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KAIDO LÄTT

Singular fractional differential equations and cordial Volterra integral operators

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Chapter 1

Introduction

The present thesis is devoted to the analysis of some classes of singular differential equations involving fractional (non-integer) order derivatives of an unknown function. Also finding the Hadamard finite part of a class of divergent integrals is under consideration.

The concept of fractional derivatives has interested mathematicians since at least the famous correspondence between L'Hospital and Leibniz in 1695. The question, raised by L'Hospital, sought the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ for $n = 1/2$. Over the years several prominent mathematicians have contributed to this field, for example Euler, Laplace, Riemann, Liouville, Grünwald, Letnikov, Hadamard and Caputo.

Recently, fractional derivatives have seen a remarkable growth in popularity mainly because of interesting new applications in physics, chemistry, mechanics, biology, economics, signal and image processing, aerodynamics, etc. Especially, fractional calculus has turned out to be one of the best tools to describe long-memory processes that are of great interest to engineers and physicist but also to pure mathematicians. For the main results regarding fractional derivatives and their applications see, for example, [3, 8, 17, 25, 30, 31] and the references cited in these monographs. Additionally, various existence and uniqueness results for fractional differential equations are given in [1, 2, 7, 33] and some recent results regarding the numerical solution of such equations can be found in [11, 12, 13, 27, 28, 29].

In this thesis we shall examine the unique solvability of singular fractional differential equations of the form (1.1.2) and (1.1.3) in $C^m[0,T]$ ($m \in \mathbb{N}_0 = \{0, 1, \ldots\}$), the space of m times continuously differentiable functions on $[0,T]$ (naturally, $C^0[0,T] = C[0,T]$). Our study has been motivated by the paper [38], where the singular system of ordinary differential equations

$$tu'(t) = A(t)u(t) + f(t) \ (0 < t \leq T < \infty) \tag{1.1.1}$$

was considered. Here $A = (a_{p,q})_{p,q=1}^n$ is a matrix function, where $a_{p,q} \in C^m[0,T]$ for $p, q = 1, 2, \ldots, n$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, $m \in \mathbb{N}_0$, and $f = (f_1, \ldots, f_n)^{\mathrm{T}}$ is a vector function, where $f_p \in C^m[0,T]$, $p = 1, 2, \ldots, n$. It was shown how the unique solvability of this problem depends on the set of eigenvalues of the matrix $A(0)$. The central idea of article [38] was the reduction of (1.1.1) to a system of cordial Volterra integral equations. The cordial Volterra integral operator V_φ with an integrable core $\varphi \in L^1(0,1)$ is defined by

$$(V_\varphi u)(t) = \int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right) u(s)ds = \int_0^1 \varphi(x)u(tx)dx, \quad 0 \le t \le T, \quad u \in C[0,T].$$

The theory of cordial Volterra integral operators was introduced by Vainikko in articles [34, 35]. For more results in this field see also [36, 37].

Singular systems (1.1.1) can also be presented in the equivalent form

$$(tu(t))' = B(t)u(t) + f(t), \quad B(t) = A(t) + I, \quad 0 < t \le T,$$

where I is the identity matrix. On the other hand, singular fractional differential equations of the form

$$(M^\beta D_0^\beta u)(t) = \sum_{k=1}^{l} a_k(t)(M^{\beta_k} D_0^{\beta_k} u)(t) + f(t), \quad 0 < t \le T, \tag{1.1.2}$$

and

$$(D_0^\alpha M^\alpha u)(t) = \sum_{k=1}^{l} b_k(t)(D_0^{\alpha_k} M^{\alpha_k} u)(t) + f(t), \quad 0 < t \le T, \tag{1.1.3}$$

are not equivalent and thus need independent treatments. Here M^ν, is the multiplication operator:

$$(M^\nu u)(t) = t^\nu u(t), \quad 0 < t \le T, \ \nu \in \mathbb{R} = (-\infty, \infty), \ u \in C[0,T].$$

One of the main goals of the present thesis is to study the unique solvabilty of (1.1.2) and (1.1.3) in the space $C^m[0,T]$ for $m \ge 0$. The basis of our discussions is the theory of cordial Volterra integral operators recalled in Section 2.3.

In equations (1.1.2) and (1.1.3) the fractional differentiation operators D_0^μ of order $\mu \in [0, \infty)$ are defined as the inverses of the Riemann-Liouville integral operator J^μ on $J^\mu C[0,T]$, i.e.

$$D_0^\mu := (J^\mu)^{-1}, \quad \mu \ge 0.$$

The Riemann-Liouville fractional integral operator J^μ is given by

$$(J^\mu u)(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u(s)ds, \quad u \in C[0,T], \quad t > 0, \ \mu > 0; \quad J^0 = I.$$

Here Γ is the Euler Gamma function and I is the identity mapping. As we explain in Section 2.2, the fractional differentiation operators D_0^μ have some useful properties lost for Riemann-Liouville and Caputo fractional differentiation operators, which are more popular in applications. Fortunately, as we detail in Section 2.2, D_0^μ and Riemann-Liouville and Caputo fractional differentiation operators have a natural connection.

This thesis is organized as follows.

Chapters 1 and 2 have an introductory character. In these chapters, we introduce definitions and present some results which we use in the present work.

In Chapter 3 we consider the equation (1.1.3) which in some aspects is simpler than (1.1.2). To be more precise, we analyze the unique solvability in $C^m[0,T]$, $m \in \mathbb{N}_0$, of a singular fractional differential equation of the form (1.1.3) where $f \in C^m[0,T]$, $\alpha, \alpha_k \in \mathbb{R}$ and

$$m < \alpha \le m+1, \quad \alpha > \alpha_k \ge 0, \quad b_k \in C^m[0,T], \quad k = 1,2,\ldots,l, \quad m \in \mathbb{N}_0. \tag{1.1.4}$$

We first consider a simplified version of equation (1.1.3) with constant coefficients and prove Theorem 3.1.1 about the unique solvability in $C^m[0,T]$ of this type of equation. The main result of Chapter 3 is Theorem 3.1.2, in which we give the unique solvability conditions for equations (1.1.3) under assumptions (1.1.4).

Note that for a unique solution $u \in C^m[0,T]$ of (1.1.3), no initial or boundary conditions are permitted: imposing them one determines, as a rule, a solution of lesser regularity. Linear fractional differential equations without singularities, but with initial conditions, have been intensively discussed e.g. in [9, 22], monographs [8, 17, 30] and references therein. Also as we explain in Section 3.1, under conditions (1.1.4), $u \in C^m[0,T]$ remains to be a solution of (1.1.3) if we replace D_0^α, $D_0^{\alpha_k}$ by either Riemann-Liouville or Caputo fractional differentiation operators.

In Chapter 4, we consider a singular fractional differential equation of the form (1.1.2), where $f \in C^m[0,T]$, $\beta, \beta_k \in \mathbb{R}$ and

$$m < \beta < m+1, \quad \beta > \beta_k \ge 0, \quad a_k \in C^m[0,T], \quad k = 1,2,\ldots,l, \quad m \in \mathbb{N}_0. \tag{1.1.5}$$

We first consider a simplified version of equation (1.1.2) with constant coefficients and treat the cases $0 < \beta < 1$ and $\beta > 1$ separately. For the case $0 < \beta < 1$ we prove Theorem 4.1.1. The case $\beta > 1$ is more complicated, we first have to extend the differentiation operators D_0^β and $D_0^{\beta_k}$ and our cordial Volterra integral operators to $C^m[0,T]$ and then prove Theorem 4.1.2 for the extended operators. To consider equations (1.1.2) we formulate and prove an auxiliary result, Lemma 4.1.2. In Theorem 4.1.4 we give the unique solvability conditions for equation (1.1.2) under conditions (1.1.5). At the end of Chapter 4 we analyze equations which we obtain from (1.1.2) replacing D_0^β, $D_0^{\beta_k}$ either by Riemann-Liouville or Caputo fractional differentiation operators.

As with (1.1.3) for a unique solution $u \in C^m[0,T]$ of (1.1.2), no initial or boundary conditions are permitted: imposing them one determines, as a rule, a solution of lesser regularity. Note that in conditions (1.1.5) we require that $\beta \notin \mathbb{N}$ but in conditions (1.1.4) no such restrictions apply to α, we will explain why this is so in Subsection 4.1.1 (p. 51). In our discussions regarding equation (1.1.2) with conditions (1.1.5) for $\beta > 1$, we have to extend our fractional differentiation and cordial Volterra integral operators. As it turns out, our extensions of cordial Volterra integral operators correspond to the Hadamard finite part interpretation of some specific divergent integrals.

Divergent integrals and equations containing them have been studied for a long time, including the principal work done by Hadamard in [15]. Equations containing divergent integrals have been useful in applications in mathematics [4, 19, 20, 32] and physics [21, 26].

One of the concepts under discussion has been the summability, i.e. finding the finite part (f.p.), of divergent integrals. Over the years numerous approaches to defining the finite part of divergent integrals have been examined (see [15, 20]). For methods on numerically finding the finite part of divergent integrals see [5], [6] and [14]. In [41], a unified approach to summability of divergent integrals was presented that covers the approaches considered before. Later, in [42], divergent integrals

$$\int_0^R a(r) r^{-\lambda-1} dr$$

depending on complex number $\lambda \in \mathbb{C}$, were examined under assumptions $\mathrm{Re}\lambda \geq 0$,

$$a \in H^{m,\alpha}[0,R], \quad m \in \mathbb{N}_0, \quad 0 < \alpha \leq 1,$$

where $H^{m,\alpha}[0,R]$ is the class of functions satisfying the conditions

$$a \in C^m[0,R], \quad \left| a^{(m)}(r) - a^{(m)}(0) \right| \leq c r^{\alpha}, \quad 0 \leq r \leq R, \ c > 0, \ c = \text{const}.$$

In Chapter 5, we consider integrals

$$\int_0^T a(t) t^{-\lambda-1} (\ln t)^n dt, \tag{1.1.6}$$

where $\lambda \in \mathbb{C}$, $n \in \mathbb{N}_0$,

$$a \in H^{m,\alpha}[0,T], \quad m \in \mathbb{N}_0, \quad 0 < \alpha \leq 1. \tag{1.1.7}$$

For $\mathrm{Re}\lambda < 0$ integral (1.1.6) converges; for $\mathrm{Re}\lambda \geq 0$ it generally diverges. The first goal of Chapter 5 is to define, for $\mathrm{Re}\lambda \geq 0$, the finite part of divergent integral (1.1.6) with $n \in \mathbb{N}_0$ so that the f.p.-integrals have the same two crucial properties f.p.-integrals had in [42]. We will also prove a formula for change of variables in the f.p.-integrals of (1.1.6).

The main contributions of the present thesis are as follows.

1) It is elaborated a methodology for the study of the existence of a special solution of maximal possible smoothness for singular fractional order differential equations. The approach is based on the concept of cordial Volterra integral operators and especially on the description of their spectra. The obtained theoretical results are useful when constructing and justifying high order numerical methods for singular fractional differential equations of the form (1.1.2) and (1.1.3). The numerics is not touched upon in the present thesis.

2) The concept of the finite part (f.p.) for a class of divergent integrals with a logarithmic factor is developed. Also the formula for change of variables for these f.p.-integrals is proved.

Most of the results given in Chapters 3-5 are published in [23, 24], the thesis also contains new results which have not been published yet.

Chapter 2

Notations and Basic Results

In this chapter we introduce notations and formulate results that we require later.

2.1 Notations

Throughout this thesis, c, c', c_1, ... denote positive constants which may have different values in different occurrences. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the set of all positive integers, by $\mathbb{N}_0 = \{0, 1, \dots\}$ the set of all non-negative integers, by $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ the set of integers, by $\mathbb{R} = (-\infty, \infty)$ the set of all real numbers and by $\mathbb{C} = \mathbb{R} + \mathrm{i}\mathbb{R}$ the set of all complex numbers, where $\lambda = \mathrm{Re}\lambda + \mathrm{i}\mathrm{Im}\lambda$ for $\lambda \in \mathbb{C}$ and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Additionally, by I we denote the identity mapping and by $D^k = \left(\frac{d}{dt}\right)^k$, $k \in \mathbb{N}$, $D^0 = I$, the differentiation operator.

By $C^m = C^m[0, T]$, $T \in (0, \infty)$, $m \in \mathbb{N}_0$, $C^0 = C$, we denote the Banach space of m times continuously differentiable functions u on $[0, T]$ with the norm

$$\|u\|_{C^m} = \max_{0 \le k \le m} \max_{0 \le t \le T} |u^{(k)}(t)|.$$

By $C^{m,r} = C^{m,r}(0, T]$ (see [37]), $m \in \mathbb{N}_0$, $r \in \mathbb{R}$, we denote the set of m times continuously differentiable functions u on $(0, T]$ such that finite limits $\lim_{t\to 0} t^{k-r} u^{(k)}(t)$ $(k = 0, 1, \dots, m)$ exist. The set $C^{m,r}$ is a Banach space with the norm

$$\|u\|_{C^{m,r}} = \max_{0 \le k \le m} \sup_{0 < t \le T} t^{k-r} |u^{(k)}(t)|.$$

In particular, $C = C^{0,0}$ and $C^m \subset C^{m,0}$. Also, the space $C^{m,m}$ has the form

$$C^{m,m} = \{u \in C^m \mid u(0) = \ldots = u^{(m-1)}(0) = 0\} \subset C^m, \quad m \in \mathbb{N}. \tag{2.1.1}$$

Indeed, let $u \in C^{m,m}$, $m \in \mathbb{N}$, then u is an m times continuously differentiable function on $(0, T]$ and the finite limits $\lim_{t\to 0} t^{k-m} u^{(k)}(t)$ $(k = 0, 1, \dots, m)$ exist.

Hence, $u^{(m)} \in C$, i.e. $u \in C^m$. Moreover, $|u^{(k)}(t)| \le ct^{m-k}$, i.e. $u^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m-1$. Thus

$$C^{m,m} \subset \{u \in C^m \mid u(0) = \ldots = u^{(m-1)}(0) = 0\}.$$

Conversely, let $u \in C^m$ and $u(0) = \ldots = u^{(m-1)}(0) = 0$. Then (see e.g. [25])

$$u(t) = \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}u^{(m)}(s)ds, \quad 0 \le t \le T.$$

This yields

$$\begin{aligned} u^{(k)}(t) &= \frac{1}{(m-k-1)!}\int_0^t (t-s)^{m-k-1}u^{(m)}(s)ds \\ &= \frac{u^{(m)}(\theta t)}{(m-k-1)!}\int_0^t (t-s)^{m-k-1}ds \\ &= -\frac{u^{(m)}(\theta t)}{(m-k)!}(t-s)^{m-k}\Big|_0^t = \frac{t^{m-k}}{(m-k)!}u^{(m)}(\theta t) \end{aligned} \tag{2.1.2}$$

for $0 \le t \le T$, $0 < \theta < 1$, $k = 0, 1, \ldots, m-1$. Consequently,

$$\lim_{t\to 0} t^{k-m}u^{(k)}(t) = \frac{1}{(m-k)!}\lim_{t\to 0} u^{(m)}(\theta t) = \frac{u^{(m)}(0)}{(m-k)!}, \quad k = 0, 1, \ldots, m-1,$$

thus finite limits $\lim_{t\to 0} t^{k-m}u^{(k)}(t)$ for $k = 0, 1, \ldots, m-1$ exist and we have that

$$\{u \in C^m \mid u(0) = \ldots = u^{(m-1)}(0) = 0\} \subset C^{m,m}.$$

Hence, (2.1.1) holds.

For $m \ge 1$, it holds that C^m is representable as a direct sum

$$C^m = C^{m,m} \oplus \mathcal{P}_{m-1}, \tag{2.1.3}$$

where $C^{m,m}$ is defined by (2.1.1) and $\mathcal{P}_{m-1}$ is the space of all polynomials with degree less than or equal to $m-1$.

Indeed, let

$$(\Pi_{m-1}u)(t) = \sum_{k=0}^{m-1} \frac{u^{(k)}(0)}{k!}t^k, \quad 0 \le t \le T, \quad u \in C^m \tag{2.1.4}$$

be the Taylor projector in C^m. For $u \in C^m$, we have that $u - \Pi_{m-1}u$ satisfies

$$(u - \Pi_{m-1}u)^{(k)}(0) = 0, \quad k = 0, 1, \ldots, m-1,$$

hence $u - \Pi_{m-1}u \in C^{m,m}$ and

$$C^m = (I - \Pi_{m-1})C^m \oplus \Pi_{m-1}C^m = C^{m,m} \oplus \mathcal{P}_{m-1}, \quad m \ge 1.$$

Moreover, for $u \in C^{m,m}$, we have (see (2.1.2))

$$\|u\|_{C^{m,m}} = \max_{0\le k\le m} \sup_{0<t\le T} t^{k-m}|u^{(k)}(t)| \le \max_{0\le k\le m} \frac{1}{(m-k)!}\|u^{(m)}\|_C \le \|u\|_{C^m};$$

on the other hand,

$$\|u\|_{C^m} = \max_{0\le k\le m} \max_{0\le t\le T} t^{m-k}t^{k-m}|u^{(k)}(t)| \le \max\{1, T^m\}\|u\|_{C^{m,m}}.$$

Thus on the subspace $C^{m,m}$ the norms of $C^{m,m}$ and C^m are equivalent and actually concur if $T < 1$.

By $H^{m,\alpha} = H^{m,\alpha}[0,T]$, $m \in \mathbb{N}_0$, $\alpha \in [0,\infty)$, we denote the class of functions $a \in C^m$ satisfying

$$\left|a^{(m)}(t) - a^{(m)}(0)\right| \le ct^\alpha, \quad 0 \le t \le T. \tag{2.1.5}$$

By $L^1(0,1)$, we denote the Banach space consisting of functions $\varphi : (0,1) \to \mathbb{R}$ such that

$$\|\varphi\|_1 = \int_0^1 |\varphi(x)| < \infty.$$

By $L^{1,r}(0,1)$, $r \in \mathbb{R}$, we denote the weighted Banach space consisting of functions $\varphi : (0,1) \to \mathbb{R}$ such that

$$\|\varphi\|_{L^{1,r}} = \int_0^1 x^r|\varphi(x)|dx < \infty.$$

Note that, $L^{1,0}(0,1) = L^1(0,1)$.

Let X and Y be Banach spaces. The notation $\mathcal{L}(X,Y)$ stands for the space of linear bounded operators from X to Y, and $\mathcal{L}(X) = \mathcal{L}(X,X)$. By $\rho_{\mathcal{L}(X)}(V)$ we denote the resolvent set of operator $V \in \mathcal{L}(X)$, and by $\sigma_{\mathcal{L}(X)}(V) = \mathbb{C} \setminus \rho_{\mathcal{L}(X)}(V)$ its spectrum. In the cases $X = C^m$ and $X = C^{m,r}$ we use abbreviated notations

$$\sigma_m(V) = \sigma_{\mathcal{L}(C^m)}(V),\ \rho_m(V) = \rho_{\mathcal{L}(C^m)}(V) \text{ for } V \in \mathcal{L}(C^m),\ m \in \mathbb{N}_0;$$
$$\sigma_{m,r}(V) = \sigma_{\mathcal{L}(C^{m,r})}(V),\ \rho_{m,r}(V) = \rho_{\mathcal{L}(C^{m,r})}(V) \text{ for } V \in \mathcal{L}(C^{m,r}),\ m \in \mathbb{N}_0,\ r \in \mathbb{R}.$$

By M^α, $\alpha \in \mathbb{R}$, we denote the multiplication operator:

$$(M^\alpha u)(t) = t^\alpha u(t), \quad 0 < t \le T, \quad u \in C \text{ (or } u \in C^{m,r}). \tag{2.1.6}$$

The notations Γ and B stand for the gamma function and beta function respectively. The Euler Gamma function Γ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \quad z \in \mathbb{C},\ \mathrm{Re}z > 0.$$

The Euler beta function B is defined by the integral

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad x,y\in\mathbb{C},\ \mathrm{Re}x>0,\ \mathrm{Re}y>0. \tag{2.1.7}$$

It holds (see e.g. [30])

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.1.8}$$

We end this section with the proof of the following result which we require later in our discussions.

Proposition 2.1.1. *Let f be two times continuously differentiable and absolutely integrable on $(0,\infty)$, i.e. $f\in C^2(0,\infty)\cap L^1(0,\infty)$. If $f''(u)\geq 0$, $0<u<\infty$, then*

$$\int_0^\infty f(u)\cos(yu)du\geq 0\quad \forall y\in\mathbb{R}\setminus\{0\}. \tag{2.1.9}$$

Proof. It is sufficient to prove (2.1.9) for $y=1$; for an arbitrary $y>0$ statement (2.1.9) follows by a suitable change of variables and for $y<0$ due to the evenness of function $\cos(yu)$. So let $y=1$. We have

$$\int_0^\infty f(u)\cos(u)du=\sum_{k=0}^\infty\int_{2k\pi}^{2(k+1)\pi} f(u)\cos(u)du. \tag{2.1.10}$$

Now,

$$\begin{aligned}\int_{2k\pi}^{2(k+1)\pi} f(u)\cos(u)du &= \int_{2k\pi}^{(2k+1)\pi} f(u)\cos(u)du+\int_{(2k+1)\pi}^{2(k+1)\pi} f(u)\cos(u)du\\ &=\int_{2k\pi}^{(2k+1)\pi}(f(u)-f(u+\pi))\cos(u)du\\ &=\int_{2k\pi}^{2k\pi+\frac{\pi}{2}}(f(u)-f(u+\pi))|\cos(u)|du-\int_{2k\pi+\frac{\pi}{2}}^{(2k+1)\pi}(f(u)-f(u+\pi))|\cos(u)|du\\ &=\int_{2k\pi}^{2k\pi+\frac{\pi}{2}}\left[(f(u)-f(u+\pi))-\left(f\left(u+\frac{\pi}{2}\right)-f\left(u+\pi+\frac{\pi}{2}\right)\right)\right]|\cos u|du.\end{aligned}$$

Under our assumptions, we get that

$$f(u)-f\left(u+\frac{\pi}{2}\right)=-f'(u_1)\frac{\pi}{2},\quad u_1\in(2k\pi,(2k+1)\pi)$$

and

$$f(u+\pi) - f\left(u+\pi+\frac{\pi}{2}\right) = -f'(u_2)\frac{\pi}{2}, \quad u_2 \in ((2k+1)\pi, 2(k+1)\pi).$$

Hence,

$$\left[(f(u) - f(u+\pi)) - \left(f\left(u+\frac{\pi}{2}\right) - f\left(u+\pi+\frac{\pi}{2}\right)\right)\right]$$
$$= \left(f'(u_2) - f'(u_1)\right)\frac{\pi}{2} = f''(\overline{u})(u_2-u_1)\frac{\pi}{2}, \quad \overline{u} \in (u_1, u_2).$$

Since $f''(u) \geq 0$, $0 < u < \infty$ and $u_1 < u_2$ we have

$$\int\limits_{2k\pi}^{2k\pi+\frac{\pi}{2}} \left[(f(u) - f(u+\pi)) - \left(f\left(u+\frac{\pi}{2}\right) - f\left(u+\pi+\frac{\pi}{2}\right)\right)\right] |\cos u| du \geq 0.$$

In conclusion, we have shown that every term in the sum on the right side of (2.1.10) is greater than or equal to 0, hence (2.1.9) holds for $y = 1$. □

2.2 Fractional differentiation operators

In this section we present definitions and some results from the theory of fractional differentiation operators. Our treatment is based on articles [39, 40]. A reader interested in additional works on fractional differentiation operators may consult for example [8, 17, 30].

In the present thesis the fractional differentiation operator D_0^α, of the order $\alpha \in [0, \infty)$, is defined as the inverse of the Riemann-Liouville integral operator J^α on $J^\alpha C$, i.e.

$$D_0^\alpha := (J^\alpha)^{-1}, \quad \alpha \geq 0. \tag{2.2.1}$$

The Riemann-Liouville fractional integral operator J^α is given by

$$(J^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad u \in C, \quad t > 0, \ \alpha > 0; \quad J^0 = I. \tag{2.2.2}$$

For $\alpha = m \in \mathbb{N}$, the operator D_0^m is the restriction of D^m to the subspace $C^{m,m}$. It is known (see e.g. [8]): if $\alpha > 0$, $\beta > 0$, then

$$\left(J^\alpha J^\beta u\right)(t) = \left(J^{\alpha+\beta} u\right)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} u(s) ds$$

where $0 < t \leq T$, $u \in C$. Consequently, also

$$D_0^\alpha D_0^\beta = D_0^{\alpha+\beta} \text{ for } \alpha > 0, \ \beta > 0 \tag{2.2.3}$$

and

$$D_0^\alpha J^\beta = J^{\beta-\alpha} \text{ for } \alpha < \beta,\ \alpha > 0,\ \beta > 0. \tag{2.2.4}$$

Property (2.2.3) is important in theoretical considerations and it does not hold for Riemann-Liouville and Caputo fractional differentiation operators.

