DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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Convex approximation properties of Banach spaces

TARTU UNIVERSITY PRESS

Faculty of Mathematics and Computer Science, University of Tartu, Tartu, Estonia

Dissertation has been accepted for the commencement of the degree of Doctor of Philosophy (PhD) in mathematics on May 4, 2011, by the Council of the Institute of Mathematics, Faculty of Mathematics and Computer Science, University of Tartu.

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Commencement will take place on June 17, 2011, at 14.15 in Liivi 2-403.

Publication of this dissertation has been granted by the Estonian Doctoral School of Mathematics and Statistics.

European Union
European Social Fund

Investing in your future

ISSN 1024-4212
ISBN 978-9949-19-665-4(trükis)
ISBN 978-9949-19-665-1(PDF)

Tartu Ülikooli Kirjastus
www.tyk.ee
Tellimus nr. 299

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Acknowledgements

First of all I would like to express my heartfelt gratitude to my supervisor Professor Eve Oja for her talent at introducing to me the marvelous theory of functional analysis, for her continuous warm support throughout my university studies, for all the advice and great ideas she has been sharing with me, for her good company, and for invaluable help in completing this thesis.

I want to thank all the colleagues whose lectures I had a pleasure to attend, as well as the secretaries of the institute and the faculty who provided their friendly help and advice on many occasions. I also want to thank other colleagues from our faculty or elsewhere with whom I have had an outstanding academical and/or extracurricular experience. I especially mention here Kati Ain, Rainis Haller, Marje Johanson, Oleg Košik, my coauthor Kristel Mikkor, Ahto Truu, and Indrek Zolk.

I would like to thank Professor Niels J. Nielsen for a very thorough answer to my question. It helped greatly during the preparation of the thesis.

I also thank my family and friends for a very wanted boost, occasional refreshing sarcasm, and believing in me, no matter what.

Finally, I would like to thank Nina Adler for showing to many children (I was one of them) the beauty of mathematics in the first place.

Chapter 1

Introduction

1.1 Background

The approximation property or "la condition d'approximation" as a term was introduced in 1955 by A. Grothendieck in his famous Memoir [G, Chapter I, p. 167]. But the origins of the notion trace back to the Lwów School of Mathematics of 1930s.

Recall that a Banach space is said to have the *approximation property* if its identity operator can be approximated by finite-rank operators uniformly on compact sets.

In fact, Grothendieck showed [G, Chapter 1, p. 165] that the question whether every Banach space has the approximation property is equivalent to several open problems emerging from the work of S. Banach, S. Mazur, and others.

One of them, the *approximation problem*, asks whether all compact operators between arbitrary Banach spaces can be approximated, in the norm topology of operators, by finite-rank operators.

The other one, like many other fine problems, has its origin in *The Scottish Coffee House* located near the University building in Lwów. It is the Problem 153 of the *Scottish book*, and it goes as follows.

Given a continuous function $f = f(s,t)$ defined on $[0,1] \times [0,1]$ and a number $\varepsilon > 0$; do there exist numbers $a_1, \dots, a_n$; $b_1, \dots, b_n$; $c_1, \dots, c_n$, such that

$$\left| f(s,t) - \sum_{k=1}^{n} a_k f(s, b_k) f(c_k, t) \right| < \varepsilon$$

for all $s, t \in [0,1]$?

The problem 153 was posed by Mazur on November 6, 1936. According to Pełczyński (see, e.g., [Pi2, p. 285]), Mazur knew that the positive answer to Problem 153 would imply the positive answer to the approximation problem. This might be the reason for the extraordinary prize he offered on Problem 153: a live goose (see, e.g., [Scottish]).

A related open question, the *basis problem* (see [B, p. 111]), asked whether every separable Banach space has a Schauder basis. A negative solution to the approximation problem is also a negative solution to the basis problem, since a Banach space with a Schauder basis satisfies the approximation property.

Although Grothendieck connected the above questions to each other, showing that their positive answers yield a series of very nice properties in Banach spaces, the questions still remained open. Grothendieck himself conjectured that the approximation problem has the negative answer in general, but every reflexive space might have the approximation property [G, Chapter II, p. 135].

The sensation came 17 years after, in May 1972, when P. Enflo discovered a separable reflexive Banach space without the approximation property [E], and therefore solved the approximation problem and the basis problem in the negative. The live goose was then indeed awarded to Enflo by Mazur (see, e.g., [Kałuża] for a photo of this event).

According to A. Pietsch [Pi2, p. 287]: *Life in Banach spaces with certain approximation properties is much easier. Thus the effect of Enflo's counterexample can be described as the banishment from Paradise.*

On the other hand, Enflo's result stimulated further research on the topic. In particular, T. Figiel and W. B. Johnson in 1973 [FJ] showed that the approximation property and the *metric approximation property* are, in general, different. (A Banach space has the metric approximation property if its identity can be approximated by finite-rank operators of norm not exceeding 1 uniformly on compact sets.)

Also, it was noticed that Enflo's space is actually a counterexample to the *compact approximation property*, i.e., to the fact that an identity operator of a Banach space can be approximated by compact operators uniformly on compact sets. The example of a Banach space failing the approximation property but having the compact approximation property was constructed in 1992 by G. Willis [W].

The field contains a number of longstanding open problems, which attract the attention of researchers. One of the most famous of them goes as follows.

Problem 1.1. For the dual space of a Banach space, does the approximation property imply the metric approximation property?

1.2 Summary of the thesis

The main aim of the thesis is to develop a unified approach to diverse versions of the approximation property, such as the compact approximation property, approximation properties defined by an operator ideal, or the positive approximation property of Banach lattices. Our principal notion is the approximation property defined by a convex set of operators containing 0, the *convex approximation property*. It turns out that this concept admits good counterparts of some important results on the classical approximation property. We also consider the approximation properties defined by linear subspaces as well as by arbitrary sets of operators.

The thesis has been organized as follows.

Chapter 1 briefly retells the historic background of the approximation problem, provides a summary of the thesis, and describes the notation used in the thesis.

In Chapter 2 we recall supplementary notation and results needed in most of the following chapters. These include operator ideals and space ideals, tensor products, weak*-to-weak continuous operators, some of the most important locally convex topologies defined on operator spaces, and the Davis–Figiel–Johnson–Pełczyński factorization lemma in its isometric version due to Lima, Nygaard, and Oja.

In Chapter 3 we consider several versions of approximation properties. We go on to define the convex approximation property and the convex approximation property with conjugate operators, and present some of the basic results concerning the approximation properties. This chapter is based on introductory parts of [L] and [LMO].

The results presented in Chapter 4 extend the classical description of the approximation property of a Banach and its dual space via approximability of compact operators by finite-rank operators [G]. We provide versions, which hold for the approximation properties defined by an arbitrary collection of bounded linear operators, as well as their improvements in the case of the convex approximation properties. This chapter is inspired by [LO4, LLN, OPe], and based on [L] and [LMO].

In Chapter 5 we prove one of the main results of the thesis. It is a description of the approximation property defined by a linear subspace of operators via the approximability of weakly compact operators. For this end, we recall the definition of the Radon–Nikodým property and the description of continuous linear functionals on the space of compact operators between Banach spaces due to Feder and Saphar, which is possible under the influence of the latter property. This chapter is based on [LMO].

In Chapter 6 we look at the strong approximation property and the weak bounded approximation property (these notions were introduced in [O3] and [LO6]). We develop a unified approach to the treatment of their convex versions, and following [O2], we observe the impact of the Radon–Nikodým property on the interplay between these notions and the convex (bounded) approximation properties. The results obtained as the product of these investigations will be important for us in Chapter 7. This chapter is based on [L].

In Chapter 7 we extend the famous Johnson's lifting theorem, which permits to lift the metric approximation property to the dual space, as well as the Lima–Oja [LO6] theorem on lifting the weak metric approximation property to the dual space. We apply these results to obtain a partial solution to a convex version of Problem 1.1, showing that under the impact of the Radon–Nikodým property, the convex approximation property of a dual space is always metric. The prototype of the latter method can be found in [O2]. This chapter is based on [LisO].

In Chapter 8 we apply the theory of convex approximation properties to the positive approximation property of Banach lattices and to the approximation property defined for pairs of Banach spaces. In order to do so, we recall some of the classical theorems on Banach lattices. This chapter is based on [LisO].

1.3 Notation

Our notation is standard.

For vector spaces X and Y (we consider vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), by $L(X, Y)$ we will denote the vector space of all linear operators between X and Y. If X and Y are normed linear spaces, then $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators between them. For an operator $T : X \to Y$ we denote $\ker T = \{x \in X : Tx = 0\}$, the kernel of T. The restriction of T to a subset $K \subset X$ will be denoted by $T|_K$. For vector spaces X, Y, Z, and sets $A \subset L(Y, X)$, $B \subset L(Z, Y)$, we denote $A \circ B = \{TS : T \in A,\ S \in B\} \subset L(Z, X)$.

A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \to X^{**}$. The identity operator on X is denoted by I_X. The closed unit ball and the unit sphere of X are denoted B_X and S_X, respectively. The closure of a set $K \subset X$ is denoted by $\overline{K}$, its linear span by $\operatorname{span} K$, its convex hull by $\operatorname{conv} K$, and its absolutely convex hull by $\operatorname{absconv} K$. The norm closures of the three latter sets are denoted by $\overline{\operatorname{span}} K$, $\overline{\operatorname{conv}} K$, and $\overline{\operatorname{absconv}} K$, respectively. For closures with respect to other topologies, we mark the topology separately, such as $\overline{\operatorname{conv}}^{w^*} K$, etc.

We assume that the reader is familiar with well-known basic notions and theorems from the theory of Banach spaces and topological vector spaces (such as dual pairs, the Hausdorff theorem, the Minkowski functional or the gauge, the Hahn-Banach theorem, the Alaoglu theorem, the Goldstine theorem, etc.), and we shall sometimes use them without proper references.

Let us also mention that additional notation will be introduced in the forthcoming chapter.

Chapter 2

Preliminaries

In this chapter we recall supplementary notation and results needed in most of the following chapters. These include operator ideals and space ideals, tensor products, weak*-to-weak continuous operators, some of the most important locally convex topologies defined on operator spaces, and the Davis–Figiel–Johnson–Pełczyński factorization lemma in its isometric version due to Lima, Nygaard, and Oja.

2.1 Operator and space ideals

Although not of immediate interest, the notions of this section are fundamental in dealing with classes of operators beyond the finite-dimensional operators.

This section is a quick overview of basic definitions and results in [Pi1].

Let $\mathcal{L}$ be the class of the bounded linear operators between arbitrary Banach spaces.

Definition 2.1. A subclass $\mathcal{A}$ of $\mathcal{L}$ is said to be an *operator ideal* if the following holds:

(i) $I_{\mathbb{K}} \in \mathcal{A}$ for the one-dimensional Banach space $\mathbb{K}$,

(ii) the *components* $\mathcal{A}(X, Y) := \mathcal{L}(X, Y) \cap \mathcal{A}$ are linear subspaces of $\mathcal{L}(X, Y)$ for all Banach spaces X and Y,

(iii) $\mathcal{A}$ is closed under compositions with operators from $\mathcal{L}$, meaning that for all Banach spaces W, X, Y, and Z, and for all operators $T \in \mathcal{L}(W,X)$, $S \in \mathcal{A}(X,Y)$, and $R \in \mathcal{L}(Y,Z)$ one has

$$RST \in \mathcal{A}(W,Z).$$

As a convenience, we also denote $\mathcal{A}(X) := \mathcal{A}(X,X)$ for an operator ideal $\mathcal{A}$ and a Banach space X.

Example 2.2. The classes of finite-rank operators $\mathcal{F}$, approximable operators $\overline{\mathcal{F}}$, compact operators $\mathcal{K}$, weakly compact operators $\mathcal{W}$, completely continuous operators $\mathcal{V}$, separable operators $\mathcal{X}$, and all bounded linear operators $\mathcal{L}$ are operator ideals.

Definition 2.3. A subclass A of Banach spaces L is said to be a *space ideal* if the following holds:

(i) $\mathbb{K} \in \mathsf{A}$ for the one-dimensional Banach space $\mathbb{K}$,

(ii) $X \times Y \in \mathsf{A}$ for all $X, Y \in \mathsf{A}$,

(iii) if $X \in \mathsf{A}$ and Y is isomorphic to a complemented subspace of X, then $Y \in \mathsf{A}$.

Example 2.4. The classes of finite-dimensional spaces F, reflexive spaces W, spaces with the Schur property V, separable spaces X, and all Banach spaces L are space ideals.

It is possible to pass from an operator ideal to a space ideal and vice versa. Let $\mathcal{A}$ be an operator ideal and let A be a space ideal. The class

$$\text{Space}(\mathcal{A}) := \{X \in \mathsf{L} : I_X \in \mathcal{A}(X)\}$$

is a space ideal and the class

$$\text{Op}(\mathsf{A}) := \{TS : X,Y \in \mathsf{L},\ Z \in \mathsf{A},\ S \in \mathcal{L}(X,Z),\ T \in \mathcal{L}(Z,Y)\}.$$

is an operator ideal. Moreover, $\text{Space}(\text{Op}(\mathsf{A})) = \mathsf{A}$ and $\text{Op}(\text{Space}(\mathcal{A})) \subset \mathcal{A}$.

The examples above are in the following correspondence:

(i) $\mathsf{F} = \text{Space}(\mathcal{F}) = \text{Space}(\overline{\mathcal{F}}) = \text{Space}(\mathcal{K})$ and $\text{Op}(\mathsf{F}) = \mathcal{F}$,

(ii) $\mathsf{W} = \text{Space}(\mathcal{W})$ and $\mathcal{W} = \text{Op}(\mathsf{W})$,

(iii) $\mathsf{X} = \mathrm{Space}(\mathcal{X})$ and $\mathcal{X} = \mathrm{Op}(\mathsf{X})$.

Given two operator ideals $\mathcal{A}$ and $\mathcal{B}$ one can compose a new operator ideal in any of the following way: *intersection* $\mathcal{A} \cap \mathcal{B}$, *product*

$$\mathcal{A} \circ \mathcal{B} := \{TS : X, Y, Z \in \mathsf{L},\ T \in \mathcal{A}(Z, Y),\ S \in \mathcal{B}(X, Z)\},$$

left-hand quotient

$$\mathcal{A} \circ \mathcal{B}^{-1} := \{T \in \mathcal{L}(X, Y) : X, Y \in \mathsf{L}, \forall Z \in \mathsf{L}\ \forall S \in \mathcal{B}(Z, X)\ TS \in \mathcal{A}(Z, Y)\},$$

and *right-hand quotient*

$$\mathcal{A}^{-1} \circ \mathcal{B} := \{T \in \mathcal{L}(X, Y) : X, Y \in \mathsf{L}, \forall Z \in \mathsf{L}\ \forall S \in \mathcal{A}(Y, Z)\ ST \in \mathcal{B}(X, Z)\}.$$

The *dual operator ideal* $\mathcal{A}^{\mathrm{dual}}$ consists of all operators $T \in \mathcal{L}$ having its conjugate operator T^* in $\mathcal{A}$. The *dual space ideal* $\mathsf{A}^{\mathrm{dual}}$ consists of all Banach spaces X having its dual space X^* in A. An operator ideal $\mathcal{A}$ is called *symmetric* if $\mathcal{A} \subset \mathcal{A}^{\mathrm{dual}}$, and *completely symmetric* if $\mathcal{A} = \mathcal{A}^{\mathrm{dual}}$. The *symmetric* and *completely symmetric* space ideals are defined in an analogous way. Moreover, if an operator ideal $\mathcal{A}$ is (completely) symmetric, then so is the space ideal $\mathrm{Space}(\mathcal{A})$.

Among the examples above, the operator ideals $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{K}$, $\mathcal{W}$, and the space ideal $\mathsf{X} \cap \mathsf{W}$ are completely symmetric. The operator ideals $\mathcal{X}$ and $\mathcal{V}$ are not symmetric.

Definition 2.5. An operator ideal $\mathcal{A}$ together with a mapping $\|\cdot\|_{\mathcal{A}}$ from $\mathcal{A}$ to non-negative real numbers is called a *normed operator ideal* if

(i) the components $[\mathcal{A}(X, Y), \|\cdot\|_{\mathcal{A}}]$ are normed spaces for all Banach spaces X and Y,

(ii) for all Banach spaces W, X, Y, and Z, and for all operators $T \in \mathcal{L}(W, X)$, $S \in \mathcal{A}(X, Y)$, and $R \in \mathcal{L}(Y, Z)$ one has

$$\|RST\|_{\mathcal{A}} \leq \|R\| \, \|S\|_{\mathcal{A}} \, \|T\| .$$

If, in addition, in (i) the components $[\mathcal{A}(X, Y), \|\cdot\|_{\mathcal{A}}]$ are Banach spaces, then $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is called a *Banach operator ideal.* If $\|\cdot\|_{\mathcal{A}}$ coincides with the usual operator norm $\|\cdot\|$, then $\mathcal{A}$ is called a *classical* (normed or Banach) operator ideal.

2.2 Weak*-to-weak continuous operators

Let X and Y be Banach spaces.

Recall that a net $(x_\alpha) \subset X$ converges to a point $x \in X$ in the weak topology $\sigma(X, X^*)$ if $x^*(x_\alpha) \to x^*(x)$ for each $x^* \in X^*$. Similarly, a net $(x^*_\alpha) \subset X^*$ converges to a functional $x^* \in X^*$ in the weak* topology $\sigma(X^*, X)$ if $x^*_\alpha(x) \to x^*_\alpha(x)$ for every $x \in X$. The dual pair $\langle X, X^* \rangle$ gives the equality $(X^*, \sigma(X^*, X))^* = X$, which can be rewritten as

Proposition 2.6. *A functional $x^{**} \in X^{**}$ is weakly* continuous if and only if $x^{**} \in j_X(X)$.*

Definition 2.7. An operator $T \in \mathcal{L}(X^*, Y)$ is said to be *weak*-to-weak continuous* if it is continuous as a mapping between locally convex spaces $(X^*, \sigma(X^*, X))$ and $(Y, \sigma(Y, Y^*))$. The subspace of $\mathcal{L}(X^*, Y)$ consisting of all such operators is denoted $\mathcal{L}_{w^*}(X^*, Y)$.

Let $\mathcal{A}$ be an operator ideal. We will denote $\mathcal{A}_{w^*}(X^*, Y) := \mathcal{L}_{w^*}(X^*, Y) \cap \mathcal{A}$.

Clearly, $\mathcal{L}_{w^*}(X^*, Y) = \mathcal{W}_{w^*}(X^*, Y)$. Indeed, by the Alaoglu theorem the unit ball B_{X^*} is compact in the weak* topology. Therefore, if $T \in \mathcal{L}(X^*, Y)$ is weak*-to-weak continuous, then $T(B_{X^*})$ is weakly compact.

The following is a well-known criterion of weak*-to-weak continuous operators. We provide a short proof to this folklore result.

Proposition 2.8. *An operator $T \in \mathcal{L}(X^*, Y)$ is weak*-to-weak continuous if and only if $T^*(Y^*) \subset j_X(X)$.*

Proof. Take $T \in \mathcal{L}(X^*, Y)$ and observe that by proposition 2.6 the condition $T^*(Y^*) \subset j_X(X)$ is equivalent to the condition that the functional $T^* y^* = y^* \circ T$ is weakly* continuous for every $y^* \in Y^*$. Clearly, the latter is exactly the weak*-to-weak continuity of T. □

Let U and V be vector spaces, and let W be a subspace of V. If $T : U \to V$ is a linear operator such that $T(U) \subset W$ then one can consider a linear operator $T_W : U \to W$ with the same values as T. The operator T_W is called the *astriction* of T to the space W. Clearly, if U and V are Banach spaces and T is continuous, then so is T_W.

Definition 2.9. For an operator $T \in \mathcal{L}_{w^*}(X^*, Y)$ by its *dual astriction* we will denote the operator $T^\diamond \in \mathcal{L}(Y^*, X)$ such that $T^* = j_X T^\diamond$.

Note. In the literature the notion is addressed simply as the dual operator T^*. We make a distinction for the sake of clarity.

We state some simple yet useful properties of dual astrictions.

Proposition 2.10. *Let $T, U \in \mathcal{L}_{w^*}(X^*, Y)$, $S \in \mathcal{L}(Y)$, and $R \in \mathcal{L}_{w^*}(X^*, Y^*)$. Then the following holds.*

(i) $T^{\diamond *} = j_Y T$,

(ii) $T^\diamond \in \mathcal{L}_{w^*}(Y^*, X)$,

(iii) $T^{\diamond\diamond} = T$,

(iv) $\|T^\diamond\| = \|T\|$,

(v) $(\lambda T + U)^\diamond = \lambda T^\diamond + U^\diamond$,

(vi) $(ST)^\diamond = T^\diamond S^*$,

(vii) $(R^\diamond j_Y)^* = R$.

Proof. (i). For every $x^* \in X^*$ and $y^* \in Y^*$ we have

$$(j_Y(Tx^*))(y^*) = y^*(Tx^*) = (T^*y^*)(x^*) = (j_X T^\diamond y^*)(x^*) = (T^{\diamond *}x^*)(y^*).$$

Property (ii) is equivalent to $T^{\diamond *}(X^*) \subset j_Y(Y)$, which follows from (i). We have (iii) because (i) implies $j_Y T = j_Y T^{\diamond\diamond}$ and j_Y is isometric. Properties (iv), (v), and (vi) follow from the matching properties of dual operators.

(vii). Since $j_Y^* j_{Y^*} = I_{Y^*}$, we have $(R^\diamond j_Y)^* = j_Y^* R^{\diamond *} = j_Y^* j_{Y^*} R = R$. □

In particular, these properties show that the operation of taking the dual astriction establishes an isometric isomorphism between $\mathcal{A}_{w^*}(X^*, Y)$ and $\mathcal{A}_{w^*}(Y^*, X)$ for any completely symmetric operator ideal $\mathcal{A}$ (e.g., $\mathcal{F}$, $\mathcal{K}$ or $\mathcal{W}$). Indeed, if $T \in \mathcal{A}_{w^*}(X^*, Y)$ then $T^{\diamond *} = j_Y T \in \mathcal{A}$, so that $T^\diamond \in \mathcal{A}_{w^*}(Y^*, X)$ because $\mathcal{A}$ is completely symmetric.

