from the theory of Banach spaces and topological vector spaces (such as Auerbach's lemma, Schauder's theorem, Gantmacher's theorem, the bipolar theorem, the Eberlein–Šmulian theorem, the Banach–Alaoglu theorem, and the Hahn–Banach theorem and its corollaries (e.g., the Tukey–Klee separation theorem)).

Chapter 2

Operator ideals and generating systems of sets and sequences

In this chapter, we study the class OI of all operator ideals, the class GSet of all generating systems of sets, the class GSeq of all generating systems of sequences, and various relations between these classes. Stephani considered a domination relation on the class GSeq, which we denote as $\lesssim$. This relation is a preorder, which induces an equivalence relation $\sim$. Using a standard procedure, the class $\mathrm{GSeq}/_\sim$ becomes an ordered class with the order induced from the preorder $\lesssim$. One of the main results of this thesis states that there is a Galois connection between the ordered classes GSet and $\mathrm{GSeq}/_\sim$ (the order on the class GSet is defined in a natural way via inclusion). We say that a generating system of sets is sequentially generatable if it can be obtained from a generating systems of sequences. The Galois connection provides a useful characterization for sequentially generatable systems of sets. This chapter is mainly based on [Lil1].

2.1 Galois connections

In this section, we recall the definition and several results about Galois connections. The results about complete lattices and Galois connections are taken from the book [DP] and they are easily provable without prior knowledge about Galois connections. Although these results are stated in the context of sets in [DP], it can be verified that they hold also in the context of classes.

Definition 2.1.1 (see [DP, 7.23])**.** Let A and B be ordered classes. A pair (R, S)

of operators $R\colon A \to B$ and $S\colon B \to A$ is a *Galois connection* between A and B if, for all $a \in A$ and $b \in B$,

$$R(a) \leq b \Leftrightarrow a \leq S(b).$$

Lemma 2.1.2 (see [DP, Lemma 7.26]). *Assume that the pair (R, S) is a Galois connection between ordered classes A and B. Let $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then*

(Gal1) $a \leq SR(a)$ *and* $RS(b) \leq b$;

(Gal2) $a_1 \leq a_2 \Rightarrow R(a_1) \leq R(a_2)$ *and* $b_1 \leq b_2 \Rightarrow S(b_1) \leq S(b_2)$;

(Gal3) $R(a) = RSR(a)$ *and* $S(b) = SRS(b)$.

Conversely, a pair of maps $R\colon A \to B$ and $S\colon B \to A$ satisfying (Gal1) and (Gal2) for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$ sets up a Galois connection between A and B.

Let $(A, \leq)$ be an ordered class. Recall that the symbol A^∂ denotes the class A equipped with the reverse order $\geq$.

Definition 2.1.3 (see [DP, 7.1]). Let A be an ordered class. An operator $T : A \to A$ is called a *closure operator* if, for all $a, a_1, a_2 \in A$,

(i) $a \leq T(a)$;

(ii) $a_1 \leq a_2 \Rightarrow T(a_1) \leq T(a_2)$;

(iii) $T(T(a)) = T(a)$.

An element $a \in A$ is said to be *closed* (with respect to the closure operator T) if $T(a) = a$.

Definition 2.1.4 (see [DP, 1.34]). Let A and B ordered classes. An operator $T\colon A \to B$ is said to be an *order-embedding* provided that $a \leq b$ if and only if $T(a) \leq T(b)$. T is said to be an *order-isomorphism* if it is an order-embedding which maps A onto B.

Proposition 2.1.5 (see [DP, 7.27]). *Assume that the pair (R, S) is a Galois connection between ordered classes A and B. Denote $\overline{A} = \{a \in A \mid a = SR(a)\}$ and $\overline{B} = \{b \in B \mid b = RS(b)\}$. Then*

(i) *operators* $SR\colon A \to A$ *and* $RS\colon B^\partial \to B^\partial$ *are closure operators;*

(ii) *operators* $R\colon \overline{A} \to \overline{B}$ *and* $S\colon \overline{B} \to \overline{A}$ *are mutually inverse order-isomorphisms.*

The next lemma follows directly from the property (Gal3).

Lemma 2.1.6. *Let* (R, S) *be a Galois connection between ordered classes* A *and* B*. Let* $a \in A, b \in B$*. Then the following are equivalent:*

(i) $a \in \overline{A}$*;*

(ii) $\exists b_1 \in B$ *such that* $a = S(b_1)$*.*

Similarly, the following are equivalent:

(i) $b \in \overline{B}$*;*

(ii) $\exists a_1 \in A$ *such that* $b = R(a_1)$*.*

2.2 Operator ideals

In this section, we recall some basic facts and properties about operator ideals. We also recall some of the classical examples. Denote by $\mathcal{L}$ the class of all bounded linear operators between arbitrary Banach spaces.

Definition 2.2.1 (see [Pi2, 2.6.6.1])**.** *An operator ideal* $\mathcal{A}$ is a subclass of $\mathcal{L}$ such that the components

$$\mathcal{A}(X, Y) := \mathcal{A} \cap \mathcal{L}(X, Y)$$

satisfy the following conditions:

(OI_0) $I_{\mathbb{K}} \in \mathcal{A}$, where $\mathbb{K}$ denotes the 1-dimensional Banach space;

(OI_1) $S + T \in \mathcal{A}(X, Y)$ for any $S, T \in \mathcal{A}(X, Y)$;

(OI_2) if $T \in \mathcal{L}(X, Y), S \in \mathcal{A}(Y, Z)$, and $R \in \mathcal{L}(Z, W)$, then $RST \in \mathcal{A}(X, W)$.

We denote the class of all operator ideals by OI.

Recall (see, e.g. [Pi2, 3.2.5.1]) that a map from a linear space X to non-negative numbers is a *quasi-norm* if the following conditions are satisfied.

(QN_0) $\|x\| = 0$ implies $x = 0$;

(QN_1) there exists $\varkappa \geq 1$ such that $\|x+y\| \leq \varkappa\,(\|x\| + \|y\|)$, where $x, y \in X$;

(QN_2) $\|\lambda x\| = |\lambda|\,\|x\|$, where $x \in X$ and $\lambda \in \mathbb{K}$.

A quasi-norm is called a *p-norm* if (QN_1) is replaced with the *p-triangle inequality*

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p .$$

(see, e.g. [Pi2, 3.2.5.2]). The following facts are well-known.

Remark 2.2.2. A 1-norm is just a norm, and an s-norm is also a t-norm if $0 < t < s \leq 1$. Every s-norm is a quasi-norm, since the constant $\varkappa := 2^{\frac{1}{p}-1}$ can be used to satisfy condition (QN_1). To see this, start from the p-triangle inequality and apply the generalized mean inequality (see, e.g., [HLP, (2.9.1)]) for the exponents p and 1.

An s-norm induces a metric on X defined by $d(x,y) = \|x-y\|^s$. A space X is said to be s-Banach if it is complete for this metric (see, e.g., [Kal]). In the case of a quasi-norm, X is endowed with a metrizable topology with the base of neighborhoods consisting of the sets

$$\{x \in X \mid \|x\| \leq \varepsilon\},$$

where $\varepsilon > 0$. A complete quasi-normed space is called a *quasi-Banach* space.

Definition 2.2.3 (see [DF, 9.3])**.** A *quasi-normed operator ideal* $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an operator ideal $\mathcal{A}$ together with a function $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to [0,\infty)$ such that

(QOI_0) $\|I_{\mathbb{K}}\|_{\mathcal{A}} = 1$;

(QOI_1) there exists a constant $\varkappa \geq 1$ such that

$$\|S_1 + S_2\|_{\mathcal{A}} \leq \varkappa\,(\|S_1\|_{\mathcal{A}} + \|S_2\|_{\mathcal{A}}) .$$

(QOI_2) If $T \in \mathcal{L}(X_0, X)$, $S \in \mathcal{A}(X,Y)$, and $R \in \mathcal{L}(Y, Y_0)$, then

$$\|RST\|_{\mathcal{A}} \leq \|R\|\,\|S\|_{\mathcal{A}}\,\|T\| .$$

If all of the components $\mathcal{A}(X,Y)$ are quasi-Banach spaces (with respect to the quasi-norm $\|\cdot\|_{\mathcal{A}}$), then $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called a *quasi-Banach operator ideal.*

As shown by the following theorem, it suffices to show that all of the components $\mathcal{A}(X,Y)$ are sequentially complete in order to show that they are complete.

Theorem 2.2.4 (see [KA, Part III, Theorem 3]). *A Hausdorff topological vector space X, which is sequentially complete and has a countable base of neighborhoods of 0, is complete.*

Proposition 2.2.5 (see [DF, Proposition(1), p. 109]). *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a quasi-normed operator ideal. Then $\|T\| \leq \|T\|_{\mathcal{A}}$ for every $T \in \mathcal{A}(X,Y)$, where X and Y are arbitrary Banach spaces.*

Definition 2.2.6 (see [DF, p. 109]). A quasi-normed operator ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a *p-normed operator ideal* (where $0 < p \leq 1$) if the p-triangle inequality holds:

$$(p\text{-}OI)\quad \|S_1 + S_2\|_{\mathcal{A}}^p \leq \|S_1\|_{\mathcal{A}}^p + \|S_2\|_{\mathcal{A}}^p \text{ for } S_1, S_2 \in \mathcal{A}(X,Y).$$

If $p = 1$, then $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called a *normed operator ideal*.

Remark 2.2.2 immediately yields the following observation.

Proposition 2.2.7 (see Remark after [Pi1, 6.2.1]). *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a p-normed operator ideal (where $0 < p \leq 1$). Then it is a quasi-normed operator ideal, where the constant $\varkappa := 2^{\frac{1}{p}-1}$ can be used to satisfy condition (QOI_1).*

Definition 2.2.8 (see [DF, p. 109]). A p-normed operator ideal (where $0 < p \leq 1$) is said to be a *p-Banach operator ideal* if all components $\mathcal{A}(X,Y)$ are p-Banach spaces. If $p = 1$, then $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called a *Banach operator ideal*.

Definition 2.2.9 (see [DF, p. 110]). An operator ideal $\mathcal{A}$ is said to be closed if it is a Banach operator ideal when equipped with the operator norm $\|\cdot\|$.

Recall that an operator $T \in \mathcal{L}(X,Y)$ is said to be (*weakly*) *compact* if it maps bounded subsets of X to relatively (weakly) compact subsets of Y. An operator $T \in \mathcal{L}(X,Y)$ is said to be *completely continuous* if it maps weakly null sequences in X to null sequences in Y. The following operator ideals are well known and studied.

(i) The operator ideal $\mathcal{F}$ of finite rank operators [Pi1, 1.2.1];

(ii) the operator ideal $\mathcal{K}$ of compact operators [Pi1, 1.4.2];

(iii) the operator ideal $\mathcal{W}$ of weakly compact operators [Pi1, 1.5.2];

(iv) the operator ideal $\mathcal{V}$ of completely continuous operators [Pi1, 1.6.2];

(v) the operator ideal $\mathcal{L}$ of bounded linear operators.

In [Pi1], the notations $\mathfrak{F}$, $\mathfrak{K}$, $\mathfrak{W}$, $\mathfrak{V}$, and $\mathfrak{L}$ are used, respectively. It is proven in [Pi1, 4.2.5] that the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{V}$, and $\mathcal{L}$ are closed.

Let $\mathcal{A}, \mathcal{B} \in$ OI. Recall that the *inclusion* $\mathcal{A} \subset \mathcal{B}$ means that $\mathcal{A}(X,Y) \subset \mathcal{B}(X,Y)$ for all Banach spaces X and Y. The *equality* of $\mathcal{A}$ and $\mathcal{B}$ is denoted as $\mathcal{A} = \mathcal{B}$ and means that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$. It is a well-known and easily verifiable that the class OI is an ordered class with respect to the relation "$\subset$".

Similar terminology is used for a quasi-Banach operator ideals $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. The *inclusion* is denoted by $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \subset (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, or shortly, $\mathcal{A} \subset \mathcal{B}$, if $\mathcal{A} \subset \mathcal{B}$ as operator ideals, and additionally, $\|T\|_{\mathcal{A}} \geq \|T\|_{\mathcal{B}}$ for all Banach spaces X and Y, and for all $T \in \mathcal{A}(X,Y)$. Two quasi-Banach operator ideals $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ are *equal*, i.e., they are equal as operator ideals and their quasi-norms coincide, if and only if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$ as quasi-Banach operator ideals.

The components of $\mathcal{A}^{\mathrm{sur}}$, the *surjective hull* of $\mathcal{A}$ are defined by

$$\mathcal{A}^{\mathrm{sur}}(X,Y) = \{T \in \mathcal{L}(X,Y) \mid TQ_X \in \mathcal{A}(\ell_1[B_X], Y)\}.$$

If $\mathcal{A} = \mathcal{A}^{\mathrm{sur}}$, then $\mathcal{A}$ is *surjective.* If $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a quasi-Banach operator ideal, then $\mathcal{A}^{\mathrm{sur}}$ is also a quasi-Banach operator ideal with $\|T\|_{\mathcal{A}^{\mathrm{sur}}} = \|TQ_X\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{\mathrm{sur}}(X,Y)$ (see, e.g., [Pi2, 6.3.2.7]). Clearly, $\mathcal{A} \subset \mathcal{A}^{\mathrm{sur}}$, and if $\mathcal{A}$ is a quasi-Banach operator ideal, then this inclusion holds in the sense of quasi-Banach operator ideals [Pi1, 8.5.3].

Proposition 2.2.10 (see [Pi1, 8.5.3])**.** *Let $\mathcal{A}, \mathcal{B}$ be (quasi-Banach) operator ideals with $\mathcal{A} \subset \mathcal{B}$. Then $\mathcal{A}^{\mathrm{sur\,sur}} = \mathcal{A}^{\mathrm{sur}}$ and $\mathcal{A}^{\mathrm{sur}} \subset \mathcal{B}^{\mathrm{sur}}$ as (quasi-Banach) operator ideals.*

Recall that the *right-hand quotient* $\mathcal{A} \circ \mathcal{B}^{-1}$ of two operator ideals $\mathcal{A}$ and $\mathcal{B}$ is the operator ideal that consists of all operators $T \in \mathcal{L}(X,Y)$ such that $TS \in \mathcal{A}(X_0, Y)$ whenever $S \in \mathcal{B}(X_0, X)$ for some Banach space X_0 (see [Pi1, 3.1.1]).

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be quasi-Banach operator ideals. The quotient $\mathcal{A} \circ \mathcal{B}^{-1}$ becomes a quasi-Banach operator ideal if for every operator $T \in \mathcal{A} \circ \mathcal{B}^{-1}(X,Y)$ one puts

$$\|T\|_{\mathcal{A}\circ\mathcal{B}^{-1}} = \sup\{\|TS\|_{\mathcal{A}} \mid S \in \mathcal{B}(X_0, X), \|S\|_{\mathcal{B}} \leq 1\},$$

where the supremum is taken over all Banach spaces X_0 (see [Pi1, 7.2.1]).

The inclusion $\mathcal{A} \subset \mathcal{A} \circ \mathcal{B}^{-1}$ holds for any two (quasi-Banach) operator ideals $\mathcal{A}$ and $\mathcal{B}$.

2.3 Galois connection between operator ideals and generating systems of sets

In this section, we recall the basic definition of a generating system of sets. We prove that there is a Galois connection between the classes of all operator ideals and all generating systems of sets. We observe how this Galois connection relates to the classical notions of surjective operator ideals and ideal systems of sets (see Propositions 2.3.3 and 2.3.7). The results in this section are not really new, as the essential parts of most of the proofs are from [S]; rather, we provide a new perspective for seeing the known results.

We remark that our notation for generating systems of sets and sequences differs from Stephani's for certain convenience reasons.

By a *system of sets* $\mathbf{G}$ we mean a rule which for every Banach space X fixes a family $\mathbf{G}(X)$ of subsets of X. The latter family is called a *component* of $\mathbf{G}$ (in X). We denote the class of all systems of sets by SSet.

By $\mathbf{B}$, we denote the system of all bounded sets in all Banach spaces (notations $\mathfrak{B}$ and $\boldsymbol{b}$ are used in [S] and [AO2], respectively).

Definition 2.3.1 (see [S, Definition 1.1]). System of sets $\mathbf{G}$ is called a *generating system of sets* if for every Banach space X the following conditions are satisfied:

(G_0) $\mathbf{G}(X) \subset \mathbf{B}(X)$;

(G_1) the component $\mathbf{G}(\mathbb{K})$ contains the unit ball $B_{\mathbb{K}}$ of the space $\mathbb{K}$;

(G_2) $\mathbf{G}(X)$ is closed under algebraic operations: if $G, H \in \mathbf{G}(X)$ and $a \in \mathbb{K}$, then $aG + H \in \mathbf{G}(X)$;

(G_3) $\mathbf{G}(X)$ is closed under taking subsets: if $G \in \mathbf{G}(X)$ and $H \subset G$, then $H \in \mathbf{G}(X)$;

(G_4) if $G \in \mathbf{G}(X)$ and $T \in \mathcal{L}(X, Y)$, then $T(G) \in \mathbf{G}(Y)$.

We denote the class of all generating systems of sets by GSet.

Let $\mathbf{G}, \mathbf{H} \in$ SSet. Define $\mathbf{G} \leq \mathbf{H}$ if $\mathbf{G}(X) \subset \mathbf{H}(X)$ for every Banach space X. Clearly, the relation "$\leq$" is an order on the classes SSet and GSet.

It is easy to see that the following are examples of generating systems of sets:

(i) the system $\mathbf{B}$;

(ii) the system $\mathbf{W}$ of all relatively weakly compact sets in all Banach spaces (notations $\mathfrak{M}_{wc}$ and $\boldsymbol{w}$ are used in [S] and [AO2], respectively);

(iii) the system $\mathbf{K}$ of all relatively compact sets in all Banach spaces (notations $\mathfrak{M}_c$ and $\boldsymbol{k}$ are used in [S] and [AO2], respectively);

(iv) the system $\mathbf{F}$ such that its component $\mathbf{F}(X)$ consists of all bounded subsets of finite-dimensional subspaces of X.

For more examples of generating systems of sets, consult [S, Section 1].

It was proven by Stephani that an operator ideal $\mathcal{A}$ is surjective if and only if $T(B_X) \subset S(B_Z)$ implies $T \in \mathcal{A}(X,Y)$ for every $S \in \mathcal{A}(Z,Y)$ and $T \in \mathcal{L}(X,Y)$ (see [S, Section 0]).

In [S, Section 1], Stephani showed that given systems $\mathbf{G}, \mathbf{H} \in \mathrm{GSet}$ such that $\mathbf{H} \leq \mathbf{G}$, one obtains an operator ideal $[\mathbf{G} \to \mathbf{H}]$ by considering all of the operators which map sets from the system $\mathbf{G}$ to sets of the system $\mathbf{H}$. In other words, an operator $T \in \mathcal{L}(X,Y)$ belongs to the component

$$[\mathbf{G} \to \mathbf{H}](X,Y)$$

if $T(G) \in \mathbf{H}(Y)$ whenever $G \in \mathbf{G}(X)$. Notice that Stephani used the notation $\mathbf{A}[\mathbf{H}/\mathbf{G}]$ instead of $[\mathbf{G} \to \mathbf{H}]$.

In the given thesis, we use this definition mostly in the special case $\mathbf{G} := \mathbf{B}$. Put

$$\Theta(\mathbf{H}) = [\mathbf{B} \to \mathbf{H}].$$

It is easy to see that $T \in \Theta(\mathbf{H})(X,Y)$ if and only if $T(B_X) \in \mathbf{H}(Y)$. Stephani remarked that the operator ideal $\Theta(\mathbf{H})$ is always surjective. Classical examples are $\mathcal{F} = \Theta(\mathbf{F})$, $\mathcal{K} = \Theta(\mathbf{K})$, $\mathcal{W} = \Theta(\mathbf{W})$, and $\mathcal{L} = \Theta(\mathbf{B})$.

Let $\mathbf{G} \in \mathrm{GSet}$ and $\mathcal{A} \in \mathrm{OI}$. Stephani gave in [S, Section 2] a method for constructing a new generating system of sets as a product $\mathcal{A} \circ \mathbf{G}$, which is defined in the following way:

$$\mathcal{A} \circ \mathbf{G}(Y) = \{G \subset Y \mid G \subset T(H), \text{ where } H \in \mathbf{G}(X) \text{ and } T \in \mathcal{A}(X,Y)\}.$$

This means that the component $\mathcal{A} \circ \mathbf{G}(Y)$ consists of those sets that are contained in the image $T(H)$ of a set from the class $\mathbf{G}$, where T is an appropriate operator from $\mathcal{A}$. In the given thesis we only consider symbols of the form $\mathcal{A} \circ \mathbf{B}$, which we denote by $\Gamma(\mathcal{A})$.

To summarize, we have a mapping Θ from operator ideals to generating systems of sets, and vice versa, a mapping Γ from generating systems of sets to operator ideals. The following proposition demonstrates how the classes OI and GSet are related to each other via those mappings.

Proposition 2.3.2. *The pair* (Γ, Θ) *is a Galois connection between the classes* OI *and* GSet.

Proof. We need to show that given $\mathcal{A} \in \mathrm{OI}$ and $\mathbf{G} \in \mathrm{GSet}$, it holds that $\Gamma(\mathcal{A}) \leq \mathbf{G}$ if and only if $\mathcal{A} \subset \Theta(\mathbf{G})$. By definitions, it is easy to see that both the former and the latter statement mean that an operator $T \in \mathcal{A}(X, Y)$ maps each bounded set G in X to a set $T(G) \in \mathbf{G}(Y)$. □

Denote the class of all surjective operator ideals by surOI. According to Proposition 2.1.5, the operator $\Theta \circ \Gamma \colon \mathrm{OI} \to \mathrm{OI}$ is a closure operator. We have the following proposition, which describes the corresponding class of closed elements.

Proposition 2.3.3. *The following classes coincide.*

(i) *The class* surOI *of all surjective operator ideals;*

(ii) *the class* $\overline{\mathrm{OI}}$ *of operator ideals that are mapped to themselves by the closure operator* $\Theta \circ \Gamma$ *of the class* OI*;*

(iii) *the class* $\Theta(\mathrm{GSet})$.

Proof. It is straightforward to verify from the definitions that $\mathcal{A} = \Theta \circ \Gamma(\mathcal{A})$ for every surjective operator ideal $\mathcal{A}$. Thus (i) $\Rightarrow$ (ii).

Clearly (iii) $\Rightarrow$ (i), since every operator ideal of the form $\mathcal{A} = \Theta(\mathbf{G})$ is surjective.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from Lemma 2.1.6. □

Note that $\Theta \circ \Gamma(\mathcal{A})$ corresponds to the smallest surjective operator ideal which contains $\mathcal{A}$. Stephani used the notation $\mathcal{A}^S$ in [S] for this concept; however, the notation $\mathcal{A}^{\mathrm{sur}}$ from [Pi1] seems to be prevalent in the literature.

Recall that $\mathcal{V} \notin \mathrm{surOI}$, since $\mathcal{V}^{\mathrm{sur}} = \mathcal{L}$ (see [Pi1, 4.7.13]).

According to Proposition 2.1.5, the operator $\Gamma \circ \Theta \colon \mathrm{GSet}^{\partial} \to \mathrm{GSet}^{\partial}$ is a closure operator. Let us denote by $\overline{\mathrm{GSet}}^{\mathrm{OI}}$ the corresponding class of closed systems of sets. In order to describe elements of this class, we need the following definitions.

Definition 2.3.4 (see [S, Section 2])**.** Let $G \subset X$ be a bounded set. The *σ-absolutely convex hull* of G is defined by

$$\sigma\text{-conv}(G) = \left\{ x \in X \,\middle|\, x = \sum_{k=1}^{\infty} a_k x_k, \text{ where } (x_k) \subset G \text{ and } (a_k) \in B_{\ell_1} \right\}.$$

Definition 2.3.5 (see [S, Section 2]). Let $\mathbf{G} \in \mathrm{GSet}$. System $\mathbf{G}$ is said to be an *ideal system of sets* if it fulfills an additional property (G_5) saying that

(G_5) $G \in \mathbf{G}(X)$ implies $\sigma\text{-conv}(G) \in \mathbf{G}(X)$.

We denote the class of all ideal systems of sets by idGSet.

Definition 2.3.6 (see [S, Section 2]). Let G be a bounded set in X. Consider the operator

$$R_G\big((\lambda_x)_{x\in G}\big) = \sum_{x\in G} \lambda_x x$$

of $\ell_1[G]$ into X.

In [S], notation Q_G is used instead of R_G. We have avoided using the notation Q_G to avoid confusion with the notation Q_X from [Pi1], which was defined earlier.

The proof of the next result is essentially due to Stephani.

Proposition 2.3.7 (cf. [S, Section 2]). *The following classes coincide.*

(i) *The class* idGSet*;*

(ii) *the class* $\overline{\mathrm{GSet}}^{\mathrm{OI}}$*;*

(iii) *the class* $\Gamma(\mathrm{OI})$.

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Lemma 2.1.6.

To show (i) $\Rightarrow$ (ii), let $\mathbf{G} \in \mathrm{GSet}$ satisfy (G_5), let X be a Banach space, and let $G \in \mathbf{G}(X)$. Observe that $\sigma\text{-conv}(G) = R_G(B_{\ell_1[G]})$. Therefore $R_G \in \Theta(\mathbf{G})$ and thus $G \in \Gamma \circ \Theta(\mathbf{G})(X)$. We have shown that $\mathbf{G} = \Gamma \circ \Theta(\mathbf{G})$.

To show (iii) $\Rightarrow$ (i), let a system of sets $\Gamma(\mathcal{A})$ be given, let X be a Banach space and let $G \in \Gamma(\mathcal{A})(X)$. We verify that $\sigma\text{-conv}(G) \in \Gamma(\mathcal{A})(X)$. There exists a Banach space Y and an operator $T \in \mathcal{A}(Y, X)$ such that $G \subset T(B_Y)$. Observe that

$$\sigma\text{-conv}(G) \subset \sigma\text{-conv}(T(B_Y)) = T(B_Y).$$

Since $T(B_Y) \in \Gamma(\mathcal{A})(X)$, we conclude that $\sigma\text{-conv}(G) \in \Gamma(\mathcal{A})(X)$. □

As a consequence of Propositions 2.1.5, 2.3.3, and 2.3.7, we have the following:

Corollary 2.3.8. *The following classes are order-isomorphic to each other:*

(i) *class* surOI *of surjective operator ideals;*

(ii) *class* idGSet *of ideal systems of sets.*

2.4 Generating systems of sequences

In this section we recall the basic definition and provide several examples of generating systems of sequences. We consider certain mappings between the classes of generating systems of sets and sequences and prove preliminary results about these classes and mappings.

By a *system of sequences* $\mathbf{g}$ we mean a rule which for every Banach space X fixes a family $\mathbf{g}(X)$ of sequences in X. The latter family is called a *component* of $\mathbf{g}$ (in X). By SSeq, we denote the class of all systems of sequences.

By $\mathbf{m}$, we denote the system of all bounded sequences in all Banach spaces (in [S], notation $\boldsymbol{B}$ is used).

Definition 2.4.1 (see [S, Definition 1.2]). A system of sequences $\mathbf{g}$ is said to be a *generating system of sequences* if for every Banach space X the following conditions are satisfied:

(S_0) $\mathbf{g}(X) \subset \mathbf{m}(X)$;

(S_1) every sequence $\overline{x} = (x_k) \subset B_{\mathbb{K}}$ contains a subsequence $\overline{y}$ of $\overline{x}$ such that $\overline{y} \in \mathbf{g}(\mathbb{K})$;

(S_2) $\mathbf{g}(X)$ is a linear subspace of $\mathbf{m}(X)$;

(S_3) $\mathbf{g}(X)$ is closed under passing to subsequences;

(S_4) if $\overline{x} = (x_k) \in \mathbf{g}(X)$ and $T \in \mathcal{L}(X, Y)$, then $(Tx_k) \in \mathbf{g}(Y)$.

We denote the class of all generating systems of sequences by GSeq.

It is easy to see that the following are examples of generating systems of sequences:

(i) the system $\mathbf{m}$;

(ii) the system $\mathbf{c}$ of all convergent sequences in all Banach spaces (in [S], notation Φ_c is used);

(iii) the system $\mathbf{f}$ whose every component $\mathbf{f}(X)$ consists of all bounded sequences which span a finite-dimensional subspace of X;

(iv) the system $\mathbf{fc}$ whose every component $\mathbf{fc}(X)$ consists of all convergent sequences which span a finite-dimensional subspace of X.

For more examples of generating systems of sequences, consult [S, Section 1].

Let $1 \le p < \infty$. Recall that a sequence (x_k) in X is said to be *absolutely p-summable* if the scalar sequence $(\|x_k\|)$ belongs to ℓ_p. Consider the following well-known systems of sequences.

(i) the system $\mathbf{c}_0$ of all null (i.e., zero-convergent) sequences in all Banach spaces;

(ii) the system $\boldsymbol{\ell}_p$ of all absolutely p-summable sequences in all Banach spaces.

Note that both of the aforementioned systems of sequences fail the property (S_1), since their components in the space $\mathbb{K}$ do not contain the constant scalar sequences.

In the following we view the classical spaces ℓ_p, c_0, and m respectively as components $\boldsymbol{\ell}_p(\mathbb{K})$, $\mathbf{c}_0(\mathbb{K})$, and $\mathbf{m}(\mathbb{K})$, where it is convenient. For notational purposes, we sometimes use the symbol ℓ_∞ instead of m, and more generally, $\boldsymbol{\ell}_\infty(X)$ instead of $\mathbf{m}(X)$.

Definition 2.4.2 (see [S, Definition 1.3])**.** Let $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$. The system $\mathbf{h}$ is said to *dominate* the system $\mathbf{g}$, written $\mathbf{g} \lesssim \mathbf{h}$, if every sequence from $\mathbf{g}(X)$ has a subsequence in $\mathbf{h}(X)$. (In [S], the symbol $\prec$ is used instead of $\lesssim$.)

It is easy to verify that the relation $\lesssim$ is a preorder on the class GSeq. However, it is not an order. Consider, for example, the systems $\mathbf{fc}$ and $\mathbf{f}$. It is straightforward to verify that $\mathbf{fc} \lesssim \mathbf{f}$ and $\mathbf{f} \lesssim \mathbf{fc}$, yet $\mathbf{f} \neq \mathbf{fc}$.

We now follow a standard process to make this preorder into an order. Given $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$, we write $\mathbf{g} \sim \mathbf{h}$ if $\mathbf{g} \lesssim \mathbf{h}$ and $\mathbf{h} \lesssim \mathbf{g}$. It is easy to see that this is an equivalence relation on GSeq, and that the preorder on GSeq induces an order on $\mathrm{GSeq}/_\sim$ via $[\mathbf{g}] \le [\mathbf{h}]$ whenever $\mathbf{g} \lesssim \mathbf{h}$.

Stephani showed in [S, Section 1] that given a system $\mathbf{g} \in \mathrm{GSeq}$, one may define a generating system of sets $\Psi(\mathbf{g})$ in the following way: every component $\Psi(\mathbf{g})(X)$ contains all subsets G of X such that each sequence $(x_j) \subset G$ contains a subsequence $(x_{j_k}) \in \mathbf{g}(X)$. This construction gives us an operator $\Psi\colon \mathrm{GSeq} \to \mathrm{GSet}$ (Stephani used the notation $\mathfrak{M}_{\mathbf{g}}$ instead of $\Psi(\mathbf{g})$).

Conversely, we define an operator $\Phi\colon \mathrm{GSet} \to \mathrm{GSeq}$ as follows.

Definition 2.4.3. Let $\mathbf{G} \in \mathrm{GSet}$. Define $\Phi(\mathbf{G})$ to be the system of all sequences that are contained in the sets of $\mathbf{G}$. That is, a sequence is in the component $\Phi(\mathbf{G})(X)$ if and only if it is contained in G for some $G \in \mathbf{G}(X)$.

It is easy to verify that $\Phi(\mathbf{G})$ indeed is a generating system of sequences.

Definition 2.4.4. We say that a system $\mathbf{G} \in \mathrm{GSet}$ is *sequentially generatable* if there exists a system $\mathbf{g} \in \mathrm{GSeq}$ such that $\Psi(\mathbf{g}) = \mathbf{G}$. Denote the class of all sequentially generatable systems of sets by seqGSet.

As Corollary 4.1.10 below shows, the class seqGSet is strictly smaller than GSet. Since $\Psi(\mathbf{f}) = \mathbf{F}, \Psi(\mathbf{c}) = \mathbf{K}$, and $\Psi(\mathbf{m}) = \mathbf{B}$, we have $\mathbf{F}, \mathbf{K}, \mathbf{B} \in \mathrm{seqGSet}$. Note that, in general, the operator Ψ is not one-to-one. Indeed, $\Psi(\mathbf{f}) = \Psi(\mathbf{fc}) = \mathbf{F}$, yet, as mentioned before, $\mathbf{f} \neq \mathbf{fc}$.

Corollary 2.4.7 below demonstrates the relation between the operator Ψ and the equivalence relation "$\sim$". To prove it, we use the following lemma from [S], for which we have included a proof for completeness.

Lemma 2.4.5 (see [S, Lemma 1.1])**.** *Let* $\mathbf{g} \in \mathrm{GSeq}$ *and let* $(x_k) \in \mathbf{g}(X)$. *Then* $\{x_k\} \in \Psi(\mathbf{g})(X)$.

Proof. In order to show that $\{x_k\} \in \Psi(\mathbf{g})(X)$, let a sequence (x_{n_k}) be given. We need to prove that it contains a subsequence belonging to $\mathbf{g}(X)$.

Assume that the sequence (n_k) contains an increasing subsequence (m_k). Then (x_{m_k}) is the needed subsequence of (x_{n_k}), since $(x_{m_k}) \in \mathbf{g}(X)$ by property (S_3).

On the other hand, if the sequence (n_k) does not contain any increasing subsequences, then it must contain a constant subsequence (r_k). Then (x_{r_k}) is the needed subsequence of (x_{n_k}), since $(x_{r_k}) \in \mathbf{g}(X)$ by properties (S_1) and (S_4). □

Lemma 2.4.6. *Let* $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$. *Then* $\mathbf{g} \lesssim \mathbf{h}$ *if and only if* $\Psi(\mathbf{g}) \leq \Psi(\mathbf{h})$.

Proof. Let $\mathbf{g} \lesssim \mathbf{h}$. Take a set $G \in \Psi(\mathbf{g})(X)$. By definition, for every sequence $\overline{x} = (x_k) \subset G$ there exists a subsequence $\overline{y}$ of $\overline{x}$ such that $\overline{y} \in \mathbf{g}(X)$. According to the assumption, there exists a subsequence $\overline{z}$ of $\overline{y}$ such that $\overline{z} \in \mathbf{h}(X)$. This proves that $G \in \Psi(\mathbf{h})(X)$.

Conversely, let $\Psi(\mathbf{g}) \leq \Psi(\mathbf{h})$ and take a sequence $\overline{x} \in \mathbf{g}(X)$. By Lemma 2.4.5,

$$\{x_k\} \in \Psi(\mathbf{g})(X) \subset \Psi(\mathbf{h})(X).$$

Therefore the sequence $\overline{x}$ contains a subsequence $\overline{y} \in \mathbf{h}(X)$. □

Corollary 2.4.7. *Let* $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$. *Then* $\mathbf{g} \sim \mathbf{h}$ *if and only if* $\Psi(\mathbf{g}) = \Psi(\mathbf{h})$.

Stephani also considered the following way to construct operator ideals from two given systems of sequences.

Definition 2.4.8 (see [S, Section 1])**.** Let $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$ be such that $\mathbf{h} \lesssim \mathbf{g}$. Denote by $[\mathbf{g} \to \mathbf{h}]$ the class of operators $T \in \mathcal{L}(X, Y)$, which map each sequence $(x_k) \in \mathbf{g}(X)$ onto a sequence $(y_k) := (Tx_k)$ having a subsequence $(y_{j_k}) \in \mathbf{h}(Y)$. (Stephani used the notation $\mathbf{A}[\mathbf{h}/\mathbf{g}]$ instead of $[\mathbf{g} \to \mathbf{h}]$.)

Proposition 2.4.9 (see [S, Theorem 1.1])**.** *Let* $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$ *be such that* $\mathbf{h} \lesssim \mathbf{g}$. *Then*

$$[\mathbf{g} \to \mathbf{h}] = [\Psi(\mathbf{g}) \to \Psi(\mathbf{h})] .$$

Proposition 2.4.10. *Let* $\mathbf{G}, \mathbf{H} \in \mathrm{GSet}$ *be such that* $\mathbf{H} \leq \mathbf{G}$. *Then*

$$[\mathbf{G} \to \mathbf{H}] \subset [\Phi(\mathbf{G}) \to \Phi(\mathbf{H})] .$$

Proof. Let $T \in [\mathbf{G} \to \mathbf{H}](X, Y)$ and let $(x_k) \in \Phi(\mathbf{G})(X)$. This means that $(x_k) \subset G \in \mathbf{G}(X)$ and we may conclude that $(Tx_k) \subset T(G) \in \mathbf{H}(Y)$. □

Remark 2.4.11. In contrast with Proposition 2.4.9, it does not always hold that

$$[\mathbf{G} \to \mathbf{H}] = [\Phi(\mathbf{G}) \to \Phi(\mathbf{H})] .$$

For a counterexample, see Proposition 4.1.13 below.

2.5 Saturated systems of sequences

We say that a system $\mathbf{g} \in \mathrm{GSeq}$ is *saturated* if for every system $\mathbf{h} \in \mathrm{GSeq}$ that satisfies $\mathbf{h} \sim \mathbf{g}$ it holds that $\mathbf{h}(X) \subset \mathbf{g}(X)$ for every Banach space X. Denote the class of all saturated systems of sequences by satGSeq. As Corollary 2.5.2 below shows, the class satGSeq coincides with $\Phi \circ \Psi(\mathrm{GSeq})$. For the sake of an example, we show that the system $\mathbf{c}$ is not saturated (see Proposition 2.5.6 below). We conclude this section by proving that the relation $\lesssim$ is an order when restricted to satGSeq (see Proposition 2.5.8 below).

Proposition 2.5.1. *Let* $\mathbf{g} \in \mathrm{GSeq}$. *Then*

(i) $\mathbf{g} \sim \Phi(\Psi(\mathbf{g}))$;

(ii) *the system* $\Phi(\Psi(\mathbf{g}))$ *is saturated*;

(iii) *if* $\mathbf{g}$ *is saturated, then* $\mathbf{g} = \Phi(\Psi(\mathbf{g}))$.

Proof. Let $\mathbf{g} \in \mathrm{GSeq}$ and let X be a Banach space. We prove that $\mathbf{g}(X) \subset \Phi(\Psi(\mathbf{g}))(X)$. Take a sequence $\overline{x} \in \mathbf{g}(X)$. By Lemma 2.4.5, $\{x_k\} \in \Psi(\mathbf{g})(X)$ and therefore $\overline{x} \in \Phi(\Psi(\mathbf{g}))(X)$.

(i). Let $\overline{x} \in \mathbf{g}(X)$. Since $\overline{x}$ is a subsequence of itself and $\overline{x} \in \Phi(\Psi(\mathbf{g}))(X)$, we have $\mathbf{g} \lesssim \Phi(\Psi(\mathbf{g}))$. Let $\overline{x} \in \Phi(\Psi(\mathbf{g}))(X)$. By definition, $\{x_k\} \in \Psi(\mathbf{g})(X)$. Consequently, there exists a subsequence $\overline{y}$ of $\overline{x}$ such that $\overline{y} \in \mathbf{g}(X)$ and therefore $\Phi(\Psi(\mathbf{g})) \lesssim \mathbf{g}$.

(ii). Let $\mathbf{h} \in \mathrm{GSeq}$ be such that $\mathbf{h} \sim \mathbf{g}$. Since $\Psi(\mathbf{h}) = \Psi(\mathbf{g})$ by Corollary 2.4.7, we have $\Phi(\Psi(\mathbf{h})) = \Phi(\Psi(\mathbf{g}))$. Therefore $\mathbf{h}(X) \subset \Phi(\Psi(\mathbf{h}))(X) = \Phi(\Psi(\mathbf{g}))(X)$ for every Banach space X.

(iii). Let X be a Banach space. On the one hand, $\Phi(\Psi(\mathbf{g}))(X) \subset \mathbf{g}(X)$ because $\mathbf{g}$ is saturated. Conversely, $\mathbf{g}(X) \subset \Phi(\Psi(\mathbf{g}))(X)$ by (ii). $\square$

Corollary 2.5.2. *It holds that* $\mathrm{satGSeq} = \Phi \circ \Psi(\mathrm{GSeq})$.

To see that satGSeq is strictly smaller than GSeq, observe that $\Phi(\Psi(\mathbf{fc})) = \mathbf{f}$ and therefore $\mathbf{fc} \notin \mathrm{satGSeq}$, but $\mathbf{f} \in \mathrm{satGSeq}$ according to Proposition 2.5.1(ii). Another example is provided by Proposition 2.5.6 below. The next two propositions further characterize saturated systems of sequences.

Proposition 2.5.3. *Let* $\mathbf{g} \in \mathrm{GSeq}$*, let* X *be a Banach space, and let* $\overline{x} \in \mathbf{m}(X)$*. Then* $\overline{x} \in \Phi(\Psi(\mathbf{g}))(X)$ *if and only if every subsequence* $\overline{y}$ *of* $\overline{x}$ *contains a subsequence* $\overline{z}$ *of* $\overline{y}$ *such that* $\overline{z} \in \mathbf{g}(X)$*.*

Proof. By definition, $(x_k) \in \Phi(\Psi(\mathbf{g}))(X)$ if and only if for every sequence $(y_k) \subset \{x_k\}$ there exists a subsequence (z_k) of (y_k) such that $(z_k) \in \mathbf{g}(X)$.

For necessity, let (y_k) be a subsequence of (x_k). Since $(y_k) \subset \{x_k\}$, there exists a subsequence (z_k) of (y_k) such that $(z_k) \in \mathbf{g}(X)$.

For sufficiency, let $(y_k) \subset \{x_k\}$. If the set $\{y_k\}$ contains only finitely many different elements, then there exists a constant subsequence (z_k) of (y_k), in which case $(z_k) \in \mathbf{g}(X)$ because of properties (S_1) and (S_4). On the other hand, let us assume that the set $\{y_k\}$ contains infinitely many different elements from the set $\{x_k\}$. Let $(j_k) \subset \mathbb{N}$ be a sequence of indices such that $y_k = x_{j_k}$ for all $k \in \mathbb{N}$. It is easy to see that there exists an increasing subsequence (h_k) of (j_k). Define $z_k = x_{h_k}$ for all $k \in \mathbb{N}$ and observe that the sequence (z_k) is a subsequence of both sequences (y_k) and (x_k). By assumption, there exists a subsequence (w_k) of (z_k) such that $(w_k) \in \mathbf{g}(X)$. $\square$

Proposition 2.5.4. *Let* $\mathbf{g}, \mathbf{h} \in \mathrm{GSeq}$*. Then* $\mathbf{g} \lesssim \mathbf{h}$ *if and only if*

$$\Phi(\Psi(\mathbf{g}))(X) \subset \Phi(\Psi(\mathbf{h}))(X)$$

for every Banach space X.

Proof. Let $\mathbf{g} \lesssim \mathbf{h}$. Take $x \in \Phi(\Psi(\mathbf{g}))(X)$ and let $\overline{y}$ be a subsequence of $\overline{x}$. Since $\Phi(\Psi(\mathbf{g})) \sim \mathbf{g} \lesssim \mathbf{h}$, there exists a subsequence $\overline{z}$ of $\overline{y}$ such that $\overline{z} \in \mathbf{h}(X)$. Thus $\overline{x} \in \Phi(\Psi(\mathbf{h}))(X)$ by the "if" part of Proposition 2.5.3.

Conversely, let $\Phi(\Psi(\mathbf{g}))(X) \subset \Phi(\Psi(\mathbf{h}))(X)$ for a Banach space X. Take $\overline{x} \in \mathbf{g}(X) \subset \Phi(\Psi(\mathbf{g}))(X) \subset \Phi(\Psi(\mathbf{h}))(X)$. By the "only if" part of Proposition 2.5.3, there exists a subsequence $\overline{y}$ of $\overline{x}$ such that $\overline{y} \in \mathbf{h}(X)$. □

Corollary 2.5.5. *Let $\mathbf{g}, \mathbf{h} \in$ GSeq. Then $\mathbf{g} \sim \mathbf{h}$ if and only if $\Phi(\Psi(\mathbf{g})) = \Phi(\Psi(\mathbf{h}))$.*

If follows from the next result that the system of sequences $\mathbf{c}$ is not saturated (because that would be a contradiction with Proposition 2.5.1(ii)).

Proposition 2.5.6. *It holds that $\mathbf{c}(\mathbb{K}) \neq \Phi(\Psi(\mathbf{c}))(\mathbb{K})$.*

Proof. Consider an alternating sequence $\overline{\alpha} = (1, 0, 1, 0, \ldots) \subset \mathbb{K}$. This sequence does not converge, and therefore $\overline{\alpha} \notin \mathbf{c}(\mathbb{K})$, but $\overline{\alpha} \in \Phi(\Psi(\mathbf{c}))(\mathbb{K})$ by Proposition 2.5.3.

Alternatively, observe that the set $\{0, 1\}$ is relatively compact and that $\Phi(\Psi(\mathbf{c})) = \Phi(\mathbf{K})$, i.e., the latter system consists of all sequences that are contained in a relatively compact set. □

Remark 2.5.7. Saturated systems of sequences can be alternatively characterized as being of the form $\Phi(\mathbf{G})$, where $\mathbf{G} \in$ seqGSet. As Corollary 4.1.12 below shows, a system of the form $\Phi(\mathbf{G})$ may fail to be saturated if $\mathbf{G} \notin$ seqGSet.

The following proposition shows that the class satGSeq is ordered. This result is of critical importance because it enables us to prove below that satGSeq and GSeq$/_\sim$ are order-isomorphic to each other (see Corollary 2.6.10).

Proposition 2.5.8. *The restriction of the relation $\lesssim$ to* satGSeq *is an order.*

Proof. We only need to show that the relation $\lesssim$ is antisymmetric on satGSeq. Let $\mathbf{g}, \mathbf{h} \in$ satGSeq be such that $\mathbf{g} \sim \mathbf{h}$. By Proposition 2.5.1(iii) and Corollary 2.5.5, $\mathbf{g} = \Phi(\Psi(\mathbf{g})) = \Phi(\Psi(\mathbf{h})) = \mathbf{h}$. □

Remark 2.5.9. A relation $\lll$ was introduced in [S, Definition 1.4]. We do not use this relation in the given thesis, but for the sake of completeness, let us observe that it can be characterized in the following way. Let $\mathbf{g}, \mathbf{h} \in$ GSeq. Then $\mathbf{g} \lll \mathbf{h}$ if and only if $\mathbf{g} \lesssim \mathbf{h}$ and

$$\Phi(\Psi(\mathbf{g}))(X) \cap \mathbf{h}(X) \subset \mathbf{g}(X)$$

for every Banach space X.

2.6 Galois connection between generating systems of sets and sequences

We begin this section by proving one of the main results of this chapter: there is a Galois connection between the ordered classes GSet and $\mathrm{GSeq}/_\sim$ (see Theorem 2.6.4). Corollary 2.6.5 shows that a generating system of sets $\mathbf{G}$ is sequentially generatable if and only if $\mathbf{G} = \Psi\left(\Phi\left(\mathbf{G}\right)\right)$. We prove that the operator $\Psi \circ \Phi$ is a closure operator on GSet (see Theorem 2.6.6) and conclude this chapter with the result that the classes $\mathrm{GSeq}/_\sim$, satGSeq, and seqGSet are order-isomorphic to each other (see Theorems 2.6.6 and 2.6.9).

Definition 2.6.1. Define the operator ϕ: GSet $\to \mathrm{GSeq}/_\sim$ by

$$\phi\left(\mathbf{G}\right) = \left[\Phi\left(\mathbf{G}\right)\right], \text{ where } \mathbf{G} \in \mathrm{GSet}.$$

Define the operator ψ: $\mathrm{GSeq}/_\sim \to$ GSet by

$$\psi\left([\mathbf{g}]\right) = \Psi\left(\mathbf{g}\right), \text{ where } [\mathbf{g}] \in \mathrm{GSeq}/_\sim.$$

The correctness of the above definition is clear from Corollary 2.4.7.

The following two corollaries are due to Lemma 2.4.6 and Proposition 2.5.1, respectively.

Corollary 2.6.2. *Let* $[\mathbf{g}], [\mathbf{h}] \in \mathrm{GSeq}/_\sim$. *Then* $[\mathbf{g}] \leq [\mathbf{h}]$ *if and only if* $\psi\left([\mathbf{g}]\right) \leq \psi\left([\mathbf{h}]\right)$.

Corollary 2.6.3. *Let* $[\mathbf{g}] \in \mathrm{GSeq}/_\sim$. *Then* $\phi\left(\psi\left([\mathbf{g}]\right)\right) = [\mathbf{g}]$.

Theorem 2.6.4. *The pair* (ϕ, ψ) *is a Galois connection between the ordered classes* GSet *and* $\mathrm{GSeq}/_\sim$.

Proof. Let $\mathbf{G} \in$ GSet and $[\mathbf{g}] \in \mathrm{GSeq}/_\sim$. We need to prove that $\phi\left(\mathbf{G}\right) \leq [\mathbf{g}]$ if and only if $\mathbf{G} \leq \psi\left([\mathbf{g}]\right)$.

By definition, $\phi\left(\mathbf{G}\right) \leq [\mathbf{g}]$ if for every Banach space X and for every sequence $\overline{x} \in \mathbf{m}(X)$ which satisfies the condition $\{x_k\} \in \mathbf{G}(X)$ there exists a subsequence $\overline{y}$ of $\overline{x}$ such that $\overline{y} \in \mathbf{g}(X)$.

On the other hand, $\mathbf{G} \leq \psi\left([\mathbf{g}]\right)$ if for every Banach space X, set $K \in \mathbf{G}(X)$, and sequence $\overline{x} = (x_k) \subset K$ there exists a subsequence $\overline{y}$ of $\overline{x}$ such that $\overline{y} \in \mathbf{g}(X)$.

It is straightforward to verify that the two statements are equivalent. $\square$

The existence of a Galois connection provides us with the following two results. The first one is immediate from Lemmas 2.1.2 and 2.1.6.

Corollary 2.6.5. *Let* $\mathbf{G}, \mathbf{H} \in \mathrm{GSet}$ *and let* $[\mathbf{g}] \in \mathrm{GSeq}/_{\sim}$. *Then*

(i) $\mathbf{G} \leq \psi(\phi(\mathbf{G})) = \Psi(\Phi(\mathbf{G}))$;

(ii) $\mathbf{G} \leq \mathbf{H} \Rightarrow \phi(\mathbf{G}) \leq \phi(\mathbf{H}) \Leftrightarrow \Phi(\mathbf{G}) \lesssim \Phi(\mathbf{H})$;

(iii) $\phi(\mathbf{G}) = \phi(\psi(\phi(\mathbf{G})))$ *and* $\psi([\mathbf{g}]) = \psi(\phi(\psi([\mathbf{g}])))$.

Let $\mathbf{G} \in \mathrm{GSet}$. *Then* $\mathbf{G} \in \mathrm{seqGSet}$ *if and only if* $\mathbf{G} = \psi(\phi(\mathbf{G}))$ *(equivalently,* $\mathbf{G} = \Psi(\Phi(\mathbf{G}))$*). To put it another way,* $\mathrm{seqGSet} = \Psi \circ \Phi(\mathrm{GSet})$.

Notice that condition (iii) above also follows from Corollary 2.6.3.

We define the operators $\psi\colon \mathrm{GSeq}/_{\sim} \to \mathrm{seqGSet}$ and $\phi\colon \mathrm{seqGSet} \to \mathrm{GSeq}/_{\sim}$ in a natural way (by restriction). The next theorem follows immediately from Proposition 2.1.5.

Theorem 2.6.6. *The following holds:*

(i) *The operator* $\psi \circ \phi$ *(which equals* $\Psi \circ \Phi$*) is a closure operator on* GSet *and the class* $\Psi \circ \Phi(\mathrm{GSet})$ *is the corresponding subclass of closed elements*;

(ii) *The operator* $\psi\colon \mathrm{GSeq}/_{\sim} \to \mathrm{seqGSet}$ *is an order-isomorphism and the operator* $\phi\colon \mathrm{seqGSet} \to \mathrm{GSeq}/_{\sim}$ *is its inverse.*

At first sight, Proposition 2.1.5 seems to yield more than we formulated in the previous theorem: it additionally asserts that the operator $\phi \circ \psi$ is a closure operator on $(\mathrm{GSeq}/_{\sim})^{\partial}$. However, this does not provide any new information since we already know from Corollary 2.6.3 that operator $\phi \circ \psi$ is the identity operator of $\mathrm{GSeq}/_{\sim}$.

Corollary 2.6.5(iii) yields the following result.