The Riemann-Liouville fractional differentiation operator $D^\alpha_{\text{R-L}}$ of order $\alpha > 0$, $m < \alpha \le m+1$, $m \in \mathbb{N}_0$, is determined by the formula (see e.g. [8, 30])

$$D^\alpha_{\text{R-L}} u = D^{m+1} J^{m+1-\alpha} u \text{ provided that } J^{m+1-\alpha} u \in C^{m+1},\ u \in C^m. \tag{2.2.5}$$

For $\alpha > 0$, $m < \alpha \le m+1$, $m \in \mathbb{N}_0$, it holds (see [39, 40])

$$D_0^\alpha u = D^{m+1} J^{m+1-\alpha} u \text{ for } u \in C \text{ such that } J^{m+1-\alpha} u \in C^{m+1,m+1}. \tag{2.2.6}$$

This can be considered as a definition equivalent to (2.2.1).

Note, that the difference between (2.2.6) and (2.2.5) is that in (2.2.5) it is assumed that $J^{m+1-\alpha}u \in C^{m+1}$ instead of $J^{m+1-\alpha}u \in C^{m+1,m+1}$ in (2.2.6). Hence a D_0^α-differentiable function u is also $D^\alpha_{\text{R-L}}$-differentiable and $D_0^\alpha u = D^\alpha_{\text{R-L}} u$. Proposition 2.2.1 below shows that the inverse is also true for functions $u \in C^m$. For $u \in C^k$, $k < m$, the situation is more complicated [40].

Proposition 2.2.1 (See [39, 40])**.** *For $m < \alpha \le m+1$, $m \in \mathbb{N}_0$, a function $u \in C^m$ is $D^\alpha_{\text{R-L}}$-differentiable if and only if u is D_0^α-differentiable. Besides $D^\alpha_{\text{R-L}} u = D_0^\alpha u$.*

Proof. We represent the proof given in [40]. Let $u \in C^m$ be $D^\alpha_{\text{R-L}}$-differentiable, i.e. $J^{m+1-\alpha}u \in C^{m+1}$ (see (2.2.5)). We have to establish that $J^{m+1-\alpha}u \in C^{m+1,m+1}$, i.e. $(J^{m+1-\alpha}u)^{(k)}(0) = 0$, $k = 0, 1, \ldots, m$; then $J^{m+1-\alpha}u \in C^{m+1,m+1}$ and according to (2.2.6) the function u is D_0^α-differentiable. It holds

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}u)(t) = \int_0^t (t-s)^{m-\alpha} u(s)ds = \int_0^t s^{m-\alpha} u(t-s)ds,$$

where $0 \le t \le T$. Clearly, $(J^{1-\alpha}u)(0) = 0$. For $m \ge 1$, $u \in C^1$ the last integral admits to differentiation, and we obtain

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}u)'(t) = u(0)t^{m-\alpha} + \int_0^t s^{m-\alpha} u'(t-s)ds, \quad 0 \le t \le T.$$

The function $t^{m-\alpha}$ has a singularity at $t = 0$, whereas the integral term belongs to C. Since $(J^{m+1-\alpha}u)' \in C$ by definition, we conclude that $u(0) = 0$ and

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}u)'(t) = \int_0^t s^{m-\alpha} v'(t-s)ds, \quad 0 \le t \le T.$$

If $m \geq 2$, we can repeat the argument with differentiation. We obtain recursively that $u^{(k-1)}(0) = 0$, $k = 1, 2, \ldots, m$, and

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}u)^{(k)}(t) = \int_0^t s^{m-\alpha}u^{(k)}(t-s)ds = \int_0^t (t-s)^{m-\alpha}u^{(k)}(s)ds$$

for $0 \leq t \leq T$, $k = 0, 1, \ldots, m$. We conclude that $(J^{m+1-\alpha}u)(0) = 0$ holds for $k = 0, 1, \ldots, m$. □

The Caputo fractional differentiation operator D^{α}_{Cap} of order $\alpha > 0$, where $m < \alpha \leq m+1$, $m \in \mathbb{N}_0$, is defined by (see e.g. [8])

$$D^{\alpha}_{\text{Cap}}u = D^{m+1}J^{m+1-\alpha}(u - \Pi_m u). \tag{2.2.7}$$

Here $u \in C^m$ is such that $J^{m+1-\alpha}(u - \Pi_m u) \in C^{m+1}$ with Π_m determined by (2.1.4). For $u \in C^{m+1}$, this is equivalent to $D^{\alpha}_{\text{Cap}}u = J^{m+1-\alpha}D^{m+1}u$ (c.f. [22, 30]).

Proposition 2.2.2 (See [39, 40])**.** *A function $u \in C^m$ has the Caputo fractional derivative $D^{\alpha}_{\text{Cap}}u \in C$, $m < \alpha \leq m+1$, $m \in \mathbb{N}_0$, if and only if $u - \Pi_m u$ has the fractional derivative $D^{\alpha}_0(u - \Pi_m u) \in C$. Besides $D^{\alpha}_{\text{Cap}}u = D^{\alpha}_0(u - \Pi_m u)$.*

Proof. We represent the proof given in [40]. For $m < \alpha \leq m+1$, conditions $u \in C^m$, $u^{(k)}(0) = 0$, $k = 0, 1, \ldots, m$, imply that,

$$(J^{m+1-\alpha}u)(0) = 0, \quad k = 0, 1, \ldots, m.$$

Therefore the presumption $J^{m+1-\alpha}(u - \Pi_m u) \in C^{m+1}$ (see (2.2.7)) is equivalent to $J^{m+1-\alpha}(u - \Pi_m u) \in C^{m+1,m+1}$. Thus $D^{\alpha}_0(u - \Pi_m u)$ is well defined by (2.2.6), and (2.2.7) can be continued as follows:

$$D^{\alpha}_{\text{Cap}}u = D^{m+1}J^{m+1-\alpha}(u - \Pi_m u) = D^{\alpha}_0(u - \Pi_m u).$$

□

2.3 Cordial Volterra integral operators

In this section we introduce definitions and some required results from the theory of cordial Volterra integral operators (see [34, 35, 37]).

The cordial Volterra integral operator V_φ with a core $\varphi \in L^1(0,1)$ is defined by

$$(V_\varphi u)(t) = \int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right)u(s)ds = \int_0^1 \varphi(x)u(tx)dx, \quad 0 \leq t \leq T, \quad u \in C. \tag{2.3.1}$$

Denote

$$\widehat{\varphi}(\lambda) = \int_0^1 x^\lambda \varphi(x)dx \tag{2.3.2}$$

for such $\lambda \in \mathbb{C}$ where the integral converges. From the second form of (2.3.1), it follows that

$$V_\varphi w_\lambda = \widehat{\varphi}(\lambda) w_\lambda, \quad \text{where } w_\lambda(t) = t^\lambda,\ 0 < t \leq T. \tag{2.3.3}$$

By differentiating the second form of (2.3.1) we have

$$(V_\varphi u)^{(m)}(t) = \int_0^1 \varphi(x) x^m u^{(m)}(tx)dx, \quad 0 \leq t \leq T,\ u \in C^m,\ m \geq 0. \tag{2.3.4}$$

In particular,

$$(V_\varphi u)^{(m)}(0) = \widehat{\varphi}(m) u^{(m)}(0), \quad u \in C^m,\ m \geq 0. \tag{2.3.5}$$

We also get

$$(aV_\varphi u)^{(m)}(t) = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} a^{(m-i)}(t) \int_0^1 \varphi(x) x^i u^{(i)}(tx)dx,\ 0 \leq t \leq T, \tag{2.3.6}$$

for $a, u \in C^m$, $m \geq 0$.

Denote

$$(M_a u)(t) = a(t)u(t), \quad 0 \leq t \leq T,\ a, u \in C^m,\ m \geq 0.$$

Theorem 2.3.1 (See [34, 35])**.** *For $\varphi \in L^1(0,1)$, $a \in C^m$, $m \geq 0$, it holds that $V_\varphi, V_\varphi M_a, M_a V_\varphi \in \mathcal{L}(C^m)$ and*

$$\sigma_0(V_\varphi) = \{0\} \cup \{\widehat{\varphi}(\lambda) \,|\, \mathrm{Re}\lambda \geq 0\}, \tag{2.3.7}$$

$$\sigma_m(V_\varphi) = \{0\} \cup \{\widehat{\varphi}(\lambda) \,|\, \mathrm{Re}\lambda \geq m\} \cup \{\widehat{\varphi}(j) \,|\, j = 0, 1, \ldots, m-1\} \text{ for } m \geq 1, \tag{2.3.8}$$

$$\sigma_m(V_\varphi M_a) = a(0)\sigma_m(V_\varphi) = \sigma_m(M_a V_\varphi) \text{ for } m \geq 0. \tag{2.3.9}$$

Moreover, $\|V_\varphi\|_{\mathcal{L}(C)} = \|\varphi\|_1$ and $\|V_\varphi M_a\|_{\mathcal{L}(C)} \leq \|a\|_\infty \|\varphi\|_1$ with

$$\|\varphi\|_1 = \int_0^1 |\varphi(x)|dx, \quad \|a\|_\infty = \max_{0 \leq t \leq T} |a(t)|.$$

If $a(0) = 0$, then operators $V_\varphi M_a, M_a V_\varphi \in \mathcal{L}(C^m)$ are compact and it holds $\sigma_m(V_\varphi M_a) = \sigma_m(M_a V_\varphi) = \{0\}$.

Theorem 2.3.2 (See [37])**.** *Assume that $\varphi \in L^{1,r}(0,1)$, $a \in C^m$, $m \geq 0$, $r \in \mathbb{R}$. Then $V_\varphi, V_\varphi M_a, M_a V_\varphi \in \mathcal{L}(C^{m,r})$ and*

$$\sigma_{m,r}(V_\varphi) = \{0\} \cup \{\widehat{\varphi}(\lambda) \mid \mathrm{Re}\lambda \geq r\}, \tag{2.3.10}$$

$$\sigma_{m,r}(V_\varphi M_a) = a(0)\sigma_{m,r}(V_\varphi) = \sigma_{m,r}(M_a V_\varphi), \tag{2.3.11}$$

also (2.3.4), (2.3.5) and (2.3.6) hold true for $u \in C^{m,r}$. If $a(0) = 0$, then operators $V_\varphi M_a, M_a V_\varphi \in \mathcal{L}(C^{m,r})$ are compact and $\sigma_{m,r}(V_\varphi M_a) = \sigma_{m,r}(M_a V_\varphi) = \{0\}$.

Proposition 2.3.1 (See [34]). *For $\varphi \in L^1(0,1)$, $\mu \notin \sigma_0(V_\varphi)$ it holds*

$$(\mu I - V_\varphi)^{-1} = \mu^{-1} I + V_\psi$$

where $\psi \in L^1(0,1)$ is uniquely determined by μ and φ.

Proposition 2.3.2 (See [34]). *For $\varphi \in L^1(0,1)$, $\mu \in \sigma_0(V_\varphi)$, $\mu \neq \widehat{\varphi}(0)$, the set $(\mu I - V_\varphi)C$ is dense in C. For $\mu = \widehat{\varphi}(0)$, the functions $f \in (\mu I - V_\varphi)C$ satisfy $f(0) = 0$, hence the set $(\widehat{\varphi}(0)I - V_\varphi)C$ is not dense in C.*

Proposition 2.3.3 (See [34]). *For $\varphi \in L^1(0,1)$, the operator*

$$\mu I - V_\varphi : C \to C$$

has the right hand inverse if and only if

$$\mu - \widehat{\varphi}(\mathrm{i}\xi) \neq 0 \ for\ any\ \xi \in \mathbb{R};$$

further, $\mu I - V_\varphi : C \to C$ has the (two side) inverse if and only if, in addition,

$$\arg[\mu - \widehat{\varphi}(\mathrm{i}\xi)]_{\xi=-\infty}^{\infty} = 0.$$

For $\alpha > 0$ and $u \in C$ it holds that

$$(J^\alpha u)(t) = \frac{t^\alpha}{\Gamma(\alpha)} \int_0^t \frac{1}{t} \left(1 - \frac{s}{t}\right)^{\alpha-1} u(s)ds, \quad 0 < t \leq T,$$

hence $M^{-\alpha} J^\alpha$ with $M^{-\alpha}$ given by (2.1.6) *is for any $\alpha > 0$ a cordial Volterra integral operator with the core*

$$\varphi(x) = \frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1} \in L^1(0,1).$$

2.4 Fredholm type operators

In this section we recall the definition and some results from the theory of Fredholm operators of index $\kappa \in \mathbb{Z}$ in a Banach space (see e.g. [32]).

Definition 2.4.1. For a Banach space X, an operator $A \in \mathcal{L}(X)$ is called Fredholm (or, Noether) if its null-space $\mathcal{N}(A) := \{u \in X \mid Au = 0\}$ is finite dimensional, and its range $\mathcal{R}(A) = AX$ is closed and of a finite codimension in X; the integer $\dim \mathcal{N}(A) - \operatorname{codim} \mathcal{R}(A)$ is called index of A. By $\Phi_\kappa(X)$ we denote the class of Fredholm operators of index $\kappa \in \mathbb{Z}$.

Here, $\operatorname{codim} \mathcal{R}(A) = \dim(X/\mathcal{R}(A))$ and $X/\mathcal{R}(A)$ is the factor space of X over $\mathcal{R}(A)$.

Proposition 2.4.1 (See e.g. [32]). *For $A \in \mathcal{L}(X)$ the following conditions are equivalent:*

a) $A \in \Phi_0(X)$;

b) *A admits a representation $A = B + K$ where $B \in \mathcal{L}(X)$ possesses the inverse $B^{-1} \in \mathcal{L}(X)$ and $K \in \mathcal{L}(X)$ is compact.*

Corollary 2.4.1. *Let $\mu I - A \in \Phi_0(X)$ for a $\mu \in \mathbb{C}$. If $\mathcal{N}(\mu I - A) = \{0\}$ then $\mu \in \rho_{\mathcal{L}(X)}(A)$.*

Corollary 2.4.2. *Suppose that $\mu I - A \notin \Phi_0(X)$ for a $\mu \in \mathbb{C}$. Then $\mu \in \sigma_{\mathcal{L}(X)}(A)$.*

Corollary 2.4.3. *The set $\Phi_0(X)$ is open in $\mathcal{L}(X)$.*

The following proposition is a simple consequence of Definition 2.4.1.

Proposition 2.4.2. *Let X be representable as a direct sum $X = X_0 \oplus X_1$, where X_0 and X_1 are closed subspaces of X. Let $A \in \mathcal{L}(X)$ be such that $AX_0 \subset X_0$, $AX_1 \subset X_1$. Then $\sigma_{\mathcal{L}(X)}(A) = \sigma_{\mathcal{L}(X_0)}(A_0) \cup \sigma_{\mathcal{L}(X_1)}(A_1)$, where $A_0 = A|_{X_0} \in \mathcal{L}(X_0)$ is the restriction of A onto X_0 and $A_1 = A|_{X_1} \in \mathcal{L}(X_1)$ is the restriction of A onto X_1. Furthermore, $A \in \Phi_\kappa(X)$ if and only if $A_1 \in \Phi_\kappa(X_1)$, provided that X_0 is finite dimensional.*

2.5 Finite part of a divergent integral

In this section we present some definitions and results from the theory of divergent integrals (see e.g. [41, 42])

Consider an integral

$$\int_0^b g(y)v(y)dy, \tag{2.5.1}$$

which may be divergent due to a singularity of $g(y)v(y)$ at $y = 0$. Here g is continuous on $(0, b]$ and satisfies the inequality

$$|g(y)| \leq c|y|^{-\mu}, \quad \mu \geq 1, \quad \mu \in \mathbb{R},$$

whereas $v \in C^{m-1}[0, b) \cap L^1(0, b)$ $(m \in \mathbb{N})$ satisfies the conditions

$$|v^{(m-1)}(y) - v^{(m-1)}(0)| \leq c'y^\alpha, \ 0 \leq y < b, \ 0 < \alpha \leq 1, \ m + \alpha > \mu.$$

Let $g_j(y)$ be an anti-derivative of function $y^j g(y)$, $j = 0, 1, \ldots, m-1$. The

Hadamard finite part of integral (2.5.1) can be defined as (see e.g. [41])

$$\begin{aligned}
&\text{f.p.} \int_0^b g(y)v(y)dy \\
&\quad = \int_0^b g(y)\left[v(y) - \sum_{j=0}^{m-1} \frac{v^j(0)}{j!} y^j\right] dy + \sum_{j=0}^{m-1} \frac{v^j(0)}{j!} \text{f.p.} \int_0^b y^j g(y)dy \\
&\quad = \int_0^b g(y)\left[v(y) - \sum_{j=0}^{m-1} \frac{v^j(0)}{j!} y^j\right] dy + \sum_{j=0}^{m-1} \frac{v^j(0)}{j!} [g_j(b) - \overline{g}_j(0)],
\end{aligned} \tag{2.5.2}$$

where $\overline{g}_j(0) = g_j(0)$ if the finite limit $g_j(0) = \lim_{y\to 0} g_j(y)$ exists and otherwise $\overline{g}_j(0) = 0$.

Consider an integral

$$\int_0^T a(t)t^{-\lambda-1}dt \quad (0 < T < \infty) \tag{2.5.3}$$

depending on $\lambda \in \mathbb{C}$, under assumptions

$$a \in H^{m,\alpha}, \quad m \in \mathbb{N}_0, \quad 0 < \alpha \leq 1. \tag{2.5.4}$$

For $\text{Re}\lambda < 0$ integral (2.5.3) converges; for $\text{Re}\lambda \geq 0$ it generally diverges.

For $\lambda \in \mathbb{C}$ with $\text{Re}\lambda < m + \alpha$ the finite part of (2.5.3) is defined in terms of Taylor expansions (see [42]): if $\lambda \in \mathbb{C} \setminus \mathbb{N}_0$, then

$$\begin{aligned}
\text{f.p.} \int_0^T a(t)t^{-\lambda-1}dt = &\int_0^T \left[a(t) - \sum_{k=0}^{m} \frac{1}{k!} a^{(k)}(0)t^k\right] t^{-\lambda-1}dt \\
&+ \sum_{k=0}^{m} \frac{1}{k!} a^{(k)}(0) \frac{T^{k-\lambda}}{k-\lambda};
\end{aligned} \tag{2.5.5}$$

while if $\lambda = l \in \mathbb{N}_0$, $l < m$, then

$$\begin{aligned}
\text{f.p.} \int_0^T a(t)t^{-l-1}dt = &\int_0^T \left[a(t) - \sum_{k=0}^{m} \frac{1}{k!} a^{(k)}(0)t^k\right] t^{-l-1}dt \\
&+ \sum_{\substack{k=0 \\ k\neq l}}^{m} \frac{1}{k!} a^{(k)}(0) \frac{T^{k-l}}{k-l} + \frac{1}{l!} a^{(l)}(0) \ln T.
\end{aligned}$$

These definitions have two crucial consequences (see [42]). Firstly, the finite part integral defined by (2.5.5) is the analytic continuation of integral (2.5.3) from $\text{Re}\lambda < 0$ into

$$\{\lambda \in \mathbb{C} \setminus \mathbb{N}_0 \mid \text{Re}\lambda < m + \alpha\}.$$

Secondly, for $\lambda = l \in \mathbb{N}_0$, it holds

$$\text{f.p.} \int_0^T a(t)t^{-l-1}dt = \lim_{\substack{\lambda \to l \\ \lambda \notin \mathbb{N}_0}} \frac{d}{d\lambda}\left[(\lambda - l)\,\text{f.p.} \int_0^T a(t)t^{-\lambda-1}dt\right].$$

For the change of variables $t = g(\rho)$ with

$$g \in H^{m+1,\alpha}[0, T_*], \quad g(0) = 0, \quad g(T_*) = T, \quad g'(\rho) > 0, \quad 0 \leq \rho \leq T_* \tag{2.5.6}$$

the following result holds (see [42]).

Theorem 2.5.1. *Assume that* (2.5.4) *and* (2.5.6) *hold. Then for* $\text{Re}\lambda < m + \alpha$,

$$\text{f.p.} \int_0^T a(t)t^{-\lambda-1}dt = \text{f.p.} \int_0^{T_*} a_*(\rho, \lambda)\rho^{-\lambda-1}d\rho + \Pi_*(\lambda),$$

where

$$a_*(\rho, \lambda) = a(g(\rho))\left(\frac{g(\rho)}{\rho}\right)^{-\lambda-1} g'(\rho)$$

and

$$\Pi_*(\lambda) = \begin{cases} 0, & \lambda \in \mathbb{C} \setminus \mathbb{N}_0, \\ -\dfrac{1}{l!}\dfrac{\partial}{\partial\lambda}\left(\dfrac{\partial}{\partial\rho}\right)^l a_*(\rho, \lambda)\Big|_{\rho=0, \lambda=l}, & \lambda = l \in \mathbb{N}_0. \end{cases}$$

Chapter 3

Equation with main term $D_0^\alpha M^\alpha$

In this chapter, we study the unique solvability of a singular fractional differential equation of the form

$$(D_0^\alpha M^\alpha u)(t) = \sum_{k=1}^{l} b_k(t)(D_0^{\alpha_k} M^{\alpha_k} u)(t) + f(t), \quad 0 < t \le T, \tag{1.1.3}$$

where $f \in C^m$, $\alpha, \alpha_k \in \mathbb{R}$ and

$$m < \alpha \le m+1, \quad \alpha > \alpha_k \ge 0, \quad b_k \in C^m, \quad k = 1, 2, \ldots, l, \quad m \in \mathbb{N}_0. \tag{1.1.4}$$

Our approach is based on ideas and results of [24].

In Section 3.1 we first consider a simplified version of equation (1.1.3) with constant coefficients and prove Theorem 3.1.1 about the unique solvability in C^m of this type of equation. To consider equations (1.1.3) we formulate an auxiliary result, Lemma 3.1.1. In Theorem 3.1.2 we give the unique solvability conditions for equations (1.1.3) under assumptions (1.1.4). We conclude Section 3.1 with two simple examples (Examples 3.1.1 and 3.1.2) of equations that have the form (1.1.3).

Section 3.2 is dedicated to the proof of Lemma 3.1.1.