Proposition 2.8 is to be compared with the classical criterion of weakly compact operators:

Proposition 2.11 (see, e.g., [DS, page 482])**.** *An operator $T \in \mathcal{L}(X, Y)$ is weakly compact if and only if $T^{**}(X^{**}) \subset j_Y(Y)$.*

This and property (vii) imply that the operators in $\mathcal{L}_{w^*}(Y^*, X^*)$ are exactly all the adjoints to operators in $\mathcal{W}(X, Y)$. Therefore, for any operator ideal $\mathcal{A} \subset \mathcal{W}$, the adjoint operation is a natural isometry between the spaces $\mathcal{A}^{\mathrm{dual}}(X, Y)$ and $\mathcal{A}_{w^*}(Y^*, X^*)$.

2.3 Tensor product

Let X and Y be Banach spaces. The algebraic tensor product $X \otimes Y$ can be constructed in many equivalent ways. See [G] or [Ryan] for some of them. For our aims, we prefer the following quick definition, and refer the reader to [Ryan] for a modern introduction to tensor products of Banach spaces and proofs of the statements in this section.

Let $x \in X$ and $y \in Y$. By the *simple tensor* $x \otimes y \in \mathcal{L}(X^*, Y)$ we mean an operator of rank one defined as

$$(x \otimes y)(x^*) = x^*(x)y$$

for all $x^* \in X^*$. A subspace

$$X \otimes Y := \operatorname{span}\{x \otimes y : x \in X,\ y \in Y\} = \mathcal{F}_{w^*}(X^*, Y)$$

is called the *algebraic tensor product* of X and Y.

Clearly, any tensor $u \in X \otimes Y$ has can be represented as

$$u = \sum_{i=1}^{n} x_i \otimes y_i, \quad n \in \mathbb{N},\ x_i \in X,\ y_i \in Y,\ i = 1, \dots, n,$$

in many different ways.

Observe that $X^* \otimes Y = \mathcal{F}_{w^*}(X^{**}, Y) \cong \mathcal{F}(X, Y)$ with

$$\left(\sum_{i=1}^{n} x_i^* \otimes y_i \right)(x) = \sum_{i=1}^{n} x_i^*(x) y_i$$

for $\sum_{i=1}^{n} x_i^* \otimes y_i \in X^* \otimes Y$ and $x \in X$.

The *injective tensor product* $X \check{\otimes} Y := \overline{X \otimes Y}$ is the completion of $X \otimes Y$ under the norm induced by $\mathcal{L}(X^*, Y)$.

The *projective norm* $\|\cdot\|_\pi$ on $X \otimes Y$ is defined as

$$\|u\|_\pi = \inf \sum_{i=1}^{n} \|x_i\| \, \|y_i\|$$

for all $u \in X \otimes Y$, where infimum is taken over all possible representations $u = \sum_{i=1}^{n} x_i \otimes y_i$. The *projective tensor product* $X \hat{\otimes} Y$ is the completion of

$X \otimes Y$ under the projective norm. For any element $u \in X\hat{\otimes}Y$ and every $\varepsilon > 0$, there exists a representation

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \|u\|_\pi + \varepsilon$. Note that we can actually choose a representation, where $\sum_{n=1}^{\infty} \|x_n\| < \|u\|_\pi + \varepsilon$, $\sup_n \|y_n\| \leq 1$, and $y_n \to 0$, or vice versa.

Let X_1 and Y_1 be Banach spaces, let $S \in \mathcal{L}(X, X_1)$, and let $T \in \mathcal{L}(Y, Y_1)$. Then a bounded linear operator $S \otimes T$ between $X\hat{\otimes}Y$ and $X_1\hat{\otimes}Y_1$ is well defined by

$$(S \otimes T)u = \sum_{n=1}^{\infty} (Sx_n) \otimes (Ty_n) \in X_1\hat{\otimes}Y_1$$

for $u = \sum_{n=1}^{\infty} x_n \otimes y_n \in X\hat{\otimes}Y$. If S happens to be an identity operator of some Banach space, we just write T in place of $S \otimes T$, and vice versa. The *trace functional* on $X^*\hat{\otimes}X$ is defined as

$$\operatorname{trace}(u) = \sum_{n=1}^{\infty} x_n^*(x_n)$$

for $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n \in X^*\hat{\otimes}X$ with $\sum_{n=1}^{\infty} \|x_n^*\| \|x_n\| < \infty$. It is well defined and does not depend on the representation of u.

2.4 Topologies on $\mathcal{L}(X,Y)$

Apart from the usual norm topology, the operator space $\mathcal{L}(X,Y)$ possesses three natural locally convex topologies which play an important role in the study of the approximation properties.

The *strong operator topology* (*SOT*) is defined by the system of seminorms

$$\{p_F : F \subset X, F \text{ is finite}\}$$

with

$$p_F(S) = \sup_{x \in F} \|Sx\|, \quad S \in \mathcal{L}(X,Y).$$

In $\mathcal{L}(X,Y)$, a net (S_α) converges to S in the SOT if and only if it converges *pointwise*, i.e., $S_\alpha x \to Sx$ for all $x \in X$. We will denote this topology by $\tau_s(X,Y)$, or simply by τ_s, when no confusion is likely to arise.

The *weak operator topology* (*WOT*) is defined by the system of seminorms

$$\{p_{F,H} : F \subset X, H \subset Y^*, F \text{ and } H \text{ are finite}\}$$

with

$$p_{F,H}(S) = \sup\{|y^*(Sx)| : x \in F, y^* \in H\}, \quad S \in \mathcal{L}(X,Y).$$

In $\mathcal{L}(X,Y)$, a net (S_α) converges to S in the WOT if and only if it converges *weakly pointwise*, i.e., $y^*(S_\alpha x) \to y^*(Sx)$ for all $x \in X$ and $y^* \in Y^*$. We will denote this topology by $\tau_w(X,Y)$, or simply by τ_w.

The *topology of uniform convergence on compact sets*, also known as the *topology of compact convergence*, is defined by the system of seminorms

$$\{p_K : K \subset X, K \text{ is compact}\}$$

with

$$p_K(S) = \sup_{x \in K} \|Sx\|, \quad S \in \mathcal{L}(X,Y).$$

We will denote this topology by τ_c. The nets converging in τ_c will be said to converge *compactly*.

It is clear that τ_w is weaker than τ_s, τ_s is weaker than τ_c, and τ_c is weaker than the norm topology. The following three propositions date back to Grothendieck's memoir [G, Chapter I, Lemma 20, Proposition 22, and Proposition 23].

Proposition 2.12 (see, e.g., [DS, Corollary VI.1.5])**.** *A convex subset of $\mathcal{L}(X,Y)$ has the same closure in τ_w as it does in τ_s.*

Proposition 2.13. *On bounded subsets of $\mathcal{L}(X,Y)$, τ_s coincides with τ_c.*

Proof. Since $\tau_c \geq \tau_s$, it is enough to show that any element T in the τ_s-closure of a norm-bounded set $A \subset \mathcal{L}(X,Y)$ is also in its τ_c-closure.

Let A be norm-bounded by $M > 0$ and let $T \in \overline{A}^{\tau_s}$. We may assume that $\|T\| \leq M$. Fix a compact set $K \subset X$ and $\varepsilon > 0$. Let F be a finite $\frac{\varepsilon}{4M}$-net of K. By assumption, we can find $S \in A$ such that $p_F(S - T) = \sup_{x \in F} \|Sx - Tx\| < \varepsilon/2$. Then for all $x \in K$ there is $x_0 \in F$ such that $\|x - x_0\| < \frac{\varepsilon}{4M}$, and we have

$$\begin{aligned} \|Sx - Tx\| &\leq \|Sx - Sx_0\| + \|Sx_0 - Tx_0\| + \|Tx_0 - Tx\| \\ &< \frac{\varepsilon(\|S\| + \|T\|)}{4M} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Hence, $p_K(T - S) = \sup_{x \in K} \|Tx - Sx\| \leq \varepsilon$, so that $T \in \overline{A}^{\tau_c}$. □

Proposition 2.14 (see, e.g., [LT, Proposition 1.e.3]). *Let X and Y be Banach spaces. There is a surjective linear operator V from $Y^*\hat{\otimes}X$ to the space $(\mathcal{L}(X,Y),\tau_c)^*$ of τ_c-continuous linear functionals on $\mathcal{L}(X,Y)$ defined by*

$$(Vu)(T) = \mathrm{trace}(Tu)$$

for $u \in Y^*\hat{\otimes}X$ and $T \in \mathcal{L}(X,Y)$.

2.5 Factorization lemma

In this section we introduce one of the main tools used throughout the thesis. It is the famous Davis–Figiel–Johnson–Pełczyński factorization lemma [DFJP, Lemma 1] in its isometric version due to Lima, Nygaard, and Oja [LNO, Lemmas 1.1, 1.2]. Let us recall the relevant construction.

Let a be the unique solution of the equation

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1, \ \ a > 1.$$

Let X be a Banach space and let K be a closed absolutely convex subset of B_X. For each $n \in \mathbb{N}$, put $B_n = a^{n/2}K + a^{-n/2}B_X$. The gauge of B_n gives an equivalent norm $\|\cdot\|_n$ on X. Set

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{1/2},$$

define $X_K = \{x \in X : \|x\|_K < \infty\}$, and let $J_K : X_K \to X$ denote the identity embedding.

Lemma 2.15 (see [DFJP] and [LNO]). *With notation as above, the following holds.*
(i) $X_K = (X_K, \|\cdot\|_K)$ *is a Banach space and* $\|J_K\| \le 1$.
(ii) $K \subset B_{X_K} \subset B_X$.
(iii) $B_{X_K} \subset B_n$ *for all* $n \in \mathbb{N}$.
(iv) $J_K^*(X^*)$ *is norm dense in* X_K^*.
(v) J_K *is compact if and only if* K *is compact; in this case* X_K *is separable.*
(vi) X_K *is reflexive if and only if* K *is weakly compact.*

For future applications of this lemma, it is useful to recall a classical result on compactness of closed absolutely convex hulls (see, e.g., [M, p. 254]).

Proposition 2.16 (Mazur, Krein, Šmulian). *The closed absolutely convex hull of a (weakly) compact subset of a Banach space is (weakly) compact.*

In many cases, we use Lemma 2.15 in the following special form.

Lemma 2.17. *Let X be a Banach space and let $K \subset B_X$ be a compact set. There exists a linear subspace X_K of the space X equipped with a norm $\|\cdot\|_K$ such that X_K is a separable reflexive Banach space with respect to this norm and the following holds.*

(i) *The identity embedding $J_K : X_K \to X$ is compact and $\|J_K\| = 1$.*

(ii) *$K \subset B_{X_K} \subset B_X$.*

Proof of Lemma 2.17. Let $K' = \overline{\operatorname{absconv}}(K)$ be a closed absolutely convex hull of K. Then $K' \subset B_X$. Hence, there is a Banach space $X_{K'}$ such that the statements of Lemma 2.15 are true for the set K'. We define the space X_K to be the space $X_{K'}$. Now since $K \subset K'$ and K' is compact by Proposition 2.16, we obtain the required conditions. □

As the name suggests, the lemma can be used to factorize operators. Let T be a bounded linear operator between Banach spaces Y and X and let $\|T\| = 1$. Denote $K := \overline{\operatorname{absconv}}\, T(B_Y)$ and construct X_K and J_K as in Lemma 2.15.

Lemma 2.18 (see [LNO, Theorem 2.2]). *With respect to the above notation there exists an operator $t \in \mathcal{L}(Y, X_K)$ such that $\|t\| = 1$ and $T = J_K t$.*

Proof. The inclusion $T(B_Y) \subset K \subset B_{X_K}$ implies $T(Y) \subset X_K$ and we can define linear operator t to have the same values as T. Now for $y \in B_Y$ we have $Ty \in K$, so that $t(y) \in B_{X_K}$. That is, $\|t\| \leq 1$. On the other hand, $1 = \|T\| = \|J_K t\| \leq \|J_K\|\,\|t\| = \|t\|$. □

Chapter 3

Approximation property

In this chapter we consider several versions of approximation properties. We go on to define the convex approximation property and the convex approximation property with conjugate operators, and present some of the basic results concerning the approximation properties. This chapter is based on introductory parts of [L] and [LMO].

3.1 Classical notions and examples

Throughout the thesis, by shorthand "AP" we mean "approximation property". We still continue to use the latter variant whenever we find appropriate; for example, inside definitions.

Definition 3.1. A Banach space X is said to have the *approximation property* (AP) if for every compact subset K of X and for every $\varepsilon > 0$, there is a finite-rank operator S on X such that $\|Sx - x\| < \varepsilon$ for all $x \in K$.

Let $\lambda \geq 1$. If operator S above can be chosen with $\|S\| \leq \lambda$, then X is said to have the *bounded approximation property* (BAP) and, in particular, the *λ-bounded approximation property.*

The 1-bounded AP is called the *metric approximation property* (MAP).

If operator S above is allowed to be compact, then X is said to have the *compact approximation property* (CAP). Its bounded and metric versions are defined in an obvious way.

Definition 3.2. A sequence (e_n) in a Banach space X is called a *Schauder* *basis* if for every element $x \in X$, there exists a unique sequence $(a_n) \subset \mathbb{K}$ such that

$$x = \sum_{k=1}^{\infty} a_k e_k.$$

Observe that a Banach space X with a Schauder basis (e_n) satisfies the AP with the basis projections $P_n \in \mathcal{F}(X)$, $P_n x = \sum_{k=1}^{n} a_k e_k$, $x \in X$, playing the role of finite-rank operators S in the definition of the AP.

A Schauder basis is called *monotone* if its basis projections are of norm 1.

Naturally, as both the approximation problem and the basis problem were open for such a long time, most classical spaces, such as c_0, $C[0,1]$, ℓ_p, $L_p[0,1]$, where $1 \leq p \leq \infty$, are known to have a monotone Schauder basis (if the space is separable) or to satisfy the MAP.

We postpone the discussion of any positive results on the AP to the forthcoming section, and focus here on the negative examples.

The most notable of this examples is the original construction by Enflo [E]. Note that, while being discovered as a counterexample to the AP, Enflo's space actually failed the CAP.

Example 3.3. There exists a separable reflexive Banach space failing the CAP.

Other examples followed Enflo's construction. In fact, the space ℓ_p with $1 \leq p < \infty$, $p \neq 2$, contains a closed subspace failing the CAP. This was proven by Davie [Davie] for $p \in (2, \infty)$, and by Szankowski [Sz2] for $p \in [1, 2)$. Szankowski [Sz3] also proved that the space $\mathcal{L}(\ell_2)$ fails the AP .

The next example is due to Figiel and Johnson [FJ].

Example 3.4. There exists a Banach space having the AP but failing the BAP.

Very recently, Figiel, Johnson, and Pełczyński [FJP] constructed a closed subspace of c_0 with properties as in Example 3.4.

In [FJ], authors also showed that the BAP does not imply the MAP. Johnson and Schechtman constructed a closed subspace of c_0 having the BAP but failing the MAP. As shown by Godefroy [Go], the Johnson–Schechtman space has the 8-bounded AP. This result was subsequently improved by Zolk [Z] to the following version.

Example 3.5. There exists a Banach space having the 6-bounded AP but failing the MAP.

Lindenstrauss [Lin] noticed that, in general, the AP cannot be lifted to the dual space.

Example 3.6. There exists a Banach space having the monotone Schauder basis such that its dual space fails the AP.

Willis [W] showed that the CAP and the AP are different properties.

Example 3.7. There exists a Banach space having the CAP but failing the AP.

3.2 Convex approximation property

In this section, let X be a Banach space, let $A \subset \mathcal{L}(X)$ be a collection of bounded linear operators on X, and let $\mathcal{A}$ be an operator ideal.

Definition 3.8. We say that X has the *A-approximation property* (A-AP) if for every compact subset $K \subset X$ and for every $\varepsilon > 0$, there is an operator $S \in A$ such that $\|Sx - x\| < \varepsilon$ for all $x \in K$.

In other words, X has the A-AP if and only if I_X is in the τ_c-closure of A.

Definition 3.9. We emphasize that the A-approximation property of X is *convex* if A is convex and contains 0.

Definition 3.10. A Banach space X is said to have the *$\mathcal{A}$-approximation property* ($\mathcal{A}$-AP) if it has the $\mathcal{A}(X)$-AP.

Observe that the $\mathcal{F}$-AP is exactly the AP, and the $\mathcal{K}$-AP is the CAP. Also, all the notions mentioned in Definition 3.1 are convex approximation properties.

In general, the bounded versions of convex approximation properties are convex approximation properties. Hence, they can be treated using the same methods.

Definition 3.11. Let $\lambda \in [1, \infty)$. We say that X has the *λ-bounded A-AP* if X has the $(\lambda B_{\mathcal{L}(X)} \cap A)$-AP.

We say that X has the *metric A-AP* if X has the 1-bounded A-AP.

We say that X has the *bounded A-AP* if X has the μ-bounded A-AP for some $\mu \in [1, \infty)$.

The bounded and metric versions of the $\mathcal{A}$-AP are defined in a similar way.

The $\mathcal{A}$-approximation property, and also the A-approximation property with $A \subset \mathcal{L}(X)$ being a linear subspace, were studied, for instance, by Reinov [R2] and by Grønbœk and Willis [GW]. Its bounded version has recently been studied in [LO5] and [O1], and it has been proven useful in the studies on the duality of the distance to closed operator ideals due to Tylli [T1, T3]. In particular, the non-self-duality was established for the essential norm of bounded linear operators on Banach spaces (see [T1]).

The convex A-approximation property was introduced in [LMO] and studied in [LMO] and [LisO]. It is the central notion of the thesis. However, a number of results we present are universal enough to work in the case, when A is an arbitrary subset of $\mathcal{L}(X)$. The following is one of them (Section 4.1 is devoted to this case).

Proposition 3.12. *Let X be a Banach space and let $A, B \subset \mathcal{L}(X)$. If X has both the A-AP and the B-AP, then X also has the $A \circ B$-AP.*

Proof. Let us check the definition. Fix a compact set $K \subset X$ and $\varepsilon > 0$. Find $T \in A$ such that $\|Tx - x\| < \varepsilon/2$ for all $x \in K$. Find $S \in B$ such that

$$\|Sx - x\| < \frac{\varepsilon}{2\,\|T\|}$$

for all $x \in K$. Then for all $x \in K$ one has

$$\|TSx - x\| = \|T(Sx - x) + Tx - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. □

In particular, the next observation points out the $\mathcal{A}$-AP passes on to complemented subspaces.

Proposition 3.13. *Let X be a Banach space, let Y be a subspace of X complemented by a projection $P \in \mathcal{L}(X)$, and let A be a subset of $\mathcal{L}(X)$. If X has the A-AP, then Y has the $\{PS|_Y : S \in A\}$-AP.*

Proof. Take a compact set $K \subset Y$ and $\varepsilon > 0$. Since Y is closed, K is compact in X. By the assumption, we can find $S \in A$ such that $\|Sx - x\| < \varepsilon / \|P\|$ for all $x \in K$. Then $\|PSx - x\| = \|PSx - Px\| < \varepsilon$. □

If A is bounded, Proposition 2.13 implies that X has the A-AP if and only if I_X can be pointwise approximated by operators in A. However, in the general case (when A is unbounded), this is not so.

Proposition 3.14. *Let X be a Banach space. Then I_X is in the τ_s-closure of $\mathcal{F}(X)$.*

Proof. Fix a finite set $F \subset X$ and $\varepsilon > 0$. We have to find $S \in \mathcal{F}(X)$ such that $\|Sx - x\| < \varepsilon$ for all $x \in F$. Since every finite-dimensional subspace of a Banach space is complemented, we can find a projection $P \in \mathcal{F}(X)$ onto $\operatorname{span} F$. That is, we actually have much more than needed: $Px = x$ for all $x \in \operatorname{span} F$. □

Let $A \subset \mathcal{L}(X)$ be convex (Section 4.2 is also devoted to this case). In this case, we can use the Hahn–Banach theorem to employ Grothendieck's description of linear functionals on $\mathcal{L}(X)$ which are continuous in the topology of compact convergence τ_c.

Lemma 3.15. *Let X be Banach space and let $A \subset \mathcal{L}(X)$ be a convex set. The space X has the A-AP if and only if for all sequences $(x_n^*) \subset X^*$ and $(x_n) \subset X$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n\| < \infty$ one has*

$$\inf_{S \in A} \left| \sum_{n=1}^{\infty} x_n^*(Sx_n - x_n) \right| = 0.$$

Proof. Note that the A-AP of the space X means that the identity operator I_X is in the closure of the convex set A in the space $(\mathcal{L}(X), \tau_c)$. This happens if and only if, for every τ_c-continuous linear functional φ on $\mathcal{L}(X)$, one has

$$\mathcal{R}e\, \varphi(I_X) \leq \sup_{S \in A} \mathcal{R}e\, \varphi(S),$$

which is implied by

$$\inf_{S \in A} |\varphi(I_X - S)| = 0.$$

Since the latter condition clearly follows from the A-AP of X, applying the description of $(\mathcal{L}(X), \tau_c)^*$ (see Proposition 2.14) completes the proof. □

Remark 3.1. Let us notice that in the assumption of Lemma 3.15 one may put $\sum_{n=1}^{\infty} \|x_n^*\| \leq M$, $x_n \to 0$, and $\sup \|x_n\| \leq N$ for any $M > 0$ and $N > 0$, or vice versa.

Lemma 3.16. *Let X be a Banach space and let $A \subset \mathcal{L}(X)$ be a convex set. The space X has the A-AP if and only if for all sequences $(x_n^*) \subset X^*$ and $(x_n) \subset X$ with $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n\| < \infty$, there is a net $(S_\alpha) \subset A$ such that*

(i) *for every $n \in \mathbb{N}$ one has*

$$x_n^*(S_\alpha x_n - x_n) \underset{\alpha}{\longrightarrow} 0,$$

(ii)

$$\sup_\alpha \left| \sum_{n>N} x_n^*(S_\alpha x_n - x_n) \right| \underset{N\to\infty}{\longrightarrow} 0.$$

Proof. Necessity. Let sequences $(x_n^*) \subset X^*$ and $(x_n) \subset X$ be such that $\sum \|x_n^*\| \le 1$ and $x_n \to 0$. The A-approximation property provides a net $(S_\alpha) \subset A$ converging to I_X in the topology of compact convergence. Since the set $K := \{0, x_1, x_2, \dots\}$ is compact, there is a subnet (S_ν) such that $\|S_\nu x - x\| \le 1$ for all $x \in K$ and for all indexes ν. Clearly, this subnet satisfies both (i) and (ii).