Corollary 2.6.7. *Let* $\mathbf{g} \in \mathrm{GSeq}$. *Then* $\Psi(\mathbf{g}) = \Psi(\Phi(\Psi(\mathbf{g})))$.

Proof. Observe that $\Psi(\mathbf{g}) = \psi([\mathbf{g}]) = \psi(\phi(\psi([\mathbf{g}]))) = \Psi(\Phi(\Psi(\mathbf{g})))$. □

Remark 2.6.8. Let $\mathbf{G} \in \mathrm{GSet}$. In contrast with the above corollary, it does not always hold that $\Phi(\mathbf{G}) = \Phi(\Psi(\Phi(\mathbf{G})))$, although Proposition 2.5.1 guarantees the inclusion $\Phi(\mathbf{G})(X) \subset \Phi(\Psi(\Phi(\mathbf{G})))(X)$ for every Banach space X. For a counterexample, see Proposition 4.1.11 below.

We define the operators $\Phi\colon \mathrm{seqGSet} \to \mathrm{satGSeq}$ and $\Psi\colon \mathrm{satGSeq} \to \mathrm{seqGSet}$ in a natural way (by restriction). Observe that Proposition 2.5.1(ii) shows the correctness of the former definition.

Theorem 2.6.9. *The operator* $\Phi\colon \mathrm{seqGSet} \to \mathrm{satGSeq}$ *is an order-isomorphism and* $\Psi\colon \mathrm{satGSeq} \to \mathrm{seqGSet}$ *is its inverse.*

Proof. In order to show that the operator $\Phi\colon \mathrm{seqGSet} \to \mathrm{satGSeq}$ is an order-isomorphism it suffices to show that it is a surjective order-embedding. Surjectivity follows from Proposition 2.5.1(iii). Let $\Psi(\mathbf{g}), \Psi(\mathbf{h}) \in \mathrm{seqGSet}$. Let us show that $\Psi(\mathbf{g}) \leq \Psi(\mathbf{h})$ if and only if $\Phi(\Psi(\mathbf{g})) \lesssim \Phi(\Psi(\mathbf{h}))$. According to Lemma 2.4.6, $\Psi(\mathbf{g}) \leq \Psi(\mathbf{h})$ if and only if $\mathbf{g} \lesssim \mathbf{h}$. Since $\mathbf{g} \sim \Phi(\Psi(\mathbf{g}))$, $\mathbf{h} \sim \Phi(\Psi(\mathbf{h}))$, and the relation $\lesssim$ is a preorder, we have $\mathbf{g} \lesssim \mathbf{h}$ if and only if $\Phi(\Psi(\mathbf{g})) \lesssim \Phi(\Psi(\mathbf{h}))$.

Corollary 2.6.7 shows that $\Psi \circ \Phi$ is the identity on seqGSet. To see that $\Phi \circ \Psi$ is the identity on the class satGSeq, take $\Phi(\Psi(\mathbf{g})) \in \mathrm{satGSeq}$ and apply Corollary 2.6.7 to observe that $\Phi(\Psi(\Phi(\Psi(\mathbf{g})))) = \Phi(\Psi(\mathbf{g}))$. $\square$

Combining Theorems 2.6.6 and 2.6.9, we obtain the following corollary.

Corollary 2.6.10. *The operator*

$$\Phi \circ \psi\colon \mathrm{GSeq}/_{\sim} \to \mathrm{satGSeq}$$

is an order-isomorphism.

Chapter 3

Lattice structures of operators, sets, and sequences

In this chapter, we study the lattice structure on the class of operator ideals, the classes of generating systems of sets and sequences, and related classes. We then study the order properties of the various mappings between these classes. This chapter is mainly based on [Lil1].

3.1 Smallest operator ideals, systems of sets and sequences

It is a well-known fact that $\mathcal{F}$ is the smallest operator ideal (see [Pi1, 1.2.2]). Similarly, we prove that the systems $\mathbf{F}$ and $[\mathbf{f}]$ are the least elements of the classes GSet and GSeq$/_\sim$, respectively (see Proposition 3.1.2). To do so, we need the following lemma.

Lemma 3.1.1. *Let* $\mathbf{h}_1, \ldots, \mathbf{h}_n \in$ GSeq. *Let sequences* $\overline{y}_j = (y_k^j)_{k\in\mathbb{N}}$ *be given for each* $j \in \{1, \ldots, n\}$ *such that every subsequence* $\overline{z}$ *of* $\overline{y}_j$ *contains a subsequence* $\overline{w}$ *of* $\overline{z}$ *such that* $\overline{w} \in \mathbf{h}_j(X)$. *Then there exists a subsequence* N *of* $\mathbb{N}$ *such that* $(y_k^j)_{k\in N} \in \mathbf{h}_j(X)$ *for each* $j \in \{1, \ldots, n\}$.

Proof. The sequence $(y_k^1)_{k\in\mathbb{N}}$ contains a subsequence $(y_k^1)_{k\in N_1} \in \mathbf{h}_1(X)$. We continue inductively. On the j-th step (where $j \in \{2, \ldots, n\}$) we use property (S_3) and the assumption to show that the sequence $(y_k^j)_{k\in N_{j-1}}$ contains a subsequence

$(y_k^j)_{k\in N_j} \in \mathbf{h}_j(X)$. It follows from property (S_3) that $(y_k^j)_{k\in N_n} \in \mathbf{h}_j(X)$ for each $j \in \{1,\dots,n\}$. The sequence $N := N_n$ is the needed subsequence of $\mathbb{N}$. □

Proposition 3.1.2. *Let* $\mathbf{G} \in \mathrm{GSet}$ *and* $\mathbf{g} \in \mathrm{GSeq}$. *Then*

(i) $\mathbf{F} \le \mathbf{G}$;

(ii) $\mathbf{f} \lesssim \mathbf{g}$ *(equivalently,* $[\mathbf{f}] \le [\mathbf{g}]$*).*

Proof. Let X be a Banach space. Take $G \in \mathbf{F}(X)$ and take a sequence $(x_k) \subset G$ (i.e., $\overline{x} \in \mathbf{f}(X)$). We prove the first statement by showing that $G \in \mathbf{G}(X)$; then we show that there exists a subsequence $\overline{z}$ of $\overline{x}$ such that $\overline{z} \in \mathbf{g}(X)$, which proves the second statement.

By assumption, G is a bounded subset of an n-dimensional subspace of X, where $n \in \mathbb{N}$. Since G is bounded and the finite-dimensional spaces $\operatorname{span} G$ and m_n are isomorphic, there exist elements $e_1,\dots,e_n$, and a constant $c > 0$ such that

$$G \subset c\left\{\sum_{k=1}^{n} \alpha_k e_k \mid (\alpha_k)_{k=1}^{n} \in B_{m_n}\right\}.$$

Define operators $T_k \in \mathcal{L}(\mathbb{K}, X)$ by

$$T_k(a) = cae_k,$$

where $k \in \{1,\dots,n\}$ and $a \in \mathbb{K}$. Then

$$G \subset \sum_{k=1}^{n} T_k(B_{\mathbb{K}})$$

and $G \in \mathbf{G}(X)$ by properties (G_1) – (G_4).

Since $(x_k) \subset G$, we have

$$x_k = \sum_{j=1}^{n} T_j(\alpha_k^j)$$

with $\alpha_k^j \in B_{\mathbb{K}}$ for each $j \in \{1,\dots,n\}$ and $k \in \mathbb{N}$. Let us define the sequence $\overline{y}_j = (\alpha_k^j)_{k\in\mathbb{N}}$ for each $j \in \{1,\dots,n\}$ and apply Lemma 3.1.1 to the sequences $\overline{y}_1,\dots,\overline{y}_n$ (we put $\mathbf{h}_j := \mathbf{g}$ for each $j \in \{1,\dots,n\}$). Note that the assumptions of the lemma are satisfied because of property (S_1). By Lemma 3.1.1, there exists a subsequence N of $\mathbb{N}$ such that $(\alpha_k^j)_{k\in N} \in \mathbf{g}(\mathbb{K})$ for each $j \in \{1,\dots,n\}$. According to properties (S_2) and (S_4),

$$\overline{z} := \sum_{j=1}^{n} \left(T_j \alpha_k^j\right)_{k\in N} = (x_k)_{k\in N} \in \mathbf{g}(X).$$

We have shown that there exists a subsequence $\overline{z}$ of $\overline{x}$ such that $\overline{z} \in \mathbf{g}(X)$. □

Let us also prove the following (easy) characterization.

Proposition 3.1.3. *Every sequence* $(x_k) \in \mathbf{f}(X)$ *can be expressed as a sum* $(x_k) = (y_k^1) + \ldots + (y_k^n)$*, where* $n \in \mathbb{N}$ *and each of the sequences* $(y_k^1), \ldots, (y_k^n)$ *is bounded in* X *and spans a* 1*-dimensional subspace of* X*.*

Proof. Obviously, every sequence of the form $(x_k) = (y_k^1) + \ldots + (y_k^n)$ belongs to the component $\mathbf{f}(X)$.

For the converse, let $(x_k) \in \mathbf{f}(X)$. By the proof of Proposition 3.1.2,

$$x_k = \sum_{j=1}^{n} T_j(\alpha_k^j)$$

with $T_j \in \mathcal{L}(\mathbb{K}, X)$ and $(\alpha_k^j) \subset B_{\mathbb{K}}$ for each $j \in \{1, \ldots, n\}$. It remains to put

$$(y_k^j)_{k\in\mathbb{N}} = \big(T_j(\alpha_k^j)\big)_{k\in\mathbb{N}}$$

for each $j \in \{1, \ldots, n\}$. □

3.2 Complete lattices related to operator ideals

In this section, we study the lattice structures of the classes $\mathrm{OI}, \mathrm{surOI}, \mathrm{GSet}$, and idGSet. Pietsch mentions in [Pi1, 1.11.1] that "the collection of all operator ideals is something like a complete lattice with respect to the natural ordering". We add to this by showing that the classes $\mathrm{surOI}, \mathrm{GSet}$, and idGSet are also complete lattices (see Theorems 3.2.4, 3.2.9, and 3.2.10).

In this chapter, we repeatedly use the following two results from [DP]. Although these results have been proven for sets, it can be verified that they hold in the context of classes.

In the remainder of this chapter, let I be a non-empty class of indices.

Proposition 3.2.1 (cf. [DP, Theorem 2.31; Lemma 2.30])**.** *Let* G *be an ordered class with a greatest element, where* $\bigwedge_{\alpha\in I} g_\alpha$ *exists for every subclass* $\{g_\alpha \mid \alpha \in I\}$ *of* G*. Then* G *is a complete lattice, where*

$$\bigvee_{\alpha\in I} g_\alpha = \bigwedge\{g \in \mathrm{G} \mid \mathrm{g} \geq \mathrm{g}_\alpha \ \forall\alpha \in I\}.$$

Proposition 3.2.2 (see [DP, Proposition 7.2])**.** *Let* G *be a complete lattice and let a closure operator* $T : \mathrm{G} \to \mathrm{G}$ *be given. Then for every* $x \in \mathrm{G}$ *and every subclass* S *of* $T(\mathrm{G})$ *it holds that*

(i) $T(\mathrm{G})$ *is a complete lattice under the order inherited from* G*;*

(ii) *infima* $\bigwedge S$ *coincide in lattices* $T(G)$ *and* G*;*

(iii) $\bigvee S = T(\bigvee S)$*, where the former supremum is taken in* $T(G)$ *and the latter in* G*;*

(iv) $T(x) = \bigwedge\{y \in T(G) \mid y \geq x\}$.

The following result is well known; we include a proof for completeness.

Theorem 3.2.3 (cf. [Pi1, 1.11.1])**.** *The ordered class* OI *is a complete lattice with the least element* $\mathcal{F}$ *and the greatest element* $\mathcal{L}$*. Let* $\{\mathcal{A}_\alpha \mid \alpha \in I\} \subset \mathrm{OI}$*. Then*

$$\bigwedge_{\alpha\in I} \mathcal{A}_\alpha = \bigcap_{\alpha\in I} \mathcal{A}_\alpha.$$

Proof. It is easy to check that $\bigcap_{\alpha\in I} \mathcal{A}_\alpha \in \mathrm{OI}$ and $\bigwedge_{\alpha\in I} \mathcal{A}_\alpha = \bigcap_{\alpha\in I} \mathcal{A}_\alpha$. Therefore OI is a complete lattice according to Proposition 3.2.1. □

Recall that $\Theta \circ \Gamma$ is a closure operator on $\mathcal{A}$ and the corresponding class of closed operator ideals coincides with the class surOI of all surjective operator ideals (see Proposition 2.3.3). Proposition 3.2.2 yields the following result.

Theorem 3.2.4. *The ordered class* surOI *is a complete lattice with the least element* $\mathcal{F}$ *and the greatest element* $\mathcal{L}$*. Let* $\mathcal{A} \in \mathrm{GSet}$ *and let* $\{\mathcal{A}_\alpha \mid \alpha \in I\} \subset \mathrm{surOI}$*. Then*

$$\mathcal{A}^{\mathrm{sur}} = \Theta \circ \Gamma(\mathcal{A}) = \bigcap \{\mathcal{B} \in \mathrm{surOI} \,|\, \mathcal{B} \supset \mathcal{A}\},$$

$$\bigwedge_{\alpha\in I} \mathcal{A}_\alpha = \bigcap_{\alpha\in I} \mathcal{A}_\alpha.$$

Although Proposition 3.2.1 provides a formula for finding suprema in the classes OI and surOI, it might be rather cumbersome for practical usage. Propositions 3.2.6 and 3.2.7 below give a direct way for finding suprema instead.

Definition 3.2.5. Let $\{\mathcal{A}_\alpha \mid \alpha \in I\} \subset \mathrm{OI}$. We define the collection of operators $\sum_{\alpha\in I} \mathcal{A}_\alpha$ by letting $T \in \sum_{\alpha\in I} \mathcal{A}_\alpha(X, Y)$ whenever T can be expressed as a finite sum of operators from any of the components $\mathcal{A}_\alpha(X, Y)$.

Proposition 3.2.6. *Let* $\{\mathcal{A}_\alpha \mid \alpha \in I\} \subset \mathrm{OI}$*. Then* $\sum_{\alpha\in I} \mathcal{A}_\alpha \in \mathrm{OI}$ *and*

$$\bigvee_{\alpha\in I} \mathcal{A}_\alpha = \sum_{\alpha\in I} \mathcal{A}_\alpha.$$

Proof. It is easy to see that $\sum_{\alpha\in I}\mathcal{A}_\alpha$ satisfies (OI_0) and (OI_1). To show (OI_2), let $T \in \mathcal{L}(X,Y), S \in \sum_{\alpha\in I}\mathcal{A}_\alpha(Y,Z)$, and $R \in \mathcal{L}(Z,W)$. Then $S = S_1 + \ldots + S_n$, where $S_j \in \mathcal{A}_{\beta_j}(X,Y)$, $\beta_j \in I$, and $j \in \{1,\ldots,n\}$. It holds that

$$RST = R(S_1 + \ldots + S_n)T = RS_1T + \ldots + RS_nT \in \sum_{\alpha\in I}\mathcal{A}_\alpha(X,W).$$

By (OI_1), $\bigvee_{\alpha\in I}\mathcal{A}_\alpha \supset \sum_{\alpha\in I}\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha \subset \sum_{\alpha\in I}\mathcal{A}_\alpha$ for every $\alpha \in I$, we see that $\bigvee_{\alpha\in I}\mathcal{A}_\alpha = \sum_{\alpha\in I}\mathcal{A}_\alpha$. □

Proposition 3.2.7. *Let* $\{\mathcal{A}_\alpha \mid \alpha \in I\} \subset \mathrm{surOI}$. *Then*

$$\bigvee_{\alpha\in I}\mathcal{A}_\alpha = \Theta\circ\Gamma\left(\sum_{\alpha\in I}\mathcal{A}_\alpha\right).$$

Proof. According to Proposition 3.2.2(iii), the supremum $\bigvee_{\alpha\in I}\mathcal{A}_\alpha$ in lattice surOI is equal to $\Theta\circ\Gamma\left(\sum_{\alpha\in I}\mathcal{A}_\alpha\right)$. □

Let us examine the structure of the class GSet.

Definition 3.2.8. Let $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{SSet}$. Define $\bigcap_{\alpha\in I}\mathbf{G}_\alpha \in \mathrm{SSet}$ by

$$\left(\bigcap_{\alpha\in I}\mathbf{G}_\alpha\right)(X) := \bigcap_{\alpha\in I}(\mathbf{G}_\alpha(X)).$$

Theorem 3.2.9. *The class* GSet *is a complete lattice with the least element* $\mathbf{F}$ *and the greatest element* $\mathbf{B}$. *Let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{GSet}$. *Then*

$$\bigwedge_{\alpha\in I}\mathbf{G}_\alpha = \bigcap_{\alpha\in I}\mathbf{G}_\alpha.$$

Proof. It is easy to check that $\bigcap_{\alpha\in I}\mathbf{G}_\alpha \in \mathrm{GSet}$ and $\bigwedge_{\alpha\in I}\mathbf{G}_\alpha = \bigcap_{\alpha\in I}\mathbf{G}_\alpha$. Therefore GSet is a complete lattice according to Proposition 3.2.1. □

Theorem 3.2.10. *The class* idGSet *is a complete lattice with the least element* $\mathbf{F}$ *and the greatest element* $\mathbf{B}$. *Let* $\mathbf{G} \in \mathrm{GSet}$ *and let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{idGSet}$. *Then*

$$\Gamma\circ\Theta(\mathbf{G}) = \bigvee\{\mathbf{H} \in \mathrm{idGSet} \,|\, \mathbf{H} \le \mathbf{G}\},$$

$$\bigwedge_{\alpha\in I}\mathbf{G}_\alpha = \bigcap_{\alpha\in I}\mathbf{G}_\alpha.$$

Proof. Recall that $\Gamma \circ \Theta$ is a closure operator on the class GSet^∂ (see Proposition 2.1.5). According to Proposition 3.2.2, $\Gamma \circ \Theta(\mathrm{GSet}^\partial) = (\Gamma \circ \Theta(\mathrm{GSet}))^\partial = \mathrm{idGSet}^\partial$ is a complete lattice. But this means that idGSet is also a complete lattice. Applying Proposition 3.2.2(iv) for the class idGSet^∂ and re-writing it for the class idGSet gives that

$$\Gamma \circ \Theta(\mathbf{G}) = \bigvee \{\mathbf{H} \in \mathrm{idGSet} \,|\, \mathbf{H} \leq \mathbf{G}\}\,.$$

Proposition 3.2.2(iii) proves that the infimum $\bigwedge_{\alpha\in I} \mathbf{G}_\alpha$ in the class idGSet is equal to $\Gamma \circ \Theta(\bigcap_{\alpha\in I} \mathbf{G}_\alpha)$. It remains to prove that $\bigcap_{\alpha\in I} \mathbf{G}_\alpha$ is an ideal system of sets, i.e., belongs to the class $\Gamma \circ \Theta(\mathrm{GSet})$.

We need to prove that $\bigcap_{\alpha\in I} \mathbf{G}_\alpha$ satisfies the property (G_5) from Definition 2.3.5. Let $G \in \bigcap_{\alpha\in I} \mathbf{G}_\alpha(X)$. To show that $\sigma\text{-conv}(G) \in \bigcap_{\alpha\in I} \mathbf{G}_\alpha(X)$, it suffices to observe that each of the systems $\mathbf{G}_\alpha$ is an ideal system of sets by assumption and therefore contains the set $\sigma\text{-conv}(G)$. □

Propositions 3.2.12 and 3.2.13 below give a direct way for finding suprema in the classes GSet and idGSet.

Definition 3.2.11. Let $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{SSet}$. Define $\mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha \in \mathrm{SSet}$ by

$$\begin{aligned}\Big(\mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha\Big)(X) = \{H \subset X \;\mid\; &\exists n \in \mathbb{N},\ \exists \beta_1, \ldots, \beta_n \in I,\\ &\exists G_1 \in \mathbf{G}_{\beta_1}(X), \ldots, G_n \in \mathbf{G}_{\beta_n}(X), H \subset G_1 + \ldots + G_n\}.\end{aligned}$$

Notice that if I is finite, then the above formula simplifies to

$$\Big(\mathrm{Sub}\sum_{1\leq j\leq n} \mathbf{G}_j\Big)(X) = \{H \subset X \mid \exists G_1 \in \mathbf{G}_1(X), \ldots, G_n \in \mathbf{G}_n(X), H \subset G_1 + \ldots + G_n\}.$$

Proposition 3.2.12. *Let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{GSet}$. *Then* $\mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha \in \mathrm{GSet}$ *and*

$$\bigvee_{\alpha\in I} \mathbf{G}_\alpha = \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha.$$

Proof. It is easy to see that the system $\mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha$ satisfies (G_1) and (G_3). To show properties (G_2) and (G_4), let $a \in \mathbb{K}$, let X, Y be Banach spaces, let $G, H \in \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha(X)$, and let $T \in \mathcal{L}(X, Y)$. Then $G \subset \sum_{k=1}^m G_k$ and $H \subset \sum_{k=1}^n H_k$, where $G_1 \in \mathbf{G}_{\beta_1}(X), \ldots, G_m \in \mathbf{G}_{\beta_m}(X)$ and $H_1 \in \mathbf{G}_{\gamma_1}(X), \ldots, H_n \in \mathbf{G}_{\gamma_n}(X)$.

(G_2) We have $aG + H \subset aG_1 + \ldots + aG_m + H_1 + \ldots + H_n \in \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha(X)$.

(G_4) It holds that $T(G) \subset T\left(\sum_{k=1}^{m} G_k\right) = \sum_{k=1}^{m} T(G_k) \in \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha(Y)$.

We apply properties (G_2) and (G_3) to see that $\bigvee_{\alpha\in I} \mathbf{G}_\alpha \geq \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha$. Since $\mathbf{G}_\alpha \leq \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha$ for every $\alpha \in I$, we get that $\bigvee_{\alpha\in I} \mathbf{G}_\alpha = \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha$. □

Proposition 3.2.13. *Let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{idGSet}$. *Then*

$$\bigvee_{\alpha\in I} \mathbf{G}_\alpha = \mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha.$$

Proof. Recall that $\Gamma\circ\Theta$ is a closure operator on the class GSet^∂. Applying Proposition 3.2.2(ii) for the class idGSet^∂ and reversing the order proves that the suprema $\bigvee_{\alpha\in I} \mathbf{G}_\alpha$ coincide in the lattices GSet and idGSet. □

We propose the following question. Affirmative answer to it would help to simplify Proposition 3.2.7.

Question 3.2.14. Let $\{\mathcal{A}_\alpha \mid \alpha \in I\} \subset \mathrm{surOI}$. Is it always true that the operator ideal

$$\sum_{\alpha\in I} \mathcal{A}_\alpha$$

is surjective?

Currently, we do not even know if this is so for any finite collection of operator ideals $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$.

3.3 Complete lattices related to systems of sets and sequences

In this section, we show that the classes $\mathrm{seqGSet}, \mathrm{satGSeq}$, and $\mathrm{GSeq}/_\sim$ are complete lattices (see Theorems, 3.3.1, 3.3.5, and 3.3.6). We provide a simpler formula for finding infima over finite sets in the class $\mathrm{GSeq}/_\sim$ (see Proposition 3.3.7). We then prove explicit formulas for finding suprema in the classes seqGSet, satGSeq, and $\mathrm{GSeq}/_\sim$ (see Propositions 3.3.8, 3.3.11, and 3.3.12).

Let us begin by examining the structure of the class seqGSet.

Theorem 3.3.1. *The ordered class* seqGSet *is a complete lattice with the least element* $\mathbf{F}$ *and the greatest element* $\mathbf{B}$. *Let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{seqGSet}$ *and let* $\mathbf{G} \in \mathrm{GSet}$. *Then*

$$\psi\left(\phi\left(\mathbf{G}\right)\right) = \bigcap \left\{\mathbf{H} \in \mathrm{seqGSet} \,|\, \mathbf{H} \geq \mathbf{G}\right\},$$

$$\bigwedge_{\alpha\in I} \mathbf{G}_\alpha = \bigcap_{\alpha\in I} \mathbf{G}_\alpha.$$

Proof. As Theorem 2.6.6 shows, the operator $\psi \circ \phi$: GSet $\to$ GSet is a closure operator. Proposition 3.2.2 yields that $\psi \circ \phi(\mathrm{GSet})$ is a complete lattice and that the above formulas hold (note that $\psi \circ \phi(\mathrm{GSet}) = \mathrm{seqGSet}$ by Corollary 2.6.5). □

Let us investigate the structure of the class satGSeq.

Proposition 3.3.2. *Let* $\mathbf{g} \in$ satGSeq. *Then inclusion* $\mathbf{f}(X) \subset \mathbf{g}(X)$ *holds for every Banach space* X.

Proof. According to Proposition 3.1.2, $\mathbf{f} \lesssim \mathbf{g}$. Since $\mathbf{f}, \mathbf{g} \in$ satGSeq, Proposition 2.5.4 yields that

$$\mathbf{f}(X) = \Phi\left(\Psi\left(\mathbf{f}\right)\right)(X) \subset \Phi\left(\Psi\left(\mathbf{g}\right)\right)(X) = \mathbf{g}(X).$$ □

Definition 3.3.3. Let $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset$ SSeq. Define $\bigcap_{\alpha\in I} \mathbf{g}_\alpha \in$ SSeq by

$$\left(\bigcap_{\alpha\in I} \mathbf{g}_\alpha\right)(X) := \bigcap_{\alpha\in I} (\mathbf{g}_\alpha(X)).$$

Proposition 3.3.4. *Let* $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset$ satGSeq. *Then* $\bigcap_{\alpha\in I} \mathbf{g}_\alpha \in$ satGSeq.

Proof. To prove that $\bigcap_{\alpha\in I} \mathbf{g}_\alpha \in$ GSeq, observe that property (S_1) follows from Proposition 3.3.2, because every sequence $\overline{x} = (x_k) \subset B_{\mathbb{K}}$ satisfies $\overline{x} \in \mathbf{f}(\mathbb{K})$. Other properties are trivial to check.

By Proposition 2.5.1 $\bigcap_{\alpha\in I} \mathbf{g}_\alpha \in$ satGSeq if

$$\bigcap_{\alpha\in I} \mathbf{g}_\alpha = \Phi\left(\Psi\left(\bigcap_{\alpha\in I} \mathbf{g}_\alpha\right)\right).$$

Let X be a Banach space and let $\overline{x} \in \Phi\left(\Psi\left(\bigcap_{\alpha\in I} \mathbf{g}_\alpha\right)\right)(X)$. Applying Proposition 2.5.3 to the system $\bigcap_{\alpha\in I} \mathbf{g}_\alpha$, we find that for every subsequence $\overline{y}$ of $\overline{x}$ there exists a subsequence $\overline{z}$ of $\overline{y}$ such that $\overline{z} \in \bigcap_{\alpha\in I} \mathbf{g}_\alpha(X)$. Once more, we apply Proposition 2.5.3 to the systems $\mathbf{g}_\alpha$ and conclude that $\overline{x} \in \Phi\left(\Psi\left(\mathbf{g}_\alpha\right)\right)(X)$ for every $\alpha \in I$. We have shown that

$$\overline{x} \in \bigcap_{\alpha\in I} \Phi\left(\Psi\left(\mathbf{g}_\alpha\right)\right)(X) = \bigcap_{\alpha\in I} \mathbf{g}_\alpha(X).$$ □

Theorem 3.3.5. *The class* satGSeq *is a complete lattice with the least element* $\mathbf{f}$ *and the greatest element* $\mathbf{m}$. *Let* $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset \mathrm{satGSeq}$. *Then*

$$\bigwedge_{\alpha\in I} \mathbf{g}_\alpha = \bigcap_{\alpha\in I} \mathbf{g}_\alpha.$$

Proof. Since $\bigcap_{\alpha\in I} \mathbf{g}_\alpha \in \mathrm{satGSeq}$, we have $\bigwedge_{\alpha\in I} \mathbf{g}_\alpha = \bigcap_{\alpha\in I} \mathbf{g}_\alpha$. Therefore satGSeq is a complete lattice according to Proposition 3.2.1. Since $\Phi(\Psi(\mathbf{f})) = \mathbf{f}$ and $\Phi(\Psi(\mathbf{m})) = \mathbf{m}$, it holds that $\mathbf{f}$ and $\mathbf{m}$ are the least element and the greatest element of the class satGSeq, respectively. $\square$

Using Corollary 2.6.10 and Theorem 3.3.5, we obtain the following result.

Theorem 3.3.6. *The class* $\mathrm{GSeq}/_\sim$ *is a complete lattice with the least element* $[\mathbf{f}]$ *and the greatest element* $[\mathbf{m}]$. *Let* $\{[\mathbf{g}_\alpha] \mid \alpha \in I\} \subset \mathrm{GSeq}/_\sim$. *Then*

$$\bigwedge_{\alpha\in I} [\mathbf{g}_\alpha] = \left[\bigcap_{\alpha\in I} \Phi(\Psi(\mathbf{g}_\alpha))\right]$$

and

$$\bigvee_{\alpha\in I} [\mathbf{g}_\alpha] = \left[\bigcap \{\mathbf{h} \in \mathrm{satGSeq} \mid \forall \alpha \in I \;\; \mathbf{h} \geq \Phi(\Psi(\mathbf{g}_\alpha))\}\right].$$

The next proposition improves the previous theorem in the case when the family I is finite.

Proposition 3.3.7. *Let* $\{[\mathbf{g}_j] \mid 1 \leq j \leq n\} \subset \mathrm{GSeq}/_\sim$, *where* $n \in \mathbb{N}$. *Then*

$$\bigwedge_{1\leq j\leq n} [\mathbf{g}_j] = \left[\bigcap_{1\leq j\leq n} \mathbf{g}_j\right]$$

Proof. By Theorem 3.3.6, it suffices to prove that $\bigcap_{1\leq j\leq n} \mathbf{g}_j \sim \bigcap_{1\leq j\leq n} \Phi(\Psi(\mathbf{g}_j))$. It is easy to see that the "$\lesssim$" part of the above equivalence holds. We show that

$$\bigcap_{1\leq j\leq n} \mathbf{g}_j \gtrsim \bigcap_{1\leq j\leq n} \Phi(\Psi(\mathbf{g}_j)).$$

Let $\overline{x} \in \bigcap_{1\leq j\leq n} \Phi(\Psi(\mathbf{g}_j))(X)$. Then for each $j \in \{1, \dots, n\}$ and every subsequence $\overline{y}$ of $\overline{x}$ there exists a subsequence $\overline{z}$ of $\overline{y}$ such that $\overline{z} \in \mathbf{g}_j(X)$. We apply Lemma 3.1.1 (we put $\overline{y}_j := \overline{x}$ and $\mathbf{h}_j := \mathbf{g}_j$ for each $j \in \{1, \dots, n\}$) to find a subsequence N of $\mathbb{N}$ such that $(x_k)_{k\in N} \in \mathbf{g}_j(X)$ for each $j \in \{1, \dots, n\}$. $\square$

Let us prove explicit formulas for finding suprema in the classes seqGSet, satGSeq, and GSeq$/_{\sim}$ (see Propositions 3.3.8, 3.3.11, and 3.3.12).

Proposition 3.3.8. *Let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{seqGSet}$. *Then*

$$\bigvee_{\alpha\in I} \mathbf{G}_\alpha = \psi\left(\phi\left(\mathrm{Sub}\sum_{\alpha\in I} \mathbf{G}_\alpha\right)\right).$$

Proof. Recall that $\psi\circ\phi(\mathrm{GSet}) = \mathrm{seqGSet}$ and that the operator $\psi\circ\phi\colon \mathrm{GSet}\to\mathrm{GSet}$ is a closure operator. By Proposition 3.2.2(iii),

$$\bigvee_{\alpha\in I} \mathbf{G}_\alpha = \psi\left(\phi\left(\bigvee_{\alpha\in I} \mathbf{G}_\alpha\right)\right),$$

where the former supremum is taken in seqGSet and the latter in GSet. □

Definition 3.3.9. Let $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset \mathrm{SSeq}$. Define the system of sequences $\sum_{\alpha\in I} \mathbf{g}_\alpha$ by letting $\overline{x} \in \sum_{\alpha\in I} \mathbf{g}_\alpha(X)$ whenever $\overline{x}$ can be expressed as a finite sum of sequences from any of the components $\mathbf{g}_\alpha(X)$.

Proposition 3.3.10. *Let* $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset \mathrm{GSeq}$. *Then* $\sum_{\alpha\in I} \mathbf{g}_\alpha \in \mathrm{GSeq}$.

Proof. It is easy to verify properties (S_0) and (S_1). Properties (S_2) and (S_4) are proved similarly to properties (G_2) and (G_4) in the proof of Proposition 3.2.12.

(S_3) Take a sequence $\overline{x} \in \sum_{\alpha\in I} \mathbf{g}_\alpha(X)$. Then $\overline{x} = \overline{y}_1 + \ldots + \overline{y}_n$, where $\overline{y}_j = (y^j_k)_{k\in\mathbb{N}} \in \mathbf{g}_{\alpha_j}(X)$ for each $j \in \{1,\ldots,n\}$. Take a subsequence (x_{p_k}) of (x_k). Since $(y^j_{p_k})_{k\in\mathbb{N}} \in \mathbf{g}_{\alpha_j}(X)$ for each $j \in \{1,\ldots,n\}$,

$$(x_{p_k}) = (y^1_{p_k}) + \ldots + (y^n_{p_k}) \in \sum_{\alpha\in I} \mathbf{g}_\alpha(X).$$ □

Proposition 3.3.11. *Let* $\{[\mathbf{g}_\alpha] \mid \alpha \in I\} \subset \mathrm{GSeq}/_{\sim}$. *Then*

$$\bigvee_{\alpha\in I} [\mathbf{g}_\alpha] = \left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right].$$

Proof. Let us first prove that the formula $\left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right]$ does not depend on the choice of representatives. Take systems $\mathbf{g}_\alpha, \mathbf{h}_\alpha \in \mathrm{GSeq}$ such that $\mathbf{g}_\alpha \sim \mathbf{h}_\alpha$ for every $\alpha \in I$. We need to show that

$$\sum_{\alpha\in I} \mathbf{g}_\alpha \sim \sum_{\alpha\in I} \mathbf{h}_\alpha.$$

Let $\overline{x} \in \sum_{\alpha\in I} \mathbf{g}_\alpha(X)$. Then $\overline{x} = \overline{y}_1 + \ldots + \overline{y}_n$, where

$$\overline{y}_1 = (y_k^1) \in \mathbf{g}_{\beta_1}(X), \ldots, \overline{y}_n = (y_k^n) \in \mathbf{g}_{\beta_n}(X).$$

Applying Lemma 3.1.1 to the sequences $\overline{y}_1 \in \mathbf{g}_{\beta_1}(X), \ldots, \overline{y}_n \in \mathbf{g}_{\beta_n}(X)$ (put $\mathbf{h}_j := \mathbf{h}_{\beta_j}$), we find a subsequence N of $\mathbb{N}$ such that $(y_k^j)_{k\in N} \in \mathbf{h}_{\beta_j}$ for each $j \in \{1, \ldots, n\}$. Therefore

$$(x_k)_{k\in N} = \sum_{j=1}^{n} (y_k^j)_{k\in N} \in \sum_{\alpha\in I} \mathbf{h}_\alpha(X).$$

This proves that $\sum_{\alpha\in I} \mathbf{g}_\alpha \lesssim \sum_{\alpha\in I} \mathbf{h}_\alpha$. The opposite relation is proven by swapping the systems $\mathbf{g}_\alpha$ and $\mathbf{h}_\alpha$.

It remains to show that

$$\bigvee_{\alpha\in I} [\mathbf{g}_\alpha] = \left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right].$$

Obviously $[\mathbf{g}_\alpha] \leq \left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right]$ for every $\alpha \in I$. Let $[\mathbf{h}] \in \mathrm{GSeq}/_\sim$ be such that $[\mathbf{g}_\alpha] \leq [\mathbf{h}]$ for every $\alpha \in I$. We need to show that

$$\left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right] \leq [\mathbf{h}].$$

Let $\overline{x} = \overline{y}_1 + \ldots + \overline{y}_n$, where

$$\overline{y}_1 = (y_k^1)_{k\in\mathbb{N}} \in \mathbf{g}_{\beta_1}(X), \ldots, \overline{y}_n = (y_k^n)_{k\in\mathbb{N}} \in \mathbf{g}_{\beta_n}(X).$$

Applying Lemma 3.1.1 to the sequences $\overline{y}_1 \in \mathbf{g}_{\beta_1}(X), \ldots, \overline{y}_n \in \mathbf{g}_{\beta_n}(X)$ (put $\mathbf{h}_j := \mathbf{h}$), we find a subsequence N of $\mathbb{N}$ such that $(x_k)_{k\in N} = \sum_{j=1}^{n} (y_k^j)_{k\in N} \in \mathbf{h}(X)$. Therefore $\sum_{\alpha\in I} \mathbf{g}_\alpha \lesssim \mathbf{h}$ and $\bigvee_{\alpha\in I} [\mathbf{g}_\alpha] = \left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right]$. □

Proposition 3.3.12. *Let* $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset \mathrm{satGSeq}$. *Then*

$$\bigvee_{\alpha\in I} \mathbf{g}_\alpha = \Phi\left(\Psi\left(\sum_{\alpha\in I} \mathbf{g}_\alpha\right)\right).$$

Proof. By using isomorphism between the classes satGSeq and $\mathrm{GSeq}/_\sim$ and applying Proposition 3.3.11, we obtain that

$$\bigvee_{\alpha\in I} \mathbf{g}_\alpha = \Phi\left(\psi\left(\bigvee_{\alpha\in I} [\mathbf{g}_\alpha]\right)\right) = \Phi\left(\psi\left(\left[\sum_{\alpha\in I} \mathbf{g}_\alpha\right]\right)\right) = \Phi\left(\Psi\left(\sum_{\alpha\in I} \mathbf{g}_\alpha\right)\right). \quad \square$$

3.4 Remarks about systems of sets and sequences

In Section 2.6, we proved that the operator $\psi\colon \mathrm{GSeq}/_\sim \to \mathrm{seqGSet}$ is an order isomorphism and therefore it preserves infima and suprema. The next proposition shows that although the operator $\psi\colon \mathrm{GSeq}/_\sim \to \mathrm{GSet}$ preserves infima, it preserves suprema if and only if the class $\mathrm{seqGSet} = \psi \circ \phi(\mathrm{GSet})$ is closed with respect to the operation $\bigvee$ of GSet.

Proposition 3.4.1. *Let* $\{[\mathbf{g}_\alpha] \mid \alpha \in I\} \subset \mathrm{GSeq}/_\sim$. *Then*

(i) $\psi\left(\bigwedge_{\alpha\in I}[\mathbf{g}_\alpha]\right) = \bigwedge_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right)$,

(ii) $\psi\left(\bigvee_{\alpha\in I}[\mathbf{g}_\alpha]\right) = \psi\left(\phi\left(\bigvee_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right)\right)\right)$.

Proof. (i). Using the order-isomorphism between $\mathrm{GSeq}/_\sim$ and seqGSet, we get that $\psi\left(\bigwedge_{\alpha\in I}[\mathbf{g}_\alpha]\right) = \bigwedge_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right)$, where the latter infimum is taken in seqGSet. Comparison of Theorem 3.2.9 and Theorem 3.3.1 reveals that the formulas for calculating infima in GSet and seqGSet coincide.

(ii). Since $\mathrm{GSeq}/_\sim$ and seqGSet are order-isomorphic, it holds that $\psi\left(\bigvee_{\alpha\in I}[\mathbf{g}_\alpha]\right) = \bigvee_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right)$, where the latter supremum is taken in seqGSet. Propositions 3.3.8 and 3.2.12 show that

$$\bigvee_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right) = \psi\left(\phi\left(\mathrm{Sub}\sum_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right)\right)\right) = \psi\left(\phi\left(\bigvee_{\alpha\in I}\psi\left([\mathbf{g}_\alpha]\right)\right)\right),$$

where the former supremum is taken in seqGSet and the latter in GSet. □

The next proposition shows that the operator $\phi\colon \mathrm{GSet} \to \mathrm{GSeq}/_\sim$ always preserves suprema and that it preserves infima if the infimum is taken over a finite family.

Proposition 3.4.2. *Let* $\{\mathbf{G}_\alpha \mid \alpha \in I\} \subset \mathrm{GSet}$. *Then*

(i) $\phi\left(\bigwedge_{\alpha\in I}\mathbf{G}_\alpha\right) \le \bigwedge_{\alpha\in I}\phi\left(\mathbf{G}_\alpha\right)$; *if I is finite, then* $\phi\left(\bigwedge_{\alpha\in I}\mathbf{G}_\alpha\right) = \bigwedge_{\alpha\in I}\phi\left(\mathbf{G}_\alpha\right)$;

(ii) $\phi\left(\bigvee_{\alpha\in I}\mathbf{G}_\alpha\right) = \bigvee_{\alpha\in I}\phi\left(\mathbf{G}_\alpha\right)$.

Proof. (i). According to Theorems 3.2.9 and 3.3.6, it suffices to show

$$\Phi\left(\bigcap_{\alpha\in I}\mathbf{G}_\alpha\right) \lesssim \bigcap_{\alpha\in I}\Phi\left(\Psi\left(\Phi\left(\mathbf{G}_\alpha\right)\right)\right)$$

to prove that $\phi\left(\bigwedge_{\alpha\in I}\mathbf{G}_\alpha\right) \leq \bigwedge_{\alpha\in I}\phi\left(\mathbf{G}_\alpha\right)$. Let $\overline{x}\in\Phi\left(\bigcap_{\alpha\in I}\mathbf{G}_\alpha\right)$. By definition, $\{x_k\}\subset\mathbf{G}_\alpha(X)$ for every $\alpha\in I$. Therefore

$$\overline{x}\in\Phi\left(\mathbf{G}_\alpha\right)(X)\subset\Phi\left(\Psi\left(\Phi\left(\mathbf{G}_\alpha\right)\right)\right)(X)$$

for every $\alpha\in I$ and $\overline{x}\in\bigcap_{\alpha\in I}\Phi\left(\Psi\left(\Phi\left(\mathbf{G}_\alpha\right)\right)\right)$.

Assume that I is finite. It is easy to see that

$$\Phi\left(\bigcap_{\alpha\in I}\mathbf{G}_\alpha\right)\gtrsim\bigcap_{\alpha\in I}\Phi\left(\mathbf{G}_\alpha\right).$$

Theorem 3.2.9 and Proposition 3.3.7 yield that $\phi\left(\bigwedge_{\alpha\in I}\mathbf{G}_\alpha\right)\geq\bigwedge_{\alpha\in I}\phi\left(\mathbf{G}_\alpha\right)$.

(ii). It suffices to prove that $\Phi\left(\operatorname{Sub}\sum_{\alpha\in I}\mathbf{G}_\alpha\right)\sim\sum_{\alpha\in I}\Phi\left(\mathbf{G}_\alpha\right)$ by Propositions 3.2.12 and 3.3.11. In fact, we even prove that

$$\Phi\left(\operatorname{Sub}\sum_{\alpha\in I}\mathbf{G}_\alpha\right)=\sum_{\alpha\in I}\Phi\left(\mathbf{G}_\alpha\right).$$

By definition, $\overline{x}\in\Phi\left(\operatorname{Sub}\sum_{\alpha\in I}\mathbf{G}_\alpha\right)(X)$ if and only if there exist sets $G_1\in\mathbf{G}_{\alpha_1}(X),\ldots,G_n\in\mathbf{G}_{\alpha_n}(X)$ such that $\{x_k\}\subset G_1+\ldots+G_n$.

On the other hand, $\overline{x}\in\sum_{\alpha\in I}\Phi\left(\mathbf{G}_\alpha\right)(X)$ if and only if there exist sequences $\overline{y}_j=(y_k^j)_{k\in\mathbb{N}}$ such that $\overline{x}=\overline{y}_1+\ldots+\overline{y}_n$ and the property $\{y_k^j\mid k\in\mathbb{N}\}\in\mathbf{G}_{\alpha_j}(X)$ is satisfied for each $j\in\{1,\ldots,n\}$.

Let $(x_k)\in\Phi\left(\operatorname{Sub}\sum_{\alpha\in I}\mathbf{G}_\alpha\right)(X)$. Take $k\in\mathbb{N}$. Since $x_k\in G_1+\ldots+G_n$, we get that $x_k=y_k^1+\ldots+y_k^n$, where $y_k^j\in G_j$ for each $j\in\{1,\ldots,n\}$. Put $\overline{y}_j=(y_k^j)_{k\in\mathbb{N}}$ and observe that $\{y_k^j\mid k\in\mathbb{N}\}\in\mathbf{G}_{\alpha_j}(X)$. Therefore $(x_k)\in\sum_{\alpha\in I}\Phi\left(\mathbf{G}_\alpha\right)(X)$.

Let $\overline{x}\in\sum_{\alpha\in I}\Phi\left(\mathbf{G}_\alpha\right)(X)$ and $\overline{x}=\overline{y}_1+\ldots+\overline{y}_n$, where the sequences $\overline{y}_j=(y_k^j)_{k\in\mathbb{N}}$ satisfy $\{y_k^j\mid k\in\mathbb{N}\}\in\mathbf{G}_{\alpha_j}(X)$ for each $j\in\{1,\ldots,n\}$. Put $G_j=\{y_k^j\mid k\in\mathbb{N}\}$. Then $\{x_k\}\subset G_1+\ldots+G_n$, which proves that $\overline{x}\in\Phi\left(\operatorname{Sub}\sum_{\alpha\in I}\mathbf{G}_\alpha\right)(X)$. □

If the family I is finite, then Proposition 3.3.7 gives a simpler formula for calculating infima than Theorem 3.3.6. Although the same proof technique does not work if I is infinite, we do not know of a counterexample to show that the simpler formula does not hold in the infinite case. This raises the following question.

Question 3.4.3. Do there exist systems $\{\mathbf{g}_\alpha\mid\alpha\in I\}\subset\mathrm{GSeq}$, where I is infinite, such that

$$\bigcap_{\alpha\in I}\Phi\left(\Psi\left(\mathbf{g}_\alpha\right)\right)\not\sim\bigcap_{\alpha\in I}\mathbf{g}_\alpha?$$

The next question concerns Proposition 3.4.1. If for every collection of systems $\{\mathbf{g}_\alpha \mid \alpha \in I\} \subset \mathrm{GSeq}$ it were true that $\bigvee_{\alpha\in I} \psi([\mathbf{g}_\alpha]) \in \psi \circ \phi(\mathrm{GSet}) = \mathrm{seqGSet}$, then it would mean that the operator $\psi\colon \mathrm{GSeq}/_\sim \to \mathrm{GSet}$ preserves both infima and suprema. Currently, we do not even know if it preserves suprema of finite collections.

Question 3.4.4. Do there exist systems $\{[\mathbf{g}_\alpha] \mid \alpha \in I\} \subset \mathrm{GSeq}/_\sim$ such that

$$\bigvee_{\alpha\in I} \psi([\mathbf{g}_\alpha]) \notin \mathrm{seqGSet}?$$

Chapter 4

The notion of (p, r)-compactness

In this chapter, we study the generating system $\mathbf{K}_{(p,r)}$ of all relatively (p,r)-compact sets and the operator ideal $\mathcal{K}_{(p,r)}$ of all (p,r)-compact operators. Relying on Chapter 2, we show that the system $\mathbf{K}_{(p,r)}$ is sequentially generatable if and only if $p = \infty$ and $r = 1$, in which case $\mathbf{K}_{(p,r)}$ coincides with the system of all relatively compact sets $\mathbf{K}$. In turn, the system $\mathbf{K}_{(p,r)}$ provides answers and counterexamples to certain questions posed in the general context of Chapter 2. We prove that $\mathcal{K}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$. This allows us to equip $\mathcal{K}_{(p,r)}$ with the corresponding s-norm of $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ and to prove that $\mathcal{K}_{(p,r)}$ is an s-Banach operator ideal. This chapter is based on [ALO] and [Lil1].

4.1 The system of relatively (p, r)-compact sets

A. Grothendieck proved in 1955 in his famous Memoir [G2], among other results, the following result (see, e.g. [LiT, p. 30]). Following [DFLORT], we call this result the Grothendieck compactness principle.

Theorem 4.1.1 (Grothendieck compactness principle)**.** *Let X be a Banach space and let $K \subset X$. Then K is relatively compact if and only if there exists a sequence $(x_k) \in \mathbf{c}_0(X)$ such that*

$$K \subset \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_1} \right\}.$$

If one replaces $\mathbf{c}_0(X)$ with $\boldsymbol{\ell}_p(X)$, for some fixed $p \geq 1$, then one obtains a stronger form of relative compactness. This form of compactness was occasionally considered in the 1980s by Reinov [Re1] and Bourgain and Reinov [BR] in the study of approximation properties of order $s \leq 1$. Let us say, in this case, that K is *relatively p-compact in the sense of Bourgain–Reinov.*

In 2002, another strong form of compactness (but weaker than the Bourgain–Reinov one) was introduced by Sinha and Karn [SK1] through the requirement that

$$K \subset \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_{p^*}} \right\}$$

for some $(x_k) \in \boldsymbol{\ell}_p(X)$, where $1 \leq p < \infty$. In this case, let us say that K is *relatively p-compact in the sense of Sinha–Karn.*

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. In [ALO], we introduced the notion of relatively (p,r)-compact sets. A subset K of X is called *relatively (p,r)-compact* if

$$K \subset \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_r} \right\}$$

for some $(x_k) \in \boldsymbol{\ell}_p(X)$ (where $(x_k) \in \mathbf{c}_0(X)$ if $p = \infty$). As for the "extremal" cases, the $(p,1)$-compactness is precisely the Bourgain–Reinov p-compactness, and the (p,p^*)-compactness is precisely the Sinha–Karn p-compactness. We denote the system of all relatively (p,r)-compact sets by $\mathbf{K}_{(p,r)}$ (in [AO2], notation $\boldsymbol{k}_{(p,r)}$ is used). Note that $\mathbf{K}_{(\infty,1)} = \mathbf{K}$ according to the Grothendieck compactness principle.

In this section, we show that the system $\mathbf{K}_{(p,r)}$ is sequentially generatable only in the case when $p = \infty$ and $r = 1$ (see Corollary 4.1.10). Relying on [DPS1, Theorem 3.14], we also provide several counterexamples (see Propositions 4.1.11, Corollary 4.1.12, and Proposition 4.1.13) to various remarks from Chapter 2 (see Remarks 2.6.8, 2.5.7, and 2.4.9).

Let us begin with the following definition.

Definition 4.1.2. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. For every $\overline{x} = (x_k) \in \boldsymbol{\ell}_p(X)$, define an operator $E_{\overline{x}} \in \mathcal{L}(\ell_r, X)$ by

$$E_{\overline{x}}(\overline{\alpha}) = \sum_{k=1}^{\infty} \alpha_k x_k, \ \overline{\alpha} \in \ell_r.$$

In [ALO], [JLO], and [Lil2], we used the notation $\Phi_{(x_k)}$ instead of $E_{\overline{x}}$.

Proposition 4.1.3 (see, e.g., [A, Section 2.5])**.** *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let X be a Banach space and let $\overline{x} \in \boldsymbol{\ell}_p(X)$ (where $\overline{x} \in \mathbf{c}_0(X)$ if $p = \infty$). Then the operator $E_{\overline{x}} \in \mathcal{L}(\ell_r, X)$ is compact.*

Clearly, a subset K of X is relatively (p,r)-compact if and only if $K \subset E_{\overline{x}}(B_{\ell_r})$ for some $\overline{x} \in \boldsymbol{\ell}_p(X)$ (where $\overline{x} \in \mathbf{c}_0(X)$ if $p=\infty$).

Remark 4.1.4. In the literature, the definition of relatively $(1,\infty)$-compact set K is sometimes given by the requirement that $K \subset E_{\overline{x}}(B_{c_0})$, where $\overline{x} \in \boldsymbol{\ell}_1(X)$ (instead of the requirement $K \subset E_{\overline{x}}(B_{\ell_\infty})$). However, as shown by the line of thought given below, these notions are equivalent; we have adopted the latter variant for notational purposes.

Proof. Clearly, the former notion implies the latter. For the converse, let a set K satisfy $K \subset E_{\overline{x}}(B_{\ell_\infty})$, where $\overline{x} \in \boldsymbol{\ell}_1(X)$. It is a well-known fact that there exists a scalar sequence (λ_k) tending to infinity so that

$$(\lambda_k x_k) \in \boldsymbol{\ell}_1(X)$$

and $\lambda_k \geq 1$ for each $k \in \mathbb{N}$. Therefore we may conclude that

$$K \subset E_{(\lambda_k x_k)}(B_{c_0}). \qquad \square$$

The next lemma is well known (see, e.g., [FHHMPZ, pp. 22, 33]).