3.1 The main results for equation with main term $D_0^\alpha M^\alpha$

Our aim is analyze the unique solvability of (1.1.3) in C^m under conditions (1.1.4). To this end we start off by considering the simplified version of equation (1.1.3) with constant coefficients:

$$(D_0^\alpha M^\alpha u)(t) = \sum_{k=1}^{l} b_k(0)(D_0^{\alpha_k} M^{\alpha_k} u)(t) + f(t), \quad 0 < t \le T. \tag{3.1.1}$$

Here $\alpha, \alpha_k \in \mathbb{R}$, $f \in C^m$ and

$$m < \alpha \le m+1, \quad \alpha > \alpha_k \ge 0, \quad k = 1, 2, \dots, l, \quad m \in \mathbb{N}_0. \tag{3.1.2}$$

We make in (3.1.1) the change of variables $v = D_0^\alpha M^\alpha u$, i.e.

$$u = (D_0^\alpha M^\alpha)^{-1} v = M^{-\alpha} J^\alpha v,$$

obtaining with respect to v the equation

$$v = \sum_{k=1}^{l} b_k(0)[D_0^{\alpha_k} M^{\alpha_k}][M^{-\alpha} J^\alpha] v + f. \tag{3.1.3}$$

Note that for any $v \in C$ and $k = 1, 2, \dots, l$, function $M^{-\alpha} J^\alpha v$ belongs to the domain of operator $D_0^{\alpha_k} M^{\alpha_k}$, or to the range of $(D_0^{\alpha_k} M^{\alpha_k})^{-1} = M^{-\alpha_k} J^{\alpha_k}$, i.e., there exists a $w \in C$ such that

$$M^{-\alpha} J^\alpha v = M^{-\alpha_k} J^{\alpha_k} w. \tag{3.1.4}$$

Namely, we claim below that the last equality holds for $w = V_{\varphi_{\alpha,\alpha_k}} v$, where $V_{\varphi_{\alpha,\alpha_k}}$ is a cordial Volterra integral operator with the core

$$\varphi_{\alpha,\alpha_k}(x) = \frac{1}{\Gamma(\alpha - \alpha_k)} (1-x)^{\alpha - \alpha_k - 1} x^{\alpha_k}, \quad \varphi_{\alpha,\alpha_k} \in L^1(0,1). \tag{3.1.5}$$

In other words, we claim that there holds the equality of cordial Volterra integral operators:

$$M^{-\alpha} J^\alpha = [M^{-\alpha_k} J^{\alpha_k}] V_{\varphi_{\alpha,\alpha_k}}. \tag{3.1.6}$$

The three cordial Volterra integral operators $M^{-\alpha} J^\alpha$, $M^{-\alpha_k} J^{\alpha_k}$ and $V_{\varphi_{\alpha,\alpha_k}}$ in (3.1.6) are well-defined and bounded in C, hence (3.1.6) holds if

$$M^{-\alpha} J^\alpha w_n = [M^{-\alpha_k} J^{\alpha_k}] V_{\varphi_{\alpha,\alpha_k}} w_n, \quad \forall n \in \mathbb{N}_0, \tag{3.1.7}$$

where $w_n(t) = t^n$, $0 \le t \le T$. By (2.2.2), (2.1.7) and (2.1.8), we have

$$\begin{aligned}(M^{-\alpha} J^\alpha w_n)(t) &= \frac{t^{-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^n ds = \frac{t^n}{\Gamma(\alpha)} \int_0^t \frac{1}{t} \left(1 - \frac{s}{t}\right)^{\alpha-1} \left(\frac{s}{t}\right)^n ds \\ &= \frac{t^n}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} x^n dx = \frac{1}{\Gamma(\alpha)} B(\alpha, n+1) w_n(t) \\ &= \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} w_n(t), \quad 0 \le t \le T, \ n \in \mathbb{N}_0.\end{aligned} \tag{3.1.8}$$

Similarly,

$$M^{-\alpha_k} J^{\alpha_k} w_n = \frac{\Gamma(n+1)}{\Gamma(\alpha_k+n+1)} w_n. \tag{3.1.9}$$

According to (2.3.3),

$$V_{\varphi_{\alpha,\alpha_k}} w_n = \widehat{\varphi}_{\alpha,\alpha_k}(n) w_n,$$

where $\widehat{\varphi}_{\alpha,\alpha_k}$ is defined by (see (2.3.2)):

$$\widehat{\varphi}_{\alpha,\alpha_k}(\lambda) = \int_0^1 x^\lambda \varphi_{\alpha,\alpha_k}(x) dx.$$

Due to (3.1.5), we get

$$\begin{aligned} \widehat{\varphi}_{\alpha,\alpha_k}(\lambda) &= \frac{1}{\Gamma(\alpha-\alpha_k)} \int_0^1 (1-x)^{\alpha-\alpha_k-1} x^{\alpha_k+\lambda} dx \\ &= \frac{1}{\Gamma(\alpha-\alpha_k)} B(\alpha-\alpha_k, \alpha_k+\lambda+1) = \frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)}, \end{aligned} \tag{3.1.10}$$

for $\mathrm{Re}\lambda > -\alpha_k - 1$, consequently,

$$V_{\varphi_{\alpha,\alpha_k}} w_n = \frac{\Gamma(\alpha_k+n+1)}{\Gamma(\alpha+n+1)} w_n, \quad n \in \mathbb{N}_0. \tag{3.1.11}$$

Putting together (3.1.8), (3.1.9) and (3.1.11), we arrive at

$$\begin{aligned} [M^{-\alpha_k} J^{\alpha_k}] V_{\varphi_{\alpha,\alpha_k}} w_n &= \frac{\Gamma(n+1)}{\Gamma(\alpha_k+n+1)} \frac{\Gamma(\alpha_k+n+1)}{\Gamma(\alpha+n+1)} w_n \\ &= \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} w_n = M^{-\alpha} J^\alpha w_n, \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Hence, we see that (3.1.7), (3.1.6) and (3.1.4) hold.

We can now rewrite equation (3.1.3) in the form of a cordial Volterra integral equation

$$v = \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} v + f.$$

Under conditions (3.1.2), we have that $\varphi_{\alpha,\alpha_k} \in L^1(0,1)$, thus the spectrum of operator $\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}$ can be characterized by Theorem 2.3.1 and (3.1.10) as follows:

$$\begin{aligned} \sigma_0\left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right) &= \{0\} \cup \left\{\sum_{k=1}^{l} b_k(0) \widehat{\varphi}_{\alpha,\alpha_k}(\lambda) \mid \mathrm{Re}\lambda \geq 0\right\} \\ &= \{0\} \cup \left\{\sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)} \mid \mathrm{Re}\lambda \geq 0\right\} \end{aligned} \tag{3.1.12}$$

and

$$\begin{aligned}\sigma_m\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right) =&\{0\}\cup\left\{\sum_{k=1}^{l} b_k(0)\widehat{\varphi}_{\alpha,\alpha_k}(q)\mid q=0,1,\dots,m-1\right\}\\ &\cup\left\{\sum_{k=1}^{l} b_k(0)\widehat{\varphi}_{\alpha,\alpha_k}(\lambda)\mid \mathrm{Re}\lambda\ge m\right\}\\ =&\{0\}\cup\left\{\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k+q+1)}{\Gamma(\alpha+q+1)}\mid q=0,1,\dots,m-1\right\}\\ &\cup\left\{\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)}\mid \mathrm{Re}\lambda\ge m\right\},\quad m\ge 1.\end{aligned} \tag{3.1.13}$$

The following result is a consequence of (3.1.12) and (3.1.13).

Theorem 3.1.1. *Let $\alpha,\alpha_k\in\mathbb{R}$, and conditions* (3.1.2) *hold. Equation* (3.1.3) *has a unique solution $v\in C$ for any $f\in C$, i.e. $1\notin\sigma_0\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right)$, if and only if*

$$\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)}\ne 1,\ \forall\lambda\in\mathbb{C}\ \textit{with}\ \mathrm{Re}\lambda\ge 0.$$

Equation (3.1.3) *has a unique solution $v\in C^m$ for any $f\in C^m$, $m\ge 1$, i.e. $1\notin\sigma_m\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right)$, if and only if*

$$\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k+q+1)}{\Gamma(\alpha+q+1)}\ne 1,\quad q=0,1,\dots,m-1,$$

and

$$\sum_{k=1}^{l} b_k(0)\frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)}\ne 1,\ \forall\lambda\in\mathbb{C}\ \textit{with}\ \mathrm{Re}\lambda\ge m.$$

Having found the solution $v\in C$ ($v\in C^m$) of equation (3.1.3), the solution of equation (3.1.1) has the form $u=M^{-\alpha}J^\alpha v$.

To study the unique solvability of the singular fractional differential equation (1.1.3) with conditions (1.1.4), we prove (see Section 3.2) the following result.

Lemma 3.1.1. *Under conditions $b_k\in C^m$, $m\ge 0$, $k=1,2,\dots,l$, it holds that*

$$\sigma_m\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}\right)=\sigma_m\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right),$$

where $\varphi_{\alpha,\alpha_k}$ is defined by (3.1.5).

Similarly as for equation (3.1.1), equation (1.1.3) can be rewritten as

$$v = \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} v + f, \tag{3.1.14}$$

where $v = D_0^\alpha M^\alpha u$ is the unknown and $V_{\varphi_{\alpha,\alpha_k}}$ is the cordial integral operator with core $\varphi_{\alpha,\alpha_k}$ defined by (3.1.5). We can now formulate our main result for equation (1.1.3).

Theorem 3.1.2. *Let $\alpha, \alpha_k \in \mathbb{R}$, and (1.1.4) hold. For any $f \in C$, equation (3.1.14) has a unique solution $v \in C$ (hence also equation (1.1.3) has a unique solution $u = M^{-\alpha} J^\alpha v \in C$) if and only if $1 \notin \sigma_0 \left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right)$, i.e.*

$$\sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \neq 1, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq 0.$$

For any $f \in C^m$, $m \geq 1$, equation (3.1.14) has a unique solution $v \in C^m$ (hence also equation (1.1.3) has a unique solution $u = M^{-\alpha} J^\alpha v \in C^m$) if and only if $1 \notin \sigma_m \left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right)$, i.e.

$$\sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \neq 1, \quad q = 0, 1, \dots, m-1,$$

and

$$\sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \neq 1, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq m.$$

Proof. The claims of Theorem 3.1.2 regarding the solution v of (3.1.14) are direct consequences of Lemma 3.1.1 and (3.1.12), (3.1.13). Furthermore, $M^{-\alpha} J^\alpha$ is a cordial Volterra integral operator and thus according to Theorem 2.3.1, $v \in C^m$, $m \in \mathbb{N}_0$, implies $u = M^{-\alpha} J^\alpha v \in C^m$; recall that $M^{-\alpha} J^\alpha v$ belongs to the domain of $D_0^{\alpha_k} M^{\alpha_k}$ and $u = M^{-\alpha} J^\alpha v$ really satisfies (1.1.3). Theorem 3.1.2 is proved. □

Let us remember that according to Propositions 2.2.1 and 2.2.2 we have that, for $m < \alpha \leq m+1$, $m \in \mathbb{N}_0$,

$$D^\alpha_{\text{R-L}} u = D_0^\alpha u, \quad D^\alpha_{\text{Cap}} u = D_0^\alpha (u - \Pi_m u), \quad u \in C^m.$$

For $u \in C^m$, $m < \alpha \leq m+1$, $m \in \mathbb{N}_0$, it holds $M^\alpha u \in C^m$, $\Pi_m M^\alpha u = 0$, hence

$$D^\alpha_{\text{R-L}} (M^\alpha u) = D_0^\alpha (M^\alpha u) = D^\alpha_{\text{Cap}} (M^\alpha u).$$

Also, for $u \in C^{m_k}$, $m_k < \alpha_k \leq m_k + 1$, $m_k \in \mathbb{N}_0$, $k = 1, 2, \ldots, l$, it holds that $M^{\alpha_k} u \in C^{m_k}$, $\Pi_{m_k} M^{\alpha_k} u = 0$, hence

$$D^{\alpha_k}_{\text{R-L}}(M^{\alpha_k} u) = D^{\alpha_k}_0 (M^{\alpha_k} u) = D^{\alpha_k}_{\text{Cap}}(M^{\alpha_k} u).$$

Thus, for $u \in C^m$, equation (1.1.3) with (1.1.4) is equivalent to the equation which we obtain from (1.1.3) replacing D_0^α, $D_0^{\alpha_k}$ either by $D^\alpha_{\text{R-L}}$, $D^{\alpha_k}_{\text{R-L}}$ or by D^α_{Cap}, $D^{\alpha_k}_{\text{Cap}}$, where $k = 1, 2, \ldots, l$, consequently our results remain to be true also in the case with Riemann-Liouville or Caputo fractional derivatives.

To illustrate the results of Theorem 3.1.2, we end this section by presenting two simple examples of equations of the form (1.1.3).

Example 3.1.1. Consider the equation

$$D_0^\alpha M^\alpha u = bu + f, \quad m < \alpha \leq m + 1, \quad b, f \in C^m, \quad m \in \mathbb{N}_0. \tag{3.1.15}$$

By Theorem 3.1.2, equation (3.1.15) has a unique solution $u \in C$ for any $f \in C$ if and only if $1 \notin \sigma_0 \left(b(0) V_{\varphi_{\alpha,0}}\right)$, i.e.

$$b(0) \neq \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1)}, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq 0;$$

for real $b(0)$ and α belonging to $(0, 1]$, i.e. $m = 0$, this condition, or the condition $1 \notin \sigma_0 \left(b(0) V_{\varphi_{\alpha,0}}\right)$, takes the form

$$b(0) < \Gamma(\alpha + 1),$$

since

$$\sigma_0 \left(V_{\varphi_{\alpha,0}}\right) \cap \mathbb{R} = \left[0, \frac{1}{\Gamma(\alpha + 1)}\right]. \tag{3.1.16}$$

To prove the last equality, we first note that the curve $\widehat{\varphi}_{\alpha,0}(\mathrm{i}\rho)$, $\rho \in \mathbb{R}$ (see (3.1.10)), encloses the spectrum $\sigma_0 \left(V_{\varphi_{\alpha,0}}\right)$ and

$$\sigma_0 \left(V_{\varphi_{\alpha,0}}\right) \cap \mathbb{R} = \left[0, \widehat{\varphi}_{\alpha,0}(0)\right].$$

We show

$$\sigma_0 \left(V_{\varphi_{\alpha,0}}\right) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geq 0\}$$

by establishing that $\mathrm{Re}(\widehat{\varphi}_{\alpha,0}(\mathrm{i}\rho)) \geq 0$, $\rho \in \mathbb{R}$, where (see (3.1.10))

$$\begin{aligned}
\mathrm{Re}(\widehat{\varphi}_{\alpha,0}(\mathrm{i}\rho)) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - x)^{\alpha - 1} \mathrm{Re}(x^{\mathrm{i}\rho}) dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - x)^{\alpha - 1} \cos(\rho \ln x) dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty (1 - e^{-u})^{\alpha - 1} e^{-u} \cos(\rho u) du, \ \rho \in \mathbb{R}.
\end{aligned}$$

For $\rho = 0$, we have

$$\mathrm{Re}(\widehat{\varphi}_{\alpha,0}(0)) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} dx = \frac{1}{\Gamma(\alpha+1)} \geq 0.$$

Denote

$$f(u) = (1-e^{-u})^{\alpha-1} e^{-u}, \quad 0 < u < \infty.$$

According to Proposition 2.1.1, for $\mathrm{Re}(\widehat{\varphi}_{\alpha,0}(\mathrm{i}\rho)) \geq 0$, $\rho \in \mathbb{R} \setminus \{0\}$ to hold, it is sufficient that $f''(u) \geq 0$, $0 < u < \infty$. For $0 < u < \infty$, we have

$$f'(u) = e^{-u}(1-e^{-u})^{\alpha-2}(\alpha e^{-u} - 1)$$

and

$$f''(u) = e^{-u}(1-e^{-u})^{\alpha-3}(1 + (1-3\alpha)e^{-u} + \alpha^2 e^{-2u}).$$

Obviously, $e^{-u}(1-e^{-u})^{\alpha-3} > 0$, $0 < u < \infty$, hence for $f''(u) \geq 0$, $0 < u < \infty$ to hold, we have to check that

$$g(\alpha, u) = 1 + (1-3\alpha)e^{-u} + \alpha^2 e^{-2u} \geq 0, \ 0 < \alpha \leq 1, \ 0 < u < \infty.$$

It holds

$$\frac{\partial g(\alpha, u)}{\partial \alpha} = -3e^{-u} + 2\alpha e^{-2u} < 0 \text{ for } \alpha < \frac{3}{2} e^u.$$

Also $g(0,u) = 1 + e^{-u} > 0$ and $g(1,u) = (1-e^{-u})^2 > 0$ for $0 < u < \infty$, hence $g(\alpha, u) \geq 0$, for $0 < \alpha \leq 1$, $0 < u < \infty$.

Consequently, we have shown that, for $0 < \alpha \leq 1$, we have $\mathrm{Re}(\widehat{\varphi}_{\alpha,0}(\mathrm{i}\rho)) \geq 0$, $\rho \in \mathbb{R}$, and thus (3.1.16) holds.

By Theorem 3.1.2, equation (3.1.15) has a unique solution $u \in C^m$ for any $f \in C^m$, $m \geq 1$, if and only if $1 \notin \sigma_m\left(b(0)V_{\varphi_{\alpha,0}}\right)$, i.e.

$$b(0) \neq \frac{\Gamma(q+\alpha+1)}{q!}, \quad q = 0, 1, \ldots, m-1$$

and

$$b(0) \neq \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)}, \quad \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq m.$$

Example 3.1.2. Equations

$$tu'(t) = a(t)u(t) + f(t), \quad 0 < t \leq T$$

and

$$(tu(t))' = b(t)u(t) + f(t), \quad 0 < t \leq T \tag{3.1.17}$$

where $b(t) = a(t) + 1$, $0 < t \le T$, are equivalent. Thus, according to [38], if

$$\operatorname{Re} b(0) < 1, \tag{3.1.18}$$

then equation (3.1.17) has for any $f \in C$ a unique solution in C. Additionally, if $b \in C$ is such that a finite limit

$$\lim_{t \to 0} \frac{b(t) - b(0)}{t^\beta}$$

exists for a $\beta > 0$, then condition (3.1.18) is also necessary for the unique solution of (3.1.17) in C for all $f \in C$.

According to Theorem 3.1.2, equation

$$D_0^\alpha(t^\alpha u(t)) = b(t)u(t) + f(t), \quad 0 < t \le T,\ 0 < \alpha < 1 \tag{3.1.19}$$

(equation (3.1.15) for $0 < \alpha < 1$) has for any $f \in C$ a unique solution in C if and only if

$$b(0) \ne \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1)} \quad \text{for any } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \ge 0.$$

As $\alpha \to 1$, the last condition takes the form

$$b(0) \ne \lambda + 1 \quad \text{for any } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \ge 0,$$

i.e.

$$\operatorname{Re} b(0) < 1.$$

Note that, the necessary and sufficient conditions for the unique solvability in C, of equation (3.1.17) and equation (3.1.19) as $\alpha \to 1$, are close to one another but do not fully coincide.

3.2 Proof of Lemma 3.1.1

To prove Lemma 3.1.1, we show that for $b_k \in C^m$, $k = 1, 2, \dots, l$ and $m \ge 0$ the following relations hold:

$$\rho_m\left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right) \subset \rho_m\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}\right), \tag{3.2.1}$$

$$\sigma_m\left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right) \subset \sigma_m\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}\right). \tag{3.2.2}$$

The proof of (3.2.1) and (3.2.2) is presented in six parts. In part a) we show that (3.2.1) holds for $m = 0$ under stricter conditions $\varphi_{\alpha,\alpha_k} \in C^1[0,1]$, and $b_k \in C^1$ for $k = 1, 2, \dots, l$. Part b) is dedicated to extending the results from part a) to $m = 0$, $\varphi_{\alpha,\alpha_k} \in L^1(0,1)$ and $b_k \in C$, $k = 1, 2, \dots, l$. With part c) the inclusion (3.2.1) is proved for $m \ge 1$. Part d) shows that (3.2.2) holds for $m = 0$ and prepares the proof for $m \ge 1$, the last two parts e) and f) complete the proof of (3.2.2) for $m \ge 1$.

Part a)

Let $m = 0$, $\varphi_{\alpha,\alpha_k} \in C^1[0,1]$, and $b_k \in C^1$ for $k = 1,2,\dots,l$. We show that $\mu \in \rho_0\left(\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right)$, $\mu \in \sigma_0\left(\sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}}\right)$ leads to a contradiction, i.e. relation (3.2.1) holds for $m = 0$.

According to (3.1.12), we have that $0 \in \sigma_0\left(\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right)$, hence we may assume that $\mu \neq 0$. Now,

$$\mu I - \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} = \mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} + \sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}}.$$

By Theorem 2.3.1, we have that operator

$$\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}} : C \to C$$

is compact. Since $\mu \in \rho_0\left(\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right)$, we get that $\mu I - \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}$ is invertible in C. Hence, by Proposition 2.4.1, we have

$$\mu I - \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \in \Phi_0(C).$$

Now, based upon Corollary 2.4.1, $\mu \in \sigma_0\left(\sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}}\right)$ is the eigenvalue of operator $\sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}}$. Let $u_0 \in C$, $\|u_0\|_\infty = 1$, be the corresponding eigenfunction:

$$\left(\mu I - \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}\right) u_0 = 0$$

or

$$u_0 = \left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right)^{-1} \left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}}\right) u_0.$$

Since, by Proposition 2.3.1,

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right)^{-1} = \mu^{-1} I + V_\psi \text{ with a } \psi \in L^1(0,1),$$

it holds

$$u_0 = (\mu^{-1} I + V_\psi) \left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}}\right) u_0. \tag{3.2.3}$$

We assumed that $b_k \in C^1$ for $k = 1, 2, \dots, l$, consequently

$$|b_k(t) - b_k(0)| \le c_k\, t, \quad k = 1, 2, \dots, l, \quad 0 \le t \le T, \tag{3.2.4}$$

with some constants $c_k > 0$, $k = 1, 2, \dots, l$. Using (3.2.3) and (3.2.4), we now evaluate $|u_0(t)|$ step-by-step. The first step is

$$\left| \left(\left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)(t) \right| \le \sum_{k=1}^{l} |b_k(t) - b_k(0)| \int_0^t t^{-1} |\varphi_{\alpha,\alpha_k}(t^{-1}s)|\, |u_0(s)| ds$$
$$\le t \sum_{k=1}^{l} c_k \|\varphi_{\alpha,\alpha_k}\|_1 \le ct \sum_{k=1}^{l} \|\varphi_{\alpha,\alpha_k}\|_1,$$
$$\left| \left(V_\psi \left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)(t) \right| \le c\, t \|\psi\|_1 \sum_{k=1}^{l} \|\varphi_{\alpha,\alpha_k}\|_1,$$
$$|u_0(t)| \le c\, t (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^{l} \|\varphi_{\alpha,\alpha_k}\|_1,$$

with a constant $c > 0$ independent of $t \in [0, T]$. Let us assume that after the n-th step we have the estimate $|u_0(t)| \le \widetilde{c}_n t^n$, $0 \le t \le T$, with a constant $\widetilde{c}_n$. Then

$$\left| \left(\left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)(t) \right| \le \sum_{k=1}^{l} |b_k(t) - b_k(0)| \int_0^t t^{-1} |\varphi_{\alpha,\alpha_k}(t^{-1}s)|\, \widetilde{c}_n s^n ds$$
$$\le c\, \widetilde{c}_n\, t^{n+1} \sum_{k=1}^{l} \|\varphi^{[n]}_{\alpha,\alpha_k}\|_1,$$
$$\left| \left(V_\psi \left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)(t) \right| \le c\, \widetilde{c}_n\, t^{n+1} \|\psi\|_1 \sum_{k=1}^{l} \|\varphi^{[n]}_{\alpha,\alpha_k}\|_1,$$
$$|u_0(t)| \le c\, \widetilde{c}_n\, t^{n+1} (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^{l} \|\varphi^{[n]}_{\alpha,\alpha_k}\|_1,$$

$0 \le t \le T$, where

$$\varphi^{[n]}_{\alpha,\alpha_k}(x) = \varphi_{\alpha,\alpha_k}(x) x^n, \quad 0 < x < 1. \tag{3.2.5}$$

Therefore,

$$|u_0(t)| \le \widetilde{c}_{n+1} t^{n+1}, \ 0 \le t \le T,$$

with a constant

$$\widetilde{c}_{n+1} = c\, \widetilde{c}_n (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^{l} \|\varphi^{[n]}_{\alpha,\alpha_k}\|_1.$$

Since

$$\|\varphi^{[n]}_{\alpha,\alpha_k}\|_1 = \int_0^1 x^n |\varphi_{\alpha,\alpha_k}(x)| dx \le \|\varphi^{[1]}_{\alpha,\alpha_k}\|_\infty \int_0^1 x^{n-1} dx = \frac{\|\varphi^{[1]}_{\alpha,\alpha_k}\|_\infty}{n},$$

we have

$$\begin{aligned}
\widetilde{c}_{n+1} \le& \frac{c\,(|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^{l} \|\varphi^{[1]}_{\alpha,\alpha_k}\|_\infty}{n}\,\widetilde{c}_n \\
\le& \frac{c^2\,(|\mu^{-1}| + \|\psi\|_1)^2 \left(\sum_{k=1}^{l} \|\varphi^{[1]}_{\alpha,\alpha_k}\|_\infty\right)^2}{n(n-1)}\,\widetilde{c}_{n-1} \\
&\vdots \\
\le& \frac{c^n\,(|\mu^{-1}| + \|\psi\|_1)^n \left(\sum_{k=1}^{l} \|\varphi^{[1]}_{\alpha,\alpha_k}\|_\infty\right)^n}{n!}\,\widetilde{c}_1.
\end{aligned}$$

Hence, for $n \in \mathbb{N}_0$, $0 \le t \le T$, we get

$$|u_0(t)| \le \frac{c^n\,(|\mu^{-1}| + \|\psi\|_1)^n \left(\sum_{k=1}^{l} \|\varphi^{[1]}_{\alpha,\alpha_k}\|_\infty\right)^n T^{n+1}}{n!}\widetilde{c}_1.$$

Next, we replace $n!$ in the last inequality with the help of Stirling formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

and consider the limit process $n \to \infty$. We get that $u_0(t) = 0$ for $0 \le t \le T$. This contradicts the fact that $u_0 \in C$ is an eigenfunction with $\|u_0\|_\infty = 1$.