Sufficiency. Let us employ Lemma 3.15 to show that X has the A-AP. Take sequences $(x_n^*) \subset X^*$ and $(x_n) \subset X$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n\| < \infty$. Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\left| \sum_{n>N} x_n^*(S_\alpha x_n - x_n) \right| < \frac{\varepsilon}{2}$$

for all α. Choose α such that $x_n^*(S_\alpha x_n - x_n) < \frac{\varepsilon}{2N}$ for $n = 1, \dots, N$. Then

$$\left| \sum_{n=1}^{\infty} x_n^*(S_\alpha x_n - x_n) \right| \le \sum_{n=1}^{N} \frac{\varepsilon}{2N} + \left| \sum_{n>N} x_n^*(S_\alpha x_n - x_n) \right| < \varepsilon,$$

as required. □

The following proposition expresses the fact that the convex λ-bounded AP is "continuous" with respect to λ.

Proposition 3.17. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. Let $\lambda \ge 1$. If X has the $(1+\varepsilon)\lambda$-bounded A-AP for every $\varepsilon > 0$, then X has the λ-bounded A-AP.*

Proof. Fix a compact set $K \subset B_X$ and $\varepsilon > 0$. Since X has the $(1+\varepsilon)\lambda$-bounded A-AP, there is $T \in A$ such that $\|T\| \leq (1+\varepsilon)\lambda$ and $\|Tx - x\| < \varepsilon^2$ for all $x \in K$. Let $S = \frac{1}{1+\varepsilon}T$. Then $\|S\| \leq \lambda$ and $S \in A$, since A is convex and contains 0. Also, for all $x \in K$ one has

$$\|Sx - x\| = \frac{1}{1+\varepsilon}\,\|Tx - x - \varepsilon x\| < \frac{\varepsilon^2 + \varepsilon}{1+\varepsilon} = \varepsilon,$$

as needed. □

3.3 AP with conjugate operators

Let X be a Banach space.

In order to investigate the interplay between the approximation properties of a Banach space and the approximation properties of its dual space, it is convenient to introduce the notion of the approximation property with conjugate operators.

Definition 3.18. Let $A \subset \mathcal{L}(X)$. We say that the space X^* has the *A-approximation property with conjugate operators* if it has the A^{a}-AP, where $A^{\mathrm{a}} \subset \mathcal{L}(X^*)$ is defined as

$$A^{\mathrm{a}} = \{S^* \colon S \in A\}.$$

Let us notice that as a simple application of Lemma 3.15 we have the following.

Proposition 3.19. *Let $A \subset \mathcal{L}(X)$ be convex. If X^* has the A-AP with conjugate operators, then X has the A-AP.*

Proof. Take a tensor $\sum_{n=1}^{\infty} x_n^* \otimes x_n \in X^* \hat{\otimes} X$ with $\sum_{n=1}^{\infty} \|x_n^*\|\,\|x_n\| < \infty$. And observe that we can regard it as an element of $X^{**}\hat{\otimes} X^*$. Therefore, by assumption

$$\inf_{S\in A}\left|\sum_{n=1}^{\infty} x_n^*(Sx_n - x_n)\right| = \inf_{S\in A}\left|\sum_{n=1}^{\infty} \jmath_X x_n(S^*x_n^* - x_n^*)\right| = 0.$$

□

It is obvious but still worth mentioning that Proposition 3.19 also applies to λ-bounded A-AP for some $\lambda \geq 1$.

Let $\mathcal{A}$ be an operator ideal. The *$\mathcal{A}$-AP with conjugate operators* of X^*, as well as the *AP with conjugate operators* and other related notions, are defined in an obvious manner.

The AP with conjugate operators is actually the same property as the AP (see Proposition 3.22 below). However, already in the case, when $\mathcal{A} = \mathcal{K}$, the $\mathcal{A}$-AP and the $\mathcal{A}$-AP with conjugate operators differ. In fact, Grønbœk and Willis [GW] exhibit a Banach space X_{GW} having a basis such that X_{GW}^* has the bounded $\mathcal{K}$-AP but does not have the $\mathcal{K}$-AP with conjugate operators.

It is easy to see that X_{GW}^* does not have the $\mathcal{W}$-AP with conjugate operators either. Indeed, for a Banach space X, we have $\mathcal{W}(X)^{\mathrm{a}} = \mathcal{L}_{w^*}(X^*)$ and $\mathcal{L}(X^*) \circ \mathcal{L}_{w^*}(X^*) \subset \mathcal{L}_{w^*}(X^*)$ (see Section 2.2). Therefore,

$$\mathcal{K}(X^*) \circ \mathcal{W}(X)^{\mathrm{a}} \subset \mathcal{K}_{w^*}(X^*) = \mathcal{K}(X)^{\mathrm{a}},$$

so that by Proposition 3.12, for X^*, the $\mathcal{W}$-AP with conjugate operators and the $\mathcal{K}$-AP would imply the $\mathcal{K}$-AP with conjugate operators.

Let us turn our attention to the equivalence of the (λ-bounded) AP and the (λ-bounded) AP with conjugate operators (see Proposition 3.22). This is a well-known fact due to Johnson [J1]. Similar results hold also for positive AP of Banach lattices and the AP for pairs of Banach spaces (see Chapter 8), and are important for application of our main theorems to those cases. We will provide a fairly detailed proof here, which will serve as a template in the sequel.

For the proof, we shall employ, as is usual by now, the *principle of local reflexivity*. It might be of interest that alternative proofs not using the principle of local reflexivity also exist. See [GW, Theorem 3.3] for a proof involving Banach algebra techniques, and [O1, Corollary 2.3] for a proof employing the description of $(\mathcal{F}(X))^*$ as the space of integral operators on X^* due to Grothendieck [G].

Let us mention that the principle of local reflexivity is due to Lindenstrauss and Rosenthal [LR]. Their lemma was subsequently improved by Johnson, Rosenthal, and Zippin [JRZ]. Many other variants and proofs are presented in the literature. We refer the reader to [OP] for a good summary of the topic.

Lemma 3.20 (Principle of local reflexivity (as in [JRZ]))**.** *Let X be a Banach space, let $E \subset X^{**}$ and $F \subset X^*$ be finite-dimensional subspaces, and let $\varepsilon > 0$. Then there exists a one-to-one operator $J \in \mathcal{F}(E, X)$ such that $\|J\|$, $\|J^{-1}\| < 1 + \varepsilon$, $Jx = x$ for all $x \in E \cap X$, and $x^*(Jx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$.*

In effect, the claim of the promised Proposition 3.22 is contained in the next result which tells us that finite-rank operators between dual Banach spaces are "locally conjugate". This fact is essentially due to Johnson, Rosenthal, and Zippin [JRZ, Lemma 3.1 and Corollary 3.2] (where X is assumed to be finite-dimensional; see [OP, Theorem 2.1] for the general result and an easier proof).

Lemma 3.21. *Let X and Y be a Banach spaces. Let $T \in \mathcal{F}(Y^*, X^*)$ and $\varepsilon > 0$. Then T is in the closure of the set*

$$\{S \in \mathcal{F}(X, Y) : \|S\| \leq (1+\varepsilon)\,\|T\|\}^{\mathrm{a}}$$

in the topology of compact convergence.

Proof. Let A denote the set above. Since A is bounded, Proposition 2.13 implies that it is enough to prove that T can be approximated by operators from A in the strong operator topology. Let us do so. Fix a finite set $G \subset Y^*$. We shall prove that there is $S \in A$ such that $Sy^* = Ty^*$ for all $y^* \in G$. Consider a representation

$$T = \sum_{i=1}^{n} y_n^{**} \otimes x_n^*$$

with $y_n^{**} \in Y^{**}$ and $x_n^* \in X^*$ for $i = 1, \dots, n$, and $n \in \mathbb{N}$. Put $F = \operatorname{span} G$ and $E = T^*(X^{**}) = \operatorname{span}\{y_1^{**}, \dots, y_n^{**}\}$. Lemma 3.20 gives us an operator $J \in \mathcal{F}(E, Y)$ such that $\|J\| < 1+\varepsilon$ and $y^*(Jy^{**}) = y^{**}(y^*)$ for all $y^{**} \in E$ and $y^* \in F$. Consider an operator $S = (J(T^*)_E j_X)^*$, where $(T^*)_E \in \mathcal{F}(X^{**}, E)$ is the astriction of T^* to E. Clearly, $S \in A$ and

$$S = \sum_{i=1}^{n} Jy_n^{**} \otimes x_n^*.$$

Then for any $y^* \in G$ we have

$$Ty^* - Sy^* = \sum_{i=1}^{n} (y_n^{**}(y^*) - y^*(Jy_n^{**}))x_n^* = 0,$$

as required.

□

Proposition 3.22 (Johnson)**.** *Let X be a Banach space and let $\lambda \geq 1$. Then X^* has the (λ-bounded) AP with conjugate operators if and only if it has the (λ-bounded) AP.*

Proof. Necessity is obvious.

Sufficiency. The assumption means that I_{X^*} is in the τ_c-closure of $\mathcal{F}(X^*)$ (respectively $\lambda B_{\mathcal{F}(X^*)}$). Lemma 3.21 yields that the latter set (and therefore also I_{X^*}) is in the τ_c-closure of $\mathcal{F}(X)^{\mathrm{a}}$ (respectively $(1+\varepsilon)\lambda B^{\mathrm{a}}_{\mathcal{F}(X)}$ for every $\varepsilon > 0$). The claim now follows from Proposition 3.17.

□

Observe that Proposition 3.22 and Proposition 3.19 enable us to say that the AP of the dual space X^* implies the AP of X. It is an open question whether the same is true for the compact AP (see [C, Problem 8.5]).

Chapter 4

Approximability of compact operators

The results of this chapter extend the classical description of the approximation property of a Banach and its dual space via approximability of compact operators by finite-rank operators. We present results for the approximation properties defined by an arbitrary collection of bounded linear operators, as well as their improvements in the case of the convex approximation properties. This chapter is based on [L] and [LMO].

4.1 Approximability in norm

The next theorem is the main source of inspiration for the results of this section. It says that a Banach space X has the AP when certain compact operators are approximated by the operators of finite rank. In other words, it establishes the relation between the AP and the approximation problem.

Theorem 4.1 (Grothendieck [G]). *Let X be a Banach space. The following conditions are equivalent.*

(a) *The space X has the AP.*

(b) *For every Banach space Y one has $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$.*

(c) *For every Banach space Y one has $\mathcal{K}_{w^*}(X^*, Y) = \overline{\mathcal{F}_{w^*}(X^*, Y)}$.*

Naturally, the straightforward generalization of this criterion is not possible even to the case of the compact approximation property. Indeed, for instance, condition (b) of such a generalization, i.e., $\mathcal{K}(Y,X) = \overline{\mathcal{K}(Y,X)}$ for any Y, would be satisfied for every space X. But, as we mentioned above, Enflo's original example (see [E]) was a Banach space without the compact approximation property.

However, modified criteria with seemingly stronger conditions (b) and (c) are possible for any A-approximation property.

Theorem 4.2. *Let X be a Banach space and let $A \subset \mathcal{L}(X)$ be an arbitrary collection of bounded linear operators on X. The following statements are equivalent.*

(a) *The space X has the A-AP.*

(b) *For every Banach space Y and for every $T \in \mathcal{K}(Y,X)$ one has $T \in \overline{\{ST\colon S \in A\}}$.*

(b′) *For every separable reflexive Banach space Z and for every $T \in \mathcal{K}(Z,X)$ one has $T \in \overline{\{ST\colon S \in A\}}$.*

(c) *For every Banach space Y and for every $T \in \mathcal{K}_{w^*}(X^*,Y)$ one has $T \in \overline{\{TS^*\colon S \in A\}}$.*

(c′) *For every separable reflexive Banach space Z and for every $T \in \mathcal{K}_{w^*}(X^*,Z)$ one has $T \in \overline{\{TS^*\colon S \in A\}}$.*

Proof. (a) $\Rightarrow$ (b). Take $T \in \mathcal{K}(Y,X)$. Then the set $K := \overline{T(B_Y)}$ is compact. Since the space X has the A-AP, for every $\varepsilon > 0$ there is an operator $S \in A$ such that $\|Sx - x\| < \varepsilon$ for all $x \in K$. Now we have

$$\|ST - T\| = \sup_{y \in B_Y} \|STy - Ty\| \leq \sup_{x \in K} \|Sx - x\| \leq \varepsilon,$$

which means that $T \in \overline{\{ST : S \in A\}}$.

(b) $\Rightarrow$ (c). Take $T \in \mathcal{K}_{w^*}(X^*,Y)$. Then $T^\diamond \in \mathcal{K}_{w^*}(Y^*,X)$, and condition (b) implies

$$T^\diamond \in \overline{\{ST^\diamond : S \in A\}}.$$

Since $TS^* - T = (ST^\diamond - T^\diamond)^\diamond$ for every $S \in A$, the previous condition is equivalent to

$$T \in \overline{\{TS^* : S \in A\}}.$$

The same proof is suitable for the implication (b′) ⇒ (c′) because $\mathsf{X} \cap \mathsf{W}$ is symmetric. The implications (b) ⇒ (b′) and (c) ⇒ (c′) are obvious.

(c′) ⇒ (a). We verify the definition of the A-AP. Take a compact set $K \subset X$. Then for some $\lambda > 0$ the set λK is a compact subset of the unit ball B_X. By Lemma 2.17 there is a separable and reflexive space Z linearly included in X such that $\lambda K \subset B_Z$ and the identity embedding $J : Z \to X$ is compact. Then $J^* \in \mathcal{K}_{w^*}(X^*, Z^*)$, and we know that the space Z^* is reflexive and separable because Z is. Therefore condition (c′) implies

$$J^* \in \overline{\{J^*S^* : S \in A\}},$$

which is equivalent to the condition

$$J \in \overline{\{SJ : S \in A\}}.$$

Fix $\varepsilon > 0$. Then there is an operator $S \in A$ such that

$$\|Jz - SJz\| \leq \|J - SJ\| < \varepsilon\lambda$$

for every $z \in B_Z$. Take an arbitrary point $x \in K$. Then $\lambda x \in \lambda K \subset B_Z$. It remains to observe that

$$\|Sx - x\| = \lambda^{-1} \|S(\lambda x) - (\lambda x)\| = \lambda^{-1} \|SJ(\lambda x) - J(\lambda x)\| < \lambda^{-1}\varepsilon\lambda = \varepsilon,$$

as needed. □

Remark 4.1. In the special case when $A = \mathcal{K}(X)$ the equivalence (a) ⇔ (b) of Theorem 4.2 has been established by Lima, Lima, and Nygaard [LLN, Theorem 2.1].

We can similarly describe the approximation property of the dual space X^* by approximating compact operators acting from the original space X.

Corollary 4.3. *Let X be a Banach space and let $A \subset \mathcal{L}(X^*)$ be a collection of bounded linear operators on the dual space X^*. Then the following statements are equivalent.*

(a) *The space X^* has the A-AP.*

(b) *For every Banach space Y and for every $T \in \mathcal{K}(X, Y)$, one has $T^* \in \overline{\{ST^* : S \in A\}}$.*

(b′) *For every separable reflexive Banach space Z and for every $T \in \mathcal{K}(X, Z)$, one has $T^* \in \overline{\{ST^* : S \in A\}}$.*

Proof. (a) $\Rightarrow$ (b). Take $T \in \mathcal{K}(X, Y)$. Then $T^* \in \mathcal{K}(Y^*, X^*)$. The claim is immediate from (a) $\Rightarrow$ (b) of Theorem 4.2 applied to the A-AP of the space X^*.

(b) $\Rightarrow$ (b$'$) is obvious.

(b$'$) $\Rightarrow$ (a). We apply Theorem 4.2 to the A-AP of the space X^* and verify its condition (b$'$), i.e., for every separable and reflexive space Z and for every operator $T \in \mathcal{K}(Z, X^*)$ one has

$$T \in \overline{\{ST : S \in A\}}.$$

Take $T \in \mathcal{K}(Z, X^*)$. Then $T^* \in \mathcal{K}(X^{**}, Z^*)$ and $T^* j_X \in \mathcal{K}(X, Z^*)$. Since Z^* is separable and reflexive, the hypothesis implies

$$(T^* j_X)^* \in \overline{\{S(T^* j_X)^* : S \in A\}}.$$

But

$$(T^* j_X)^* = j_X^* T^{**} = j_X^* j_{X^*} T = I_{X^*} T = T,$$

because Z is reflexive. Hence,

$$T \in \overline{\{ST : S \in A\}},$$

as needed. □

Remark 4.2. In the special case when $A = \mathcal{K}(X^*)$ it has been shown in [LLN, Theorem 3.1] that condition (a) is equivalent to the following condition: for every space Y and for every operator $T \in \mathcal{K}(X, Y)$ one has

$$j_Y T \in \overline{\{T^{**} S : S \in \mathcal{K}(X, X^{**})\}}.$$

It is easy to see that this condition is equivalent to condition (b) of Corollary 4.3.

Remark 4.3. Observe that in Corollary 4.3 we only provided the analogues for conditions (b) and (b$'$) of Theorem 4.2. Actually, conditions (c) and (c$'$) would yield the same analogues, since $\mathcal{K}(X, Y)$ is naturally isometric to $\mathcal{K}_{w^*}(X^{**}, Y)$.

The next corollary is a straightforward application of Corollary 4.3 to the A-approximation property with conjugate operators.

Corollary 4.4. *Let X be a Banach space and let $A \subset \mathcal{L}(X)$ be a collection of bounded linear operators on X. The following statements are equivalent.*

(a) *The space X^* has the A-AP with conjugate operators.*

(b) *For every space Y and for every $T \in \mathcal{K}(X,Y)$ one has $T \in \overline{\{TS\colon S \in A\}}$.*

(b′) *For every separable reflexive Banach space Z and for every $T \in \mathcal{K}(X,Z)$ one has $T \in \overline{\{TS\colon S \in A\}}$.*

Remark 4.4. In the special case when $A = \mathcal{K}(X)$ the equivalence (a) $\Leftrightarrow$ (b) of Corollary 4.4 has been proven in [LLN, Theorem 3.3].

Remark 4.5. It is trivial but still worth mentioning that in the case of convex approximation properties, the approximability of compact operators is always somewhat “metric”, meaning that we can choose the approximating operators with norms not exceeding the norm of their limit (see Proposition 4.5 below). This fact becomes harder to establish once we go beyond compact operators (see Section 5.3).

Proposition 4.5. *Let $C \subset X$ be a convex subset of a Banach space X containing 0 and let $x \in \overline{C}$. Then there is a sequence $(x_n) \subset C$ such that $x_n \to x$ and $\sup_n \|x_n\| \leq \|x\|$.*

Proof. We can assume that $x \neq 0$. There is a sequence $(y_n) \subset C$ such that $y_n \to x$. In that case $\|y_n\| \to \|x\|$, so that there is a null sequence $(\varepsilon_n) \subset (0,\infty)$ such that $\|y_n\| \leq \|x\| + \varepsilon_n$ for any $n \in \mathbb{N}$. Denoting

$$x_n := \frac{\|x\|}{\|x\| + \varepsilon_n} y_n \in C$$

for $n \in \mathbb{N}$, we get $\sup_n \|x_n\| \leq \|x\|$ and $x_n \to x$, as needed. □

4.2 Approximability in other topologies

When $A \subset \mathcal{L}(X)$ is convex, we can relax sufficient conditions for the A-approximation property.

We prove the implication (c′) $\Rightarrow$ (a) of the following theorem by developing the approach of Oja and Pelander from [OPe, Theorem 3, (c_3) $\Rightarrow$ (a)].

Theorem 4.6. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$. The following statements are equivalent.*

(a) *The space X has the A-AP.*

(b′) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(Z, X)$ there is a net $(S_\alpha) \subset A$ such that* $\sup_\alpha \|S_\alpha T\| < \infty$ *and* $S_\alpha T \to T$ *in the weak operator topology on* $\mathcal{L}(Y, X)$.

(c′) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}_{w^*}(X^*, Z)$ there is a net $(S_\alpha) \subset A$ such that* $\sup_\alpha \|TS^*_\alpha\| < \infty$ *and* $TS^*_\alpha \to T$ *in the weak operator topology on* $\mathcal{L}(X^*, Y)$.

Proof. (a) ⇒ (b′). This is immediate from implication (a) ⇒ (b) of Theorem 4.2.

(b′) ⇒ (c′). Take $T \in \mathcal{K}_{w^*}(X^*, Z)$, then $T^\diamond \in \mathcal{K}_{w^*}(Z^*, X)$. Hence there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|S_\alpha T^\diamond\| < \infty$ and $S_\alpha T^\diamond \to T^\diamond$ in $\tau_w(Z^*, X)$. This means that $(S_\alpha T^\diamond)^\diamond \to T^{\diamond\diamond}$ or, what is the same (see Proposition 2.10), $TS^*_\alpha \to T$ in $\tau_w(X^*, Z)$, as needed.

(c′) ⇒ (a). We apply Lemma 3.16 to show that X has the A-approximation property. Take $(x_n) \subset X$ and $(x^*_n) \subset X^*$ such that $x_n \to 0$, $\|x_n\| \le 1$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^\infty \|x^*_n\| < \infty$. We have to show that

$$\sup_\alpha \left| \sum_{n>N} x^*_n(S_\alpha x_n - x_n) \right| \underset{N\to\infty}{\longrightarrow} 0,$$

and for all $n \in \mathbb{N}$ one has

$$x^*_n(S_\alpha x_n - x_n) \underset{\alpha}{\longrightarrow} 0.$$

By adding the limit value 0 to the sequence (x_n) we get a compact set $K = \{0, x_1, x_2, \dots, \}$ which resides in the unit ball B_X. By Lemma 2.17, we can now obtain a separable reflexive Banach space Z linearly included in X such that $K \subset B_Z \subset B_X$, the identity embedding $J : Z \to X$ is compact and $\|J\| = 1$. Let $(z_n) \subset B_Z$ be such that $Jz_n = x_n$ for all $n \in \mathbb{N}$.

Since $J^* \in \mathcal{K}_{w^*}(X^*, Z^*)$ (see Section 2.2) and Z^* is separable and reflexive, there is a net $(S_\alpha) \subset A$ such that $M := \sup_\alpha \|S_\alpha J\| < \infty$ and $J^* S^*_\alpha \to J^*$ in $\tau_w(X^*, Z^*)$. This clearly implies the convergence $S_\alpha J \to J$ in $\tau_w(Z, X)$.