Lemma 4.1.5. *Let $\overline{x} \in \mathbf{c}_0(X)$. Then $E_{\overline{x}}(B_{\ell_1}) = \overline{\operatorname{absconv}}\{x_k \mid k \in \mathbb{N}\}$.*

Proposition 4.1.6. *Let $1 \leq p \leq \infty$ and let $1 \leq r \leq p^*$. Then $\mathbf{K}_{(p,r)} \in \mathrm{GSet}$ and $\mathbf{K}_{(p,r)} \leq \mathbf{K}$.*

Proof. We may assume that $p \neq \infty$, because there is nothing to prove for the case $\mathbf{K}_{(\infty,1)} = \mathbf{K}$. Let $G, H \in \mathbf{K}_{(p,r)}(X)$ be such that $G \subset E_{\overline{x}}(B_{\ell_r})$ and $H \subset E_{\overline{y}}(B_{\ell_r})$ for some $\overline{x}$ and $\overline{y}$ in $\boldsymbol{\ell}_p(X)$. Since $E_{\overline{x}}$ is a compact operator, we have $G \in \mathbf{K}(X)$ and therefore $\mathbf{K}_{(p,r)} \leq \mathbf{K}$. Let us verify the conditions of Definition 2.3.1.

(G_0) This follows from the fact that any set $E_{\overline{x}}(B_{\ell_r})$ is norm bounded by $\|E_{\overline{x}}\|$.

(G_1) Let $\overline{\beta} = (1,0,0,\ldots)$. Then $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_{\ell_r})$.

(G_2) Let $a \in \mathbb{K}$. Put

$$z_k = \begin{cases} 2^{1/r} a x_{(k+1)/2} & \text{if } k \text{ is odd,} \\ 2^{1/r} y_{k/2} & \text{if } k \text{ is even.} \end{cases}$$

We have $(z_k) \in \boldsymbol{\ell}_p(X)$ and $aG+H \subset E_{\overline{z}}(B_{\ell_r})$. Therefore $aG+H \in \mathbf{K}_{(p,r)}(X)$.

(G_3) This is obvious from the definition of a relatively (p,r)-compact set.

(G_4) Let $T \in \mathcal{L}(X,Y)$. Put $(y_k) = (Tx_k) \in \boldsymbol{\ell}_p(Y)$. Then $T(G) \subset E_{\overline{y}}(B_{\ell_r})$. $\square$

The following result describes the relations between the various forms of the relative (p,r)-compactness.

Proposition 4.1.7 (see [A, Theorem 3.6])**.** *Let X be a Banach space. Let $1 \leq p \leq q \leq \infty$, $1 \leq r \leq p^*$, and $1 \leq s \leq q^*$. Assume that*

$$\frac{1}{q} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{r}.$$

Then every relatively (p,r)-compact subset of X is relatively (q,s)-compact. Put another way, $\mathbf{K}_{(p,r)} \leq \mathbf{K}_{(q,s)}$.

The next result is rather immediate from [DPS1, Theorem 3.14].

Proposition 4.1.8. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$ and let X be an infinite-dimensional Banach space X. Then there exists a compact set $K \subset X$ such that $K \notin \mathbf{K}_{(p,r)}(X)$.*

Proof. According to [DPS1, Theorem 3.14], there exists a compact set $K \subset X$ such that $K \notin \mathbf{K}_{(p,p^*)}(X)$. But then $K \notin \mathbf{K}_{(p,r)}(X)$, because $\mathbf{K}_{(p,r)} \leq \mathbf{K}_{(p,p^*)}$. $\square$

Proposition 4.1.9. *Let $1 \leq p \leq \infty$ and let $1 \leq r \leq p^*$. Then $\Psi\left(\Phi\left(\mathbf{K}_{(p,r)}\right)\right) = \mathbf{K}$.*

Proof. According to Corollary 2.4.7, it suffices to show that $\Phi\left(\mathbf{K}_{(p,r)}\right) \sim \mathbf{c}$. We may assume that $p \neq \infty$, because $\Phi\left(\mathbf{K}_{(\infty,1)}\right) = \Phi\left(\mathbf{K}\right) = \Phi\left(\Psi\left(\mathbf{c}\right)\right) \sim \mathbf{c}$. Since $\mathbf{K}_{(p,r)} \leq \mathbf{K}$, we get from Corollary 2.6.5(ii) that $\Phi\left(\mathbf{K}_{(p,r)}\right) \lesssim \Phi\left(\mathbf{K}\right) \sim \mathbf{c}$. It remains to show that $\Phi\left(\mathbf{K}_{(p,r)}\right) \gtrsim \mathbf{c}$.

Let X be a Banach space and let $(x_k) \in \mathbf{c}(X)$ with $\lim_{k\to\infty} x_k = z$. Then $(x_k - z) \in \mathbf{c}_0(X)$ and there exists a subsequence (y_k) of $(x_k - z)$ such that $(y_k) \in \boldsymbol{\ell}_p(X)$. Obviously then $\{y_k\} \in \mathbf{K}_{(p,r)}(X)$. According to properties (G_1), (G_2), and (G_4), it holds that $\{y_k + z \mid k \in \mathbb{N}\} \in \mathbf{K}_{(p,r)}(X)$. Therefore we have found a subsequence $(y_k + z)$ of (x_k), which satisfies

$$(y_k + z) \in \Phi\left(\mathbf{K}_{(p,r)}\right)(X). \qquad \square$$

The following result is immediate from Corollary 2.6.5 and Propositions 4.1.8 and 4.1.9.

Corollary 4.1.10. *Let $1 \leq p \leq \infty$ and let $1 \leq r \leq p^*$. Then $\mathbf{K}_{(p,r)} \in \mathrm{seqGSet}$ if and only if $p = \infty$ and $r = 1$.*

Proposition 4.1.11. *Let X be an infinite-dimensional Banach space. Let $1 \leq p < \infty$. Then there exists a sequence $\overline{x} \in \Phi\left(\Psi\left(\Phi\left(\mathbf{K}_{(p,1)}\right)\right)\right)(X)$, but such that $\overline{x} \notin \Phi\left(\mathbf{K}_{(p,1)}\right)(X)$.*

Proof. According to Proposition 4.1.8, there exists a compact set $K \subset X$ such that $K \notin \mathbf{K}_{(p,1)}(X)$. Since K is compact, there exists a sequence $\overline{x} \in \mathbf{c}_0(X)$ such that $K \subset E_{\overline{x}}(B_{\ell_1})$. Since $\mathbf{c}_0(X) \subset \Phi\left(\Psi\left(\mathbf{c}\right)\right)(X)$, it follows from Proposition 4.1.9 that

$$\overline{x} \in \Phi\left(\Psi\left(\mathbf{c}\right)\right)(X) = \Phi\left(\mathbf{K}\right)(X) = \Phi\left(\Psi\left(\Phi\left(\mathbf{K}_{(p,1)}\right)\right)\right)(X).$$

Assume to the contrary that $\overline{x} \in \Phi\left(\mathbf{K}_{(p,1)}\right)(X)$. Then

$$\{x_k\} \in \mathbf{K}_{(p,1)}(X).$$

This yields a sequence $\overline{y} \in \boldsymbol{\ell}_p(X)$ such that

$$\{x_k\} \subset E_{\overline{y}}(B_{\ell_1}).$$

But then

$$K \subset E_{\overline{x}}(B_{\ell_1}) = \overline{\operatorname{absconv}}\{x_k \mid k \in \mathbb{N}\} \subset \overline{\operatorname{absconv}}(E_{\overline{y}}(B_{\ell_1})) = E_{\overline{y}}(B_{\ell_1}),$$

which is a contradiction because $E_{\overline{y}}(B_{\ell_1}) \in \mathbf{K}_{(p,1)}(X)$. □

Corollary 4.1.12. *Let $1 \leq p < \infty$. Then $\Phi\left(\mathbf{K}_{(p,1)}\right) \notin \mathrm{satGSeq}$.*

Proof. Assume to the contrary that the system $\Phi\left(\mathbf{K}_{(p,1)}\right)$ is saturated. Then, by Proposition 2.5.1(iii), we have $\Phi\left(\mathbf{K}_{(p,1)}\right) = \Phi\left(\Psi\left(\Phi\left(\mathbf{K}_{(p,1)}\right)\right)\right)$, which is in contradiction with Proposition 4.1.11. □

Proposition 4.1.13. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. Then*

$$[\mathbf{B} \to \mathbf{K}_{(p,r)}] \neq [\Phi\left(\mathbf{B}\right) \to \Phi\left(\mathbf{K}_{(p,r)}\right)].$$

Proof. We have $[\mathbf{B} \to \mathbf{K}_{(p,r)}] = \mathcal{K}_{(p,r)}$. According to Proposition 2.4.9, it holds that

$$\left[\Phi\left(\mathbf{B}\right) \to \Phi\left(\mathbf{K}_{(p,r)}\right)\right] = \left[\Psi\left(\Phi\left(\mathbf{B}\right)\right) \to \Psi\left(\Phi\left(\mathbf{K}_{(p,r)}\right)\right)\right] = [\mathbf{B} \to \mathbf{K}] = \mathcal{K}.$$

□

4.2 Description of $\mathcal{K}(p,r)$ as an s-Banach operator ideal

Let $1 \leq p \leq \infty$. Recall that $\mathcal{K} = \Theta(\mathbf{K})$, since a linear operator $T\colon X \to Y$ is said to be compact if $T(B_X)$ is a relatively compact subset of Y. Using relatively

p-compact subsets of Y instead of relatively compact ones, Sinha and Karn [SK1] obtained the concept of *p-compact operators.* If one uses relatively p-compact subsets of Y in the sense of Bourgain–Reinov instead of relatively compact ones, then one obtains the notion of *p-compact operators in the sense of Bourgain–Reinov.*

Following [SK1], denote the class of all p-compact operators (in the sense of Sinha–Karn) by $\mathcal{K}_p$. Properties of $\mathcal{K}_p$ were studied in [SK1] and, e.g., in the papers [DOPS], [DPS1], [DPS2], [GLT], [LiT], [O2], [Pi3], and [SK2].

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Define the operator ideal $\mathcal{K}_{(p,r)}$ of *(p, r)-compact operators* by $\mathcal{K}_{(p,r)} = \Theta(\mathbf{K}_{(p,r)})$. The "extremal" cases, $\mathcal{K}_p = \mathcal{K}_{(p,p^*)}$ and $\mathcal{K}_{(p,1)}$ coincide with the classes of p-compact operators in the sense of Sinha–Karn and Bourgain–Reinov, respectively.

The main objective of this section is to prove that the operator ideal $\mathcal{K}_{(p,r)}$ is equal to $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ (see Theorem 4.2.10). Note that the definition of the operator ideal $\mathcal{N}_{(t,u,v)}$ is recalled in Definition 4.2.7 below. In this section, we use the conventions $1\infty = \infty 1 = 1 + \infty = \infty + 1 = \infty$ and $\infty/\infty = 1$.

Proposition 4.1.7 immediately yields the following corollary.

Corollary 4.2.1. *Let X be a Banach space. Let $1 \leq p \leq q \leq \infty$, $1 \leq r \leq p^*$, and $1 \leq s \leq q^*$. Assume that*

$$\frac{1}{q} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{r}.$$

Then $\mathcal{K}_{(p,r)} \subset \mathcal{K}_{(q,s)}$ as operator ideals.

Let us start with the following (well-known) notions.

Definition 4.2.2. Let $1 \leq p < \infty$ and let X be a Banach space. The space $\boldsymbol{\ell}_p(X)$ becomes a Banach space with respect to the norm

$$\|\overline{x}\|_p = \left(\sum_{k=1}^{\infty} \|x_k\|^p \right)^{1/p} .$$

Definition 4.2.3. Let X be a Banach space. The space $\mathbf{m}(X) = \boldsymbol{\ell}_\infty(X)$ becomes a Banach space with respect to the norm

$$\|\overline{x}\|_\infty = \sup_{k \in \mathbb{N}} \|x_k\| .$$

The space $\mathbf{c}_0(X)$ becomes a Banach space with respect to the norm of $\boldsymbol{\ell}_\infty(X)$.

Definition 4.2.4 (see, e.g., [DJT, pp. 32-33]). Let $1 \leq p < \infty$ and let X be a Banach space. A sequence $\overline{x} = (x_k)$ in X is said to be *weakly p-summable* if

$$\sum_{k=1}^{\infty} |x^*(x_k)|^p < \infty$$

for every $x^* \in X^*$.

The space $\boldsymbol{\ell}_p^w(X)$ of weakly p-summable sequences in X becomes a Banach space when equipped with the norm

$$\|\overline{x}\|_p^w = \sup_{x^* \in B_{X^*}} \big\| \big(x^*(x_k)\big) \big\|_p.$$

We mention that the finiteness of the norm $\|\overline{x}\|_p^w$ follows from the closed graph theorem (for details, see, e.g., [DJT, p. 32]). We denote the system of all weakly p-summable sequences in all Banach spaces by $\boldsymbol{\ell}_p^w$.

Consider the analogue of the previous definition for the case $p = \infty$. Then the space $\boldsymbol{\ell}_\infty^w(X)$ of "weakly bounded" sequences would coincide with the space $\boldsymbol{\ell}_\infty(X)$ and the norm

$$\|\overline{x}\|_\infty^w = \sup_{x^* \in B_{X^*}} \sup_{k \in \mathbb{N}} |x^*(x_k)|$$

would coincide with the norm $\|\overline{x}\|_{\boldsymbol{\ell}_\infty(X)}$ (see, e.g., [DJT, p. 33]). Therefore we use the notations $\boldsymbol{\ell}_\infty^w(X)$ and $\boldsymbol{\ell}_\infty(X)$ interchangeably in the sequel.

Definition 4.2.5 (see, e.g., [DJT, p. 33]). Let X be a Banach space. A sequence $\overline{x}$ in X is said to be *weakly null* if

$$\lim_{k \to \infty} x^*(x_k) = 0$$

for every $x^* \in X^*$.

The space $\mathbf{c}_0^w(X)$ of weakly null sequences in X is a closed subspace of $\boldsymbol{\ell}_\infty(X)$; therefore it is a Banach space with the supremum norm of $\boldsymbol{\ell}_\infty(X)$.

Denote the system of all weakly null sequences in all Banach spaces by $\mathbf{c}_0^w$. The following lemma is well known and straightforward to verify.

Lemma 4.2.6. *Let $1 \leq p < \infty$ and let X be a Banach space. Then $\boldsymbol{\ell}_p^w(X) \subset \mathbf{c}_0^w(X)$ and*

$$\|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)} \geq \|\overline{x}\|_{\mathbf{c}_0^w(X)}$$

for every sequence $\overline{x} \in \boldsymbol{\ell}_p^w(X)$.

Let us recall the definition of the s-Banach operator ideal $\mathcal{N}_{(t,u,v)}$ of (t,u,v)-nuclear operators.

Definition 4.2.7 (see [Pi1, 18.1.1]). Let $0 < t \le \infty$, $1 \le u, v \le \infty$, and $1/u + 1/v \le 1 + 1/t$. An operator $T \in \mathcal{L}(X,Y)$ is called (t,u,v)-*nuclear* if

$$T = \sum_{n=1}^{\infty} \delta_n x_n^* \otimes y_n \tag{4.1}$$

with $(\delta_n) \in \ell_t$, $(x_n^*) \in \boldsymbol{\ell}_{v^*}^w(X^*)$, and $(y_n) \in \boldsymbol{\ell}_{u^*}^w(Y)$. Denote

$$\|T\|_{\mathcal{N}_{(t,u,v)}} = \inf \|(\delta_n)\|_t \, \|(x_n^*)\|_{v^*}^w \, \|(y_n)\|_{u^*}^w ,$$

where the infimum is taken over all (t,u,v)-*nuclear representations* (4.1) of T.

Set $1/s = 1/t + 1/u^* + 1/v^*$. Then $s \in (0,1]$. It is well known (see [Pi1, 18.1.2]), that $(\mathcal{N}_{(t,u,v)}, \|\cdot\|_{\mathcal{N}_{(t,u,v)}})$ is an s-Banach operator ideal.

The proof of the next result follows [ALO, p. 149]. For a more detailed version of the same proof, see [A, Proposition 2.16].

Proposition 4.2.8. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Let $\overline{x} \in \boldsymbol{\ell}_p(X)$. Then $E_{\overline{x}} \in \mathcal{N}_{(p,1,r^*)}(\ell_r, X)$.*

Proof. Clearly, for $E_{\overline{x}} \in \mathcal{K}(\ell_r, X)$, we have

$$E_{\overline{x}} = \sum_{n=1}^{\infty} e_n \otimes x_n,$$

where the unit vectors $e_n \in \ell_{r^*} \subset (\ell_r)^*$ are considered as coordinate functionals for ℓ_r. It is well known and easy to verify that $(e_n) \in S_{\boldsymbol{\ell}_r^w(\ell_r^*)}$. Therefore, from $e_n \otimes x_n = \|x_n\| e_n \otimes (\|x_n\|^{-1} x_n)$, it is clear that

$$E_{\overline{x}} \in \mathcal{N}_{(p,1,r^*)}(\ell_r, X)$$

and

$$\|E_{\overline{x}}\|_{\mathcal{N}_{(p,1,r^*)}} \le \|\overline{x}\|_p . \qquad \square$$

The *key observation* for our approach to the study of $\mathcal{K}_{(p,r)}$ is that the injective associate of $E_{\overline{x}}$ belongs to $\mathcal{N}_{(p,1,r^*)}^{\text{sur}}$. Let us denote it by $\overline{E}_{\overline{x}}$ and recall that

$$E_{\overline{x}} = \overline{E}_{\overline{x}} q,$$

where $q : \ell_r \to Z := \ell_r / \ker E_{\overline{x}}$ is the quotient mapping.

Proposition 4.2.9. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Let $\overline{x} \in \boldsymbol{\ell}_p(X)$. Then $\overline{E}_{\overline{x}} \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(Z,X)$, where $Z = \ell_r/\ker E_{\overline{x}}$, and $\left\|\overline{E}_{\overline{x}}\right\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \le \|\overline{x}\|_p$.*

Proof. Let $\varepsilon > 0$ be arbitrary. For every $z \in B_Z$ choose $\alpha_z \in \ell_r$ with $\|\alpha_z\| \le 1+\varepsilon$ such that $q\alpha_z = z$. Define $\hat{Q}\colon \ell_1[B_Z] \to \ell_r$ by $\hat{Q}((\lambda_z)_{z\in B_Z}) = \sum_{z\in B_Z} \lambda_z \alpha_z$. Then $\hat{Q} \in \mathcal{L}(\ell_1[B_Z], \ell_r)$, $\|\hat{Q}\| \le 1+\varepsilon$, and $q\hat{Q} = Q_Z$:

$$\begin{tikzcd}
& \ell_r \arrow[rr, "E_{\overline{x}}"] \arrow[dr, "q"] & & X \\
\ell_1[B_Z] \arrow[ur, "\hat{Q}"] \arrow[rr, "Q_Z"] & & Z \arrow[ur, "\overline{E}_{\overline{x}}"'] &
\end{tikzcd}$$

In fact, we have here explicitly written down the lifting property of $\ell_1[B_Z]$ (see [Pi1, C.3.5, C.3.6]). Therefore,

$$\overline{E}_{\overline{x}} Q_Z = \overline{E}_{\overline{x}} q \hat{Q} = E_{\overline{x}} \hat{Q} \in \mathcal{N}_{(p,1,r^*)}(\ell_1[B_Z], X),$$

meaning that $\overline{E}_{\overline{x}} \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(Z,X)$. Moreover,

$$\begin{aligned}
\|\overline{E}_{\overline{x}}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} &= \|E_{\overline{x}}\hat{Q}\|_{\mathcal{N}_{(p,1,r^*)}} \le \|E_{\overline{x}}\|_{\mathcal{N}_{(p,1,r^*)}} \|\hat{Q}\| \\
&\le (1+\varepsilon)\|E_{\overline{x}}\|_{\mathcal{N}_{(p,1,r^*)}} \le (1+\varepsilon)\|\overline{x}\|_p.
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we have $\|\overline{E}_{\overline{x}}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \le \|\overline{x}\|_p$, as desired. □

Let X and Y be Banach spaces, and let $T \in \mathcal{K}_{(p,r)}(X,Y)$. Similarly to the case of $\mathcal{K}_p = \mathcal{K}_{(p,p^*)}$ (see [SK1, pp. 20–21]), we have the *natural factorization* of T as follows.

Let $\overline{y} \in \boldsymbol{\ell}_p(Y)$ such that $T(B_X) \subset E_{\overline{y}}(B_{\ell_r})$. Denote, as before, $Z = \ell_r/\ker E_{\overline{y}}$. Then

$$E_{\overline{y}} = \overline{E}_{\overline{y}} q,$$

where $q\colon \ell_r \to Z$ is the quotient mapping and $\overline{E}_{\overline{y}}\colon Z \to Y$ is the injective associate of $E_{\overline{y}}$. Let $x \in B_X$ and let $\overline{\alpha} \in B_{\ell_r}$ satisfy $Tx = E_{\overline{y}}\overline{\alpha}$. If $Tx = E_{\overline{y}}\overline{\beta}$ for some (other) $\overline{\beta} \in \ell_r$, then clearly $\overline{\alpha} - \overline{\beta} \in \ker E_{\overline{y}}$. Therefore one can define $T_{\overline{y}}\colon X \to Z$ by $T_{\overline{y}}x = q\overline{\alpha}$, $x \in X$, where $\overline{\alpha} \in \ell_r$ satisfies $\|\overline{\alpha}\| \le \|x\|$ and $Tx = E_{\overline{y}}\overline{\alpha}$:

$$\begin{tikzcd}
\ell_r \arrow[rr, "E_{\overline{y}}"] \arrow[dr, "q"'] & & Y & \\
& Z \arrow[ur, "\overline{E}_{\overline{y}}"] & & X \arrow[ul, "T"'] \arrow[ll, "T_{\overline{y}}"]
\end{tikzcd}$$

Since $E_{\overline{y}} = \overline{E}_{\overline{y}} q$, one immediately obtains the factorization

$$T = \overline{E}_{\overline{y}} T_{\overline{y}},$$

with $T_{\overline{y}} \in \mathcal{L}(X, Z)$, $\|T_{\overline{y}}\| \leq 1$, and $\ker T_{\overline{y}} = \ker T$.

Theorem 4.2.10. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then the operator ideal $\mathcal{K}_{(p,r)}$ is equal to $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$.*

Proof. Let $T \in \mathcal{K}_{(p,r)}(X, Y)$. Since, by the natural factorization, $T = \overline{E}_{\overline{y}} T_{\overline{y}}$, and $\overline{E}_{\overline{y}} \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(Z, Y)$, we have $T \in \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}(X, Y)$.

Conversely, to see that $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)} \subset \mathcal{K}_{(p,r)}$, it suffices to prove that $\mathcal{N}_{(p,1,r^*)} \subset \mathcal{K}_{(p,r)}$, because $\mathcal{K}_{(p,r)}$ is surjective and $\mathcal{A}^{\mathrm{sur}} \subset \mathcal{B}^{\mathrm{sur}}$ whenever $\mathcal{A} \subset \mathcal{B}$ (see Proposition 2.2.10).

Consider $T \in \mathcal{N}_{(p,1,r^*)}(X, Y)$. Then $T = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n$, where $(\sigma_n) \in \ell_p$, $(x_n^*) \in \boldsymbol{\ell}_r^w(X^*)$, $(y_n) \in \boldsymbol{\ell}_\infty(Y)$. We clearly may assume that $\|(x_n^*)\|_r^w = 1$. Indeed,

$$\sigma_n x_n^* \otimes y_n = \sigma_n \frac{x_n^*}{\|(x_n^*)\|_r^w} \otimes \|(x_n^*)\|_r^w \, y_n$$

and $(\|(x_n^*)\|_r^w \, y_n) \in \boldsymbol{\ell}_\infty(Y)$. Observe that $(\sigma_n y_n) \in \boldsymbol{\ell}_p(Y)$. Together with the assumption $\|(x_n^*)\|_r^w = 1$, we have

$$Tx = \sum_{n=1}^{\infty} x_n^*(x) \sigma_n y_n \in E_{(\sigma_n y_n)}(B_{\ell_r})$$

for every $x \in B_X$. Thus we have shown that $T \in \mathcal{K}_{(p,r)}(X, Y)$. □

Since $\mathcal{K}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ as operator ideals, *defining*

$$\|\cdot\|_{\mathcal{K}_{(p,r)}} := \|\cdot\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}},$$

we immediately get that $(\mathcal{K}_{(p,r)}, \|\cdot\|_{\mathcal{K}_{(p,r)}})$ is an *s-Banach operator ideal where*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{r},$$

i.e., $s = pr/(p + r)$. Note that $\frac{1}{2} \leq s \leq 1$, since $\frac{1}{s} = \frac{1}{p} + \frac{1}{r} \leq 2$. The only case when $s = pr/(p + r) = 1$ is precisely when $r = p^*$. This suggests that from the whole family of s-Banach operator ideals $\mathcal{K}_{(p,r)}$, only $\mathcal{K}_{(p,p^*)}$ is a Banach operator ideal. As was mentioned earlier, $\mathcal{K}_{(p,p^*)} = \mathcal{K}_p$ is the ideal of p-compact operators introduced in [SK1] by Sinha and Karn. For the other extreme, $s = \frac{1}{2}$ if and only if $p = r = 1$. In the language of quasi-norms, this means that $\mathcal{K}_{p,r}$ is a quasi-norm with the quasi-constant $\varkappa = 2^{\frac{1}{s}-1}$ satisfying $1 \le \varkappa \le 2$ (recall Proposition 2.2.7).

Since $\mathcal{A}^{\mathrm{sur}} \subset \mathcal{B}^{\mathrm{sur}}$ whenever $\mathcal{A} \subset \mathcal{B}$ (see Proposition 2.2.10), Theorem 4.2.10 together with the inclusion theorem for (t,u,v)-nuclear operators [Pi1, 18.1.5] immediately yield the following result.

Corollary 4.2.11. *Let $1 \le p \le q \le \infty$, $1 \le r \le p^*$, and $1 \le s \le q^*$. Assume that $s \le r$ and*

$$\frac{1}{q} + \frac{1}{s} \le \frac{1}{p} + \frac{1}{r}.$$

Then

$$(\mathcal{K}_{(p,r)}, \|\cdot\|_{\mathcal{K}_{(p,r)}}) \subset (\mathcal{K}_{(q,s)}, \|\cdot\|_{\mathcal{K}_{(q,s)}}).$$

In particular, $(\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}) \subset (\mathcal{K}_q, \|\cdot\|_{\mathcal{K}_q})$ and $(\mathcal{K}_{(p,1)}, \|\cdot\|_{\mathcal{K}_{(p,1)}}) \subset (\mathcal{K}_{(q,1)}, \|\cdot\|_{\mathcal{K}_{(q,1)}})$ if $1 \le p \le q \le \infty$.

It is important that we can explicitly calculate the s-norm $\|\cdot\|_{\mathcal{K}_{(p,r)}}$ as follows. Among other uses, this shows that the norm $\|\cdot\|_{\mathcal{K}_p}$ coincides with the norms introduced in [SK1] and [DPS1] (see Remarks 4.2.14 and 4.2.15 below).

Theorem 4.2.12 (see [ALO, Theorem 3.4])**.** *Let $T \in \mathcal{K}_{(p,r)}(X,Y)$. Then*

$$\|T\|_{\mathcal{K}_{(p,r)}} = \inf \|T_{\overline{y}}\| \|\overline{y}\| = \inf \|\overline{y}\|,$$

where both infima are taken over all sequences $\overline{y} \in \boldsymbol{\ell}_p(Y)$ such that

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n y_n \mid \overline{\alpha} \in B_{\ell_r} \right\}.$$

Proof. Let $\overline{y} \in \boldsymbol{\ell}_p(Y)$ be such that $T(B_X) \subset E_{\overline{y}}(B_{\ell_r})$. We know that $T = \overline{E}_{\overline{y}} T_{\overline{y}}$, $\|T_{\overline{y}}\| \le 1$, and $\|\overline{E}_{\overline{y}}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \le \|\overline{y}\|$. Hence,

$$\begin{aligned} \|T\|_{\mathcal{K}_{(p,r)}} &= \|T\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \le \|\overline{E}_{\overline{y}}\|_{\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}} \|T_{\overline{y}}\| \\ &\le \|T_{\overline{y}}\| \|\overline{y}\| \le \|\overline{y}\|. \end{aligned}$$

Consequently,

$$\|T\|_{\mathcal{K}_{(p,r)}} \le \inf \|T_{\overline{y}}\| \|\overline{y}\| \le \inf \|\overline{y}\|.$$

On the other hand, from the factorization theorem of (t,u,v)-nuclear operators (see [Pi1, 18.1.3]), we know that the $(p,1,r^*)$-nuclear operator TQ_X factorizes as follows:

$$\begin{tikzcd} Z := \ell_1[B_X] \arrow[r, "Q_X"] \arrow[d, "A"] & X \arrow[r, "T"] & Y \\ \ell_r \arrow[rr, "\Delta"] & & \ell_1 \arrow[u, "B"] \end{tikzcd}$$

where $\Delta \in \mathcal{L}(\ell_r, \ell_1)$ is a diagonal operator of the form $\Delta(\overline{\alpha}) = (\sigma_n a_n)$ with $\overline{\sigma} \in \ell_p$, $A \in \mathcal{L}(Z, \ell_r)$, $B \in \mathcal{L}(\ell_1, Y)$, and

$$\|TQ_X\|_{\mathcal{N}_{(p,1,r^*)}} = \inf \|B\| \|\overline{\sigma}\| \|A\|,$$

where the infimum is taken over the all possible factorizations.

Let $\varepsilon > 0$. Choose A, $\overline{\sigma}$, and B as above so that

$$\varepsilon + \|T\|_{\mathcal{K}_{(p,r)}} = \varepsilon + \|TQ_X\|_{\mathcal{N}_{(p,1,r^*)}} \geq \|B\| \|\overline{\sigma}\| \|A\| = \|\overline{\sigma}\|,$$

because we clearly may assume that $\|A\| = \|B\| = 1$. Since $B_X \subset Q_X(B_Z)$, we have

$$T(B_X) \subset (B\Delta A)(B_Z) \subset (B\Delta)(B_{\ell_r}) = \left\{ \sum_{n=1}^{\infty} \alpha_n \sigma_n B e_n \mid \overline{\alpha} \in B_{\ell_r} \right\},$$

where e_n, $n \in \mathbb{N}$, are the unit vectors of ℓ_r. Put $y_n = \sigma_n B e_n$. Then $\overline{y} \in \boldsymbol{\ell}_p(Y)$, $T(B_X) \subset E_{\overline{y}}(B_{\ell_r})$, and $\|\overline{y}\| \leq \|\overline{\sigma}\|$. Therefore,

$$\|T\|_{\mathcal{K}_{(p,r)}} \geq \inf \|\overline{y}\|.$$

This concludes the proof. $\square$

Let us spell out the immediate special case of Theorems 4.2.10 and 4.2.12 which characterizes the p-compact operators $\mathcal{K}_{(p,1)}$ in the Bourgain–Reinov sense. Concerning the p-compact operators in the Sinha–Karn sense, see Remarks 4.2.14–4.2.16 below.

Theorem 4.2.13. *The operator ideal $\mathcal{K}_{(p,1)} = \mathcal{N}^{\mathrm{sur}}_{(p,1,\infty)}$ is a $\frac{p}{p+1}$-Banach operator ideal. The $\frac{p}{p+1}$-norm of $T \in \mathcal{K}_{(p,1)}(X,Y)$ is calculated as follows:*

$$\|T\|_{\mathcal{K}_{(p,1)}} = \inf \|T_{\overline{y}}\| \|\overline{y}\| = \inf \|\overline{y}\|,$$

where both infima are taken over all sequences $\overline{y} \in \boldsymbol{\ell}_p(Y)$ such that

$$T(B_X) \subset \left\{ \sum_{n=1}^{\infty} a_n y_n \mid \overline{\alpha} \in B_{\ell_1} \right\}.$$

Remark 4.2.14. In [SK1, Theorem 4.2], the Banach operator ideal norm was introduced on the operator ideal $\mathcal{K}_p$ through the formula $\|T\|_{\kappa_p} := \inf \|T_{(y_n)}\|\|y_n\|$, which is the special case with $r = p^*$ of the first equality in Theorem 4.2.12. Thus Corollary 4.2.11 extends the inclusion result [SK1, Proposition 4.3] for Banach operator ideals $\mathcal{K}_p$.

Remark 4.2.15. Delgado, Piñero, and Serrano [DPS1] made a thorough study of the operator ideal $\mathcal{K}_p$, but they defined the Banach operator ideal norm in $\mathcal{K}_p$ through the formula $\|T\|_{k_p} := \inf \|y_n\|$, which is the special case (with $r = p^*$) of the second equality in Theorem 4.2.12. They proved (see [DPS1, Proposition 3.15]) that the Banach operator ideal norms from [SK1] and [DPS1] coincide: $\|T\|_{k_p} = \|T\|_{\kappa_p}$ whenever $T \in \mathcal{K}_p(X,Y)$; this equality is also contained in Theorem 4.2.12.

Remark 4.2.16. One of the main results of [DPS1](see [DPS1, Proposition 3.11]) is that $(\mathcal{K}_p, \|\cdot\|_{k_p}) = (\mathcal{N}^p, \|\cdot\|_{\mathcal{N}^p})^{\mathrm{sur}}$, the Banach operator ideal of right p-nuclear operators. Since, by definition, $(\mathcal{N}^p, \|\cdot\|_{\mathcal{N}^p}) = (\mathcal{N}_{(p,1,p)}, \|\cdot\|_{\mathcal{N}_{(p,1,p)}})$ (cf. [Pi1, 18.1.1] and, e.g., [Ry, p. 140]), this result is contained as the special case with $r = p^*$ in Theorems 4.2.10 and 4.2.12. In [DPS1], to prove this result, the authors used a roundabout approach, first describing $\mathcal{K}_p^{\mathrm{dual}}$, and relied on Reinov's recent study [Re2] on operators with p-nuclear adjoints.

Chapter 5

Sequentially generated subclasses of the system $\mathbf{K}$

In this chapter, we study sequentially generatable systems of sets $\Psi(\mathrm{g})$, which satisfy $\mathbf{F} \leq \Psi(\mathrm{g}) \leq \mathbf{K}$. We introduce the notion of a hereditarily almost autoapproximable sequence. Using this notion, we prove that the latter inequality $\Psi(\mathrm{g}) \leq \mathbf{K}$ is strict if and only if the system g consists entirely of hereditarily almost autoapproximable sequences. We also provide an example of such a system of sequences g.

5.1 Motivation

In Section 2.6, we proved that a system $\mathbf{G} \in \mathrm{GSet}$ is sequentially generatable if and only if $\Psi(\Phi(\mathbf{G})) = \mathbf{G}$ (see Corollary 2.6.5). Proposition 4.1.9 and Corollary 4.1.10 demonstrated that the systems $\mathbf{K}_{(p,r)}$, where $1 \leq p < \infty$ and $1 \leq r \leq p^*$, satisfy $\Psi\left(\Phi\left(\mathbf{K}_{(p,r)}\right)\right) = \mathbf{K}$, and therefore fail to be sequentially generatable. These examples, together with the proof of Proposition 4.1.9, leave the impression that it is somewhat difficult for a sequentially generatable system $\mathbf{G} \in \mathrm{seqGSet}$ to satisfy simultaneously $\mathbf{F} < \mathbf{G}$ and $\mathbf{G} < \mathbf{K}$.

This chapter is devoted to investigating such systems $\mathbf{G} \in \mathrm{seqGSet}$, which indeed satisfy $\mathbf{F} < \mathbf{G} < \mathbf{K}$. For this, we introduce the notion of hereditarily almost autoapproximable sequences (see Definition 5.2.5) and prove the following characterization (see Theorem 5.2.6): a system $\Psi(\mathbf{g}) \in \mathrm{seqGSet}$ satisfying $\Psi(\mathbf{g}) \leq \mathbf{K}$ is strictly smaller than $\mathbf{K}$ if and only if every sequence in every component $\mathbf{g}(X)$ is hereditarily almost autoapproximable.

However, this does not answer the question whether such systems $\mathbf{G} \in \text{seqGSet}$ exist (other than the trivial example $\mathbf{F}$). Section 5.3 is devoted to constructing such a system. The construction utilizes the notion and an example of a hypercomplete sequence, taken from [GL].

5.2 Sequentially generated subsystems of K

We begin this section by proving the following result, which uses essentially the same idea as Proposition 4.1.9.

Proposition 5.2.1. *Let* $\mathbf{g} \in \text{GSeq}$ *and let the component* $\mathbf{g}(\ell_1)$ *contain a sequence of the form* $(\alpha_k e_k)$*, where* $|\alpha_k| > 0$ *for each* $k \in \mathbb{N}$*. Then* $\Psi(\mathbf{g}) \geq \mathbf{K}$*.*

Proof. Take a sequence $(\alpha_k e_k) \in \mathbf{g}(\ell_1)$, where $|\alpha_k| > 0$ for each $k \in \mathbb{N}$. Let K be any relatively compact set in any Banach space X. We aim to show that $K \in \Psi(\mathbf{g})(X)$.

Since K is relatively compact, every sequence $(x_k) \subset K$ contains a convergent subsequence (y_k). Put $x = \lim_{k\to\infty} y_k$ and $z_k = y_k - x$. Fix an index k_1 such that $\|z_{k_1}\| \leq |\alpha_1|$. As a next step, fix an index $k_2 > k_1$ such that $\|z_{k_2}\| \leq |\alpha_2|$. Continuing like this, we obtain a sequence (w_j), where $w_j = z_{k_j}$. Define an operator $T \in \mathcal{L}(\ell_1, X)$ by

$$T(\alpha_k e_k) = w_k.$$

According to the assumption and property (G_4), we have $(w_k) \in \mathbf{g}(X)$. The constant sequence $(x, x, \ldots)$ belongs to the component $\mathbf{g}(X)$ according to properties (S_1) and (S_4). Therefore we have shown that the original sequence (x_k) contains a subsequence

$$(y_{k_j})_{j\in\mathbb{N}} = (z_{k_j} + x)_{j\in\mathbb{N}} = (w_j + x)_{j\in\mathbb{N}} \in \mathbf{g}(X). \qquad \square$$

The importance of the above proposition is made apparent by Proposition 5.2.4 and Theorem 5.2.6 below.

Definition 5.2.2. Let (x_k) be any sequence. Define

$$[x_j]_{j\in\mathbb{N}}^{j\neq k} = \text{span}\{x_j \mid j \in \mathbb{N}, j \neq k\}$$

for each $k \in \mathbb{N}$. Denote by $d(x, Y)$ the distance of an element $x \in X$ to a subset Y of X. Clearly,

$$d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) = \inf_{\substack{\alpha_1,\ldots,\alpha_n\in\mathbb{K},\\ n\in\mathbb{N}}} \left\| x_k - \sum_{\substack{j=1,\\ j\neq k}}^{n} \alpha_j x_j \right\|.$$

For the proof of Proposition 5.2.4, we need the following result, which is due to R. S. Phillips.

Theorem 5.2.3 (see [Di2, p. 71 (Theorem 3)]). *Let Y be a linear subspace of the Banach space X and suppose that $T\colon Y \to m$ is a bounded linear operator. Then T may be extended to a bounded linear operator $S\colon X \to m$ having the same norm as T.*

Proposition 5.2.4. *Let X be a Banach space and let (x_k) be a sequence in X which satisfies $d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) > 0$ for all $k \in \mathbb{N}$. Then there exists a bounded linear operator $R\colon X \to \ell_1$ such that $R(x_k) = \alpha_k e_k$, where $|\alpha_k| > 0$ for each $k \in \mathbb{N}$.*

Proof. Put

$$\gamma_k = d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) > 0$$

for each $k \in \mathbb{N}$. Define

$$Y = \operatorname{span}\{x_k \mid k \in \mathbb{N}\} \subset X$$

and $T\colon Y \to m$ by

$$T(x_k) = \gamma_k e_k,$$

where $k \in \mathbb{N}$. To see that T is well defined, observe that the elements x_k, where $k \in \mathbb{N}$, are linearly independent (since $d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) > 0$ for each $k \in \mathbb{N}$). Clearly, T is a linear operator. Let $x = \sum_{k=1}^{n} \beta_k x_k \in Y$, where $\beta_1, \ldots, \beta_n \in \mathbb{K} \setminus \{0\}$ and $x \neq 0$. Observe that

$$|\beta_k \gamma_k| = |\beta_k|\, d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) \leq |\beta_k| \left\| x_k - \sum_{\substack{j=1,\\ j\neq k}}^{n} \frac{-\beta_j}{|\beta_k| \operatorname{sign}\beta_k} x_j \right\|_Y =$$

$$= \left\| |\beta_k|\, x_k - \sum_{\substack{j=1,\\ j\neq k}}^{n} \frac{-\beta_j}{\operatorname{sign}\beta_k} x_j \right\|_Y = \left\| \beta_k x_k + \sum_{\substack{j=1,\\ j\neq k}}^{n} \beta_j x_j \right\|_Y = \left\| \sum_{j=1}^{n} \beta_j x_j \right\|_Y$$

for each $k \in \mathbb{N}$. Therefore the operator T is bounded, since

$$\left\| T\left(\sum_{k=1}^{n} \beta_k x_k \right) \right\|_m = \left\| \sum_{k=1}^{n} \beta_k \gamma_k e_k \right\|_m = \sup_k |\beta_k \gamma_k| \leq \left\| \sum_{j=1}^{n} \beta_j x_j \right\|_Y .$$

By Theorem 5.2.3, there exists a linear norm-preserving extension $S\colon X \to m$ of T. Define $J \in \mathcal{L}(m, \ell_1)$ by

$$Je_k = \frac{1}{2^k} e_k.$$

Put

$$R = JS \in \mathcal{L}(X, \ell_1)$$

and

$$\alpha_k = \frac{1}{2^k}\gamma_k$$

for each $k \in \mathbb{N}$. We complete the proof by observing that

$$Rx_k = JSx_k = JTx_k = \gamma_k Je_k = \frac{1}{2^k}\gamma_k e_k = \alpha_k e_k$$

for each $k \in \mathbb{N}$. □

Definition 5.2.5. Let X be a Banach space. We say that a sequence (x_k) in X is

(i) *autoapproximable* if $d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) = 0$ for each $k \in \mathbb{N}$;

(ii) *autoapproximable modulo* n if there exist indices $\{k_1, \dots, k_n\} \subset \mathbb{N}$ such that $d(x_k, [x_j]_{j\in\mathbb{N}}^{j\neq k}) = 0$ for each $k \in \mathbb{N} \setminus \{k_1, \dots, k_n\}$;

(iii) *almost autoapproximable* if it is autoapproximable modulo n for some $n \in \mathbb{N}$;

(iv) *hereditarily autoapproximable* if every subsequence (y_k) of (x_k) is autoapproximable;

(v) *hereditarily autoapproximable modulo* n if every subsequence (y_k) of (x_k) is autoapproximable modulo n;

(vi) *hereditarily almost autoapproximable* if every subsequence (y_k) of (x_k) is almost autoapproximable.

Theorem 5.2.6. *Take* $\mathbf{g} \in \mathrm{GSeq}$ *such that* $\Psi(\mathbf{g}) \leq \mathbf{K}$. *Then* $\Psi(\mathbf{g}) < \mathbf{K}$ *if and only if every sequence in every component* $\mathbf{g}(X)$ *is hereditarily almost autoapproximable.*

Proof. For the "if" part, let $\mathbf{g} \in \mathrm{GSeq}$ be such that $\Psi(\mathbf{g}) \leq \mathbf{K}$, where every sequence in every component $\mathbf{g}(X)$ is hereditarily almost autoapproximable. Consider the set $A = \left\{\frac{1}{k}e_k\right\} \in \mathbf{K}(\ell_1)$. We show that $A \notin \Psi(\mathbf{g})(\ell_1)$. Assume to the contrary that $A \in \Psi(\mathbf{g})(\ell_1)$. Then the sequence $\left(\frac{1}{k}e_k\right) \subset A$ must contain a hereditarily almost autoapproximable subsequence $\left(\frac{1}{n_k}e_{n_k}\right) \in \mathbf{g}(\ell_1)$. However, this is a contradiction, since it fails to be almost autoapproximable.

For the "only if" part, assume that there exists a Banach space X and a sequence $(x_k) \in \mathbf{g}(X)$ that is not hereditarily almost autoapproximable. Then it contains a subsequence $(y_k) \in \mathbf{g}(X)$ which is not almost autoapproximable. It follows that there exists a countable set of indices I such that

$$d(y_k, [y_j]_{j\in\mathbb{N}}^{j\neq k}) > 0, \text{ if } k \in I.$$

Consider the subsequence $(z_k) = (y_k)_{k\in I}$ of (y_k). Observe that

$$d(z_k, [z_j]_{j\in\mathbb{N}}^{j\neq k}) > 0$$

for each $k \in \mathbb{N}$. But then $\Psi(\mathbf{g}) \geq \mathbf{K}$ by Propositions 5.2.4 and 5.2.1. □

5.3 A system of hereditarily almost autoapproximable sequences

In this section, we provide an example of a system $\mathbf{g} \in \mathrm{GSet}$ which consists of hereditarily almost autoapproximable sequences and satisfies $\mathbf{F} < \Psi(\mathbf{g}) < \mathbf{K}$.

We begin by recalling the notion of a hypercomplete sequence (Definition 5.3.1). We then show that hereditarily autoapproximable sequences are exactly the sequences that are hypercomplete in their own closed span (see Proposition 5.3.2). For our purposes, we prove (using essentially the same proof as in [GL, Theorem 2.1.2]) that a certain sequence $(x_k) \in \mathbf{c}_0(\ell_1)$ is hypercomplete (see Proposition 5.3.5).

We consider a certain system $\mathbf{g} \in \mathrm{GSet}$ which contains both the system $\mathbf{f}$ and the aforementioned sequence (x_k). We observe that $\Psi(\mathbf{g}) \leq \mathbf{K}$ (see Proposition 5.3.8). The rest of this section is devoted to proving that such a system $\mathbf{g}$ indeed consists of only hereditarily almost autoapproximable sequences, which, by Theorem 5.2.6, proves that $\Psi(\mathbf{g}) < \mathbf{K}$.

Definition 5.3.1 ([GL, Definition 2.1.1]). Let X be a Banach space and let (x_k) be a sequence in X. Sequence (x_k) is said to be *hypercomplete* if $\overline{\mathrm{span}}\{x_{n_k}\} = X$ for each subsequence (x_{n_k}). We say that (x_k) is *hypercomplete in its own closed span* if $\overline{\mathrm{span}}\{x_{n_k}\} = \overline{\mathrm{span}}\{x_k\}$ for each subsequence (x_{n_k}).

Proposition 5.3.2. *Let X be a Banach space and let (x_k) be a sequence in X. Then (x_k) is hypercomplete in its own closed span if and only if (x_k) is hereditarily autoapproximable.*

Proof. Let (x_k) be hypercomplete in its own closed span. Take a subsequence (y_k) of (x_k) and fix an index $j \in \mathbb{N}$. By assumption, $\overline{\mathrm{span}}\{y_k \mid k \in \mathbb{N}, k \neq j\} = \overline{\mathrm{span}}\{x_k\}$. But then $y_j \in \overline{\mathrm{span}}\{y_k \mid k \in \mathbb{N}, k \neq j\}$ and therefore

$$d(y_j, [y_k]_{k\in\mathbb{N}}^{k\neq j}) = 0.$$

On the other hand, let (x_k) be hereditarily autoapproximable. It suffices to prove that

$$x_j \in \overline{\mathrm{span}}\{x_{p_k}\},$$

for every subsequence (x_{p_k}) of (x_k) and every element x_j, which does not belong to the sequence (x_{p_k}). Denote by (x_{r_k}) the subsequence of (x_k) satisfying

$$\{r_k \mid k \in \mathbb{N}\} = \{j\} \cup \{p_k \mid k \in \mathbb{N}\}.$$

But then

$$d(x_j, [x_{r_k}]_{k\in\mathbb{N}}^{r_k\neq j}) = 0,$$

since (x_{r_k}) is autoapproximable. We may conclude that

$$x_j \in \overline{\mathrm{span}}\{x_{p_k}\}. \qquad \square$$

It is proven in [GL, Theorem 2.1.2] that every infinite-dimensional Banach space X contains a hypercomplete sequence. Inspired by that result, we consider the following sequence in $\mathbf{c}_0(\ell_1)$, which suits our purposes.

Definition 5.3.3. Put

$$\alpha_j^k = 2^{-k^3-(k+j-1)^2}$$

for each $j, k \in \mathbb{N}$. Define

$$x_k = \sum_{j=1}^{k} \alpha_j^k e_j \in \ell_1$$

for each $k \in \mathbb{N}$.

In the remainder of this section, the above two notations (α_k^j and x_k) are always meant to be understood as defined above. For the sake of clarity, we write out the first few elements of the sequence $(x_k) \in \mathbf{c}_0(\ell_1)$.

$$\begin{array}{rccccc}
x_1 = & (2^{-1-1}, & 0, & 0, & 0, & \ldots). \\
x_2 = & (2^{-8-4}, & 2^{-8-9}, & 0, & 0, & \ldots). \\
x_3 = & (2^{-27-9}, & 2^{-27-16}, & 2^{-27-25}, & 0, & \ldots). \\
x_4 = & (2^{-64-16}, & 2^{-64-25}, & 2^{-64-36}, & 2^{-64-49}, & \ldots). \\
 & & & \ldots & &
\end{array}$$

We will make frequent use of the following lemma.

Lemma 5.3.4. *Let $j \in \mathbb{N}$ and let $j < k$. Then*

$$\sum_{p=j+1}^{k} \frac{\alpha_p^k}{\alpha_j^k} \xrightarrow{k\to\infty} 0.$$

Proof. Observe that

$$\frac{\alpha_{j+1}^k}{\alpha_j^k} = 2^{-(k+j)^2+(k+j-1)^2} = 2^{-(2k+2j-1)}.$$

Let $j+1 \leq p \leq k$. Then

$$\frac{\alpha_p^k}{\alpha_j^k} = \frac{\alpha_{j+1}^k}{\alpha_j^k} \cdot \ldots \cdot \frac{\alpha_p^k}{\alpha_{p-1}^k} = 2^{-(2k+2j-1)} \cdot \ldots \cdot 2^{-(2k+2p-3)} \leq$$
$$\leq 2^{-(2k+2j-1)} \cdot 2^{-1} \cdot \ldots \cdot 2^{-1} = 2^{-(2k+2j-1)} \cdot 2^{-(p-j-1)} = 2^{-(2k+p+j-2)}$$

Therefore

$$\sum_{p=j+1}^{k} \frac{\alpha_p^k}{\alpha_j^k} = 2^{-(2k+2j-1)} + \ldots + 2^{-(3k+j-2)} \leq 2^{-(2k+2j-2)} \xrightarrow{k\to\infty} 0. \qquad \square$$

We prove the following result (using essentially the same proof as in [GL, Theorem 2.1.2]).

Proposition 5.3.5. *The sequence $\overline{x} = (x_k)$ is a hypercomplete sequence in ℓ_1.*

Proof. Let (x_{n_k}) be an arbitrary subsequence of (x_k). Assume that $f \in \ell_1^*$ satisfies $f(x_{n_k}) = 0$ for each $k \in \mathbb{N}$. Hence

$$|\alpha_1^{n_k} f(e_1)| = \left| -\sum_{j=2}^{n_k} \alpha_j^{n_k} f(e_j) \right| \leq \|f\| \sum_{j=2}^{n_k} \alpha_j^{n_k}$$

for each $k \in \mathbb{N}$. According to Lemma 5.3.4,

$$|f(e_1)| \leq \|f\| \sum_{j=2}^{n_k} \frac{\alpha_j^{n_k}}{\alpha_1^{n_k}} \xrightarrow{k\to\infty} 0.$$

Therefore $f(e_1) = 0$. Similarly, we see that

$$|f(e_2)| \leq \|f\| \sum_{j=3}^{n_k} \frac{\alpha_j^{n_k}}{\alpha_2^{n_k}} \xrightarrow{k\to\infty} 0.$$

and thus $f(e_2) = 0$. Continuing in this fashion, we arrive at $f(e_k) = 0$ for each $k \in \mathbb{N}$. By a well-known corollary to the Hahn–Banach theorem, we have $\ell_1 = \overline{\operatorname{span}}\{e_k\} \subset \overline{\operatorname{span}}\{x_{n_k}\}$. In other words, the sequence (x_k) is hypercomplete, as desired. $\square$

Let us define a system $\mathbf{g} \in \mathrm{GSeq}$ which contains both the system $\mathbf{f}$ and the sequence $\overline{x} \in \mathbf{c}_0(\ell_1)$.

Definition 5.3.6. Let X be a Banach space. The component $\mathbf{g}(X)$ consists of sequences of the form

$$(y_k) = \left(T_1 x_{p_k^1} + \ldots + T_N x_{p_k^N} + \beta_k\right),$$

where $N \in \mathbb{N}_0$, $T_1, \ldots, T_N \in \mathcal{L}(\ell_1, X)$, $(\beta_k) \in \mathbf{f}(X)$, and $(p_k^1), \ldots, (p_k^N)$ are increasing sequences of natural numbers.

Throughout the remainder of this section, the system $\mathbf{g}$ is meant to be understood as defined above.

Proposition 5.3.7. *We have* $\mathbf{g} \in \mathrm{GSeq}$.