Part b)

Let $m = 0$, $\varphi_{\alpha,\alpha_k} \in L^1(0,1)$ and $b_k \in C$, $k = 1, 2, \ldots, l$. We begin the discussion as in part a) and then interpret relation (3.2.3) as follows: the eigenvalue problem

$$\lambda u = (\mu^{-1} I + V_\psi)\left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}}\right) u$$

with compact operator

$$(\mu^{-1} I + V_\psi)\left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}}\right)$$

has an eigensolution (λ_0, u_0), $\lambda_0 = 1$, $u_0 \in C$, $\|u_0\|_\infty = 1$.

We approximate functions $\varphi_{\alpha,\alpha_k}$ and b_k by $\varphi^{\varepsilon}_{\alpha,\alpha_k} \in C^1[0,1]$ and $b^{\varepsilon}_k \in C^1$ so that

$$\|\varphi_{\alpha,\alpha_k} - \varphi^{\varepsilon}_{\alpha,\alpha_k}\|_1 \leq \varepsilon,\ b^{\varepsilon}_k(0) = b_k(0),\ \|b_k - b^{\varepsilon}_k\|_\infty \leq \varepsilon,\ k = 1, 2, \ldots, l,$$

where $\varepsilon > 0$ is a given small number. The operator

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi^{\varepsilon}_{\alpha,\alpha_k}} : C \to C$$

is still invertible and, by Proposition 2.3.1, can be expressed as

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi^{\varepsilon}_{\alpha,\alpha_k}}\right)^{-1} = \mu^{-1} I + V_{\psi_\varepsilon} \text{ with a } \psi_\varepsilon \in L^1(0,1),$$

where $\|\psi - \psi_\varepsilon\|_1 \leq c'\varepsilon$, $c' > 0$. We get

$$\left\|(\mu^{-1}I+V_\psi)\left(\sum_{k=1}^{l}[b_k-b_k(0)]V_{\varphi_{\alpha,\alpha_k}}\right)-(\mu^{-1}I+V_{\psi_\varepsilon})\left(\sum_{k=1}^{l}[b^{\varepsilon}_k-b^{\varepsilon}_k(0)]V_{\varphi^{\varepsilon}_{\alpha,\alpha_k}}\right)\right\|_{\mathcal{L}(C)} \leq c''\varepsilon.$$

For a sufficiently small $\varepsilon > 0$, the perturbed eigenvalue problem

$$\lambda u = (\mu^{-1} I + V_{\psi_\varepsilon}) \left(\sum_{k=1}^{l} [b^{\varepsilon}_k - b^{\varepsilon}_k(0)] V_{\varphi^{\varepsilon}_{\alpha,\alpha_k}}\right) u$$

has a solution $(\lambda_\varepsilon, u_\varepsilon)$, $\|u_\varepsilon\|_\infty = 1$ such that, $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0$. Using a similar discussion as in part a) we get that $u_\varepsilon = 0$. This is a contradiction since $\|u_\varepsilon\|_\infty = 1$. Consequently, for $m = 0$, $\varphi_{\alpha,\alpha_k} \in L^1(0,1)$ and $b_k \in C$, $k = 1, 2, \ldots, l$, relation (3.2.1) holds.

Part c)

This part of the proof will show that inclusion (3.2.1) holds for $m \geq 1$. Let $\mu \in \rho_m\left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right)$. We may assume that, $\mu \neq 0$ since according to (3.1.12) we have $0 \in \sigma_0\left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right)$. Proposition 2.4.1 yields

$$\mu I - \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \in \Phi_0(C^m).$$

To prove that $\mu \in \rho_m\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}\right)$, by Corollary 2.4.1, it is sufficient to show that the homogeneous equation

$$\mu u = \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} u$$

has in C^m only the trivial solution. Let $u_0 \in C^m$ be a solution:

$$\mu u_0 = \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right) u_0. \tag{3.2.6}$$

We first show by induction that

$$u_0^{(q)}(0) = 0, \quad q = 0, 1, \ldots, m-1. \tag{3.2.7}$$

Note that, for $u_\lambda(t) = t^\lambda$, $\mathrm{Re}\lambda > 0$,

$$\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} u_\lambda = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} u_\lambda. \tag{3.2.8}$$

For $t = 0$, (3.2.6) takes the form

$$\mu u_0(0) = u_0(0) \sum_{k=1}^{l} b_k(0) \int_0^1 \varphi_{\alpha,\alpha_k}(x) dx. \tag{3.2.9}$$

If $u_0(0) \neq 0$, then (3.2.9) can be interpreted as follows: μ is an eigenvalue of

$$\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}$$

corresponding to the eigenfunction 1 (see (3.2.8)). This contradicts the assumption $\mu \in \rho_m \left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right)$. Hence $u_0(0) = 0$. We show that the induction hypothesis

$$u_0^{(j)}(0) = 0, \quad j = 0, 1, \ldots, n-1,$$

where $n \leq m-1$, leads to $u_0^{(n)}(0) = 0$. Indeed, from (2.3.6) it follows that

$$\begin{aligned} \left(\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)^{(n)} &= \sum_{k=1}^{l} b_k V_{\varphi^{[n]}_{\alpha,\alpha_k}} u_0^{(n)} \\ &+ \sum_{q=0}^{n-1} \frac{n!}{q!(n-q)!} \sum_{k=1}^{l} b_k^{(n-q)} V_{\varphi^{[q]}_{\alpha,\alpha_k}} u_0^{(q)}, \end{aligned} \tag{3.2.10}$$

where $\varphi^{[q]}_{\alpha,\alpha_k}$, $q = 0, 1, \ldots, n$, is given by the formula (3.2.5). Since $u_0^{(j)}(0) = 0$ for $j = 0, 1, \ldots, n-1$, we get (see (3.2.6) and (2.3.5))

$$\mu u_0^{(n)}(0) = u_0^{(n)}(0) \sum_{k=1}^{l} b_k(0) \int_0^1 x^n \varphi_{\alpha,\alpha_k}(x) dx.$$

Now, $u_0^{(n)}(0) = 0$, since otherwise μ would be an eigenvalue of $\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}}$ corresponding to the eigenfunction t^n (see (3.2.8)). This completes the proof of (3.2.7).

Next, to obtain $u_0 = 0$, it is sufficient to show that $u_0^{(m)} = 0$ (see (3.2.7)). We know that

$$\mu u_0^{(m)} = \left(\left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}\right) u_0\right)^{(m)},$$

thus (see (3.2.10))

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}^{[m]}}\right) u_0^{(m)} = \sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}^{[m]}} u_0^{(m)} + \sum_{q=0}^{m-1} \frac{m!}{q!(m-q)!} \sum_{k=1}^{l} b_k^{(m-q)} V_{\varphi_{\alpha,\alpha_k}^{[q]}} u_0^{(q)}.$$

Since we have $\mu \in \rho_m\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}}\right)$ for $m \geq 1$, then, by (2.3.8), we get that $\mu \in \rho_0\left(\sum_{k=1}^{l} b_k(0)V_{\varphi_{\alpha,\alpha_k}^{[m]}}\right)$. Hence, operator

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}^{[m]}} : C \to C$$

is invertible, and according to Proposition 2.3.1 the inverse has the form

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}^{[m]}}\right)^{-1} = \mu^{-1} I + V_{\psi_m}, \text{ with a } \psi_m \in L^1(0,1).$$

Relations (3.2.7) imply that

$$u_0^{(q)}(t) = \frac{1}{(m-q-1)!} \int_0^t (t-s)^{m-q-1} u_0^{(m)}(s) ds, \quad 0 \leq t \leq T,\ q = 0, 1, \ldots, m-1.$$

In conclusion,

$$u_0{}^{(m)} = (\mu^{-1} I + V_{\psi_m}) \times \left\{ \sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}^{[m]}} u_0^{(m)} + \sum_{q=0}^{m-1} \frac{m!}{q!(m-q)!} \sum_{k=1}^{l} b_k^{(m-q)} V_{\varphi_{\alpha,\alpha_k}^{[q]}} G_q u_0^{(m)} \right\}, \tag{3.2.11}$$

where

$$(G_q v)(t) = \frac{1}{(m-q-1)!} \int_0^t (t-s)^{m-q-1} v(s) ds, \quad q = 0, 1, \ldots, m-1.$$

Note that,

$$|v(t)| \le ct^p,\ p \ge 0 \Rightarrow |(G_q v)(t)| \le c_1 t^{p+m-q}, \quad \left| \left(\sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi^{[m]}_{\alpha,\alpha_k}} v \right)(t) \right| \le c_2 t^{p+1}.$$

Also, operators $\sum_{k=1}^{l} b_k^{(m-q)} V_{\varphi^{[q]}_{\alpha,\alpha_k}}$ and V_{ψ_m} preserve the convergence order of $v(t)$ for $t \to 0$. Approximating $u_0^{(m)}(t)$, with the help of (3.2.11), step-by-step as in part a), we obtain that $u_0^{(m)} = 0$. Therefore (3.2.1) holds for $m \ge 1$.

Part d)

We now turn to the proof of (3.2.2). By (3.1.12) and (3.1.13), the inclusion (3.2.2) is equivalent to the following inclusions: for $m = 0$,

$$\{0\} \cup \left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \mathrm{Re}\lambda \ge 0 \right\} \subset \sigma_0 \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right); \tag{3.2.12}$$

for $m \ge 1$,

$$\begin{aligned} \{0\} \cup \left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \mid q = 0, 1, \ldots, m-1 \right\} \\ \cup \left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \mathrm{Re}\lambda \ge m \right\} \subset \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right). \end{aligned} \tag{3.2.13}$$

First off all, the fact $0 \in \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right)$ for $m \ge 0$ follows directly from the closedness of the spectrum $\sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right)$: in accordance with (3.2.1) the spectrum $\sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right)$ contains points $\sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k+\lambda+1)}{\Gamma(\alpha+\lambda+1)}$ (see (3.1.12) and (3.1.13)) for arbitrary large $\lambda \in \mathbb{R}$ and

$$\sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \to 0$$

as $\lambda \to \infty$, since $\alpha_k < \alpha$.

Next, to establish the inclusions

$$\left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \mathrm{Re}\lambda \ge m \right\} \subset \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right), \quad m \ge 0,$$

it is sufficient to show that

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \notin \Phi_0(C^m), \ \forall \mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}, \text{ with } \mathrm{Re}\lambda \geq m. \tag{3.2.14}$$

Indeed,

$$\mu I - \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} = \mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} + \sum_{k=1}^{l} [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}},$$

hence, by Proposition 2.4.1, relation (3.2.14) implies that also

$$\mu I - \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \notin \Phi_0(C^m)$$

and thus, by Corollary 2.4.2 we get that

$$\mu \in \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right) \text{ for } \mathrm{Re}\lambda \geq m.$$

Let us establish (3.2.14) for $m = 0$. For

$$\mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}, \quad \mathrm{Re}\lambda > 0,$$

according to Propositions 2.3.2 and 2.3.3,

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right) C = C$$

holds and μ is an eigenvalue of operator $\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}$ with eigenfunction t^λ in C (see (3.2.8)), so

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \notin \Phi_0(C^m).$$

For

$$\mu_0 = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda_0 + 1)}{\Gamma(\alpha + \lambda_0 + 1)}, \quad \mathrm{Re}\lambda_0 = 0,$$

relation

$$\mu_0 I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \in \Phi_0(C)$$

cannot hold since otherwise it would also be true for a λ with $\mathrm{Re}\lambda > 0$ that is close to λ_0 (see Corollary 2.4.3), but this is not the case. Thus (3.2.14), and as a consequence (3.2.12) and (3.2.2) hold for $m = 0$.

Part e)

In this part we prove (3.2.14) for $m \geq 1$. It holds (see (2.1.3))

$$C^m = C^{m,m} \oplus \mathcal{P}_{m-1},$$

where $C^{m,m}$ is defined in (2.1.1) and $\mathcal{P}_{m-1}$ is the space of all polynomials with degree equal to or less than $m-1$. Our aim is to use Proposition 2.4.2: we show that $C^{m,m}$ and $\mathcal{P}_{m-1}$ fulfill the presumptions about X_1 and X_0 respectively and

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \notin \Phi_0(C^{m,m}) \ \forall \mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \text{ with } \mathrm{Re}\lambda > m.$$

In such case, by Proposition 2.4.2

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}$$

belongs to

$$\Phi_\kappa(C^m), \ \kappa > 0 \text{ for } \mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \text{ with } \mathrm{Re}\lambda > m.$$

We start our discussion by noting that according to (2.3.4), for any $\mu \in \mathbb{C}$ and $u \in C^{m,m}$ it holds that

$$\left(\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right) u \right)^{(n)} = \left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi^{[n]}_{\alpha,\alpha_k}} \right) u^{(n)}$$

for $n = 1, 2, \ldots, m$. Thus

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right) C^{m,m} \subset C^{m,m}.$$

In accordance with (3.2.8), the inclusion

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right) \mathcal{P}_{m-1} \subset \mathcal{P}_{m-1}$$

also holds. Furthermore, we claim that for $\mathrm{Re}\lambda > m$ we have

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right) C^{m,m} = C^{m,m} \text{ for every } \mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)},$$

i.e., that equation

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right) u = v$$

has for every $v \in C^{m,m}$ and every

$$\mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \text{ with } \mathrm{Re}\lambda > m$$

a solution $u \in C^{m,m}$. Since operator $D^m : C^{m,m} \to C$ is an isomorphism, the last statement is equivalent to the following: equation

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi^m_{\alpha,\alpha_k}}\right) \overline{u} = \overline{v}$$

has for every $\overline{v} \in C$ and every

$$\mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda' + 1)}{\Gamma(\alpha + \lambda' + 1)} \text{ with } \mathrm{Re}\lambda' > 0 \ (\lambda' = \lambda - m)$$

a solution $\overline{u} \in C$ ($\overline{u} = u^{(m)}$, $\overline{v} = v^{(m)}$). Now, according to Propositions 2.3.2 and 2.3.3, we get

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi^{[m]}_{\alpha,\alpha_k}}\right) C = C$$

and

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right) C^{m,m} = C^{m,m}.$$

Also, due to (3.2.8), for $u_\lambda(t) = t^\lambda$, $0 < t \leq T$, which belongs for $\mathrm{Re}\lambda > m$ (also for $\lambda = m$) to $C^{m,m}$, we have

$$\left(\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}\right) u_\lambda = 0 \text{ if and only if } \mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)},$$

thus

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \in \Phi_\kappa(C^{m,m}) \text{ with } \kappa > 0 \text{ for } \mathrm{Re}\lambda > m.$$

For us it is important that

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \notin \Phi_0(C^{m,m}) \text{ for } \mathrm{Re}\lambda > m.$$

In conclusion, by Proposition 2.4.2 we get that

$$\mu I - \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}$$

belongs to $\Phi_\kappa(C^m)$, $\kappa > 0$ for

$$\mu = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \text{ with } \mathrm{Re}\lambda > m.$$

Therefore, for $\mathrm{Re}\lambda > m$, relation (3.2.14) is established. If $\mathrm{Re}\lambda = m$, then a similar approximation argument as at the end of d) can be applied.

Part f)

To conclude the proof of Lemma 3.1.1, it remains to show that for $m \geq 1$,

$$\left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \mid q = 0, 1, \ldots, m-1 \right\} \subset \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right).$$

Denote

$$\mu_q = \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)}, \quad q = 0, 1, \ldots, m-1.$$

We may assume, that

$$\mu_q \notin \left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \mathrm{Re}\lambda \geq m \right\},$$

since otherwise it follows from previous discussions that

$$\mu_q \in \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right).$$

Let us fix a sufficiently small $\delta > 0$ such that: 1) the ball $|\mu - \mu_q| \leq \delta$ does not contain μ_j different from μ_q, $j = 0, 1, \ldots, m-1$ and 2) the intersection of

$$\left\{ \sum_{k=1}^{l} b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \mathrm{Re}\lambda \geq m \right\}$$

and the ball $|\mu - \mu_q| \leq \delta$ is empty. Note that, under our assumptions, the sphere $|\mu - \mu_q| = \delta$ is contained in $\rho_m \left(\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \right)$ and as a consequence of (3.2.1) also in $\rho_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right)$.

Let V_ψ^θ be an operator depending on parameter $\theta \in [0,1]$ defined by

$$V_\psi^\theta := \theta \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} + (1-\theta) \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \in \mathcal{L}(C^m), \quad 0 \le \theta \le 1.$$

Obviously,

$$V_\psi^0 = \sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}} \text{ and } V_\psi^1 = \sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}}.$$

The inclusion (3.2.1) for operator V_ψ^θ implies that the sphere $|\mu - \mu_q| = \delta$ is in $\rho_m(V_\psi^\theta)$ for $0 \le \theta \le 1$.

Furthermore, the Riesz projector defined for V_ψ^θ by (see [10])

$$P_q^\theta := \frac{1}{2\pi \mathrm{i}} \int_{|\mu-\mu_q|=\delta} (\mu I - V_\psi^\theta)^{-1} d\mu \in \mathcal{L}(C^m), \quad 0 \le \theta \le 1,$$

projects the space C^m onto an invariant subspace of operator V_ψ^θ corresponding to its spectrum part in the ball $|\mu - \mu_q| \le \delta$. Since, by assumptions 1) and 2) above, the only possible point from this spectrum part in ball $|\mu - \mu_q| \le \delta$ is μ_q, we have that $\mu_q \in \sigma_m(V_\psi^\theta)$ if and only if $P_q^\theta \ne 0$.

The operator V_ψ^θ, $0 \le \theta \le 1$, is continuously dependent on parameter θ on the sphere $|\mu - \mu_q| = \delta$, therefore there exists a constant c_δ, such that

$$\|(\mu I - V_\psi^\theta)^{-1}\|_{\mathcal{L}(C^m)} \le c_\delta, \quad \text{for } |\mu - \mu_q| = \delta, \quad 0 \le \theta \le 1.$$

Since $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$, there exists a constant c'_δ such that

$$\|P_q^\theta - P_q^{\theta'}\|_{\mathcal{L}(C^m)} \le c_\delta^2 \delta \|V_\psi^\theta - V_\psi^{\theta'}\|_{\mathcal{L}(C^m)} \le c'_\delta |\theta - \theta'|$$

for $0 \le \theta \le \theta' \le 1$.

Consider the gap [18, 16] between subspaces $X^\theta = P_q^\theta X$ and $X^{\theta'} = P_q^{\theta'} X$ of $X = C^m$:

$$\begin{aligned} \operatorname{gap}(X^\theta, X^{\theta'}) &:= \max\left\{ \sup_{u \in X^\theta, \|u\|_X = 1} \inf_{v \in X^{\theta'}} \|u - v\|_X, \sup_{u \in X^{\theta'}, \|u\|_X = 1} \inf_{v \in X^\theta} \|u - v\|_X \right\} \\ &\le \max\left\{ \sup_{u \in X, \|u\|_X = 1} \|P_q^\theta u - P_q^{\theta'} u\|_X, \sup_{u \in X, \|u\|_X = 1} \|P_q^{\theta'} u - P_q^\theta u\|_X \right\} \\ &\le \|P_q^\theta - P_q^{\theta'}\|_{\mathcal{L}(X)} \\ &\le c'_\delta |\theta - \theta'| \to 0 \quad \text{as } |\theta - \theta'| \to 0, \ 0 \le \theta \le \theta' \le 1. \end{aligned}$$

It is known [16] that for a Banach space X and its closed subspaces X_1, X_2 the inequality $\operatorname{gap}(X_1, X_2) < 1$ implies that $\dim X_1 = \dim X_2$. Thus $\dim X^\theta = \dim X^{\theta'}$

for $\theta, \theta' \in [0,1]$ such that $c'_\delta|\theta - \theta'| < 1$, hence also for all $\theta, \theta' \in [0,1]$. As μ_q is an eigenvalue of operator $\sum_{k=1}^{l} b_k(0) V_{\varphi_{\alpha,\alpha_k}}$, then $\dim(P_q^\theta C^m) \geq 1$. Consequently, $P_q^\theta \neq 0$ for every $\theta \in [0,1]$, in particular for $\theta = 1$, i.e.

$$\mu_q \in \sigma_m \left(\sum_{k=1}^{l} b_k V_{\varphi_{\alpha,\alpha_k}} \right).$$

With parts d), e) and f) we have shown that (3.2.2) holds for $m \geq 1$. This concludes the proof of Lemma 3.1.1.

Chapter 4

Equation with main term $M^\beta D_0^\beta$

In this chapter, our main goal is to analyze the unique solvability of a singular fractional differential equation of the form

$$(M^\beta D_0^\beta u)(t) = \sum_{k=1}^{l} a_k(t)(M^{\beta_k} D_0^{\beta_k} u)(t) + f(t), \quad 0 < t \leq T, \tag{1.1.2}$$

where $\beta, \beta_k \in \mathbb{R}$, $f \in C^m$ and

$$m < \beta < m+1, \quad \beta > \beta_k \geq 0, \quad a_k \in C^m, \quad k = 1, 2, \ldots, l, \quad m \in \mathbb{N}_0. \tag{1.1.5}$$

In Subsection 4.1.1 we first consider a simplified version of equation (1.1.2) with constant coefficients. As we explain in more detail later, we have to consider the cases $0 < \beta < 1$ and $\beta > 1$ separately. For case $0 < \beta < 1$, we prove Theorem 4.1.1. The case $\beta > 1$ is more complicated, we first have to extend the fractional differentiation operators D_0^β and $D_0^{\beta_k}$ and our cordial Volterra integral operators to the space C^m and then prove Theorem 4.1.2 for the extended operators. Subsection 4.1.2 is devoted to equation (1.1.2) with possibly non-constant coefficients. We first formulate and prove an auxiliary result, Lemma 4.1.2. In Theorem 4.1.4 we give the unique solvability conditions for equation (1.1.2) in C^m, $m \geq 1$, under conditions (1.1.5).

Section 4.2 is dedicated to analyzing equations which we obtain from (1.1.2) replacing D_0^β, $D_0^{\beta_k}$ either by $D_{\text{R-L}}^\beta$, $D_{\text{R-L}}^{\beta_k}$ or by D_{Cap}^β, $D_{\text{Cap}}^{\beta_k}$, where $k = 1, 2, \ldots, l$. Our treatment is based on Propositions 2.2.1 and 2.2.2. As it turns out, in the case with Caputo fractional differentiation operators no extensions are necessary.

Note that in conditions (1.1.5) we require that $\beta \notin \mathbb{N}$ but in conditions (1.1.4) no such restrictions apply to α, we will explain this difference in Subsection 4.1.1 on page 51.

4.1 The main results for equation with main term $M^\beta D_0^\beta$

4.1.1 Equation with constant coefficients

We first study a simpler equation having constant coefficients:

$$(M^\beta D_0^\beta u)(t) = \sum_{k=1}^{l} a_k(0)(M^{\beta_k} D_0^{\beta_k} u)(t) + f(t), \quad 0 < t \le T, \tag{4.1.1}$$

where $\beta, \beta_k \in \mathbb{R}$, $f \in C^m$ and

$$m < \beta < m+1, \quad \beta > \beta_k \ge 0, \quad k = 1, 2, \dots, l, \quad m \in \mathbb{N}_0. \tag{4.1.2}$$

We make in (4.1.1) the change of variables $v = M^\beta D_0^\beta u$, i.e.

$$u = (M^\beta D_0^\beta)^{-1} v = J^\beta M^{-\beta} v,$$

obtaining with respect to v the equation

$$v = \sum_{k=1}^{l} a_k(0)[M^{\beta_k} D_0^{\beta_k}][J^\beta M^{-\beta}]v + f. \tag{4.1.3}$$

By (2.2.4) and (2.2.2), for $0 < t \le T$, $u \in C$,

$$\begin{aligned}
([M^{\beta_k} D_0^{\beta_k}][J^\beta M^{-\beta}]u)(t) &= (M^{\beta_k} J^{\beta-\beta_k} M^{-\beta} u)(t) \\
&= t^{\beta_k} \frac{1}{\Gamma(\beta-\beta_k)} \int_0^t (t-s)^{\beta-\beta_k-1} s^{-\beta} u(s) ds \\
&= \frac{1}{\Gamma(\beta-\beta_k)} \int_0^t \frac{1}{t} \left(1 - \frac{s}{t}\right)^{\beta-\beta_k-1} \left(\frac{s}{t}\right)^{-\beta} u(s) ds = (V_{\psi_{\beta,\beta_k}} u)(t).
\end{aligned}$$

Thus operator $[M^{\beta_k} D_0^{\beta_k}][J^\beta M^{-\beta}]$ is a cordial Volterra integral operator with the core

$$\psi_{\beta,\beta_k}(x) = \frac{1}{\Gamma(\beta-\beta_k)} (1-x)^{\beta-\beta_k-1} x^{-\beta}, \quad 0 < x < 1. \tag{4.1.4}$$

For $0 < \beta < 1$, it holds $\psi_{\beta,\beta_k} \in L^1(0,1)$, hence it follows from Theorem 2.3.1 that

$$[M^{\beta_k} D_0^{\beta_k}][J^\beta M^{-\beta}] \in \mathcal{L}(C).$$

If $\beta > 1$, then $\psi_{\beta,\beta_k} \notin L^1(0,1)$, but we obtain with the help of Theorem 2.3.2 that

$$[M^{\beta_k} D_0^{\beta_k}][J^\beta M^{-\beta}] \in \mathcal{L}(C^{m,r}), \quad m \ge 1, \quad r > \beta - 1,$$

since $\psi_{\beta,\beta_k} \in L^{1,r}(0,1)$ for $r > \beta - 1$.