Observe that, for every $n \in \mathbb{N}$ and for every index α one has

$$x^*_n(S_\alpha x_n - x_n) = x^*_n(S_\alpha J z_n - J z_n),$$

so that

$$x^*_n(S_\alpha x_n - x_n) \underset{\alpha}{\longrightarrow} 0$$

and

$$\begin{aligned}\sup_\alpha \left|\sum_{n>N} x_n^*(S_\alpha x_n - x_n)\right| &\le \sup_\alpha \sum_{n>N} (\|J^* S_\alpha^*\| + \|J^*\|)\, \|x_n^*\| \\ &\le (M+1) \sum_{n>N} \|x_n^*\| \underset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

as needed. □

Remark 4.6. In the special case when $A = \mathcal{K}(X)$ in [LLN, Theorem 2.3] it is shown that the A-AP is equivalent to a condition similar to (b′) of Theorem 4.6 where the convergence is in the strong operator topology and the approximability is "metric" (see Remark 4.5). Clearly, in this special case, this condition follows from (b) of Theorem 4.2 and implies (b′) of Theorem 4.6.

By the uniform boundedness principle any sequence converging in the weak operator topology is bounded in norm. This enables us to state the following result concerning sequences.

Corollary 4.7. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$. The following statements are equivalent.*

(a) *The space X has the A-AP.*

(b) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(Z, X)$ there is a sequence $(S_n) \subset A$ such that $S_n T \to T$ in the weak operator topology on $\mathcal{L}(Z, X)$.*

(c) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}_{w^*}(X^*, Z)$ there is a sequence $(S_n) \subset A$ such that $TS_n^* \to T$ in the weak operator topology on $\mathcal{L}(X^*, Z)$.*

(d) *For every compact subset $K \subset X$ there is a sequence $(S_n) \subset A$ such that $S_n x \to x$ weakly for every $x \in K$.*

Proof. The equivalences (a) ⇔ (b) ⇔ (c) are clear from Theorems 4.2 and 4.6, and the uniform boundedness principle. The implication (a) ⇒ (d) follows directly from the definition of the A-AP. To prove (d) ⇒ (b), take $\overline{T(B_Z)}$ as the compact set in (d). □

Remark 4.7. In the special case when $A = \mathcal{F}(X)$, the equivalence (a) ⇔ (d) has been pointed out in [FJPP]. In the same special case when $A = \mathcal{F}(X)$ the sequence $(S_n T)$ in condition (c) of Corollary 4.7 can be replaced by a sequence $(T_n) \subset \mathcal{F}(Z, X)$ (see [Pe, Corollary 2]).

A simple argument based on Proposition 3.14 yields that, in general, the boundedness of the approximating net in conditions (b) and (c) of Theorem 4.6 cannot be removed.

Proposition 4.8. *Let X be a Banach space.*

(b) *For every Banach space Y and for every operator $T \in \mathcal{K}(Y, X)$ there is a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha T \to T$ in the strong operator topology on $\mathcal{L}(Y, X)$.*

(c) *For every Banach space Y and for every operator $T \in \mathcal{K}_{w^*}(X^*, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $TS_\alpha^* \to T$ in the strong operator topology on $\mathcal{L}(X^*, Y)$.*

Proof. Let Y be a Banach space.

(b). Proposition 3.14 gives us a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_X$ in τ_s. Then also $S_\alpha T \to T$ in τ_s for all $T \in \mathcal{K}(Y, X)$.

(c). Proposition 3.14 applied to X^* and Lemma 3.21 give us a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha^* \to I_{X^*}$ in τ_s. The rest follows as in (b). □

Corollary 4.9. *Let X be a Banach space and let $A \subset \mathcal{L}(X^*)$ be a convex set. Then the following statements are equivalent.*

(a) *The space X^* has the A-AP.*

(b) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$ there is a sequence $(S_n) \subset A$ such that $S_n T^* \to T^*$ in the weak operator topology on $\mathcal{L}(Y^*, X^*)$.*

(c) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$ there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|S_\alpha T^*\| < \infty$ and $S_\alpha T^* \to T^*$ in the weak operator topology on $\mathcal{L}(Y^*, X^*)$.*

Proof. The implication (a) ⇒ (b) follows from Corollary 4.3 and the implication (b) ⇒ (c) follows from the uniform boundedness principle.

(c) ⇒ (a). We apply Theorem 4.6 to the space X^* and check its condition (b′). Let Z be a separable reflexive Banach space. Take an operator $T \in \mathcal{K}(Z, X^*)$. Similarly to the proof of the implication (b′) ⇒ (a) of Corollary 4.3, there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|S_\alpha T\| < \infty$ and $S_\alpha T \to T$ in the weak operator topology, as needed. □

Remark 4.8. In the special case when $A = \mathcal{K}(X^*)$ in [LLN, Theorem 3.2, (i) $\Leftrightarrow$ (v)] it is shown that the A-AP of X^* is equivalent to a condition similar to (c) of Corollary 4.9 where the approximability is in the strong operator topology and "metric". Clearly, Corollaries 4.3 and 4.9 imply the equivalence (i) $\Leftrightarrow$ (v) of [LLN, Theorem 3.2].

The following is a direct application of Corollary 4.9 to the A-approximation property with conjugate operators.

Corollary 4.10. *Let X be a Banach space and let $A \subset \mathcal{L}(X)$ be a convex set. The following statements are equivalent.*

(a) *The space X^* has the A-AP with conjugate operators.*

(b) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$ there is a sequence $(S_n) \subset A$ such that $S_n^* T^* \to T^*$ in the weak operator topology on $\mathcal{L}(Z^*, X^*)$.*

(c) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$ there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|TS_\alpha\| < \infty$ and $S_\alpha^* T^* \to T^*$ in the weak operator topology on $\mathcal{L}(Z^*, X^*)$.*

Remark 4.9. In the special case when $A = \mathcal{K}(X)$ in [LLN, Theorem 3.4, (i) $\Leftrightarrow$ (v)] it is shown that A-AP with conjugate operators of X is equivalent to a condition similar to (c) of Corollary 4.10 where the approximability is in the strong operator topology and "metric". Clearly, Corollaries 4.4 and 4.10 imply the equivalence (i) $\Leftrightarrow$ (v) of [LLN, Theorem 3.4].

Chapter 5

Approximating larger class of operators

In this chapter we present one of the main results of the thesis. It is a description of the approximation property defined by a linear subspace via the approximability of weakly compact operators. For this end, we recall the definition of the Radon–Nikodým property and the description of $(\mathcal{K}(X,Y))^*$ due to Feder and Saphar, which is possible under the influence of the latter property. This chapter is based on [LMO].

5.1 Radon–Nikodým property

The Radon–Nikodým property of Banach spaces has many equivalent definitions and as a concept combines the properties of reflexive spaces and separable duals. See [DU] for a fairly complete review of this property. In the sequel, we are not so much interested in the inner workings of the Radon–Nikodým property itself, but rather in the effects it has on the ability to describe functionals on certain spaces of operators (see Section 5.2 below). Therefore, we follow [Pi1, pp. 335–338] in our brief introduction of the topic and refer to the latter for proofs of any claims in this section.

Let $\Omega = (\Omega, \mu)$ be a measure space. Let X and Y be Banach spaces.

A function $\boldsymbol{x} : \Omega \to X$ is said to be *μ-simple* if

$$\boldsymbol{x}(\omega) = \sum_{i=1}^{n} \mathbb{1}_{\Omega_i}(\omega) x_i,$$

where $x_1, \ldots, x_n \in X$ and $\mathbb{1}_{\Omega_1}, \ldots, \mathbb{1}_{\Omega_n}$ are indicator functions of μ-measurable subsets $\Omega_1, \ldots, \Omega_n$ of Ω. In this case, the *μ-integral* of $\boldsymbol{x}$ is defined as

$$\int_\Omega \boldsymbol{x}(\omega) d\mu(\omega) := \sum_{i=1}^{n} \mu_i(\Omega_i) x_i \in X.$$

A function $\boldsymbol{x} : \Omega \to X$ is called *μ-measurable* if there exists a sequence of μ-simple functions $\boldsymbol{x}_n : \Omega \to X$ such that $\boldsymbol{x}(\omega) = \lim \boldsymbol{x}_n(\omega)$ almost everywhere. Observe that then the function $\|\cdot\| \circ \boldsymbol{x} : \Omega \to \mathbb{R}$ is also μ-measurable.

A μ-measurable function $\boldsymbol{x} : \Omega \to X$ is said to be *μ-integrable* if there exists a sequence of μ-simple functions $\boldsymbol{x}_n : \Omega \to X$ such that

$$\lim_n \int_\Omega \|\boldsymbol{x}_n(\omega) - \boldsymbol{x}(\omega)\| \, d\mu(\omega) = 0.$$

If that is the case, then we can define the *μ-integral* of $\boldsymbol{x}$ by

$$\int_\Omega \boldsymbol{x}(\omega) d\mu(\omega) := \lim_n \int_\Omega \boldsymbol{x}_n(\omega) d\mu(\omega).$$

The latter definition is correct: if $\boldsymbol{x}$ is μ-integrable, then the limit

$$\lim_n \int_\Omega \boldsymbol{x}_n(\omega) d\mu(\omega)$$

exists and does not depend on the choice of the sequence $(\boldsymbol{x}_n)$.

Recall that $L_1(\Omega, \mu)$ denotes the Banach space of all μ-integrable functions $f : \Omega \to \mathbb{K}$ with the norm

$$\|f\|_1 := \int_\Omega |f(\omega)| \, d\mu(\omega).$$

To be more precise, the elements of $L_1(\Omega, \mu)$ are equivalence classes consisting of μ-integrable functions $f : \Omega \to \mathbb{K}$ which coincide almost everywhere.

Definition 5.1. An operator $T \in \mathcal{L}(X, Y)$ is called a *Radon–Nikodým operator* if for every measure space (Ω, μ) and for every operator $S \in \mathcal{L}(L_1(\Omega, \mu), X)$ there exists a μ-measurable function $\boldsymbol{y} : \Omega \to Y$ such that

$$TSf = \int_\Omega f(\omega) \boldsymbol{y}(\omega) d\mu(\omega)$$

for all $f \in L_1(\Omega, \mu)$.

The class $\mathcal{RN}$ of all Radon–Nikodým operators between Banach spaces is an operator ideal.

Definition 5.2. A Banach space X is said to have the *Radon–Nikodým property* (RNP) if X belongs to the space ideal $\mathsf{RN} := \text{Space}(\mathcal{RN})$.

Examples of spaces having the RNP include all separable dual spaces and all reflexive spaces. Observe that the latter and $\mathcal{W} = \text{Op}(\mathsf{W})$ also imply $\mathcal{W} \subset \mathcal{RN}$.

Let us note that $\mathsf{RN} \neq \mathsf{W}$ and $\mathsf{RN} \neq \mathsf{L}$ because, for instance, ℓ_1 is a non-reflexive separable dual space and c_0 does not have the RNP.

5.2 Description of $(\mathcal{K}(X,Y))^*$

The following theorem due to Feder and Saphar [FS] tells us that the sufficient impact of the Radon–Nikodým property enables one to interpret the bounded linear functionals on $\mathcal{K}(X,Y)$ as elements of the projective tensor product $Y^*\hat{\otimes}X^{**}$.

Lemma 5.3 (Feder and Saphar)**.** *Let X and Y be Banach spaces such that X^{**} or Y^* has the RNP. Let V denote the linear operator from $Y^*\hat{\otimes}X^{**}$ to $(\mathcal{K}(X,Y))^*$ defined by*

$$(Vu)(T) = \text{trace}(T^{**}u)$$

for all $u \in Y^*\hat{\otimes}X^{**}$ and for all $T \in \mathcal{K}(X,Y)$. Then V is surjective. Moreover, for all $\varphi \in (\mathcal{K}(X,Y))^*$, there exists $u \in Y^*\hat{\otimes}X^{**}$ such that $\varphi = Vu$ and $\|\varphi\| = \|u\|_\pi$.

5.3 Main result

So far, we have considered criteria for approximation properties via approximability of compact operators. Actually, Grothendieck showed that the AP is closely related to the approximability of weakly compact operators in the strong operator topology. Namely, he proved (see [G, Chapter I, p. 141]) that if the dual space X^* of X has the approximation property, then for every Banach space Y, the closed unit ball $B_{\mathcal{F}(Y,X)}$ of $\mathcal{F}(Y,X)$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology. Grothendieck also claimed a stronger result (see [G, Chapter I, p. 184]) that the latter condition would be implied by the approximation property of X. But his proof only goes through for the particular case, when Y is complemented in Y^{**} by a norm one projection. (This proof was thoroughly analyzed by Reinov [R1]; for a discussion of related questions see [R3].)

The Grothendieck's result was strengthened by Lima, Nygaard, and Oja [LNO] who proved that X has the AP if and only if $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology. This may be considered as a "metric" characterization of the AP. Recently also the CAP was described in terms of the approximability of weakly compact operators in the strong operator topology by Lima, Lima, and Nygaard [LLN] using similar "metric" conditions.

With Theorems 5.4 and 5.6 below we aim to provide the analogues of the discussed results in the case of the approximation property defined by a linear subspace of operators.

Theorem 5.4. *Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let $\mathcal{B}$ be an operator ideal containing $\mathcal{K}$ such that*

$$\{ST^\diamond : S \in A,\ T \in \mathcal{B}_{w^*}(X^*, Y)\} \subset \mathcal{K}(Y^*, X)$$

for all Banach spaces Y. The following assertions are equivalent.

(a) *The space X has the A-AP.*

(b) *For every Banach space Y and for every operator $T \in \mathcal{B}_{w^*}(X^*, Y)$, there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|TS_\alpha^*\| \le \|T\|$ and $S_\alpha T^\diamond \to T^\diamond$ in the strong operator topology on $\mathcal{L}(Y^*, X)$.*

Proof. The implication (b) $\Rightarrow$ (a) immediately follows from Theorem 4.6, (c′) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). We present the outline of the proof and then prove the required steps.

We clearly may assume that $\|T\| = 1$. Denote $K = \overline{T^\diamond(B_{Y^*})}$. Then K is a closed absolutely convex subset of B_X. Since K is also weakly compact, the space X_K is reflexive (see Lemma 2.15) and $T^\diamond$ factorizes through X_K as $T^\diamond = J_K t$ so that $\|t\| = \|T^\diamond\| = \|J_K\| = 1$ (see Lemma 2.18).

Consider the linear subspace $\mathcal{Z} := \{SJ_K : S \in A\} \subset \mathcal{L}(X_K, X)$.

Step 1. We show that $\mathcal{Z} \subset \mathcal{K}(X_K, X)$.

Since X_K is reflexive, the space $(\mathcal{K}(X_K, X))^*$ admits the description due to Feder and Saphar (see Lemma 5.3).

According to this description, the trace mapping V from the projective tensor product $X^*\widehat{\otimes}X_K$ to $(\mathcal{K}(X_K, X))^*$, defined by

$$(Vu)(S) = \text{trace }(Su),\ u \in X^*\widehat{\otimes}X_K,\ S \in \mathcal{K}(X_K, X),$$

is surjective and, moreover, for all $g \in (\mathcal{K}(X_K, X))^*$, there exists $u_g \in X^*\widehat{\otimes}X_K$ such that $g = Vu_g$ and $\|g\| = \|u_g\|_\pi$.

Let $g \in \mathcal{Z}^*$. By passing to a norm-preserving extension, we may assume that $g \in (\mathcal{K}(X_K, X))^*$. Define

$$(\Phi g)(\lambda J_K) = \lambda(Vu_g)(J_K),\ \lambda \in \mathbb{K}.$$

Step 2. Using the A-AP of X, we show that $\Phi : \mathcal{Z}^* \to (\text{span}\{J_K\})^*$ is a correctly defined linear mapping with $\|\Phi\| \le 1$.

Since $\Phi^*(J_K) \in B_{\mathcal{Z}^{**}}$, the Goldstine theorem implies that there is a net $(S_\alpha) \subset A$ such that $(S_\alpha J_K) \subset B_{\mathcal{Z}}$ and $S_\alpha J_K \to \Phi^*(J_K)$ in the weak* topology of $\mathcal{Z}^{**}$.

Step 3. We show (b) using the net (S_α).

Proof of Step 1. Let $S \in A$. We show that $SJ_K \in \mathcal{K}(X_K, X)$. By assumption, $ST^\diamond \in \mathcal{K}(Y^*, X)$. We know (see Lemma 2.15) that

$$J_K(B_{X_K}) \subset B_n = a^{n/2}\overline{T^\diamond(B_{Y^*})} + a^{-n/2}B_X$$

for all $n \in \mathbb{N}$. Hence

$$(SJ_K)(B_{X_K}) \subset a^{n/2}\overline{(ST^\diamond)(B_{Y^*})} + a^{-n/2}\|S\|B_X$$

for all $n \in \mathbb{N}$. This implies that $(SJ_K)(B_{X_K})$ has, for all $\varepsilon > 0$, a compact ε-net, and therefore it is relatively compact in X. Hence $SJ_K \in \mathcal{K}(X_K, X)$ as needed. □

Proof of Step 2. By (a), there is a net $(S_\beta) \subset A$ such that $S_\beta \to I_X$ uniformly on compact subsets of X.

Fix $g \in \mathcal{Z}^*$. We may assume that

$$u_g = \sum_{n=1}^{\infty} x_n^* \otimes z_n,$$

where $x_n^* \in X^*$ and $z_n \in X_K$ satisfy $\sum_{n=1}^{\infty} \|x_n^*\| = 1$ and $z_n \to 0$. Let us show that

$$(\Phi g)(J_K) = \lim_{\beta} g(S_\beta J_K).$$

This can be done similarly to [LNO, proof of Theorem 1.2]. Indeed,

$$\begin{aligned} |(Vu_g)(J_K) - g(S_\beta J_K)| = |(Vu_g)(J_K - S_\beta J_K)| = \Big|\sum_{n=1}^{\infty} x_n^*((I_X - S_\beta)J_K z_n)\Big| \\ \le \sup_n \|(I_X - S_\beta)(J_K z_n)\| \to_\beta 0 \end{aligned}$$

because $\{0, J_K z_1, J_K z_2, \ldots\}$ is a compact subset of X. This means that the value of Φg does not depend on the choice of u_g. Hence, Φ is correctly defined and linear. Since

$$|(\Phi g)(J_K)| = |(Vu_g)(J_K)| \le \|u_g\|_\pi = \|g\|,$$

we also have $\|\Phi\| \le 1$. □

Proof of Step 3. For all $x^* \in X^*$ and $z \in X_K$, consider the functional $x^* \otimes z \in (\mathcal{K}(X_K, X))^*$. Let $g = (x^* \otimes z)|_{\mathcal{Z}} \in \mathcal{Z}^*$. Since $X^* \otimes X = \mathcal{F}(X) \subset \mathcal{A}(X)$, we clearly have $J_K^*(X^*) \otimes X \subset \mathcal{Z}$. But $X_K^* = \overline{J_K^*(X^*)}$ (see Lemma 2.15). Hence $X_K^* \otimes X \subset \overline{\mathcal{Z}}$ in $\mathcal{K}(X_K, X)$. This implies that

$$\|g\| = \|(x^* \otimes z)|_{\overline{\mathcal{Z}}}\| \ge \|(x^* \otimes z)|_{X_K^* \otimes X}\| = \|x^*\|\|z\| = \|x^* \otimes z\|.$$

Consequently, $x^* \otimes z$ is a norm-preserving extension of g, and we may take $u_g = x^* \otimes z \in X^* \widehat{\otimes} X_K$. But then, since $S_\alpha J_K \to \Phi^*(J_K)$ in the weak* topology of Z^{**},

$$\begin{aligned} x^*(S_\alpha J_K z) &= g(S_\alpha J_K) \to_\alpha (\Phi^*(J_K))(g) \\ &= (\Phi g)(J_K) = (Vu_g)(J_K) = x^*(J_K z). \end{aligned}$$

This means that $(S_\alpha J_K)$ converges to J_K in the weak operator topology on $\mathcal{L}(X_K, X)$. By passing to convex combinations, we may assume that $(S_\alpha J_K)$ converges to J_K in the strong operator topology on $\mathcal{L}(X_K, X)$. Recalling that $T^\diamond = J_K t$, this implies that $(S_\alpha T^\diamond)$ converges to $T^\diamond$ in the strong operator topology on $\mathcal{L}(Y^*, X)$. Moreover, since $TS_\alpha^* = (S_\alpha T^\diamond)^\diamond$, we also have $\sup_\alpha \|TS_\alpha^*\| = \sup_\alpha \|S_\alpha J_K t\| \le \|t\| = 1$, because $(S_\alpha J_K) \subset B_Z$. □

Since $\mathcal{B}_{w^*}(X^*, Y) \subset \mathcal{W}(X^*, Y)$, Theorem 5.4 essentially concerns those operator ideals $\mathcal{B}$ which are contained in $\mathcal{W}$. (However, there are cases when

$\mathcal{B}(X^*, Y) \subset \mathcal{W}(X^*, Y)$ for any Y without assuming that $\mathcal{B} \subset \mathcal{W}$. For instance, this is the case when $A = \mathcal{V}(X)$ and X^* contains no copy of ℓ_1, because then $\mathcal{V}(X^*, Y) = \mathcal{K}(X^*, Y)$.) Keeping this in mind, let us observe that the hypothesis of Theorem 5.4 could be reformulated as follows.

Proposition 5.5. *Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$. Let $\mathcal{B}$ be an operator ideal. If*

$$\{ST : S \in A,\ T \in \mathcal{B}^{\text{dual}}(Y, X)\} \subset \mathcal{K}(Y, X)$$

for all Banach spaces Y, then

$$\{ST^{\diamond} : S \in A,\ T \in \mathcal{B}_{w^*}(X^*, Y)\} \subset \mathcal{K}(Y^*, X)$$

for all Banach spaces Y. The converse holds whenever $\mathcal{B} \subset \mathcal{W}$.

Proof. Let $S \in A$ and $T \in \mathcal{B}_{w^*}(X^*, Y)$. Then $(T^{\diamond})^* = j_Y T \in \mathcal{B}(X^*, Y^{**})$, so that $T^{\diamond} \in \mathcal{B}^{\text{dual}}(Y^*, X)$. Hence $ST^{\diamond} \in \mathcal{K}(Y^*, X)$.