Proof. Let us verify that $\mathbf{g}$ satisfies the properties $(S_0), \ldots, (S_4)$. The system $\mathbf{g}$ satisfies (S_1), since every sequence in $B_{\mathbb{K}}$ can be written as $(\beta_k \cdot 1)$, where $(\beta_k) \in B_m$ and $1 \in \mathbb{K}$. Clearly, the system $\mathbf{g}$ satisfies (S_0), (S_2), and (S_3). Let $S \in \mathcal{L}(X, Y)$ and $(y_k) \in \mathbf{g}(X)$. Then

$$(Sy_k) = \left(ST_1 x_{p_k^1} + \ldots + ST_N x_{p_k^N} + S\beta_k\right),$$

where $N \in \mathbb{N}_0$, $ST_1, \ldots, ST_N \in \mathcal{L}(\ell_1, Y)$, $(S\beta_k) \in \mathbf{f}(Y)$, and $(p_k^1), \ldots, (p_k^N)$ are increasing sequences of natural numbers. This means the sequence (Sy_k) belongs to the component $\mathbf{g}(Y)$. □

Proposition 5.3.8. *It holds that* $\Psi(\mathbf{g}) \leq \mathbf{K}$.

Proof. By Lemma 2.4.6, it suffices to show that $\mathbf{g} \lesssim \mathbf{c}$, since $\mathbf{K} = \Psi(\mathbf{c})$. Let

$$(y_k) = \left(T_1 x_{p_k^1} + \ldots + T_N x_{p_k^N} + \beta_k\right),$$

where $N \in \mathbb{N}_0$, $T_1, \ldots, T_N \in \mathcal{L}(\ell_1, X)$, $(\beta_k) \in \mathbf{f}(X)$, and $(p_k^1), \ldots, (p_k^N)$ are increasing sequences of natural numbers.

We need to show that the sequence (y_k) contains a convergent subsequence. Observe that

$$(z_k) = (T_1 x_{p_k^1} + \ldots + T_N x_{p_k^N}) \in \mathbf{c}_0(X),$$

because $(x_k) \in \mathbf{c}_0(\ell_1)$. Since $\mathbf{f} \lesssim \mathbf{fc}$, $(\beta_k) \in \mathbf{f}(X)$ contains a convergent subsequence $(\beta_{r_k})_{k\in\mathbb{N}} \in \mathbf{fc}(X)$. It remains to notice that the sequence $(y_{r_k})_{k\in\mathbb{N}} = (z_{r_k} + \beta_{r_k})_{k\in\mathbb{N}}$ is convergent in X. □

It remains to prove the most difficult part of this section: that every sequence (y_k) in $\mathbf{g}(X)$ is indeed hereditarily almost autoapproximable. For this purpose, we define two intermediate systems of sequences $\mathbf{g}_1$ and $\mathbf{g}_2$ such that $\mathbf{g}_1 \subsetneq \mathbf{g}_2 \subsetneq \mathbf{g}$ (see Definitions 5.3.9 and 5.3.10 below).

The remainder of this section is structured as follows. We verify that every sequence in the system $\mathbf{g}_1$ is hereditarily autoapproximable (Corollary 5.3.16). Then we show that every sequence in the system $\mathbf{g}_2$ is hereditarily almost autoapproximable (Corollary 5.3.22). Finally we prove that every sequence in the system $\mathbf{g}$ is hereditarily almost autoapproximable (Theorem 5.3.26).

Let us define of the system of sequences $\mathbf{g}_1$.

Definition 5.3.9. Let X be a Banach space. The component $\mathbf{g}_1(X)$ consists of sequences of the form

$$(y_k) = \left(T_1 x_{s_k^1} + \ldots + T_N x_{s_k^N}\right),$$

where $N \in \mathbb{N}_0$, $T_1, \ldots, T_N \in \mathcal{L}(\ell_1, X)$, and increasing sequences $(s_k^1), \ldots, (s_k^N)$ of natural numbers satisfy $s_k^1 < s_k^2 < \ldots < s_k^N$ for each $k \in \mathbb{N}$.

Note that $\mathbf{g}_1$ fails to be a generating system of sequences, since it does not satisfy property (S_1). Indeed, the constant sequence $e = (1, 1, \ldots, 1, \ldots) \subset B_{\mathbb{K}}$ does not contain a subsequence from the component $\mathbf{g}_1(\mathbb{K})$ (recall that the sequence (x_k) is a null sequence in ℓ_1).

Additionally, $\mathbf{g}_1$ fails to satisfy property (S_2) (see Example 5.3.23 below). It is easy to verify that $\mathbf{g}_1$ satisfies properties (S_0), (S_3), and (S_4).

Next, let us define the system of sequences $\mathbf{g}_2$. Recall that the sum $\mathbf{h} + \mathbf{k}$ for any two systems of sequences $\mathbf{h}$ and $\mathbf{k}$ was defined in 3.3.9.

Definition 5.3.10. $\mathbf{g}_2 = \mathbf{g}_1 + \mathbf{f}$.

Note that the system $\mathbf{g}_2$ resides inside the system $\mathbf{g}$. Clearly, $\mathbf{g}_2$ contains the system $\mathbf{f}$ and therefore satisfies property (S_1). Likewise, it is easy to verify that $\mathbf{g}_2$ satisfies properties (S_0), (S_3), and (S_4). However, it fails to satisfy property (S_2) (see Example 5.3.23 below).

In the forthcoming, it is convenient to consider the systems of sequences $\mathbf{f}^\infty$ and $\mathbf{g}_2^*$, which are the unbounded version of $\mathbf{f}$ and the corresponding version of $\mathbf{g}_2$, respectively.

Definition 5.3.11. Every component $\mathbf{f}^\infty(X)$ of the system $\mathbf{f}^\infty$ consists of *all* sequences which span a finite-dimensional subspace of X.

Clearly, $\mathbf{f}^\infty$ contains the system $\mathbf{f}$. Note that the system $\mathbf{f}^\infty$ violates property (S_0) and therefore $\mathbf{f}^\infty \notin \mathrm{GSeq}$.

Definition 5.3.12. $\mathbf{g}_2^* = \mathbf{g}_1 + \mathbf{f}^\infty$.

Clearly, $\mathbf{g}_2^*$ is not a generating system of sets and $\mathbf{g}_2^* \not\subset \mathbf{g}$; nonetheless, it will be advantageous to employ this system for proving Proposition 5.3.21.

We begin our journey with the following lemma.

Lemma 5.3.13. *Take increasing sequences (v_k) and (w_k) of natural numbers such that $v_k < w_k$ for all $k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Then*

$$\sum_{p=1}^{w_k} \frac{\alpha_p^{w_k}}{\alpha_j^{v_k}} \xrightarrow{k\to\infty} 0.$$

Proof. By definition,

$$\sum_{p=1}^{w_k} \frac{\alpha_p^{w_k}}{\alpha_j^{v_k}} = \frac{2^{-(w_k)^3} \cdot \sum_{p=1}^{w_k} 2^{-(w_k+p-1)^2}}{2^{-(v_k)^3} \cdot 2^{-(v_k+j-1)^2}}.$$

Observe that

$$\sum_{p=1}^{w_k} 2^{-(w_k+p-1)^2} = 2^{-w_k^2} + \ldots + 2^{-(2w_k-1)^2} \leq$$
$$\leq 2^{-w_k^2}(1 + 2^{-1} + \ldots + 2^{-(w_k-1)}) \leq 2 \cdot 2^{-w_k^2} \leq 2 \cdot 2^{-(v_k+1)^2},$$

Therefore

$$\sum_{p=1}^{w_k} \frac{\alpha_p^{w_k}}{\alpha_j^{v_k}} \leq \frac{2^{-(v_k+1)^3} \cdot 2 \cdot 2^{-(v_k+1)^2}}{2^{-(v_k)^3} \cdot 2^{-(v_k+j-1)^2}} = 2^{-(v_k+1)^3+(v_k)^3+1+(v_k+j-1)^2-(v_k+1)^2} =$$
$$2^{-3v_k^2-3v_k-1+1-(2v_k+j)(j-2)} = 2^{-3v_k^2+(1-2j)v_k+2j} \xrightarrow{k\to\infty} 0. \qquad \square$$

Corollary 5.3.14. *Take increasing sequences (v_k) and (w_k) of natural numbers such that $v_k < w_k$ for all $k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Then*

$$\frac{x_{w_k}}{\alpha_j^{v_k}} \xrightarrow{k\to\infty} 0.$$

Proof. By Lemma 5.3.13,

$$\left\| \frac{x_{w_k}}{\alpha_j^{v_k}} \right\| = \left\| \sum_{p=1}^{w_k} \frac{\alpha_p^{w_k}}{\alpha_j^{v_k}} e_j \right\| \leq \sum_{p=1}^{w_k} \frac{\alpha_p^{w_k}}{\alpha_j^{v_k}} \xrightarrow{k\to\infty} 0. \qquad \square$$

The following result is the cornerstone for proving that every sequence in $\mathbf{g}_1(X)$ is hereditarily autoapproximable (Corollary 5.3.16).

Proposition 5.3.15. *Let X be a Banach space and let $(y_k) \in \mathbf{g}_1(X)$. To be specific, let*

$$(y_k) = \left(T_1 x_{s_k^1} + \ldots + T_N x_{s_k^N}\right)$$

be given as in the definition of the system $\mathbf{g}_1$. Then

$$\{T_n e_j \mid 1 \le n \le N,\, j \in \mathbb{N}\} \subset \overline{\operatorname{span}}\{y_k\}.$$

Proof. We use induction over N. Clearly, there is nothing to prove for the base case $N = 0$. We assume that the claim holds for the case $N - 1$ and prove that it holds for the case $N \ge 1$.

Assume that $f \in X^*$ vanishes on the sequence (y_k). Put another way, assume that

$$f(y_k) = \sum_{n=1}^{N} f(T_n x_{s_k^n}) = 0 \tag{5.1}$$

for each $k \in \mathbb{N}$. Recall that

$$f(T_n x_{s_k^n}) = \sum_{j=1}^{s_k^n} \alpha_j^{s_k^n} f(T_n e_j).$$

Hence

$$\left|\alpha_1^{s_k^1} f(T_1 e_1)\right| = \left| -\sum_{j=2}^{s_k^1} \alpha_j^{s_k^1} f(T_1 e_j) - \sum_{n=2}^{N} \sum_{j=1}^{s_k^n} \alpha_j^{s_k^n} f(T_n e_j) \right|$$

for all $k \in \mathbb{N}$. By Lemma 5.3.4 and Lemma 5.3.13 (recall that $s_k^1 < s_k^2 < \ldots < s_k^N$),

$$|f(T_1 e_1)| \le \|f\| \sup_{1 \le n \le N} \|T_n\| \left(\sum_{j=2}^{s_k^1} \frac{\alpha_j^{s_k^1}}{\alpha_1^{s_k^1}} + \sum_{n=2}^{N} \sum_{j=1}^{s_k^n} \frac{\alpha_j^{s_k^n}}{\alpha_1^{s_k^1}} \right) \xrightarrow{k \to \infty} 0 + \sum_{n=2}^{N} 0 = 0.$$

Therefore $f(T_1 e_1) = 0$. Similarly, we see that

$$|f(T_1 e_2)| \le \|f\| \sup_{1 \le n \le N} \|T_n\| \left(\sum_{j=3}^{s_k^1} \frac{\alpha_j^{s_k^1}}{\alpha_2^{s_k^1}} + \sum_{n=2}^{N} \sum_{j=1}^{s_k^n} \frac{\alpha_j^{s_k^n}}{\alpha_2^{s_k^1}} \right) \xrightarrow{k \to \infty} 0 + \sum_{n=2}^{N} 0 = 0$$

and thus $f(T_1 e_2) = 0$. Continuing in this fashion, we arrive at $f(T_1 e_k) = 0$ for each $k \in \mathbb{N}$. We have shown that every functional $f \in X^*$, which vanishes on the sequence (y_k), vanishes also on the set

$$\{T_1 e_j \mid j \in \mathbb{N}\}.$$

By a corollary to the Hahn–Banach theorem, we have

$$\{T_1 e_j \mid j \in \mathbb{N}\} \subset \overline{\operatorname{span}}\{y_k\}. \tag{5.2}$$

If $N = 1$, we have completed the proof.

Assume that $N > 1$. Put

$$(z_k) = \left(T_2 x_{s_k^1} + \ldots + T_N x_{s_k^N}\right) = \left(y_k - T_1 x_{s_k^1}\right) = \left(y_k - \sum_{j=1}^{s_k^1} \alpha_j^{s_k^1} T_1 e_j\right). \tag{5.3}$$

The sequence (z_k) satisfies the assumptions of the induction hypothesis for the case $N - 1$. Therefore

$$\{T_n e_j \mid 2 \le n \le N,\ j \in \mathbb{N}\} \subset \overline{\operatorname{span}}\{y_k\}.$$

Combining the latter observation with (5.2) concludes the proof. $\square$

We are ready to prove the first step in the path to Theorem 5.3.26.

Corollary 5.3.16. *Let X be a Banach space and let $(y_k) \in \mathbf{g}_1(X)$. Then the sequence (y_k) is hereditarily autoapproximable.*

Proof. Take $(y_k) \in \mathbf{g}_1(X)$. To be more specific, take

$$(y_k) = \left(T_1 x_{s_k^1} + \ldots + T_N x_{s_k^N}\right)$$

such that the assumptions from the definition of the system $\mathbf{g}_1$ are satisfied. Let (y_{p_k}) be an arbitrary subsequence of (y_k). Then $(y_{p_k}) \in \mathbf{g}_1(X)$, where

$$(y_{p_k}) = \left(T_1 x_{s_{p_k}^1} + \ldots + T_N x_{s_{p_k}^N}\right).$$

Proposition 5.3.15 yields that

$$Z = \{T_n e_j \mid 1 \le n \le N,\ j \in \mathbb{N}\} \subset \overline{\operatorname{span}}\{y_{p_k}\}.$$

On the other hand, clearly $y_k \in \overline{\operatorname{span}}\, Z$ for each $k \in \mathbb{N}$. Combining the previous two observations yields that

$$\overline{\operatorname{span}}\{y_k\} \subset \overline{\operatorname{span}}\{y_{p_k}\}.$$

Since the converse inclusion holds trivially, we have shown that (y_k) is hypercomplete in its own closed span. By Proposition 5.3.2, it is hereditarily autoapproximable. $\square$

The following example demonstrates that if we drop the assumption that

$$s_k^1 < s_k^2 < \ldots < s_k^N, \ k \in \mathbb{N}$$

from the system $\mathbf{g}_1$ (and correspondingly, from Corollary 5.3.16), then we cannot guarantee that the sequence (y_k) will be hereditarily autoapproximable. However, as Theorem 5.3.26 below demonstrates, we can still prove that any such sequence is hereditarily almost autoapproximable.

Example 5.3.17. Let $X = \ell_1$. Put $T_1 = I_{\ell_1} \in \mathcal{L}(\ell_1, \ell_1)$ and put $T_2 = -I_{\ell_1}$. Let $(s_k^1)_{k\in\mathbb{N}} = (2, 3, 4, \ldots)$ and let $(s_k^2)_{k\in\mathbb{N}} = (1, 3, 4, \ldots)$. Then the sequence $(y_k) = (T_1 x_{s_k^1} + T_2 x_{s_k^2})$ is hereditarily almost autoapproximable, but not hereditarily autoapproximable.

Proof. Write

$$(y_k) = (T_1 x_{s_k^1} + T_2 x_{s_k^2}) = (x_2 - x_1, 0, 0, \ldots).$$

Obviously, this sequence is not autoapproximable – its first element cannot be approximated by the other elements of the sequence. Therefore it is also not hereditarily autoapproximable (since the sequence (y_k) is also its own subsequence).

To see that it is hereditarily almost autoapproximable, take any subsequence (z_k) of (y_k). Clearly, (z_k) becomes autoapproximable after dropping the first element of the sequence (z_k). $\square$

We prove the following easy result.

Lemma 5.3.18. *Let (y_k) be a sequence in a finite-dimensional space Y. Then there exists a subsequence (y_{p_k}) of (y_k) which satisfies one of the following conditions.*

(*a*) *the sequence $(\|y_{p_k}\|)$ tends to infinity and the sequence $\left(\frac{y_{p_k}}{\|y_{p_k}\|}\right)$ converges to some element $z \in Y$;*

(*b*) *the sequence (y_{p_k}) converges to some element $z \in Y$.*

Proof. Suppose that the sequence (y_k) is unbounded. Then it contains a subsequence (y_{n_k}) such that the sequence $(\|y_{n_k}\|)$ tends to infinity and $\|y_{n_k}\| \neq 0$ for each $k \in \mathbb{N}$. Clearly, the sequence $(y_{n_k}/\,\|y_{n_k}\|)$ belongs to a compact set B_Y. We arrive at (a) by noticing that the latter sequence contains a convergent subsequence.

On the other hand, let the sequence (y_k) be bounded; then it indeed contains a convergent subsequence (y_{p_k}), since Y is finite-dimensional. Thus we have arrived at (b). $\square$

We need the following definition and result for proving that each sequence in $\mathbf{g}_2$ is hereditarily almost autoapproximable (Corollary 5.3.22).

Definition 5.3.19. Let X be a Banach space and let W be a finite-dimensional subspace of X. We say that a sequence (y_k) in X is *hypercomplete modulo W in its own closed span*, if every subsequence (y_{p_k}) of (y_k) satisfies

$$\{y_k\} \subset \overline{\operatorname{span}}\{y_{p_k}\} + W.$$

Proposition 5.3.20. *Let X be a Banach space and let W be a subspace of X, where* $\dim W \leq M$. *Let (y_k) be a sequence in X. If (y_k) is hypercomplete modulo W in its own closed span, then (y_k) is hereditarily autoapproximable modulo M.*

Proof. Consider a subsequence (y_{p_k}) of (y_k). Assume to the contrary that there exist indices $j_1, \ldots, j_{M+1}$ such that

$$d(y_{p_{j_1}}, [y_{p_k}]_{k\in\mathbb{N}}^{k\neq j_1}), \ldots, d(y_{p_{j_{M+1}}}, [y_{p_k}]_{k\in\mathbb{N}}^{k\neq j_{M+1}}) > 0.$$

Consider the sequence

$$(y_{p_k})_{k\in\mathbb{N}\setminus\{j_1,\ldots,j_{M+1}\}}.$$

Let us denote

$$Z = \overline{\operatorname{span}}\{y_{p_k} \mid k \in \mathbb{N},\, k \neq j_1, \ldots, k \neq j_{M+1}\}$$

By assumption,

$$\{y_k\} \subset Z + W.$$

Therefore we may write

$$y_{p_{j_m}} = z_m + \mu_m \in Z + W$$

for each $1 \leq m \leq M+1$.

The set $\{\mu_1, \ldots, \mu_{M+1}\} \subset W$ must be linearly dependent, since $\dim W \leq M$. Thus there exists an index $1 \leq r \leq M+1$ such that

$$\mu_r = \sum_{\substack{m=1\\ m\neq r}}^{M+1} \beta_m \mu_m.$$

Put

$$w = y_{p_{j_r}} - \sum_{\substack{m=1\\ m\neq r}}^{M+1} \beta_m y_{p_{j_m}} = z_r + \mu_r - \sum_{\substack{m=1\\ m\neq r}}^{M+1} \beta_m (z_m + \mu_m) = z_r - \sum_{\substack{m=1\\ m\neq r}}^{M+1} \beta_m z_m \in Z.$$

Therefore

$$y_{p_{j_r}} = w + \sum_{\substack{m=1\\ m\neq r}}^{M+1} \beta_m y_{p_{j_m}} \in \overline{\operatorname{span}}\{y_{p_k} \mid k \in \mathbb{N},\, k \neq j_r\},$$

which is a contradiction with our assumption that $d(y_{p_{j_r}}, [y_{p_k}]_{k\in\mathbb{N}}^{k\neq j_r}) > 0$. $\square$

Let us take the next step in the path to proving Theorem 5.3.26.

Proposition 5.3.21. *Let X be a Banach space and let $(y_k) \in \mathbf{g}_2^*(X)$. To be more specific, let*

$$(y_k) = \left(T_1 x_{p_k^1} + \ldots + T_N x_{p_k^N} + \beta_k\right),$$

where $N \in \mathbb{N}_0$, $T_1, \ldots, T_N \in \mathcal{L}(\ell_1, X)$, increasing sequences $(p_k^1), \ldots, (p_k^N)$ of natural numbers satisfy $p_k^1 < p_k^2 < \ldots < p_k^N$ for each $k \in \mathbb{N}$, and the sequence (β_k) spans a finite-dimensional subspace W of X. Then the sequence (y_k) is hypercomplete modulo W in its own closed span.

Proof. We use induction over the pair of variables (M, N), where $\dim W \leq M$. It is fairly obvious that the base case $N = 0$ holds (for any $M \geq 0$). Now consider the base case $M = 0$ (for any $N \geq 0$). Then the assumption simplifies to $(y_k) \in \mathbf{g}_1(X)$ and it becomes necessary to prove that (y_k) is hypercomplete in its own closed span. To do so, it suffices to apply Corollary 5.3.16 and Proposition 5.3.2.

We use the following form of induction. We assume that the claim holds for the cases $(M-1, N)$ and $(M, N-1)$, and show that the claim holds for the case (M, N), where $M, N \geq 1$. Consider a subsequence (y_{s_k}) of (y_k). Then

$$(y_{s_k}) = \left(T_1 x_{p_{s_k}^1} + \ldots + T_N x_{p_{s_k}^N} + \beta_{s_k}\right).$$

We need to prove that

$$\{y_k\} \subset \overline{\operatorname{span}}\{y_{s_k}\} + W.$$

Consider the sequence

$$(\eta_{s_k}^1) = \left(\frac{\beta_{s_k}}{\alpha_1^{p_{s_k}^1}}\right).$$

By Lemma 5.3.18, there exists a subsequence $(r_{1,k})$ of (s_k) such that either

(*a*) the sequence $\left(\left\|\eta_{r_{1,k}}^1\right\|\right)$ tends to infinity and the sequence $\left(\eta_{r_{1,k}}^1 / \left\|\eta_{r_{1,k}}^1\right\|\right)$ converges to some element μ_1 in W;

(*b*) the sequence $\left(\eta^1_{r_{1,k}}\right)$ converges to some element $\mu_1 \in W$.

Let us first consider the case (*a*). Assume that a functional $f \in X^*$ vanishes on the sequence (y_{s_k}). In other words, it satisfies

$$f(y_{s_k}) = \sum_{n=1}^{N} f(T_n x_{p^n_{s_k}}) + f(\beta_{s_k}) = 0$$

for each $k \in \mathbb{N}$.

Then

$$\left|\frac{f\left(\beta_{r_{1,k}}\right)}{\left\|\beta_{r_{1,k}}\right\|}\right| = \frac{\left|f\left(\beta_{r_{1,k}}\right)\right|}{\alpha_1^{p^1_{r_{1,k}}}} \frac{\alpha_1^{p^1_{r_{1,k}}}}{\left\|\beta_{r_{1,k}}\right\|} \leq \|f\| \sup_{1\leq n\leq N} \|T_n\| \left\| \sum_{n=1}^{N} \frac{x_{p^n_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}} \right\| \frac{\alpha_1^{p^1_{r_{1,k}}}}{\left\|\beta_{r_{1,k}}\right\|}. \tag{5.4}$$

Observe that

$$\frac{\alpha_1^{p^1_{r_{1,k}}}}{\left\|\beta_{r_{1,k}}\right\|} \xrightarrow{k\to\infty} 0.$$

By Lemma 5.3.4,

$$\frac{x_{p^1_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}} \xrightarrow{k\to\infty} e_1$$

and by Corollary 5.3.14,

$$\sum_{n=2}^{N} \frac{x_{p^n_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}} \xrightarrow{k\to\infty} 0.$$

Therefore

$$\left\| \sum_{n=1}^{N} \frac{x_{p^n_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}} \right\| \xrightarrow{k\to\infty} \|e_1 + 0\| = 1$$

and (5.4) demonstrates that

$$\frac{f\left(\beta_{r_{1,k}}\right)}{\left\|\beta_{r_{1,k}}\right\|} \xrightarrow{k\to\infty} 0.$$

On the other hand,

$$\frac{f\left(\beta_{r_{1,k}}\right)}{\left\|\beta_{r_{1,k}}\right\|} = \frac{f\left(\eta^1_{r_{1,k}}\right)}{\left\|\eta^1_{r_{1,k}}\right\|} \xrightarrow{k\to\infty} f(\mu_1).$$

Therefore $f(\mu_1) = 0$.

Let W' be a subspace of W satisfying $\dim W' \leq M - 1$ and

$$W = W' + \operatorname{span}\{\mu_1\}.$$

Clearly, there exists a sequence (σ_k) in W' and a scalar sequence (γ_k) such that

$$(\beta_k) = (\sigma_k + \gamma_k \mu_1).$$

Put

$$(w_k) = \left(T_1 x_{p_k^1} + \ldots + T_N x_{p_k^N} + \sigma_k\right).$$

Each functional $f \in X^*$, which vanishes on the sequence (y_{s_k}), vanishes also on the sequence (w_{s_k}), since $f(\mu_1) = 0$. By a corollary to the Hahn–Banach theorem, we have

$$\overline{\operatorname{span}}\{w_{s_k}\} \subset \overline{\operatorname{span}}\{y_{s_k}\}.$$

By the induction hypothesis for the case $(M-1, N)$, (w_k) is hypercomplete modulo W' in its own closed span. Therefore

$$\{w_k\} \subset \overline{\operatorname{span}}\{w_{s_k}\} + W'.$$

We conclude the proof for the case (a) by combining the last two observations and noticing that

$$\{y_k\} \subset \{w_k\} + W \subset \overline{\operatorname{span}}\{w_{s_k}\} + W' + W \subset \overline{\operatorname{span}}\{y_{s_k}\} + W.$$

For this reason, let us only consider the case (b) in the following. Then

$$\frac{\beta_{r_{1,k}}}{\alpha_1^{p^1_{r_{1,k}}}} \xrightarrow{k\to\infty} \mu_1.$$

Assume that a functional $f \in X^*$ vanishes on the sequence (y_{s_k}). In other words, it satisfies

$$f(y_{s_k}) = \sum_{n=1}^{N} f(T_n x_{p^n_{s_k}}) + f(\beta_{s_k}) = 0 \tag{5.5}$$

for each $k \in \mathbb{N}$.

Equation (5.5) and Corollary 5.3.14 allow us to write

$$\left| f\left(\frac{\beta_{r_{1,k}} + T_1 x_{p^1_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}}\right)\right| \leq \|f\| \sup_{1\leq n\leq N} \|T_n\| \sum_{n=2}^{N} \left\|\frac{x_{p^n_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}}\right\| \xrightarrow{k\to\infty} 0.$$

By Lemma 5.3.4,

$$\lim_{k\to\infty} f\left(\frac{\beta_{r_{1,k}} + T_1 x_{p^1_{r_{1,k}}}}{\alpha_1^{p^1_{r_{1,k}}}}\right) = f(\mu_1 + T_1 e_1).$$

Therefore

$$f(\mu_1 + T_1 e_1) = 0. \tag{5.6}$$

Put

$$(z_{1,k}) = \left(y_k - T_1\left(\alpha_1^{p^1_k} e_1\right) - \alpha_1^{p^1_k}\mu_1\right)$$

and

$$(\beta^2_k) = (\beta_k - \alpha_1^{p^1_k}\mu_1).$$

Equations (5.5) and (5.6) demonstrate that every functional $f \in X^*$, which vanishes on the sequence (y_{s_k}), vanishes also on the sequence (z_{1,s_k}). Observe that

$$\begin{aligned} f(z_{1,k}) &= f\left(T_1\left(x_{p^1_k} - \alpha_1^{p^1_k} e_1\right)\right) + \sum_{n=2}^{N} f\left(T_n x_{p^n_k}\right) + f\left(\beta_k - \alpha_1^{p^1_k}\mu_1\right) = \\ &= f\left(\sum_{j=2}^{p^1_k} \alpha_j^{p^1_k} T_1(e_j)\right) + \sum_{n=2}^{N} f\left(T_n x_{p^n_k}\right) + f\left(\beta^2_k\right) \end{aligned} \tag{5.7}$$

for each $k \in \mathbb{N}$.

Consider the sequence

$$(\eta^2_k) = \left(\frac{\beta^2_k}{\alpha_2^{p^1_k}}\right).$$

By Lemma 5.3.18, there exists a subsequence $(r_{2,k})$ of (s_k) such that either

(a) the sequence $\left(\left\|\eta^2_{r_{2,k}}\right\|\right)$ tends to infinity and the sequence $\left(\eta^2_{r_{2,k}}/\left\|\eta^2_{r_{2,k}}\right\|\right)$ converges to some element μ_2 in W;

(b) the sequence $\left(\eta^2_{r_{2,k}}\right)$ converges to some element $\mu_2 \in W$.

Let us first consider the case (a). Assume that a functional $f \in X^*$ vanishes on the sequence $(z_{1,k})$.

Then (5.7) allows us to write

$$\left|\frac{f\left(\beta^2_{r_{2,k}}\right)}{\left\|\beta^2_{r_{2,k}}\right\|}\right| \le \|f\| \sup_{1\le n\le N} \|T_n\| \frac{\alpha_2^{p^1_{r_{2,k}}}}{\left\|\beta^2_{r_{2,k}}\right\|} \left\|\frac{\sum_{j=2}^{p^1_{r_{2,k}}} \alpha_j^{p^1_{r_{2,k}}} e_j + \sum_{n=2}^{N} x_{p^n_{r_{2,k}}}}{\alpha_2^{p^1_{r_{2,k}}}}\right\|. \tag{5.8}$$

Observe that

$$\frac{\alpha_2^{p^1_{r_{2,k}}}}{\left\|\beta^2_{r_{2,k}}\right\|} \xrightarrow{k\to\infty} 0$$

and, by Lemma 5.3.4 and Corollary 5.3.14,

$$\left\|\frac{\sum_{j=2}^{p^1_{r_{2,k}}} \alpha_j^{p^1_{r_{2,k}}} e_j + \sum_{n=2}^{N} x_{p^n_{r_{2,k}}}}{\alpha_2^{p^1_{r_{2,k}}}}\right\| \xrightarrow{k\to\infty} \|e_2 + 0\| = 1.$$

Therefore (5.8) shows that

$$\frac{f\left(\beta^2_{r_{2,k}}\right)}{\left\|\beta^2_{r_{2,k}}\right\|} \xrightarrow{k\to\infty} 0.$$

On the other hand,

$$\frac{f\left(\beta^2_{r_{2,k}}\right)}{\left\|\beta^2_{r_{2,k}}\right\|} = \frac{f\left(\eta^2_{r_{2,k}}\right)}{\left\|\eta^2_{r_{2,k}}\right\|} \xrightarrow{k\to\infty} f(\mu_2).$$

Therefore $f(\mu_2) = 0$.

Similarly to the procedure that we followed in the previous step when we encountered the case (a), we may conclude the proof by applying the induction hypothesis for the case $(M-1, N)$. Because of this, let us only consider the case (b) in the following.

If (b) holds, then

$$\frac{\beta^2_{r_{2,k}}}{\alpha_2^{p^1_{r_{2,k}}}} \xrightarrow{k\to\infty} \mu_2.$$

Assume that a functional $f \in X^*$ vanishes on the sequence (z_{1,s_k}). Observe that (5.7) and Corollary 5.3.14 allow us to write

$$\left| f\left(\frac{\beta^2_{r_{2,k}} + \sum_{j=2}^{p^1_{r_{2,k}}} \alpha_j^{p^1_{r_{2,k}}} T_1 e_j}{\alpha_2^{p^1_{r_{2,k}}}}\right)\right| \le \|f\| \sup_{1\le n\le N} \|T_n\| \sum_{n=2}^{N} \left\|\frac{x_{p^n_{r_{2,k}}}}{\alpha_2^{p^1_{r_{2,k}}}}\right\| \xrightarrow{k\to\infty} 0.$$

By Lemma 5.3.4,

$$\lim_{k\to\infty} f\left(\frac{\beta^2_{r_{2,k}} + \sum_{j=2}^{p^1_{r_{2,k}}} \alpha_j^{p^1_{r_{2,k}}} T_1 e_j}{\alpha_2^{p^1_{r_{2,k}}}}\right) = f(\mu_2 + T_1 e_2).$$

Therefore

$$f(\mu_2 + T_1 e_2) = 0.$$

Continuing in this fashion, we arrive at one of the following results.

(*A*) There exists $\mu \in W$ such that every functional, which vanishes on (y_{s_k}), vanishes on the element μ.

(*B*) For each $j \in \mathbb{N}$, there exists an element $\mu_j \in W$ such that every functional, which vanishes on the sequence (y_{s_k}), vanishes also on the set

$$G = \{\mu_j + T_1 e_j \mid j \in \mathbb{N}\}. \tag{5.9}$$

If (A) holds, then we may complete the proof by referring to the induction hypothesis for the case $(M-1, N)$. Consider the case (B). Then, by the Hahn–Banach theorem, we have

$$\overline{\operatorname{span}}\, G = \overline{\operatorname{span}}\{\mu_j + T_1 e_j \mid j \in \mathbb{N}\} \subset \overline{\operatorname{span}}\{y_{s_k}\}. \tag{5.10}$$

By definition of the sequence x_j and (5.10),

$$\operatorname{span}\{T_1 x_j \mid j \in \mathbb{N}\} + W \subset \operatorname{span}\{\mu_j + T_1 e_j \mid j \in \mathbb{N}\} + W \subset \overline{\operatorname{span}}\{y_{s_k}\} + W.$$

If $N = 1$, then we may complete the proof by observing that

$$\{y_k\} \subset \operatorname{span}\{T_1 x_j \mid j \in \mathbb{N}\} + W.$$

Let us assume that $N > 1$. Recall that

$$T_1 x_{p_k^1} = \sum_{j=1}^{p_k^1} \alpha_j^{p_k^1} T_1 e_j.$$

Put

$$(z_{\infty,k}) = \left(y_k - \sum_{j=1}^{p_k^1} \alpha_j^{p_k^1} T_1 e_j - \sum_{j=1}^{p_k^1} \alpha_j^{p_k^1} \mu_j \right) \tag{5.11}$$

and

$$(\beta_k^\infty) = \left(\beta_k - \sum_{j=1}^{p_k^1} \alpha_j^{p_k^1} \mu_j \right).$$

Then

$$(z_{\infty,k}) = \left(\sum_{n=2}^{N} T_n x_{p_k^n} + \beta_k^\infty \right). \tag{5.12}$$

The sequence $(z_{\infty,k})$ satisfies all of the assumptions of the induction hypothesis for the case $(M, N-1)$, as demonstrated by (5.12). Therefore $(z_{\infty,k})$ is hypercomplete modulo W in its own closed span and we may write

$$\{z_{\infty,k}\} \subset \overline{\operatorname{span}}\{z_{\infty,s_k}\} + W. \tag{5.13}$$

Equation (5.11) with (B) yield that every functional $f \in X^*$, which vanishes on the sequence (y_{s_k}), vanishes also on the sequence (z_{∞,s_k}). By the Hahn–Banach theorem, we have

$$\overline{\operatorname{span}}\{z_{\infty,s_k}\} \subset \overline{\operatorname{span}}\{y_{s_k}\}. \tag{5.14}$$

Combining the observations (5.13) and (5.14), we obtain that

$$\{z_{\infty,k}\} \subset \overline{\operatorname{span}}\{y_{s_k}\} + W. \tag{5.15}$$

By (5.11),

$$\{y_k\} \subset \operatorname{span}\{z_{\infty,k}\} + \operatorname{span} G.$$

It remains to observe that by (5.15) and (5.10),

$$\operatorname{span}\{z_{\infty,k}\} + \operatorname{span} G \subset \overline{\operatorname{span}}\{y_{s_k}\} + W. \qquad \square$$

The following result follows immediately from Propositions 5.3.21 and 5.3.20.

Corollary 5.3.22. *Let X be a Banach space and let $(y_k) \in \mathbf{g}_2(X)$. Then the sequence (y_k) is hereditarily autoapproximable modulo M for some $M \in \mathbb{N}_0$.*

The following example demonstrates that the systems $\mathbf{g}_2$ and $\mathbf{g}$ do not coincide and that the systems $\mathbf{g}_1$ and $\mathbf{g}_2$ fail to satisfy property (S_2).

Example 5.3.23. Define $S\colon \ell_1 \to \ell_1$ and $T\colon \ell_1 \to \ell_1$ by

$$S(e_k) = e_{2k-1}$$

and

$$T(e_k) = e_{2k},$$

where $k \in \mathbb{N}$. Next, define the sequences $(y_k), (z_k) \in \ell_1$ by

$$y_k = Sx_{3k-2} + Tx_{3k-1}$$

and

$$z_k = Sx_{3k} + Tx_{3k},$$

where $k \in \mathbb{N}$.

Then the alternating sequence $(y_1, z_1, y_2, z_2, \dots)$ fails to be hereditarily autoapproximable modulo M for every $M \in \mathbb{N}_0$ and therefore does not belong to $\mathbf{g}_2(\ell_1)$. However, it can be represented as

$$\begin{aligned}(y_1, z_1, y_2, z_2, \dots) &= (Sx_1 + Tx_2, Sx_3 + Tx_3, Sx_4 + Tx_5, Sx_6 + Tx_6, \dots) = \\ &= (Sx_1, Sx_3, Sx_4, Sx_5, \dots) + (Tx_2, Tx_3, Tx_5, Tx_6, \dots) \in \mathbf{g}(\ell_1),\end{aligned}$$

where

$$(Sx_1, Sx_3, Sx_4, Sx_5, \dots) \in \mathbf{g}_1(\ell_1) \subset \mathbf{g}_2(\ell_1)$$

and

$$(Tx_2, Tx_3, Tx_5, Tx_6, \dots) \in \mathbf{g}_1(\ell_1) \subset \mathbf{g}_2(\ell_1).$$

Proof. The only thing to prove is that the sequence $(y_1, z_1, y_2, z_2, \dots)$ fails to be hereditarily autoapproximable modulo M for every $M \in \mathbb{N}$. Assume to the contrary that it is hereditarily autoapproximable modulo M for some $M \in \mathbb{N}$. Consider the subsequence (w_k), defined by

$$w_k = \begin{cases} y_k, & \text{if } k \le M+1, \\ z_k, & \text{if } k > M+1. \end{cases}$$

We will reach the contradiction by proving that

$$d(y_1, [w_k]_{k\in\mathbb{N}}^{k\neq 1}), \dots, d(y_{M+1}, [w_k]_{k\in\mathbb{N}}^{k\neq M+1}) > 0,$$

since this shows that we cannot arrive at an autoapproximable sequence by removing M elements.

Let $m \in \{1, \dots, M+1\}$. We need to prove that

$$y_m \notin \overline{\operatorname{span}}\{w_k \mid k \in \mathbb{N}, k \neq m\}.$$

By the Hahn–Banach theorem, this is equivalent to the fact there exists a functional $f \in \ell_1^*$ such that

$$f(y_1) = \dots = f(y_{m-1}) = 0, \tag{5.16}$$
$$f(y_{m+1}) = \dots = f(y_{M+1}) = 0, \tag{5.17}$$
$$f(z_k) = 0 \text{ for all } k > M+1, \tag{5.18}$$
$$\text{yet } f(y_m) \neq 0. \tag{5.19}$$

Note that condition (5.18) is satisfied if

$$\begin{aligned} f(e_1) &= c_1, \\ f(e_2) &= -c_1, \\ f(e_3) &= c_2, \\ f(e_4) &= -c_2, \\ &\dots \end{aligned}$$

Indeed,

$$f(z_k) = f(Sx_{3k} + Tx_{3k}) = f\left(S\left(\sum_{j=1}^{3k} \alpha_j^{3k} e_j\right) + T\left(\sum_{j=1}^{3k} \alpha_j^{3k} e_j\right)\right) =$$
$$= \sum_{j=1}^{3k} \alpha_j^{3k} f(Se_j + Te_j) = 0,$$

since $f(Se_j + Te_j) = f(e_{2k-1}) + f(e_{2k}) = c_k - c_k = 0$. Furthermore, $f \in \ell_1^* = m$ whenever the scalar sequence (c_k) is bounded. Observe that

$$f(y_k) = f(Sx_{3k-2} + Tx_{3k-1}) = f\left(S\left(\sum_{j=1}^{3k-2} \alpha_j^{3k-2} e_j\right) + T\left(\sum_{j=1}^{3k-1} \alpha_j^{3k-1} e_j\right)\right) =$$
$$= \sum_{j=1}^{3k-2} \alpha_j^{3k-2} f(Se_j) + \sum_{j=1}^{3k-1} \alpha_j^{3k-1} f(Te_j) = \sum_{j=1}^{3k-2} \alpha_j^{3k-2} c_j - \sum_{j=1}^{3k-1} \alpha_j^{3k-1} c_j.$$

To satisfy (5.16), put

$$c_k = 0,$$

where $1 \le k \le 3(m-1) - 1 = 3m - 4$. Then

$$f(y_m) = \sum_{j=1}^{3m-2} \alpha_j^{3m-2} c_j - \sum_{j=1}^{3m-1} \alpha_j^{3m-1} c_j =$$
$$= \alpha_{3m-3}^{3m-2} c_{3m-3} + \alpha_{3m-2}^{3m-2} c_{3m-2} - \alpha_{3m-3}^{3m-1} c_{3m-3} - \alpha_{3m-2}^{3m-1} c_{3m-2} - \alpha_{3m-1}^{3m-1} c_{3m-1}.$$

Clearly, (5.19) is satisfied if we put

$$c_{3m-3} = c_{3m-2} = 0, \;\; c_{3m-1} \neq 0.$$

In order to satisfy (5.17), we follow the iterative process below.

As a first step, put $c_{3m} = c_{3m+1} = 0$ and define the constant c_{3m+2} by

$$c_{3m+2} = \frac{\alpha_{3m-1}^{3m+1} c_{3m-1} - \alpha_{3m-1}^{3m+2} c_{3m-1}}{\alpha_{3m+2}^{3m+2}}.$$

Then

$$f(y_{m+1}) = \alpha_{3m-1}^{3m+1} c_{3m-1} - \alpha_{3m-1}^{3m+2} c_{3m-1} - \alpha_{3m+2}^{3m+2} c_{3m+2} = 0.$$

Similarly, $f(y_{m+2}) = 0$ for $c_{3m+3} = c_{3m+4} = 0$ and a suitable choice of the constant c_{3m+5}. Continuing in this way, we arrive at (5.17), having fixed the constants $c_{3m+6}, \ldots, c_{3M+2}$. It remains to notice that $f \in \ell_1^* = m$ if we put $c_k = 0$ for each $k \ge 3M + 3$. □

For the proof of Theorem 5.3.26, we need to consider the set of all possible *orderings* over the indices $\{1, \ldots, n\}$.

Definition 5.3.24. A finite sequence

$$\omega = (d_1, \chi_1, d_2, \chi_2, \ldots, \chi_{n-1}, d_n)$$

is called an *ordering* (of length n) if the following conditions are satisfied.

(i) $\{d_1, \ldots, d_n\} = \{1, \ldots, n\}$;

(ii) the symbol χ_j is either "$<$" or "$=$" for each $1 \leq j \leq n-1$;

(iii) $d_j < d_{j+1}$ if χ_j is "$=$" for some $1 \leq j \leq n-1$.

Denote the set of all such orderings (of length n) by Ω_n.

An ordering $\omega = (d_1, \chi_1, d_2, \chi_2, \ldots, \chi_{n-1}, d_n)$ is said to *correspond* to a finite sequence $s = (s_1, \ldots, s_n)$ if the following conditions are satisfied.

(i) $s_{d_j} < s_{d_{j+1}}$ whenever χ_j is "$<$";

(ii) $s_{d_j} = s_{d_{j+1}}$ whenever χ_j is "$=$".

Let $\iota \colon \mathbb{N}^n \to \Omega_n$ denote the function which maps each sequence $s = (s_1, \ldots, s_n)$ to a (uniquely determined) ordering $\iota(s)$, which corresponds to s.

Example 5.3.25. Let $s = (7, 3, 3)$. Then $\iota(s) = (2, \text{"="}, 3, \text{"<"}, 1)$.

We are now prepared to tackle the most general case.

Theorem 5.3.26. *Let X be a Banach space and let $(y_k) \in \mathbf{g}(X)$. Then the sequence (y_k) is hereditarily almost autoapproximable.*

Proof. Let (z_k) be an arbitrary subsequence of (y_k). Then $(z_k) \in \mathbf{g}(X)$ and

$$(z_k) = \left(T_1 x_{s_k^1} + \ldots + T_N x_{s_k^N} + \beta_k\right),$$

where $N \in \mathbb{N}_0$, $T_1, \ldots, T_N \in \mathcal{L}(\ell_1, X)$, $\beta_k \in \mathbf{f}(X)$, and $(s_k^1), \ldots, (s_k^N)$ are increasing sequences of natural numbers.

Our goal is to prove that (z_k) is almost autoapproximable. Put

$$s_k = (s_k^1, \ldots, s_k^N).$$

Denote by I_ω the set of indices satisfying the condition that the ordering ω corresponds to the finite sequence s_k. Put another way,

$$I_\omega = \{k \in \mathbb{N} \mid \iota(s_k) = \omega\}.$$

Note that the sets I_ω, where $\omega \in \Omega_N$, form a pairwise disjoint covering of $\mathbb{N}$. In the following, we prove that each set I_ω, where $\omega \in \Omega_N$, contains only a finite amount of indices $\{j_1, \dots, j_r\}$ such that

$$d(z_{j_1}, [z_k]_{k\in\mathbb{N}}^{k\neq j_1}), \dots, d(z_{j_r}, [z_k]_{k\in\mathbb{N}}^{k\neq j_r}) > 0.$$

This completes the proof, since there is only a finite amount of possible orderings of length N.

Let $\omega \in \Omega_N$. There is nothing to prove if $|I_\omega| < \infty$. Therefore let us assume that $|I_\omega| = \infty$. Arrange the elements of I_ω in an increasing order and denote the sequence obtained in this way by (t_k). Put

$$v_k^1 = s_{t_k}^1, \dots, v_k^N = s_{t_k}^N$$

for each $k \in \mathbb{N}$. We aim to prove that the sequence (z_{t_k}) is almost autoapproximable. By definition,

$$\omega = (d_1, \chi_1, d_2, \chi_2, \dots, \chi_{N-1}, d_N).$$

Partition the set $\{d_1, \dots, d_N\}$ into the sets

$$A_1 = \{d_1, \dots, d_{a_1}\}, A_2 = \{d_{a_1+1}, \dots, d_{a_2}\}, \dots, A_l = \{d_{a_{l-1}+1}, \dots, d_{a_l}\},$$

so that the indices d_h and d_{h+1} belong to the same partition if and only if χ_h is "=". This means that

$$x_{v_k^{c_1}} = x_{s_{t_k}^{c_1}} = x_{s_{t_k}^{c_2}} = x_{v_k^{c_2}}$$

whenever the indices c_1 and c_2 belong to the same partition (recall that $t_k \in I_\omega$ for each $k \in \mathbb{N}$). Furthermore,

$$v_k^{c_1} = s_{t_k}^{c_1} < s_{t_k}^{c_2} = v_k^{c_2}$$

for every index $c_1 \in A_h$ and $c_2 \in A_{h+1}$, where $1 \leq h \leq l-1$ and $k \in \mathbb{N}$.

Define the operators $S_1, \dots, S_l \in \mathcal{L}(\ell_1, X)$ by

$$\begin{aligned} S_1 &= T_{d_1} + \ldots + T_{d_{a_1}}, \\ S_2 &= T_{d_{a_1+1}} + \ldots + T_{d_{a_2}}, \\ &\cdots \\ S_l &= T_{d_{a_{l-1}+1}} + \ldots + T_{d_{a_l}}. \end{aligned}$$

Observe that, for each $k \in \mathbb{N}$,

$$T_{d_1} x_{v_k^{d_1}} + \ldots + T_{d_N} x_{v_k^{d_N}} =$$
$$= \left(T_{d_1} x_{v_k^{d_1}} + \ldots + T_{d_{a_1}} x_{v_k^{d_{a_1}}} \right) + \ldots + \left(T_{d_{a_{l-1}+1}} x_{v_k^{d_{a_{l-1}+1}}} + \ldots + T_{d_{a_l}} x_{v_k^{d_{a_l}}} \right) =$$
$$= S_1 x_{v_k^{d_{a_1}}} + \ldots + S_l x_{v_k^{d_{a_l}}},$$

since $x^{c_1}_{v_k} = x^{c_2}_{v_k}$ whenever the indices c_1 and c_2 belong to the same partition. Put $(w_k) = (z_{t_k})$. Note that

$$(w_k) = (S_1 x_{v_k^{d_{a_1}}} + \ldots + S_l x_{v_k^{d_{a_l}}} + \beta_{t_k}) \in \mathbf{g}_2.$$

Indeed, $S_1, \ldots, S_l \in \mathcal{L}(\ell_1, X)$ and $(v_k^{d_{a_1}}), \ldots, (v_k^{d_{a_l}})$ are increasing sequences of natural numbers such that

$$v_k^{d_{a_1}} < v_k^{d_{a_2}} < \ldots < v_k^{d_{a_l}}$$

for each $k \in \mathbb{N}$. By Corollary 5.3.22, the sequence $(w_k) = (z_{t_k})$ is hereditarily almost autoapproximable. Clearly, it is then almost autoapproximable. $\square$

Chapter 6

Representing completely continuous operators through weakly ∞-compact operators

In this chapter, we study the generating system $\mathbf{W}_\infty$ of all relatively weakly ∞-compact sets and the operator ideal $\mathcal{W}_\infty$ of all weakly ∞-compact operators. We show that the operator ideal $\mathcal{W}_\infty$ is a Banach operator ideal. We prove that the equality $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$ holds (even in the context of Banach operator ideals). As a consequence, this provides an alternative proof for the following result from [DFLORT, Theorem 1]: the weak Grothendieck compactness principle holds only in Schur spaces. This chapter is based on [JLO] and [Lil2].

6.1 The Banach operator ideal $\mathcal{W}_\infty$ of weakly ∞-compact operators

Let us begin this section by extending Definition 4.1.2.

Definition 6.1.1. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. For every $\overline{x} = (x_k) \in \boldsymbol{\ell}_p^w(X)$, define an operator $E_{\overline{x}} \in \mathcal{L}(\ell_r, X)$ by

$$E_{\overline{x}}(\overline{\alpha}) = \sum_{k=1}^{\infty} \alpha_k x_k,\ \overline{\alpha} \in \ell_r.$$

Proposition 6.1.2. *Let $1 \leq p \leq \infty$, let $1 \leq r \leq p^*$, and let $\overline{x} = (x_k) \in \boldsymbol{\ell}_p^w(X)$. Then $E_{\overline{x}} \in \mathcal{L}(\ell_r, X)$ is well defined and bounded. Moreover, $\|E_{\overline{x}}\| \leq \|\overline{x}\|_p^w$.*

Proof. By a corollary to the Hahn–Banach theorem, it suffices to show that the set $E_{\overline{x}}(B_{\ell_r})$ is weakly bounded by the constant $\|\overline{x}\|_p^w$. Let $\overline{\alpha} \in B_{\ell_r}$ and let $x^* \in B_{X^*}$. By Hölder's inequality,

$$\|x^*(E_{\overline{x}}(\overline{\alpha}))\| \leq \sum_{k=1}^{\infty} |\alpha_k| \, \|x^*(x_k)\| \leq \|\overline{\alpha}\|_{p^*} \left\|\left(x^*(x_k)\right)\right\|_p \leq \|\overline{\alpha}\|_{p^*} \|\overline{x}\|_p^w . \qquad \square$$

Definition 6.1.3 (see [AO2, Section 4.1])**.** Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A subset K of X is *relatively weakly* (p,r)-compact if

$$K \subset E_{\overline{x}}(B_{\ell_r})$$

for some $(x_k) \in \boldsymbol{\ell}_p^w(X)$ (where $(x_k) \in \mathbf{c}_0^w(X)$ if $p = \infty$).

If $r = p^*$, then we arrive at the notion of a *relatively weakly p-compact* set, which originates from [SK1]. Let $\mathbf{W}_{(p,r)}$ denote the system of all relatively weakly (p,r)-compact sets in all Banach spaces (in [AO2], the notation $\boldsymbol{w}_{(p,r)}$ was used).

Put $\mathcal{W}_{(p,r)} = \Theta(\mathbf{W}_{(p,r)})$. In this chapter, we focus only on the operator ideal $\mathcal{W}_{(\infty,1)}$. Because of this, let us denote $\mathbf{W}_\infty = \mathbf{W}_{(\infty,1)}$ and $\mathcal{W}_\infty = \mathcal{W}_{(\infty,1)}$ for conciseness. We prove that $\mathbf{W}_\infty$ is a generating system of sets (see Proposition 6.1.4) and that $\mathcal{W}_\infty$ is a Banach operator ideal (see Theorem 6.1.6).

Concerning $\mathcal{W}_{(p,1)}$, where $1 \leq p < \infty$, we prove in the next chapter that it is a quasi-Banach operator ideal (see Theorem 7.7.4) via a more general approach (see Theorem 7.6.10).

