Enlargements of rings

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Introduction

I will consider associative not necessarily unital rings.

Definition

A set R is called a **ring**, if it is equipped with two binary operations, addition $+\colon\ R \times R \to R$ and multiplication $\cdot\colon\ R \times R \to R$, such that the following properties hold:

1. $(R, +)$ is an abelian group;
2. $(R, \cdot)$ is a semigroup,
3. $\forall r, s, t \in R\colon \quad (r + s)t = rt + st \quad \& \quad r(s + t) = rs + rt.$

Idempotent rings

Definition

A ring R is called **idempotent** if $RR = R$, where

$$RR = \left\{ \sum_{k=1}^{k^*} r_k r'_k \,\middle|\, r_1, r'_1, \ldots, r_{k^*}, r'_{k^*} \in R \right\}.$$

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The inclusion $RR \subseteq R$ is always satisfied.

A ring R is idempotent iff for every $r \in R$ there exist elements $r_1, r'_1, \ldots, r_{k^*}, r'_{k^*} \in R$ such that

$$r = \sum_{k=1}^{k^*} r_k r'_k.$$

Example (idempotent ring 1):

Let X be a set. Denote $\mathcal{P}_{\text{fin}}(X)$ as the set of all finite subsets of X. $(\mathcal{P}_{\text{fin}}(X); \Delta, \cap)$ is an idempotent ring, which does not have an identity element, because

$$\forall A \in \mathcal{P}_{\text{fin}}(X): \quad A \cap A = A.$$

(Δ – symmetric difference) $\square$

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Consider the direct sum $\bigoplus_{k=1}^{\infty} \mathbb{Z}_n$. There is no unit-element in this ring, but it is idempotent, because every element can be written as a sum of idempotents.

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Example (idempotent ring 2):

Consider the direct sum $\bigoplus_{k=1}^{\infty} \mathbb{Z}_n$. There is no unit-element in this ring, but it is idempotent, because every element can be written as a sum of idempotents. E.g.

$$\begin{aligned}(\overline{0},\overline{3},\overline{2},\overline{0},\overline{0},\dots) &= (\overline{0},\overline{3},\overline{0},\overline{0},\dots) + (\overline{0},\overline{0},\overline{2},\overline{0},\overline{0},\dots) = \\ &= (\overline{0},\overline{1},\overline{0},\overline{0},\dots) + (\overline{0},\overline{1},\overline{0},\overline{0},\dots) + (\overline{0},\overline{1},\overline{0},\overline{0},\dots) + \\ &\quad + (\overline{0},\overline{0},\overline{1},\overline{0},\overline{0},\dots) + (\overline{0},\overline{0},\overline{1},\overline{0},\overline{0},\dots).\end{aligned}$$

□

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Definition

We call a ring R an **enlargement** of its subring S if the conditions $R = RSR$ and $S = SRS$ hold, where

$$RSR = \left\{ \sum_{k=1}^{k^*} r_k s_k r'_k \,\middle|\, k^* \in \mathbb{N},\ r_1, r'_1, \ldots, r_{k^*}, r'_{k^*} \in R,\ s_1, \ldots, s_{k^*} \in S \right\},$$

$$SRS = \left\{ \sum_{k=1}^{k^*} s_k r_k s'_k \,\middle|\, k^* \in \mathbb{N},\ s_1, s'_1, \ldots, s_{k^*}, s'_{k^*} \in S,\ r_1, \ldots, r_{k^*} \in R \right\}.$$

We also say that R is an **enlargement** of all rings isomorphic to such S.

Denote $S \sqsubseteq R$, if R is an enlargement of S.

Properties of enlargements

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Proposition

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Idempotency of R:

$$R = RSR = R(SR) \subseteq RR \subseteq R \implies RR = R.$$

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Relation $\sqsubseteq$ is a (partial) order relation on the class of idempotent rings.