For the converse, assume that $\mathcal{B} \subset \mathcal{W}$. Let $S \in A$ and $T \in \mathcal{B}^{\text{dual}}(Y, X)$. Then $T^* \in \mathcal{B}_{w^*}(X^*, Y^*)$ (see Section 2.2). Hence, by assumption, $S(T^*)^{\diamond} \in \mathcal{K}(Y^{**}, X)$. But then $ST = S(T^*)^{\diamond} j_Y \in \mathcal{K}(Y, X)$ as needed. □

Proposition 5.5 allows us to obtain the following version of Theorem 5.4.

Theorem 5.6. *Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let $\mathcal{B}$ be an operator ideal such that $\mathcal{K} \subset \mathcal{B} \subset \mathcal{W}$ and*

$$\{ST : S \in A,\ T \in \mathcal{B}^{\text{dual}}(Y, X)\} \subset \mathcal{K}(Y, X)$$

for all Banach spaces Y. The following assertions are equivalent.

(a) *The space X has the A-AP.*

(b) *For every Banach space Y and for every operator $T \in \mathcal{B}^{\text{dual}}(Y, X)$, there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|S_\alpha T\| \le \|T\|$ and $T^* S_\alpha^* \to T^*$ in the strong operator topology on $\mathcal{L}(X^*, Y^*)$.*

Proof. Notice that, by Proposition 5.5, the hypothesis of Theorem 5.4 is satisfied.

The implication (b) ⇒ (a) immediately follows from Theorem 4.6, (b′) ⇒ (a).

(a) ⇒ (b). It suffices to show that (b) is implied by condition (b) of Theorem 5.4. Let $T \in \mathcal{B}^{\text{dual}}(Y, X)$. Then $T^* \in \mathcal{B}_{w^*}(X^*, Y^*)$ (see Section 2.2). Hence, there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|T^* S_\alpha^*\| \le \|T^*\|$

and $S_\alpha(T^*)^\diamond \to (T^*)^\diamond$ in the strong operator topology on $\mathcal{L}(Y^{**}, X)$. This implies $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ and $T^* S_\alpha^* \to T^*$ in the weak operator topology on $\mathcal{L}(X^*, Y^*)$. By passing to convex combinations we can acquire the needed net. □

Remark 5.1. In the special case when $A = \mathcal{F}(X)$, the net $(S_\alpha T)$ in condition (b) of Theorem 5.6 can be replaced by a net $(T_\alpha) \subset \mathcal{F}(Y, X)$ (see [LO4, Theorem 3.1]). However, already in the case when $A = \mathcal{K}(X)$, this is no longer possible. In fact, by [LO4, Corollary 2.4], the condition "for every Banach space Y and for every operator $T \in \mathcal{W}(Y, X)$, there exists a net $(T_\alpha) \subset \mathcal{K}(Y, X)$ such that $\sup_\alpha \|T_\alpha\| \leq \|T\|$ and $T_\alpha^* \to T^*$ in the strong operator topology" is equivalent to the condition "$\mathcal{K}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y". But the latter condition may be satisfied even when X does not have the compact approximation property (see [LNO, p. 340]).

Remark 5.2. In the special case when $A = \mathcal{K}(X)$, equivalence (a) ⇔ (b) of Theorem 5.6 has been established in [LLN, Theorem 2.3]. The proof in [LLN] uses a roundabout way that relies on criteria of the compact approximation property in terms of ideals (see [LLN, Theorem 2.2]).

Remark 5.3. In the case when $\mathcal{A}$ and $\mathcal{B}$ are operator ideals such that $\mathcal{A} = \mathcal{B}^{\mathrm{dual}}$, and $A = \mathcal{A}(X)$, condition (b) of Theorem 5.6 represents a weakening of the *outer $\mathcal{A}$-approximation property*. This notion was introduced in [T3] and studied in [T1], [T2], and [T3].

To conclude this section, let us discuss some situations when Theorems 5.4 and 5.6 can be applied. A general case is when $\mathcal{A}$ and $\mathcal{B}$ are operator ideals satisfying

$$\mathcal{A} \circ \mathcal{B}^{\mathrm{dual}} \subset \mathcal{K} \qquad (\circ)$$

and $A = \mathcal{A}(X)$.

Condition (∘) holds always if $\mathcal{A} \subset \mathcal{K}$; in particular, if $\mathcal{A} = \mathcal{F}$ or $\mathcal{A} = \mathcal{K}$.

Let us now also consider the following operator ideals: $\mathcal{AC}-$ absolutely continuous operators, $\mathcal{BS}-$ Banach–Saks operators, $\mathcal{H}-$ Hilbert operators, $\mathcal{J}-$ integral operators, $\mathcal{P}_p-$ absolutely p-summing operators (p-summing in [DJT]) (see [Pi1] or [DJT]; for $\mathcal{AC}$ and $\mathcal{BS}$, see [Ni] and [DSe]).

If we take $\mathcal{B} = \mathcal{W}$ (recall that $\mathcal{W}^{\mathrm{dual}} = \mathcal{W}$), then condition (∘) is satisfied for several important operator ideals $\mathcal{A}$ which are *not* contained in $\mathcal{K}$, for instance, if $\mathcal{A}$ equals $\mathcal{AC}$, $\mathcal{J}$, $\mathcal{P}_p$ with $1 \leq p < \infty$, or $\mathcal{V}$. (Indeed, all of them are contained in $\mathcal{V}$ and $\mathcal{V} \circ \mathcal{W} \subset \mathcal{K}$, which is a well-known and an easy-to-see fact.) Notice that $\mathcal{AC}$ and $\mathcal{V}$ are even larger than $\mathcal{K}$. Let us recall that before, in the literature (as we discussed earlier in this section), the only operator ideals $\mathcal{A}$, for which one had been able to characterize the $\mathcal{A}(X)$-approximation property through a "metric" condition like (b) in Theorems 5.4 and 5.6 (with $\mathcal{B}$ larger than $\mathcal{K}$), were $\mathcal{F}$ and $\mathcal{K}$.

Moreover, there are some other interesting pairs of operator ideals $\mathcal{A}$ and $\mathcal{B}$ that satisfy ($\circ$), and Theorems 5.4 and 5.6 apply. For instance, take $\mathcal{B} = \mathcal{J}$. Then $\mathcal{J}^{\mathrm{dual}} = \mathcal{J}$ and ($\circ$) is satisfied for any $\mathcal{A} \subset (\mathcal{RN}^{\mathrm{dual}})^{\mathrm{dual}}$ (in fact, if $T \in (\mathcal{RN}^{\mathrm{dual}})^{\mathrm{dual}} \circ \mathcal{J}$, then T^* is a nuclear operator). Here, important cases are $(\mathcal{RN}^{\mathrm{dual}})^{\mathrm{dual}}$, $\mathcal{W}$, and, of course, any operator ideal contained in $\mathcal{W}$, like $\mathcal{AC}$, $\mathcal{BS}$, $\mathcal{H}$, $\mathcal{J}$, $\mathcal{P}_p$, etc.

Chapter 6

The strong AP and the weak bounded AP

In this chapter we look at the strong approximation property and the weak bounded approximation property. We develop a unified approach to the treatment of their convex versions and observe the impact of the RNP (originally discovered by Oja) on the interplay between these notions and the convex (bounded) approximation properties. This chapter is based on [L].

6.1 Unified approach

Let X be a Banach space, let $A \subset \mathcal{L}(X)$ be a convex set containing 0, and let $\lambda \in [1, \infty)$.

It is apparent that our method of proof does not allow to replace the convergence $S_\alpha^* T^* \to T^*$ with $TS_\alpha \to T$ in condition (b′) of Corollary 4.10. Oja showed in [O3, see Theorem 2.1 and Propositions 4.5 and 4.6] that, in the case of the classical AP, it cannot be done, and the new condition actually yields a property, which is different from both the AP of X and the AP of X^*.

Definition 6.1 (see [O3]). A Banach space X is said to have the *strong approximation property* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, there exists a bounded net $(T_\alpha) \subset \mathcal{F}(X, Z)$ such that $T_\alpha x \to Tx$ for all $x \in X$.

We are interested in the following characterization of the strong approximation property.

Proposition (see [O3, Proposition 4.6])**.** *A Banach space X has the strong approximation property if and only if for every Banach space Y and for every operator $T \in \mathcal{K}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $\sup_\alpha \|TS_\alpha\| < \infty$ and $TS_\alpha x \to Tx$ for all $x \in X$.*

This description allows us to extend the notion to the convex approximation properties (for which $A = \mathcal{F}(X)$ below).

Definition 6.2. We say that X has the *strong A-approximation property* (strong A-AP) if for every Banach space Y and for every operator $T \in \mathcal{K}(X,Y)$ there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|TS_\alpha\| < \infty$ and $TS_\alpha x \to Tx$ for all $x \in X$.

In the same paper [O3], it was noticed that the strong AP shares similar characterizations with the *weak bounded AP*, which was introduced and studied in [LO6]. The following definition is based on [LO6, Theorem 2.4].

Definition 6.3. A Banach space X has the *weak λ-bounded AP* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X,Z)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $\sup_\alpha \|TS_\alpha\| \le \lambda \|T\|$ and $S_\alpha \to I_X$ in the topology of compact convergence.

The generalization of the latter concept to the case of convex approximation properties, the *convex weak bounded AP*, is of critical importance for us in the sequel.

Definition 6.4. We say that X has the *weak λ-bounded A-approximation property* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X,Z)$ there is a net $(S_\alpha) \subset A$ such that $\sup_\alpha \|TS_\alpha\| \le \lambda \|T\|$ and $S_\alpha \to I_X$ in the topology of compact convergence.

We say that X has the *weak metric A-AP* if it has the weak 1-bounded A-AP, and that X has the *weak bounded A-AP* if it has the weak μ-bounded A-AP for some $\mu \ge 1$.

Observe that the weak λ-bounded A-AP of X means that for every separable reflexive Banach space Z and for every $T \in \mathcal{K}(X,Z)$ the space X has the A_T^λ-AP, where

$$A_T^\lambda := \{S \in A : \|TS\| \le \lambda \|T\|\}.$$

We would like to investigate both notions in a unified fashion. For this, it is convenient to extend Definition 6.4 as follows.

Definition 6.5. Let $\Lambda \subset [1,\infty)$. We say that X has the *weak Λ-bounded A-approximation property* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X,Z)$ there is $\lambda \in \Lambda$ such that the space X has the A_T^λ-AP.

The following theorem explains why Definition 6.5 extends both Definition 6.2 and Definition 6.4.

Theorem 6.6. *Let X be a Banach space, let $A \subset \mathcal{L}(X)$ be a convex set containing 0, and let $\Lambda \subset [1,\infty)$. The following statements are equivalent.*

(a) *For every Banach space Y and for every $T \in \mathcal{K}(X,Y)$ there are $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A_T^\lambda$ such that $S_\alpha \to I_X$ in $\tau_c(X,Y)$.*

(a′) *X has the weak Λ-bounded A-AP.*

(b) *For every Banach space Y and for every operator $T \in \mathcal{K}(X,Y)$ there are $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A_T^\lambda$ such that $TS_\alpha \to T$ in $\tau_s(X,Y)$.*

(b′) *For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X,Z)$ there are $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A_T^\lambda$ such that $TS_\alpha \to T$ in $\tau_w(X,Z)$.*

The proof of Theorem 6.6 is almost identical to the proof of Theorem 6.9 below. We will provide a combined proof after the statement of Theorem 6.9.

It is now clear that the strong A-AP coincides with the weak Λ-bounded A-AP for any unbounded $\Lambda \subset [1,\infty)$, while the weak λ-bounded A-AP is exactly the weak Λ-bounded A-AP, when $\max \Lambda = \lambda$. Let us stress that the reason for Definition 6.5 is purely technical, and the term "weak Λ-bounded A-AP" may be regarded simply as a substitute for both the weak λ-bounded A-AP and the strong A-AP.

Corollary 6.7. *Let X be a Banach space and let $A \subset \mathcal{L}(X)$ be a convex set containing 0. Consider the following conditions:*

(a) *X^* has the A-AP with conjugate operators,*

(b) *X has the bounded A-AP,*

(c_1) *X has the weak metric A-AP,*

(c) *X has the weak bounded A-AP,*

(d) *X has the strong A-AP,*

(e) *X has the A-AP.*

Then (a) $\Rightarrow$ (c$_1$) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) *and* (b) $\Rightarrow$ (c)*. If $A = \mathcal{F}(X)$, then the implications* (a) $\Rightarrow$ (c$_1$) *and* (d) $\Rightarrow$ (e) *are strict.*

Proof. The chain (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) and the implication (c$_1$) $\Rightarrow$ (c) are clear from the definitions and Theorem 6.6. The implication (a) $\Rightarrow$ (c$_1$) follows from (a) $\Rightarrow$ (b) of Corollary 4.10 and Definition 6.4. For (e) $\not\Rightarrow$ (d), see [O3, Theorem 2.1, (a) $\not\Rightarrow$ (c*), and Proposition 4.6]. For (c$_1$) $\not\Rightarrow$ (a), observe that the weak metric AP is implied by the MAP but there is a Banach space having a monotone basis (hence, the MAP) such that its dual space fails the AP (hence, also the AP with conjugate operators). □

We do not know whether the implications (c) $\Rightarrow$ (d), (c$_1$) $\Rightarrow$ (c), or (b) $\Rightarrow$ (c) can be reversed. Chapter 7 provides some hints on why it is hard to check some of these conditions.

For the further investigation of the topic, primarily in the case when $A \subset \mathcal{K}(X)$, we introduce the following even more general definition.

Definition 6.8. Let $\mathcal{A}$ be an operator ideal, and let A be a space ideal (see Section 2.1). We say that X has the *Λ-bounded A-AP for the pair* $(\mathcal{A}, \mathsf{A})$ if for every space $Y \in \mathsf{A}$ and for every operator $T \in \mathcal{A}(X, Y)$ there is $\lambda \in \Lambda$ such that X has the A_T^λ-AP.

By the term the *Λ-bounded A-AP for $\mathcal{A}$* we mean “the Λ-bounded A-AP for $(\mathcal{A}, \mathsf{L})$”.

Also, we say that X has the *Λ-bounded A-AP* if it has the Λ-bounded A-AP for some $\lambda \in \Lambda$.

If $\Lambda = \{\lambda\}$, we replace Λ with λ in the above notions.

The definition above is modelled after the definition of “λ-bounded AP for $\mathcal{B}$” in [LLO1] (see also [O5]), where $\mathcal{B}$ is a Banach operator ideal. Our definition is not consistent with the original Lima–Lima–Oja definition where the approximating sets are given using the operator ideal norm. For simplicity, we instead use the usual operator norm. However, for classical Banach operator ideals these two norms coincide. Therefore, in such a case, the definitions are still consistent.

Let us point out some observations from Definition 6.8.

- The weak Λ-bounded A-AP is the Λ-bounded A-AP for $(\mathcal{K}, \mathsf{X} \cap \mathsf{W})$ (or just for $\mathcal{K}$, by Theorem 6.6).
- The Λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$ implies the A-AP. Indeed, $0 \in \mathcal{A}(X, Y)$ for every space $Y \in \mathsf{A}$, so for some $\lambda \in \Lambda$ the space X has the A_0^λ-AP. But $A_0^\lambda = A$.
- The Λ-bounded A-AP coincides with the Λ-bounded A-AP for $\mathcal{L}$. Indeed, take $T = I_X \in \mathcal{L}(X)$ in Definition 6.8, then for any $\lambda \geq 1$ one has
$$A_T^\lambda = \lambda B_{\mathcal{L}(X)} \cap A.$$
- Let $\mathcal{B}$ be an operator ideal such that $\mathcal{B} \subset \mathcal{A}$ and let B be a space ideal such that $\mathsf{B} \subset \mathsf{A}$. Then the Λ-bounded A-AP for $(\mathcal{A}, \mathsf{A})$ implies the Λ-bounded A-AP for $(\mathcal{B}, \mathsf{B})$.

The weak Λ-bounded A-AP for $\mathcal{W}$ also admits a version of Theorem 6.6.

Theorem 6.9. *Let X be a Banach space, let $A \subset \mathcal{L}(X)$ be a convex set containing* 0, *and let $\Lambda \subset [1, \infty)$. The following statements are equivalent.*

(a) *X has the Λ-bounded A-AP for $\mathcal{W}$.*

(a′) *X has the Λ-bounded A-AP for $(\mathcal{W}, \mathsf{W})$.*

(b) *For every Banach space Y and for every operator $T \in \mathcal{W}(X, Y)$ there is $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A_T^\lambda$ such that $TS_\alpha \to T$ in $\tau_s(X, Y)$.*

(b′) *For every reflexive space Z and for every operator $T \in \mathcal{W}(X, Z)$ there is $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A_T^\lambda$ such that $TS_\alpha \to T$ in $\tau_w(X, Z)$.*

Proof of Theorems 6.6 and 6.9. Below, we prove Theorem 6.9. Modifications for the proof of Theorem 6.6 will be enclosed in brackets.

Implications (a) $\Rightarrow$ (a′) and (b) $\Rightarrow$ (b′), as well as (a) $\Rightarrow$ (b) and (a′) $\Rightarrow$ (b′), are obvious.

(b′) $\Rightarrow$ (a). Let Y be a Banach space and let $T \in \mathcal{W}(X, Y)$ (respectively, $T \in \mathcal{K}(X, Y)$). We shall use Lemma 3.16 to show that X has the A_T^λ-AP for some $\lambda \in \Lambda$.

Take a tensor $\sum_{n=1}^\infty x_n^* \otimes x_n \in X^* \hat{\otimes} X$ such that $\sum_{n=1}^\infty \|x_n\| \leq 1$, $x_n^* \to 0$, and $\|x_n^*\| \leq 1$ for all $n \in \mathbb{N}$. Let

$$K := \overline{\text{absconv}} \left(\{x_1^*, x_2^*, \dots\} \cup T^*(B_{Y^*}) \right) \subset B_{X^*}.$$

Since K is weakly compact (respectively, compact), by Lemma 2.17, there is a space $Z := (X^*)_K$ belonging to the space ideal W (respectively, $\mathsf{X}\cap\mathsf{W}$) and a weakly compact (respectively, compact) operator $J := J_K : Z \to X^*$ such that $K \subset J(B_Z)$ and $\|J\| \le 1$. For every $n \in \mathbb{N}$ there is $z_n \in B_Z$ such that $J_K z_n = x_n^*$ Moreover, for the astriction $t \in \mathcal{L}(Y^*, Z)$ of T^* to Z, we have $T^* = Jt$ and $\|t\| \le \|T\|$.

Since $J^* j_X \in \mathcal{K}(X, Z^*)$ and $Z^* \in \mathsf{W}$ (respectively, $Z^* \in \mathsf{X}\cap\mathsf{W}$), condition (b$'$) gives us $\lambda \in \Lambda$ and a net $(S_\alpha) \subset A^\lambda_{(J^* j_X)}$ such that $J^* j_X S_\alpha \to J^* j_X$ in $\tau_w(X, Z^*)$. Since $A^\lambda_{(J^* j_X)}$ is convex, by Proposition 2.12, we can assume that the latter convergence is pointwise. Fix α. Note that

$$(J^* j_X S_\alpha)^* j_Z = S_\alpha^* j_X^* J^{**} j_Z = S_\alpha^* j_X^* j_{X^*} J = S_\alpha^* J,$$

so that the inclusion $J^* j_X S_\alpha \in A^\lambda_{(J^* j_X)}$ implies

$$\|S_\alpha^* J\| = \|(J^* j_X S_\alpha)^* j_Z\| = \|J^* j_X S_\alpha\| \le \lambda$$

because the reflexivity of Z means that j_Z is an isometry. Therefore,

$$\|T S_\alpha\| = \|S_\alpha^* T^*\| = \|S_\alpha^* J t\| \le \lambda \|T\|.$$

That is, $(S_\alpha) \subset A^\lambda_T$. For every $n \in \mathbb{N}$, we also have

$$x_n^*(S_\alpha x_n - x_n) = J z_n(S_\alpha x_n - x_n) = (J^* j_X(S_\alpha x_n - x_n))(z_n) \to 0$$

and

$$|x_n^*(S_\alpha x_n - x_n)| = |(J^* j_X(S_\alpha x_n - x_n))(z_n)| \le \|x_n\| (\lambda + 1),$$

so that

$$\sup_\alpha \left| \sum_{n>N} x_n^*(S_\alpha x_n - x_n) \right| \le (\lambda + 1) \sum_{n>N} \|x_n\| \underset{N\to\infty}{\longrightarrow} 0,$$

as needed. □

Let us end this section with the following observation that "the boundedness" of convex approximation properties is "defined on compact level". This is inspired by the well-known fact that the AP and the bounded CAP imply the bounded AP (see, e.g., [C, Proposition 8.2]).

Proposition 6.10. *Let X be a Banach space, let A be a convex subset of $\mathcal{L}(X)$ containing 0, let B be a convex subset of $\mathcal{K}(X)$ containing 0, let $\mathcal{A}$ be an operator ideal, let A be a space ideal, and let $\Lambda \subset [1, \infty)$. If X has both the A-AP and the Λ-bounded B-AP for $(\mathcal{A}, \mathsf{A})$, then X also has the Λ-bounded $A \circ B$-AP for $(\mathcal{A}, \mathsf{A})$.*

Proof. Let $Y \in \mathsf{A}$ and let $T \in \mathcal{A}(X,Y)$. We need to show that X has the $(A \circ B)_T^\lambda$-AP for some $\lambda \in \Lambda$. By assumption, there is $\lambda \in \Lambda$ such that X has the B_T^λ-AP. Take $R \in B_T^\lambda$. Since X has the A-AP, Theorem 4.1 gives us a sequence $(S_n) \subset A$ such that $S_n R \to R$. Then $TS_n R \to TR$. Since A is convex and contains 0, we can assume that $\|TS_n R\| \le \|TR\|$ for all $n \in \mathbb{N}$. But $\|TR\| \le \lambda \|T\|$ because $R \in B_T^\lambda$. Therefore, $(S_n R) \subset (A \circ B)_T^\lambda$ and $S_n R \to R$, so that $B_T^\lambda \subset \overline{(A \circ B)_T^\lambda} \subset \overline{(A \circ B)_T^\lambda}^{\tau_c}$. Now we have

$$I_X \in \overline{B_T^\lambda}^{\tau_c} \subset \overline{(A \circ B)_T^\lambda}^{\tau_c},$$

as desired. □

In particular, Proposition 6.10 states that if X has both the AP and the strong (respectively, weak λ-bounded or λ-bounded) $\mathcal{K}(X)$-AP, then X actually has the strong (respectively, weak λ-bounded or λ-bounded) AP.