Proposition 6.1.4. $\mathbf{W}_\infty \in$ GSet.

Proof. Let us verify the conditions of Definition 2.3.1.

(G_0) This follows from Proposition 6.1.2.

(G_1) Put $\overline{\beta} = (1, 0, 0, \ldots) \in \mathbf{c}_0^w(\mathbb{K})$. Then $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_{\ell_1})$.

(G_2) Let $a \in \mathbb{K}$. Put

$$z_k = \begin{cases} 2^{1/r} a x_{(k+1)/2} & \text{if } k \text{ is odd,} \\ 2^{1/r} y_{k/2} & \text{if } k \text{ is even.} \end{cases}$$

We have $(z_k) \in \mathbf{c}_0^w(X)$ and $aG + H \subset E_{\overline{z}}(B_{\ell_1})$. Therefore $aG + H \in \mathbf{W}_\infty(X)$.

(G_3) This is obvious from the definition of a relatively weakly ∞-compact set.

(G_4) Let $T \in \mathcal{L}(X,Y)$. Put $(y_k) = (Tx_k)$. To see that $(y_k) \in \mathbf{c}_0^w(Y)$, let $y^* \in Y^*$. Then $\lim_{k\to\infty} y^*(Tx_k) = 0$, since

$$y^*(Tx_k) = (T^*y^*)(x_k),$$

$(x_k) \in \mathbf{c}_0^w(X)$, and $T^*y^* \in X^*$. Therefore $T(G) \subset E_{\overline{y}}(B_{\ell_1})$. $\square$

It follows that $\mathcal{W}_\infty = \Theta(\mathbf{W}_\infty)$ is a surjective operator ideal. Let us introduce a norm on it.

Definition 6.1.5. Let $T \in \mathcal{W}_\infty(X,Y)$ and put

$$\|T\|_{\mathcal{W}_\infty} = \inf\{\|\overline{x}\|_{\mathbf{c}_0^w(Y)} \mid \overline{x} \in \mathbf{c}_0^w(Y), T(B_X) \subset E_{\overline{x}}(B_{\ell_1})\}.$$

We are ready to prove the main result of this section.

Proposition 6.1.6. *$\mathcal{W}_\infty$ is a Banach operator ideal with the norm $\|\cdot\|_{\mathcal{W}_\infty}$.*

Proof. It is easy to see that $\|I_{\mathbb{K}}\|_{\mathcal{W}_\infty} = 1$. Indeed, put $\overline{\beta} = (1,0,0,\ldots) \in \mathbf{c}_0^w(\mathbb{K})$ and observe that $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_{\ell_1})$. Therefore $\|I_{\mathbb{K}}\|_{\mathcal{W}_\infty} \leq 1$. For the opposite inequality, let $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_{\ell_1})$ for some $\overline{\beta} \in \mathbf{c}_0^w(\mathbb{K})$. Then there exists a sequence $\overline{\alpha} \in B_{\ell_1}$ so that $1 = \sum_{n=1}^\infty \alpha_n\beta_n$. Therefore

$$1 \leq \left|\sum_{n=1}^\infty \alpha_n\beta_n\right| \leq \sum_{n=1}^\infty |\alpha_n\beta_n| \leq \sup_{n\in\mathbb{N}} |\beta_n| \sum_{n=1}^\infty |\alpha_n| \leq \sup_{n\in\mathbb{N}} |\beta_n|,$$

and we have shown that $\|I_{\mathbb{K}}\|_{\mathcal{W}_\infty} \geq 1$.

Let $S,T \in \mathcal{W}_\infty(X,Y)$. We need to prove that $\|S+T\|_{\mathcal{W}_\infty} \leq \|S\|_{\mathcal{W}_\infty} + \|T\|_{\mathcal{W}_\infty}$. For this, take $\varepsilon > 0$ and sequences $\overline{x}$ and $\overline{y}$ from $\mathbf{c}_0^w(Y)$ such that $S(B_X) \subset E_{\overline{x}}(B_{\ell_1})$ and $T(B_X) \subset E_{\overline{y}}(B_{\ell_1})$ with $\|\overline{x}\| \leq (1+\varepsilon)\|S\|_{\mathcal{W}_\infty}$ and $\|\overline{y}\| \leq (1+\varepsilon)\|T\|_{\mathcal{W}_\infty}$.

Assume that $\sup_{n\in\mathbb{N}} \|x_n\| \neq 0$ and that $\sup_{n\in\mathbb{N}} \|y_n\| \neq 0$ (otherwise, either $S = 0$ or $T = 0$, and the proof is trivial). Put

$$q := \frac{\sup_{n\in\mathbb{N}} \|y_n\|}{\sup_{n\in\mathbb{N}} \|x_n\|}.$$

Define $\overline{z} \in \mathbf{c}_0^w(Y)$ by

$$z_n = \begin{cases} (q+1)x_k & \text{if } n = 2k-1, \\ \frac{q+1}{q} y_k & \text{if } n = 2k. \end{cases}$$

We check that

$$\sup_{n\in\mathbb{N}} \|z_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| + \sup_{n\in\mathbb{N}} \|y_n\|.$$

For this purpose, we use the fact that

$$(q+1)\sup_{n\in\mathbb{N}} \|x_n\| = \frac{q+1}{q}\sup_{n\in\mathbb{N}} \|y_n\|.$$

We have

$$\sup_{n\in\mathbb{N}} \|z_n\| = \max\left\{(q+1)\sup_{n\in\mathbb{N}} \|x_n\|, \frac{q+1}{q}\sup_{n\in\mathbb{N}} \|y_n\|\right\} =$$
$$= (q+1)\sup_{n\in\mathbb{N}} \|x_n\| = \sup_{n\in\mathbb{N}} \|x_n\| + q\sup_{n\in\mathbb{N}} \|x_n\| = \sup_{n\in\mathbb{N}} \|x_n\| + \sup_{n\in\mathbb{N}} \|y_n\|.$$

It remains to show that $(S+T)(B_X) \subset E_{\overline{z}}(B_{\ell_1})$. Let $z \in B_X$, let $Sz = E_{\overline{z}}(\overline{\alpha})$ and let $Tz = E_{\overline{z}}(\overline{\beta})$. Define $\overline{\gamma} \in B_{\ell_1}$ by

$$\gamma_n = \begin{cases} \frac{1}{q+1}\alpha_k & \text{if } n = 2k-1, \\ \frac{q}{q+1}\beta_k & \text{if } n = 2k. \end{cases}$$

Then

$$(S+T)(x) = \sum_{n=1}^{\infty} \alpha_n x_n + \sum_{n=1}^{\infty} \beta_n y_n = \sum_{\substack{n=2k-1,\\ k\in\mathbb{N}}} \gamma_n z_n + \sum_{\substack{n=2k,\\ k\in\mathbb{N}}} \gamma_n z_n = \sum_{n=1}^{\infty} \gamma_n z_n = E_{\overline{z}}(\overline{\gamma}).$$

Fix operators $T \in \mathcal{L}(X_0, X)$, $S \in \mathcal{W}_\infty(X, Y)$, and $R \in \mathcal{L}(Y, Y_0)$. We prove that $\|RST\|_{\mathcal{W}_\infty} \leq \|R\| \, \|S\|_{\mathcal{W}_\infty} \|T\|$. Let $\varepsilon > 0$ and let $\overline{y} \in \mathbf{c}_0^w(Y)$ be such that $\sup_{n\in\mathbb{N}} \|y_n\| \leq \|S\|_{\mathcal{W}_\infty} + \varepsilon$ and $S(B_X) \subset E_{\overline{y}}(B_{\ell_1})$. Put $(z_n) := (\|T\| \, Ry_n)$. Then $\overline{z} \in \mathbf{c}_0^w(Y_0)$ and

$$\|\overline{z}\|_{\mathbf{c}_0^w(Y_0)} = \sup_{n\in\mathbb{N}} \|z_n\| \leq \|T\| \, \|R\| \sup_{n\in\mathbb{N}} \|y_n\| \leq \|R\| \left(\|S\|_{\mathcal{W}_\infty} + \varepsilon\right) \|T\| \, .$$

Since

$$RST(B_{X_0}) \subset \|T\| \, RS(B_X) \subset \|T\| \, R(E_{(y_n)}(B_{\ell_1})) = \|T\| \, E_{(Ry_n)}(B_{\ell_1}) = E_{(z_n)}(B_{\ell_1}),$$

we have $\|RST\|_{\mathcal{W}_\infty} \leq \|\overline{z}\|_{\mathbf{c}_0^w(Y_0)}$ and therefore $\|RST\|_{\mathcal{W}_\infty} \leq \|R\| \, \|S\|_{\mathcal{W}_\infty} \|T\|$.

We have shown that $(\mathcal{W}_\infty, \|\cdot\|_{\mathcal{W}_\infty})$ is a normed operator ideal. To prove that it is a Banach operator ideal, we need to show that each of the quasi-normed components $\mathcal{W}_\infty(X, Y)$ is complete. By Theorem 2.2.4, it suffices to show that they are sequentially complete. For this, we show that a series $\sum_{k=1}^{\infty} R_k$ converges in $\big(\mathcal{W}_\infty(X,Y), \|\cdot\|_{\mathcal{W}_\infty}\big)$ whenever $\sum_{k=1}^{\infty} \|R_k\|_{\mathcal{W}_\infty} < \infty$.

By Proposition 2.2.5,

$$\sum_{k=1}^{\infty} \|R_k\| \le \sum_{k=1}^{\infty} \|R_k\|_{\mathcal{W}_\infty} < \infty.$$

Therefore we may define

$$R = \sum_{k=1}^{\infty} R_k \in \mathcal{L}(X,Y).$$

It remains to show that

$$R \in \mathcal{W}_\infty(X,Y) \tag{6.1}$$

and

$$\lim_{n\to\infty} \left\| R - \sum_{k=1}^{n} R_k \right\|_{\mathcal{W}_\infty} = \lim_{n\to\infty} \left\| \sum_{k=n+1}^{\infty} R_k \right\|_{\mathcal{W}_\infty} = 0. \tag{6.2}$$

Let $\varepsilon > 0$. Since the series $\sum_{k=1}^{\infty} R_k$ is absolutely convergent, there exists an increasing sequence (p_m) of natural numbers such that

$$\sum_{k=p_m}^{\infty} \|R_k\|_{\mathcal{W}_\infty} < \frac{\varepsilon}{4^m}$$

for each $m \in \mathbb{N}$. Put

$$S_m = \sum_{k=p_m}^{p_{m+1}-1} R_k$$

and note that $\|S_m\|_{\mathcal{W}_\infty} < \frac{\varepsilon}{4^m}$ for each $m \in \mathbb{N}$.

Clearly,

$$S := \sum_{k=1}^{\infty} S_k = \sum_{k=m_1}^{\infty} R_k = R - \sum_{k=1}^{m_1-1} R_k.$$

We prove both (6.1) and (6.2) by showing that $S \in \mathcal{W}_\infty(X,Y)$ and $\|S\|_{\mathcal{W}_\infty} \le \varepsilon$.

Let $m \in \mathbb{N}$. Since $S_m \in \mathcal{W}_\infty(X,Y)$ and $\|S_m\|_{\mathcal{W}_\infty} \le \frac{\varepsilon}{4^m}$, there exists a sequence $\overline{y}_m = (y_k^m) \in \mathbf{c}_0^w(Y)$ such that $S_m(B_X) \subset E_{\overline{y}_m}(B_{\ell_1})$ and

$$\|\overline{y}_m\| = \sup_{k\in\mathbb{N}} \|y_k^m\| \le \frac{\varepsilon}{4^m}.$$

Define the sequence $\overline{z}$ as any permutation of the following elements:

$$\begin{gathered} 2y_1^1, 2y_2^1, \dots, 2y_n^1, \dots, \\ 4y_1^2, 4y_2^2, \dots, 4y_n^2, \dots, \\ \dots, \\ 2^m y_1^m, 2^m y_2^m, \dots, 2^m y_n^m, \dots, \\ \dots, \end{gathered}$$

where $z_n = 2^{j_n} y_{i_n}^{j_n}$. To prove that $\overline{z} \in \mathbf{c}_0^w(Y)$, we take any $y^* \in Y^*$, let $\delta > 0$, and show that the set $\{n \in \mathbb{N} \mid |y^*(z_n)| > \delta\}$ is finite. It is so because

$$2^m \sup_{k \in \mathbb{N}} \|y_k^m\| \leq \frac{2^m}{4^m} \xrightarrow[m \to \infty]{} 0$$

and each of the sequences $(y_k^m)_{k \in \mathbb{N}}$ contains only finite number of elements such that $|2^m y^*(y_k^m)| > \delta$.

We claim that $S(B_X) \subset E_{\overline{z}}(B_{\ell_1})$. Let $x \in B_X$. For every $m \in \mathbb{N}$, we have $S_m x = \sum_{k \in \mathbb{N}} \alpha_k^m y_k^m$ for some sequence $(\alpha_k^m)_{k \in \mathbb{N}} \in B_{\ell_1}$. Put

$$\beta_n := \frac{1}{2^{j_n}} \alpha_{i_n}^{j_n}.$$

Then $\overline{\beta} \in B_{\ell_1}$, because

$$\sum_{n=1}^{\infty} |\beta_n| = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^m} |\alpha_k^m| \leq 1.$$

We have

$$Sx = \sum_{m=1}^{\infty} S_m x = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k^m y_k^m.$$

The double sequence $(\alpha_k^m y_k^m)_{m,k \in \mathbb{N}}$ is absolutely convergent. Indeed,

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_k^m| \, \|y_k^m\| \leq \sum_{m=1}^{\infty} \|\overline{y}_m\| \sum_{k=1}^{\infty} |\alpha_k^m| \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{4^m} = \frac{\varepsilon}{3}.$$

Therefore it is unconditionally convergent and we may write

$$Sx = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k^m y_k^m = \sum_{n=1}^{\infty} \frac{\alpha_{i_n}^{j_n}}{2^{j_n}} \left(2^{j_n} y_{i_n}^{j_n}\right) = \sum_{n=1}^{\infty} \beta_n z_n.$$

We have shown that $S \in \mathcal{W}_\infty(X, Y)$. Also,

$$\|S\|_{\mathcal{W}_\infty} \leq \|\overline{z}\| = \sup_{m \in \mathbb{N}} \|z_m\| = \sup_{n,k \in \mathbb{N}} \|2^n y_k^n\| \leq \sup_{n \in \mathbb{N}} 2^n \frac{\varepsilon}{4^n} \leq \varepsilon. \qquad \square$$

6.2 Representing completely continuous operators through weakly ∞-compact operators

Recall that $\mathcal{K} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$, but $\mathcal{V}$ and $\mathcal{W}$ are incomparable [Pi1, 1.11.8]. Consider the following well-known equality:

$$\mathcal{V} = \mathcal{K} \circ \mathcal{W}^{-1},$$

which holds in the context of Banach operator ideals (see [Pi1, 3.2.3]). The main tool in the proof of the latter formula in [Pi1, 3.2.3] is the simple fact that *$E_{\overline{x}}\colon c_0^* \to X$ is weak*-to-weak continuous* if $\overline{x} \in \mathbf{c}_0^w(X)$.

The main result of this chapter is that $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. We first prove that this equality holds as operator ideals (see Theorem 6.2.4) and then prove that it holds as Banach operator ideals (see Theorem 6.2.8). As an immediate consequence, this provides an alternative proof for the result [DFLORT, Theorem 1] below. Recall that X has the *Schur property* (is a Schur space) if weakly null sequences in X are norm null.

Theorem 6.2.1 (see [DFLORT, Theorem 1])**.** *Every weakly compact subset of a Banach space X is contained in the closed convex hull of a weakly null sequence if and only if X has the Schur property.*

In [DFLORT], this result was concisely described as follows: *the weak Grothendieck compactness principle holds only in Schur spaces.* In [DFLORT], Schauder basis theory was used for the proof. In contrast, our method of proof relies on the Davis–Figiel–Johnson–Pełczyński factorization theorem [DFJP].

Let us start with the following well-known fact, for which we include a proof for completeness.

Proposition 6.2.2. *If (x_n) is a weakly null sequence in a Banach space X, then $E_{\overline{x}}(B_{\ell_1})$ is weakly compact and coincides with the closed absolutely convex hull of (x_n).*

Proof. (cf. [AO1, proof of the "if" part of Theorem 3]). The set $E_{\overline{x}}(B_{\ell_1})$ is clearly absolutely convex. It is also weakly compact because $E_{\overline{x}}\colon c_0^* \to X$ is weak*-to-weak continuous and $B_{\ell_1} = B_{c_0^*}$ is weak* compact by the Banach–Alaoglu theorem. It is closed, since weakly closed convex sets are closed in the norm topology. Hence, $E_{\overline{x}}(B_{\ell_1})$ is a closed absolutely convex subset of X containing (x_n). Since $E_{\overline{x}}(B_{\ell_1})$ is obviously contained in the closed absolutely convex hull of (x_n), it coincides with the latter set. $\square$

Let X be a Banach space. By the Grothendieck compactness principle, every relatively compact set is relatively weakly ∞-compact. This observation along with Proposition 6.2.2 yields the following (known) result.

Corollary 6.2.3. *$\mathcal{K} \subset \mathcal{W}_\infty \subset \mathcal{W}$ as operator ideals.*

From the proof of Proposition 6.3.1 below, it can be seen that these inclusions are strict.

Theorem 6.2.4. *The equality $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$ holds in the context of operator ideals.*

Proof. Since $\mathcal{K} \subset \mathcal{W}_\infty$, we clearly have that $\mathcal{K} \circ \mathcal{W}^{-1} \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. Since also $\mathcal{V} \subset \mathcal{K} \circ \mathcal{W}^{-1}$ (this is the obvious "part" of the equality $\mathcal{V} = \mathcal{K} \circ \mathcal{W}^{-1}$),

$$\mathcal{V} \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1}.$$

It remains to show that

$$\mathcal{W}_\infty \circ \mathcal{W}^{-1} \subset \mathcal{V}.$$

Let X and Y be Banach spaces and let $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X, Y)$. Assume for contradiction that $T \notin \mathcal{V}(X, Y)$. Then there exists a weakly null sequence (x_n) in X such that (Tx_n) is not a null sequence in Y. Passing to a subsequence of (x_n), we may assume that $\|Tx_n\| \geq \delta, n \in \mathbb{N}$, for some $\delta > 0$. Hence, (Tx_n) is not relatively compact.

Since $E_{\overline{x}} \in \mathcal{W}(\ell_1, X)$ (see Proposition 6.2.2), the Davis–Figiel–Johnson–Pełczyński factorization theorem [DFJP] yields a reflexive space R and weakly compact operators $S\colon \ell_1 \to R$ with $\|S\| = 1$ and $J\colon R \to X$ such that $E_{\overline{x}} = JS$. From the definition of $\mathcal{W}_\infty \circ \mathcal{W}^{-1}$, we get that $TJ \in \mathcal{W}_\infty(R, Y)$ because $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X, Y)$ and $J \in \mathcal{W}(R, X)$. Hence, there exists a weakly null sequence $\overline{y}$ in Y such that $TJ(B_R) \subset E_{\overline{y}}(B_{\ell_1})$. In particular, $Tx_n = TE_{\overline{x}}e_n = TJSe_n \in E_{\overline{y}}(B_{\ell_1})$, $n \in \mathbb{N}$, where (e_n) is the unit vector basis in ℓ_1.

Denote by $\overline{E}_{\overline{y}}$ the injective associate of $E_{\overline{y}}$, which means that $E_{\overline{y}} = \overline{E}_{\overline{y}}q$, where $q\colon \ell_1 \to Z := \ell_1 / \ker E_{\overline{y}}$ is the quotient mapping. Since $\operatorname{ran} TJ \subset \operatorname{ran} E_{\overline{y}} = \operatorname{ran} \overline{E}_{\overline{y}}$, we can consider the linear operator $\overline{E}_{\overline{y}}^{-1}TJ\colon R \to Z$. This operator is bounded: if $r \in B_R$, then $TJr = E_{\overline{y}}\alpha$ for some $\alpha \in B_{\ell_1}$ and $\|\overline{E}_{\overline{y}}^{-1}TJr\| = \|q\alpha\| \leq 1$.

We prove that Z has the Schur property. Observe that $\ker E_{\overline{y}}$ is weak* closed in $\ell_1 = c_0^*$ (because $E_{\overline{y}}$ is weak*-to-weak continuous). Put $W = (\ker E_{\overline{y}})_\perp$. Then $W^\perp = \ker E_{\overline{y}}$, since $((\ker E_{\overline{y}})_\perp)^\perp = \overline{\ker E_{\overline{y}}}^{w^*}$ by the bipolar theorem and $\ker E_{\overline{y}}$ is w^*-closed. Therefore $W^* = \ell_1 / W^\perp = \ell_1 / \ker E_{\overline{y}} = Z$. This proves that Z has the

Schur property, since by Grothendieck's result [G1, Theorem 10] (see also Remark 6.2.6 below), the dual W^* of any closed subspace W of c_0 has the Schur property.

Since R is reflexive and Z has the Schur property, $\mathcal{L}(R, Z) = \mathcal{K}(R, Z)$. In particular, $\overline{E}_{\overline{y}}^{-1}TJ$ and therefore also $\overline{E}_{\overline{y}}\overline{E}_{\overline{y}}^{-1}TJ = TJ$ are compact operators. It follows that $(Tx_n) = (TJSe_n) \subset (TJ)(B_R)$ is relatively compact, a contradiction that completes the proof. $\square$

Recall the following definition.

Definition 6.2.5. A Banach space X is said to have the *Dunford–Pettis property* if for each Banach space Y, every weakly compact linear operator $T\colon X \to Y$ is completely continuous.

For equivalent characterizations of the Dunford–Pettis property, see, e.g., [Di1, Theorem 1].

Remark 6.2.6. Let W be a closed subspace of c_0. To prove that the dual W^* has the Schur property, Grothendieck [G1, Theorem 10] first establishes that W has the Dunford–Pettis property (DPP). Grothendieck's easy and beautiful proof can be found in Diestel's survey article [Di1, pp. 25–26, see also Theorem 4]. Since W does not contain a copy of ℓ_1, relying on Rosenthal's ℓ_1 theorem, Diestel [Di1, Theorem 3] quickly concludes that W^* has the Schur property. Let us provide a version of Grothendieck's proof [G1, pp. 171–172], showing that the DPP of W implies that W^* has the Schur property.

Proof. To show that W^* has the Schur property, let (w_n^*) be a weakly null sequence in W^*. Consider the operator $S \in \mathcal{L}(W, c_0)$ defined by $Sw = (\langle w_n^*, w\rangle)$. It is straightforward to verify that $S^* = E_{(w_n^*)}$. Since (w_n^*) is weakly null in W^*, $E_{(w_n^*)}$ is weak*-to-weak continuous and thus $S^* \in \mathcal{W}(c_0^*, W^*)$. By Gantmacher's theorem, $S \in \mathcal{W}(W, c_0)$.

The Dunford–Pettis property of W yields that $S \in \mathcal{V}(W, c_0)$. By [Sw, p. 398, Proposition 3], S is compact, since it is a completely continuous operator and the dual of its domain, W^*, is separable. Therefore, by Schauder's theorem, we have $S^* \in \mathcal{K}(\ell_1, W^*)$.

Assume to the contrary that the sequence (w_n^*) is not null. Then it contains a subsequence $(w_{n_k}^*)$ satisfying that $\|w_{n_k}^*\| > \delta$ for each $k \in \mathbb{N}$, where $\delta > 0$. Since $(w_n^*) \subset E_{(w_n^*)}(B_{\ell_1})$, this subsequence $(w_{n_k}^*)$ must contain a subsequence converging to an element $w^* \in W^*$, where $\|w^*\| \geq \delta$. But this is a contradiction with the fact that the sequence (w_n^*) is weakly null. $\square$

Let $\varepsilon > 0$ and let K be a relatively compact set in X. It is well known and easy to see that in the proof of the Grothendieck compactness principle, one may choose the sequence $\overline{x} = (x_k) \in \mathbf{c}_0(X)$ so that

$$\sup_{n\in\mathbb{N}} \|x_n\| \le \sup_{x\in K} \|x\| + \varepsilon,$$

where $K \subset E_{\overline{x}}(B_{\ell_1})$.

This allows us to make the following observation.

Proposition 6.2.7. *Let $T \in \mathcal{K}(X,Y)$. Then $\|T\|_{\mathcal{W}_\infty} = \|T\|$.*

Proof. The Grothendieck compactness principle yields that

$$\|T\| = \inf\{\sup_{n\in\mathbb{N}} \|x_n\| \mid \overline{x} \in \mathbf{c}_0(Y),\, T(B_X) \subset E_{\overline{x}}(B_{\ell_1})\}.$$

Therefore $\|T\|_{\mathcal{W}_\infty} \le \|T\|$, since infimum in the definition of $\|T\|_{\mathcal{W}_\infty}$ is taken over a larger set than in the previous formula. On the other hand, $\|T\| \le \|T\|_{\mathcal{W}_\infty}$ because $\mathcal{W}_\infty$ is a Banach operator ideal. □

For the proof of the next theorem, recall that $\mathcal{K} = \mathcal{V} \circ \mathcal{W}$ as Banach operator ideals (see [Pi1, 3.1.3]).

Theorem 6.2.8. *The equality $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$ holds in the context of Banach operator ideals.*

Proof. Fix an operator $T \in \mathcal{V}(X,Y) = \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X,Y)$. By definition,

$$\|T\|_{\mathcal{W}_\infty \circ \mathcal{W}^{-1}} = \sup\{\|TW\|_{\mathcal{W}_\infty} \mid W \in \mathcal{W}(X_0, X), \|W\|_{\mathcal{W}} \le 1\},$$

where the supremum is taken over all Banach spaces X_0.

Therefore $TW \in \mathcal{V} \circ \mathcal{W}(X_0, Y) = \mathcal{K}(X_0, Y)$ for any $W \in \mathcal{W}(X_0, X)$. According to Proposition 6.2.7,

$$\|TW\|_{\mathcal{W}_\infty} = \|TW\| = \|TW\|_{\mathcal{K}}.$$

Therefore

$$\begin{aligned}\|T\|_{\mathcal{W}_\infty \circ \mathcal{W}^{-1}} &= \sup\{\|TW\|_{\mathcal{W}_\infty} \mid W \in \mathcal{W}(X_0, X), \|W\|_{\mathcal{W}} \le 1\} = \\ &= \sup\{\|TW\|_{\mathcal{K}} \mid W \in \mathcal{W}(X_0, X), \|W\|_{\mathcal{W}} \le 1\} = \|T\|_{\mathcal{K}\circ\mathcal{W}^{-1}} = \|T\|_{\mathcal{V}}.\end{aligned}$$

□

6.3 Applications

It is well known that $\mathcal{K} \subset \mathcal{V}$. As we see now, $\mathcal{W}_\infty$ lies strictly between $\mathcal{K}$ and $\mathcal{V}$.

Proposition 6.3.1. *$\mathcal{K} \subset \mathcal{W}_\infty \subset \mathcal{V}$ as Banach operator ideals, and both of the inclusions are strict.*

Proof. Observe that $\mathcal{W}_\infty \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1} = \mathcal{V}$ by Theorems 6.2.4 and 6.2.8. The inclusion $\mathcal{K} \subset \mathcal{W}_\infty$ was observed in Corollary 6.2.3 (as operator ideals) and Proposition 6.2.7 (as Banach operator ideals).

To see that $\mathcal{K} \neq \mathcal{W}_\infty$, consider the identity embedding $j\colon \ell_1 \to c_0$ that is not compact but is weakly ∞-compact, because $j = E_{(e_n)}$, where (e_n) is the unit vector basis of c_0. On the other hand, the identity operator on ℓ_1 is completely continuous (because ℓ_1 has the Schur property) but since it is not weakly compact, it is not weakly ∞-compact either (recall that $\mathcal{W}_\infty \subset \mathcal{W}$). (Another way to see that $\mathcal{W}_\infty \neq \mathcal{V}$ is to use that $\mathcal{W}_\infty = \mathcal{W}_\infty^{\mathrm{sur}}$, but $\mathcal{V} \neq \mathcal{V}^{\mathrm{sur}} = \mathcal{L}$.) Now it is easy to see that the inclusion $\mathcal{W}_\infty \subset \mathcal{W}$ in Corollary 6.2.3 is strict: the identity operator on ℓ_2 is weakly compact but since it is not completely continuous, it is not weakly ∞-compact. $\square$

Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. It is well known (and clear thanks to the Eberlein–Šmulian theorem) that $T \in \mathcal{V}(X, Y)$ *if and only if T takes relatively weakly compact subsets of X into relatively compact subsets of Y.*

Theorem 6.3.2. *Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Then $T \in \mathcal{V}(X, Y)$ if and only if T takes relatively weakly compact subsets of X into relatively weakly ∞-compact subsets of Y.*

Proof. The "only if" part is obvious because relatively compact sets are relatively weakly ∞-compact. From the definition of $\mathcal{W}_\infty \circ \mathcal{W}^{-1}$, it is clear that if T takes relatively weakly compact sets into relatively weakly ∞-compact sets, then $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X, Y)$. By Theorem 6.2.4, this means that $T \in \mathcal{V}(X, Y)$. $\square$

Let $\mathcal{A}$ be an operator ideal. Recall that the *space ideal* $\mathrm{Space}(\mathcal{A})$ is defined as the class of all Banach spaces X such that the identity operator on X belongs to $\mathcal{A}(X, X)$. If $\mathcal{A}$ and $\mathcal{B}$ are operator ideals, then obviously $X \in \mathrm{Space}(\mathcal{A} \circ \mathcal{B}^{-1})$ if and only if $\mathcal{B}(Z, X) \subset \mathcal{A}(Z, X)$ for all Banach spaces Z.

From the definitions, it is clear that $\mathrm{Space}(\mathcal{V})$ is the class of all Banach spaces with the Schur property. Theorem 6.2.4 yields that

$$\mathrm{Space}(\mathcal{V}) = \mathrm{Space}(\mathcal{W}_\infty \circ \mathcal{W}^{-1}).$$

This can be reformulated as follows. Note that the equivalence (i) $\Leftrightarrow$ (ii) below is precisely Corollary 6.2.1 and, as was mentioned in the beginning of Section 6.2, it is due to [DFLORT, Theorem 1].

Theorem 6.3.3. *For a Banach space X, the following statements are equivalent:*

(i) *X has the Schur property;*

(ii) *the weak Grothendieck compactness principle holds in X;*

(iii) $\mathcal{W}(Z, X) \subset \mathcal{W}_\infty(Z, X)$ *for all Banach spaces Z.*

Proof. We already observed that (i) $\Leftrightarrow$ (iii) thanks to Theorem 6.2.4. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. (But (i) $\Leftrightarrow$ (ii) is also the special case of Theorem 6.3.2, when T is the identity operator on X.) $\square$

Remark 6.3.4. By the Davis–Figiel–Johnson–Pełczyński factorization theorem, (iii) is equivalent to

(iv) *all injective operators from reflexive Banach spaces to X are weakly ∞-compact.*

Chapter 7

Constructing quasi-Banach operator ideals from sequence spaces and systems of sequences

In this chapter, we propose a general method for constructing generating systems of sets and quasi-Banach operator ideals. This method is inspired by the construction of generating systems of sets $\mathbf{K}_{(p,r)}$ and $\mathbf{W}_\infty$, and the corresponding operator ideals $\mathcal{K}_{(p,r)}$ and $\mathcal{W}_\infty$. This construction produces a quasi-Banach operator ideal from a BK-space g and a normed system of sequences $\mathbf{h}$, provided that g and $\mathbf{h}$ satisfy certain criteria. Among other examples, we prove that the operator ideals $\mathcal{W}_{(p,1)}$ and $\mathcal{U}_{(p,1)}$ are quasi-Banach operator ideals (for $1 \leq p < \infty$).

7.1 Summary of this chapter

The aim of this chapter is to provide a new method for constructing generating systems of sets and quasi-Banach operator ideals. This method relies on the classical theory of BK-spaces and their Köthe-duals. The definitions and relevant aspects of the latter theory are given in Section 7.2.

In Section 7.3, we give the definition of a normed system of sequences $\mathbf{h}$ (see Definition 7.3.8) and provide several examples of them (see Examples 7.3.11, 7.3.12, and 7.3.13).

Section 7.4 starts with the definition of a g-compatible (for a BK-space g) system of sequences $\mathbf{h}$ (see Definition 7.4.3). Then we construct the system of sets $\Lambda[g, \mathbf{h}]$ (see Definition 7.4.4) and prove that it is a generating system of sets, provided that g and $\mathbf{h}$ satisfy certain properties (see Theorem 7.4.9). We observe that $\mathbf{K}_{(\infty,1)} = \Lambda[\ell_1, \mathbf{c}_0]$ and $\mathbf{K}_{(p,r)} = \Lambda[\ell_r, \boldsymbol{\ell}_p]$, where $1 \leq p < \infty$ and $1 \leq r \leq p^*$ (see Proposition 7.4.5).

Another method for constructing systems of sets was put forth by M. Gupta and A. Bhar in [GB], encompassing the system $\mathbf{K}_{(p,p^*)}$ of relatively (p, p^*)-compact sets. Although there are similarities between these approaches, there are also substantial differences (see Remark 7.4.6). Nonetheless, we prove in Proposition 7.4.7 that our construction encompasses all of the systems of sets produced by the approach in [GB].

In Section 7.5, we study the surjective operator ideal $\Theta(\Lambda[g, \mathbf{h}])$ (we use the notation $\Theta_\Lambda[g, \mathbf{h}]$ for conciseness). We can equip it with a quasi-norm $\|\cdot\|_{[g,\mathbf{h}]}$, provided that g and $\mathbf{h}$ satisfy certain properties (see Theorem 7.5.1). For the operator ideal $\mathcal{K}_{(p,r)}$, this quasi-norm coincides with the s-norm $\|\cdot\|_{\mathcal{K}_{(p,r)}}$ from Theorem 4.2.12 and for the operator ideal $\mathcal{W}_\infty$, it coincides with the norm $\|\cdot\|_{\mathcal{W}_\infty}$ from Definition 6.1.5.

It was proven in [GB, Theorem 3.10], that under suitable assumptions on a BK-space λ, the system K_λ of λ-compact operators (a notion, which encompasses the Banach operator ideal $\mathcal{K}_{(p,p^*)}$, where $1 \leq p \leq \infty$) is a quasi-normed operator ideal with a quasi-constant of 8. In Section 7.8, we prove that this result is encompassed by Theorem 7.5.1.

In Section 7.6, we prove that under some additional assumptions on g and $\mathbf{h}$, the operator ideal $\Theta_\Lambda[g, \mathbf{h}]$ becomes a quasi-Banach operator ideal (see Theorem 7.6.10).

In Section 7.7, we provide several examples of quasi-Banach operator ideals produced by our construction. We start with the prototypical example of $\mathcal{K}_{(p,r)}$ (see Proposition 7.7.1). Another example, the quasi-Banach operator ideal $\Theta_\Lambda[\ell_1, \mathbf{m}]$ (see Theorem 7.7.2), is different from any of the operator ideals $\mathcal{K}_{(p,r)}$, since it contains non-compact operators (see Proposition 7.4.12).

In [AO2], the notion of unconditionally (p, r)-compact operators was introduced. The operator ideal of unconditionally (p, r)-compact operators is denoted by $\mathcal{U}_{(p,r)}$. We prove that $\mathcal{W}_{(p,1)} = \Theta_\Lambda[\ell_1, \boldsymbol{\ell}_p^w]$ and $\mathcal{U}_{(p,1)} = \Theta_\Lambda[\ell_1, \boldsymbol{\ell}_p^u]$ are quasi-Banach operator ideals (see Theorems 7.7.4 and 7.7.5, respectively). Finally, we recall some known results about the Banach operator ideals $\mathcal{W}_{(p,p^*)}$ and $\mathcal{U}_{(p,p^*)}$ (see Remarks 7.7.6 and 7.7.7).

We conclude this summary with a few general remarks. On the positive side, the construction provided in this chapter simplifies the process of showing that a given collection of operators $\Theta_\Lambda[g, \mathbf{h}]$ is a quasi-Banach operator ideal. Indeed, what remains is to ensure that the given g and $\mathbf{h}$ satisfy the necessary properties (see the various examples in Section 7.7), while the "heavy-weight lifting" is done by Theorems 7.4.9, 7.5.1, and 7.6.10. However, there is a catch. In all of the examples provided in Section 7.7, the obtained quasi-constant $\varkappa$ is equal to 2. However, this is not optimal in general; for example, in the case of the operator ideal $\mathcal{K}_{(p,r)}$ of (p,r)-compact operators, this constant seems to be optimal only if $p = r = 1$ (see remarks following Theorem 4.2.10). The quasi-constant $\varkappa = 2$ is also suboptimal for the operator ideal $\mathcal{W}_{(\infty,1)}$, since it is even a Banach operator ideal (recall Proposition 6.1.6 and compare it to Proposition 7.7.3). For the time being, we are not aware of a method for calculating the best quasi-constant for a given operator ideal $\Theta_\Lambda[g, \mathbf{h}]$ which would only depend on the immediate properties of g and $\mathbf{h}$.

7.2 Sequence spaces and their Köthe-duals

We begin by giving some basic definitions related to sequence spaces. Let ω denote the vector space of all sequences over the field $\mathbb{K}$. Denote by fin the smallest subspace of ω containing all of the unit vectors e_n; i.e., $fin = \operatorname{span}\{e_n \mid n \in \mathbb{N}\}$ (in literature, the symbol ϕ is commonly used instead). A linear subspace g of ω is called a *sequence space*, if it contains the vector space fin (the latter assumption is for our convenience only, as we will need to consider only such spaces; in literature, the definitions are usually given without this assumption).

The following definition is due to G. Köthe and O. Toeplitz [KT]. We refer the reader to [K, §30] for an overview of properties of sequence spaces and their α-duals. A wide range of examples of sequence spaces may be found, e.g., in [K, §30], [KG, p. 48], and [Bo].

Definition 7.2.1. Let g be a sequence space. The *α-dual* or *Köthe-dual* of g is the space $g^\times$ defined as

$$g^\times = \left\{ \overline{\beta} = (\beta_k) \in \omega \colon \sum_{k=1}^{\infty} |\alpha_k \beta_k| < \infty, \text{ for all } \overline{\alpha} = (\alpha_k) \in g \right\}.$$

In the literature, the notations g^α and $g^\times$ are used interchangeably.

Definition 7.2.2. A sequence space g is said to be *solid* (or normal) if for every $(\alpha_k) \in g$ it also contains every sequence (β_k) satisfying $|\beta_k| \leq |\alpha_k|$ for each $k \in \mathbb{N}$.

The following result is well known and easy to see.

Proposition 7.2.3. *Let g be a sequence space. Then the space $g^\times$ is a solid sequence space.*

Definition 7.2.4. The k-th *coordinate map* $P_k\colon g \to \mathbb{K}$ is defined by $P_k(\overline{\alpha}) = \alpha_k$, where $\overline{\alpha} \in g$ and $k \in \mathbb{N}$.

A sequence space g equipped with a norm is called a *normed sequence space.* If g is complete with respect to this norm, then it is called a *Banach sequence space.*

Definition 7.2.5 (see [Bo, Definition 7.3.1])**.** A Banach sequence space is said to be a *BK-space* provided that each of the projection maps P_k is continuous.

The following definition and theorem are given in [W, Theorem 4.3.15] in the more general setting of multiplier spaces $M(g,h)$. The latter notion encompasses the notion of the Köthe-dual $g^\times$ for $h = \ell_1$.

Definition 7.2.6. For a BK-space $(g, \|\cdot\|_g)$, the *dual norm* on $g^\times$ is defined as

$$\left\|\overline{\beta}\right\|_{g^\times} = \sup\left\{ \sum_{k=1}^{\infty} |\alpha_k|\,|\beta_k| \;\middle|\; \overline{\alpha} \in g,\ \|\overline{\alpha}\|_g \leq 1 \right\}.$$

Theorem 7.2.7 (see [W, Theorem 4.3.15])**.** *Let $(g, \|\cdot\|_g)$ be a BK-space. Then the norm $\|\cdot\|_{g^\times}$ is well defined and the space $(g^\times, \|\cdot\|_{g^\times})$ is a BK-space.*

Definition 7.2.8 (see [GB, p. 357])**.** Let g be a sequence space. A norm $\|\cdot\|_g$ on g is said to be *monotone* if $\|\overline{\alpha}\|_g \leq \left\|\overline{\beta}\right\|_g$ for every $\overline{\alpha}$, $\overline{\beta} \in g$ with $|\alpha_k| \leq |\beta_k|$ for each $k \in \mathbb{N}$.

The following result is well known and easy to see.

Proposition 7.2.9. *Let g be a BK-space. The norm $\|\cdot\|_{g^\times}$ of $g^\times$ is monotone.*

Each of the classical sequence spaces ℓ_p (where $1 \leq p \leq \infty$) and c_0 is a BK-space. Their Köthe-duals are given as follows.

Example 7.2.10. Let $1 \leq p \leq \infty$. Then

(i) $\ell_p^\times = \ell_{p^*}$;

(ii) $c_0^\times = \ell_1$

as BK-spaces.

In [K, §30], the following observations and definition are given.

Proposition 7.2.11. *Let g and h be sequence spaces. It holds that*

(i) *if $g \subset h$, then $h^\times \subset g^\times$;*

(ii) *$g^{\times\times} := (g^\times)^\times \supset g$.*

Definition 7.2.12. A sequence space g is said to be *perfect* if $g = g^{\times\times}$.

Proposition 7.2.13. *Let g be a sequence space. Then*

(i) *$g^\times$ is perfect;*

(ii) *$g^{\times\times}$ is the smallest perfect space containing g.*

Example 7.2.14. Let $1 \leq p \leq \infty$. Each of the BK-spaces ℓ_p and c_0 is a solid space with a monotone norm. Moreover, the spaces ℓ_p are perfect.

The following result is well known. We include a proof for completeness.

Proposition 7.2.15. *Let g be a sequence space and let $\overline{\beta} \in g$. Then*

$$\left\|\overline{\beta}\right\|_{g^{\times\times}} \leq \left\|\overline{\beta}\right\|_g .$$

Proof. Clearly, the claim holds if $\overline{\beta} = 0$. Assume that $\overline{\beta} \neq 0$. Recall that

$$\left\|\overline{\beta}\right\|_{g^{\times\times}} = \sup\left\{ \sum_{k=1}^{\infty} |\alpha_k|\, |\beta_k| \;\middle|\; \overline{\alpha} \in g^\times,\ \|\overline{\alpha}\|_{g^\times} \leq 1 \right\}.$$

It suffices to prove that

$$\sum_{k=1}^{\infty} |\alpha_k|\, |\beta_k| \leq \left\|\overline{\beta}\right\|_g$$

for each $\overline{\alpha} \in B_{g^\times}$. This follows from the definition of the norm $\|\overline{\alpha}\|_{g^\times}$. Indeed,

$$\sum_{k=1}^{\infty} |\alpha_k| \frac{|\beta_k|}{\left\|\overline{\beta}\right\|_g} \leq \|\overline{\alpha}\|_{g^\times} \leq 1. \qquad \square$$

The normed vector-valued sequence space $g^s(X)$, defined below, has been introduced and studied in [Ra1], [Ra2], under different notation; we follow the notation used in [GB].

Definition 7.2.16 (cf. [GB, p. 358])**.** Let g be a solid sequence space and let X be a Banach space. Define the space $g^s(X)$ by

$$g^s(X) = \{(x_k) \subset X \mid (\|x_k\|) \in g\}.$$

Denote by g^s the corresponding system of sequences.

Proposition 7.2.17 (cf. [GB, p. 358])**.** *Let g be a solid sequence space with a monotone norm $\|\cdot\|_g$ and let X be a Banach space. The space $g^s(X)$ becomes a normed space with respect to the norm*

$$\|\overline{x}\|_{g^s(X)} = \left\|(\|x_k\|)_{k\in\mathbb{N}}\right\|_g,$$

where $\overline{x} \in g^s(X)$. Moreover, the space $g^s(X)$ contains the space $fin^s(X)$.

Proof. Let X be a Banach space, let $\overline{x}, \overline{y} \in g^s(X)$, and let $\alpha \in \mathbb{K}$. Let us prove that $\alpha\overline{x} + \overline{y} \in g^s(X)$ and

$$\|\alpha\overline{x} + \overline{y}\|_{g^s(X)} \leq |\alpha| \, \|\overline{x}\|_{g^s(X)} + \|\overline{y}\|_{g^s(X)} \, .$$

Clearly,

$$(\|\alpha x_k\| + \|y_k\|)_{k\in\mathbb{N}} \in g.$$

Since g is solid and the norm $\|\cdot\|_g$ is monotone, we have

$$(\|\alpha x_k + y_k\|)_{k\in\mathbb{N}} \in g$$

and

$$\|\alpha\overline{x} + \overline{y}\|_{g^s(X)} = \|(\|\alpha x_k + y_k\|)_k\|_g \leq \|(\|\alpha x_k\| + \|y_k\|)_k\|_g \leq |\alpha| \, \|\overline{x}\|_{g^s(X)} + \|\overline{y}\|_{g^s(X)} \, .$$

To see that the space $g^s(X)$ contains the space $fin^s(X)$, recall that the space g contains the space fin. □

We denote $\mathbf{fin}(X) = fin^s(X)$. For conciseness, we use the notation $g^{\times s} := (g^{\times})^s$ in the remainder of this chapter. We have the following (obvious) examples.

Example 7.2.18. Let $1 \leq p \leq \infty$. Then

(i) $\boldsymbol{\ell}_p = \ell_p^s$;

(ii) $\mathbf{c}_0 = c_0^s$.

Remark 7.2.19. In [GB, p. 358], the normed space $g^s(X)$ is defined without the assumption that g is solid. However, this seems to be an oversight – without the assumption of, e.g., solidness, the component $g^s(X)$ may fail to be a linear space for a given space X. To see this, consider the space

$$m_0 = \mathrm{span}\{A\},$$

where A is the set of all sequences of zeros and ones (see [KG, p. 48]). Clearly, it is a normed sequence space with respect to the supremum norm of m. Consider $X = \mathbb{C}$. Take elements $\alpha_k, \beta_k \in S_{\mathbb{C}}$ so that $\alpha_k + \beta_k = \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Then

$$\overline{\alpha} = (\alpha_1, \alpha_2, \ldots) \in m_0^s(\mathbb{C})$$

and

$$\overline{\beta} = (\beta_1, \beta_2, \ldots) \in m_0^s(\mathbb{C}).$$

However,

$$\overline{\alpha} + \overline{\beta} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right) \notin m_0^s(\mathbb{C}).$$

7.3 Normed systems of sequences

In this section, we give the definition of a normed systems of sequences $\mathbf{h}$ (see Definition 7.3.8) and provide several examples of them (see Examples 7.3.11, 7.3.12, and 7.3.13). For a BK-space g, we give the definition of a g-compatible system of sequences $\mathbf{h}$ (see Definition 7.4.3).

Denote the set of all increasing functions from $\mathbb{N}$ to itself by $\mathrm{Inc}(\mathbb{N})$. Let us start with the following definition (the notation is inspired by [Pi1, A.4.3]).

Definition 7.3.1. Let $\pi \in \mathrm{Inc}(\mathbb{N})$ and let g be a sequence space. Define the operator $Q_\pi^g \colon g \to \omega$ by

$$Q_\pi^g(\overline{\alpha}) = (\alpha_{\pi(k)}).$$

The operator $J_\pi^g \colon g \to \omega$ is given by

$$J_\pi^g(\overline{\alpha}) = (\alpha_{\pi^{-1}(k)}),$$

where $\alpha_\emptyset := 0$.

For the sake of clarity, we include the following example.

Example 7.3.2. Let g be a sequence space and let $\pi \in \mathrm{Inc}(\mathbb{N})$ be given by $\pi(k) = 2k$. Then

$$Q_\pi^g(\overline{\alpha}) = (\alpha_2, \alpha_4, \dots)$$

and

$$J_\pi^g(\overline{\alpha}) = (0, \alpha_1, 0, \alpha_2, \dots).$$

Clearly, the operators Q_π^g and J_π^g are always linear.

Definition 7.3.3. Let $\mathbf{h}$ be a system of sequences and let X be a Banach space. The operators $Q_\pi^{\mathbf{h}(X)}$ and $J_\pi^{\mathbf{h}(X)}$, acting from the component $\mathbf{h}(X)$ to $\omega^s(X)$, are defined analogously to their scalar counterparts Q_π^g and J_π^g.

Definition 7.3.4. Let $(g, \|\cdot\|)$ be a normed sequence space and let $c \geq 1$. Then

(i) g is *contractible* if $Q_\pi^g(g) \subset g$ and $Q_\pi^g \in \mathcal{L}(g)$ for every $\pi \in \mathrm{Inc}(\mathbb{N})$;

(ii) moreover, g is *c-boundedly contractible* if $\|Q_\pi^g\| \leq c$ for every $\pi \in \mathrm{Inc}(\mathbb{N})$;

(iii) g is *expandable* if $J_\pi^g(g) \subset g$ and $J_\pi^g \in \mathcal{L}(g)$ for every $\pi \in \mathrm{Inc}(\mathbb{N})$;

(iv) moreover, g is *c-boundedly expandable* if $\|J_\pi^g\| \leq c$ for every $\pi \in \mathrm{Inc}(\mathbb{N})$;

(v) g is *shiftable* if it is contractible and expandable;

(vi) g is *c-boundedly shiftable* if it is c-boundedly contractible and c-boundedly expandable.

If c may be taken to be 1 in the above definitions, then we say that g is *metrically contractible*, *metrically expandable*, or *metrically shiftable*, respectively.

In the context of systems of sequences, let us give the following definition.

Definition 7.3.5. Let $c \geq 1$ and let $\mathbf{h}$ be a system of sequences, where every component $\mathbf{h}(X)$ is a linear subspace of $\omega^s(X)$, equipped with a norm $\|\cdot\|_{\mathbf{h}(X)}$. Then

(i) $\mathbf{h}$ is *expandable* if for every $\pi \in \mathrm{Inc}(\mathbb{N})$ there exists a constant $C_\pi \geq 1$ such that $J_\pi^{\mathbf{h}(X)}(\mathbf{h}(X)) \subset \mathbf{h}(X)$ and $\left\| J_\pi^{\mathbf{h}(X)} \right\| \leq C_\pi$ for every Banach space X;

(ii) moreover, $\mathbf{h}$ is *c-boundedly expandable* if $C_\pi \leq c$ for every $\pi \in \mathrm{Inc}(\mathbb{N})$.

The system of sequences $\mathbf{h}$ is said to be *metrically expandable* if it is 1-boundedly expandable.

Example 7.3.6. Let $1 \leq p \leq \infty$. The following BK-spaces are metrically shiftable.

(i) ℓ_p;

(ii) c_0.

Proposition 7.3.7. *Let g be a BK-space.*

(i) *If g is c-boundedly contractible, then $g^\times$ is c-boundedly expandable;*

(ii) *if g is c-boundedly expandable, then g^s is c-boundedly expandable.*

Proof. (i). Let $\overline{\beta} \in g^\times$. Let us show that

$$\left\| J_\pi^{g^\times}(\overline{\beta}) \right\|_{g^\times} = \left\| (\beta_{\pi^{-1}(k)}) \right\|_{g^\times} \leq c \left\| \overline{\beta} \right\|_{g^\times}.$$

Observe that

$$\left\| (\beta_{\pi^{-1}(k)}) \right\|_{g^\times} = \sup_{(\alpha_j) \in B_g} \sum_{j=1}^{\infty} \left| \beta_{\pi^{-1}(j)} \right| |\alpha_j| = \sup_{(\alpha_k) \in B_g} \sum_{k=1}^{\infty} |\beta_k| \left| \alpha_{\pi(k)} \right| =$$

$$= \sup_{(\gamma_k) \in Q_\pi^g(B_g)} \sum_{k=1}^{\infty} |\beta_k| |\gamma_k| \leq \sup_{(\gamma_k) \in \left\| Q_\pi^g \right\| \cdot B_g} \sum_{k=1}^{\infty} |\beta_k| |\gamma_k| = \| Q_\pi^g \| \left\| \overline{\beta} \right\|_{g^\times} \leq c \left\| \overline{\beta} \right\|_{g^\times}.$$

Clearly, (ii) follows from the definitions. □

Let us give the main definition of this section.

Definition 7.3.8. We say that a system of sequences $\mathbf{h}$ is a *normed system of sequences* if:

(NS_1) every component $\mathbf{h}(X)$ is a linear subspace of $\omega^s(X)$ equipped with a norm $\|\cdot\|_{\mathbf{h}(X)}$;

(NS_2) $\mathbf{fin}(\mathbb{K}) \subset \mathbf{h}(\mathbb{K})$;

(NS_3) if $(x_k) \in \mathbf{h}(X)$, Y is a Banach space, and $T \in \mathcal{L}(X, Y)$, then $(Tx_k) \in \mathbf{h}(Y)$ and

$$\|(Tx_k)\|_{\mathbf{h}(Y)} \leq \|T\| \, \|(x_k)\|_{\mathbf{h}(X)}.$$

Proposition 7.3.9. *Let g be a solid BK-space with a monotone norm $\|\cdot\|_g$. Then the system of sequences g^s is a normed system of sequences.*

Proof. Let us verify the properties (NS_1)–(NS_3).