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Example (enlargement 1):

A ring of square matrices $\mathrm{Mat}_n(S)$ over an idempotent ring S. Then

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$$S \sqsubseteq \mathrm{Mat}_n(S).$$

Denote $S' := \{A_{11}(s) \mid s \in S\}$, where $A_{hk}(s)$ is an $n \times n$-matrix with entry s at the intersection of h-th row and k-th column, and zeroes elsewhere.

$$S \cong S' \subseteq \mathrm{Mat}_n(S).$$

Ring $\mathrm{Mat}_n(S)$ can be expressed as

$$R := \mathrm{Mat}_n(S) = \sum \{A_{hk}(s) \mid h, k \in \{1, \ldots, n\},\ s \in S\}.$$

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$$A_{hk}(s) = \sum_{t=1}^{t^*} A_{hk}(u_t s_t v_t) = \sum_{t=1}^{t^*} A_{h1}(u_t) \cdot A_{11}(s_t) \cdot A_{1k}(v_t) \in RS'R\,.$$

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Also,

$$A_{11}(s) \cdot A \cdot A_{11}(s') = A_{11}(s a_{11} s') \in S' \implies S'RS' \subseteq S'.$$

where $s, s' \in S$ and matrix $A = [a_{hk}] \in R$.

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$\therefore S \cong S' \sqsubseteq \mathrm{Mat}_n(S) =: R$ $\square$

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Definition

A **Rees matrix ring over a ring** S is the set $\mathcal{M}(S;U,V;X)$ of all $U \times V$-matrices (U and V are nonempty sets) over S having a finite number of nonzero entries, endowed with the usual addition of matrices and with the multiplication $\circ$ defined by

$$A \circ B = AXB,$$

where X is a $V \times U$-matrix over S (that is, a mapping $V \times U \to S$) and AXB is the usual matrix product.

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We call a Rees matrix ring $\mathcal{M}(S;U,V;X)$ **unital** if S is a ring with identity 1 and 1 is an entry of X.

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Example (enlargement 2):

Let S be a ring with identity 1 and $\mathcal{M} = \mathcal{M}(S;U,V;X)$ a unital Rees matrix ring such that $X(v_0,u_0) = 1$. Then

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Denote $S' := \{A_{u_0v_0}(s) \mid s \in S\}$, where $A_{u_0v_0}(s) \in \mathcal{M}$ is such a Rees matrix that $A_{u_0v_0}(s)(u_0, v_0) = s$ and $A_{u_0v_0}(s)(u, v) = 0$ for all other pairs $(u, v) \in U \times V$. Clearly $S \cong S' \in \mathcal{M}$.

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The ring S' is idempotent because for every $s \in S$, we have

$$A_{u_0 v_0}(s) = A_{u_0 v_0}(s) X A_{u_0 v_0}(1) = A_{u_0 v_0}(s) \circ A_{u_0 v_0}(1) \in S' \circ S'.$$

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Also the equalities $S' \circ \mathcal{M} \circ S' = S'$ and $\mathcal{M} \circ S' \circ \mathcal{M} = \mathcal{M}$ hold. □

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Definition

A six-tuple $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, where R and S are rings and ${}_RP_S$, ${}_SQ_R$ are bimodules, is called a **Morita context**, if

$$\theta\colon\ {}_R(P \otimes_S Q)_R \to {}_RR_R, \quad \phi\colon\ {}_S(Q \otimes_R P)_S \to {}_SS_S$$

are bimodule homomorphisms

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$$\theta\colon\ {}_R(P \otimes_S Q)_R \to {}_R R_R, \quad \phi\colon\ {}_S(Q \otimes_R P)_S \to {}_S S_S$$

are bimodule homomorphisms such that

$$\theta(p \otimes q)p' = p\phi(q \otimes p'),$$
$$q\theta(p \otimes q') = \phi(q \otimes p)q'$$

for every $p, p' \in P$ and $q, q' \in Q$

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for every $p, p' \in P$ and $q, q' \in Q$

We say that a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is **unitary**, if the bimodules ${}_RP_S$ and ${}_SQ_R$ are unitary (i.e. $P = RPS$ and $Q = SQR$); and **surjective**, if the homomorphisms θ and ϕ are surjective.

For the purposes of this presentation, we say that two idempotent rings R and S are **Morita equivalent**, if they are connected by a unitary surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$.

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Proposition

If two rings R and S are connected by an unitary surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ then they are idempotent.