6.2 The weak bounded AP and the RNP

Let us now consider the case, when $\Lambda = \{\lambda\}$. The prototype of the following result is [O2, Theorem 2].

Lemma 6.11. *Let X and Y be Banach spaces, let $A \subset \mathcal{L}(X)$ be convex and contain 0, and let $\lambda \ge 1$. Let X have the weak λ-bounded A-AP. Let $T \in \mathcal{L}(X,Y)$ be such that $\{TS : S \in A\} \subset \mathcal{K}(X,Y)$. If X^{**} or Y^* has the RNP, then X has the A_T^λ-AP.*

Proof. We may assume that $\|T\| = 1$. We show that X has the $A_T^{\lambda+\delta}$-AP for every $\delta > 0$. Since A is convex and contains 0, the claim would then follow as in Proposition 3.17.

Fix $\delta > 0$, a compact set $C \subset X$, and $\varepsilon > 0$. Define

$$\mathcal{C} = \{TS : S \in A,\ \|Sa - a\| < \varepsilon\ \forall a \in C\} \subset \mathcal{K}(X,Y).$$

We need to show that $\mathcal{C} \cap B_{\mathcal{K}(X,Y)}(0, \lambda + \delta)$ is not empty. Observe that $\mathcal{C}$ is convex and not empty because X has the A-AP, while $B_{\mathcal{K}(X,Y)}(0, \lambda + \delta)$ is convex with non-empty interior. Therefore by the Hahn–Banach separation theorem (see, e.g., [M, p. 179]), it remains to show that

$$\inf_{TS \in \mathcal{C}} \mathcal{R}e\, \varphi(TS) < \sup\{\mathcal{R}e\, \varphi(R) : R \in \mathcal{K}(X,Y),\ \|R\| \le \lambda + \delta\} = \lambda + \delta$$

for every $\varphi \in (\mathcal{K}(X,Y))^*$ with $\|\varphi\| = 1$.

Let $\varphi \in \mathcal{K}(X,Y)^*$ with $\|\varphi\| = 1$. Since X^{**} or Y^* has the RNP, from the theorem of Feder and Saphar (see Lemma 5.3), there is $u \in Y^* \hat{\otimes} X^{**}$ such that $\|u\|_\pi = 1$ and

$$\varphi(R) = \operatorname{trace}(R^{**}u)$$

for all $R \in \mathcal{K}(X,Y)$. Pick a representation

$$u = \sum_{n=1}^{\infty} y_n^* \otimes x_n^{**} \in Y^* \hat{\otimes} X^{**}$$

such that $1 \geq \|y_n^*\| \to 0$ and $\sum_{n=1}^{\infty} \|x_n^{**}\| < 1 + \frac{\delta}{\lambda}$.

Let $K := \overline{\{T^*y_1^*, T^*y_2^*, \dots\}} \subset B_{X^*}$. Since K is compact, by Lemma 2.17, we can construct a separable reflexive Banach space Z, sitting inside X^*, such that the embedding operator $J \in \mathcal{K}(Z, X^*)$ has norm 1, and $K \subset J(B_Z)$. For all $n \in \mathbb{N}$ let $z_n \in B_Z$ be such that $Jz_n = T^*y_n^*$. We have $J^*j_X \in \mathcal{K}(X, Z^*)$. By assumption we can find $S \in A$ such that $\|J^*j_X S\| \leq \lambda$ and $\|Sa - a\| < \varepsilon$ for all $a \in C$. Since Z^* is reflexive, we get

$$\|J^*S^{**}\| = \|J^{***}j_X^{**}S^{**}\| \leq \lambda$$

and

$$\begin{aligned} |\varphi(TS)| &= \left|\sum_{n=1}^{\infty}(S^{**}x_n^{**})(T^*y_n^*)\right| = \left|\sum_{n=1}^{\infty}(J^*S^{**}x_n^{**})(z_n)\right| \\ &\leq \lambda \sum_{n=1}^{\infty} \|x_n^{**}\| < \lambda(1 + \frac{\delta}{\lambda}) = \lambda + \delta, \end{aligned}$$

as required. □

As an immediate consequence of Lemma 6.11 we obtain the next theorem. Recall that $(\mathcal{A}^{-1} \circ \mathcal{K})(X)$ consists of all operators $S \in \mathcal{L}(X)$ such that for every Banach space Y and for every $T \in \mathcal{A}(X,Y)$ one has $TS \in \mathcal{K}(X,Y)$.

Theorem 6.12. *Let X be a Banach space, let $\mathcal{A}$ be an operator ideal, let A be a convex subset of $(\mathcal{A}^{-1} \circ \mathcal{K})(X)$ containing 0, and let $\lambda \geq 1$. Let X have the weak λ-bounded A-AP. Then:*

(i) *X has the λ-bounded A-AP for $(\mathcal{A}, \mathsf{RN}^{\mathrm{dual}})$;*

(ii) *if X^{**} has the RNP, then X has the λ-bounded A-AP for $\mathcal{A}$.*

In the case, when $A \subset \mathcal{K}(X)$, Theorem 6.12 allows us to nicely describe the weak λ-bounded A-AP.

Corollary 6.13. *Let X be a Banach space, let A be a convex subset of $\mathcal{K}(X)$ containing 0, and let $\lambda \geq 1$. The following properties are equivalent for X:*

(a) *weak λ-bounded A-AP,*

(b) *λ-bounded A-AP for $\mathcal{K}$.*

(c) *λ-bounded A-AP for $\mathcal{W}$,*

(d) *λ-bounded A-AP for $(\mathcal{L}, \mathsf{RN}^{\mathrm{dual}})$.*

Proof. The implication (a) $\Rightarrow$ (d) follows from (i) of Theorem 6.12, (d) $\Rightarrow$ (c) follows from Theorem 6.9 and the inclusion $\mathsf{W} \subset \mathsf{RN}$, and (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious. □

Remark 6.1. In the case, when $A = \mathcal{F}(X)$, the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) have been established in [LO6, Theorem 2.4]. In the case, when $A = \mathcal{K}(X)$, the equivalence (a) $\Leftrightarrow$ (b) has been proven in [LL2, Theorem 4.1].

The following result is one of the main steps needed for our partial solution of Problem 1.1 (see Theorem 7.10 below). Its prototype in the case when $A = \mathcal{F}(X)$ can be found in [O2, Corollary 1].

Corollary 6.14. *Let X be a Banach space and let A be a convex subset of $\mathcal{K}(X)$ containing 0. If X^* or X^{**} has the RNP, then the weak λ-bounded A-AP and the λ-bounded A-AP are equivalent for X.*

Proof. The case, when X^{**} has the RNP, follows from (ii) of Theorem 6.12. The case, when X^* has the RNP, follows from Lemma 6.11 applied to $T = I_X$. □

Apart from the case, when A consists of compact operators, Theorem 6.12 can be applied, for instance, when $\mathcal{A} = \mathcal{V}$. Operators of the ideal $\mathcal{V}^{-1} \circ \mathcal{K}$ are called *Rosenthal operators.* Observe that $\mathcal{W} \subset \mathcal{V}^{-1} \circ \mathcal{K}$ (see [Pi1, p. 61]).

Chapter 7

Lifting the AP to the dual space

In this chapter we provide an analogue of the famous Johnson's lifting theorem, which permits to lift the metric approximation property to the dual space, as well as an analogue of the Lima–Oja theorem on lifting the weak metric approximation property to the dual space. We apply these results to obtain a partial solution to a convex version of Problem 1.1. This chapter is based on [LisO].

7.1 Johnson's theorem

The results of this chapter are inspired by the famous lifting theorem of Johnson [J2].

Theorem 7.1 (Johnson)**.** *Let X be a Banach space. If X has the metric AP in every equivalent norm, then the dual space X^* has the metric AP.*

7.2 Equivalent dual norms

Let U be a vector space and let $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ be norms on U. These norms are said to be *equivalent* if there is $C > 0$ such that $1/C\,\|u\| \leq |\!|\!|u|\!|\!| \leq C\,\|u\|$ for all $u \in U$. In other words, the norms are equivalent if and only if they induce the same topology on U.

Let X be a Banach space.

Definition 7.2. A norm $|\!|\!|\cdot|\!|\!|$ on X is called an *equivalent norm* if it is equivalent to the natural norm $\|\cdot\|$. An equivalent norm $|\!|\!|\cdot|\!|\!|$ on X^* is called a *dual norm* if it coincides with the norm of the dual space $(X, p)^*$ for some equivalent norm p on X.

Consider an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X^*. The norm $|\!|\!|\cdot|\!|\!|^*$ of the dual space $(X^*, |\!|\!|\cdot|\!|\!|)^*$ is an equivalent norm on X^{**} because the spaces $(X^*, |\!|\!|\cdot|\!|\!|)$ and X^* have the same topologies, and so do their duals. Therefore, also the restriction p of $|\!|\!|\cdot|\!|\!|^*$ to X is an equivalent norm on X. Note that for all $x \in X$ we have

$$p(x) = \sup_{|\!|\!|x^*|\!|\!| \le 1} |x^*(x)| .$$

Clearly, the norm p^* of the dual space $(X, p)^*$ is also an equivalent norm on X^*. Actually, p^* coincides with $|\!|\!|\cdot|\!|\!|$ if and only if the unit ball $B_{(X^*, |\!|\!|\cdot|\!|\!|)}$ is weakly* closed (see, e.g., [Wi]). For completeness, we shall provide a proof of this description (see Proposition 7.4).

In fact, it is a simple application of the *bipolar theorem*. Recall that given a dual pair $\langle U, V\rangle$ and a set $C \subset U$, the *polar* $C^\circ \subset V$ of C is defined as follows:

$$C^\circ = \{v \in V : |\langle u, v\rangle| \le 1 \ \forall u \in C\}.$$

Proposition 7.3 (Bipolar theorem (see, e.g., [AB, Theorem 5.103]))**.** *Let $\langle U, V\rangle$ be a dual pair, and let C be a nonempty subset of U. The bipolar $C^{\circ\circ}$ is the $\sigma(U, V)$-closed absolutely convex hull of C.*

Also, recall that the natural norm of X^* is *weakly* lower semi-continuous* (see, e.g., [AB, Lemma 6.22]), that is,

$$\|x^*\| \le \liminf_{\alpha} \|x^*_\alpha\|$$

for any net $(x^*_\alpha) \subset X^*$ converging to $x^* \in X^*$ in the weak* topology.

Proposition 7.4. *Let $|\!|\!|\cdot|\!|\!|$ be an equivalent norm on X^*. The following statements are equivalent:*

(i) *$|\!|\!|\cdot|\!|\!|$ is a dual norm,*

(ii) *the unit ball $B_{(X^*, |\!|\!|\cdot|\!|\!|)}$ is weakly* closed,*

(iii) *$|\!|\!|\cdot|\!|\!|$ is weakly* lower semi-continuous.*

Proof. We just discussed the implication (i) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let us use the same notation as above. Denote B to be the unit ball $B_{(X^*,|||\cdot|||)}$. Then for the polar B° with respect to the dual pair $\langle X^*, X\rangle$ we have

$$B^\circ = \{x \in X : |x^*(x)| \le 1 \ \forall x^* \in B\} = \{x \in X : p(x) \le 1\}.$$

Similarly,

$$B^{\circ\circ} = \{x^* \in X^* : p^*(x^*) \le 1\} = B_{(X^*,p^*)}.$$

This means that $|||\cdot|||$ coincides with p^* whenever $B = B^{\circ\circ}$. Since B is absolutely convex (because it is a unit ball) and weakly* closed, the latter equality follows from the bipolar theorem. $\square$

7.3 Johnson's norms

In the proof of the lifting theorem (Theorem 7.1 above), Johnson used the following norms on X^* defined for all finite-dimensional subspaces F of X^* and for all positive numbers ε:

$$|||x^*||| = \|x^*\| + \varepsilon \operatorname{dist}(x^*, F), \quad x^* \in X^*, \tag{7.1}$$

where $\operatorname{dist}(x^*, F) = \inf\{\|x^* - f\| : f \in F\}$ denotes the $\|\cdot\|$-distance of x^* to F.

These *Johnson's norms* were then applied by Figiel and Johnson [FJ] to prove that there exists a Banach space which has the AP but fails the BAP (see [FJ, p. 199] or, e.g., [C, p. 290]), and also by Reinov [R2] who showed that a Banach space with the AP may fail the $\mathcal{A}$-BAP not only for $\mathcal{A} = \mathcal{F}$ but also for some other classical operator ideals $\mathcal{A}$ (for instance, when $\mathcal{A} = \mathcal{W}$ or $\mathcal{A} = \mathcal{RN}^{\text{dual}}$).

It is well known (see [J2, p. 308]) that $|||\cdot|||$ is the dual norm to an equivalent norm on X, likewise denoted by $|||\cdot|||$. We call these norms on X also *Johnson's norms.*

For completeness, let us discuss why Johnson's norms $|||\cdot|||$ on X^* are dual norms.

Fix $\varepsilon > 0$, a finite-dimensional subspace $F \subset X$, and let $|||\cdot|||$ be a Johnson's norm defined by ε and F. Note that for $x^* \in X^*$ we actually have

$$\operatorname{dist}(x^*, F) = \inf\{\|x^* + f\| : f \in F\} = \|x^* + F\|,$$

where $x^* + F$ is an element of the quotient space X^*/F. By Proposition 7.4, the duality of a norm is equivalent to its weak* lower semi-continuity. Since linear combinations of functions clearly respect the latter property, it would suffice if both the natural norms of X^* and X^*/F were dual norms. Hence, it remains to notice that X^*/F is isometric to a dual Banach space. For this, let us recall several facts about quotient spaces and annihilators.

Let $Y \subset X$ and $W \subset X^*$ be linear subspaces. The *annihilators* $Y^\perp \subset X^*$ and $W_\perp \subset X$ are defined as

$$Y^\perp = \{x^* \in X^* : x^*(y) = 0 \ \forall y \in Y\}$$

and

$$W_\perp = \{x \in X : w(x) = 0 \ \forall w \in W\}.$$

It is easy to see that $Y^\perp = Y^\circ$ and $W_\perp = W^\circ$, where the polars are taken with respect to the dual pair $\langle X, X^* \rangle$. Hence, the bipolar theorem implies that $(Y^\perp)_\perp = \overline{Y}^w = \overline{Y}$ and $(W_\perp)^\perp = \overline{W}^{w^*}$. Moreover, if $Y \subset X$ is a closed subspace, then $X^*/Y^\perp$ is isometrically isomorphic to Y^* (through the identification $x^* + Y^\perp \mapsto x^*|_Y$) (see, e.g., [Day, p. 30]).

Now we have $(F_\perp)^\perp = \overline{F}^{w^*} = F$ because a finite-dimensional subspace is closed in any linear Hausdorff topology. Therefore $X^*/F = X^*/(F_\perp)^\perp$ is isometrically isomorphic to the dual space $(F_\perp)^*$, as desired.

Let us now point out a property of the norm (7.1) which will be needed in the sequel to prove our main Theorems 7.7 and 7.9.

Proposition 7.5. *Let $X = (X, \|\cdot\|)$ be a Banach space and let $|||\cdot|||$ be a Johnson's norm defined by (7.1). Then the relative $|||\cdot|||$ and weak* topologies on $B_{(X^*,|||\cdot|||)}$ agree at each point $x^* \in S_{(F,|||\cdot|||)}$, meaning that the identity mapping from $(B_{(X^*,|||\cdot|||)}, weak^*)$ to $(B_{(X^*,|||\cdot|||)}, |||\cdot|||)$ is continuous at each point $x^* \in F$ with $|||x^*||| = 1$.*

Proof. Let $x^* \in F$ with $|||x^*||| = 1$. Then also $\|x^*\| = 1$. Let (x^*_α) be a net in X^* such that $|||x^*_\alpha||| \le 1$ for all α and $x^*_\alpha \to x^*$ pointwise on X. By the weak* lower semi-continuity of dual norms,

$$1 = \|x^*\| \le \liminf_\alpha \|x^*_\alpha\| \le \limsup_\alpha \|x^*_\alpha\| \le \limsup_\alpha |||x^*_\alpha||| \le 1$$

and

$$1 = |||x^*||| \le \liminf_\alpha |||x^*_\alpha||| \le \limsup_\alpha |||x^*_\alpha||| \le 1.$$

Hence, $\lim |||x^*_\alpha||| = \lim \|x^*_\alpha\|$ and therefore $\operatorname{dist}(x^*_\alpha, F) \to 0$, i.e., for every α there is $f_\alpha \in F$ so that $\|x^*_\alpha - f_\alpha\| \to 0$. It follows that $f_\alpha \to x^*$

pointwise on X. Hence $\|f_\alpha - x^*\| \to 0$ because $\dim F < \infty$, and therefore also $\|x^*_\alpha - x^*\| \to 0$. This is equivalent to the desired convergence $|||x^*_\alpha - x^*||| \to 0$. □

7.4 Lifting the MAP

For the proof of Theorem 7.7 below, it is convenient to combine the "continuity" of bounded convex approximation properties (see Proposition 3.17) and the equivalence of τ_c and τ_s on bounded sets of operators (see Proposition 2.13) into the following criterion.

Proposition 7.6. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. The space X has the metric A-AP if and only if for every finite subset $G \subset B_X$ and for every $\varepsilon > 0$, there is an operator $S \in A$ with $\|S\| \le 1 + \varepsilon$ such that $\|Sx - x\| \le \varepsilon$ for all $x \in G$.*

The following theorem is the main result of this section.

Theorem 7.7. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. If X has the metric A-AP in every equivalent norm (or just in every Johnson's norm), then X^* has the metric A-AP with conjugate operators.*

Proof. Let $X = (X, \|\cdot\|)$. We use Proposition 7.6 to show that X^* has the metric $\{S^* : S \in A\}$-AP.

Fix a finite set $G \subset B_{X^*}$ and $\varepsilon > 0$. Let $|||\cdot||| := |||\cdot|||_{F,\varepsilon}$ be the Johnson's norm for $F := \operatorname{span} G$ and ε. Since $(X, |||\cdot|||)$ has the metric A-AP, there is a net $(S_\alpha) \subset A$, $|||S_\alpha||| \le 1$, such that $S_\alpha x \to x$ for every $x \in X$. Hence, $S^*_\alpha x^* \to x^*$ weakly* for every $x^* \in X^*$. By Proposition 7.5, $S^*_\alpha x^* \to x^*$ in norm for every $x^* \in S_{(F,|||\cdot|||)}$, and therefore also for every $x^* \in F$. Choose $S \in A$, $|||S||| \le 1$, so that

$$\max_{x^* \in G} \|S^* x^* - x^*\| < \varepsilon.$$

By definition, $\|S\| \le (1 + \varepsilon)\, |||S||| \le 1 + \varepsilon$, as needed. □

Putting $A = \mathcal{F}(X)$ in Theorem 7.7, one obtains Johnson's lifting theorem (see Theorem 7.1).

Reinov in [R2, Theorem 2.1] extended Johnson's theorem from $\mathcal{F}$ to an operator ideal $\mathcal{A}$ under the following restrictions on the pair $(X, \mathcal{A})$:

- X^* has the bounded $\mathcal{A}$-AP, or
- for any operators $T \in \mathcal{A}(X^*)$ and $U \in \mathcal{L}(X^*, X^{***})$ such that $\|UT\| \leq 1$ there exists a net $(S_\alpha) \subset \mathcal{A}(X^*)$ such that $\|S_\alpha\| \leq 1$ and $S_\alpha x^* \to UTx^*$ for all $x^* \in X^*$.

Our Corollary 7.8 below improves his result, showing that no restriction is needed.

Corollary 7.8. *Let X be a Banach space and let $\mathcal{A}$ be an operator ideal. If X has the metric $\mathcal{A}^{\text{dual}}$-AP in every equivalent norm (or just in every Johnson's norm), then the dual space X^* has the metric $\mathcal{A}$-AP.*

Proof. By Theorem 7.7, X^* has the metric $\mathcal{A}^{\text{dual}}(X)$-AP with conjugate operators. This implies that X^* has the metric $\mathcal{A}(X^*)$-AP. □

7.5 Lifting the weak metric AP

The weak metric A-approximation property clearly implies the A-approximation property (which is just the A_0-approximation property). The converse is not true in general (see [LO6] or [O2] for examples in the case when $A = \mathcal{F}(X)$).

The weak metric convex approximation property admits a version of the lifting theorem (cf. Theorem 7.7), and in this case also the converse holds.

Theorem 7.9. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. The following statements are equivalent.*

(a) *The space X has the weak metric A-AP in every equivalent norm (or just in every Johnson's norm).*

(b) *The dual space X^* has the A-AP with conjugate operators.*

For the weak metric approximation property, i.e. when $A = \mathcal{F}(X)$, the result (without the claim about Johnson's norms) was proven in [LO6, Theorem 4.2]. The proof of the implication (a) ⇒ (b) below will develop the idea of the proof of the corresponding implication in [LO6], but our proof is more elementary, since instead of locally uniformly rotund renormings it uses Johnson's norms (see Section 7.3). The proof of (b) ⇒ (a) below is different from the proof of its prototype in [LO6], which relied on a trace characterization of the weak metric approximation property [LO6, Theorem 3.2].

Proof of Theorem 7.9. (b) $\Rightarrow$ (a). Obviously, the A-AP with conjugate operators is stable with respect to renormings. Therefore, for every equivalent norm on $(X, |||\cdot|||)$, the space $(X, |||\cdot|||)^*$ has the A-AP with conjugate operators. Hence, the claim follows from (a) $\Rightarrow$ (c$_1$) of Corollary 6.7.

(a) $\Rightarrow$ (b). By Lemma 3.15, we have to show that for any $\sum_{n=1}^{\infty} x_n^{**} \otimes x_n^* \in X^{**}\hat{\otimes} X^*$ with $\sum_{n=1}^{\infty} \|x_n^{**}\|\,\|x_n^*\| < \infty$ one has

$$\inf_{S\in A} \Big| \sum_{n=1}^{\infty} x_n^{**}(S^* x_n^* - x_n^*) \Big| = 0.$$

We may clearly assume that $\sum_{n=1}^{\infty} \|x_n^{**}\| \leq 1$ and $\frac{1}{1+\tau} \geq \|x_n^*\| \to 0$ for some $\tau > 0$.

Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\sum_{n>N} \|x_n^{**}\| < \frac{\varepsilon}{4(1+\tau)}.$$

Let $F := \operatorname{span}\{x_1^*, \ldots, x_N^*\}$ and let $|||\cdot|||$ denote Johnson's norm $|||\cdot|||_{F,\tau}$ on X^* as well as its predual norm on X and dual norm on X^{**}. Let us write $\hat{X}$ for $(X, |||\cdot|||)$. Then $|||x_n^*||| \leq 1$ for all n and

$$\sum_{n>N} |||x_n^{**}||| < \frac{\varepsilon}{4}.$$

Note that

$$K := \overline{\{x_1^*, x_2^*, \ldots\} \cup S_{(F,|||\cdot|||)}} \subset B_{\hat{X}^*}$$

is a compact set. Using Lemma 2.17 we can find a separable reflexive Banach space Z and a compact operator $J : Z \to \hat{X}^*$ with $\|J\| = 1$ and $K \subset J(B_Z)$. Since the space $\hat{X}$ has the weak metric A-AP, for $J^* j_{\hat{X}} \in \mathcal{K}(\hat{X}, Z^*)$, there exists a net $(S_\alpha) \subset A$ such that $\sup \|J^* j_{\hat{X}} S_\alpha\| \leq \|J^* j_{\hat{X}}\| \leq 1$ and $S_\alpha x \to x$ for all $x \in X$. Hence, $S_\alpha^* x^* \to x^*$ weakly* for all $x^* \in X^*$. Fix an index α. Observe that, by reflexivity of Z,

$$(J^* j_{\hat{X}} S_\alpha)^* = S_\alpha^* j_{\hat{X}}^* J^{**} = S_\alpha^* j_{\hat{X}}^* j_{\hat{X}^*} J = S_\alpha^* J,$$

so that

$$\|S_\alpha^* J\| = \|(J^* j_{\hat{X}} S_\alpha)^*\| = \|J^* j_{\hat{X}} S_\alpha\| \leq 1.$$

If $x^* \in S_{(F,|||\cdot|||)}$, then $x^* = Jz$ for some $z \in B_Z$. Hence,

$$|||S_\alpha^* x^*||| = |||S_\alpha^* J z||| \leq \|S_\alpha^* J\| \leq 1.$$

Therefore, Proposition 7.5 implies the convergence $S_\alpha^* x^* \to x^*$ in norm for all $x^* \in F$. Thus we can find $S := S_\alpha$ for some α such that

$$\max_{1\le n\le N} \|S^* x_n^* - x_n^*\| < \frac{\varepsilon}{2}.$$

Noting that $x_n^* = Jz_n$ with some $z_n \in B_Z$ for all n, we get

$$\begin{aligned}\Big|\sum_{n=1}^{\infty} x_n^{**}(S^*x_n^* - x_n^*)\Big| &\le \sum_{n=1}^{N} \|x_n^{**}\|\,\|S^*x_n^* - x_n^*\| + \sum_{n>N} |\!|\!|x_n^{**}|\!|\!|\,|\!|\!|S^*Jz_n - Jz_n|\!|\!| \\ &< \frac{\varepsilon}{2} + \sum_{n>N} |\!|\!|x_n^{**}|\!|\!|(\|S^*J\| + \|J\|) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

Remark 7.1. The special cases of Theorem 7.9 (without the claim about Johnson's norms) have been proven in [LL2, Theorem 4.9] for $A = \mathcal{K}(X)$ and in [LMO, Theorem 13] for a linear subspace A of $\mathcal{L}(X)$. The proof in [LL2] used a characterization of the weak metric compact approximation property that involves Hahn–Banach extension operators (see [LL2, Theorem 4.3]). The proof in [LMO] was essentially modelled after the proof of (a) $\Rightarrow$ (c) in [LO6, Theorem 4.2].

7.6 The RNP impact

Let X be a Banach space.

The case $A = \mathcal{F}(X)$ of Theorem 7.10 below is well known and goes back to Grothendieck's Memoir [G]. It asserts that the Problem 1.1 has the affirmative answer whenever X^* or X^{**} has the Radon–Nikodým property. There have been many different proofs of this result (see [O5] for more details). As written in [C, p. 289], the proofs have "always been a little mysterious". An alternative proof of this classical result was recently given in [O2]. According to the proof in [O2], *the reason*, why the metric approximation property appears in the dual space X^* with the approximation property, *is that the space X has the metric approximation property in all equivalent norms.* Using Corollary 6.14 and our main Theorems 7.7 and 7.9, we shall now repeat this proof nearly verbatim for the convex case.

Theorem 7.10. *Let X be a Banach space and let A be a convex subset of $\mathcal{K}(X)$ containing 0. If X^* or X^{**} has the Radon–Nikodým property, then the A-AP with conjugate operators of X^* implies the metric A-AP with conjugate operators.*

Proof. Assume that X^* has the A-AP with conjugate operators. By Theorem 7.9, X has the weak metric A-AP in every equivalent norm. Since the Radon–Nikodým property is preserved under changes to equivalent norms, by Corollary 6.14, X has the metric A-AP in every equivalent norm. Now Theorem 7.7 implies the metric A-AP with conjugate operators of X^*. □

Let us spell out Grothendieck's version of the theorem (without the conjugate operators). We can do so because of Proposition 3.22.

Corollary 7.11. *Let X be a Banach space. If X^* or X^{**} has the Radon–Nikodým property, then the AP of X^* implies the MAP.*

Remark 7.2. The case $A = \mathcal{K}(X)$ of Theorem 7.10 was established in [GS, Corollary 1.6] and extended to the linear subspaces A of $\mathcal{K}(X)$ in [LMO, Corollary 15].

Theorem 7.10 immediately implies the following result, the particular case of which with X being a Banach lattice having the positive approximation property is due to Nielsen [N2, Corollary 2.8].

Corollary 7.12. *Let X be a reflexive Banach space and let A be a convex subset of $\mathcal{K}(X)$ containing 0. If X has the A-AP, then X has the metric A-AP.*

Chapter 8

Applications

In this chapter we apply the theory of convex approximation properties to the positive approximation property of Banach lattices and to the approximation property defined for pairs of Banach spaces. In order to do so, we recall some of the classical theorems on Banach lattices. This chapter is based on [LisO].

8.1 Positive approximation property

In order to apply our framework to the case of the positive approximation property, let us introduce the necessary background. We recall only the basic notions of Banach lattice theory, and refer the reader to some of the excellent classical [LZ], or more modern [M-N], textbooks on the topic.

Recall that a set V is a *lattice* if it is partially ordered and for every pair $x, y \in V$ of its elements there exists the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$.

A partially ordered vector space V over $\mathbb{R}$ is called an *ordered vector space* if its order and linear structure are compatible in the following sense: if $x, y \in V$ are such that $x \leq y$, then $x + z \leq y + z$ for all $z \in V$ and $ax \leq ay$ for all real $a > 0$. If V happens to be a lattice, then V is called a *Riesz space*.

The *positive cone* V_+ of an ordered vector space V consists of all $x \in V$ such that $x \geq 0$. Let us notice that V_+ is convex and contains 0.

For every $x \in V$, the *positive part*, the *negative part*, and the *absolute value* of x are defined as $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$, respectively. These values may fail to exist unless V is a Riesz space.

Let U and V be Riesz spaces. A linear operator $T : U \to V$ is called a *positive operator* if $T(U_+) \subset V_+$. The space $L(U, V)$ is an ordered vector space (but in general not a Riesz space) with respect to relation:

$$T \leq S \iff S - T \text{ is positive.}$$

A norm $\|\cdot\|$ on a Riesz space V is called a *lattice norm* if $\|x\| \leq \|y\|$ whenever $x, y \in V$ satisfy $|x| \leq |y|$. In the latter case the space $(V, \|\cdot\|)$ is called a *normed Riesz space*. A complete normed Riesz space is called a *Banach lattice*.

Let X and Y be Banach lattices. Then every positive operator $T \in L(X, Y)$ is continuous (see, e.g., [M-N, Proposition 1.3.5]). We shall therefore write $\mathcal{A}(X, Y)_+ = \mathcal{A} \cap L(X, Y)_+$ and $\mathcal{A}(X)_+ = \mathcal{A} \cap L(X, X)_+$ for an operator ideal $\mathcal{A}$.

If X is a Banach lattice, then X^* is also a Banach lattice with the positive cone $X^*_+ = \mathcal{L}(X, \mathbb{R})_+$ (see, e.g., [M-N, Proposition 1.3.7]).

We introduce the principal definition of this section.

Definition 8.1. A Banach lattice X is said to have the *positive approximation property* (pAP) if for every compact set $K \subset X$ and for every $\varepsilon > 0$ there is a positive finite-rank operator T on X such that $\|Tx - x\| < \varepsilon$ for all $x \in K$.

By Szankowski [Sz1], there is a Banach lattice without the pAP.

In the terminology of convex approximation properties, the pAP of X means the $\mathcal{F}(X)_+$-AP. The related notions (the *bounded pAP*, the *metric pAP*, the *pAP with conjugate operators*, etc.) are defined in a standard way.

The following longstanding problem makes the investigation of the pAP an intriguing occupation (see [C, Problem 2.18]).

Problem 8.1. Let X be a Banach lattice. Does the (bounded) AP of X imply the (bounded) pAP of X?

The most recent results in the direction of this problem are due to Nielsen [N2]. He also obtained the pAP versions for a number of important results on the AP. His methods deeply involve the structure of finite-dimensional operators on Banach lattices.

We aim to show that several results for the pAP theory can be (at least partially) accomplished in the general setting of the convex approximation properties, as opposed to methods specific to Banach lattices.

Consider an immediate application of Theorem 7.7.

Corollary 8.2. *Let X be a Banach lattice and let A be a convex set of positive operators containing* 0. *If X has the metric A-AP in every equivalent norm (or just in every Johnson's norm), then X^* has the metric A-AP with conjugate operators. In particular, if A is the cone of positive finite-rank operators, then X^* has the metric pAP with conjugate operators.*

Apart from the classical case, Corollary 8.2 applies to approximation properties defined by larger classes of positive operators. Some such classes (e.g., compact, weakly compact, or Dunford–Pettis positive operators) are studied and found to be important in connection to the domination problem in [AA].

Remark 8.1. Although Corollary 8.2 provides some means to study the positive approximation property, for instance, essentially it will be needed for proving Corollary 8.8 below, it would be natural to require the metric A-approximation property only in equivalent lattice norms. We do not know whether the statement of Corollary 8.2 holds in that case.

Remark 8.2. In general, the bounded pAP cannot be lifted from a Banach lattice X to its dual lattice X^* (see Corollary 8.7 for the converse). Indeed, let U be the Pełczyński's universal space for unconditional bases (see [P], [LT, p. 92], or [C, pp. 279–280]). By definition, U has a basis (e_n) whose unconditional constant is equal to one. Therefore U is a Banach lattice ($\sum_{n=1}^{\infty} a_n e_n \geq 0$ if and only if $a_n \geq 0$ for all n) and the partial sum projections $P_m : \sum_{n=1}^{\infty} a_n e_n \mapsto \sum_{n=1}^{m} a_n e_n$ provide the metric pAP for U. However, as observed in [C, p. 285] basing on [J2] and [Sz1], U^* fails the AP.

By Corollary 7.12, if X is a reflexive Banach lattice, then a convex AP defined by a set of compact positive operators is always metric.

In the case of pAP, we can obtain a version of Corollary 7.11 (see Corollary 8.8) simply because the pAP and the pAP with conjugate operators coincide (see Proposition 8.6 below). To prove the latter, we follow the template in Section 3.3.

As a first step, we need the principle of local reflexivity for Banach lattices due to Conroy and Moore [CM] and Bernau [Ber].

We recall the necessary notions. A subspace U of a Riesz space V is called a *sublattice* if it is closed under the lattice operations $\vee$ and $\wedge$. Then U

is a Riesz space on its own. A *lattice homomorphism* between Riesz spaces is a linear operator that respects the lattice operations. Clearly, the lattice homomorphism is a positive operator. A *lattice isomorphism* is a bijective lattice homomorphism.

Lemma 8.3 (see [Ber, Theorem 2])**.** *Let X be a Banach lattice. Suppose that $\delta > 0$ and V is a weak* neighbourhood of 0 in X^{**}. If G is a finite-dimensional sublattice of X^{**}, then there is a lattice isomorphism J from G onto $J(G) \subset X$ such that $\|J\|, \|J^{-1}\| < 1 + \delta$ and $x^{**} - Jx^{**} \in \|x^{**}\| V$ for all $x^{**} \in G$.*

A key for applying Lemma 8.3 is provided by the following general result on existence of finite-dimensional sublattices in order complete Banach lattices.

A Riesz space V is *order complete* if all its order bounded subsets have both a greatest lower bound and a least upper bound. If it is the case and U is a Riesz space, then $L^r(U, V) := \operatorname{span} L(U, V)_+$ is an order complete Riesz space (see, e.g., [M-N, Theorem 1.3.2]). Therefore, if X is a Banach lattice, then its dual Banach lattice X^* is necessarily order complete.

Lemma 8.4. *Let E be a finite-dimensional subspace of an order complete Banach lattice X and let $\varepsilon > 0$. Then there exist a sublattice Z of X containing E, a finite-dimensional sublattice G of Z, and a positive linear projection $P \in \mathcal{L}(Z)$ onto G such that $\|Px - x\| \le \varepsilon \|x\|$ for all $x \in E$.*

The method behind Lemma 8.4 is well known in the Banach lattice theory. However, the published results seem to be lacking the positivity of projection P (see, e.g., [N1, Proposition 2.9] or [JL, p. 23]), which is important for us. For completeness, we shall provide a proof. However, as this proof is based on a series of classical results from the Banach lattice theory, we present it in the next separate section.

Let us continue with our template.

The following analogue of Proposition 3.21 tells us that positive finite-rank operators between dual Banach lattices are "locally conjugate". This fact seems to be new for positive operators.

Theorem 8.5. *Let X and Y be a Banach lattices. Let $T \in \mathcal{F}(Y^*, X^*)_+$ and $\varepsilon > 0$. Then T is in the closure of the set*

$$\{S \in \mathcal{F}(X, Y)_+ : \|S\| \le (1 + \varepsilon)\|T\|\}^{\mathrm{a}}$$

in the topology of compact convergence.

Since the set defined above is bounded, the topology of compact convergence can be replaced with the strong operator topology, and for a cleaner proof, we can restate Theorem 8.5 in the following seemingly weaker way.

Let X and Y be Banach lattices, let $\varepsilon > 0$, and let F be a finite subset of Y^*. If $T \in \mathcal{F}(Y^*, X^*)$ is positive, then there exists a positive operator $S \in \mathcal{F}(X, Y)$ such that $\|S\| \leq (1+\varepsilon)\|T\|$ and $\|S^* y^* - Ty^*\| \leq \varepsilon$ for all $y^* \in F$.

Proof of Theorem 8.5. We clearly may assume that $F \subset B_{Y^*}$ and $\|S\| = 1$.

To apply Lemma 8.4, let $E = T^*(X^{**}) \subset Y^{**}$ (note that Y^{**} is order complete as a dual Banach lattice) and let $\delta > 0$ be such that $(1+\delta)^2 = 1+\varepsilon$. Lemma 8.4 gives us a sublattice Z of Y^{**} containing E, a finite-dimensional sublattice G of Z, and a positive linear projection $P \in \mathcal{L}(Z)$ with $P(Z) = G$ such that $\|PT^* x^{**} - T^* x^{**}\| \leq \delta \|T^* x^{**}\| \leq \delta \|x^{**}\|$ for all $x^{**} \in X^{**}$. This clearly implies that $\|PT^* - T^*\| \leq \delta$ and $\|PT^*\| \leq 1+\delta$.

To apply Lemma 8.3, let V be defined by F and δ. Lemma 8.3 gives us a lattice homomorphism J from G into Y such that $\|J\| \leq 1+\delta$ and $y^{**} - Jy^{**} \in \|y^{**}\| V$ for all $y^{**} \in G$, meaning that

$$|y^{**}(y^*) - y^*(Jy^{**})| < \delta \|y^{**}\| \quad \forall\, y^* \in F,\ \forall\, y^{**} \in P(Z).$$

Let $S = JPT^* j_X$. Then $S \in \mathcal{F}(X, Y)$, S is positive, and $\|S\| \leq (1+\delta)^2 = 1+\varepsilon$. Moreover, for every $y^* \in F$ and for every $x \in B_X$, we have

$$\begin{aligned} |(S^* y^* - Ty^*)x| &\leq |y^*(JPT^* x) - (PT^* x)(y^*)| + \|PT^* - T^*\| \\ &< \delta \|PT^* x\| + \delta \leq \delta(1+\delta) + \delta = \varepsilon, \end{aligned}$$

as desired. □

Remark 8.3. Clearly, replacing the condition "$\|S^* y^* - Ty^*\| \leq \varepsilon$ for all $y^* \in F$" with "$\|S^* y^* - Ty^*\| \leq \varepsilon \|y^*\|$ for all $y^* \in F$" will provide a version of Theorem 8.5 for a finite-dimensional subspace F of Y^*.

Proposition 8.6. *Let X be a Banach lattice and let $\lambda \geq 1$. Then, for the dual Banach lattice X^*, the (λ-bounded) pAP and the (λ-bounded) pAP with conjugate operators are equivalent.*

Proof. Observe that by Theorem 8.5, $\mathcal{F}(X^*)_+$ (respectively, $\lambda B_{\mathcal{L}(X^*)} \cap \mathcal{F}(X^*)_+$) is in the τ_c-closure of $\mathcal{F}(X)_+^{\mathrm{a}}$ (respectively, $(1+\varepsilon)\lambda(B_{\mathcal{L}(X)} \cap \mathcal{F}(X)_+)^{\mathrm{a}}$ for every $\varepsilon > 0$) and proceed as in the proof of Proposition 3.22. □

By Propositions 3.19 and 8.6 we can state the following observation.

Corollary 8.7. *Let X be a Banach lattice and let $\lambda \geq 1$. If X^* has the (λ-bounded) pAP, then X has the (λ-bounded) pAP.*

Proposition 8.6 and Theorem 7.10 immediately imply the next result. Note that for a Banach lattice X, the Radon–Nikodým property of X^{**} implies the reflexivity of X (see, e.g., [M-N, Theorem 5.4.13]), hence also the Radon–Nikodým property of X^*.

Corollary 8.8. *Let X be a Banach lattice such that X^* has the Radon–Nikodým property. Then the pAP and the metric pAP with conjugate operators are equivalent for X^*.*

In [N2, Corollary 2.8], Nielsen proved that the positive approximation property and the metric positive approximation property are equivalent for an *order continuous* Banach lattice X whenever X has the Radon–Nikodým property, and there is a positive contractive projection from X^{**} onto X. Since a dual Banach lattice with the Radon–Nikodým property is always order continuous (see, e.g., [M-N, Theorems 2.4.14 and 5.4.14]), the statement of Corollary 8.8 clearly follows from Nielsen's result (modulo "with conjugate operators").

Unlike our unified approach, Nielsen's method of proof seems to be specific to the pAP of Banach lattices. It relies on the following fact.

Theorem 8.9 ([N2, Theorem 2.6]). *Let X be an order continuous Banach lattice and let $\lambda \geq 1$. Then the λ-bounded pAP of X is equivalent to the $\{T \in \mathcal{F}(X) : \| \, |T| \, \| \leq \lambda\}$-AP of X.*

The proof of Theorem 8.9 in [N2] is quite technical and involves non-trivial representations of finite-rank operators on Banach lattices.

8.2 An excursion into Riesz Spaces and Banach lattices

The purpose of this section is to prove Lemma 8.4, which expresses the fact that order complete Banach lattices are "rich" in finite-dimensional sublattices. As we already mentioned, the method behind the proof is well known in the Banach lattice theory, but the statements of published results we found (see, e.g., [N1, Proposition 2.9] or [JL, p. 23]) lack the positivity of projection P. Although, for example, the proof in [N1] almost does not need to be modified to provide this property.

Nevertheless, we approach Lemma 8.4 in detail because the positivity of projection P is important for the application of our results to the pAP. Moreover, since the main subject of the thesis lies outside the realm of Banach lattices, we feel it is reasonable to present a number of classical results as the theoretical foundation.

8.2.1 Finite-dimensional sublattices and disjointness

Let x and y be elements of a Riesz space V.

Observe that the identities $2(x \vee y) = x + y + |x - y|$ and $2(x \wedge y) = x + y - |x - y|$ imply that a subspace U of V is a sublattice if and only if $x \in U$ implies $|x| \in U$.

Elements x and y are called *disjoint* if $|x| \wedge |y| = 0$.

If x and y are disjoint, then so are ax and by for all reals a and b, because $0 \leq |ax| \wedge |ay| \leq [(|a| + |b|)\,|x|] \wedge [(|a| + |b|)\,|y|] = 0$.

Proposition. *Let $x_1, \dots, x_n$ be mutually disjoint vectors in a Riesz space. Then these vectors are linearly independent, and*

$$\left| \sum_{i=1}^{n} x_i \right| = \sum_{i=1}^{n} |x_i| \,.$$

For the neccesity part of the following proposition, see [LZ, Theorem 26.11], while the sufficiency part easily follows from the above remarks.

Proposition. *A finite-dimensional subspace of a Riesz space is a sublattice if and only if it has a basis consisting of positive mutually disjoint vectors.*

8.2.2 Stonian spaces and $C(K)$

Let us recall some facts concerning topological spaces.

A topological space is called *Hausdorff* if any two distinct points have disjoint neighbourhoods. A topological space is called *normal* if any two disjoint closed sets have disjoint neighbourhoods.

Clearly, every compact Hausdorff space is normal.

Proposition (Urysohn's lemma). *A topological space X is normal if and only if any two disjoint closed subsets A and B of X can be separated by a continuous function, i.e., there exists a continuous function $f : X \to [0,1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Let X be a topological space and $A \subset X$. Recall that the indicator function $\mathbb{1}_A : X \to [0,1]$ is defined as

$$\mathbb{1}_A(t) = \begin{cases} 0, & t \notin A \\ 1, & t \in A \end{cases}, \quad t \in X,$$

and observe that $\mathbb{1}_A$ is continuous if and only if A is *clopen*, i.e., both closed and open.

A normal Hausdorff space is called *Stonian* or *extremely disconnected* if the closure of every open set is open. Clearly, in a Stonian space any two distinct points have disjoint clopen neighbourhoods.

Let K be a compact Hausdorff space. Recall that the collection $C(K)$ of all continuous functions $f : K \to \mathbb{R}$ is a Banach space with respect to pointwise vector space operations and the maximum-norm. It is also a Banach lattice with $f \geq 0$ if and only if $f(t) \geq 0$ for all $t \in K$.