Proposition 7.2.17 yields that $g^s(X)$ is a normed space and that $\mathbf{fin}(\mathbb{K}) \subset g^s(\mathbb{K})$ (i.e., (NS_1) and (NS_2)). To show (NS_3), let $\overline{x} \in g^s(X)$ and $T \in \mathcal{L}(X, Y)$. Then $(\|Tx_k\|) \in g$, since

$$(\|T\| \, \|x_k\|)_{k\in\mathbb{N}} = \|T\| \, (\|x_k\|)_{k\in\mathbb{N}} \in g$$

and g is solid. Therefore $(Tx_k) \in g^s(Y)$. Similarly,

$$\|(Tx_k)\|_{g^s(Y)} = \left\|(\|Tx_k\|)_{k\in\mathbb{N}}\right\|_g \leq \|T\| \left\|(\|x_k\|)_{k\in\mathbb{N}}\right\|_g = \|T\| \, \|(x_k)\|_{g^s(Y)} ,$$

since $\|\cdot\|_g$ is monotone. □

Corollary 7.3.10. *Let g be a BK-space. Then the system of sequences $g^{\times s}$ is a normed system of sequences.*

Proof. It suffices to recall that $g^{\times}$ is a solid space with a monotone norm $\|\cdot\|_{g^{\times}}$ (see Propositions 7.2.3 and 7.2.9) and apply Proposition 7.3.9. □

Example 7.3.11. Let $1 \leq p \leq \infty$. The following are metrically expandable normed systems of sequences.

(i) $\boldsymbol{\ell}_p$;

(ii) $\mathbf{c}_0$.

Proof. By Example 7.2.14 and Proposition 7.3.9, $\boldsymbol{\ell}_p$ and $\mathbf{c}_0$ are normed systems of sequences. Proposition 7.3.7 and Example 7.3.6 yield that they are metrically expandable. □

Consider also the following examples.

Example 7.3.12. Let $1 \leq p < \infty$. The systems $\mathbf{c}_0^w$ and $\boldsymbol{\ell}_p^w$ are metrically expandable normed systems of sequences.

Proof. Let us first prove the claim for the system $\boldsymbol{\ell}_p^w$.

Property (NS_1) follows from the fact that $\boldsymbol{\ell}_p^w(X)$ is a subspace of $\boldsymbol{\ell}_\infty(X)$ for every Banach space X.

Property (NS_2) holds, since $\mathbf{fin}(\mathbb{K}) \subset \boldsymbol{\ell}_p(\mathbb{K}) = \boldsymbol{\ell}_p^w(\mathbb{K})$.

To prove (NS_3), let $\overline{x} \in \boldsymbol{\ell}_p^w(X)$ and $T \in \mathcal{L}(X, Y)$. We show that $(Tx_k) \in \boldsymbol{\ell}_p^w(Y)$ and

$$\|(Tx_k)\|_{\boldsymbol{\ell}_p^w(Y)} \leq \|T\| \, \|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)} .$$

Clearly, it suffices to prove that $\left\|\left(y^*(Tx_k)\right)\right\|_{\ell_p} \leq \|T\| \, \|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)}$ for each $y^* \in B_{Y^*}$. Let $y^* \in B_{Y^*}$. Then

$$\left\|\left(y^*(Tx_k)\right)\right\|_{\ell_p} = \|(T^*y^*x_k)\|_{\ell_p} \leq \|T^*y^*\| \, \|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)} \leq \|T\| \, \|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)} ,$$

since $T^*y^* \in X^*$ and $\|T^*y^*\| \leq \|T^*\| = \|T\|$.

To see that $\boldsymbol{\ell}_p^w$ is metrically expandable, let $\overline{x} \in \boldsymbol{\ell}_p^w(X)$ and $\pi \in \mathrm{Inc}(\mathbb{N})$. It follows easily from the definitions that $(x_{\pi^{-1}(k)}) \in \boldsymbol{\ell}_p^w(X)$ and

$$\left\|\left(x_{\pi^{-1}(k)}\right)\right\|_{\boldsymbol{\ell}_p^w(X)} = \|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)} .$$

The proof for the system $\mathbf{c}_0^w$ is essentially the same. The only substantial difference is in the proof of property (NS_3). For this, let $\overline{x} \in \mathbf{c}_0^w(X)$ and $T \in \mathcal{L}(X,Y)$ be given. Then $(Tx_k) \in \mathbf{c}_0^w(Y)$ and

$$\|(Tx_k)\|_{\mathbf{c}_0^w(Y)} = \sup_{k\in\mathbb{N}} \|Tx_k\| \leq \|T\| \sup_{k\in\mathbb{N}} \|x_k\| = \|T\| \, \|\overline{x}\|_{\mathbf{c}_0^w(X)} . \qquad \square$$

As a final example of this section, let us consider the system $\boldsymbol{\ell}_p^u$ of *unconditionally p-summable sequences*, which resides "between" the systems $\boldsymbol{\ell}_p$ and $\boldsymbol{\ell}_p^w$ (see, e.g., [DF, 8.2, 8.3]; we follow [BCFP] in our terminology). The space $\boldsymbol{\ell}_p^u(X)$ is defined as the (closed) subspace of $\boldsymbol{\ell}_p^w(X)$, equipped with the norm of $\boldsymbol{\ell}_p^w(X)$, and formed by $(x_n) \in \boldsymbol{\ell}_p^w(X)$ satisfying $(x_n) = \lim_{N\to\infty}(x_1, \ldots, x_N, 0, 0, \ldots)$ in $\boldsymbol{\ell}_p^w(X)$. The space $\boldsymbol{\ell}_p^u(X)$ was introduced and thoroughly studied by Fourie and Swart [FS] in 1979.

Example 7.3.13. Let $1 \leq p < \infty$. The system $\boldsymbol{\ell}_p^u$ is a metrically expandable normed system of sequences.

Proof. Clearly, properties (NS_1) and (NS_2) are satisfied.

To show (NS_3), let $\overline{x} \in \boldsymbol{\ell}_p^u(X)$ and $T \in \mathcal{L}(X,Y)$. Note that we only need to prove that $(Tx_k) \in \boldsymbol{\ell}_p^u(Y)$, since the norms of $\boldsymbol{\ell}_p^u(X)$ and $\boldsymbol{\ell}_p^w(X)$ coincide. By Example 7.3.12,

$$\|(0, \ldots, 0, Tx_{N+1}, Tx_{N+2}, \ldots)\|_{\boldsymbol{\ell}_p^w(Y)} \leq \|T\| \, \|(0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots)\|_{\boldsymbol{\ell}_p^w(X)}$$

This proves that $(Tx_k) \in \boldsymbol{\ell}_p^u(Y)$.

To see that $\boldsymbol{\ell}_p^u$ is metrically expandable, let $\overline{x} \in \boldsymbol{\ell}_p^u(X)$ and $\pi \in \mathrm{Inc}(\mathbb{N})$. It follows easily from the definitions that $(x_{\pi^{-1}(k)}) \in \boldsymbol{\ell}_p^u(X)$. $\square$

7.4 Constructing systems of sets

In this section, we give the notion of a g-compatible system of sequences (see Definition 7.4.3) and show how to construct a system of sets $\Lambda[g, \mathbf{h}]$ from a BK-space g and a g-compatible normed system of sequences $\mathbf{h}$ (see Definition 7.4.4). We prove that the system of sets $\Lambda[g, \mathbf{h}]$ produced in this way is a generating system of sets, provided that g and $\mathbf{h}$ satisfy certain properties (see Theorem 7.4.9).

In Remark 7.4.6, we compare the method of constructing systems of sets from [GB] with our approach. We prove in Proposition 7.4.7 that our construction encompasses all of the examples produced by their approach. We end this section with several examples.

Let us extend Definition 4.1.2.

Definition 7.4.1. Let g be a BK-space. For every $\overline{x} \in g^{\times s}(X)$, define an operator $E_{\overline{x}} \colon g \to X$ by

$$E_{\overline{x}}(\overline{\alpha}) = \sum_{k=1}^{\infty} \alpha_k x_k.$$

Lemma 7.4.2. *Let g be a BK-space and let $\overline{x} \in g^{\times s}(X)$. Then $E_{\overline{x}} \in \mathcal{L}(g, X)$.*

Proof. Clearly, $E_{\overline{x}}$ is a linear operator. To show that it is bounded, let $\overline{\alpha} \in B_g$. Then

$$\|E_{\overline{x}}(\overline{\alpha})\| = \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| \leq \sum_{k=1}^{\infty} |\alpha_k| \, \|x_k\| \leq \left\| (\|x_k\|)_{k \in \mathbb{N}} \right\|_{g^\times} = \|\overline{x}\|_{g^{\times s}(X)} . \qquad \square$$

Definition 7.4.3. Let g be a BK-space. We say that a system of sequences $\mathbf{h}$ is *g-compatible* if $\mathbf{h}(X) \subset g^{\times s}(X)$ for every Banach space X.

Definition 7.4.4. Let a BK-space g be given along with a g-compatible system of sequences $\mathbf{h}$. Define a system of sets $\Lambda[g, \mathbf{h}]$ in the following way: a set K belongs to the component $\Lambda[g, \mathbf{h}](X)$ whenever there exists a sequence $\overline{x} \in \mathbf{h}(X)$ such that

$$K \subset E_{\overline{x}}(B_g).$$

The familiar system $\mathbf{K}_{(p,r)}$ is a special case of the above construction, as demonstrated by the following proposition.

Proposition 7.4.5. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Put $g = \ell_r$ and $\mathbf{h} = \boldsymbol{\ell}_p$ (where $\mathbf{h} = \mathbf{c}_0$, if $p = \infty$). Then $\mathbf{K}_{(p,r)} = \Lambda[g, \mathbf{h}]$.*

Proof. Essentially, we only need to check that the system $\mathbf{h}$ is g-compatible. If $p = \infty$, then this follows from the fact that $c_0 \subset \ell_1^\times$. Similarly, if $1 \leq p < \infty$, then this is clear from the fact that $\ell_r \subset \ell_{p^*} = \ell_p^\times$. □

Having given a method for constructing systems of sets, we compare it with the method for constructing λ-compact systems of sets introduced in [GB].

Remark 7.4.6. In [GB, Definition 3.1], Gupta and Bhar started from a BK-space λ and defined for every $\overline{x} \in \lambda^s(X)$ the λ-*convex hull* of the sequence $\overline{x}$ by

$$\lambda - co(\overline{x}) = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \,\middle|\, \overline{\alpha} \in B_{\lambda^\times} \right\}.$$

A set K was defined to be λ-*compact* if there exists a sequence $\overline{x} \in \lambda^s(X)$ such that

$$K \subset \lambda - co(\overline{x}).$$

Comparing the previous remark with Definition 7.4.1, one observes that we required $\overline{x} \in \lambda^{\times s}(X)$ and $\overline{\alpha} \in \lambda$ instead of $\overline{x} \in \lambda^s(X)$ and $\overline{\alpha} \in \lambda^\times$. Nonetheless, the systems of λ-compact sets (in the sense of Gupta and Bhar) are encompassed by our construction, as demonstrated by the following result.

Proposition 7.4.7. *Let λ be a BK-space. Then the system $\Lambda[\lambda^\times, \lambda^s]$ coincides with the system of λ-compact sets.*

Proof. According to Theorem 7.2.7, $\lambda^\times$ is a BK-space. Recall that $\lambda \subset \lambda^{\times\times}$; therefore the system $\lambda^s \subset \lambda^{\times\times s}$ is $\lambda^\times$-compatible and the system $\Lambda[\lambda^\times, \lambda^s]$ is well defined. It remains to observe that the sets of the system $\Lambda[\lambda^\times, \lambda^s]$ are exactly the λ-compact sets. □

Proposition 7.4.8. *Let a BK-space g be given along with a g-compatible system of sequences $\mathbf{h}$. Then $\Lambda[g, \mathbf{h}] \leq \mathbf{B}$.*

Proof. Let us verify that a set $K \in \Lambda[g, \mathbf{h}](X)$ is bounded. By definition, there exists a sequence $\overline{x} \in \mathbf{h}(X) \subset g^{\times s}(X)$ so that $K \subset E_{\overline{x}}(B_g)$. Take an element y from the set K; then $y = \sum_{k=1}^{\infty} \alpha_k x_k$ for some $\overline{\alpha} \in B_g$. Therefore

$$\|y\| = \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| \leq \sum_{k=1}^{\infty} |\alpha_k| \, \|x_k\| \leq \|\overline{x}\|_{g^{\times s}(X)},$$

which proves that the set K is bounded. □

Theorem 7.4.9. *Let an expandable BK-space g be given along with a g-compatible expandable normed system of sequences $\mathbf{h}$. Then $\Lambda[g, \mathbf{h}]$ is a generating system of sets.*

Proof. We prove that the system of sets $\Lambda[g, \mathbf{h}]$ satisfies properties (G_0)–(G_4).

(G_0). This has been proved in Proposition 7.4.8.

(G_1). We need to show that $B_{\mathbb{K}} \in \Lambda[g, \mathbf{h}](\mathbb{K})$. Define

$$\overline{\beta} = (\|e_1\|_g, 0, 0, \ldots) \in \mathbf{fin}(\mathbb{K}) \subset \mathbf{h}(\mathbb{K}) \subset g^{\times s}(\mathbb{K}).$$

Take $c \in B_{\mathbb{K}}$ and denote

$$\overline{\alpha} = \frac{c}{\|e_1\|_g} e_1 \in B_g.$$

Then

$$c = \sum_{k=1}^{\infty} \alpha_k \beta_k \in E_{\overline{\beta}}(B_g).$$

Thus we have shown that $B_{\mathbb{K}} \in E_{\overline{\beta}}(B_g)$.

(G_2). Let $c \in \mathbb{K}$ and let $G, H \in \Lambda[g, \mathbf{h}](X)$. By definition, there exist sequences $\overline{x} \in \mathbf{h}(X)$ and $\overline{y} \in \mathbf{h}(X)$ such that $G \subset E_{\overline{x}}(B_g)$ and $H \subset E_{\overline{y}}(B_g)$. Put $\pi_1(k) = 2k - 1$ and $\pi_2(k) = 2k$ (where $k = 1, 2, \ldots$). Define

$$\overline{z} = 2\left(c \left\| J^g_{\pi_1} \right\| J^{\mathbf{h}(X)}_{\pi_1}(\overline{x}) + \left\| J^g_{\pi_2} \right\| J^{\mathbf{h}(X)}_{\pi_2}(\overline{y})\right).$$

Then $\overline{z} \in \mathbf{h}(X)$. Let us verify that

$$cG + H \subset E_{\overline{z}}(B_g) = \left\{ \sum_{k=1}^{\infty} \gamma_k z_k \mid \overline{\gamma} \in B_g \right\}.$$

Take $w = cw_1 + w_2$, where $w_1 = \sum_{k=1}^{\infty} \alpha_k x_k$ and $w_2 = \sum_{k=1}^{\infty} \beta_k y_k$ with $\overline{\alpha}, \overline{\beta} \in B_g$. Put

$$\overline{\gamma} = \frac{1}{2}\left(\frac{J^g_{\pi_1}(\overline{\alpha})}{\|J^g_{\pi_1}\|} + \frac{J^g_{\pi_2}(\overline{\beta})}{\|J^g_{\pi_2}\|} \right).$$

Then

$$w = c\sum_{k=1}^{\infty} \alpha_k x_k + \sum_{k=1}^{\infty} \beta_k y_k = \sum_{k=1}^{\infty} c\alpha_{\pi_1^{-1}(k)} x_{\pi_1^{-1}(k)} + \sum_{k=1}^{\infty} \beta_{\pi_2^{-1}(k)} y_{\pi_2^{-1}(k)} = \sum_{k=1}^{\infty} \gamma_k z_k.$$

It remains to notice that

$$\|\overline{\gamma}\|_g \leq \frac{1}{2}\big(\|\overline{\alpha}\|_g + \|\overline{\beta}\|_g \big) \leq 1.$$

(G_3). Trivial and omitted.

(G_4). Let $G \in \Lambda[g, \mathbf{h}](X)$. By definition, $G \subset E_{\overline{x}}(B_g)$, where $\overline{x} \in \mathbf{h}(X)$. Therefore

$$T(G) \subset TE_{\overline{x}}(B_g) = E_{(Tx_k)}(B_g),$$

where $(Tx_k) \in \mathbf{h}(Y)$ by (NS_3). We have shown that $T(G) \in \Lambda[g, \mathbf{h}](Y)$. □

Corollary 7.4.10. *Let g be a shiftable BK-space. Then $\Lambda[g, g^{\times s}] \in \mathrm{GSet}$.*

Proof. By Corollary 7.3.10 and Proposition 7.3.7, $g^{\times s}$ is an expandable normed system of sequences. The claim follows from Theorem 7.4.9, since $g^{\times s}$ is clearly g-compatible. □

Let us re-prove the prototypical example that $\mathbf{K}_{(p,r)} \in \mathrm{GSet}$.

Proposition 7.4.11. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Put $g = \ell_r$ and $\mathbf{h} = \boldsymbol{\ell}_p$ (where $\mathbf{h} = \mathbf{c}_0$, if $p = \infty$). Then $\mathbf{K}_{(p,r)} = \Lambda[g, \mathbf{h}] \in \mathrm{GSet}$.*

Proof. By Proposition 7.4.5, $\mathbf{K}_{(p,r)} = \Lambda[g, \mathbf{h}]$. It remains to observe that by Example 7.3.6, g is an expandable BK-space and by Example 7.3.11, $\mathbf{h}$ is an expandable normed system of sequences. □

Recall that, by Remark 4.1.4, $\Lambda[m, \boldsymbol{\ell}_1] = \Lambda[c_0, \boldsymbol{\ell}_1]$. Observe that we have already considered all of the possible combinations that can be obtained from the "classical" sequence spaces ℓ_p, c_0, m, and their vector counterparts, except for the system $\Lambda[\ell_1, \mathbf{m}]$. Let us consider it now.

Proposition 7.4.12. *The generating system of sets $\Lambda[\ell_1, \mathbf{m}]$ contains the system $\mathbf{K} = \mathbf{K}_{(\infty,1)} = \Lambda[\ell_1, \mathbf{c}_0]$ of all relatively compact sets, but does not coincide with it.*

Proof. Clearly, $\Lambda[\ell_1, \mathbf{c}_0] \le \Lambda[\ell_1, \mathbf{m}]$. Observe that

$$B_{\ell_1} = E_e(B_{\ell_1}) \in \Lambda[\ell_1, \mathbf{m}](\ell_1),$$

where $e = (1, 1, \ldots, 1, \ldots) \in \mathbf{m}(\ell_1)$. However, B_{ℓ_1} is not relatively compact. □

Let $1 \le p < \infty$. Observe that $\mathbf{W}_{(p,1)} = \Lambda[\ell_1, \boldsymbol{\ell}_p^w]$ and $\mathbf{W}_\infty = \Lambda[\ell_1, \mathbf{c}_0^w]$.

Theorem 7.4.9 together with Example 7.3.12 yields the following result. For proof, it suffices to observe that $\boldsymbol{\ell}_p^w(X) \subset \mathbf{m}(X)$ for every Banach space X.

Proposition 7.4.13. *Let $1 \le p < \infty$. Then $\mathbf{W}_{(p,1)} = \Lambda[\ell_1, \boldsymbol{\ell}_p^w] \in \mathrm{GSet}$.*

Observe that Theorem 7.4.9 together with 7.3.12 proves once again that $\mathbf{W}_\infty$ is a generating system of sets (it is only necessary to notice that $\mathbf{c}_0^w(X) \subset \mathbf{m}(X)$ for every Banach space X).

In [AO2, p. 1574], the notion of a *relatively unconditionally* (p,r)*-compact set* was introduced, where $1 \le p < \infty$ and $1 \le r \le p^*$. In our terminology, the system $\mathbf{U}_{(p,1)}$ of relatively unconditionally $(p,1)$-compact sets can be expressed as $\mathbf{U}_{(p,1)} = \Lambda[\ell_1, \boldsymbol{\ell}_p^u]$ (in [AO2], the notation $\boldsymbol{u}_{(p,1)}$ was used instead).

Theorem 7.4.9 together with Example 7.3.13 yields the following result.

Proposition 7.4.14. *Let* $1 \le p < \infty$*. Then* $\mathbf{U}_{(p,1)} = \Lambda[\ell_1, \boldsymbol{\ell}_p^u] \in \mathrm{GSet}$.

7.5 Constructing quasi-normed operator ideals

In this section, we show that under suitable assumptions, the operator ideal $\Theta(\Lambda[g, \mathbf{h}])$ is quasi-normed. If both g and $\mathbf{h}$ are metrically expandable, then property (QOI_1) is satisfied for a quasi-constant $\varkappa = 2$. This quasi-constant is not optimal in general; consider the quasi-Banach operator ideal $\mathcal{K}_{(p,r)} = \Theta(\Lambda[\ell_r, \boldsymbol{\ell}_p])$ from Section 4.2 (where $1 \le p < \infty$ and $1 \le r \le p^*$). We showed that property (QOI_1) for $\mathcal{K}_{(p,r)}$ is satisfied for a quasi-constant $1 \le \varkappa \le 2$, where the one extreme $\varkappa = 1$ occurs if $r = p^*$, and the other extreme $\varkappa = 2$ occurs if $r = p = 1$. In the following, we use the notation $\Theta_\Lambda[g, \mathbf{h}] := \Theta(\Lambda[g, \mathbf{h}])$ for conciseness.

Note that it was proven in [GB] that the class K_λ of λ-compact operators is a quasi-Banach operator ideal with a quasi-constant of 8. We prove in Section 7.8 that their result is encompassed by the next theorem.

The quasi-norm $\|\cdot\|_{[g,\mathbf{h}]}$ defined in the following theorem is a generalization of the norms $\|\cdot\|_{\mathcal{K}_{(p,r)}}$ (recall Theorem 4.2.12) and $\|\cdot\|_{\mathcal{W}_\infty}$.

Theorem 7.5.1. *Let an expandable BK-space g be given along with a g-compatible expandable normed system of sequences $\mathbf{h}$ so that*

(i) *for each $\overline{\beta} \in \mathbf{h}(\mathbb{K})$, it holds that* $\|\overline{\beta}\|_{\mathbf{h}(\mathbb{K})} \ge \|\overline{\beta}\|_{g^\times}$*;*

(ii) $\|e_1\|_{\mathbf{h}(\mathbb{K})} = \|e_1\|_{g^\times}$.

Then $\Theta_\Lambda[g, \mathbf{h}]$ is a quasi-normed operator ideal with respect to the quasi-norm

$$\|T\|_{[g,\mathbf{h}]} = \inf\{\|\overline{x}\|_{\mathbf{h}(Y)} \mid \overline{x} \in \mathbf{h}(Y),\, T(B_X) \subset E_{\overline{x}}(B_g)\},$$

where $T \in \Theta_\Lambda[g, \mathbf{h}](X, Y)$. If both g and $\mathbf{h}$ are metrically expandable, then property (QOI_1) is satisfied for the quasi-constant $\varkappa = 2$.

Proof. By Theorem 7.4.9, $\Theta(\Lambda[g, \mathbf{h}])$ is an operator ideal. Let us verify (QOI_0) by showing that $\|I_{\mathbb{K}}\|_{[g,\mathbf{h}]} = 1$.

Observe that $g^{\times} = g^{\times s}(\mathbb{K})$. By assumption (i) and the fact that $\mathbf{h}$ is g-compatible,

$$\|I_{\mathbb{K}}\|_{[g,\mathbf{h}]} = \inf\{\|\overline{\beta}\|_{\mathbf{h}(\mathbb{K})} \mid \overline{\beta} \in \mathbf{h}(\mathbb{K}),\, B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_g)\} \geq$$
$$\geq \inf\{\|\overline{\beta}\|_{g^{\times}} \mid \overline{\beta} \in g^{\times},\, B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_g)\}.$$

Let $\overline{\beta} \in g^{\times}$ be given so that $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_g)$. Then there exists $\overline{\alpha} \in B_g$ such that $1 = \sum_{k=1}^{\infty} \beta_k \alpha_k$. Therefore

$$\|\overline{\beta}\|_{g^{\times}} = \sup\left\{\sum_{k=1}^{\infty} |\gamma_k|\,|\beta_k| \;\middle|\; \overline{\gamma} \in B_g\right\} \geq \sum_{k=1}^{\infty} |\beta_k|\,|\alpha_k| \geq \left|\sum_{k=1}^{\infty} \beta_k \alpha_k\right| = 1.$$

We have shown that $\|\overline{\beta}\|_{g^{\times}} \geq 1$ for any $\overline{\beta} \in g^{\times}$ satisfying $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_g)$. Therefore

$$\|I_{\mathbb{K}}\|_{[g,\mathbf{h}]} \geq \inf\{\|\overline{\beta}\|_{g^{\times}} \mid \overline{\beta} \in g^{\times} \text{ and } B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_g)\} \geq 1.$$

For the opposite inequality, let us check that $\|I_{\mathbb{K}}\|_{[g,\mathbf{h}]} \leq 1 + \varepsilon$ for each $\varepsilon > 0$. By definition,

$$\|e_1\|_{g^{\times}} = \sup\{|\alpha_1| \mid \overline{\alpha} \in B_g\}.$$

Therefore there exists a sequence $\overline{\alpha} \in B_g$ such that $(1 + \delta)\alpha_1 = \|e_1\|_{g^{\times}}$, where $0 < \delta \leq \varepsilon$. Put

$$\overline{\beta} = \frac{1 + \delta}{\|e_1\|_{g^{\times}}} e_1 \in \mathbf{fin}(\mathbb{K}) \subset \mathbf{h}(\mathbb{K}).$$

Then

$$\|\overline{\beta}\|_{\mathbf{h}(\mathbb{K})} = \|\overline{\beta}\|_{g^{\times}} = 1 + \delta,$$

since $\|e_1\|_{\mathbf{h}(\mathbb{K})} = \|e_1\|_{g^{\times}}$ by assumption (ii).

We will have shown that $\|I_{\mathbb{K}}\|_{[g,\mathbf{h}]} \leq 1 + \delta \leq 1 + \varepsilon$ if we prove that $B_{\mathbb{K}} \subset E_{\overline{\beta}}(B_g)$. Take $c \in B_{\mathbb{K}}$. Since $(c \cdot \alpha_k) \in B_g$, we may conclude that, indeed,

$$c = \frac{c(1 + \delta)\alpha_1}{\|e_1\|_{g^{\times}}} = \sum_{k=1}^{\infty} c\alpha_k \beta_k \in E_{\overline{\beta}}(B_g).$$

In order to verify property (QOI_1), let $S, T \in \Theta_{\Lambda}[g, \mathbf{h}](X, Y)$. We will prove that

$$\|S + T\|_{[g,\mathbf{h}]} \leq \varkappa[\|S\|_{[g,\mathbf{h}]} + \|T\|_{[g,\mathbf{h}]}],$$

for a certain constant $\varkappa$ (see (7.1) below). For this, take $\varepsilon > 0$ and sequences $\overline{x}$ and $\overline{y}$ from $\mathbf{h}(Y)$ such that $S(B_X) \subset E_{\overline{x}}(B_g)$ and $T(B_X) \subset E_{\overline{y}}(B_g)$ with $\|\overline{x}\|_{\mathbf{h}(Y)} \le (1+\varepsilon)\|S\|_{[g,\mathbf{h}]}$ and $\|\overline{y}\|_{\mathbf{h}(Y)} \le (1+\varepsilon)\|T\|_{[g,\mathbf{h}]}$. As in the proof of Theorem 7.4.9, put $\pi_1(k) = 2k-1$ and $\pi_2(k) = 2k$ ($k = 1, 2, \ldots$), and define

$$\overline{z} = 2\big(\left\|J^g_{\pi_1}\right\| J^{\mathbf{h}(X)}_{\pi_1}(\overline{x}) + \left\|J^g_{\pi_2}\right\| J^{\mathbf{h}(X)}_{\pi_2}(\overline{y})\big).$$

Then, similarly to the proof of Theorem 7.4.9, we may verify that

$$(S+T)(B_X) \subset S(B_X) + T(B_X) \subset E_{\overline{z}}(B_g).$$

Since $\mathbf{h}$ is extendable, there exist constants C_{π_1} and C_{π_2} (independent of the space Y) such that $\left\|J^{\mathbf{h}(Y)}_{\pi_1}\right\| \le C_{\pi_1}$ and $\left\|J^{\mathbf{h}(Y)}_{\pi_2}\right\| \le C_{\pi_2}$. By (NS_1),

$$\begin{aligned}\|\overline{z}\|_{\mathbf{h}(Y)} &\le 2\left[\left\|J^g_{\pi_1}\right\| C_{\pi_1}\|\overline{x}\|_{\mathbf{h}(Y)} + \left\|J^g_{\pi_2}\right\| C_{\pi_2}\|\overline{y}\|_{\mathbf{h}(Y)}\right] \le \\ &\le 2\max_{k\in\{1,2\}}\left\{\left\|J^g_{\pi_k}\right\| C_{\pi_k}\right\}\left[\|\overline{x}\|_{\mathbf{h}(Y)} + \|\overline{y}\|_{\mathbf{h}(Y)}\right].\end{aligned}$$

Put

$$\varkappa = 2\max_{k\in\{1,2\}}\left\{\left\|J^g_{\pi_k}\right\| C_{\pi_k}\right\}. \tag{7.1}$$

If both g and $\mathbf{h}$ are metrically expandable, then $\varkappa = 2$.

Recall that $(S+T)(B_X) \subset E_{\overline{z}}(B_g)$. Therefore

$$\|S+T\|_{[g,\mathbf{h}]} \le \|\overline{z}\|_{\mathbf{h}(Y)} \le \varkappa\left[\|\overline{x}\|_{\mathbf{h}(Y)} + \|\overline{y}\|_{\mathbf{h}(Y)}\right] \le (1+\varepsilon)\varkappa\left[\|S\|_{[g,\mathbf{h}]} + \|T\|_{[g,\mathbf{h}]}\right].$$

Since the choice of ε was arbitrary, we have

$$\|S+T\|_{[g,\mathbf{h}]} \le \varkappa\left[\|S\|_{[g,\mathbf{h}]} + \|T\|_{[g,\mathbf{h}]}\right].$$

It remains to check (QOI_2). Fix operators $T \in \mathcal{L}(X_0, X)$, $S \in \Theta_\Lambda[g,\mathbf{h}](X,Y)$, and $R \in \mathcal{L}(Y, Y_0)$. We need to prove that

$$\|RST\|_{[g,\mathbf{h}]} \le \|R\|\,\|S\|_{[g,\mathbf{h}]}\,\|T\|.$$

Let $\varepsilon > 0$. Take $\overline{y} \in \mathbf{h}(Y)$ satisfying $\|\overline{y}\|_{\mathbf{h}(Y)} \le \|S\|_{[g,\mathbf{h}]} + \varepsilon$ and $S(B_X) \subset E_{\overline{y}}(B_g)$. Put $\overline{z} = (\|T\|\,Ry_k)$. By (NS_3), $\overline{z} \in \mathbf{h}(Y_0)$ and

$$\|\overline{z}\|_{\mathbf{h}(Y_0)} \le \|T\|\,\|R\|\,\|\overline{y}\|_{\mathbf{h}(Y)} \le \|R\|\left(\|S\|_{[g,\mathbf{h}]} + \varepsilon\right)\|T\|.$$

Since

$$RST(B_{X_0}) \subset \|T\|\,RS(B_X) \subset \|T\|\,R(E_{\overline{y}}(B_g)) = E_{(\|T\|Ry_k)}(B_g) = E_{\overline{z}}(B_g),$$

we have $\|RST\|_{[g,\mathbf{h}]} \le \|\overline{z}\|_{\mathbf{h}(Y_0)}$ and therefore $\|RST\|_{[g,\mathbf{h}]} \le \|R\|\,\|S\|_{[g,\mathbf{h}]}\,\|T\|$. □

Corollary 7.5.2. *Let g be a shiftable BK-space. Then $\Theta_\Lambda[g, g^{\times s}]$ is a quasi-normed operator ideal. If g is metrically shiftable, then property (QOI_1) is satisfied for the quasi-constant $\varkappa = 2$.*

Proof. By Corollary 7.3.10, $g^{\times s}$ is a normed system of sequences. To see that assumptions (i) and (ii) of Theorem 7.5.1 are satisfied, observe that $g^\times = g^{\times s}(\mathbb{K})$. Additionally, Proposition 7.3.7 yields that $g^{\times s}$ is (metrically) expandable provided that g is (metrically) contractible. □

7.6 Constructing quasi-Banach operator ideals

In this section, we prove that the quasi-normed operator ideal $\Theta_\Lambda[g, \mathbf{h}]$ is complete (i.e., a quasi-Banach operator ideal) whenever certain assumptions are fulfilled.

Definition 7.6.1. We call any bijection $\varrho\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ a *rearrangement.* We denote the set of all rearrangements by Arr.

Definition 7.6.2. Let g be a normed sequence space and let ϱ be a rearrangement. We define the *rearrangement operator*

$$H^g_\varrho\colon \boldsymbol{\ell}_1(g) \to \omega$$

in the following way. Let $(\overline{\alpha}_n)_{n\in\mathbb{N}}$ be an absolutely summable sequence of scalar sequences, where $\overline{\alpha}_n = (\alpha^n_k)_{k\in\mathbb{N}} \in g$ for each $n \in \mathbb{N}$. Put

$$H^g_\varrho\big((\overline{\alpha}_n)_{n\in\mathbb{N}}\big) = (\beta_m)_{m\in\mathbb{N}},$$

where $\beta_{\varrho(n,k)} = \alpha^n_k$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

Clearly, every rearrangement operator is linear.

Definition 7.6.3. Let $\varrho \in$ Arr. A normed sequence space g is said to be *ϱ-rearrangeable*, if

$$H^g_\varrho\big(\boldsymbol{\ell}_1(g)\big) \subset g$$

and the operator $H^g_\varrho\colon \boldsymbol{\ell}_1(g) \to g$ is bounded.

If $\big\|H^g_\varrho\big\| \leq 1$, then g is said to be *metrically ϱ-rearrangeable.*

Example 7.6.4. Let $\varrho \in$ Arr. The following BK-spaces are metrically ϱ-rearrangeable.

(i) ℓ_p, where $1 \leq p \leq \infty$;

(ii) c_0.

Proof. (i). Let $1 \le p < \infty$. Consider sequences $\overline{\alpha}_n = (\alpha_k^n)_{k\in\mathbb{N}} \in \ell_p$, where $n \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} \|\overline{\alpha}_n\|_{\ell_p} < \infty.$$

Put

$$\overline{\alpha} = (\overline{\alpha}_n) \in \boldsymbol{\ell}_1(\ell_p)$$

and

$$\overline{\beta} = H_\varrho^{\ell_p}(\overline{\alpha}).$$

The double sequence $\left(|\alpha_k^n|^p\right)_{n,k\in\mathbb{N}}$ is absolutely convergent. Indeed,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_k^n|^p = \sum_{n=1}^{\infty} \|\overline{\alpha}_n\|_{\ell_p}^p = \|\overline{\alpha}\|_{\boldsymbol{\ell}_p(\ell_p)}^p \le \|\overline{\alpha}\|_{\boldsymbol{\ell}_1(\ell_p)}^p = \left(\sum_{n=1}^{\infty} \|\overline{\alpha}_n\|_{\ell_p}\right)^p < \infty.$$

Therefore it is unconditionally convergent and we may write

$$\|\overline{\beta}\|_{\ell_p} = \left(\sum_{m=1}^{\infty} |\beta_m|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_k^n|^p\right)^{\frac{1}{p}} \le \sum_{n=1}^{\infty} \|\overline{\alpha}_n\|_{\ell_p}.$$

We have shown that $\left\|H_\varrho^{\ell_p}\right\| \le 1$.

Let $p = \infty$. To prove that m is metrically ϱ-rearrangeable, consider sequences $\overline{\alpha}_n = (\alpha_k^n)_k \in m$ such that

$$\sum_{n=1}^{\infty} \|\overline{\alpha}_n\|_m < \infty.$$

Put

$$\overline{\alpha} = (\overline{\alpha}_n) \in \boldsymbol{\ell}_1(m)$$

and

$$\overline{\beta} = H_\varrho^{m}(\overline{\alpha}).$$

Observe that

$$\|\overline{\beta}\|_m = \sup_{m\in\mathbb{N}} |\beta_m| = \sup_{n,k\in\mathbb{N}} |\alpha_k^n| \le \sum_{n=1}^{\infty} \sup_{k\in\mathbb{N}} |\alpha_k^n| = \sum_{n=1}^{\infty} \|\overline{\alpha}_n\|_m = \|\overline{\alpha}\|_{\boldsymbol{\ell}_1(m)}.$$

(ii). Since $c_0 \subset m$ and the norms agree, we only need to prove that

$$H_\varrho^{c_0}\left(\boldsymbol{\ell}_1(c_0)\right) \subset c_0.$$

Let $\overline{\alpha} = (\overline{\alpha}_n)_n = ((\alpha^n_k)_{k\in\mathbb{N}})_{n\in\mathbb{N}} \in \boldsymbol{\ell}_1(c_0)$. Clearly, $\overline{\alpha} \in \mathbf{c}_0(c_0)$. Put

$$\overline{\beta} = H^{c_0}_{\varrho}(\overline{\alpha}).$$

We need to prove that $\lim_{m\to\infty} |\beta_m| = 0$. Let $\varepsilon > 0$. Observe that the double sequence $\overline{\alpha} = ((\alpha^n_k)_{k\in\mathbb{N}})_{n\in\mathbb{N}} \in \mathbf{c}_0(c_0)$ contains only finitely many elements satisfying $|\alpha^n_k| > \varepsilon$. Put

$$M = \max\{\varrho(n,k) \,\big|\, \left|\beta_{\varrho(n,k)}\right| = |\alpha^n_k| > \varepsilon\}.$$

Then $|\beta_m| \leq \varepsilon$ for each $m > M$. □

Let us repeat Definitions 7.6.2 and 7.6.3 in the context of normed systems of sequences.

Definition 7.6.5. Let $\mathbf{h}$ be a normed system of sequences, let X be a Banach space, and let ϱ be a rearrangement. We define the *rearrangement operator*

$$H^{\mathbf{h}(X)}_{\varrho}\colon \boldsymbol{\ell}_1(\mathbf{h}(X)) \to \omega^s(X)$$

in the following way. Let $(\overline{x}_n)_{n\in\mathbb{N}}$ be an absolutely summable sequence of X-valued sequences, where $\overline{x}_n = (x^n_k)_{k\in\mathbb{N}} \in \mathbf{h}(X)$ for each $n \in \mathbb{N}$. Put

$$H^{\mathbf{h}(X)}_{\varrho}\big((\overline{x}_n)_{n\in\mathbb{N}}\big) = (y_m)_{m\in\mathbb{N}},$$

where $y_{\varrho(n,k)} = x^n_k$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

Definition 7.6.6. Let $\varrho \in \mathrm{Arr}$. A normed system of sequences $\mathbf{h}$ is said to be *ϱ-rearrangeable* if

$$H^{\mathbf{h}(X)}_{\varrho} \in \mathcal{L}\big(\boldsymbol{\ell}_1(\mathbf{h}(X)), \mathbf{h}(X)\big)$$

for every Banach space X.

If $\big\|H^{\mathbf{h}(X)}_{\varrho}\big\| \leq 1$ for every Banach space X, then $\mathbf{h}$ is said to be *metrically ϱ-*rearrangeable.

It is straightforward to prove the following proposition from the definitions.

Proposition 7.6.7. *Let $\varrho \in$ Arr and let g be a normed sequence space. If g is (metrically) ϱ-rearrangeable, then g^s is also (metrically) ϱ-rearrangeable.*

The previous proposition yields the following examples.

Example 7.6.8. Let $\varrho \in$ Arr. The following normed systems of sequences are metrically ϱ-rearrangeable.

(i) $\boldsymbol{\ell}_p$, where $1 \le p \le \infty$;

(ii) $\mathbf{c}_0$.

Consider also the following examples.

Example 7.6.9. Let $\varrho \in \mathrm{Arr}$ and let $1 \le p < \infty$. The following normed systems of sequences are metrically ϱ-rearrangeable.

(i) $\mathbf{c}_0^w$;

(ii) $\boldsymbol{\ell}_p^w$;

(iii) $\boldsymbol{\ell}_p^u$.

Proof. (i). Let X be a Banach space. We prove that $\mathbf{c}_0^w(X)$ is metrically ϱ-rearrangeable. Since $\mathbf{c}_0^w(X) \subset \boldsymbol{\ell}_\infty(X)$ and the norms agree, it suffices to show that

$$H_\varrho^{\mathbf{c}_0^w(X)}\big(\boldsymbol{\ell}_1(\mathbf{c}_0^w(X))\big) \subset \mathbf{c}_0^w(X).$$

Let $\overline{x} = (\overline{x}_n)_n = ((x_k^n)_{k\in\mathbb{N}})_{n\in\mathbb{N}} \in \boldsymbol{\ell}_1(\mathbf{c}_0^w(X))$. Clearly, $\overline{x} \in \mathbf{c}_0(\mathbf{c}_0^w(X))$. Put

$$\overline{y} = H_\varrho^{\mathbf{c}_0^w(X)}(\overline{x}).$$

We need to prove that $\lim_{m\to\infty} |x^*(y_m)| = 0$ for every $x^* \in X^*$. Let $x^* \in X^*$ and let $\varepsilon > 0$. Observe that the double sequence

$$\Big(\big(x^*(x_k^n)\big)_{k\in\mathbb{N}}\Big)_{n\in\mathbb{N}} \in \mathbf{c}_0(c_0)$$

contains only finitely many elements which satisfy $|x^*(x_k^n)| > \varepsilon$. Put

$$M = \max\{\varrho(n,k) \mid \big|x^*(y_{\varrho(n,k)})\big| = |x^*(x_k^n)| > \varepsilon\}.$$

Then $|x^*(y_m)| \le \varepsilon$ for each $m > M$.

(ii). Let X be a Banach space. To prove that $\boldsymbol{\ell}_p^w(X)$ is metrically ϱ-rearrangeable, consider sequences $\overline{x}_n = (x_k^n)_{k\in\mathbb{N}} \in \boldsymbol{\ell}_p^w(X)$ such that

$$\sum_{n=1}^{\infty} \|\overline{x}_n\|_{\boldsymbol{\ell}_p^w(X)} < \infty.$$

Put

$$\overline{x} = (\overline{x}_n) \in \boldsymbol{\ell}_1\left(\boldsymbol{\ell}_p^w(X)\right)$$

and

$$\overline{y} = H_{\varrho}^{\boldsymbol{\ell}_p^w(X)}(\overline{x}).$$

We need to prove that $\overline{y} \in \boldsymbol{\ell}_p^w(X)$ and that

$$\|\overline{y}\|_{\boldsymbol{\ell}_p^w(X)} = \sup_{x^* \in B_{X^*}} \|(x^*(y_m))\|_{\ell_p} \leq \|\overline{x}\|_{\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^w(X))}.$$

Take $x^* \in B_X$ and put

$$\overline{\beta} = (x^*(y_m))_{m \in \mathbb{N}}.$$

To complete the proof, it suffices to prove that $\overline{\beta} \in \ell_p$ and that

$$\left\|\overline{\beta}\right\|_{\ell_p} \leq \|\overline{x}\|_{\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^w(X))}.$$

Put

$$\overline{\alpha}_n = (x^*(x_k^n))_{k \in \mathbb{N}} \in \ell_p$$

for each $n \in \mathbb{N}$. Then

$$\|\overline{\alpha}_n\|_{\ell_p} \leq \|\overline{x}_n\|_{\boldsymbol{\ell}_p^w(X)}.$$

Therefore $\overline{\alpha} = (\overline{\alpha}_n)_{n \in \mathbb{N}} \in \boldsymbol{\ell}_1(\ell_p)$ and

$$\|\overline{\alpha}\|_{\boldsymbol{\ell}_1(\ell_p)} \leq \|\overline{x}\|_{\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^w(X))}.$$

Observe that

$$\overline{\beta} = H_{\varrho}^{\ell_p}(\overline{\alpha}) \in \ell_p$$

and

$$\left\|\overline{\beta}\right\|_{\ell_p} \leq \left\|H_{\varrho}^{\ell_p}\right\| \|\overline{\alpha}\|_{\boldsymbol{\ell}_1(\ell_p)} \leq 1 \cdot \|\overline{x}\|_{\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^w(X))}.$$

(iii). Let X be a Banach space. Since $\boldsymbol{\ell}_p^u(X) \subset \boldsymbol{\ell}_p^w(X)$ and the norms agree, it suffices to show that

$$H_{\varrho}^{\boldsymbol{\ell}_p^u(X)}\big(\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^u(X))\big) \subset \boldsymbol{\ell}_p^u(X).$$

Consider sequences $\overline{x}_n = (x_k^n)_{k \in \mathbb{N}} \in \boldsymbol{\ell}_p^u(X)$ such that

$$\sum_{n=1}^{\infty} \|\overline{x}_n\|_{\boldsymbol{\ell}_p^u(X)} < \infty.$$

Put

$$\overline{x} = (\overline{x}_n) \in \boldsymbol{\ell}_1\left(\boldsymbol{\ell}_p^u(X)\right)$$

and

$$\overline{y} = H_{\varrho}^{\boldsymbol{\ell}_p^u(X)}(\overline{x}) \in \boldsymbol{\ell}_p^w(X).$$

Since the norms of $\boldsymbol{\ell}_p^u(X)$ and $\boldsymbol{\ell}_p^w(X)$ agree, it suffices to prove that $\overline{y} \in \boldsymbol{\ell}_p^u(X)$.
Let $\varepsilon > 0$. Fix $N \in \mathbb{N}$ so that

$$\|(\overline{x}_n)_{n>N}\|_{\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^u(X))} \leq \frac{\varepsilon}{2}. \tag{7.2}$$

Since $(\overline{x}_n) \in \boldsymbol{\ell}_p^u(X)$ for each $1 \leq n \leq N$, we may find an index K so that

$$\left\|(x_{K+1}^n, x_{K+2}^n, \ldots)\right\|_{\boldsymbol{\ell}_p^w(X)} = \left\|(\underbrace{0, \ldots, 0}_{K}, x_{K+1}^n, x_{K+2}^n, \ldots)\right\|_{\boldsymbol{\ell}_p^w(X)} \leq \frac{\varepsilon}{2N} \tag{7.3}$$

for each $1 \leq n \leq N$. Put

$$M = \max\{\varrho(n,k) \,\big|\, 1 \leq n \leq N,\, 1 \leq k \leq K\}.$$

It suffices to prove that

$$\|(y_m)_{m>M}\|_{\boldsymbol{\ell}_p^w(X)} = \left\|(\underbrace{0, \ldots, 0}_{M}, y_{M+1}, y_{M+2}, \ldots)\right\|_{\boldsymbol{\ell}_p^w(X)} \leq \varepsilon.$$

Take $x^* \in B_{X^*}$ and denote

$$\overline{\beta} = (x^*(y_m))_{m\in\mathbb{N}} \in \ell_p.$$

Let us prove that $\|(\beta_m)_{m>M}\|_{\ell_p} \leq \varepsilon$. Denote

$$\alpha_k^n = x^*(x_k^n)$$

for each $k, n \in \mathbb{N}$. Then

$$\overline{\alpha}_n := (\alpha_k^n)_{k\in\mathbb{N}} \in \ell_p$$

for each $n \in \mathbb{N}$, since $(x_k^n)_{k\in\mathbb{N}} \in \boldsymbol{\ell}_p^w(X)$.
Let us write

$$\begin{aligned}
\|(\beta_m)_{m>M}\|_{\ell_p} &= \left(\sum_{m=M+1}^{\infty} |x^*(y_m)|^p\right)^{\frac{1}{p}} = \left(\sum_{\substack{n,k=1,\\ \varrho(n,k)>M}}^{\infty} |x^*(y_k^n)|^p\right)^{\frac{1}{p}} \leq \\
&\leq \left(\sum_{n=N+1}^{\infty}\sum_{k=1}^{\infty} |\alpha_k^n|^p + \sum_{n=1}^{\infty}\sum_{k=K+1}^{\infty} |\alpha_k^n|^p\right)^{\frac{1}{p}} \leq \\
&\leq \left(\sum_{n=N+1}^{\infty}\sum_{k=1}^{\infty} |\alpha_k^n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N}\sum_{k=K+1}^{\infty} |\alpha_k^n|^p\right)^{\frac{1}{p}} = \\
&= \left(\sum_{n=N+1}^{\infty} \|\overline{\alpha}_n\|_{\ell_p}^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} \|(\alpha_k^n)_{k>K}\|_{\ell_p}^p\right)^{\frac{1}{p}}.
\end{aligned}$$

According to (7.2),

$$\left(\sum_{n=N+1}^{\infty}\|\overline{\alpha}_n\|_{\ell_p}^p\right)^{\frac{1}{p}} = \|(\overline{\alpha}_n)_{n>N}\|_{\boldsymbol{\ell}_p(\ell_p)} \leq \|(\overline{\alpha}_n)_{n>N}\|_{\boldsymbol{\ell}_1(\ell_p)} \leq \|(\overline{x}_n)_{n>N}\|_{\boldsymbol{\ell}_1(\boldsymbol{\ell}_p^w(X))} \leq \frac{\varepsilon}{2},$$

since $\|\overline{\alpha}_n\|_{\ell_p} \leq \|\overline{x}_n\|_{\boldsymbol{\ell}_p^w(X)}$ for each $n \in \mathbb{N}$. By (7.3),

$$\left(\sum_{n=1}^{N}\|(\alpha_k^n)_{k>K}\|_{\ell_p}^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{N}\|(x_k^n)_{k>K}\|_{\boldsymbol{\ell}_p^w(X)}^p\right)^{\frac{1}{p}} \leq \sum_{n=1}^{N}\frac{\varepsilon}{2N} \leq \frac{\varepsilon}{2}.$$

Combining the last two observations yields that $\|(\beta_m)_{m>M}\|_{\ell_p} \leq \varepsilon$. □

Let us prove the main result of this section.

Theorem 7.6.10. *Let an expandable BK-space g be given along with a g-compatible expandable normed system of sequences $\mathbf{h}$, which satisfy the following assumptions.*

(i*) *For each $\overline{x} \in \mathbf{h}(X)$, it holds that $\|\overline{x}\|_{\mathbf{h}(X)} \geq \|\overline{x}\|_{g^{\times s}(X)}$;*

(ii) $\|e_1\|_{\mathbf{h}(\mathbb{K})} = \|e_1\|_{g^\times}$*;*

(iii) *there exists a rearrangement $\varrho \in \mathrm{Arr}$ such that both g and $\mathbf{h}$ are ϱ-rearrangeable.*

Then the operator ideal $\Theta_\Lambda[g, \mathbf{h}]$ is a quasi-Banach operator ideal.

Proof. By Theorem 7.5.1, $\Theta_\Lambda[g, \mathbf{h}]$ is a quasi-normed operator ideal (observe that assumption (i*) is stronger than assumption (i) of Theorem 7.5.1).

To prove that $\Theta_\Lambda[g, \mathbf{h}]$ is a quasi-Banach operator ideal, we need to show that each of the quasi-normed components $\Theta_\Lambda[g, \mathbf{h}](X, Y)$ is complete. By Theorem 2.2.4, it suffices to show that they are sequentially complete. For this, we show that a series $\sum_{k=1}^\infty R_k$ in $\big(\Theta_\Lambda[g, \mathbf{h}], \|\cdot\|_{[g,\mathbf{h}]}\big)$ converges whenever $\sum_{k=1}^\infty \|R_k\|_{[g,\mathbf{h}]} < \infty$. By Proposition 2.2.5,

$$\sum_{k=1}^{\infty}\|R_k\| \leq \sum_{k=1}^{\infty}\|R_k\|_{[g,\mathbf{h}]} < \infty.$$

Therefore we may define

$$R = \sum_{k=1}^{\infty} R_k \in \mathcal{L}(X, Y).$$

It remains to show that

$$R \in \Theta_\Lambda[g, \mathbf{h}](X, Y) \tag{7.4}$$

and

$$\lim_{n\to\infty}\left\|R-\sum_{k=1}^{n}R_k\right\|_{[g,\mathbf{h}]}=\lim_{n\to\infty}\left\|\sum_{k=n+1}^{\infty}R_k\right\|_{[g,\mathbf{h}]}=0. \tag{7.5}$$

Let $\varepsilon>0$. By assumption (iii), there exists $\varrho\in\mathrm{Arr}$ such that the rearrangement operators $H_\varrho^g\colon \boldsymbol{\ell}_1(g)\to g$ and $H_\varrho^{\mathbf{h}(Y)}\colon \boldsymbol{\ell}_1(\mathbf{h}(Y))\to\mathbf{h}(Y)$ are well defined, linear, and bounded. Put

$$c_m=2^m\left\|H_\varrho^g\right\|$$

and

$$d_m=\frac{\varepsilon}{4^m\left\|H_\varrho^g\right\|\left\|H_\varrho^{\mathbf{h}(X)}\right\|}.$$

Since the series $\sum_{k=1}^{\infty}R_k$ is absolutely convergent, there exists an increasing sequence (p_m) of natural numbers such that

$$\sum_{k=p_m}^{\infty}\|R_k\|_{[g,\mathbf{h}]}<d_m$$

for each $m\in\mathbb{N}$. Put

$$S_m=\sum_{k=p_m}^{p_{m+1}-1}R_k$$

and note that $\|S_m\|_{[g,\mathbf{h}]}<d_m$ for each $m\in\mathbb{N}$.