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Every $r \in R$ can be expressed as

$$r = \theta\left(\sum_{h=1}^{h^*} p_h \otimes q_h\right) = \sum_{h=1}^{h^*} \theta(p_h \otimes q_h) = \sum_{h=1}^{h^*} \theta\left(\sum_{k=1}^{k^*} r_{hk} p_{hk} \otimes q_h\right) =$$

$$= \sum_{h=1}^{h^*} \sum_{k=1}^{k^*} r_{hk} \theta(p_{hk} \otimes q_h) \in RR.$$

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Theorem

If R is an enlargement of an idempotent ring S then R and S are Morita equivalent.

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If R is an enlargement of an idempotent ring S then R and S are Morita equivalent.

This Morita equivalence is established by the Morita context

$$(R, S, RS, SR, \theta, \phi),$$

where

$$\theta\colon\ RS \otimes_S SR \to R, \qquad \sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k \mapsto \sum_{k=1}^{k^*} r_k s_k s'_k r'_k,$$

$$\phi\colon\ SR \otimes_R RS \to S, \qquad \sum_{k=1}^{k^*} s_k r_k \otimes r'_k s'_k \mapsto \sum_{k=1}^{k^*} s_k r_k r'_k s'_k$$

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Corollary

A full matrix ring $\mathrm{Mat}_n(S)$ *over an idempotent ring* S *is Morita equivalent to* S*.*

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A full matrix ring $\mathrm{Mat}_n(S)$ *over an idempotent ring* S *is Morita equivalent to* S*.*

Corollary

A unital Rees matrix ring $\mathcal{M}(S;U,V;X)$ *over a ring* S *with identity is Morita equivalent to* S*.*

Definition

Let $\Gamma = (R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be a Morita context. Then **Morita ring** $\overline{\Gamma}$ **of the context** Γ is defined as the matrix set

$$\overline{\Gamma} = \left\{ \begin{bmatrix} r & p \\ q & s \end{bmatrix} \middle| r \in R, s \in S, p \in P, q \in Q \right\}$$

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with componentwise addition and with the multiplication

$$\begin{bmatrix} r & p \\ q & s \end{bmatrix} \begin{bmatrix} r' & p' \\ q' & s' \end{bmatrix} = \begin{bmatrix} rr' + \theta(p \otimes q') & rp' + ps' \\ qr' + sq' & \phi(q \otimes p') + ss' \end{bmatrix}.$$

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A ring T is a **joint enlargement** of rings R and S if it is an enlargement of both R and S.

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Theorem

If idempotent rings R and S are connected by a unitary surjective Morita context Γ then the Morita ring $\overline{\Gamma}$ is a joint enlargement of R and S.

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If idempotent rings R and S are connected by a unitary surjective Morita context Γ then the Morita ring $\overline{\Gamma}$ is a joint enlargement of R and S.

Theorem

Idempotent rings are Morita equivalent if and only if they have a joint enlargement.

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Theorem

Let R and S be two idempotent rings. The following assertions are equivalent:

1. *R and S are Morita equivalent;*
2. $\mathsf{FMod}_R \approx \mathsf{FMod}_S$;
3. $\mathsf{CMod}_R \approx \mathsf{CMod}_S$;
4. $\mathsf{UTfMod}_R \approx \mathsf{UTfMod}_S$;
5. *R and S are connected by a unitary surjective Morita context* $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$;
6. *rings R and S have a joint enlargement;*
7. *R is isomorphic to a bijectively defined tensor product ring $P \otimes_S Q$ (for firm rings).*

Corollary

Two rings with identity (two rings with local units) are Morita equivalent if and only if they have a joint enlargement which has identity (has local units).

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Theorem

A ring R with left local units is Morita equivalent to a ring with identity if and only if there exists an idempotent element $e \in R$ such that $R = ReR$. In that case R is Morita equivalent to its subring eRe.

We say that a ring R has left local units if for every finite subset $\{r_1, \dots, r_n\} \subseteq R$ there exists an idempotent element $e \in R$ such that $r_1 = er_1, \dots, r_n = er_n$.

A few problems for additional research

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1. Do two finite idempotent Morita equivalent rings have a finite joint enlargement?

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2. Is the problem of Morita equivalence of finite idempotent rings decidable?

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1. Do two finite idempotent Morita equivalent rings have a finite joint enlargement?
2. Is the problem of Morita equivalence of finite idempotent rings decidable?

We know that the answer to the first question is positive in the case of semigroups with local units and the answer to the second question is positive for arbitrary finite semigroups.

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End

Thank you for listening!

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