We can rewrite equation (4.1.3) in the form of a cordial Volterra integral equation

$$v = \sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}} v + f, \tag{4.1.5}$$

where ψ_{β,β_k} is defined by (4.1.4) and belongs to $L^1(0,1)$ for $0 < \beta < 1$ and to $L^{1,r}(0,1)$ for $r > \beta - 1$, $\beta > 1$. In our discussions below we also need the formula (see (2.3.2), (2.1.7) and (2.1.8))

$$\begin{aligned} \widehat{\psi}_{\beta,\beta_k}(\lambda) &= \int_0^1 x^{\lambda}\psi_{\beta,\beta_k}(x)dx = \frac{1}{\Gamma(\beta-\beta_k)}\int_0^1 x^{\lambda-\beta}(1-x)^{\beta-\beta_k-1}dx \\ &= \frac{1}{\Gamma(\beta-\beta_k)}B(\lambda-\beta+1,\beta-\beta_k) = \frac{\Gamma(\lambda-\beta+1)}{\Gamma(\lambda-\beta_k+1)}, \end{aligned} \tag{4.1.6}$$

where $\mathrm{Re}\lambda > \beta - 1$.

Theorem 4.1.1. *Let $\beta, \beta_k \in \mathbb{R}$, $0 < \beta < 1$, and $\beta > \beta_k \geq 0$, $k = 1, 2, \ldots, l$. For any $f \in C$, equation (4.1.5) has a unique solution $v \in C$ (hence equation (4.1.1) has a unique solution $u = J^{\beta}M^{-\beta}v \in C$) if and only if*

$$\sum_{k=1}^{l} a_k(0)\frac{\Gamma(\lambda-\beta+1)}{\Gamma(\lambda-\beta_k+1)} \neq 1, \ \forall \lambda \in \mathbb{C} \textit{ with } \mathrm{Re}\lambda \geq 0.$$

Proof. Under formulated conditions, it follows from (4.1.4) that $\psi_{\beta,\beta_k} \in L^1(0,1)$, $k = 1, 2, \ldots, l$. Thus the spectrum of $\sum_{k=1}^{l} a_k(0)V_{\psi_{\beta,\beta_k}}$ as an operator from C into C is given by the following formula (see Theorem 2.3.1 and (4.1.6)):

$$\begin{aligned} \sigma_0\left(\sum_{k=1}^{l} a_k(0)V_{\psi_{\beta,\beta_k}}\right) &= \{0\} \cup \left\{\sum_{k=1}^{l} a_k(0)\widehat{\psi}_{\beta,\beta_k}(\lambda) \mid \mathrm{Re}\lambda \geq 0\right\} \\ &= \{0\} \cup \left\{\sum_{k=1}^{l} a_k(0)\frac{\Gamma(\lambda-\beta+1)}{\Gamma(\lambda-\beta_k+1)} \mid \mathrm{Re}\lambda \geq 0\right\}. \end{aligned}$$

The claims of the theorem follow directly from the last equality. □

By Theorem 2.3.2 and (4.1.6), the spectrum of $\sum_{k=1}^{l} a_k(0)V_{\psi_{\beta,\beta_k}}$ as an operator from $C^{m,r}$ into $C^{m,r}$ $(r > \beta - 1)$ is given by the formula

$$\begin{aligned} \sigma_{m,r}\left(\sum_{k=1}^{l} a_k(0)V_{\psi_{\beta,\beta_k}}\right) &= \{0\} \cup \left\{\sum_{k=1}^{l} a_k(0)\widehat{\psi}_{\beta,\beta_k}(\lambda) \mid \mathrm{Re}\lambda \geq r\right\} \\ &= \{0\} \cup \left\{\sum_{k=1}^{l} a_k(0)\frac{\Gamma(\lambda-\beta+1)}{\Gamma(\lambda-\beta_k+1)} \mid \mathrm{Re}\lambda \geq r\right\}. \end{aligned} \tag{4.1.7}$$

The following lemma is a consequence of (4.1.7).

Lemma 4.1.1. *Let $\beta, \beta_k \in \mathbb{R}$, $\beta > 1$, $r > \beta - 1$ and conditions* (4.1.2) *hold. For any $f \in C^{m,r}$, equation* (4.1.5) *has a unique solution $v \in C^{m,r}$ (hence equation* (4.1.1) *has a unique solution $u = J^\beta M^{-\beta} v \in C^{m,r}$) if and only if*

$$\sum_{k=1}^{l} a_k(0) \frac{\Gamma(\lambda - \beta + 1)}{\Gamma(\lambda - \beta_k + 1)} \neq 1, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq r. \tag{4.1.8}$$

To consider equations (4.1.1) and (4.1.5) in the space C^m, $m \geq 1$, under conditions (4.1.2), we have to extend the operators D_0^β, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ to C^m. Indeed, for $\mu > -1$ we have (see (2.2.2), (2.1.7) and (2.1.8))

$$\begin{aligned}(J^\beta w_\mu)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^\mu ds = \frac{t^{\mu+\beta}}{\Gamma(\beta)} \int_0^1 (1-x)^{\beta-1} x^\mu dx \\ &= \frac{t^{\mu+\beta}}{\Gamma(\beta)} B(\beta, \mu+1) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)} w_{\mu+\beta}(t), \ 0 < t \leq T,\end{aligned} \tag{4.1.9}$$

where

$$w_\mu(t) = t^\mu, \quad 0 < t \leq T, \ \mu \in \mathbb{R}.$$

The definition of D_0^β (see (2.2.1)) and (4.1.9) lead us to

$$w_\mu = D_0^\beta J^\beta w_\mu = D_0^\beta \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)} w_{\mu+\beta}, \ \mu > -1,$$

i.e.

$$D_0^\beta w_{\mu+\beta} = \frac{\Gamma(\mu+\beta+1)}{\Gamma(\mu+1)} w_\mu, \ \mu > -1.$$

Hence

$$D_0^\beta w_\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\beta+1)} w_{\mu-\beta} \tag{4.1.10}$$

holds only for $\mu > \beta - 1$. Additionally,

$$\begin{aligned}(V_{\psi_{\beta,\beta_k}} w_\mu)(t) &= \frac{1}{\Gamma(\beta-\beta_k)} \int_0^t \frac{1}{t} \left(1 - \frac{s}{t}\right)^{\beta-\beta_k-1} \left(\frac{s}{t}\right)^{-\beta} s^\mu ds \\ &= \frac{t^\mu}{\Gamma(\beta-\beta_k)} \int_0^1 (1-x)^{\beta-\beta_k-1} x^{\mu-\beta} dx, \quad 0 \leq t \leq T,\end{aligned} \tag{4.1.11}$$

is well-defined only for $\mu > \beta - 1$. To extend operators D_0^β, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ to the space C^m, $m \geq 1$, we use the formula

$$C^m = C^{m,m} \oplus \mathcal{P}_{m-1},$$

where $\mathcal{P}_{m-1}$ is the space of all polynomials with degree less than or equal to $m-1$ and $C^{m,m}$ is defined by (2.1.1).

We start with fractional differentiation operators D_0^β and $D_0^{\beta_k}$. Under conditions (4.1.2), $m \geq 1$, operators D_0^β and $D_0^{\beta_k}$ are well-defined in the space $C^{m,m}$. In particular,

$$D_0^\beta w_n = \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} w_{n-\beta}, \ n > \beta - 1 \ (n \geq m).$$

Operators D_0^β and $D_0^{\beta_k}$ for $\beta_k > 0$ are not defined on the space of polynomials $\mathcal{P}_{m-1}$, therefore, we somehow have to define the extensions $\overline{D}_0^\beta$ and $\overline{D}_0^{\beta_k}$ of D_0^β and $D_0^{\beta_k}$ on the whole $\mathcal{P}_{m-1}$. We put

$$\begin{aligned}
\overline{D}_0^\beta w_n &= \lambda_{n,\beta} w_{n-\beta}, \ \text{for } n = 0, 1, \ldots, m-1, \ m < \beta < m+1, \\
\overline{D}_0^{\beta_k} w_n &= \lambda_{n,\beta_k} w_{n-\beta_k}, \ \text{for } n = 0, 1, \ldots, m_k - 1, \\
&\qquad\qquad m_k < \beta_k < m_k + 1, \ n - \beta_k < -1, \\
\overline{D}_0^{\beta_k} w_n &= D^{\beta_k} w_n, \ \text{for } \beta_k \in \mathbb{N}, \ \beta_k < m,
\end{aligned}$$

where $\lambda_{n,\beta}$ and λ_{n,β_k} are some fixed constants. For $n - \beta_k > -1$, $\beta_k \notin \mathbb{N}_0$, it holds

$$\overline{D}_0^{\beta_k} w_n = D_0^{\beta_k} w_n = \frac{\Gamma(n+1)}{\Gamma(n-\beta_k+1)} w_{n-\beta_k}.$$

Define also

$$\begin{aligned}
\overline{D}_0^\beta u &= D_0^\beta (u - \Pi_{m-1} u) + \overline{D}_0^\beta (\Pi_{m-1} u), \\
&\qquad m < \beta < m+1, \ u \in C^m \text{ with } u - \Pi_{m-1} u \in J^\beta C, \\
\overline{D}_0^{\beta_k} u &= D_0^{\beta_k} (u - \Pi_{m_k-1} u) + \overline{D}_0^{\beta_k} (\Pi_{m_k-1} u), \\
&\qquad m_k < \beta_k < m_k + 1, \ n - \beta_k < -1, \ u \in C^{m_k}, \\
\overline{D}_0^{\beta_k} u &= D_0^{\beta_k} (u - \Pi_{m_k-1} u) + \overline{D}_0^{\beta_k} (\Pi_{m_k-1} u) = D_0^{\beta_k} (u - \Pi_{m_k-1} u) + D^{\beta_k} (\Pi_{m_k-1} u), \\
&\qquad \beta_k \in \mathbb{N}, \ \beta_k < m.
\end{aligned} \tag{4.1.12}$$

There are several different ways to fix constants $\lambda_{n,\beta}$ and λ_{n,β_k}; consequently we also get different extensions $\overline{D}_0^\beta$ and $\overline{D}_0^{\beta_k}$ to C^m, $m \geq 1$. One way to fix the constants would be to take

$$\begin{aligned}
\lambda_{n,\beta} &= \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)}, \ \text{for } n = 0, 1, \ldots, m-1, \ m < \beta < m+1, \\
\lambda_{n,\beta_k} &= \frac{\Gamma(n+1)}{\Gamma(n-\beta_k+1)}, \ \text{for } n = 0, 1, \ldots, m_k - 1, \\
&\qquad\qquad m_k < \beta_k < m_k + 1, \ n - \beta_k < -1,
\end{aligned} \tag{4.1.13}$$

where we interpret

$$\frac{\Gamma(n+1)}{\Gamma(n-\beta_k+1)} = \lim_{y\to n-\beta_k+1} \frac{\Gamma(n+1)}{\Gamma(y)} = 0 \text{ for } n \in \beta_k - \mathbb{N}.$$

If $\lambda_{n,\beta} \neq 0$, $n = 0, 1, \ldots, m-1$, we can extend also the operator $J^\beta = (D_0^\beta)^{-1}$ into operator $\overline{J}^\beta$ by setting

$$\overline{J}^\beta := (\overline{D}_0^\beta)^{-1}.$$

For this we have to check that $\overline{D}_0^\beta$ is invertible on its range, i.e.

$$\left\{ \begin{aligned} &\overline{D}_0^\beta u = D_0^\beta(u - \Pi_{m-1}u) + \overline{D}_0^\beta(\Pi_{m-1}u) = 0 \\ &\qquad\qquad \text{for an } u \in C^m,\ u - \Pi_{m-1}u \in J^\beta C \end{aligned} \right\} \Rightarrow u = 0.$$

Indeed, $u - \Pi_{m-1}u \in J^\beta C$ implies that $D_0^\beta(u - \Pi_{m-1}u) \in C$, hence we also have $\overline{D}_0^\beta(\Pi_{m-1}u) \in C$, which ensues that in

$$\overline{D}_0^\beta(\Pi_{m-1}u) = \sum_{n=0}^{m-1} \frac{u^{(n)}(0)}{n!} \lambda_{n,\beta} w_{n-\beta}$$

the coefficients $u^{(n)}(0)$ vanish if $\lambda_{n,\beta} \neq 0$:

$$u^{(n)}(0) = 0, \quad n = 0, 1, \ldots, m-1.$$

Thus $u \in C^{m,m}$ and $\overline{D}_0^\beta u = D_0^\beta u = 0$, hence $u = J^\beta D_0^\beta u = 0$ as claimed.

On the other hand if $\lambda_{n,\beta} = 0$ for some n, where $0 \leq n \leq m-1$, then $\overline{D}_0^\beta w_n = 0$ and $\overline{D}_0^\beta$ is not invertible in C^m. For instance, for the Caputo fractional differentiation operator D_{Cap}^β we have that $D_{\text{Cap}}^\beta w_n = D_0^\beta(I - \Pi_m)w_n = 0$ for all $n = 0, 1, \ldots, m-1$, and hence D_{Cap}^β is not invertible. Therefore equations with Caputo fractional differentiation operators need a special treatment presented in Section 4.2.

By extending the fractional differentiation operators D_0^β and $D_0^{\beta_k}$ to operators $\overline{D}_0^\beta$ and $\overline{D}_0^{\beta_k}$ in C^m, $m \geq 1$, equation (4.1.1) extends into the differential equation

$$M^\beta \overline{D}_0^\beta u = \sum_{k=1}^{l} a_k(0) M^{\beta_k} \overline{D}_0^{\beta_k} u + f. \tag{4.1.14}$$

Assuming that $\lambda_{n,\beta} \neq 0$, $n = 0, 1, \ldots, m-1$, we make in (4.1.14) the change of variables $v = M^\beta \overline{D}_0^\beta u$, i.e. $u = \overline{J}_0^\beta M^{-\beta} v$ and get the equation

$$v = \sum_{k=1}^{l} a_k(0) \left[M^{\beta_k} \overline{D}_0^{\beta_k}\right] \left[\overline{J}_0^\beta M^{-\beta}\right] v + f. \tag{4.1.15}$$

We now turn to extending the operators $V_{\psi_{\beta,\beta_k}}$ to C^m, $m \geq 1$. Under conditions (4.1.2), operator $V_{\psi_{\beta,\beta_k}}$ belongs to $\mathcal{L}(C^{m,m})$ since $\psi_{\beta,\beta_k} \in L^{1,m}(0,1)$ ($m > \beta - 1$). It follows from (4.1.11) and (2.1.8) that

$$V_{\psi_{\beta,\beta_k}} w_n = \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)} w_n,$$

holds only for $n > \beta - 1$. By (4.1.2), $m > \beta - 1$, hence operator $V_{\psi_{\beta,\beta_k}}$ is completely undefined on the space $\mathcal{P}_{m-1}$ since $V_{\psi_{\beta,\beta_k}} w_n$, $n = 0, 1, \ldots, m-1$, do not exist. To extend the operator $V_{\psi_{\beta,\beta_k}}$ to operator $\overline{V}_{\psi_{\beta,\beta_k}}$ on $\mathcal{P}_{m-1}$ one has somehow to define $\overline{V}_{\psi_{\beta,\beta_k}} w_n$ also for $n < \beta - 1$. We take

$$\overline{V}_{\psi_{\beta,\beta_k}} w_n = \nu_{n,\beta,\beta_k} w_n, \quad n = 0, 1, \ldots, m-1, \tag{4.1.16}$$

where ν_{n,β,β_k} are some fixed constants. Define also

$$\overline{V}_{\psi_{\beta,\beta_k}} u = V_{\psi_{\beta,\beta_k}}(u - \Pi_{m-1} u) + \overline{V}_{\psi_{\beta,\beta_k}}(\Pi_{m-1} u), \quad u \in C^m. \tag{4.1.17}$$

There are several different ways how constants ν_{n,β,β_k} can be fixed, hence we also get different extensions $\overline{V}_{\psi_{\beta,\beta_k}}$ of $V_{\psi_{\beta,\beta_k}}$ to C^m. As we demonstrate below, the method we propose is connected with the Hadamard finite part of a divergent integral. Since the gamma function Γ is defined and analytic on the whole complex plane except the poles at $0, -1, -2, \ldots$ (see e.g. [30]) and since $n - \beta + 1$ is for $\beta \notin \mathbb{N}$ (see (4.1.2)) different from the poles, it is natural to put

$$\nu_{n,\beta,\beta_k} = \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)}, \quad n = 0, 1, \ldots, m-1. \tag{4.1.18}$$

Note that in case $n \in \beta_k - \mathbb{N}$, it holds

$$\frac{\Gamma(n-\beta+1)}{\Gamma(y)} \to 0 \text{ as } y \to n - \beta_k + 1,$$

thus actually $\nu_{n,\beta,\beta_k} = 0$ for $n \in \beta_k - \mathbb{N}$. If we were to allow $\beta \in \mathbb{N}$ in conditions (1.1.5) and (4.1.2), then the right hand side of (4.1.18) may be undefined. It is the reason we restrict ourselves for equations (1.1.2) with conditions (1.1.5) to the case $\beta \notin \mathbb{N}$.

If we fix constants ν_{n,β,β_k} by (4.1.18), then the extension (4.1.17) corresponds, for $\beta > m \geq 1$, to the Hadamard finite part interpretation (see (2.5.2)) of the diverging integral $V_{\psi_{\beta,\beta_k}}u$, $u \in C^m$. Indeed, for $0 < t \leq T$, $u \in C^m$,

$$\begin{aligned}(\text{f.p.}V_{\psi_{\beta,\beta_k}}u)(t) =&\frac{1}{\Gamma(\beta-\beta_k)}\text{f.p.}\int_0^t \frac{1}{t}\left(1-\frac{s}{t}\right)^{\beta-\beta_k-1}\left(\frac{s}{t}\right)^{-\beta} u(s)ds \\ =&\frac{1}{\Gamma(\beta-\beta_k)}\text{f.p.}\int_0^1 (1-x)^{\beta-\beta_k-1}x^{-\beta}u(tx)dx \\ =&\frac{1}{\Gamma(\beta-\beta_k)}\int_0^1 (1-x)^{\beta-\beta_k-1}x^{-\beta}\left[u(tx)-\sum_{j=0}^{m-1}\frac{u^{(j)}(0)}{j!}(tx)^j\right]dx \\ &+\frac{1}{\Gamma(\beta-\beta_k)}\sum_{j=0}^{m-1}\frac{u^{(j)}(0)}{j!}t^j\text{f.p.}\int_0^1 (1-x)^{\beta-\beta_k-1}x^{j-\beta}dx.\end{aligned}$$

If the integral

$$\int_0^1 (1-x)^{\beta-\beta_k-1}x^{j-\beta}dx, \quad j=0,1,\ldots,m-1,\ \beta>1,$$

converges, then it results to $B(\beta-\beta_k, j-\beta+1)$; if the integral diverges, we take it to equal zero (omitting the corresponding term). Consequently,

$$\begin{aligned}(\text{f.p.}V_{\psi_{\beta,\beta_k}}u)(t) =&\frac{1}{\Gamma(\beta-\beta_k)}\int_0^1 (1-x)^{\beta-\beta_k-1}x^{-\beta}\left[u(tx)-\sum_{j=0}^{m-1}\frac{u^{(j)}(0)}{j!}(tx)^j\right]dx \\ &+\sum_{j=0}^{m-1}\frac{u^{(j)}(0)}{j!}\frac{\Gamma(j-\beta+1)}{\Gamma(j-\beta_k+1)}t^j, \quad 0<t\leq T,\ u\in C^m.\end{aligned}$$

i.e.

$$\text{f.p.}V_{\psi_{\beta,\beta_k}}u = V_{\psi_{\beta,\beta_k}}(u-\Pi_{m-1}u)+\overline{V}_{\psi_{\beta,\beta_k}}(\Pi_{m-1}u)=\overline{V}_{\psi_{\beta,\beta_k}}u, \quad u\in C^m.$$

In this sense, choice (4.1.18) of constants ν_{n,β,β_k} is quite natural.

By extending operator $V_{\psi_{\beta,\beta_k}}$ to operator $\overline{V}_{\psi_{\beta,\beta_k}}$ in C^m, $m \geq 1$, using (4.1.16) and (4.1.17), we get an extended cordial Volterra integral equation

$$v=\sum_{k=1}^{l}a_k(0)\overline{V}_{\psi_{\beta,\beta_k}}v+f, \tag{4.1.19}$$

where ψ_{β,β_k} is defined by (4.1.4).

Theorem 4.1.2. *Let $\beta, \beta_k \in \mathbb{R}$, and conditions* (4.1.2) *hold. Also let the operators D_0^{β}, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ be extended to C^m with the help of* $\{(4.1.12),(4.1.13)\}$ *and* $\{(4.1.17),(4.1.18)\}$ *respectively. For any $f \in C^m$, $m \geq 1$, equation* (4.1.19) *has* *a unique solution* $v \in C^m$ *(hence equation* (4.1.14) *also has a unique solution* $u = \left(M^\beta \overline{D}_0^\beta\right)^{-1} v = \overline{J}^\beta M^{-\beta} v$ *in* C^m*) if and only if*

$$\sum_{k=1}^{l} a_k(0) \frac{\Gamma(\lambda - \beta + 1)}{\Gamma(\lambda - \beta_k + 1)} \neq 1, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq m \tag{4.1.20}$$

and

$$\sum_{k=1}^{l} a_k(0) \frac{\Gamma(n - \beta + 1)}{\Gamma(n - \beta_k + 1)} \neq 1, \quad n = 0, 1, \ldots, m-1. \tag{4.1.21}$$

Proof. Under the formulated condition, by Proposition 2.4.2, the spectrum of $\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}}$ as an operator from C^m into C^m $(m \geq 1)$ is given by the formula

$$\begin{aligned}\sigma_m\left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}}\right) = \{0\} &\cup \left\{\sum_{k=1}^{l} a_k(0) \frac{\Gamma(\lambda - \beta + 1)}{\Gamma(\lambda - \beta_k + 1)} \mid \mathrm{Re}\lambda \geq m\right\} \\ &\cup \left\{\sum_{k=1}^{l} a_k(0) \frac{\Gamma(n - \beta + 1)}{\Gamma(n - \beta_k + 1)} \mid n = 0, 1, \ldots, m-1\right\}.\end{aligned} \tag{4.1.22}$$

The claims concerning equation (4.1.19) follow directly from (4.1.22). For the claims regarding equation (4.1.14) we note that $\overline{J}^\beta = (\overline{D}_0^\beta)^{-1}$. □

Note that, by using other extensions of D_0^β, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ to C^m we also obtain different extended differential and integral equations. Theorem 4.1.2 remains to be true but (4.1.21) takes the form which corresponds to the invertibility of the extended operators on the space $\mathcal{P}_{m-1}$.

4.1.2 Equation with non-constant coefficients

Let us now turn to equation (1.1.2) with possibly non-constant coefficients. We make the change of variables $v = M^\beta D_0^\beta u$ in equation (1.1.2) and get the cordial Volterra integral equation

$$v = \sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}} v + f, \tag{4.1.23}$$

where ψ_{β,β_k} is defined by (4.1.4) and belongs to $L^1(0,1)$ for $0 < \beta < 1$ and to $L^{1,r}(0,1)$ for $r > \beta - 1$, $\beta > 1$.

For $0 < \beta < 1$ (by (1.1.5), $m = 0$), we have $\psi_{\beta,\beta_k} \in L^1(0,1)$, hence by Theorem 2.3.1 we get $\sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}} \in \mathcal{L}(C)$. Similarly as with Lemma 3.1.1, it can be shown that for $a_k \in C$, $k = 1, 2, \ldots, l$,

$$\sigma_0\left(\sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}}\right) = \sigma_0\left(\sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}}\right).$$

The last equality allows us to formulate the following result.