Clearly,

$$S:=\sum_{k=1}^{\infty}S_k=\sum_{k=p_1}^{\infty}R_k=R-\sum_{k=1}^{p_1-1}R_k.$$

We prove both (7.4) and (7.5) by showing that $S\in\Theta_\Lambda[g,\mathbf{h}](X,Y)$ and $\|S\|_{[g,\mathbf{h}]}\le\varepsilon$.

Let $m\in\mathbb{N}$. By the definition of the quasi-norm $\|\cdot\|_{[g,\mathbf{h}]}$, there exists a sequence $\overline{y}_m=(y_k^m)_{k\in\mathbb{N}}\in\mathbf{h}(Y)$ with $\|\overline{y}_m\|_{\mathbf{h}(Y)}\le d_m$ such that

$$S_m(B_X)\subset E_{\overline{y}_m}(B_g).$$

Put

$$\overline{y}=(c_n\overline{y}_n)_{n\in\mathbb{N}}\in\boldsymbol{\ell}_1(\mathbf{h}(Y)).$$

The latter definition is correct, because

$$\|\overline{y}\|_{\boldsymbol{\ell}_1(\mathbf{h}(Y))}=\sum_{n=1}^{\infty}c_n\|\overline{y}_n\|_{\mathbf{h}(Y)}\le\sum_{n=1}^{\infty}c_nd_n=\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n\left\|H_\varrho^{\mathbf{h}(X)}\right\|}=\frac{\varepsilon}{\left\|H_\varrho^{\mathbf{h}(X)}\right\|}<\infty.$$

Denote
$$\overline{z} = H_{\varrho}^{\mathbf{h}(X)}(\overline{y}) \in \mathbf{h}(Y).$$
Then
$$\|\overline{z}\|_{\mathbf{h}(Y)} \le \left\|H_{\varrho}^{\mathbf{h}(X)}\right\| \|\overline{y}\|_{\boldsymbol{\ell}_1(\mathbf{h}(Y))} \le \varepsilon.$$
We claim that
$$S(B_X) \subset E_{\overline{z}}(B_g).$$
Let $x \in B_X$. For each $m \ge 1$, we have
$$S_m x = \sum_{k=1}^{\infty} \alpha_k^m y_k^m$$
for some sequence $\overline{\alpha}_m = (\alpha_k^m)_{k\in\mathbb{N}} \in B_g$.

Put
$$\overline{\alpha} = \left(\frac{1}{c_n}\,\overline{\alpha}_n\right)_{n\in\mathbb{N}} \in \boldsymbol{\ell}_1(g).$$
The latter definition is correct, since
$$\|\overline{\alpha}\|_{\boldsymbol{\ell}_1(g)} = \sum_{n=1}^{\infty} \frac{1}{c_n}\,\|\overline{\alpha}_n\|_g \le \frac{1}{\|H_{\varrho}^g\|}.$$

Denote
$$\overline{\beta} = H_{\varrho}^g(\overline{\alpha}) \in g.$$
Then
$$\left\|\overline{\beta}\right\|_g = \left\|H_{\varrho}^g(\overline{\alpha})\right\|_g \le \left\|H_{\varrho}^g\right\| \|\overline{\alpha}\|_{\boldsymbol{\ell}_1(g)} \le 1.$$

We have
$$Sx = \sum_{n=1}^{\infty} S_n x = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_k^n y_k^n.$$
We prove that the double sequence $(\alpha_k^n y_k^n)_{n,k\in\mathbb{N}}$ is absolutely convergent. Since $\overline{\alpha}_n = (\alpha_k^n)_{k\in\mathbb{N}} \in B_g$ and $\overline{y}_n = (y_k^n)_{k\in\mathbb{N}} \in \mathbf{h}(Y) \subset g^{\times s}(Y)$ for each $n \in \mathbb{N}$, we have
$$\sum_{k=1}^{\infty} |\alpha_k^n|\,\|y_k^n\| \le \sup_{\overline{\gamma}\in B_g} \sum_{k=1}^{\infty} |\gamma_k|\,\|y_k^n\| = \|\overline{y}_n\|_{g^{\times s}(Y)}.$$

By assumption (i*),
$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_k^n|\,\|y_k^n\| \le \sum_{n=1}^{\infty} \|\overline{y}_n\|_{g^{\times s}(Y)} \le \sum_{n=1}^{\infty} \|\overline{y}_n\|_{\mathbf{h}(Y)} \le \sum_{n=1}^{\infty} d_m < \infty.$$

The double sequence $(\alpha_k^n y_k^n)_{n,k\in\mathbb{N}}$ is unconditionally convergent, since it is absolutely convergent. This allows us to write

$$Sx = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_k^n y_k^n = \sum_{n,k=1}^{\infty} \left(\frac{1}{c_n}\alpha_k^n\right) \cdot (c_n y_k^n) = \sum_{n,k=1}^{\infty} \beta_{\varrho(n,k)} z_{\varrho(n,k)} = \sum_{m=1}^{\infty} \beta_m z_m.$$

We have shown that $S(B_X) \subset E_{\overline{z}}(B_g)$ and therefore $S \in \Theta_\Lambda[g,\mathbf{h}](X,Y)$. Also,

$$\|S\|_{[g,\mathbf{h}]} \leq \|\overline{z}\|_{\mathbf{h}(Y)} \leq \varepsilon. \qquad \square$$

We have the following easy corollary.

Corollary 7.6.11. *Let g be a shiftable BK-space. Assume that there exists a rearrangement $\varrho \in \mathrm{Arr}$ so that both g and $g^\times$ are ϱ-rearrangeable. Then $\Theta_\Lambda[g, g^{\times s}]$ is a quasi-Banach operator ideal.*

Proof. Corollary 7.5.2 yields that $\Theta_\Lambda[g, g^{\times s}]$ is a quasi-normed operator ideal. Clearly, property (i*) of Theorem 7.6.10 is fulfilled. By Proposition 7.6.7 and the assumption that $g^\times$ is ϱ-rearrangeable, $g^{\times s}$ is also ϱ-rearrangeable and thus the property (iii) holds. Therefore $\Theta_\Lambda[g, g^{\times s}]$ is a quasi-Banach operator ideal by Theorem 7.6.10. $\square$

7.7 Examples of the construction

We are ready to present some examples of quasi-Banach operator ideals produced by the theory outlined in the previous sections. We begin by re-proving the prototypical example of the operator ideal $\mathcal{K}_{(p,r)}$.

Proposition 7.7.1. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Put $g = \ell_r$ and $\mathbf{h} = \boldsymbol{\ell}_p$ (where $\mathbf{h} = \mathbf{c}_0$, if $p = \infty$). Then*

$$\mathcal{K}_{(p,r)} = \Theta_\Lambda[g,\mathbf{h}]$$

is a quasi-Banach operator ideal, where the property (QOI_1) is satisfied for $\varkappa = 2$.

Proof. We proved already in Proposition 7.4.5 that the system of sequences $\mathbf{h}$ is g-compatible and $\mathbf{K}_{(p,r)} = \Lambda[g,\mathbf{h}]$. We know from Example 7.3.6 that ℓ_r is a metrically extendable BK-space. In order to apply Theorem 7.6.10, we need to show that ℓ_r and $\boldsymbol{\ell}_p$ ($\mathbf{c}_0$, if $p = \infty$) satisfy properties (i*), (ii), and (iii) of Theorem 7.6.10.

Property (iii) is proven by Examples 7.6.4 and 7.6.8.

Clearly,

$$\|e_1\|_{\mathbf{c}_0(\mathbb{K})} = \|e_1\|_{\boldsymbol{\ell}_p(\mathbb{K})} = \|e_1\|_{\ell_r^\times} = 1.$$

To show the property (i*), consider first the case when $p = \infty$. Then for each $(x_k) \in \mathbf{c}_0(X)$, it holds that

$$\|(x_k)\|_{\mathbf{c}_0(X)} = \|(x_k)\|_{\mathbf{m}(X)} = \|(x_k)\|_{\ell_1^{\times s}(X)}\,.$$

On the other hand, if $1 \leq p < \infty$, then $\ell_r^{\times s}(X) = \boldsymbol{\ell}_{r^*}(X)$ and

$$\|(x_k)\|_{\boldsymbol{\ell}_p(X)} \geq \|(x_k)\|_{\boldsymbol{\ell}_{r^*}(X)} = \|(x_k)\|_{\ell_r^{\times s}(X)}\,,$$

since $r^* \geq p$ if and only if $r \leq p^*$. □

The following result complements Proposition 7.4.12.

Theorem 7.7.2. *The operator ideal $\Theta_\Lambda[\ell_1, \mathbf{m}]$ is a quasi-Banach operator ideal, where the property (QOI_1) is satisfied for $\varkappa = 2$.*

Proof. This follows from Corollary 7.6.11, since ℓ_1 is a metrically shiftable BK-space and both ℓ_1 and m are metrically ϱ-rearrangeable for any $\varrho \in$ Arr. Corollary 7.5.2 yields that the quasi-constant $\varkappa$ does not exceed 2. □

We proved in Section 6.1 that $\mathcal{W}_\infty$ is a Banach operator ideal. Let us prove this result using the theory from the current chapter (with the shortcoming that we only obtain a quasi-Banach operator ideal).

Proposition 7.7.3. *The operator ideal $\mathcal{W}_\infty = \Theta(\mathbf{W}_\infty) = \Theta_\Lambda[\ell_1, \mathbf{c}_0^w]$ is a quasi-Banach operator ideal, where the property (QOI_1) is satisfied for $\varkappa = 2$.*

Proof. Clearly, $\|\cdot\|_{[\ell_1, \mathbf{c}_0^w]} = \|\cdot\|_{\mathcal{W}_\infty}$ (recall that the definition of the latter norm was given in Definition 6.1.5). We only need to show that ℓ_1 and $\mathbf{c}_0^w$ satisfy properties (i*), (ii), and (iii) of Theorem 7.6.10.

Property (iii) is proven by Examples 7.6.4 and 7.6.9.

Clearly, $\|e_1\|_{\mathbf{c}_0^w(\mathbb{K})} = 1 = \|e_1\|_{\ell_1^\times}$, since $\mathbf{c}_0^w(\mathbb{K}) = \mathbf{c}_0(\mathbb{K}) = c_0$.

To show property (i*), let $\overline{x} \in \mathbf{c}_0^w(X)$. Then

$$\|\overline{x}\|_{\mathbf{c}_0^w(X)} = \|\overline{x}\|_{\mathbf{m}(X)} = \|\overline{x}\|_{g^{\times s}(X)}\,.$$ □

We consider the next two results to be rather interesting examples of our construction. Recall that the operator ideals $\mathcal{W}_{(p,1)}$ and $\mathcal{U}_{(p,1)}$ are special cases of the operator ideals $\mathcal{W}_{(p,r)}$ and $\mathcal{U}_{(p,r)}$, respectively, which were considered in [AO2].

Theorem 7.7.4. *Let* $1 \le p < \infty$. *The operator ideal* $\mathcal{W}_{(p,1)} = \Theta(\mathbf{W}_{(p,1)}) = \Theta_\Lambda[\ell_1, \boldsymbol{\ell}_p^w]$ *is a quasi-Banach operator ideal, where the property* (QOI_1) *is satisfied for* $\varkappa = 2$.

Proof. We only need to show that ℓ_1 and $\boldsymbol{\ell}_p^w$ satisfy properties (i*), (ii), and (iii) of Theorem 7.6.10.

Property (iii) is proven by Examples 7.6.4 and 7.6.9.

Clearly, $\|e_1\|_{\boldsymbol{\ell}_p^w(\mathbb{K})} = 1 = \|e_1\|_{\ell_1^\times}$, since $\boldsymbol{\ell}_p^w(\mathbb{K}) = \boldsymbol{\ell}_p(\mathbb{K}) = \ell_p$.

To show the property (i*), let $\overline{x} \in \boldsymbol{\ell}_p^w(X)$. By Lemma 4.2.6,

$$\|\overline{x}\|_{\boldsymbol{\ell}_p^w(X)} \ge \|\overline{x}\|_{\mathbf{c}_0^w(X)} = \|\overline{x}\|_{\mathbf{m}(X)} = \|\overline{x}\|_{g^{\times s}(X)} . \qquad \square$$

Theorem 7.7.5. *Let* $1 \le p < \infty$. *The operator ideal* $\mathcal{U}_{(p,1)} = \Theta(\mathbf{U}_{(p,1)}) = \Theta_\Lambda[\ell_1, \boldsymbol{\ell}_p^u]$ *is a quasi-Banach operator ideal, where the property* (QOI_1) *is satisfied for* $\varkappa = 2$.

Proof. We only need to show that ℓ_1 and $\boldsymbol{\ell}_p^u$ satisfy properties (i*), (ii), and (iii) of Theorem 7.6.10. Property (iii) is proven by Examples 7.6.4 and 7.6.9. Properties (i*) and (ii) are proven exactly as in the previous theorem, since the norms of $\boldsymbol{\ell}_p^u(X)$ and $\boldsymbol{\ell}_p^w(X)$ coincide. $\square$

The following remarks give an overview of the already known results from the literature about the operator ideals $\mathcal{W}_{(p,r)}$ and $\mathcal{U}_{(p,r)}$ for the special case $r = p^*$ (which our construction does not contain, since the system $\boldsymbol{\ell}_p^w$ is not ℓ_{p^*}-compatible unless $p = \infty$).

Remark 7.7.6. Let $1 \le p < \infty$. In [SK1, Theorem 4.1], it was proved that $\mathcal{W}_{(p,p^*)}$ is a Banach operator ideal when equipped with a certain factorization norm.

Remark 7.7.7. In [AO2, p. 1575], it was proven that $\mathcal{U}_{(p,r)} = \mathcal{N}^{\mathrm{sur}}_{(\infty,p^*,r^*)}$. Recall that by definition, $K_{p^*} = \mathcal{N}_{(\infty,p^*,p)}$, where K_p denotes the ideal of *classical p-compact operators* (see, e.g., [Pi1, 18.3]). (Note that $\mathcal{K}_{(p,p^*)} = \mathcal{K}_p$ and K_p are different as operator ideals (see [O2] and [Pi3])).

In [Ki], the operator ideal $\mathcal{U}_{(p,p^*)}$ was equipped with the following norm (where X and Y are Banach spaces and $T \in \mathcal{U}_{(p,p^*)}(Y,X)$):

$$\|T\|_{\mathcal{U}_{(p,p^*)}} = \inf \|(x_n)\|_p^w ,$$

where the infimum is taken over all sequences $(x_n) \in \boldsymbol{\ell}_p^u(X)$ such that

$$T(B_Y) \subset E_{(x_n)}(B_{\ell_{p^*}}).$$

It was proven that $\mathcal{U}_{(p,p^*)}$ is a Banach operator ideal with respect to this norm. Notice that the norm $\|\cdot\|_{\mathcal{U}_{(p,p^*)}}$ is essentially the same norm that we have been utilizing all along in this chapter.

In [MOP, p. 2885], referring to the fact from [AO2] that $\mathcal{U}_{(p,p^*)} = K_{p^*}^{\mathrm{sur}}$ as operator ideals, $\mathcal{U}_{(p,p^*)}$ was equipped with the norm $\|\cdot\|_{K_{p^*}^{\mathrm{sur}}}$. It was stated that with respect to this norm, $\mathcal{U}_{(p,p^*)}$ becomes a Banach operator ideal. The authors then remarked that an explicit description for the norm $\|\cdot\|_{\mathcal{U}_{(p,p^*)}}$ can be given, using the same technique as in [ALO, Theorem 3.4] (which is Theorem 4.2.12 in the current thesis), and that this obtained norm coincides with the norm provided in [Ki].

7.8 A comparison with λ-compact operators

In this section, we compare the approach taken in [GB] with our construction from this chapter. Let us begin by introducing the necessary terminology and definitions from [GB].

Definition 7.8.1 (see [GB, Definition 3.1])**.** Let λ be a BK-space. An operator $T \in \mathcal{L}(X, Y)$ is said to be λ-*compact* if it maps bounded sets to λ-compact sets.

The collection of all λ-compact operators between arbitrary Banach spaces is denoted by K_λ. Keeping in mind Proposition 7.4.7, we state this by $K_\lambda = \Theta_\Lambda[\lambda^\times, \lambda^s]$ in our terminology.

It is proven in [GB, Theorem 3.10] that, under suitable assumptions, the collection of operators K_λ becomes a quasi-normed operator ideal with respect to the quasi-norm k_λ, where, in our terminology,

$$k_\lambda(T) = \|T\|_{[\lambda^\times, \lambda^s]}\,.$$

We prove that [GB, Theorem 3.10] is encompassed by Theorem 7.5.1 (see Proposition 7.8.9 and the preceding remarks). The following assumptions are made in [GB, Theorem 3.10] (see Definitions 7.8.2–7.8.4 for *emphasized* notions, which are defined below).

(i) λ is a *monotone* and *symmetric* BK-space;

(ii) $\|\cdot\|_\lambda$ is a *k-symmetric* and monotone norm.

Let us provide the necessary definitions.

Definition 7.8.2. A sequence space λ is said to be *symmetric* if $(\alpha_{\sigma(k)}) \in \lambda$ whenever $(\alpha_k) \in \lambda$ and $\sigma \in \Pi$, where Π is the collection of all permutations of the set $\mathbb{N}$.

Definition 7.8.3. Let $(\lambda, \|\cdot\|_\lambda)$ be a symmetric sequence space. The norm $\|\cdot\|_\lambda$ is said to be *k-symmetric* if $\left\|(\alpha_{\sigma(k)})\right\|_\lambda = \|(\alpha_k)\|_\lambda$ whenever $(\alpha_k) \in \lambda$ and $\sigma \in \Pi$.

The next definition is given in [GB] via *J-stepspaces* and the *canonical preimage* of a sequence. We prefer to use the following equivalent definition, using the terminology already established in this thesis.

Definition 7.8.4. A sequence space λ is said to be *monotone* if $(J_\pi^\omega \circ Q_\pi^\lambda)(\lambda) \subset \lambda$ for every $\pi \in \mathrm{Inc}(\mathbb{N})$.

For clarity, we include the following example.

Example 7.8.5. Let λ be a sequence space and let $\pi \in \mathrm{Inc}(\mathbb{N})$ be given by $\pi(k) = 2k$. Then

$$(J_\pi^\omega \circ Q_\pi^\lambda)(\overline{x}) = (0, x_2, 0, x_4, 0, \ldots).$$

Clearly, every solid sequence space is monotone.

Let us prove a few preliminary results.

Lemma 7.8.6. *If a sequence space λ is solid and symmetric with a k-symmetric and monotone norm, then it is* 2*-boundedly shiftable.*

Proof. Let λ be a solid and symmetric sequence space with a k-symmetric and monotone norm. To see that it is 2-boundedly expandable, let $\overline{x} \in \lambda$ and let $\pi \in \mathrm{Inc}(\mathbb{N})$. Consider the sequences

$$\overline{y} = (x_1, 0, x_3, 0, x_5, \ldots)$$

and

$$\overline{z} = (0, x_2, 0, x_4, 0, \ldots).$$

Then $\overline{y}, \overline{z} \in \lambda$, since λ is solid. Observe that the sequences $J_\pi^\lambda(\overline{y})$ and $J_\pi^\lambda(\overline{z})$ are permutations of the sequences $\overline{y}$ and $\overline{z}$, respectively (since there are countably many zeros which we may rearrange at will). Therefore $J_\pi^\lambda(\overline{y}) \in \lambda$ and $J_\pi^\lambda(\overline{z}) \in \lambda$, since λ is symmetric. Also note that $J_\pi^\lambda(\overline{x}) = J_\pi^\lambda(\overline{y}) + J_\pi^\lambda(\overline{z}) \in \lambda$. Since $\|\cdot\|_\lambda$ is k-symmetric and monotone, we get

$$\left\|J_\pi^\lambda(\overline{x})\right\| \leq \left\|J_\pi^\lambda(\overline{y})\right\| + \left\|J_\pi^\lambda(\overline{z})\right\| \leq \|\overline{y}\| + \|\overline{z}\| \leq 2\|\overline{x}\|.$$

To show that λ is 2-boundedly contractible, let $\overline{x} \in \lambda$ and let $\pi \in \mathrm{Inc}(\mathbb{N})$. Since λ is solid and symmetric, we have

$$\overline{y} = (x_{\pi(1)}, 0, x_{\pi(3)}, 0, \ldots) \in \lambda$$

and

$$\overline{z} = (0, x_{\pi(2)}, 0, x_{\pi(4)}, \ldots) \in \lambda.$$

Moreover, $\|\overline{y}\| \leq \|\overline{x}\|$ and $\|\overline{z}\| \leq \|\overline{x}\|$, since $\|\cdot\|_\lambda$ is k-symmetric and monotone. Therefore

$$\left\|Q_\pi^\lambda(\overline{x})\right\| = \|\overline{y} + \overline{z}\| \leq \|\overline{y}\| + \|\overline{z}\| \leq 2\,\|\overline{x}\|\,. \qquad \square$$

Lemma 7.8.7. *Let λ be a solid sequence space with a monotone norm. Then $\lambda^s(\mathbb{K}) = \lambda$ as normed spaces.*

Proof. Let $\overline{\alpha} \in \lambda^s(\mathbb{K})$. By definition, $\overline{\alpha} \in \lambda^s(\mathbb{K})$ whenever $(|\alpha_k|) \in \lambda$. Since λ is solid, this is equivalent to the fact that $\overline{\alpha} = (\alpha_k) \in \lambda$. Moreover, $\|(\alpha_k)\|_\lambda = \|(|\alpha_k|)\|_\lambda$, since $\|\cdot\|_\lambda$ is monotone. $\square$

Lemma 7.8.8. *Let λ be a normed sequence space. Then*

$$\|e_1\|_\lambda = \|e_1\|_{\lambda^{\times\times}}\,.$$

Proof. By Proposition 7.2.15, we only need to show that $\|e_1\|_{\lambda^{\times\times}} \geq \|e_1\|_\lambda$. Observe that

$$\|e_1\|_{\lambda^\times} \leq \frac{1}{\|e_1\|_\lambda},$$

since $e_1/\|e_1\| \in B_\lambda$. Put

$$\overline{\beta} = \frac{e_1}{\|e_1\|_{\lambda^\times}} \in B_{\lambda^\times}.$$

By definition,

$$\|e_1\|_{\lambda^{\times\times}} = \sup\{|\alpha_1| \mid \overline{\alpha} \in B_{\lambda^\times}\} \geq |\beta_1| = \frac{1}{\|e_1\|_{\lambda^\times}} \geq \|e_1\|_\lambda\,. \qquad \square$$

The following result shows that [GB, Theorem 3.10] is encompassed by Theorem 7.5.1. Observe that the monotonicity of λ is not sufficient for λ^s to be a linear space (this can be seen from Remark 7.2.19). To ensure correctness, it suffices to add the solidness of λ to the assumptions of [GB, Theorem 3.10]. Since every solid sequence space is monotone, it becomes unnecessary to keep the assumption that λ is monotone.

Proposition 7.8.9 (cf. [GB, Theorem 3.10])**.** *Let λ be a BK-space satisfying the following conditions.*

(i) λ *is solid and symmetric;*

(ii) $\|\cdot\|_\lambda$ *is a k-symmetric and monotone norm.*

Then

(i) $\lambda^\times$ *is a* 2*-boundedly expandable BK-space;*

(ii) λ^s *is a* 2*-boundedly expandable normed system of sequences;*

(iii) *assumptions* (i) *and* (ii) *of Theorem 7.5.1 hold.*

By Theorem 7.5.1, $K_\lambda = \Theta_\Lambda[\lambda^\times, \lambda^s]$ *is a quasi-normed operator ideal with the quasi-constant* 8*.*

Proof. By Lemma 7.8.6, λ is 2-boundedly shiftable. Proposition 7.3.7 yields that $\lambda^\times$ and λ^s are both 2-boundedly expandable. We know from Theorem 7.2.7 that $\lambda^\times$ is a BK-space. Also, we know from Proposition 7.3.9 that $\lambda^s(X)$ is a normed system of sequences.

It remains to verify that the assumptions of Theorem 7.5.1 hold. To prove (i), let $\overline{\beta} \in \lambda^s(\mathbb{K})$. By Lemma 7.8.7, $\overline{\beta} \in \lambda$ and $\|\overline{\beta}\|_\lambda = \|\overline{\beta}\|_{\lambda^s(\mathbb{K})}$. We need to show that

$$\|\overline{\beta}\|_\lambda \geq \|\overline{\beta}\|_{\lambda^{\times\times}},$$

which is proven in Proposition 7.2.15.

To show (ii) of Theorem 7.5.1, we need to prove that $\|e_1\|_{\lambda^{\times\times}} = \|e_1\|_{\lambda^s(\mathbb{K})} = \|e_1\|_\lambda$, which follows from Lemma 7.8.8. $\square$

Chapter 8

Approximable sets and sequences

We begin this chapter with an overview about some of the known results concerning the approximation property. After this, we give the definitions of an approximable set and an approximable sequence. We prove a Grothendieck-like criterion for describing the approximable sets in a Banach space via the approximable sequences in this space. We also prove that there exists a non-approximable sequence, which can be represented as a sum of three approximable sequences.

8.1 Background and motivation

We begin this chapter with an overview about some of the known results concerning the approximation property, a notion which was introduced by Grothendieck in [G2]. By any means, we do not aim to be exhaustive; rather, we aim only to cover some aspects of it that are directly related to the contents of this chapter. We refer the reader to [O1] for a more detailed overview of the history of the approximation property and its variants; for an extensive historical treatment, see [Pi2, Section 5.7].

Recall that a Banach space X is said to have the *approximation property* (AP) if, for every compact set K in X and every $\varepsilon > 0$, there exists an operator $T \in \mathcal{F}(X)$ so that for every $x \in K$ we have

$$\|(I - T)x\| \leq \varepsilon. \tag{8.1}$$

Let $\lambda \geq 1$. A Banach space X is said to have the λ-*bounded approximation property* (λ-BAP) if the operators $T \in \mathcal{F}(X)$ appearing in the above definition can be chosen so that their norm does not exceed λ. A Banach space X is said to have the *bounded approximation property* (BAP) if it has the λ-BAP for some $\lambda \geq 1$. If $\lambda = 1$, then X has the *metric approximation property* (MAP).

The *approximation problem* asks whether every compact operator between arbitrary Banach spaces can be approximated, in the norm topology of operators, by finite-rank operators. The approximation problem goes back at least to the Polish School in Lwów.

We quote [O1, p. 220]: "The AP and the MAP were deeply studied by A. Grothendieck in his famous Memoir [G2]. He found eight important criteria for the AP and five for the MAP (see [G2, Chapter I, pp. 165 and 179]). In "Proposition" 37 in [ibid, pp. 170–171], he proved among others that the approximation problem is equivalent (...) to the fact that all Banach spaces have the AP. In fact, Grothendieck's "Proposition" 37 contains 19 conditions which are all equivalent to the approximation problem.". One of those equivalent conditions may be formulated as follows: every matrix $A = (a_{j,k})_{j,k=1}^{\infty}$ of scalars, for which $A^2 = 0$, $\sum_{j=1}^{\infty} \max_k |a_{j,k}| < \infty$, and $\lim_k a_{j,k} = 0$, where $j \in \mathbb{N}$, satisfies

$$\operatorname{trace} A = \sum_{n=1}^{\infty} a_{nn} = 0. \tag{8.2}$$

Even though Grothendieck's contribution to the understanding of the approximation problem was enormous, the problem itself remained open. Only 17 years later, in 1972, did Enflo discover a separable reflexive Banach space without the approximation property [E], and therefore solved the approximation problem in the negative.

By "Proposition" 37, Enflo's counterexample demonstrates that there exists a matrix which fails to satisfy condition (8.2), which in turn shows that there exists a closed subspace of c_0 that fails the AP. A simplified construction of such a matrix and the corresponding subspace of c_0 without the AP was given by Davie [Da] in 1973.

We conclude this historical overview with a quote from [O1, p. 220] concerning the properties AP, BAP, and MAP. "Grothendieck remarked [G2, Chapter I, p. 182] that there would exist a Banach space without the BAP, provided that there exist Banach spaces having the BAP, but failing the λ-BAP for arbitrarily large λ. This idea was made explicit by T. Figiel and W. B. Johnson [FJ] in 1973. They succeeded to construct a sequence of Banach spaces X_n, $n = 1, 2, \ldots$, with the BAP but failing the n-BAP, in particular, failing the MAP, and observed that the direct ℓ_2-sum $(\sum_{n=1}^{\infty} X_n)_2$ has the AP but fails the BAP. These were the first counterexamples showing that the AP, BAP, and MAP are, in general, different notions."

Since there are Banach spaces without the AP, it makes sense to study the extent to which a given Banach space X has the AP. In the spirit of this thesis, we propose to study the *system* **AP** *of approximable sets*, i.e., sets that satisfy condition (8.1). More precisely, let a bounded set K belong to the component $\mathbf{AP}(X)$ if for every $\varepsilon > 0$ there exists an operator $T \in \mathcal{F}(X)$ so that

$$\|(I - T)x\| \leq \varepsilon \tag{8.3}$$

for every $x \in K$. Let $\lambda \geq 1$. Define the *system* $\lambda-$**BAP** *of* λ*-boundedly approximable sets* by letting $K \in \lambda-\mathbf{BAP}(X)$ if the operators $T \in \mathcal{F}(X)$ appearing in the above definition can be chosen so that their norms do not exceed λ. The *system* **BAP** *of boundedly approximable sets* consists of all λ-boundedly approximable sets for any $\lambda \geq 1$. Denote $\mathbf{MAP} = 1-\mathbf{BAP}$.

The following well-known result demonstrates that all approximable sets are relatively compact.

Lemma 8.1.1. *Let X be a Banach space. Then $\mathbf{AP}(X) \subset \mathbf{K}(X)$.*

Proof. Let $G \in \mathbf{AP}(X)$ and let $\varepsilon > 0$. Fix an operator $T \in \mathcal{F}(X)$, which satisfies condition (8.3). Then the set $T(G)$ is a relatively compact ε-net to the set G. □

It follows that a Banach space X has the AP if and only if $\mathbf{AP}(X) = \mathbf{K}(X)$. Similarly, X has the λ-AP if and only if $\mathbf{K}(X) = \lambda-\mathbf{BAP}(X)$.

The Grothendieck compactness principle characterizes relatively compact subsets of a Banach space X as those that reside in the closed absolutely convex hull of a null sequence. In our terminology, this may be stated concisely as $\mathbf{K} = \Lambda[\ell_1, \mathbf{c}_0]$. Could the approximable sets be characterized in an analogous fashion? In order to answer this question, we introduce some additional definitions.

Let us define the *system* **ap** *of approximable sequences* as follows: $(x_k) \in \mathbf{ap}(X)$ if (x_k) is a null sequence in X and for every $\varepsilon > 0$ there exists an operator $T \in \mathcal{F}(X)$ satisfying

$$\|(I - T)x_k\| \leq \varepsilon \tag{8.4}$$

for each $k \in \mathbb{N}$. Let $\lambda \geq 1$. Define the *system* $\lambda-$**bap** *of* λ*-boundedly approximable sequences* by letting $(x_k) \in \lambda-\mathbf{bap}(X)$ if the operators $T \in \mathcal{F}(X)$ appearing in the above definition can be chosen so that their norms do not exceed λ. The *system* **bap** *of boundedly approximable sequences* consists of all λ-boundedly approximable sequences for any $\lambda \geq 1$. Denote $\mathbf{map} = 1-\mathbf{bap}$.

We prove (see Theorem 8.2.3) that the (λ-bounded) approximable sets are exactly those that reside in the closed absolutely convex hull of a (λ-bounded) approximable sequence. In our terminology, these observations may be stated concisely by $\mathbf{AP} = \Lambda[\ell_1, \mathbf{ap}]$ and $\lambda-\mathbf{BAP} = \Lambda[\ell_1, \lambda-\mathbf{bap}]$. A consequence of this is that if one desires to describe the structure $\mathbf{AP}(X)$ of the approximable sets of a Banach space X, then it is sufficient to describe the (presumably simpler) structure $\mathbf{ap}(X)$ of approximable sequences in X. Note that a Banach space X has the AP if and only if $\mathbf{c}_0(X) = \mathbf{ap}(X)$, and λ-BAP if and only if $\mathbf{c}_0(X) = \lambda-\mathbf{bap}(X)$.

From the perspective of this thesis, it is natural to ask whether the systems of sets $\mathbf{AP}$, $\mathbf{BAP}$, $\lambda-\mathbf{BAP}$ are generating systems of sets? Keeping in mind that $\mathbf{AP} = \Lambda[\ell_1, \mathbf{ap}]$ and $\lambda-\mathbf{BAP} = \Lambda[\ell_1, \lambda-\mathbf{bap}]$, we ask whether the systems of sequences $\mathbf{ap}$, $\mathbf{bap}$, $\lambda-\mathbf{bap}$ are normed systems of sequences (when equipped with the supremum norm of the system $\mathbf{c}_0$)? Answer to all of these questions turns out to be no – a straightforward verification shows these systems are not closed with respect to applying bounded linear operators (see Propositions 8.5.1 and 8.5.2).

In Sections 8.3–8.5, we revisit some of the classical results due to Grothendieck and prove modified versions of them, where we concentrate on specific approximable sets and sequences, instead of looking at the whole Banach space at once. In this perspective, Theorem 8.3.1 (which is an extract of [G2, Proposition 35]) and the direction $\neg$(ii) $\Rightarrow$ $\neg$(i) of "Proposition" 8.4.1 (which is an extract of [G2, "Proposition" 37]) become Theorem 8.3.8 and Proposition 8.4.2, respectively. Our approach, combined with the matrix constructed by Davie, yields an example of a non-approximable sequence, which can be represented as a sum of three approximable sequences (see Theorem 8.5.8). This proves that the systems $\mathbf{AP}$ and $\mathbf{ap}$ are not closed with respect to addition (i.e., they do not satisfy properties (G_2) and (NS_1), respectively).

Recall that the example of Figiel and Johnson in 1973 showed that the BAP and the AP are different notions for Banach spaces. However, it is not immediately apparent from this fact that the systems $\mathbf{BAP}$ and $\mathbf{AP}$ are different. Indeed, consider a Banach space X having the AP, but failing the BAP. This means that for each $\lambda \geq 1$, there exists a set that fails the λ-BAP. But does this guarantee that $\mathbf{BAP}(X) \neq \mathbf{AP}(X)$? In order to answer this question affirmatively, we need to construct a single set (or alternatively, a single sequence) which does not belong to $\lambda-\mathbf{BAP}(X)$ (respectively, $\lambda-\mathbf{bap}(X)$) for *any* $\lambda \geq 1$. Indeed, we are able to construct such a sequence. We do it by taking, for each $n \in \mathbb{N}$, an approximable sequence which fails to be n-boundedly approximable, and manipulating and reordering these sequences into a single approximable sequence, which fails to be boundedly approximable. In fact, we show that this construction works in every Banach space X having the AP but failing the BAP (see Proposition 8.6.1).

Coming to terms with the fact that the system $\mathbf{AP}$ fails to be a generating system of sets, it is natural to ask the following questions. What is the smallest generating system of sets encompassing $\mathbf{AP}$? What is the largest generating system of sets residing in **AP**? It is relatively easy to show that the answer to the first question is the system **K** of all relatively compact sets (see Corollary 8.5.3). We propose that the answer to the second question might be the system $\mathbf{K}_{(2,2)}$ of all relatively $(2,2)$-compact sets (see Question 8.5.4).

8.2 Criterion for approximable sets

By a classical and well-known criterion (see, e.g., [LiT, p. 37]), a Banach space X has the λ-AP if and only if every finite set is λ-boundedly approximable. Using essentially the same proof as in [LiT, p. 37], we obtain the following criterion for λ-boundedly approximable sets.

Proposition 8.2.1. *Let X be a Banach space. Then $K \in \lambda{-}\mathbf{BAP}(X)$ if and only if for every $\varepsilon > 0$ there exists a finite set $L \in \lambda{-}\mathbf{BAP}(X)$ so that L is an ε-net to the set K.*

Proof. For the "only if" part, let $K \in \lambda{-}\mathbf{BAP}(X)$ and let $\varepsilon > 0$. Find an ε-net L in the set K (we may do so, since $\lambda{-}\mathbf{BAP}(X) \subset \mathbf{AP}(X) \subset \mathbf{K}(X)$). Then also $L \in \lambda{-}\mathbf{BAP}(X)$, since $L \subset K$.

To prove the "if" part, take a set K, fix $\varepsilon > 0$, and take a finite set $L = \{x_1, \dots, x_n\} \in \lambda{-}\mathbf{BAP}(X)$ so that L is an $\frac{\varepsilon}{3\lambda}$-net to the set K. Now let $T \in \mathcal{F}(X)$ be given such that $\|T\| \le \lambda$ and $\|Tx_j - x_j\| \le \frac{\varepsilon}{3}$ for every $j \in \mathbb{N}$. Fix $x \in K$ and choose an index $j \in \{1, \dots, n\}$ so that $\|x - x_j\| \le \frac{\varepsilon}{3\lambda}$. Then

$$\|Tx - x\| \le \|Tx - Tx_j\| + \|Tx_j - x_j\| + \|x_j - x\| \le \frac{\varepsilon}{3\lambda}\|T\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\lambda} \le \varepsilon. \quad \square$$

According to the following well-known result, every finite set $\{x_1, \dots, x_n\}$ is approximable. Therefore one cannot replace "$\lambda{-}\mathbf{BAP}$" with "$\mathbf{AP}$" in Proposition 8.2.1 above.

Lemma 8.2.2. *Let X be a Banach space and let $\{x_1, \dots, x_n\} \subset X$. Then*

$$\{x_1, \dots, x_n\} \in \mathbf{AP}(X).$$

Proof. Put $Y = \operatorname{span}\{x_1, \dots, x_n\}$. Auerbach's Lemma provides a projection P of X onto Y such that $\|P\| \le n$. Observe that

$$\|Px - x\| = 0$$

for every $x \in Y$. $\square$

The following result draws inspiration from the Grothendieck compactness principle. It describes approximable sets as those sets which are situated in the closed absolutely convex hulls of approximable sequences. It also gives a similar characterization for λ-boundedly approximable sets.

Theorem 8.2.3. *Let X be a Banach space and let $K \subset X$. Then K is approximable if and only if there exists a sequence $\overline{x} \in \mathbf{ap}(X)$ such that*

$$K \subset E_{\overline{x}}(B_{\ell_1}) = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \mid \overline{\alpha} \in B_{\ell_1} \right\}.$$

Let $\lambda \geq 1$. Then K is λ-boundedly approximable if and only if there exists a sequence $\overline{x} \in \lambda{-}\mathbf{bap}(X)$ such that

$$K \subset E_{\overline{x}}(B_{\ell_1}) = \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \mid \overline{\alpha} \in B_{\ell_1} \right\}.$$

These two observations can also be written as

$$\mathbf{AP} = \Lambda[\ell_1, \mathbf{ap}]$$

and

$$\lambda{-}\mathbf{BAP} = \Lambda[\ell_1, \lambda{-}\mathbf{bap}].$$

Proof. The proofs for the both cases are similar in their nature. Therefore the proof is only written once, and there are remarks in the places where the proof techniques diverge. Note that the proof of the "only if" direction is inspired by the proof of the Grothendieck compactness principle.

To prove the "if" direction, take a (λ-bounded) approximable sequence (x_k) so that

$$K \subset \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \mid \overline{\alpha} \in B_{\ell_1} \right\}.$$

Let $\varepsilon > 0$. We may choose an operator $T \in \mathcal{F}(X)$ so that $\|(I-T)(x_k)\| \leq \varepsilon$ for each $k \in \mathbb{N}$ (if $K \in \lambda{-}\mathbf{BAP}(X)$, we require $\|T\| \leq \lambda$). Let $x \in X$. Every element of the set K can be represented as a sum $\sum_{k=1}^{\infty} \alpha_k x_k$, where $\overline{\alpha} \in B_{\ell_1}$, so we have

$$\|(I-T)(x)\| = \left\| (I-T) \left(\sum_{k=1}^{\infty} \alpha_k x_k \right) \right\| \leq \sum_{k=1}^{\infty} |\alpha_k| \, \|(I-T)(x_k)\| \leq \varepsilon \sum_{k=1}^{\infty} |\alpha_k| \leq \varepsilon.$$

To prove the "only if" direction, let K be a (λ-boundedly) approximable set. Choose, for each $k \in \mathbb{N}$, an operator $T_k \in \mathcal{F}(X)$ so that

$$\|(I - T_k)(x)\| \leq \frac{1}{2^{2k+1}} \tag{8.5}$$

for every $x \in K$ (we additionally require that $\|T_k\| \leq \lambda$ if K is λ-boundedly approximable). Put

$$\varepsilon_k = \min\left\{\frac{1}{4^k}, \frac{1}{2^{k+2}(1+\|T_1\|)}, \dots, \frac{1}{2^{2k+1}(1+\|T_k\|)}\right\}. \tag{8.6}$$

Choose an ε_1-net $\{y_{1,j}\}_{j=1}^{n_1}$ from the set K. Define a relatively compact set $K_{1,j} = B(y_{1,j}, \varepsilon_1) \cap K$ for each $1 \leq j \leq n_1$ and put $I_1 = \{1, \dots, n_1\}$.

We start constructing the sequence (x_k) by putting $x_j = 2y_{1,j}$, where $1 \leq j \leq n_1$. The general idea of the proof is that in every k-th step hereafter ($k \geq 2$), we will

(a) choose a finite ε_k-net $L_{k,j} \subset K_{k-1,j}$ for each set $K_{k-1,j}$, where $1 \leq j \leq n_{k-1}$; unite all of those nets into a single set and denote it by $\{y_{k,j}\}_{j=1}^{n_k}$;

(b) denote $K_{k,j} = B(y_{k,j}, \varepsilon_k) \cap K$ for each $1 \leq j \leq n_k$;

(c) denote $I_k = \{m_{k-1}+1, \dots, m_{k-1}+n_k\}$, where $m_k = \max\{r \mid r \in I_k\}$;

(d) put $x_m = 2^k(y_{k,j} - y_{k-1,p})$ for each $1 \leq j \leq n_k$, where $m = m_{k-1}+j \in I_k$ and $y_{k,j} \in L_{k,p} \subset K_{k-1,p}$ (note that $\|x_m\| \leq 2^k\varepsilon_{k-1}$, since $y_{k,j} \in B(y_{k-1,p}, \varepsilon_{k-1})$).

This construction gives us that

(i) $(x_k) \in \mathbf{c}_0(X)$;

(ii) every $x \in K$ can be expressed as a sum $\sum_{k=1}^{\infty} \frac{1}{2^k} x_{j_k}$, where $j_k \in I_k$.

(iii) $\|(I - T_m)(x_j)\| \leq \frac{1}{2^m}$ for every $j, m \in \mathbb{N}$.

To see (i), observe that if $k \geq 2$ and $j \in I_k$, then $\|x_j\| \leq 2^k\varepsilon_{k-1} \leq \frac{1}{2^{k-2}} \xrightarrow{k\to\infty} 0$.

By Lemma 4.1.5, it suffices to verify (ii) to show that $K \subset E_{\overline{x}}(B_{\ell_1})$, since

$$\sum_{k=1}^{\infty} \frac{1}{2^k} x_{j_k} \in \left\{\sum_{k=1}^{\infty} \alpha_k x_k \mid \overline{\alpha} \in B_{\ell_1}\right\}.$$

Let $x \in K$ and take $r_1 \in \{1, \dots, n_1\}$ such that $x \in B(y_{1,r_1}, \varepsilon_1)$. Then $x \in K_{1,r_1}$. Take $r_2 \in \{1, \dots, n_2\}$ such that $x \in B(y_{2,r_2}, \varepsilon_2)$ and $y_{2,r_2} \in L_{1,r_1}$. Then $x \in K_{2,r_2}$. Continuing in this manner, we obtain a sequence (y_{k,r_k}) such that $\lim_{k\to\infty} y_{k,r_k} = x$.

Put $j_k = m_k + r_k \in I_k$ if $k \geq 2$; put $j_1 = r_1 \in I_1$. Write

$$x = \lim_{k\to\infty} y_{k,r_k} = y_{1,r_1} + \sum_{k=2}^{\infty}(y_{k,r_k} - y_{k-1,r_{k-1}}) = \frac{1}{2}x_{j_1} + \sum_{k=2}^{\infty}\frac{1}{2^k}x_{j_k} = \sum_{k=1}^{\infty}\frac{1}{2^k}x_{j_k}.$$

It remains to prove (iii), which yields that $(x_j) \in \mathbf{ap}(X)$. Furthermore, if the set K was assumed to be λ-boundedly approximable, then we chose the operators T_k so that $\|T_k\| \leq \lambda$, in which case (iii) yields that $(x_j) \in \lambda{-}\mathbf{bap}(X)$.

We use two different techniques to prove that $\|(I - T_m)(x_j)\| \leq \frac{1}{2^m}$ for every $j, m \in \mathbb{N}$. Using condition (8.5), we will prove that it holds if $1 \leq n \leq m$, where $j \in I_n$; using condition (8.6), we will prove that this inequality holds if $1 \leq m < n$, where $j \in I_n$.

Let $1 \leq n \leq m$ and let $j \in I_n$. Then $x_j = 2^n(y_{n,l} - y_{n-1,p})$, where $y_{n,l} \in K$ and $y_{n-1,p} \in K$. By condition (8.5),

$$\begin{aligned} \|(I - T_m)(x_j)\| &= \|(I - T_m)(2^n(y_{n,l} - y_{n-1,p}))\| \leq \\ \leq 2^n\|(I - T_m)(y_{n,l})\| + 2^n\|(I - T_m)(y_{n-1,p})\| &\leq 2^{n+1}\frac{1}{2^{2m+1}} \leq \frac{1}{2^m}. \end{aligned}$$

Let $1 \leq m < n$ and let $j \in I_n$. Therefore $\|x_j\| \leq 2^n\varepsilon_{n-1}$. By condition (8.6),

$$\|(I - T_m)(x_j)\| \leq (1 + \|T_m\|)\|x_j\| \leq 2^n\varepsilon_{n-1}(1 + \|T_m\|) \leq \frac{2^n}{2^{n+m}} = \frac{1}{2^m}. \qquad \square$$

Remark 8.2.4. Let $\mathcal{A}$ be an operator ideal satisfying $\mathcal{F} \subset \mathcal{A} \subset \mathcal{K}$. Consider the systems of "$\mathcal{A}$-approximable sets" and "$\mathcal{A}$-approximable sequences", obtained by replacing the operator ideal $\mathcal{F}$ with $\mathcal{A}$ in the definitions of approximable sets and sequences, respectively. It is not difficult to see that Theorem 8.2.3 will remain true in this more general context, using essentially the same proof.

8.3 A characterization of approximable sequences

In this and the following two sections we revisit some of the classical results due to Grothendieck and interpret their proofs in the context of approximable sets and sequences (instead of concentrating on the whole space). Recall the following result from [G2] (see, e.g., [LiT, Theorem 1.e.4]).

Theorem 8.3.1 ([G2, Proposition 35]). *Let X be a Banach space. The following are equivalent.*

(i) *X has the AP;*

(ii) *For every choice $(x_k) \subset X$, $(x_k^*) \subset X^*$ such that $\sum_{k=1}^{\infty} \|x_k^*\| \|x_k\| < \infty$ and $\sum_{k=1}^{\infty} x_k^*(x)x_k = 0$, for all $x \in X$, we have $\sum_{k=1}^{\infty} x_k^*(x_k) = 0$.*

In this section, we prove a more specific version of the above theorem (see Theorem 8.3.8), which allows us to characterize whether a given sequence $(x_k) \in \mathbf{c}_0(X)$ is approximable. Essentially we follow the proof of Theorem 8.3.1 as given in [LiT, pp. 31–33]. We begin by recalling the relevant topology.

Definition 8.3.2 (see [LiT, Proposition 1.e.3]). Let X and Y be Banach spaces. Let $K \in \mathbf{K}(X)$ and define the semi-norm $p_K \colon \mathcal{L}(X, Y) \to \mathbb{K}$ as follows.

$$p_K(T) = \sup_{x \in K} \|Tx\|.$$

Denote by τ_c the locally convex topology generated by the family of semi-norms

$$\{p_K \mid K \in \mathbf{K}(X)\}.$$

One can also define this topology using a different set of semi-norms.

Definition 8.3.3. Let X and Y be Banach spaces. Let $\overline{x} \in \mathbf{c}_0(X)$ and define the semi-norm $p_{\overline{x}} \colon \mathcal{L}(X, Y) \to \mathbb{K}$ as follows.

$$p_{\overline{x}}(T) = \sup_{k \in \mathbb{N}} \|Tx_k\|.$$

Put another way,

$$p_{\overline{x}} = p_{\{x_k\}}.$$

Lemma 8.3.4. *The topology τ_c coincides with the locally convex topology τ_c' generated by the family of semi-norms*

$$\{p_{\overline{x}} \mid \overline{x} \in \mathbf{c}_0(X)\}.$$

Proof. To see that the topology τ_c' is weaker than τ_c, observe that every semi-norm $p_{\overline{x}}$, where $\overline{x} \in \mathbf{c}_0(X)$, is equal to a semi-norm $p_{\{x_k\}}$, where $\{x_k\} \in \mathbf{K}(X)$.

To show that the topology τ_c' is stronger than the topology τ_c, take a semi-norm p_K, where $K \in \mathbf{K}(X)$. According to Grothendieck's compactness principle, there exists a sequence $\overline{x} \in \mathbf{c}_0(X)$ so that $K \subset E_{\overline{x}}(B_{\ell_1})$. It suffices to show that $p_K(T) \le p_{\overline{x}}(T)$ for each $T \in \mathcal{L}(X,Y)$. In other words, we need to show that

$$\sup_{x \in K} \|Tx\| \le \sup_{k \in \mathbb{N}} \|Tx_k\|.$$

Let $x \in K$. Since $K \subset E_{\overline{x}}(B_{\ell_1})$, there exists a scalar sequence $\overline{\alpha} \in B_{\ell_1}$ with $x = \sum_{k=1}^{\infty} \alpha_k x_k$. Thus

$$\|Tx\| = \left\| T\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) \right\| = \left\| \sum_{k=1}^{\infty} \alpha_k T x_k \right\| \le \sup_{k \in \mathbb{N}} \|Tx_k\| \sum_{k=1}^{\infty} |\alpha_k| \le \sup_{k \in \mathbb{N}} \|Tx_k\|. \quad \square$$

We now introduce another topology on $\mathcal{L}(X,Y)$ which, in general, is considerably weaker than the topology τ_c.

Definition 8.3.5. Let $\overline{x} \in \mathbf{c}_0(X)$. Denote by $\tau_c^{\overline{x}}$ the locally convex topology generated by the single semi-norm $p_{\overline{x}}$.

Recall the following Grothendieck's result.

Proposition 8.3.6 (see, e.g., [LiT, Proposition 1.e.3])**.** *Continuous linear functionals on $(\mathcal{L}(X,Y), \tau_c)$ coincide with the functionals f of the form*

$$f(T) = \sum_{k=1}^{\infty} y_k^*(Tx_k), \ \ (x_k) \subset X, \ \ (y_k^*) \subset Y^*, \ \ \sum_{k=1}^{\infty} \|x_k\| \, \|y_k^*\| < \infty.$$

We continue by proving a modified version of the aforementioned result. We include a proof for completeness, although it is essentially the same as in [LiT, Proposition 1.e.3].

Proposition 8.3.7. *Let $\overline{x} \in \mathbf{c}_0(X)$. Continuous linear functionals on $(\mathcal{L}(X,Y), \tau_c^{\overline{x}})$ coincide with the functionals f of the form*

$$f(T) = \sum_{k=1}^{\infty} y_k^*(Tx_k), \ \ (y_k^*) \subset \boldsymbol{\ell}_1(Y^*).$$

Proof. Assume that f has such a representation. Then

$$|f(T)| \le \sum_{k=1}^{\infty} \|y_k^*\| \, \|Tx_k\| \le \sup_{k \in \mathbb{N}} \|Tx_k\| \sum_{k=1}^{\infty} \|y_k^*\| \le \sup_{k \in \mathbb{N}} \|Tx_k\| = p_{\overline{x}}(T),$$

which proves that f is continuous.