Proposition (for the converse, see, e.g., [M-N, Proposition 2.1.4]). *If $C(K)$ is order complete, then K is Stonian.*

Proof. Let $V \subset K$ be open. By Urysohn's lemma, for every $x \in V$ there exists $f_x \in C(K)$ such that $f_x(x) = 1$, $f_x(K) = [0,1]$, and $f_x(y) = 0$ for every $y \in K \backslash V$. Since $f_x \leq \mathbb{1}_K$ for every $x \in V$, and $C(K)$ is order complete, there exists $f = \sup_{x \in V} f_x \in C(K)$. Clearly, $f|_{\overline{V}} = 1$. For every $y \notin \overline{V}$, Urysohn's lemma provides $g_y \in C(K)$ such that $g_y|_{\overline{V}} = 1$ and $g_y(y) = 0$. Then $g_y \geq f_x$ for every $x \in V$, and therefore, $g_y \geq f$, so that $f(y) = 0$. Hence, $f = \mathbb{1}_{\overline{V}}$. □

For the proof of the following classical theorem, see, e.g., [M-N, Theorem 2.1.1].

Theorem (Stone–Weierstrass). *Every sublattice U of $C(K)$ which separates the points of K and contains the constant function $\mathbb{1}_K$ is dense in $C(K)$.*

Let $J \subset C(K)$ denote the set of all continuous indicator functions.

Proposition. *The subspace* span J *is a sublattice of $C(K)$.*

Proof. We only have to check that $y \in \operatorname{span} J$ implies $|y| \in \operatorname{span} J$. Clearly, $y \in \operatorname{span}\{\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n}\}$ for some mutually disjoint clopen sets $A_1, \ldots, A_n$. Since $\operatorname{span}\{\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n}\}$ is a finite-dimensional sublattice, we have $|y| \in \operatorname{span}\{\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n}\} \subset \operatorname{span} J$.

□

Corollary 8.10. *If $C(K)$ is order complete, then* $\operatorname{span} J$ *is dense in $C(K)$.*

Proof. To apply the Stone–Weierstrass theorem, notice that if $C(K)$ is order complete, then K is Stonian. So that any two distinct points in K have disjoint clopen neighbourhoods. This means that J separates the points of K. Also, clearly $\mathbb{1}_K \in J$. □

Proposition 8.11. *Let $A_1, \ldots, A_n$ be mutually disjoint non-empty clopen subsets of K. Let $F := \operatorname{span}\{\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n}\}$ be a finite-dimensional sublattice of $C(K)$ generated by the indicator functions of these sets. Then there is a positive projection $P : C(K) \to F$ with $\|P\| = 1$.*

Proof. For every $i = 1, \ldots, n$ take the Dirac functional δ_{x_i} for some point $x_i \in A_i$ (i.e., the evaluation at x_i). Observe that $\{\delta_{x_i}; \mathbb{1}_{A_i}\}_{i=1}^n$ is a biorthogonal system, hence the operator

$$P := \sum_{i=1}^{n} \delta_{x_i} \otimes \mathbb{1}_{A_i}$$

is a projection onto F, which is clearly positive and of norm 1. □

Proposition 8.12. *Let E be a finite-dimensional subspace of an order complete $C(K)$. For every $\varepsilon > 0$ there are a finite-dimensional sublattice F of $C(K)$ and a positive linear projection $P : C(K) \to F$ such that $\|Pf - f\| \leq \varepsilon \|f\|$ for all $f \in E$ and $\|P\| = 1$.*

Proof. Let $f_1, \ldots, f_n \in C(K)$ be a basis of E. By Corollary 8.10, $\overline{\operatorname{span}}\, J = C(K)$. So there are $g_1, \ldots, g_n \in \operatorname{span} J$ such that $\|f_i - g_i\| < \varepsilon/2$. Since finite intersections of clopen subsets of K are clopen, there are mutually disjoint clopen subsets $A_1, \ldots, A_m$ of K such that

$$\{g_1, \ldots, g_n\} \subset \operatorname{span}\{\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_m}\} =: F.$$

By Proposition 8.11, there is a positive projection P from $C(K)$ onto F with $\|P\| = 1$. For $i = 1, \ldots, n$, one has

$$\|Pf_i - f_i\| \leq \|Pf_i - g_i\| + \|g_i - f_i\| = \|Pf_i - Pg_i\| + \|g_i - f_i\| < \varepsilon.$$

□

8.2.3 Abstract M-spaces

A lattice norm $\|\cdot\|$ on a Riesz space X is called an *M-norm* if $\|x \vee y\| = \max(\|x\|, \|y\|)$ for $x, y \in X$ with $x \wedge y = 0$. If, in addition, X is complete, then X is called an *abstract M-space.*

A sublattice U of a Riesz space V is called an *ideal* if $x \in U$ whenever $|x| \leq |y|$ for some $y \in U$. Clearly, an intersection of two ideals is an ideal too.

Proposition. *An ideal U of an order complete Riesz space V is an order complete Riesz space.*

Proof. Take a non-empty order bounded set $S \subset U$. This means that there are $x, y \in U$ such that $y \leq s \leq x$ for all $s \in S$. Taking $u = |x| \vee |y| \in U_+$ we have that $-u \leq s \leq u$ for all $s \in S$. Since V is order complete, there exists $\sup S \in V$. But then $-u \leq \sup S \leq u$ meaning that $|\sup S| \leq u$ and $\sup S \in U$. Similarly, $\inf S \in U$. □

A positive element u of a Banach lattice X is called an *order unit* if the norm of X is the gauge of the order interval $[-u, u] = \{x \in X : -u \leq x \leq u\}$.

Take a positive element u of a Banach lattice X. It is easy to see that the order interval $[-u, u]$ is an absolutely convex set. Hence, the gauge $\|\cdot\|_u$ of $[-u, u]$ defines a norm on $\operatorname{span}[-u, u]$. Moreover, it is clear, that the latter set is exactly the ideal X_u generated by u, and $\|\cdot\|_u$ is an M-norm on it. Since $[-u, u] \subset \|u\| B_X$, we also have $\|u\| \|\cdot\| \leq \|\cdot\|_u$.

Lemma. *A normed Riesz space is complete if and only if every positive increasing Cauchy sequence converges.*

Proof. Sufficiency. Take a Cauchy sequence (x_n). By passing to a subsequence, we may assume $\|x_{n+1} - x_n\| \leq 2^{-n}$. Now for $m \geq n$ we have

$$\left\| \sum_{i=n}^{m} (x_{i+1} - x_i)^+ \right\| \leq 2^{1-n}.$$

Therefore, by assumption, there exists $x^+ = x_1^+ + \sum_{i=1}^{\infty} (x_{i+1} - x_i)^+$. Similarly, there exists $x^- = x_1^- + \sum_{i=1}^{\infty} (x_{i+1} - x_i)^-$. Hence, $x^+ - x^- = \lim x_n$. □

Proposition. *The space $(X_u, \|\cdot\|_u)$ is a Banach lattice; hence, it is an abstract M-space with an order unit u.*

Proof. In order to use the lemma, take an increasing Cauchy sequence $(x_n) \subset (X_u)_+$. Since it is also a Cauchy sequence in X, it converges in X, and therefore there exists $x = \sup_n x_n$. Since the Cauchy sequence (x_n) is bounded in X_u, it is absorbed by $[-u, u]$, and so is x. Therefore, $x \in X_u$. Now, for any $m \in \mathbb{N}$,

$$x - x_m = \sup_n (x_n - x_m) = \sup_{n>m} (x_n - x_m),$$

so that

$$\|x - x_m\|_u \leq \sup_{n>m} \|x_n - x_m\|_u \to_n 0.$$

□

For proving Lemma 8.4, it remains to combine our construction with a famous representation theorem of Kakutani (see, e.g., [M-N, Theorem 2.1.3]).

Theorem (Kakutani)**.** *An abstract M-space with an order unit u is lattice isometric to $C(K)$ (for some compact Hausdorff space K) via an isomorphism which maps u to $\|u\| 1_K$.*

Proof of Lemma 8.4. Let E be a finite-dimensional subspace of X. We may assume that $E \subset \operatorname{span}\{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n \in X_+$. Indeed, if any x_i is not positive, replace it with x_i^+ and x_i^- (recall that $x_i = x_i^+ - x_i^-$). Let $u = \sup_i x_i$, and note that we may assume $\|u\| = 1$. Then

$$E \subset \operatorname{span}[0, u] \subset X_u.$$

Since X is order complete and X_u is an ideal in X, we also have that X_u is order complete. By Kakutani's theorem, X_u is lattice isometric to $C(K)$ for some compact K. Hence, this $C(K)$ is order complete, and we can apply Proposition 8.12. Therefore, for every $\varepsilon > 0$ there are a finite-dimensional sublattice G of $Z := X_u$ and a positive linear projection $P : Z \to G$ such that

$$\|Px - x\|_u \leq \varepsilon \|x\|_u \quad \forall\, x \in E.$$

In particular, for $i = 1, \ldots, n$, one has

$$\|Px_i - x_i\| \leq \|Px_i - x_i\|_u \leq \varepsilon.$$

□

Remark 8.4. The proof of Lemma 8.4 in [LisO] is somewhat more direct and involves neither an analogue of Proposition 8.12 nor the Stone–Weierstrass theorem.

8.3 The approximation property of pairs

Let X be a Banach space, let Y be a closed subspace of X, and let $\lambda \geq 1$.

Very recently, Figiel, Johnson, and Pełczyński [FJP] introduced the following variant of the bounded approximation property.

Definition 8.13 (see [FJP, Definition 1.1])**.** The *pair* (X, Y) *has the* λ*-bounded AP* if for every finite-dimensional subspace F of X and every $\varepsilon > 0$, there exists an operator $S \in \mathcal{F}(X)$ such that $Sx = x$ for all $x \in F$, $\|S\| \leq \lambda + \varepsilon$, and $S(Y) \subset Y$.

The *pair* (X, Y) *has the metric AP* if it has the 1-bounded AP.

Let us denote
$$\mathcal{F}(X)_Y := \{S \in \mathcal{F}(X) : S(Y) \subset Y\}.$$

Relying on [FJP, Lemma 1.5], it is easy to obtain the following description. For completeness, we provide its proof.

Proposition 8.14. *The pair* (X, Y) *has the* λ*-bounded AP if and only if* X *has the* λ*-bounded* $\mathcal{F}(X)_Y$*-AP.*

Proof. Necessity. The λ-BAP of (X, Y) clearly implies the $(\lambda + \varepsilon)$-bounded $\mathcal{F}(X)_Y$-AP of X for every $\varepsilon > 0$. The claim follows from Proposition 3.17.

Sufficiency. Fix $\varepsilon > 0$ and a finite-dimensional subspace $F \subset X$. There is a projection $P \in \mathcal{L}(X)$ such that $P(X) = F$ and $P(Y) \subset Y$. Indeed, let $E = \{x_1, \ldots, x_n\}$ be a basis of F. For all i, if $x_i \in Y$, pick $x_i^* \in (\operatorname{span}(E \setminus \{x_i\}))^\perp$ with $x_i^*(x_i) = 1$, otherwise pick $x_i^* \in (Y \cup \operatorname{span}(E \setminus \{x_i\}))^\perp$ with $x_i^*(x_i) = 1$. Then we can take
$$P = \sum_{i=1}^{n} x_i^* \otimes x_i.$$
By assumption, we can choose $T \in \mathcal{F}(X)_Y$ such that $\|T\| \leq \lambda$ and $\|Tx - x\| < \varepsilon / \|P\|$ for all $x \in B_F$. Put $S = T + P - TP = T + (I - T)P \in \mathcal{F}(X)$. Clearly, $S(Y) \subset Y$ and $Sx = x$ for all $x \in F$. Furthermore, we have
$$\|S\| \leq \|T\| + \frac{\varepsilon}{\|P\|} \|P\| \leq \lambda + \varepsilon.$$
□

Note that the set $\mathcal{F}(X)_Y$ is a subspace, so that we may apply our results to this case.

Since $S(Y) \subset Y$ implies $S^*(Y^\perp) \subset Y^\perp$, Theorem 7.7 immediately yields the following lifting result for the metric approximation property of pairs.

Corollary 8.15. *Let X be a Banach space and let Y be a closed subspace of X. If the pair (X, Y) has the metric AP in every equivalent norm of X (or just in every Johnson's norm), then the pair $(X^*, Y^\perp)$ has the metric AP.*

Remark 8.5. If $(X^*, Y^\perp)$ has the metric AP, then both X^* and Y^* have the metric AP. For X^*, this is obvious. For $Y^* \cong X^*/Y^\perp$, this is clear from [FJP, Corollary 1.2].

It is natural to define the approximation property of pairs of Banach spaces as follows.

Definition 8.16. Let X be a Banach space and let Y be a closed subspace of X. We say that the *pair* (X, Y) *has the approximation property* if for every compact subset K of X and for every $\varepsilon > 0$, there exists $S \in \mathcal{F}(X)$ with $S(Y) \subset Y$ such that $\|Sx - x\| < \varepsilon$ for all $x \in K$.

Hence, the (λ-bounded) AP of the pair (X, Y) is precisely the (λ-bounded) $\mathcal{F}(X)_Y$-AP of X. In order to apply Theorem 7.10 to such properties, we need an analogue of Lemma 3.21.

Lemma 8.17. *The subspace*

$$\{S \in \mathcal{F}(X^*) : S(Y^\perp) \subset Y^\perp\} \subset \mathcal{F}(X^*)$$

is in τ_c-closure of $(\mathcal{F}(X)_Y)^{\mathrm{a}}$.

Proof. Let $S \in \mathcal{F}(X^*)$ satisfy $S(Y^\perp) \subset Y^\perp$. Consider a representation $S = \sum_{i=1}^n x_i^{**} \otimes x_i^*$ with the linearly independent elements $x_i^* \in X^*$ such that $\{x_i^*\}_{i=1}^k \subset X^* \setminus Y^\perp$ and $\{x_i^*\}_{i=k+1}^n \subset Y^\perp$ for some $k \in \{0, \dots, n\}$. The assumption $S(Y^\perp) \subset Y^\perp$ then implies $\{x_i^{**}\}_{i=1}^k \subset (Y^\perp)^\perp$. Hence, there are operators S_1, $S_2 \in \mathcal{F}(X^*)$ such that $S = S_1 + S_2$, $S_1^*(X^{**}) \subset (Y^\perp)^\perp$, and $Y \subset \ker S_2^*$. Let I denote the isometric isomorphism between $(Y^\perp)^\perp$ and Y^{**} such that for all $x^{**} \in (Y^\perp)^\perp$ and for all $y^* \in Y^*$ one has $(Ix^{**})(y^*) = x^{**}(x^*)$ with $x^* \in X^*$ being any extension of y^* to X.

It is enough to show that S can be approximated in the strong operator topology by operators T^* with $T \in \mathcal{F}(X)_Y$ and $\|T\| \leq 2(\|S_1\| + \|S_2\|)$. Fix a finite set $F \subset X^*$ and denote $F_1 := \{f|_Y : f \in F\} \subset Y^*$. Let $G_1 := (IS_1^*)(X^{**}) \subset Y^{**}$ and $G_2 := S_2^*(X^{**}) \subset X^{**}$. By the principle of local reflexivity, there are linear injections $J_1 : G_1 \to Y$ and $J_2 : G_2 \to X$, both norm-bounded by 2, such that

$$g(J_1 y^{**}) = y^{**}(g) \quad \forall y^{**} \in G_1 \; \forall g \in F_1$$

and

$$f(J_2 x^{**}) = x^{**}(f) \quad \forall x^{**} \in G_2 \; \forall f \in F.$$

Let $T := (J_1 I S_1^* + J_2 S_2^*)\, j_X \in \mathcal{F}(X)$. Then for $y \in Y$, one has

$$Ty = J_1 I S_1^* y + J_2 S_2^* y = J_1 I S_1^* y + 0 \in Y,$$

so that $T(Y) \subset Y$. For $x \in X$ and $f \in F$, one has

$$f(J_1 I S_1^* x) = f|_Y (J_1 I S_1^* x) = (I S_1^* x)(f|_Y) = (S_1^* x)(f) = (S_1 f)(x)$$

and

$$f(J_2 S_2^* x) = (S_2 f)(x),$$

so that

$$T^* f = (S_1 + S_2) f = Sf \quad \forall f \in F.$$

Clearly, $\|T\| \leq 2(\|S_1\| + \|S_2\|)$, as required. □

Observe that Lemma 8.17 does not provide any good estimate for the norms of approximating operators. Therefore, we do not get the λ-bounded case in the naturally consequent proposition below.

Proposition 8.18. *Let X be a Banach space and let Y be a closed subspace of X. The AP of the pair $(X^*, Y^\perp)$ is equivalent to the $\mathcal{F}(X)_Y$-AP of X^* with conjugate operators.*

Proposition 8.18 and Theorem 7.10 imply the following result.

Corollary 8.19. *Let X be a Banach space and let Y be a closed subspace of X. Let X^* or X^{**} have the Radon–Nikodým property. If the pair $(X^*, Y^\perp)$ has the approximation property, then $(X^*, Y^\perp)$ has the metric approximation property; in particular, both X^* and Y^* have the metric approximation property.*

Corollary 8.19 contains the classical case discussed before Theorem 7.10 since the pair $(X^*, \{0\}^\perp) = (X^*, X^*)$ clearly has the (metric) approximation property if and only if X^* has.

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Banachi ruumide kumerad aproksimatsiooniomadused

Kokkuvõte

Käesoleva väitekirja põhieesmärk on välja arendada ühtne meetod aproksimatsiooniomaduse erisuguste versioonide käsitlemiseks. Peale klassikalise aproksimatsiooniomaduse, kuuluvad selliste versioonide hulka näiteks kompaktne aproksimatsiooniomadus, operaatorideaali poolt tekitatud aproksimatsiooniomadus ja Banachi võrede positiivne aproksimatsiooniomadus. Väitekirja põhiline mõiste on *kumer aproksimatsiooniomadus* ehk, täpsemalt, Banachi ruumil tegutsevate pidevate lineaarsete operaatorite nulli sisaldava kumera hulga poolt defineeritud aproksimatsiooniomadus. Väitekirjas loodud kumerate aproksimatsiooniomaduste teooria on heas vastavuses klassikalise aproksimatsiooniomaduse teooria oluliste tulemustega. Väitekirjas uuritakse ka lineaarse alamruumi poolt ja suvalise operaatorite hulga poolt defineeritud aproksimatsiooniomadusi, milledest esimene on kumera aproksimatsiooniomaduse erijuhtum, teine aga selle üldistus.

Väitekirja esimene peatükk sisaldab aproksimatsiooniprobleemi ajaloolise tausta lühikest tutvustust, väitekirja kokkuvõtet ning väitekirjas kasutatud tähistuste kirjeldust.

Väitekirja teises peatükis tuuakse välja järgnevate osade jaoks vajalikud lisatähistused, mõisted ja tulemused nagu operaator- ja ruumideaalid, tensorkorrutised, nõrk*-nõrgalt pidevad operaatorid, operaatorite ruumi olulisemad lokaalselt kumerad topoloogiad ning Davis–Figiel–Johnson–Pełczyński faktorisatsioonilemma Lima–Nygaard–Oja isomeetriline versioon.

Kolmandas peatükis vaadeldakse aproksimatsiooniomaduste erinevaid versioone ning tuuakse sisse kumera aproksimatsiooniomaduse ja kaasoperaatoritega kumera aproksimatsiooniomaduse mõisted. Tuletatakse kumera aproksimatsiooniomaduse põhilised kirjeldused ja tõestatakse nende järeldused. Kolmas peatükk põhineb artikli [LMO] ja artikli eelvariandi [L] sissejuhatavatel osadel.

Neljandas peatükis saadud tulemused laiendavad Banachi ruumi ja tema kaasruumi klassikalise aproksimatsiooniomaduse kirjeldust kompaktsete operaatorite lähendamise kaudu [G]. Nende tulemuste analoogid kehtivad suvalise operaatorite hulga poolt defineeritud aproksimatsiooniomaduse korral ning nende tugevamad versioonid kehtivad kumera hulga poolt defineeritud aproksimatsiooniomaduse korral. Neljas peatükk on inspireeritud artiklitest [LO4, LLN, OPe]. See peatükk põhineb artiklil [LMO] ja artikli eelvariandil [L].

Viiendas peatükis esitatakse üks väitekirja põhitulemustest. See on lineaarse alamruumi poolt defineeritud aproksimatsiooniomaduse kirjeldus nõrgalt kompaktsete operaatorite lähendamise kaudu. Abivahenditena meenutakse Radon–Nikodými omaduse mõistet ja Feder–Saphari teoreemi kompaktsete operaatorite ruumil tegutsevate pidevate lineaarsete funktsionaalide üldkujust. Viienda peatüki põhitulemused on ilmunud artiklis [LMO].

Kuuendas peatükis vaadeldakse tugeva aproksimatsiooniomaduse ja nõrgalt tõkestatud aproksimatsiooniomaduse mõisteid, mis olid sisse toodud vastavalt artiklites [O3] ja [LO6]. Nende mõistete kumerate versioonide käsitlemiseks arendatakse välja ühtne lähenemismeetod. Inspireerituna artiklist [O2], uuritakse nende mõistete ja (tõkestatud) kumerate aproksimatsiooniomaduste vahekorda Radon–Nikodými omaduse mõjuväljas. Kuues peatükk põhineb artikli eelvariandil [L].

Seitsmendas peatükis üldistatakse Johnsoni teoreem [J2] meetrilise aproksimatsiooniomaduse ülekandumisest Banachi ruumilt tema kaasruumile. Samuti üldistatakse Lima–Oja [LO6] teoreem nõrga meetrilise aproksimatsiooniomaduse ülekandumisest kaasruumile. Neid tulemusi rakendatakse näitamaks, et Radon–Nikodými omaduse mõjul on kaasruumi kumer aproksimatsiooniomadus meetriline. Selle tulemuse prototüüp sisaldub artiklis [O2]. Seitsmes peatükk põhineb artiklil [LisO].

Kaheksandas peatükis rakendatakse eelnevates peatükkides loodud kumerate aproksimatsiooniomaduste teooriat Banachi võrede positiivse aproksimatsiooniomaduse ja Banachi ruumide paari poolt defineeritud aproksimatsiooniomaduse uurimisel. Selle jaoks meenutakse klassikalisi tulemusi Banachi võrede teooriast. Kaheksas peatükk põhineb artiklil [LisO].

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