Theorem 4.1.3. *Let $\beta, \beta_k \in \mathbb{R}$, $0 < \beta < 1$, $\beta > \beta_k \geq 0$ and $a_k \in C$ with $k = 1, 2, \ldots, l$. For any $f \in C$, equation* (4.1.23) *has a unique solution $v \in C$ (hence equation* (1.1.2) *has a unique solution $u = J^\beta M^{-\beta} v \in C$) if and only if*

$$\sum_{k=1}^{l} a_k(0) \frac{\Gamma(\lambda - \beta + 1)}{\Gamma(\lambda - \beta_k + 1)} \neq 1, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \geq 0.$$

For $\beta > 1$ (by (1.1.5), $m \geq 1$), the situation is more complicated, we first extend the operators D_0^β, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ to C^m with the help of (4.1.12) and (4.1.17). Consequently, we get the extended differential equation

$$M^\beta \overline{D}_0^\beta u = \sum_{k=1}^{l} a_k M^{\beta_k} \overline{D}_0^{\beta_k} u + f, \tag{4.1.24}$$

and the extended cordial Volterra integral equation

$$v = \sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} v + f, \tag{4.1.25}$$

where ψ_{β,β_k} is defined by (4.1.4). We need the following auxiliary result.

Lemma 4.1.2. *Let the operators $V_{\psi_{\beta,\beta_k}}$ be extended to $\overline{V}_{\psi_{\beta,\beta_k}}$ with the help of* $\{(4.1.17), (4.1.18)\}$. *Under conditions $a_k \in C^m$, $m \geq 1$, $k = 1, 2, \ldots, l$, it holds*

$$\sigma_m\left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}}\right) = \sigma_m\left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}}\right), \tag{4.1.26}$$

where ψ_{β,β_k} is defined by formula (4.1.4).

Proof. We have to show that under the formulated conditions the following relations hold:

$$\rho_m\left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}}\right) \subset \rho_m\left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}}\right), \tag{4.1.27}$$

$$\sigma_m\left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}}\right) \subset \sigma_m\left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}}\right). \tag{4.1.28}$$

Below, we present in detail only the proof of (4.1.27). For the the proof of (4.1.28) note that, by (4.1.22), inclusion (4.1.28) is equivalent to the following:

$$\{0\} \cup \left\{ \sum_{k=1}^{l} a_k(0) \frac{\Gamma(q-\beta+1)}{\Gamma(q-\beta_k+1)} \mid q = 0, 1, \dots, m-1 \right\}$$
$$\cup \left\{ \sum_{k=1}^{l} a_k(0) \frac{\Gamma(\lambda-\beta+1)}{\Gamma(\lambda-\beta+1)} \mid \mathrm{Re}\lambda \geq m \right\} \subset \sigma_m \left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \right).$$

The proof of the last inclusion can be carried out in a similar way as parts d)-f) of the proof of Lemma 3.1.1.

To establish (4.1.27) we take a $\mu \in \rho_m \left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}} \right)$ and show that then $\mu \in \rho_m \left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \right)$. We may assume that $\mu \neq 0$ since according to (4.1.22) we have $0 \in \sigma_m \left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}} \right)$. Now,

$$\mu I - \sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} = \mu I - \sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}} - \sum_{k=1}^{l} (a_k - a_k(0)) \overline{V}_{\psi_{\beta,\beta_k}}.$$

Observe that the operator

$$\sum_{k=1}^{l} (a_k - a_k(0)) \overline{V}_{\psi_{\beta,\beta_k}} \in \mathcal{L}(C^m)$$

is compact. Indeed, for $u \in C^m$, we have

$$\sum_{k=1}^{l} (a_k - a_k(0)) \overline{V}_{\psi_{\beta,\beta_k}} u = \sum_{k=1}^{l} (a_k - a_k(0)) V_{\psi_{\beta,\beta_k}} (u - \Pi_{m-1} u)$$
$$- \sum_{k=1}^{l} (a_k - a_k(0)) \overline{V}_{\psi_{\beta,\beta_k}} \Pi_{m-1} u$$

and by Theorem 2.3.2 operator $\sum_{k=1}^{l} (a_k - a_k(0)) V_{\psi_{\beta,\beta_k}}$ is compact in $\mathcal{L}(C^{m,m})$, whereas $\sum_{k=1}^{l} (a_k - a_k(0)) \overline{V}_{\psi_{\beta,\beta_k}} \Pi_{m-1}$ is finite-dimensional. Thus, by Proposition 2.4.1, we get

$$\mu I - \sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \in \Phi_0(C^m).$$

To prove that $\mu \in \rho_m \left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \right)$, it is sufficient to show that the homogeneous equation

$$\mu u = \sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} u \tag{4.1.29}$$

has in C^m only the trivial solution (see Corollary 2.4.1). Let $u_0 \in C^m$ be a solution:

$$\mu u_0 = \left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \right) u_0, \tag{4.1.30}$$

i.e.

$$\mu u_0 = \left(\sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}} \right) (u_0 - \Pi_{m-1} u_0) + \left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \right) \Pi_{m-1} u_0. \tag{4.1.31}$$

We first show by induction that

$$u_0^{(q)}(0) = 0, \quad q = 0, 1, \ldots, m-1. \tag{4.1.32}$$

By $\{(4.1.17), (4.1.18)\}$, for $w_n(t) = t^n$, $0 \le t \le T$, $n = 0, 1, \ldots, m-1$, we have

$$\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}} w_n = \sum_{k=1}^{l} a_k(0) \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)} w_n. \tag{4.1.33}$$

For $t = 0$, equation (4.1.31) takes the form

$$\mu u_0(0) = \left(\sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}} \right) (u_0(0) - u_0(0)) + \sum_{k=1}^{l} a_k(0) u_0(0) \frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)},$$

since $\Pi_{m-1} u_0(0) = u_0(0)$ and

$$\overline{V}_{\psi_{\beta,\beta_k}} w_0 = \frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)}, \quad k = 1, 2, \ldots, l.$$

Thus

$$\mu u_0(0) = u_0(0) \sum_{k=1}^{l} a_k(0) \frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)}. \tag{4.1.34}$$

If $u_0(0) \neq 0$, then

$$\mu = \sum_{k=1}^{l} a_k(0) \frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)} \in \sigma_m \left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}} \right)$$

(see (4.1.22)) that contradicts our choice of μ. Hence $u_0(0) = 0$. Next, we show that the induction hypothesis

$$u_0^{(j)}(0) = 0 \text{ for } j = 0, 1, \ldots, p-1, \text{ where } p \le m-1,$$

leads to $u_0^{(p)}(0) = 0$. Indeed, by (4.1.31), we get

$$\mu u_0^{(p)} = \left(\left(\sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}} \right) (u_0 - \Pi_{m-1} u_0) \right)^{(p)} + \left(\left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} \right) \Pi_{m-1} u_0 \right)^{(p)}. \tag{4.1.35}$$

According to $\{(4.1.17), (4.1.18)\}$, we have

$$\left(\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}}\right) \Pi_{m-1}u_0 = \sum_{k=1}^{l} a_k \sum_{n=0}^{m-1} \frac{u_0^{(n)}(0)}{n!} \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)} w_n,$$

hence by (4.1.35) and (2.3.6), we get

$$\begin{aligned}
\mu u_0^{(p)} =& \sum_{q=0}^{p} \frac{p!}{q!(p-q)!} \sum_{k=1}^{l} a_k^{(p-q)} V_{\psi^{[q]}_{\beta,\beta_k}} (u_0 - \Pi_{m-1}u_0)^{(q)} \\
&+ \sum_{q=0}^{p} \frac{p!}{q!(p-q)!} \sum_{k=1}^{l} a_k^{(p-q)} \sum_{n=0}^{m-1} \frac{u_0^{(n)}(0)}{n!} \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)} w_n^{(q)} \\
=& \sum_{k=1}^{l} a_k V_{\psi^{[p]}_{\beta,\beta_k}} (u_0 - \Pi_{m-1}u_0)^{(p)} + \sum_{k=1}^{l} a_k \sum_{n=0}^{m-1} \frac{u_0^{(n)}(0)}{n!} \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)} w_n^{(p)} \\
&+ \sum_{q=0}^{p-1} \frac{p!}{q!(p-q)!} \sum_{k=1}^{l} a_k^{(p-q)} V_{\psi^{[q]}_{\beta,\beta_k}} (u_0 - \Pi_{m-1}u_0)^{(q)} \\
&+ \sum_{q=0}^{p-1} \frac{p!}{q!(p-q)!} \sum_{k=1}^{l} a_k^{(p-q)} \sum_{n=0}^{m-1} \frac{u_0^{(n)}(0)}{n!} \frac{\Gamma(n-\beta+1)}{\Gamma(n-\beta_k+1)} w_n^{(q)},
\end{aligned} \tag{4.1.36}$$

where

$$\psi^{[q]}_{\beta,\beta_k}(x) = \psi_{\beta,\beta_k}(x) x^q, \quad q = 0, 1, \ldots, p.$$

Since $u_0^{(j)}(0) = 0$ for $j = 0, 1, \ldots, p-1$, it holds that

$$(\Pi_{m-1}u_0)^{(p)}(0) = u_0^{(p)}(0), \quad (u_0 - \Pi_{m-1}u_0)^{(q)}(0) = 0 \tag{4.1.37}$$

for $0 \leq q \leq n-1 \leq m-1$, $p \leq m-1$, also

$$w_n^{(p)}(0) = (t^n)^{(p)}|_{t=0} = \begin{cases} 0, & p < n, \\ n!, & p = n. \end{cases} \tag{4.1.38}$$

By (2.3.5), (4.1.37) and (4.1.38), for $t = 0$, equation (4.1.36) takes the form

$$\mu u_0^{(p)}(0) = u_0^{(p)}(0) \sum_{k=1}^{l} a_k(0) \frac{\Gamma(p-\beta+1)}{\Gamma(p-\beta_k+1)},$$

If $u_0^{(p)}(0) \neq 0$, then

$$\mu = \sum_{k=1}^{l} a_k(0) \frac{\Gamma(p-\beta+1)}{\Gamma(p-\beta_k+1)} \in \sigma_m \left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}}\right)$$

(see (4.1.22)) that contradicts our choice of μ. Hence $u_0^{(p)}(0) = 0$. This completes the proof of (4.1.32).

Next, to obtain $u_0 = 0$, it is sufficient to show that $u_0^{(m)} = 0$ (see (4.1.32)). Note that (4.1.32) actually means that $u_0 \in C^{m,m}$, hence

$$\sum_{k=1}^{l} a_k \overline{V}_{\psi_{\beta,\beta_k}} u_0 = \sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}} u_0. \tag{4.1.39}$$

Due to (4.1.30) and (4.1.39), we get

$$\mu u_0^{(m)} = \left(\left(\sum_{k=1}^{l} a_k V_{\psi_{\beta,\beta_k}} \right) u_0 \right)^{(m)},$$

i.e. (see (2.3.6))

$$\begin{aligned} \left(\mu I - \sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}^{[m]}} \right) u_0^{(m)} = & \sum_{k=1}^{l} [a_k - a_k(0)] V_{\psi_{\beta,\beta_k}^{[m]}} u_0^{(m)} \\ & + \sum_{q=0}^{m-1} \frac{m!}{q!(m-q)!} \sum_{k=1}^{l} a_k^{(m-q)} V_{\psi_{\beta,\beta_k}^{[q]}} u_0^{(q)}, \end{aligned} \tag{4.1.40}$$

where

$$\sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}^{[m]}} \in \mathcal{L}(C) \text{ with } \psi_{\beta,\beta_k}^{[m]} \in L^1(0,1).$$

Since $\mu \in \rho_m \left(\sum_{k=1}^{l} a_k(0) \overline{V}_{\psi_{\beta,\beta_k}} \right)$, $m \geq 1$, it holds $\mu \in \rho_{m,m} \left(\sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}} \right)$, hence also $\mu \in \rho_0 \left(\sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}^{[m]}} \right)$. Consequently, operator

$$\mu I - \sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}^{[m]}} : C \to C$$

is invertible, and according to Proposition 2.3.1 the inverse has the form

$$\left(\mu I - \sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}^{[m]}} \right)^{-1} = \mu^{-1} I + V_{\psi_m}, \text{ with a } \psi_m \in L^1(0,1).$$

Also, relations (4.1.32) imply that, for $0 \leq t \leq T$, $q = 0, 1, \ldots, m-1$,

$$u_0^{(q)}(t) = \frac{1}{(m-q-1)!} \int_0^t (t-s)^{m-q-1} u_0^{(m)}(s) ds.$$

In conclusion, it follows from (4.1.40) that

$$u_0^{(m)} = (\mu^{-1} I + V_{\psi_m}) \times \left\{ \sum_{k=1}^{l} [a_k - a_k(0)] V_{\psi^{[m]}_{\beta,\beta_k}} u_0^{(m)} + \sum_{q=0}^{m-1} \frac{m!}{q!(m-q)!} \sum_{k=1}^{l} a_k^{(m-q)} V_{\psi^{[q]}_{\beta,\beta_k}} G_q u_0^{(m)} \right\}, \tag{4.1.41}$$

where

$$(G_q v)(t) = \frac{1}{(m-q-1)!} \int_0^t (t-s)^{m-q-1} v(s) ds, \quad q = 0, 1, \ldots, m-1.$$

Note that,

$$|v(t)| \le ct^p,\ p \ge 0 \Rightarrow |(G_q v)(t)| \le c_1 t^{p+m-q}, \quad \left| \left(\sum_{k=1}^{l} [a_k - a_k(0)] V_{\psi^{[m]}_{\beta,\beta_k}} v \right)(t) \right| \le c_2 t^{p+1}.$$

Also, operators $\sum_{k=1}^{l} a_k^{(m-q)} V_{\psi^{[q]}_{\beta,\beta_k}}$ and V_{ψ_m} preserve the convergence order of $v(t)$ for $t \to 0$. Approximating $u_0^{(m)}(t)$, with the help of (4.1.41), step-by-step, similarly as was done in Section 3.2 part a), we obtain that $u_0^{(m)} = 0$. Thus equation (4.1.29) possesses in C^m only the trivial solution $u = u_0 = 0$ and therefore (4.1.27) holds for $m \ge 1$. □

We can now formulate the main result of Chapter 4.

Theorem 4.1.4. *Let $\beta, \beta_k \in \mathbb{R}$, and conditions* (1.1.5) *hold. Also let the operators D_0^β, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ be extended to C^m with the help of* $\{(4.1.12), (4.1.13)\}$ *and* $\{(4.1.17), (4.1.18)\}$ *respectively. For any $f \in C^m$, $m \ge 1$, equation* (4.1.25) *has a unique solution $v \in C^m$ (hence also equation* (4.1.24) *has a unique solution $u = \left(M^\beta \overline{D}_0^\beta\right)^{-1} v = \overline{J}^\beta M^{-\beta} v$ in C^m) if and only if*

$$\sum_{k=1}^{l} a_k(0) \frac{\Gamma(\lambda - \beta + 1)}{\Gamma(\lambda - \beta_k + 1)} \ne 1,\ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda \ge m \tag{4.1.42}$$

and

$$\sum_{k=1}^{l} a_k(0) \frac{\Gamma(n - \beta + 1)}{\Gamma(n - \beta_k + 1)} \ne 1, \quad n = 0, 1, \ldots, m-1. \tag{4.1.43}$$

Proof. The claims for equation (4.1.25) follow directly from (4.1.22) and Lemma 4.1.2. For the claims of Theorem 4.1.4 regarding equation (4.1.24) we note that $\overline{J}^\beta = (\overline{D}_0^\beta)^{-1}$. □

Note that, by using other extensions of D_0^β, $D_0^{\beta_k}$ and $V_{\psi_{\beta,\beta_k}}$ to C^m we also obtain different extended differential and integral equations. Theorem 4.1.4 remains to be true but (4.1.43) takes the form which corresponds to the invertibility of the extended operators on the space $\mathcal{P}_{m-1}$.

4.2 Equations with main terms $M^\beta D^\beta_{\mathrm{R-L}}$ or $M^\beta D^\beta_{\mathrm{Cap}}$

In Section 3.1 we showed, with the help of Propositions 2.2.1 and 2.2.2, that for $u \in C^m$, equation (1.1.3) is equivalent to the equation which we obtain from (1.1.3) replacing our fractional differentiation operators D_0^α, $D_0^{\alpha_k}$ either by $D^\alpha_{\text{R-L}}$, $D^{\alpha_k}_{\text{R-L}}$ or by D^α_{Cap}, $D^{\alpha_k}_{\text{Cap}}$, where $k = 1, 2, \ldots, l$. Consequently our results regarding equation (1.1.3) with conditions (1.1.4) remained to be true also in the case with Riemann-Liouville or Caputo fractional differentiation operators. For equation (1.1.2) with conditions (1.1.5) the situation regarding Riemann-Liouville fractional differentiation operators is similar but with Caputo fractional differentiation operators it is more complicated.

Let us start with Riemann-Liouville fractional differentiation operators. According to Proposition 2.2.1, we get that for $u \in C^m$, $m < \beta < m+1$, $m \in \mathbb{N}_0$, it holds

$$M^\beta D^\beta_{\text{R-L}} u = M^\beta D_0^\beta u.$$

Also, for $u \in C^{m_k}$, $m_k < \beta_k \le m_k + 1$, $m_k \in \mathbb{N}_0$, $k = 1, 2, \ldots, l$, it holds

$$M^{\beta_k} D^{\beta_k}_{\text{R-L}} u = M^{\beta_k} D_0^{\beta_k} u.$$

Thus, for $u \in C^m$, equation (1.1.2) with (1.1.5) is equivalent to the equation which we obtain from (1.1.2) replacing D_0^β, $D_0^{\beta_k}$ by $D^\beta_{\text{R-L}}$, $D^{\beta_k}_{\text{R-L}}$, where $k = 1, 2, \ldots, l$. Consequently, our results regarding equation (1.1.2) with (1.1.5) remain to be true also in the case with Riemann-Liouville fractional differentiation operators.

The switch from D_0^β, $D_0^{\beta_k}$ to D^β_{Cap}, $D^{\beta_k}_{\text{Cap}}$ requires a more elaborate discussion. The following subsections (Subsections 4.2.1 and 4.2.2) are dedicated to this. The basis of our considerations is Proposition 2.2.2, which gives a way to move from D^β_{Cap}, $D^{\beta_k}_{\text{Cap}}$ to D_0^β, $D_0^{\beta_k}$. Fortunately, an important privilege of Caputo fractional differentiation operators is that we do not need to extend our operators.

4.2.1 Equation with main term $M^\beta D^\beta_{\text{Cap}}$ and constant coefficients

We start our discussion by considering the equation

$$M^\beta D^\beta_{\text{Cap}} u = \sum_{k=1}^{l} a_k(0) M^{\beta_k} D^{\beta_k}_{\text{Cap}} u + a(0) u + f, \tag{4.2.1}$$

where $f \in C$ and

$$0 < \beta_1 < \ldots < \beta_l < \beta < 1, \quad \beta, \beta_k \in \mathbb{R}, \quad k = 1, 2, \ldots, l. \tag{4.2.2}$$

By Proposition 2.2.2 and formula (4.1.10), we get

$$M^\beta D^\beta_{\mathrm{Cap}} u = M^\beta D^\beta_0 (u - \Pi_0 u) = M^\beta D^\beta_0 u - \frac{u(0)}{\Gamma(1-\beta)} w_0,$$
$$M^{\beta_k} D^{\beta_k}_{\mathrm{Cap}} u = M^{\beta_k} D^{\beta_k}_0 (u - \Pi_0 u) = M^{\beta_k} D^{\beta_k}_0 u - \frac{u(0)}{\Gamma(1-\beta_k)} w_0,$$

where

$$w_\mu(t) = t^\mu, \ 0 < t \leq T, \ \mu \in \mathbb{R}. \tag{4.2.3}$$

Consequently, equation (4.2.1) takes the form

$$\begin{aligned} M^\beta D^\beta_0 u = \sum_{k=1}^{l} a_k(0) M^{\beta_k} D^{\beta_k}_0 u + a(0) u + f \\ + \frac{u(0)}{\Gamma(1-\beta)} w_0 - \sum_{k=1}^{l} a_k(0) \frac{u(0)}{\Gamma(1-\beta_k)} w_0. \end{aligned} \tag{4.2.4}$$

We make in (4.2.4) the change of variables $v = M^\beta D^\beta_0 u$, i.e.

$$u = (M^\beta D^\beta_0)^{-1} v = J^\beta M^{-\beta} v, \quad u(0) = v(0)\Gamma(1-\beta),$$

obtaining with respect to v the equation

$$\begin{aligned} v = \sum_{k=1}^{l} a_k(0) [M^{\beta_k} D^{\beta_k}_0][J^\beta M^{-\beta}] v + a(0) J^\beta M^{-\beta} v \\ + f + \frac{v(0)\Gamma(1-\beta)}{\Gamma(1-\beta)} w_0 - \sum_{k=1}^{l} a_k(0) \frac{v(0)\Gamma(1-\beta)}{\Gamma(1-\beta_k)} w_0, \end{aligned}$$

i.e.

$$v = \sum_{k=1}^{l} a_k(0) V_{\psi_{\beta,\beta_k}} v + a(0) V_{\psi_{\beta,0}} v + f + v(0) - \sum_{k=1}^{l} a_k(0) v(0) \frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)}, \tag{4.2.5}$$

where $V_{\psi_{\beta,\beta_k}}$ is the cordial Volterra integral operator with the core $\psi_{\beta,\beta_k} \in L^1(0,1)$ defined by (see Subsection 4.1.1 page 46)

$$\psi_{\beta,\beta_k}(x) = \frac{1}{\Gamma(\beta-\beta_k)} (1-x)^{\beta-\beta_k-1} x^{-\beta}, \quad 0 < x < 1. \tag{4.1.4}$$

At $t = 0$, equation (4.2.5) takes the form (see (2.3.5) and (4.1.6))

$$v(0) = \sum_{k=1}^{l} a_k(0)v(0)\frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)} + a(0)v(0)\Gamma(1-\beta) + f(0) + v(0) - \sum_{k=1}^{l} a_k(0)v(0)\frac{\Gamma(1-\beta)}{\Gamma(1-\beta_k)},$$

i.e.

$$a(0)v(0)\Gamma(1-\beta) + f(0) = 0.$$

Assuming that $a(0) \neq 0$, we have

$$v(0) = -\frac{f(0)}{\Gamma(1-\beta)a(0)}. \tag{4.2.6}$$

In conclusion, equation (4.2.5) with conditions (4.2.2) takes the form

$$v = Wv + f - \frac{f(0)}{\Gamma(1-\beta)a(0)} + \sum_{k=1}^{l} a_k(0)\frac{f(0)}{a(0)\Gamma(1-\beta_k)}, \tag{4.2.7}$$

where

$$Wy = \sum_{k=1}^{l} a_k(0)V_{\psi_{\beta,\beta_k}}y + a(0)V_{\psi_{\beta,0}}y, \quad y \in C.$$

Equation (4.2.7) is a cordial Volterra integral equation in C, it was obtained under condition $a(0) \neq 0$. This equation (equation (4.2.7)) is uniquely solvable in C if and only if

$$1 \notin \sigma_0(W), \tag{4.2.8}$$

where (see (2.3.7) and (4.1.6))

$$\sigma_0(W) = \left\{\sum_{k=1}^{l} a_k(0)\widehat{\psi}_{\beta,\beta_k}(\lambda) + a(0)\widehat{\psi}_{\beta,0}(\lambda) \mid \mathrm{Re}\lambda \geq 0\right\}$$
$$= \left\{\sum_{k=1}^{l} a_k(0)\frac{\Gamma(\lambda+1-\beta)}{\Gamma(\lambda+1-\beta_k)} + a(0)\frac{\Gamma(\lambda+1-\beta)}{\Gamma(\lambda+1)} \mid \mathrm{Re}\lambda \geq 0\right\}.$$

We have shown that the following result holds.

Theorem 4.2.1. *Let conditions* (4.2.2) *hold. For any $f \in C$, equation* (4.2.5) *has a unique solution $v \in C$ if and only if $a(0) \neq 0$ and* (4.2.8) *holds. Moreover, the solution v is uniquely determined by condition* (4.2.6) *and due to* (4.2.8).

Having found the solution $v \in C$ of (4.2.5) the solution of equation (4.2.1) has the form $u = J^\beta M^{-\beta} v \in C$ and

$$u(0) = -\frac{f(0)}{a(0)}.$$

Note that, if $a(0) = 0$, then equation (4.2.5) (hence also equation (4.2.1)) with conditions (4.2.2) is solvable in C only for $f \in C$ that satisfy $f(0) = 0$.