Conversely, assume that f is a linear functional on $\mathcal{L}(X,Y)$ so that $|f(T)| \leq Cp_{\overline{x}}(T)$ for some constant C. Define $S\colon \mathcal{L}(X,Y) \to \mathbf{c}_0(Y)$ by

$$S(T) = (Tx_1, Tx_2, \ldots).$$

Observe that $\ker S \subset \ker f$. Indeed,

$$|f(T)| \leq Cp_{\overline{x}}(T) = C \sup_{k\in\mathbb{N}} \|Tx_k\| = C\,\|S(T)\|_{\mathbf{c}_0(Y)}\,.$$

It is well known that one may define (see, e.g., [M, Theorem 1.7.13, 1.7.14]) a bounded linear operator $g_0\colon \operatorname{ran} S \to \mathbb{K}$ by

$$g_0(ST) = f(T).$$

By the Hahn–Banach theorem, we may extend g_0 to a continuous linear functional g on $\mathbf{c}_0(Y)$. Denote by J the canonical isomorphism from $\mathbf{c}_0(Y)^*$ to $\boldsymbol{\ell}_1(Y^*)$ and put $(y_k^*) = J(g)$. Then

$$f(T) = g_0(ST) = g(ST) = J^{-1}Jg(ST) = (J^{-1}(y_k^*))(ST) = \sum_{k=1}^{\infty} y_k^*(Tx_k). \qquad \square$$

We are now ready to prove the main result of this section. The proof follows the proof of Theorem 8.3.1 (see [LiT, Theorem 1.e.4]) with obvious modifications.

Theorem 8.3.8. *Let X be a Banach space and let $(x_k) \in \mathbf{c}_0(X)$. The following are equivalent.*

(i) $(x_k) \in \mathbf{ap}(X)$;

(ii) $I_X \in \overline{\mathcal{F}(X,X)}^{\tau_c^{\overline{x}}}$;

(iii) *For every choice $(x_k^*) \in \boldsymbol{\ell}_1(X^*)$ such that $\sum_{k=1}^{\infty} x_k^*(x)x_k = 0$ for all $x \in X$, we have $\sum_{k=1}^{\infty} x_k^*(x_k) = 0$.*

Proof. Observe that (ii) means that for every $\varepsilon > 0$ there exists an operator $T \in \mathcal{F}(X,X)$ such that $p_{\overline{x}}(T - I_X) \leq \varepsilon$. This is equivalent to the fact that $\overline{x} \in \mathbf{ap}(X)$, since

$$p_{\overline{x}}(T - I_X) = \sup_{k\in\mathbb{N}} \|Tx_k - x_k\|.$$

By the Tukey-Klee separation theorem and the fact that every finite-rank operator can be expressed as a sum of rank 1 operators, (ii) is equivalent to the fact that every operator $f \in (\mathcal{L}(X,X), \tau_c^{\overline{x}})^*$, which vanishes on operators of rank 1, vanishes also on the identity operator. Let us show the the latter statement is equivalent to (iii).

According to Proposition 8.3.7, $f \in (\mathcal{L}(X,X), \tau_c^{\overline{x}})^*$ if and only if there exists a sequence $(x_k^*) \subset \boldsymbol{\ell}_1(X^*)$ such that

$$f(T) = \sum_{k=1}^{\infty} x_k^*(Tx_k).$$

Recall that f vanishes on all operators of rank 1 if and only if $f(x^* \otimes x) = 0$ for every $x^* \in X^*$, $x \in X$. Observe that

$$f(x^* \otimes x) = \sum_{k=1}^{\infty} x_k^*((x^* \otimes x)x_k) = \sum_{k=1}^{\infty} x^*(x_k)\, x_k^*(x) = x^* \left(\sum_{k=1}^{\infty} x_k^*(x) x_k \right).$$

Since $f(I_X) = \sum_{k=1}^{\infty} x_k^*(x_k)$, it remains to show only that

$$\forall x^* \in X^*,\, \forall x \in X,\ x^* \left(\sum_{k=1}^{\infty} x_k^*(x) x_k \right) = 0 \Leftrightarrow \forall x \in X,\, \sum_{k=1}^{\infty} x_k^*(x) x_k = 0.$$

The “only if” part follows from the Hahn–Banach theorem; the “if” part is obvious. □

We remark that having proven Theorem 8.3.8 for a single approximable sequence $\overline{x}$, one may easily obtain the original Theorem 8.3.1 from it in the following way.

Proof of Theorem 8.3.1. (ii) $\Rightarrow$ (i). For every choice $(x_k) \subset X$, $(x_k^*) \subset X^*$ such that $\sum_{k=1}^{\infty} \|x_k^*\|\, \|x_k\| < \infty$ and $\sum_{k=1}^{\infty} x_k^*(x) x_k = 0$, for all $x \in X$, we have $\sum_{k=1}^{\infty} x_k^*(x_k) = 0$. By the part (ii) $\Rightarrow$ (i) of Theorem 8.3.8, it holds that every sequence $(x_k) \in \mathbf{c}_0(X)$ is approximable and therefore X has the AP.

(i) $\Rightarrow$ (ii). Let X be a Banach space with the approximation property. Let $(x_k) \subset X$ and $(x_k^*) \subset X^*$ so that $\sum_{k=1}^{\infty} \|x_k^*\|\, \|x_k\| < \infty$ and $\sum_{k=1}^{\infty} x_k^*(x) x_k = 0$ for every $x \in X$. We need to show that $\sum_{k=1}^{\infty} x_k^*(x_k) = 0$. Take a sequence (β_k) of positive scalars tending to ∞ so that

$$\sum_{k=1}^{\infty} \beta_k \|x_k^*\|\, \|x_k\| < \infty.$$

Put

$$y_k = \begin{cases} \frac{x_k}{\beta_k \|x_k\|}, & \text{if } x_k \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$y_k^* = \beta_k \|x_k\| x_k^*.$$

Then $(y_k^*) \in \boldsymbol{\ell}_1(X^*)$ and $(y_k) \in \mathbf{c}_0(X) = \mathbf{ap}(X)$. Since $y_k^*(y_k) = x_k^*(x_k)$ and $y_k^*(x)y_k = x_k^*(x)x_k$ for every $x \in X$, then by the part (i) $\Rightarrow$ (ii) of Theorem 8.3.8,

$$\sum_{k=1}^{\infty} x_k^*(x_k) = \sum_{k=1}^{\infty} y_k^*(y_k) = 0. \qquad \square$$

8.4 Non-approximable sequences arising from infinite matrices

Recall the following result, which we mentioned in the introduction to the current chapter.

"Proposition" 8.4.1 (see [G2, "Proposition" 37])**.** *The following are equivalent.*

(i) *Every Banach space has the AP;*

(ii) *every matrix $A = (a_{jk})$ of scalars, for which $\lim_k a_{jk} = 0$ for each $j \in \mathbb{N}$,*

$$\sum_{j=1}^{\infty} \max_{k\in\mathbb{N}} |a_{jk}| < \infty \textit{ and } A^2 = 0, \textit{ satisfies } \operatorname{trace} A = \sum_{n=1}^{\infty} a_{nn} = 0.$$

Of course, this result in the given formulation is only of historical interest, since we now know that both of these statements are false. However, the simple and explicit proof enables us to transfer a counterexample of one statement to a counterexample of the other statement. Specifically, we are interested in the direction (i) $\Rightarrow$ (ii), i.e., $\neg$(ii) $\Rightarrow$ $\neg$(i). By inspecting the proof of the latter direction (see, e.g., [LiT, Proposition 1.e.8]) and modifying it slightly, we obtain the following result.

Proposition 8.4.2. *Let $A = (a_{jk})$ be an infinite matrix with the following properties.*

(i) $\lim_k a_{jk} = 0$ *for each* $j \in \mathbb{N}$*;*

(ii) $\sum_{j=1}^{\infty} \max_{k\in\mathbb{N}} |a_{jk}| < \infty$*;*

(iii) $A^2 = 0$*;*

(iv) $\operatorname{trace} A \neq 0$.

Denote the j-th row of A by $x_j = (a_{jk})_{k\in\mathbb{N}}$. Note that $x_j \in c_0$ for every $j \in \mathbb{N}$. Define $X = \overline{\operatorname{span}}\{x_j \mid j \in \mathbb{N}\} \subset c_0$. Then there exists a sequence of positive scalars (β_j) tending to ∞ so that the sequence

$$\left(\frac{x_j}{\beta_j\,\|x_j\|}\right)$$

converges to zero in X, but is not approximable in X.

Proof. Let e_k, where $k \in \mathbb{N}$, denote the k-th unit vector of $c_0^* = \ell_1$, restricted to the subspace $X \subset c_0$. Then $e_k(x_j) = a_{jk}$ and

$$\|e_k\|_{X^*} = \sup_{x\in B_X} |e_k(x)| \le \sup_{x\in B_{c_0}} |e_k(x)| = 1.$$

By assumption,

$$\sum_{j=1}^{\infty} \|e_j\|\,\|x_j\| \le \sum_{j=1}^{\infty} 1\cdot\left(\max_k |a_{jk}|\right) = \sum_{j=1}^{\infty} \max_k |a_{jk}| < \infty.$$

Fix $j \in \mathbb{N}$ and $k \in \mathbb{N}$. Then

$$e_k\left(\sum_{n=1}^{\infty} e_n(x_j)x_n\right) = \sum_{n=1}^{\infty} e_n(x_j)e_k(x_n) = \sum_{n=1}^{\infty} a_{jn}a_{nk} = 0,$$

since the latter is the entry in the j-th row and k-th column of the matrix A^2.

Let $x \in X$ be an arbitrary non-zero element. We will show that for every $k \in \mathbb{N}$,

$$e_k\left(\sum_{n=1}^{\infty} e_n(x)x_n\right) = 0.$$

Let $\varepsilon > 0$. Choose an element $\hat{x} \in \operatorname{span}\{x_j\}$ so that

$$\|x - \hat{x}\| < \varepsilon\Big/\left(\sum_{n=1}^{\infty} \max_k |a_{nk}|\right).$$

Then

$$e_k\left(\sum_{n=1}^{\infty} e_n(\hat{x})x_n\right) = \sum_{n=1}^{\infty} e_n(\hat{x})e_j(x_n) = 0.$$

Observe that

$$\left| e_k\left(\sum_{n=1}^{\infty} e_n(x)x_n\right)\right| \le \left| e_k\left(\sum_{n=1}^{\infty} e_n(x-\hat{x})x_n\right)\right| + \left| e_k\left(\sum_{n=1}^{\infty} e_n(\hat{x})x_n\right)\right| \le$$
$$\|e_k\|\left(\sum_{n=1}^{\infty}\|e_n\|\,\|x-\hat{x}\|\,\|x_n\|\right) \le \|x-\hat{x}\|\sum_{n=1}^{\infty}\max_k |a_{nk}| < \varepsilon.$$

Take $x^* \in X^*$ and find its continuous extension $\hat{x}^* \in \ell_1$. Then $\hat{x}^* = \sum_{j=1}^{\infty} a_j e_j$ and

$$x^*\left(\sum_{n=1}^{\infty} e_n(x)x_n\right) = \hat{x}^*\left(\sum_{n=1}^{\infty} e_n(x)x_n\right) = \sum_{j=1}^{\infty} a_j e_j\left(\sum_{n=1}^{\infty} e_n(x)x_n\right) = \sum_{j=1}^{\infty} 0 = 0.$$

We have shown that every functional x^* vanishes on the element $\sum_{n=1}^{\infty} e_n(x)x_n$, where $x \in X$ is arbitrarily chosen. Therefore $\sum_{n=1}^{\infty} e_n(x)x_n = 0$ for every $x \in X$. Observe that

$$\sum_{j=1}^{\infty} e_j(x_j) = \sum_{j=1}^{\infty} a_{jj} = \operatorname{trace} A \ne 0.$$

Recall that $\sum_{j=1}^{\infty}\|e_j\|\,\|x_j\| < \infty$. Take a sequence of positive scalars (β_j) tending to ∞ so that

$$\sum_{j=1}^{\infty} \beta_j\,\|e_j\|\,\|x_j\| < \infty.$$

Put

$$y_j = \begin{cases} \frac{x_j}{\beta_j\|x_j\|}, & \text{if } x_j \ne 0,\\ 0, & \text{otherwise.}\end{cases}$$

and

$$f_j = \beta_j\,\|x_j\|\,e_j,$$

where $j \in \mathbb{N}$. Then $(y_j) \in c_0(c_0)$ and $(f_j) \in \ell_1(\ell_1) = \boldsymbol{\ell}_1(c_0^*)$. Since $f_j(y_j) = e_j(x_j) \ne 0$ and $f_j(x)y_j = e_j(x)x_j = 0$ for every $x \in X$, then $(y_j) \notin \mathbf{ap}(c_0)$ by Theorem 8.3.8. □

8.5 Are approximable sets a generating system of sets?

It is natural to ask whether the systems **AP** and $\lambda-$**BAP** are generating systems of sets. Clearly, the systems of sets **AP** and $\lambda-$**BAP** satisfy properties (G_0) (each approximable set is relatively compact), (G_1) (the unit ball $B_{\mathbb{K}}$ is an approximable set), and (G_3) (a subset of an approximable set is an approximable set) and they are closed with respect to multiplication by a scalar. However, neither of the aforementioned systems is closed with respect to applying bounded linear operators, as demonstrated by Proposition 8.5.2 below. To see this, let K in Proposition 8.5.2 be any relatively compact set which fails to be (λ-boundedly) approximable.

On a related note, are the systems **ap** and $\lambda-$**bap** normed systems of sequences (when equipped with the supremum norm of $\mathbf{c}_0$)? Clearly, they satisfy property (NS_2). Indeed, any sequence $(x_k) \in \mathbf{fin}(\mathbb{K})$ is (1-boundedly) approximable, since $I_{\mathbb{K}}$ has rank 1 (and $\|I_{\mathbb{K}}\| = 1$). However, neither **ap** nor $\lambda-$**bap** is closed with respect to applying bounded linear operators, as demonstrated by Proposition 8.5.1 below. To see this, let (x_k) in Proposition 8.5.1 be any convergent sequence which fails to be (λ-boundedly) approximable.

Proposition 8.5.1. *Let X be a Banach space and let $\overline{x} \in \mathbf{c}_0(X)$. Then there exists an operator $T \in \mathcal{L}(\ell_1, X)$ and a sequence $(\overline{\alpha}_k) \in \mathbf{c}_0(\ell_1) = 1-\mathbf{bap}(\ell_1)$ such that*

$$(x_k) = (T\overline{\alpha}_k)_{k\in\mathbb{N}}.$$

Proof. Put $\overline{\alpha}_k = \|x_k\|\, e_k \in \ell_1$ for each $k \in \mathbb{N}$. Define an operator $T \in \mathcal{L}(\ell_1, X)$ by

$$T(e_k) = \frac{x_k}{\|x_k\|} \text{ (put } T(e_k) = 0 \text{ if } x_k = 0).$$

Obviously, $(\overline{\alpha}_k) \in \mathbf{c}_0(\ell_1) = 1-\mathbf{bap}(\ell_1)$. Then

$$(T\overline{\alpha}_k)_{k\in\mathbb{N}} = (\|x_k\|\, Te_k)_{k\in\mathbb{N}} = (x_k). \qquad \square$$

Proposition 8.5.2. *Let X be a Banach space and let $K \in \mathbf{K}(X)$. Then there exists an operator $T \in \mathcal{L}(\ell_1, X)$ and a set $G \in \mathbf{K}(\ell_1) = 1-\mathbf{BAP}(\ell_1)$ such that*

$$K \subset T(G).$$

Proof. Let K be a relatively compact set. Then $K \subset E_{\overline{x}}(B_{\ell_1})$ for some sequence $\overline{x} \in \mathbf{c}_0(X)$. Define $(\overline{\alpha}_k) \in \mathbf{c}_0(\ell_1)$ and $T \in \mathcal{L}(\ell_1, X)$ exactly as in the proof of the previous proposition. Denote $G = E_{(\overline{\alpha}_k)}(B_{\ell_1})$ and observe that $G \in \mathbf{K}(\ell_1) = 1-\mathbf{BAP}(\ell_1)$. Therefore

$$K \subset E_{\overline{x}}(B_{\ell_1}) = E_{(T\overline{\alpha}_k)}(B_{\ell_1}) = TE_{(\overline{\alpha}_k)}(B_{\ell_1}) = T(G). \qquad \square$$

The following corollary follows easily from Proposition 8.5.2 and the fact that every generating system of sets satisfies property (G_4).

Corollary 8.5.3. $\mathbf{K}$ *is the smallest generating system of sets which contains the system* $\mathbf{AP}$.

Coming to terms with the fact that the system $\mathbf{AP}$ fails to be a generating system of sets, it is natural to ask the following question: what is the largest generating system of sets residing in $\mathbf{AP}$? Sinha and Karn proved in [SK1, Theorem 6.4] that every Banach space has the $(2,2)$-*approximation property*. In our terminology, this result can be stated as $\mathbf{K}_{(2,2)} \subset \mathbf{AP}$. We pose the following question.

Question 8.5.4. Is $\mathbf{K}_{(2,2)}$ the largest generating system of sets which is contained by the system $\mathbf{AP}$?

It turns out that the systems $\mathbf{AP}$ and $\mathbf{ap}$ also fail to be closed under addition (of sets and sequences, respectively); this will be demonstrated in Theorem 8.5.8. However, compared to the above proposition, its proof is rather more involved. The proof is based on the "shape" of an infinite matrix, constructed by Davie in [Da], which satisfies the assumptions of Proposition 8.4.2. The latter proposition then yields an example of a non-approximable null sequence in a subspace X of c_0. We will show that this sequence can be represented as a sum of three approximable sequences (see Theorem 8.5.8).

Our proof relies on the following lemma.

Lemma 8.5.5. *Let* Y *be a Banach space and let* X *be its subspace. If* $G \in \mathbf{AP}(X)$, *then* $G \in \mathbf{AP}(Y)$. *Similarly, if* $\overline{x} \in \mathbf{ap}(X)$, *then* $\overline{x} \in \mathbf{ap}(Y)$.

Proof. Let $G \in \mathbf{AP}(X)$ and let $\varepsilon > 0$. Then there exists an operator $T \in \mathcal{F}(X)$ so that

$$\|(I-T)x\| \leq \varepsilon$$

for every $x \in G \subset X$. Since T is of finite rank, it admits a representation

$$T = \sum_{k=1}^{n} x_k^* \otimes x_k,$$

where $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*$, and $x_1, \ldots, x_n \in X$. By the Hahn–Banach theorem, each of those bounded linear functionals x_k^* can be extended to a bounded linear functional y_k^* on Y. Put $S = \sum_{k=1}^{n} y_k^* \otimes x_k$. Then $S \in \mathcal{F}(Y,Y)$ and for every $x \in G \subset Y$ we have

$$\|(I-S)x\| = \left\| \left(I - \sum_{k=1}^{n} y_k^* \otimes x_k \right) x \right\| = \|(I-T)x\| \leq \varepsilon,$$

i.e., $G \in \mathbf{AP}(Y)$. The claim is proved similarly for approximable sequences. $\square$

Remark 8.5.6. We do not know whether the above result also holds for λ-boundedly approximable sets and sequences. Notice that the line of thought employed in the above lemma cannot be used in that case; indeed, although the Hahn–Banach theorem allows us to choose extensions y_k^* such that $\|y_k^*\| = \|x_k^*\|$, it does not guarantee that $\|S\| = \|T\|$.

We also need the following result to prove Theorem 8.5.8.

Proposition 8.5.7. *Let A be an infinite diagonal block matrix of the following shape, where blocks B_k are any scalar matrices of any size $m_k \times n_k$ and zeros denote the zero matrices of suitable sizes.*

$$A = \begin{pmatrix} B_1 & 0 & 0 & \dots \\ 0 & B_2 & 0 & \dots \\ 0 & 0 & B_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let $\overline{x}_k$ denote the k-th row of A and let $X = \overline{\operatorname{span}}\{\overline{x}_k \mid k \in \mathbb{N}\} \subset c_0$. Then the space X has the 1-BAP.

Proof. Denote the linear span of first m_1 rows by X_1, the linear span of next m_2 rows by X_2, etc. For each $k \in \mathbb{N}$ define the projection $P_k \colon X \to X$ as follows.

$$P_k(x_j) = \begin{cases} x_j, & \text{if } x_j \in X_k, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the definition of P_k is correct and that $\|P_k\| \leq 1$ (if X_k is a non-trivial space, then $\|P_k\| = 1$). Since X_k is a finite-dimensional subspace of X, we have $P_k \in \mathcal{F}(X, X)$. Put

$$S_k = \sum_{j=1}^{\infty} P_j.$$

Clearly, $\|S_k\| \leq 1$ and $S_k \in \mathcal{F}(X, X)$ for each $k \in \mathbb{N}$.

Let $(\overline{y}_k) \in \mathbf{c}_0(X)$. We need to show that $(\overline{y}_k) \in 1\mathbf{-bap}(X)$. Fix $\varepsilon > 0$. Since the sequence $(\overline{y}_k)$ converges to 0 in X, there exists an index $n_0 \in \mathbb{N}$ such that $\|\overline{y}_n\| \leq \frac{\varepsilon}{2}$ for every $n > n_0$.

Let $1 \leq j \leq n_0$. Since $\overline{y}_j = (y_j^k)_{k\in\mathbb{N}} \in X$, then there exists an index m_j so that $\left\|(I - S_{m_j})\overline{y}_j\right\| \leq \varepsilon$. Put

$$M = \max_{1\leq j\leq n_0} m_j.$$

For each $n \leq n_0$, we have

$$\|(I - S_M)\overline{y}_n\| \leq \|(I - S_{m_n})\overline{y}_n\| \leq \varepsilon.$$

On the other hand, for every $n > n_0$, we have

$$\|(I - S_M)\overline{y}_n\| \leq (\|I\| + \|S_M\|)\, \|\overline{y}_n\| \leq 2\, \|\overline{y}_n\| \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Therefore we may conclude that $(\overline{y}_k) \in 1-\mathbf{bap}(X)$. $\square$

We are now ready to give the main result of this section. Denote by A the matrix constructed by Davie, which satisfies the assumptions of Proposition 8.4.2. It is a block matrix and has the following shape (where P_k is a certain $2^{k+1} \times 3 \cdot 2^k$ matrix and Q_k is a certain $2^k \times 3 \cdot 2^k$ matrix for each $k \in \mathbb{N}_0$):

$$A = \begin{pmatrix} P_0^* P_0 & P_0^* Q_1 & 0 & 0 & 0 & \cdots \\ -Q_1^* P_0 & P_1^* P_1 - Q_1^* Q_1 & P_1^* Q_2 & 0 & 0 & \cdots \\ 0 & -Q_2^* P_1 & P_2^* P_2 - Q_2^* Q_2 & P_2^* Q_3 & 0 & \cdots \\ 0 & 0 & -Q_3^* P_2 & P_3^* P_3 - Q_3^* Q_3 & P_3^* Q_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Following Proposition 8.4.2, denote the j-th row of A by $\overline{x}_j = (a_{jk})_{k \in \mathbb{N}}$. Clearly, $\overline{x}_j \in c_0$ for every $j \in \mathbb{N}$. Put $X = \overline{\operatorname{span}}\{\overline{x}_j \mid j \in \mathbb{N}\} \subset c_0$. Then there exists a sequence of positive scalars (β_j) tending to ∞ so that the sequence

$$(\overline{x}'_j) = \left(\frac{\overline{x}_j}{\beta_j \, \|\overline{x}_j\|} \right)$$

converges to zero in X, but is not approximable in X.

Define the matrices B, C, and D in the following way.

(i) The matrix B is obtained by putting rows filled with zeros instead of all rows of the matrix A, except for the rows from 1st, 4th, 7th, etc. blocks of rows.

(ii) The matrix C is obtained by putting rows filled with zeros instead of all rows of the matrix A, except for the rows from 2nd, 5th, 8th, etc. blocks of rows.

(iii) The matrix D is obtained by putting rows filled with zeros instead of all rows of the matrix A, except for the rows from 3rd, 6th, 9th, etc. blocks of rows.

For the sake of clarity, we write out a part of the matrix B.

$$\begin{pmatrix} P_0^*P_0 & P_0^*Q_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -Q_3^*P_2 & P_3^*P_3 - Q_3^*Q_3 & P_3^*Q_4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Clearly, $A = B + C + D$. Denote by $\overline{y}_j$, $\overline{z}_j$, and $\overline{w}_j$ the j-th rows of B, C, and D, respectively. It is easy to see that $\overline{x}_j = \overline{y}_j + \overline{z}_j + \overline{w}_j$ for each $j \in \mathbb{N}$. Put $Y = \overline{\operatorname{span}}\{\overline{y}_j \mid j \in \mathbb{N}\}$; $Z = \overline{\operatorname{span}}\{\overline{z}_j \mid j \in \mathbb{N}\}$; $W = \overline{\operatorname{span}}\{\overline{w}_j \mid j \in \mathbb{N}\}$. Then each of the spaces Y, Z, and W is a subspace of X. The following theorem demonstrates that the systems **ap** and **AP** are not closed with respect to addition.

Theorem 8.5.8. *Let the space $X \subset c_0$, the scalar sequence (β_j), and the sequences $(\overline{x}_j)$, $(\overline{y}_j)$, $(\overline{z}_j)$, and $(\overline{w}_j)$ be given as described above. Put*

$$(\overline{y}'_j) = \left(\frac{\overline{y}_j}{\beta_j \|\overline{x}_j\|}\right), \ (\overline{z}'_j) = \left(\frac{\overline{z}_j}{\beta_j \|\overline{x}_j\|}\right), \ \text{and } (\overline{w}'_j) = \left(\frac{\overline{w}_j}{\beta_j \|\overline{x}_j\|}\right).$$

Then each of the sequences $(\overline{y}'_j)$, $(\overline{z}'_j)$, and $(\overline{w}'_j)$ belongs to the system $\mathbf{ap}(X)$, but their sum $\left(\overline{x}'_j\right)$ does not belong to the system $\mathbf{ap}(X)$. Likewise, each of the sets $\left\{\overline{y}'_j\right\}$, $\left\{\overline{z}'_j\right\}$, and $\left\{\overline{w}'_j\right\}$ belongs to the system $\mathbf{AP}(X)$, but $\left\{\overline{x}'_j\right\} \notin \mathbf{AP}(X)$, although

$$\left\{\overline{x}'_j\right\} = \left\{\overline{y}'_j + \overline{z}'_j + \overline{w}'_j\right\} \subset \left\{\overline{y}'_j\right\} + \left\{\overline{z}'_j\right\} + \left\{\overline{w}'_j\right\}.$$

Proof. Clearly, $\left(\overline{y}'_j\right) \in \mathbf{c}_0(Y)$, $\left(\overline{z}'_j\right) \in \mathbf{c}_0(Z)$, and $\left(\overline{w}'_j\right) \in \mathbf{c}_0(W)$. According to Proposition 8.5.7, $(\overline{y}'_j) \in 1{-}\mathbf{bap}(Y)$, $(\overline{z}'_j) \in 1{-}\mathbf{bap}(Z)$, and $(\overline{w}'_j) \in 1{-}\mathbf{bap}(W)$. Lemma 8.5.5 yields that each of the sequences $(\overline{y}'_k)$, $(\overline{z}'_k)$, and $(\overline{w}'_k)$ belongs to the system $\mathbf{ap}(X)$. We conclude the proof by recalling that $(\overline{x}'_j) \notin \mathbf{ap}(X)$. □

We conclude this section by remarking that we do not know whether the systems $\lambda{-}\mathbf{BAP}$ and $\lambda{-}\mathbf{bap}$ also fail to be closed with respect to addition.

8.6 Approximable sequences and sets which are not boundedly approximable

Recall that the example of Figiel and Johnson [FJ] in 1973 showed that the BAP and the AP are different notions for Banach spaces. From the point of view of this thesis, we are interested in the question whether the systems **BAP** and **AP** are also different. The answer to this questions turns out to be yes, as demonstrated by the following result.

Proposition 8.6.1. *Let X be a Banach space which has the AP, but does not have the BAP. Then* $\mathbf{bap}(X) \neq \mathbf{ap}(X)$. *Similarly,* $\mathbf{BAP}(X) \neq \mathbf{AP}(X)$.

Proof. For each $n \in \mathbb{N}$, fix a set $K_n \in \mathbf{AP}(X)$ such that $K_n \notin n-\mathbf{BAP}(X)$. According to Theorem 8.2.3, there exists a sequence $\overline{x}_n = (x_k^n)_{k\in\mathbb{N}} \in \mathbf{ap}(X)$ such that $K_n \subset E_{\overline{x}_n}(B_{\ell_1})$, but $\overline{x}_n \notin n-\mathbf{bap}(X)$. Put $M_n = \sup_{k\in\mathbb{N}} \|x_k^n\| \neq 0$.

Define $\overline{y} = (y_n)$ as any array of the following elements:

$$\begin{gathered}
\frac{x_1^1}{2M_1}, \frac{x_2^1}{2M_1}, \dots, \frac{x_k^1}{2M_1}, \dots, \\
\frac{x_1^2}{4M_2}, \frac{x_2^2}{4M_2}, \dots, \frac{x_k^2}{4M_2}, \dots, \\
\dots, \\
\frac{x_1^n}{2^n M_n}, \frac{x_2^n}{2^n M_n}, \dots, \frac{x_k^n}{2^n M_n}, \dots, \\
\dots
\end{gathered}$$

Observe that $\overline{y} \in \mathbf{c}_0(X)$. But this means that $\overline{y} \in \mathbf{ap}(X)$, since X has the *AP*. Assume to the contrary that $\overline{y} \in \mathbf{bap}(X)$. Then $\overline{y} \in n-\mathbf{bap}(X)$ for some $n \in \mathbb{N}$. Thus also the sequence

$$\frac{\overline{x}_n}{2^n M_n} = \frac{(x_k^n)_{k\in\mathbb{N}}}{2^n M_n},$$

which is a re-ordered subsequence of $\overline{y}$, belongs to $n-\mathbf{bap}(X)$. But then the sequence $\overline{x}_n$ also belongs to $n-\mathbf{bap}(X)$, since the system $n-\mathbf{bap}$ is closed with respect to scalar multiplication. Furthermore, this yields that $K_n \in n-\mathbf{BAP}(X)$. We have reached a contradiction, as desired. $\square$

Bibliography

[A] K. Ain, *Compactness and null sequences defined by ℓ_p spaces*, Diss. Math. Univ. Tartu. **97**, Tartu University Press, Tartu, 2015.

[ALO] K. Ain, R. Lillemets, and E. Oja, *Compact operators which are defined by ℓ_p-spaces*, Quaest. Math. **35** (2012) 145–159.

[AO1] K. Ain and E. Oja, *A description of relatively (p, r)-compact sets*, Acta Comment. Univ. Tartu. Math. **16** (2012) 227–232.

[AO2] K. Ain and E. Oja, *On (p, r)-null sequences and their relatives*, Math. Nachr. **288** (2015) 1569–1580.

[BCFP] G. Botelho, D. Cariello, V. V. Fávaro, and D. Pellegrino, *Maximal spaceability in sequence spaces*, Linear Algebra Appl. **437** (2012) 2978–2985.

[BR] J. Bourgain and O. Reinov, *On the approximation properties for the space H^∞*, Math. Nachr. **122** (1985) 19–27.

[Bo] J. Boos and P. Cass, *Classical and Modern Methods in Summability*, Oxford University Press, Oxford, 2000.

[DP] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order, Second Edition*, Cambridge University Press, Cambridge, 2002.

[Da] A. M. Davie, *The approximation problem for Banach spaces*, Bull. London Math. Soc. **5** (1973) 261–266.

[DFJP] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974) 311–327.

[DF] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, Math. Studies **176**, North–Holland, Amsterdam–London–New York–Tokyo, 1993.

[DOPS] J. M. Delgado, E. Oja, C. Piñeiro, and E. Serrano, *The p-approximation property in terms of density of finite rank operators*, J. Math. Anal. Appl. **354** (2009) 159–164.

[DPS1] J. M. Delgado, C. Piñeiro, and E. Serrano, *Operators whose adjoints are quasi p-nuclear*, Studia Math. **197** (2010) 291–304.

[DPS2] J. M. Delgado, C. Piñeiro, and E. Serrano, *Density of finite rank operators in the Banach space of p-compact operators*, J. Math. Anal. Appl. **370** (2010) 498–505.

[Di1] J. Diestel, *A survey of results related to Dunford–Pettis property*, Contemp. Math. **2** (1980) 15–60.

[Di2] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.

[DJT] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge, 1995.

[DFLORT] P. N. Dowling, D. Freeman, C. J. Lennard, E. Odell, B. Randrianantoanina, and B. Turett, *A weak Grothendieck compactness principle*, J. Funct. Anal. **263** (2012) 1378–1381.

[E] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. **130** (1973) 309–317.

[FHHMPZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, Springer–Verlag, New York, 2001.

[FJ] T. Figiel and W. B. Johnson, *The approximation property does not imply the bounded approximation property*, Proc. Amer. Math. Soc. **41** (1973) 197–200.

[FS] J. Fourie and J. Swart, *Banach ideals of p-compact operators*, Manuscripta Math. **26** (1979) 349–362.

[GLT] D. Galicer, S. Lassalle, and P. Turco, *The ideal of p-compact operators: a tensor product approach*, Studia Math. **211** (2012) 269–286.

[G1] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. **5** (1953) 129–173.

[G2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[GB] M. Gupta and A. Bhar, *On λ-compact operators*, Indian J. Pure Appl. Math. **44**(3) (2013) 355–374.

[GL] V. I. Gurariy and W. Lusky, *Geometry of Müntz Spaces and Related Questions*, Springer–Verlag, Berlin–Heidelberg, 2005.

[HLP] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[JLO] W. B. Johnson, R. Lillemets, and E. Oja, *Representing completely continuous operators through weakly ∞-compact operators*, Bull. London Math. Soc. **48** (2016) 452–456.

[Kal] N. Kalton, *Quasi-Banach spaces*, in: W. B. Johnson and J. Lindenstrauss (eds.), Handbook of the Geometry of Banach Spaces. Vol. 2, North–Holland, Amsterdam (2003) 1099–1130.

[Kałuża] R. Kałuża, *Through a Reporter's Eyes: the Life of Stefan Banach*, Birkhäuser, Boston, 1996.

[KG] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Lecture Notes No. **65**, Marcel Dekker Inc., New York–Basel, 1981.

[KA] L. V. Kantorovich and G. P. Akilov, *Functional Analysis, Second Edition, revised*, Izdat. "Nauka", Moscow, 1977 (in Russian).

[Ki] J. M. Kim, *Unconditionally p-null sequences and unconditionally p-compact operators*, Studia Math. **224** (2014) 133–142.

[K] G. Köthe, *Topological Vector Spaces I*, Springer–Verlag, Berlin–Heidelberg–New York, 1969.

[KT] G. Köthe and O. Toeplitz, *Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen*, Jour. reine angew. Math. **171** (1934) 193–226.

[Lil1] R. Lillemets, *Galois connections between generating systems of sets and sequences*, Positivity (2016) (online), DOI: 10.1007/s11117-016-0451-z

[Lil2] R. Lillemets, *Representing the Banach operator ideal of completely continuous operators*, Proc. Estonian Acad. Sci. Phys. Math. **66**(2) (2017) 189–193.

[LaT] S. Lassalle and P. Turco, *On p-compact mappings and the p-approximation property*, J. Math. Anal. Appl. **389** (2012) 1204–1221.

[LiT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Ergebnisse der Mathematik und ihrer Grenzgebiete **92**, Springer–Verlag, Berlin–Heidelberg–New York, 1977.

[Scottish] R. D. Mauldin (ed.), *The Scottish Book: Mathematics from the Scottish Café*, Birkhäuser, Boston (1981).

[M] R. E. Megginson, *An Introduction to Banach Space Theory,* Graduate Texts in Mathematics **183**, Springer-Verlag, New York, 1998.

[MOP] F. Muñoz, E. Oja, and C. Piñeiro, *On α-nuclear operators with applications to vector-valued function spaces*, J. Funct. Anal. **269** (2015) 2871–2889.

[O1] E. Oja, *On bounded approximation properties of Banach spaces*, Banach Center Publ. **91.1** (2010) 219–231.

[O2] E. Oja, *A remark on the approximation of p-compact operators by finite-rank operators*, J. Math. Anal. Appl. **387** (2012) 949–952.

[Pi1] A. Pietsch, *Operator Ideals*, Deutsch. Verlag Wiss., Berlin, 1978; North–Holland Publishing Company, Amsterdam–New York–Oxford, 1980.

[Pi2] A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007.

[Pi3] A. Pietsch, *The ideal of p-compact operators and its maximal hull*, Proc. Amer. Math. Soc. **142** (2014) 519–530.

[Ra1] M. S. Ramanujan, *Absolutely λ-summing operators, λ a symmetric sequence space*, Math. Z. **114** (1970) 187–193.

[Ra2] M. S. Ramanujan, *Generalized nuclear maps in normed linear spaces*, J. Reine Angew. Math. **244** (1970) 190–197.

[Re1] O. Reinov, *A survey of some results in connection with Grothendieck approximation property*, Math. Nachr. **119** (1984) 257–264.

[Re2] O. I. Reinov, *On linear operators with p-nuclear adjoints*, Vestnik St. Petersburg Univ. Math. **33**, no. 4 (2000) 19–21; translation from Vestnik St-Petersburg Univ., Ser. I, Mat. Mekh. Astron. 2000, no. 4 (2000) 24–27 (in Russian).

[Ry] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

[SK1] D. P. Sinha and A. K. Karn, *Compact operators whose adjoints factor through subspaces of* ℓ_p, Studia Math. **150** (2002) 17–33.

[SK2] D. P. Sinha and A. K. Karn, *Compact operators which factor through subspaces of* ℓ_p, Math. Nachr. **281** (2008) 412–423.

[S] I. Stephani, *Generating systems of sets and quotients of surjective operator ideals*, Math. Nachr. **99** (1980) 13–27.

[Sw] C. Swartz, *An Introduction to Functional Analysis*, Marcel Dekker, New York, 1992.

[W] A. Wilansky, *Summability through Functional Analysis*, Math. Studies **85**, North–Holland, Amsterdam, 1984.

Genereerivate hulkade ja jadade süsteemid

Kokkuvõte

Operaatorideaalide teooria sai alguse A. Pietschi monograafiast [Pi1] ning on tänaseks saanud kaasaegse Banachi ruumide teooria lahutamatuks osaks. I. Stephani tõi artiklis [S] sisse kaks operaatorideaalidega tihedalt seotud mõistet: genereerivate hulkade süsteem ja genereerivate jadade süsteem. Nimelt, lähtudes kahest etteantud genereerivate hulkade süsteemist, saame me tekitada operaatorideaali, mis koosneb kõigist operaatoritest, mis teisendavad esimesse süsteemi kuuluvad hulgad teise süsteemi kuuluvateks hulkadeks. Genereerivate jadade süsteeme saab omakorda kasutada genereerivate hulkade süsteemide tekitamiseks. Siinkohal märgime, et genereerivate hulkade ja jadade süsteemide mõisteid on uuritud operaatorideaalidest oluliselt vähem.

Üks väitekirja põhieesmärkidest on uurida genereerivate hulkade ja jadade süsteemide klasse ning nendevahelisi seoseid. Muuhulgas tõestatakse, et leidub Galois' vastavus genereerivate hulkade süsteemide klassi ja teatava genereerivate jadade süsteemide faktorklassi vahel.

Teine väitekirja eesmärk on uurida järgmisi struktuure ning nendega seotud klasse võreteoreetilisest aspektist: operaatorideaalide klass, genereerivate hulkade süsteemide klass ja genereerivate jadade süsteemide klass.

Nagu eelnevalt mainitud, näitas Stephani, kuidas genereerivate hulkade süsteemi kaudu saab tekitada operaatorideaale. Sellele lisaks näitas ta, kuidas etteantud operaatorideaalist saab tekitada genereerivate hulkade süsteemi. Nende teisendustega seotult tõi Stephani sisse kaks mõistet: sürjektiivse operaatorideaali mõiste ja ideaalse hulkade süsteemi mõiste. Väitekirjas näidatakse, et operaatorideaalide ja genereerivate hulkade süsteemide vahel on Galois' vastavus, ning et läbi selle vaatenurga on võimalik uuesti näha ja mõtestada teatud tulemusi artiklist [S].

Üks levinud näide genereerivate hulkade süsteemide kohta on kõigi suhteliselt kompaktsete hulkade süsteem, mis on tekitatav teatud viisil kõigi koonduvate jadade süsteemi kaudu. Kõigi kompaktsete operaatorite operaatorideaal on omakorda tekitatav kõigi tõkestatud ja suhteliselt kompaktsete hulkade süsteemide kaudu järgmisel viisil: operaator on kompaktne parajasti siis, kui teisendab tõkestatud hulgad suhteliselt kompaktseteks hulkadeks. Väitekirjas uuritakse lisaks mitmeid alternatiivseid suhtelise kompaktsuse variante. Need alternatiivsed mõisted baseeruvad A. Grothendiecki poolt 1955. aastal tõestatud tulemusel [G2]: Banachi ruumi alamhulk on suhteliselt kompaktne parajasti siis, kui ta sisaldub nulli koonduva jada kinnises kumeras kattes. Tänapäeval tuntakse seda tulemust kui Grothendiecki kompaktsuse printsiipi.

Asendades Grothendiecki kompaktsuse printsiibis nulli koonduvad jadad absoluutselt p-summeeruvate jadadega (kus $1 \leq p < \infty$), saadakse tugevam variant suhtelisest kompaktsusest. Sellist kompaktsuse mõistet uuriti 1980ndatel O. Reinovi [Re1] ja J. Bourgaini ning O. Reinovi [BR] poolt. Käesolevas väitekirjas nimetatakse selliseid hulki suhteliselt p-kompaktseteks Bourgain–Reinovi mõttes. Aastal 2002 tõid D. P. Sinha ja A. K. Karn artiklis [SK1] sisse teise suhtelise kompaktsuse mõiste, mis asetseb eelmainitud mõistete vahel. Väitekirjas nimetatakse sellised hulki suhteliselt p-kompaktseteks Sinha–Karni mõttes.

Kehtigu $1 \leq p \leq \infty$ ning $1 \leq r \leq p^*$, kus p^* on indeksi p kaasindeks. Selleks, et uurida ühise käsitluse raames eelmainitud omadusi, tuuakse väitekirjas sisse suhteliselt (p, r)-kompaktse hulga mõiste, mis erijuhuna sisaldab suhteliselt p-kompaktseid hulki Bourgain–Reinovi mõttes (juhul $r = 1$) ja Sinha–Karni mõttes (juhul $r = p^*$). Seejärel veendutakse, et kõigi suhteliselt (p, r)-kompaktsete hulkade süsteem on genereerivate hulkade süsteem. Vaadeldes operaatoreid, mis teisendavad tõkestatud hulgad suhteliselt (p, r)-kompaktseteks, saadakse (p, r)-kompaktsed operaatorid. Erijuhtudena sisaldab viimane mõiste p-kompaktseid operaatoreid Bourgain–Reinovi mõttes (juhul $r = 1$) ning Sinha–Karni mõttes (juhul $r = p^*$).

Artiklis [SK1] tõestati, et (Sinha–Karni mõttes) p-kompaktsed operaatorid moodustavad Banachi operaatorideaali. Väitekirjas antakse kõigi (p, r)-kompaktsete operaatorite operaatorideaali $\mathcal{K}_{(p,r)}$ kirjeldus operaatorideaali $\mathcal{N}_{(p,1,r^*)}$ sürjektiivse katte kaudu. See võimaldab varustada operaatorideaali $\mathcal{K}_{(p,r)}$ tuntud operaatorideaali $\mathcal{N}^{\mathrm{sur}}_{(p,1,r^*)}$ s-normiga ning näidata, et $\mathcal{K}_{(p,r)}$ on s-Banachi operaatorideaal.

Artiklis [SK1] vaadeldi ka suhteliselt nõrgalt p-kompaktse hulga mõistet. Selle mõiste üldistatud variant, suhteliselt nõrgalt (p, r)-kompaktne hulk, toodi sisse artiklis [AO2]. Artiklis [AO2] toodi sisse ka tingimatult (p, r)-kompaktse hulga mõiste. Nõrgalt p-kompaktsed, nõrgalt (p, r)-kompaktsed, ning tingimatult (p, r)-kompaktsed operaatorid defineeritakse loomulikul viisil. Tähistagu $\mathcal{W}_{(p,r)}$ ja $\mathcal{U}_{(p,r)}$ vastavalt kõigi nõrgalt (p, r)-kompaktsete ja tingimatult (p, r)-kompaktsete operaatorite kogumeid. Artiklis [SK1] tõestati, et nõrgalt p-kompaktsete operaatorite klass $\mathcal{W}_p = \mathcal{W}_{(p,p^*)}$ on Banachi operaatorideaal, kui $1 \leq p < \infty$. Väitekirjas tõestatakse, et $\mathcal{W}_{(p,1)}$ ja $\mathcal{U}_{(p,1)}$ on kvaasi-Banachi operaatorideaalid. Selle saavutamiseks tuuakse sisse üldine meetod genereerivate hulkade süsteemide ja operaatorideaalide konstrueerimiseks, lähtudes BK-ruumist g ja normeeritud jadade süsteemist $\mathbf{h}$. Tõestatakse, et sellel viisil konstrueeritud operaatorideaal on kvaasi-Banachi operaatorideaal, eeldusel, et g ja $\mathbf{h}$ rahuldavad teatud tingimusi.

Väitekirjas näidatakse, et $\mathcal{W}_\infty = \mathcal{W}_{(\infty,1)}$ on Banachi operaatorideaal ning tõestatakse, et täielikult pidevate operaatorite operaatorideaal $\mathcal{V}$ on esitatav jagatisena $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. See annab uue tõestuse tulemusele [DFLORT, Theorem 1]: nõrk Grothendiecki kompaktsuse printsiip kehtib vaid Schuri omadusega ruumides.

Öeldakse, et Banachi ruumil on aproksimatsiooniomadus, kui ühikoperaatorit saab lähendada lõplikumõõtmeliste operaatoritega ühtlaselt ruumi suhteliselt kompaktsetel alamhulkadel. Käesolevas väitekirjas vaadeldakse kõigi aproksimeeritavate hulkade süsteemi, kus aproksimeeritav hulk on defineeritud kui tõkestatud hulk, millel ühikoperaator on ühtlaselt lähendatav lõplikumõõtmeliste operaatoritega. Lisaks defineeritakse aproksimeeritav jada kui nulli koonduv jada, millel ühikoperaator on ühtlaselt lähendatav lõplikumõõtmeliste operaatoritega. Grothendiecki kompaktsuse printsiibile analoogiliselt tõestatakse väitekirjas, et hulk on aproksimeeritav parajasti siis, kui ta sisaldub aproksimeeritava jada kinnises kumeras kattes. Lisaks tõestatakse, et leidub mitteaproksimeeritav jada, mis on esitatav kolme aproksimeeritava jada summana.

Väitekirja esimene peatükk sisaldab vaadeldava temaatika ülevaadet, väitekirja kokkuvõtet ning väitekirjas kasutatud tähistuste kirjeldust.

Väitekirja teises peatükis uuritakse operaatorideaalide klassi OI, genereerivate hulkade süsteemide klassi GSet ning genereerivate jadade süsteemide klassi GSeq ning nende klasside omavahelisi seoseid. Stephani vaatles klassil GSeq teatavat seost, mida me tähistame sümboliga $\lesssim$. Selle seose näol on tegemist eeljärjestusega, mis indutseerib ekvivalentsiseose $\sim$. Kasutades eeljärjestust $\lesssim$, saab faktorklassi $\mathrm{GSeq}/_\sim$ muuta järjestatud klassiks. Üks väitekirja põhitulemustest on, et järjestatud klasside GSet ja $\mathrm{GSeq}/_\sim$ vahel eksisteerib Galois' vastavus. Defineeritakse, et genereerivate hulkade süsteem on jadaliselt tekitatav, kui ta on tekitatav mingi genereerivate jadade süsteemi poolt. Eelmainitud Galois' vastavuse olemasolu annab praktilise kriteeriumi jadaliselt tekitatavate genereerivate hulkade süsteemide kirjeldamiseks. Teine peatükk põhineb enamuses artiklil [Lil1].

Kolmandas peatükis uuritakse klasside OI, GSet ja GSeq võrestruktuure. Samuti uuritakse võrestruktuure eelnevate klassidega seotud klassidel, mis tekivad loomulikul viisil nende klasside vaheliste teisenduste ja Galois' vastavuste kaudu. Antud peatükk põhineb artiklil [Lil1].

Neljandas peatükis uuritakse kõigi suhteliselt (p,r)-kompaktsete hulkade süsteemi $\mathbf{K}_{(p,r)}$ ja veendutakse, et tegemist on genereerivate hulkade süsteemiga. Toetudes teisele peatükile, tõestatakse, et hulkade süsteem $\mathbf{K}_{(p,r)}$ on jadaliselt tekitatav vaid juhul $p = \infty$ ja $r = 1$, millisel juhul süsteem $\mathbf{K}_{(p,r)}$ ühtib kõigi suhteliselt kompaktsete hulkade süsteemiga $\mathbf{K}$. Lisaks pakub hulkade süsteemi $\mathbf{K}_{(p,r)}$ uurimine vastuseid ja kontranäiteid mitmetele teises peatükis püstitatud küsimustele.

Tõestatakse, et $\mathcal{K}_{(p,r)} = \mathcal{N}^{\text{sur}}_{(p,1,r^*)}$, mis lubab varustada operaatorideaali $\mathcal{K}_{(p,r)}$ operaatorideaali $\mathcal{N}^{\text{sur}}_{(p,1,r^*)}$ s-normiga ning näha, et $\mathcal{K}_{(p,r)}$ on s-Banachi operaatorideaal. Neljas peatükk põhineb artiklitel [ALO] ja [Lil1].

Viiendas peatükis uuritakse jadaliselt tekitatavaid genereeritavate hulkade süsteeme $\mathbf{G}$, mis rahuldavad tingimust $\mathbf{G} \leq \mathbf{K}$. Selleks tuuakse sisse mitmeid mõisteid, muuhulgas päranduvalt peaaegu eneselähendava jada mõiste. Sellele mõistele toetudes tõestatakse, et võrratus $\mathbf{G} \leq \mathbf{K}$ on range parajasti siis, kui süsteem $\mathbf{G}$ on tekitatav mingi genereerivate jadade süsteemi $\mathbf{g}$ poolt, milles iga jada on päranduvalt peaaegu eneselähendav. Samuti konstrueeritakse sellise omadusega genereerivate jadade süsteem $\mathbf{g}$.

Kuuendas peatükis uuritakse kõigi suhteliselt nõrgalt ∞-kompaktsete hulkade süsteemi ning näidatakse, et tegemist on genereerivate hulkade süsteemiga. Tõestatakse, et kõigi nõrgalt ∞-kompaktsete operaatorite kogum $\mathcal{W}_\infty$ on Banachi operaatorideaal. Tõestatakse üks väitekirja põhitulemusi, et kehtib operaatorideaalide võrdus $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$ (ning et see võrdus kehtib ka Banachi operaatorideaalide kontekstis). Vahetu järeldusena saadakse sellest võrdusest uus tõestus tulemusele [DFLORT, Theorem 1], et nõrk Grothendiecki kompaktsuse printsiip kehtib vaid Schuri omadusega ruumides. Kuues peatükk põhineb artiklitel [JLO] ja [Lil2].

Seitsmendas peatükis luuakse meetod genereerivate hulkade süsteemide ja kvaasi-Banachi operaatorideaalide konstrueerimiseks. See meetod üldistab viisi, kuidas on konstrueeritud hulkade süsteemid $\mathbf{K}_{(p,r)}$ ja $\mathbf{W}_\infty$ ning vastavad operaatorideaalid $\mathcal{K}_{(p,r)}$ ja $\mathcal{W}_\infty$. Antud konstruktsioon lähtub BK-ruumist g ja normeeritud jadade süsteemist $\mathbf{h}$ ning annab tulemuseks kvaasi-Banachi operaatorideaali, eeldusel, et g ja $\mathbf{h}$ rahuldavad teatavaid lisatingimusi. Muuhulgas tõestatakse, et $\mathcal{W}_{(p,1)}$ ja $\mathcal{U}_{(p,1)}$ on kvaasi-Banachi operaatorideaalid (kus $1 \leq p < \infty$).

Kaheksas peatükk algab ülevaatega mõningatest teadaolevatest aproksimatsiooniomadust puudutavatest tulemustest. Seejärel tuuakse sisse aproksimeeritava hulga ja jada mõisted. Lähtudes neist mõistetest, tõestatakse kriteerium, mis kirjeldab Banachi ruumi aproksimeeritavaid hulki selle ruumi aproksimeeritavate jadade kaudu. Näidatakse, et leidub mitteaproksimeeritav jada, mis esitub kolme aproksimeeritava jada summana.

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List of original publications

1. K. Ain, R. Lillemets, E. Oja, *Compact operators which are defined by ℓ_p-spaces*, Quaest. Math. **35** (2012) 145–159.

2. W. B. Johnson, R. Lillemets, E. Oja, *Representing completely continuous operators through weakly ∞-compact operators*, Bull. London Math. Soc. **48** (2016) 452–456.

3. R. Lillemets, *Galois connections between generating systems of sets and sequences*, Positivity (2016) (online), DOI: 10.1007/s11117-016-0451-z

4. R. Lillemets, *Representing the Banach operator ideal of completely continuous operators*, Proc. Estonian Acad. Sci. Phys. Math. **66**(2) (2017) 189–193.

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