We now turn our attention to the case $\beta > 1$ considering the equation

$$M^\beta D^\beta_{\mathrm{Cap}} u = b(0) M^\nu D^\nu_{\mathrm{Cap}} u + a(0) u + f, \tag{4.2.9}$$

where $f \in C^m$ and

$$m < \beta < m+1, \quad n < \nu \leq n+1, \; n < m, \quad n \in \mathbb{N}_0, \; m \in \mathbb{N}. \tag{4.2.10}$$

We are interested in the unique solvability of equation (4.2.9) in C^m, $m \geq 1$, under the conditions (4.2.10).

We start our discussion by using Proposition 2.2.2 and formula (4.1.10) to move from Caputo fractional differentiation operators D^β_{Cap} and D^ν_{Cap} to fractional differentiation operators D^β_0 and D^ν_0:

$$\begin{aligned} M^\beta D^\beta_{\mathrm{Cap}} u =& M^\beta D^\beta_0 (u - \Pi_m u) = M^\beta D^\beta_0 (u - \Pi_{m-1} u) - \frac{u^{(m)}(0)}{m!} M^\beta D^\beta_0 w_m \\ =& M^\beta D^\beta_0 (u - \Pi_{m-1} u) - \frac{u^{(m)}(0)}{\Gamma(m+1-\beta)} w_m, \\ M^\nu D^\nu_{\mathrm{Cap}} u =& M^\nu D^\nu_0 (u - \Pi_n u) = M^\nu D^\nu_0 (u - \Pi_{m-1} u) + M^\nu D^\nu_0 (\Pi_{m-1} u - \Pi_n u), \end{aligned}$$

where w_m, $m \in \mathbb{N}$, is defined by (4.2.3). Consequently, equation (4.2.9) takes the form

$$\begin{aligned} M^\beta D^\beta_0 (u - \Pi_{m-1} u) =& b(0) M^\nu D^\nu_0 (u - \Pi_{m-1} u) + a(0)(u - \Pi_{m-1} u) \\ &+ (f - \Pi_{m-1} f) + \frac{u^{(m)}(0)}{\Gamma(m+1-\beta)} w_m \\ &+ \Pi_{m-1} f + a(0) \Pi_{m-1} u + b(0) M^\nu D^\nu_0 (\Pi_{m-1} u - \Pi_n u). \end{aligned} \tag{4.2.11}$$

We determine $\Pi_{m-1} u$ in $u = \Pi_{m-1} u + (u - \Pi_{m-1} u)$ so that

$$\Pi_{m-1} f + a(0) \Pi_{m-1} u + b(0) M^\nu D^\nu_0 (\Pi_{m-1} u - \Pi_n u) = 0, \tag{4.2.12}$$

then after finding also $u^{(m)}(0)$, equation (4.2.11), reduces to an equation to determine $u - \Pi_{m-1} u$.

We consider the cases $n = m-1$ and $n < m-1$ separately. If $n = m-1$, then

$$\Pi_{m-1} u - \Pi_n u = \Pi_{m-1} u - \Pi_{m-1} u = 0$$

and (4.2.12) takes the form of equation

$$\frac{f^{(m-1)}(0)}{(m-1)!}w_{m-1} + \frac{f^{(m-2)}(0)}{(m-2)!}w_{m-2} + \ldots + f'(0)w_1 + f(0)$$
$$+ a(0)\frac{u^{(m-1)}(0)}{(m-1)!}w_{m-1} + a(0)\frac{u^{(m-2)}(0)}{(m-2)!}w_{m-2} + \ldots + a(0)u'(0)w_1 + a(0)u(0) = 0$$

leading us to conditions

$$f^{(q)}(0) + a(0)u^{(q)}(0) = 0, \quad q = 0, 1, \ldots, m-1.$$

Assuming that $a(0) \neq 0$, we get

$$u^{(q)}(0) = -\frac{f^{(q)}(0)}{a(0)}, \quad q = 0, 1, \ldots, m-1. \tag{4.2.13}$$

If $n < m-1$, then (4.2.12) takes the form of equation (see (4.1.10))

$$\frac{f^{(m-1)}(0)}{(m-1)!}w_{m-1} + \frac{f^{(m-2)}(0)}{(m-2)!}w_{m-2} + \ldots + f'(0)w_1 + f(0)$$
$$+ a(0)\frac{u^{(m-1)}(0)}{(m-1)!}w_{m-1} + a(0)\frac{u^{(m-2)}(0)}{(m-2)!}w_{m-2} + \ldots + a(0)u'(0)w_1 + a(0)u(0)$$
$$+ b(0)\left[\frac{u^{(m-1)}(0)}{\Gamma(m-\nu)}w_{m-1} + \ldots + \frac{u^{(n+1)}(0)}{\Gamma(n+2-\nu)}w_{n+1}\right] = 0,$$

leading to conditions

$$f^{(q)}(0) + a(0)u^{(q)}(0) = 0, \quad q = 0, 1, \ldots, n,$$

and

$$\frac{f^{(q)}(0)}{q!} + a(0)\frac{u^{(q)}(0)}{q!} + b(0)\frac{u^{(q)}(0)}{\Gamma(q+1-\nu)} = 0, \quad q = n+1, n+2, \ldots, m-1.$$

Assuming that $a(0) \neq 0$ and

$$a(0) + b(0)\frac{q!}{\Gamma(q+1-\nu)} \neq 0, \quad q = n+1, n+2, \ldots, m-1, \tag{4.2.14}$$

we get

$$u^{(q)}(0) = -\frac{f^{(q)}(0)}{a(0)}, \quad q = 0, 1 \ldots, n, \tag{4.2.15}$$

and

$$u^{(q)}(0) = -\frac{f^{(q)}(0)}{a(0) + b(0)\frac{q!}{\Gamma(q+1-\nu)}}, \quad q = n+1, n+2, \ldots, m-1. \tag{4.2.16}$$

After determining $\Pi_{m-1}u$ by (4.2.12), equation (4.2.11) takes the form

$$\begin{aligned} M^\beta D_0^\beta(u-\Pi_{m-1}u) =& b(0)M^\nu D_0^\nu(u-\Pi_{m-1}u)+a(0)(u-\Pi_{m-1}u) \\ &+f-\Pi_{m-1}f+\frac{u^{(m)}(0)}{\Gamma(m+1-\beta)}w_m. \end{aligned}$$

We make in the last equation the change of variables $v=M^\beta D_0^\beta(u-\Pi_{m-1}u)$, i.e.

$$u-\Pi_{m-1}u=(M^\beta D_0^\beta)^{-1}v=J^\beta M^{-\beta}v, \quad u^{(m)}(0)=v^{(m)}(0)\frac{\Gamma(m+1-\beta)}{m!},$$

obtaining with respect to v the equation

$$v=b(0)[M^\nu D_0^\nu][J^\beta M^{-\beta}]v+a(0)J^\beta M^{-\beta}v+f-\Pi_{m-1}f+\frac{v^{(m)}(0)}{m!}w_m,$$

i.e.

$$v=b(0)V_{\psi_{\beta,\nu}}v+a(0)V_{\psi_{\beta,0}}v+f-\Pi_{m-1}f+\frac{v^{(m)}(0)}{m!}w_m, \tag{4.2.17}$$

where $V_{\psi_{\beta,\nu}}$ and $V_{\psi_{\beta,0}}$ are cordial Volterra integral operators with cores $\psi_{\beta,\nu}$ and $\psi_{\beta,0}$ belonging to $L^{1,m}(0,1)$ (see Subsection 4.1.1 page 46).

Assuming that (4.2.17) is solvable in $C^{m,m}$, we apply to both sides of (4.2.17) the (invertible) operator D_0^m. Note that, the fact that (4.2.17) is solvable in $C^{m,m}$ is equivalent to that (4.2.9) is solvable in C^m and for $n=m-1$ condition $a(0)\neq 0$ holds and for $n<m-1$ conditions $a(0)\neq 0$ and (4.2.14) hold. We get

$$D_0^m v=b(0)D_0^m[V_{\psi_{\beta,\nu}}v]+a(0)D_0^m[V_{\psi_{\beta,0}}v]+D_0^m(f-\Pi_{m-1}f)+\frac{v^{(m)}(0)}{m!}D_0^m w_m,$$

i.e. (see (2.3.4))

$$D^m v=b(0)V_{\psi^{[m]}_{\beta,\nu}}D^m v+a(0)V_{\psi^{[m]}_{\beta,0}}D^m v+D^m f+v^{(m)}(0),$$

where

$$\psi^{[m]}_{\beta,\nu}(x)=x^m\psi_{\beta,\nu}(x), \quad \psi^{[m]}_{\beta,0}(x)=x^m\psi_{\beta,0}(x), \quad 0<x<1.$$

In particular, at $t=0$, it holds (see (2.3.5) and (4.1.6))

$$\begin{aligned} v^{(m)}(0) =& v^{(m)}(0)b(0)\frac{\Gamma(m+1-\beta)}{\Gamma(m+1-\nu)}+v^{(m)}(0)a(0)\frac{\Gamma(m+1-\beta)}{m!} \\ &+f^{(m)}(0)+v^{(m)}(0), \end{aligned}$$

i.e.

$$v^{(m)}(0)\frac{\Gamma(m+1-\beta)}{m!}\left[a(0)+b(0)\frac{m!}{\Gamma(m+1-\nu)}\right]=-f^{(m)}(0).$$

Assuming that

$$a(0) + b(0)\frac{m!}{\Gamma(m+1-\nu)} \neq 0, \tag{4.2.18}$$

we have

$$v^{(m)}(0) = -\frac{m!}{\Gamma(m+1-\beta)} \frac{f^{(m)}(0)}{(a(0) + b(0)\frac{m!}{\Gamma(m+1-\nu)})}. \tag{4.2.19}$$

In conclusion, equation (4.2.17) takes the form

$$\begin{aligned} v =& b(0)V_{\psi_{\beta,\nu}} v + a(0)V_{\psi_{\beta,0}} v + f - \Pi_{m-1} f \\ &- \frac{f^{(m)}(0)\Gamma(m+1-\nu)}{\Gamma(m+1-\beta)(a(0)\Gamma(m+1-\nu) + b(0)m!)} w_m. \end{aligned} \tag{4.2.20}$$

Equation (4.2.20) is a cordial Volterra integral equation in $C^{m,m}$ to determine $v \in C^{m,m}$. It was obtained for $n = m-1$ under conditions $a(0) \neq 0$, (4.2.18) and for $n < m-1$ under conditions $a(0) \neq 0$, (4.2.14), (4.2.18). Equation (4.2.20) is uniquely solvable in $C^{m,m}$ if and only if

$$1 \notin \sigma_{m,m}(b(0)V_{\psi_{\beta,\nu}} + a(0)V_{\psi_{\beta,0}}), \tag{4.2.21}$$

where (see (2.3.10) and (4.1.6))

$$\begin{aligned} \sigma_{m,m}(b(0)V_{\psi_{\beta,\nu}} + a(0)V_{\psi_{\beta,0}}) =& \left\{ b(0)\widehat{\psi}_{\beta,\nu}(\lambda) + a(0)\widehat{\psi}_{\beta,0}(\lambda) \mid \mathrm{Re}\lambda \geq m \right\} \\ =& \left\{ b(0)\frac{\Gamma(\lambda+1-\beta)}{\Gamma(\lambda+1-\nu)} + a(0)\frac{\Gamma(\lambda+1-\beta)}{\Gamma(\lambda+1)} \mid \mathrm{Re}\lambda \geq m \right\}. \end{aligned}$$

The presented argument shows that the following result holds.

Theorem 4.2.2. *Let conditions* (4.2.10) *hold with* $n < m-1$. *For any* $f \in C^m$ *equation* (4.2.9) *has a unique solution* $u \in C^m$ *if and only if conditions* $a(0) \neq 0$, (4.2.14) *and* (4.2.18) *are fulfilled and* (4.2.21) *holds. Moreover, for the solution*

$$u = (u - \Pi_{m-1}u) + \Pi_{m-1}u,$$

we have that $\Pi_{m-1}u$ *is uniquely determined by conditions* (4.2.15) *and* (4.2.16),

$$u^{(m)}(0) = -\frac{f^{(m)}(0)\Gamma(m+1-\nu)}{a(0)\Gamma(m+1-\nu) + b(0)m!},$$

and $u - \Pi_{m-1}u = J^\beta M^{-\beta} v \in C^{m,m}$ *is uniquely determined due to* (4.2.21).

If some of the conditions $a(0) \neq 0$, (4.2.14) or (4.2.18) are not fulfilled, then equation (4.2.9) under conditions (4.2.10) is not solvable for some $f \in C^m$. For example, if $a(0) \neq 0$ and (4.2.14) holds, but (4.2.18) is not fulfilled, then equation (4.2.9) is solvable in C^m only for $f \in C^m$ with $f^{(m)}(0) = 0$.

Note that, if $n = m-1$, then the last theorem remains to be true but instead of conditions $a(0) \neq 0$, (4.2.14) and (4.2.18) we have conditions $a(0) \neq 0$ and (4.2.18), also in such case $\Pi_{m-1}u$ is determined by conditions (4.2.13).

4.2.2 Equation with main term $M^\beta D^\beta_{\mathrm{Cap}}$ and a non-constant coefficient

We conclude Chapter 4 by analyzing an equation with Caputo fractional differentiation operators which may have a non-constant coefficient by considering the equation

$$M^\beta D^\beta_{\mathrm{Cap}} u = au + f, \tag{4.2.22}$$

where $a, f \in C^m$ and $m < \beta < m+1$. We are interested in the unique solvability of equation (4.2.22) in the space C^m, $m \in \mathbb{N}_0$.

Let $m = 0$, i.e. $0 < \beta < 1$. According to Proposition 2.2.2 and (4.1.10), we get

$$M^\beta D^\beta_{\mathrm{Cap}} u = M^\beta D^\beta_0 (u - \Pi_0 u) = M^\beta D^\beta_0 u - \frac{u(0)}{\Gamma(1-\beta)} w_0,$$

hence equation (4.2.22) for $m = 0$ takes the form

$$M^\beta D^\beta_0 u = au + f + \frac{u(0)}{\Gamma(1-\beta)} w_0.$$

To determine $u(0)$ we apply the (invertible) operator $J^\beta M^{-\beta}$ to both sides of the last equation:

$$J^\beta M^{-\beta} M^\beta D^\beta_0 u = J^\beta M^{-\beta}(au) + J^\beta M^{-\beta} f + \frac{u(0)}{\Gamma(1-\beta)} J^\beta M^{-\beta} w_0,$$

i.e.

$$u = V_{\psi_{\beta,0}}(au) + V_{\psi_{\beta,0}} f + u(0), \tag{4.2.23}$$

where $V_{\psi_{\beta,0}}$ is the cordial Volterra integral operator with the core $\psi_{\beta,0} \in L^1(0,1)$ (see (4.1.4)). At $t = 0$, the last equation takes the form (see (2.3.5) and (4.1.6))

$$u(0) = \Gamma(1-\beta)a(0)u(0) + \Gamma(1-\beta)f(0) + u(0).$$

Assuming that $a(0) \neq 0$, we get

$$u(0) = -\frac{f(0)}{a(0)}. \tag{4.2.24}$$

In conclusion, equation (4.2.23) takes the form

$$u = V_{\psi_{\beta,0}}(au) + V_{\psi_{\beta,0}} f - \frac{f(0)}{a(0)}. \tag{4.2.25}$$

The last equation is a cordial Volterra integral equation in C to determine u, it was obtained under the condition $a(0) \neq 0$. Equation (4.2.25) is uniquely solvable in C if and only if

$$1 \notin \sigma_0(V_{\psi_{\beta,0}} a). \tag{4.2.26}$$

By Theorem 2.3.1, we get that

$$\sigma_0(V_{\psi_{\beta,0}}a) = a(0)\sigma_0(V_{\psi_{\beta,0}}),$$

where (see (2.3.7) and (4.1.6))

$$a(0)\sigma_0(V_{\psi_{\beta,0}}) = \left\{a(0)\widehat{\psi}_{\beta,0}(\lambda) \mid \mathrm{Re}\lambda \geq 0\right\} = \left\{a(0)\frac{\Gamma(\lambda+1-\beta)}{\Gamma(\lambda+1)} \mid \mathrm{Re}\lambda \geq 0\right\}.$$

We have shown that the following result holds.

Theorem 4.2.3. *Let $0 < \beta < 1$. For any $f \in C$ equation (4.2.22) has a unique solution $u \in C$, if and only if $a(0) \neq 0$ and (4.2.26) holds. Moreover, the solution u is uniquely determined by condition (4.2.24) and due to (4.2.26).*

Let now $m \geq 1$. By Proposition 2.2.2 and (4.1.10), we get

$$M^\beta D^\beta_{\mathrm{Cap}} u = M^\beta D^\beta_0(u - \Pi_{m-1}u) - \frac{u^{(m)}(0)}{\Gamma(m+1-\beta)} w_m.$$

We also have, for $a, f \in C^m$, $m \in \mathbb{N}$,

$$\begin{aligned} au &= a(u - \Pi_{m-1}u) + a\Pi_{m-1}u \\ &= a(u - \Pi_{m-1}u) + (I - \Pi_{m-1})(a\Pi_{m-1}u) + \Pi_{m-1}(a\Pi_{m-1}u). \end{aligned}$$

Consequently, equation (4.2.22) for $m \geq 1$ takes the form

$$\begin{aligned} M^\beta D^\beta_0(u - \Pi_{m-1}u) =& a(u - \Pi_{m-1}u) + (f - \Pi_{m-1}f) + \frac{u^{(m)}(0)}{\Gamma(m+1-\beta)} w_m \\ &+ (I - \Pi_{m-1})(a\Pi_{m-1}u) + \Pi_{m-1}f + \Pi_{m-1}(a\Pi_{m-1}u). \end{aligned} \tag{4.2.27}$$

We determine $\Pi_{m-1}u$ in $u = \Pi_{m-1}u + (u - \Pi_{m-1}u)$ so that

$$\Pi_{m-1}f + \Pi_{m-1}(a\Pi_{m-1}u) = 0, \tag{4.2.28}$$

then after finding also $u^{(m)}(0)$, equation (4.2.27) reduces to an equation to determine $u - \Pi_{m-1}u$. Condition (4.2.28) leads to

$$f^{(j)}(0) + \sum_{q=0}^{j} \frac{j!}{(j-q)!q!} a^{(j-q)}(0) u^{(q)}(0) = 0, \quad j = 0, 1, \ldots, m-1.$$

Assuming that $a(0) \neq 0$, we get

$$u(0) = -\frac{f(0)}{a(0)} \tag{4.2.29}$$

and

$$u^{(j)}(0) = -\frac{f^{(j)}(0) + \sum_{q=0}^{j-1} \frac{j!}{(j-q)!q!} a^{(j-q)}(0)u^{(q)}(0)}{a(0)}, \quad j = 1, 2, \dots, m-1. \tag{4.2.30}$$

Thus, equation (4.2.27) takes the form

$$\begin{aligned} M^\beta D_0^\beta (u - \Pi_{m-1} u) =& a(u - \Pi_{m-1}u) + f - \Pi_{m-1} f \\ &+ \frac{u^{(m)}(0)}{\Gamma(m+1-\beta)} w_m + (I - \Pi_{m-1})(a\Pi_{m-1}u). \end{aligned}$$

Our next objective is to determine $u^{(m)}(0)$. We start by applying the (invertible) operator $J^\beta M^{-\beta}$ to both sides of the last equation:

$$\begin{aligned} J^\beta M^{-\beta} M^\beta D_0^\beta (u - \Pi_{m-1}u) &= J^\beta M^{-\beta} a(u - \Pi_{m-1}u) + J^\beta M^{-\beta}(f - \Pi_{m-1} f) \\ &+ \frac{u^{(m)}(0)}{\Gamma(m+1-\beta)} J^\beta M^{-\beta} w_m + J^\beta M^{-\beta}((I - \Pi_{m-1})(a\Pi_{m-1}u)), \end{aligned}$$

i.e.

$$\begin{aligned} u - \Pi_{m-1}u =& V_{\psi_{\beta,0}}(a(u - \Pi_{m-1}u)) + V_{\psi_{\beta,0}}(f - \Pi_{m-1}f) + \frac{u^{(m)}(0)}{m!} w_m \\ &+ V_{\psi_{\beta,0}}((I - \Pi_{m-1})(a\Pi_{m-1}u)), \end{aligned} \tag{4.2.31}$$

where $V_{\psi_{\beta,0}}$ is the cordial Volterra integral operator with the core $\psi_{\beta,0} \in L^{1,m}(0,1)$ (see Subsection 4.1.1).

Assuming that equation (4.2.31) is solvable in $C^{m,m}$ or equivalently that equation (4.2.22) is solvable in C^m and $a(0) \neq 0$, we apply the (invertible) operator D_0^m to both sides of (4.2.31) and get

$$\begin{aligned} D_0^m(u - \Pi_{m-1}u) =& D_0^m[V_{\psi_{\beta,0}}(a(u - \Pi_{m-1}u))] + D_0^m[V_{\psi_{\beta,0}}(f - \Pi_{m-1}f)] \\ &+ \frac{u^{(m)}(0)}{m!} D_0^m w_m + D_0^m[V_{\psi_{\beta,0}}((I - \Pi_{m-1})(a\Pi_{m-1}u))]. \end{aligned}$$

The last equation enables us to determine $u^{(m)}(0)$, namely $D^m u = D_0^m(u - \Pi_{m-1}u)$ satisfies (see (2.3.4))

$$\begin{aligned} D^m u = V_{\psi^{[m]}_{\beta,0}} D^m(au) - V_{\psi^{[m]}_{\beta,0}} D^m(a\Pi_{m-1}u) + V_{\psi^{[m]}_{\beta,0}} D^m f \\ + u^{(m)}(0) + V_{\psi^{[m]}_{\beta,0}} D^m(a\Pi_{m-1}u), \end{aligned}$$

where $\psi^{[m]}_{\beta,0}(x) = x^m \psi_{\beta,0}(x)$, $0 < x < 1$. At $t = 0$, it holds (see (2.3.5) and (4.1.6))

$$\begin{aligned} u^{(m)}(0) = \sum_{q=0}^{m} \frac{m!}{(m-q)!q!} \frac{\Gamma(m+1-\beta)}{m!} a^{(m-q)}(0) u^{(q)}(0) \\ + f^{(m)}(0) \frac{\Gamma(m+1-\beta)}{m!} + u^{(m)}(0), \end{aligned}$$

i.e.

$$a(0)u^{(m)}(0) + \sum_{q=0}^{m-1} \frac{m!}{(m-q)!q!} a^{(m-q)}(0)u^{(q)}(0) + f^{(m)}(0) = 0.$$

Assuming that $a(0) \neq 0$, we have

$$u^{(m)}(0) = -\frac{f^{(m)}(0) + \sum_{q=0}^{m-1} \frac{m!}{(m-q)!q!} a^{(m-q)}(0)u^{(q)}(0)}{a(0)}. \tag{4.2.32}$$

In conclusion, equation (4.2.31) takes the form

$$u - \Pi_{m-1}u = V_{\psi_{\beta,0}}(a(u - \Pi_{m-1}u)) + g, \tag{4.2.33}$$

where

$$g = V_{\psi_{\beta,0}}(f - \Pi_{m-1}f) - \frac{f^{(m)}(0) + \sum_{q=0}^{m-1} \frac{m!}{(m-q)!q!} a^{(m-q)}(0)u^{(q)}(0)}{a(0)m!} w_m \\ + V_{\psi_{\beta,0}}((I - \Pi_{m-1})(a\Pi_{m-1}u)).$$

Obviously, $g \in C^{m,m}$ for $f \in C^m$. Equation (4.2.33) is a cordial Volterra integral equation in $C^{m,m}$ to determine $u - \Pi_{m-1}u \in C^{m,m}$, it was obtained under the condition $a(0) \neq 0$. Equation (4.2.33) is uniquely solvable in $C^{m,m}$ if and only if

$$1 \notin \sigma_{m,m}(V_{\psi_{\beta,0}}a). \tag{4.2.34}$$

By Theorem 2.3.2, we get that

$$\sigma_{m,m}(V_{\psi_{\beta,0}}a) = a(0)\sigma_{m,m}(V_{\psi_{\beta,0}}),$$

where (see (2.3.10) and (4.1.6))

$$a(0)\sigma_{m,m}(V_{\psi_{\beta,0}}) = \left\{a(0)\widehat{\psi}_{\beta,0}(\lambda) \mid \mathrm{Re}\lambda \geq m\right\} = \left\{a(0)\frac{\Gamma(\lambda+1-\beta)}{\Gamma(\lambda+1)} \mid \mathrm{Re}\lambda \geq m\right\}.$$

The presented argument shows that the following result holds.

Theorem 4.2.4. *Let $m < \beta < m+1$, $m \geq 1$. For any $f \in C^m$ equation (4.2.22) has a unique solution $u \in C^m$, if and only if $a(0) \neq 0$ and (4.2.34) holds. Moreover, for the solution*

$$u = (u - \Pi_{m-1}u) + \Pi_{m-1}u$$

we have that $\Pi_{m-1}u$ is uniquely determined by (4.2.29) and (4.2.30), $u^{(m)}(0)$ is uniquely determined by (4.2.32), and $u - \Pi_{m-1}u$ is uniquely determined due to (4.2.34).

Chapter 5

Divergent integral with a logarithmic factor

In this chapter, we consider an integral of the form

$$\int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt, \tag{1.1.6}$$

where $\lambda \in \mathbb{C}$, $n \in \mathbb{N}_0$,

$$a \in H^{m,\alpha}, \quad m \in \mathbb{N}_0, \quad 0 < \alpha \leq 1, \tag{1.1.7}$$

and $H^{m,\alpha}$ is the class of functions $a \in C^m$ that satisfy conditions

$$\left| a^{(m)}(t) - a^{(m)}(0) \right| \leq ct^{\alpha}, \quad 0 \leq t \leq T. \tag{2.1.5}$$

For Re$\lambda < 0$ integral (1.1.6) converges; for Re$\lambda \geq 0$ it generally diverges.

In Section 5.1, we first define the finite part of a divergent integral of the form (1.1.6) and then prove two important properties our definition has.

Section 5.2 focuses on the concept of change of variables in f.p.-integrals of (1.1.6); we formulate and prove the main result of this chapter, Theorem 5.2.1.

The current chapter is based on article [23] extending the results of [41, 42] described in Section 2.5.

5.1 The finite part of a divergent integral

We consider an integral of the form (1.1.6), where $\lambda \in \mathbb{C}$, $n \in \mathbb{N}_0$ and (1.1.7) holds. As we stated before for Re$\lambda < 0$ integral (1.1.6) converges; for Re$\lambda \geq 0$ it generally diverges.

Let $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda < m+\alpha$. We define the finite part of (1.1.6) in the following way: for $\lambda \in \mathbb{C} \setminus \mathbb{N}_0$

$$\text{f.p.} \int_0^T a(t) t^{-\lambda-1} (\ln t)^n dt = \int_0^T \left[a(t) - \sum_{k=0}^m \frac{1}{k!} a^{(k)}(0) t^k \right] t^{-\lambda-1} (\ln t)^n dt$$
$$+ \sum_{k=0}^m \frac{1}{k!} a^{(k)}(0) T^{k-\lambda} \sum_{j=0}^n (-1)^{n-j} \frac{n!}{j!} \frac{(\ln T)^j}{(k-\lambda)^{n-j+1}}; \tag{5.1.1}$$

while for $\lambda = l \in \mathbb{N}_0$, $l \le m$,

$$\text{f.p.} \int_0^T a(t) t^{-l-1} (\ln t)^n dt = \int_0^T \left[a(t) - \sum_{k=0}^m \frac{1}{k!} a^{(k)}(0) t^k \right] t^{-l-1} (\ln t)^n dt$$
$$+ \sum_{\substack{k=0 \\ k\neq l}}^m \frac{1}{k!} a^{(k)}(0) T^{k-l} \sum_{j=0}^n (-1)^{n-j} \frac{n!}{j!} \frac{(\ln T)^j}{(k-l)^{n-j+1}} + \frac{1}{l!} a^{(l)}(0) \frac{(\ln T)^{n+1}}{n+1}. \tag{5.1.2}$$

Both definitions (5.1.1) and (5.1.2) are based on the expansion of the absolutely integrable function a in a Taylor series with centre at the singular point. To demonstrate that the integrals on the right-hand sides of (5.1.1) and (5.1.2) converge absolutely for $\mathrm{Re}\lambda < m+\alpha$, observe that condition (2.1.5) leads to

$$\left| a(t) - \sum_{k=0}^m \frac{1}{k!} a^{(k)}(0) t^k \right| \le \frac{c}{m!} t^{m+\alpha}, \quad 0 \le t \le T. \tag{5.1.3}$$

For $m = 0$, (5.1.3) is obvious. For $m \ge 1$, by Taylor's formula with integral remainder term we have

$$a(t) - \sum_{k=0}^{m-1} \frac{1}{k!} a^{(k)}(0) t^k = \frac{1}{(m-1)!} \int_0^t (t-\rho)^{m-1} a^{(m)}(\rho) d\rho$$
$$= \frac{1}{(m-1)!} \int_0^t (t-\rho)^{m-1} [a^{(m)}(\rho) - a^{(m)}(0)] d\rho + \frac{1}{m!} a^{(m)}(0) t^m,$$

that is,

$$a(t) - \sum_{k=0}^m \frac{1}{k!} a^{(k)}(0) t^k = \frac{1}{(m-1)!} \int_0^t (t-\rho)^{m-1} [a^{(m)}(\rho) - a^{(m)}(0)] d\rho. \tag{5.1.4}$$

Now (5.1.3) follows from (2.1.5) and (5.1.4).

Note that, f.p.-integrals defined by (5.1.1) and (5.1.2) have the same two crucial properties as in the case $n = 0$ (see Section 2.5).

Firstly, the f.p.-integral defined by (5.1.1) is the analytic continuation of (1.1.6) from $\mathrm{Re}\lambda < 0$ to $D = \{\lambda \in \mathbb{C} \setminus \mathbb{N}_0 \mid \mathrm{Re}\lambda < m+\alpha\}$ because the right-hand side of

(5.1.1) is an analytic function of $\lambda \in \mathbb{C}$ in the domain D and for $\text{Re}\lambda < 0$

$$\text{f.p.} \int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt = \int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt.$$

Secondly, for integer points $\lambda = l \in \mathbb{N}_0$,

$$\begin{aligned}&\text{f.p.} \int_0^T a(t)t^{-l-1}(\ln t)^n dt \\ &\quad = \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.} \int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt\right]. \end{aligned} \tag{5.1.5}$$

To show (5.1.5) we use the fact that for a function f analytic at $\lambda = l$, it holds that

$$\lim_{\lambda\to l} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!} f(\lambda)\right] = \lim_{\lambda\to l} f(\lambda) = f(l). \tag{5.1.6}$$

Thus, by definition (5.1.1), we get

$$\begin{aligned}&\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.} \int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt\right] \\ &\quad = \int_0^T \left[a(t) - \sum_{k=0}^{m} \frac{1}{k!} a^{(k)}(0)t^k\right] t^{-l-1}(\ln t)^n dt \\ &\qquad + \sum_{\substack{k=0\\ k\neq l}}^{m} \frac{1}{k!} a^{(k)}(0)T^{k-l} \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!} \frac{(\ln T)^j}{(k-l)^{n-j+1}} + \Psi(l),\end{aligned}$$

with

$$\begin{aligned}\Psi(l) &= -\frac{1}{l!} a^{(l)}(0) \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \sum_{j=0}^{n} \frac{1}{j!(n+1)} (\ln T)^j \left(\frac{d}{d\lambda}\right)^{n+1} \left[(\lambda-l)^j T^{l-\lambda}\right] \\ &= -\frac{1}{l!} a^{(l)}(0) \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \sum_{j=0}^{n}\sum_{u=0}^{n+1} \frac{j!}{j!(n+1)(j-u)!} \binom{n+1}{u} \\ &\quad \times (\lambda-l)^{j-u}(-1)^{n+1-u}(\ln T)^{n+1-u+j} T^{l-\lambda},\end{aligned}$$

where the binomial coefficient $\binom{m}{k}$ for $k, m \in \mathbb{N}_0$ is defined by

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

Since terms with $u \neq j$ converge to zero as $\lambda \to l$, we get

$$\Psi(l) = -\frac{1}{l!}\frac{1}{n+1}a^{(l)}(0)(\ln T)^{n+1}(-1)^{n+1}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} T^{l-\lambda} \times \left[\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j - \binom{n+1}{n+1}(-1)^{n+1}\right].$$

Bearing in mind that

$$\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j = (1-1)^{n+1} = 0,$$

we obtain

$$\Psi(l) = \frac{1}{l!}a^{(l)}(0)\frac{(\ln T)^{n+1}}{n+1}.$$

Hence, (5.1.5) holds. Moreover, the proof of (5.1.5) shows us that

$$\text{f.p.}\int_0^T a(t)t^{-l-1}(\ln t)^i dt = \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\left(\frac{d}{d\lambda}\right)^{n+1}\left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.}\int_0^T a(t)t^{-\lambda-1}(\ln t)^i dt\right] \tag{5.1.7}$$

holds for $i = 0, 1, \ldots, n$.

5.2 The change of variables in f.p.-integrals

Before we consider the change of variables in f.p.-integrals, notice that f.p.-integrals defined by (5.1.1) and (5.1.2) for different $\lambda \in \mathbb{C}$ are independent of one another if $a \in H^{m,\alpha}$, $m \in \mathbb{N}_0$, $\alpha \in (0,1]$. It means we can apply these definitions also in the case of functions $a(t,\lambda)$, which are dependent of $\lambda \in \mathbb{C}$ as a parameter. For example we use

$$\left(\frac{\partial}{\partial t}\right) a(t,\lambda)\big|_{t=0}$$

instead of $a^{(k)}(0)$ on the right-hand side of (5.1.1) when defining

$$\int_0^T a(t,\lambda)t^{-\lambda-1}(\ln t)^n dt, \quad \lambda \in \mathbb{C}\setminus\mathbb{N}_0,\ n \in \mathbb{N}_0.$$

We now discuss the concept of change of variables $t = g(\rho)$ in f.p.-integrals (5.1.1) and (5.1.2), where $m \in \mathbb{N}_0$, $\alpha \in (0,1]$ and

$$g \in H^{m+1,\alpha}[0,T_*], \quad g(0) = 0, \quad g(T_*) = T, \quad g'(\rho) > 0, \quad 0 \le \rho \le T_*. \tag{5.2.1}$$

One can easily see that (5.2.1) implies

$$g(\rho) = \rho g_1(\rho)$$

with $g_1 \in H^{m,\alpha}[0, T_*]$ and $g_1(\rho) > 0$ for all $\rho \in [0, T_*]$.

Theorem 5.2.1. *Assume that* (1.1.7) *and* (5.2.1) *hold. Then for* $m \in \mathbb{N}_0$ *and* $\mathrm{Re}\lambda < m + \alpha$,

$$\text{f.p.}\int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt = \sum_{i=0}^{n} \binom{n}{i} \text{f.p.}\int_0^{T_*} a_i(\rho, \lambda)\rho^{-\lambda-1}(\ln \rho)^i d\rho + \Pi_*(\lambda), \tag{5.2.2}$$

where

$$a_i(\rho, \lambda) = a(g(\rho))g_1(\rho)^{-\lambda-1}g'(\rho)(\ln g_1(\rho))^{n-i} \ (i = 0, 1, \ldots, n)$$

and

$$\Pi_*(\lambda) = \begin{cases} 0, & \lambda \in \mathbb{C} \setminus \mathbb{N}_0, \\ -\displaystyle\sum_{i=0}^{n} \binom{n}{i} \frac{1}{l!} \frac{1}{i+1} \left(\frac{\partial}{\partial \lambda}\right)^{i+1} \left(\frac{\partial}{\partial \rho}\right)^{l} a_i(\rho, \lambda)\Big|_{\rho=0, \lambda=l}, & \lambda = l \in \mathbb{N}_0. \end{cases}$$

Proof. Observe that under our assumptions about a and g, functions $a_i(\rho, \lambda)$, $i = 0, 1, \ldots, n$, as functions of ρ belong to $H^{m,\alpha}[0, T_*]$. Therefore, according to the remark we made at the beginning of this section, the f.p.-integrals on the right-hand side of (5.2.2) are correctly defined.

Every f.p.-integral in (5.2.2), as a function of λ, is analytic in the domain $\{\lambda \in \mathbb{C} \setminus \mathbb{N}_0 \mid \mathrm{Re}\lambda < m + \alpha\}$. For $\mathrm{Re}\lambda < 0$, integral (1.1.6) is absolutely convergent and therefore we can make a change of variables $t = g(\rho)$:

$$\int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt = \sum_{i=0}^{n} \binom{n}{i} \int_0^{T_*} a_i(\rho, \lambda)\rho^{-\lambda-1}(\ln \rho)^i d\rho.$$

Hence, (5.2.2) holds for $\lambda \in \mathbb{C} \setminus \mathbb{N}_0$ with $\Pi_*(\lambda) = 0$.

Now let $\lambda = l \in \mathbb{N}_0$, $l < m + \alpha$. In such case we have the formula (5.1.5), on the right-hand side of which, for points $\lambda \in \mathbb{C} \setminus \mathbb{N}_0$, we make a change of variables according to (5.2.2) with $\Pi_*(\lambda) = 0$:

$$\begin{aligned} &\text{f.p.}\int_0^T a(t)t^{-l-1}(\ln t)^n dt \\ &\quad = \lim_{\substack{\lambda \to l \\ \lambda \notin \mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda - l)^{n+1}}{(n+1)!} \text{f.p.}\int_0^T a(t)t^{-\lambda-1}(\ln t)^n dt\right] \\ &\quad = \sum_{i=0}^{n} \binom{n}{i} \lim_{\substack{\lambda \to l \\ \lambda \notin \mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda - l)^{n+1}}{(n+1)!} \text{f.p.}\int_0^{T_*} a_i(\rho, \lambda)\rho^{-\lambda-1}(\ln \rho)^i d\rho\right]. \end{aligned}$$

Denoting

$$a_i^*(\rho,\lambda) = a_i(\rho,\lambda) - a_i(\rho,l),$$

we get

$$\begin{aligned}
&\text{f.p.}\int_0^T a(t)t^{-l-1}(\ln t)^n dt \\
&\quad = \sum_{i=0}^n \binom{n}{i} \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.}\int_0^{T_*} a_i(\rho,l)\rho^{-\lambda-1}(\ln\rho)^i d\rho\right] \\
&\quad\quad + \sum_{i=0}^n \binom{n}{i} \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.}\int_0^{T_*} a_i^*(\rho,\lambda)\rho^{-\lambda-1}(\ln\rho)^i d\rho\right].
\end{aligned}$$

Note that

$$\left(\frac{\partial}{\partial\lambda}\right)^l a_i^*(\rho,\lambda) = \left(\frac{\partial}{\partial\lambda}\right)^l a_i(\rho,\lambda) \tag{5.2.3}$$

for $l\in\mathbb{N}$. Keeping in mind the remark we made at the beginning of the present section, (5.1.7) leads us to

$$\begin{aligned}
&\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.}\int_0^{T_*} a_i(\rho,l)\rho^{-\lambda-1}(\ln\rho)^i d\rho\right] \\
&\quad = \text{f.p.}\int_0^{T_*} a_i(\rho,l)\rho^{-l-1}(\ln\rho)^i d\rho
\end{aligned}$$

for $i=0,1,\ldots,n$. Therefore we arrive at a formula in the form of (5.2.2) with

$$\Pi_*(l) = \sum_{i=0}^n \binom{n}{i} \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\text{f.p.}\int_0^{T_*} a_i^*(\rho,\lambda)\rho^{-\lambda-1}(\ln\rho)^i d\rho\right],$$

which by definition (5.1.1) is transformed to

$$\begin{aligned}
\Pi_*(l) = &\sum_{i=0}^n \binom{n}{i} \lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}} \left(\frac{d}{d\lambda}\right)^{n+1} \left\{\frac{(\lambda-l)^{n+1}}{(n+1)!}\int_0^{T_*}\left[a_i^*(\rho,\lambda)\right.\right. \\
&\left. - \sum_{k=0}^m \frac{1}{k!}\left(\frac{\partial}{\partial\rho}\right)^k a_i^*(\rho,\lambda)\Big|_{\rho=0}\rho^k\right]\rho^{-\lambda-1}(\ln\rho)^i d\rho + \frac{(\lambda-l)^{n+1}}{(n+1)!} \\
&\left.\times \sum_{k=0}^m \frac{1}{k!}\left(\frac{\partial}{\partial\rho}\right)^k a_i^*(\rho,\lambda)\Big|_{\rho=0} T_*^{k-\lambda}\sum_{j=0}^i(-1)^{i-j}\frac{i!}{j!}\frac{(\ln T_*)^j}{(k-\lambda)^{i-j+1}}\right\}.
\end{aligned}$$

The integrand

$$\left[a_i^*(\rho,\lambda) - \sum_{k=0}^m \frac{1}{k!}\left(\frac{\partial}{\partial\rho}\right)^k a_i^*(\rho,\lambda)\Big|_{\rho=0}\rho^k\right]\rho^{-\lambda-1}(\ln\rho)^i$$

approaches zero as $\lambda \to l$ for each $\rho \in [0, T_*]$ and is dominated by a function

$$c\rho^{m+\alpha-\lambda-1}(\ln\rho)^{i+n+1},$$

which is integrable for $\mathrm{Re}\lambda < m+\alpha$ (c.f. (5.1.3)). Hence, by Lebesgue's theorem the integral also converges to zero as $\lambda \to l$.

In conclusion, in a neighbourhood of $\lambda = l$ the integral and terms in the sum with $k \neq l$ are analytic functions and converge to zero as $\lambda \to l$ (c.f. (5.1.6)), thus

$$\begin{aligned}\Pi_*(l) = &\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\left(\frac{d}{d\lambda}\right)^{n+1}\left[\frac{(\lambda-l)^{n+1}}{(n+1)!}\left(\frac{\partial}{\partial\rho}\right)^{l} a_i^*(\rho,\lambda)\Big|_{\rho=0} T_*^{l-\lambda}\right.\\ &\left.\times\sum_{j=0}^{i}(-1)^{i-j}\frac{i!}{j!}\frac{(\ln T_*)^j}{(l-\lambda)^{i-j+1}}\right]\\ = -&\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\sum_{j=0}^{i}(\ln T_*)^j\frac{i!}{(n+1)!j!}\left(\frac{d}{d\lambda}\right)^{n+1}\left[(\lambda-l)^{n-i+j}\right.\\ &\left.\times\left(\frac{\partial}{\partial\rho}\right)^{l} a_i^*(\rho,\lambda)\Big|_{\rho=0} T_*^{l-\lambda}\right].\end{aligned}$$

It holds

$$\begin{aligned}&\left(\frac{d}{d\lambda}\right)^{n+1}\left[(\lambda-l)^{n-i+j}\left(\frac{\partial}{\partial\rho}\right)^{l} a_i^*(\rho,\lambda)\Big|_{\rho=0} T_*^{l-\lambda}\right]\\ &=\sum_{u=0}^{n+1}\binom{n+1}{u}\frac{(n-i+j)!}{(n-i+j-u)!}(\lambda-l)^{n-i+j-u}\sum_{v=0}^{n+1-u}\binom{n+1-u}{v}\\ &\quad\times\left(\frac{\partial}{\partial\lambda}\right)^{v}\left[\left(\frac{\partial}{\partial\rho}\right)^{l} a_i^*(\rho,\lambda)\Big|_{\rho=0}\right]\left(\frac{\partial}{\partial\lambda}\right)^{n+1-u-v}\left[T_*^{l-\lambda}\right],\end{aligned}$$

where

$$\left(\frac{\partial}{\partial\lambda}\right)^{n+1-u-v}\left[T_*^{l-\lambda}\right] = (-1)^{n+1-u-v}(\ln T_*)^{n+1-u-v}T_*^{l-\lambda}.$$

Consequently,

$$\begin{aligned}\Pi_*(l) = -&\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\sum_{j=0}^{i}\sum_{u=0}^{n+1}\binom{n+1}{u}\frac{i!(n-i+j)!}{(n+1)!j!(n-i+j-u)!}\\ &\times(\lambda-l)^{n-i+j-u}(\ln T_*)^j\sum_{v=0}^{n+1-u}\binom{n+1-u}{v}\\ &\times\left(\frac{\partial}{\partial\lambda}\right)^{v}\left[\left(\frac{\partial}{\partial\rho}\right)^{l} a_i^*(\rho,\lambda)\Big|_{\rho=0}\right](-1)^{n+1-u-v}(\ln T_*)^{n+1-u-v}T_*^{l-\lambda}.\end{aligned}$$

Since the terms in the last sum with $u \neq n-i+j$ converge to zero as $\lambda \to l$, we get

$$\begin{aligned}\Pi_*(l) = &-\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\sum_{j=0}^{i}\sum_{v=0}^{i-j+1}\binom{n+1}{n-i+j}\binom{i-j+1}{v}\frac{i!(n-i+j)!}{(n+1)!j!}\\ &\times\left(\frac{\partial}{\partial\lambda}\right)^{v}\left[\left(\frac{\partial}{\partial\rho}\right)^{l}a_i^*(\rho,\lambda)\Big|_{\rho=0}\right](-1)^{i-j+1-v}(\ln T_*)^{i+1-v}T_*^{l-\lambda}\\ = &-\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\sum_{v=1}^{i+1}T_*^{l-\lambda}\left(\frac{\partial}{\partial\lambda}\right)^{v}\left[\left(\frac{\partial}{\partial\rho}\right)^{l}a_i^*(\rho,\lambda)\Big|_{\rho=0}\right](\ln T_*)^{i+1-v}\\ &\times\sum_{j=0}^{i+1-v}\binom{n+1}{n-i+j}\binom{i-j+1}{v}\frac{i!(n-i+j)!}{(n+1)!j!}(-1)^{i-j+1-v}.\end{aligned}$$

The last equality holds because

$$\lim_{\lambda\to l,\lambda\notin\mathbb{N}_0} a_i^*(\rho,\lambda)=0.$$

As

$$\binom{n+1}{n-i+j}\binom{i-j+1}{v}\frac{i!(n-i+j)!}{(n+1)!j!}=\frac{1}{i+1}\binom{i+1}{v}\binom{i+1-v}{j},$$

we have (see also (5.2.3))

$$\begin{aligned}\Pi_*(l) = &-\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\frac{1}{i+1}\lim_{\substack{\lambda\to l\\ \lambda\notin\mathbb{N}_0}}\sum_{v=1}^{i+1}T_*^{l-\lambda}\left(\frac{\partial}{\partial\lambda}\right)^{v}\left[\left(\frac{\partial}{\partial\rho}\right)^{l}a_i(\rho,\lambda)\Big|_{\rho=0}\right]\\ &\times(\ln T_*)^{i+1-v}(-1)^{i+1-v}\binom{i+1}{v}\sum_{j=0}^{i+1-v}\binom{i+1-v}{j}(-1)^j.\end{aligned}$$

Since

$$\sum_{j=0}^{r}\binom{r}{j}(-1)^j=(1-1)^r=0$$

for $r\neq 0$, we get

$$\Pi_*(l)=-\sum_{i=0}^{n}\binom{n}{i}\frac{1}{l!}\frac{1}{i+1}\left(\frac{\partial}{\partial\lambda}\right)^{i+1}\left(\frac{\partial}{\partial\rho}\right)^{l}a_i(\rho,\lambda)\Big|_{\rho=0,\lambda=l}.$$

Thus, (5.2.2) holds also for $\lambda=l\in\mathbb{N}_0$. $\square$

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Sisukokkuvõte

Singulaarsed murrulised diferentsiaalvõrrandid ja südamlikud Volterra integraaloperaatorid

Murrulist järku tuletised (st tuletised, mille järk ei ole naturaalarv) on pakkunud huvi juba kaua aega. Kuna esialgu ei olnud näha, millised võiksid olla murruliste tuletiste rakendusvõimalused, siis käsitleti nendega seotud küsimusi tavaliselt vaid teoreetilisest vaatepunktist ja sellest huvitusid enamasti ainult matemaatikud. Viimastel aastakümnetel on aga leitud, et murrulist järku tuletisi sisaldavad diferentsiaalvõrrandid võimaldavad suurepäraselt kirjeldada mitmesuguste materjalide käitumist ja paljude protsesside modelleerimist. Võib öelda, et viimase paarikümne aasta jooksul on murruliste tuletiste populaarsus hüppeliselt kasvanud tänu uutele rakendustele füüsikas, bioloogias, keemias ja teistes valdkondades.

Käesolevas töös vaadeldakse singulaarseid murrulist järku tuletistega diferentsiaalvõrrandeid, mis on kujul

$$(M^\beta D_0^\beta u)(t) = \sum_{k=1}^{l} a_k(t)(M^{\beta_k} D_0^{\beta_k} u)(t) + f(t), \quad 0 < t \leq T, \tag{6.1.1}$$

ja

$$(D_0^\alpha M^\alpha u)(t) = \sum_{k=1}^{l} b_k(t)(D_0^{\alpha_k} M^{\alpha_k} u)(t) + f(t), \quad 0 < t \leq T, \tag{6.1.2}$$

kus D_0^μ on μ-järku ($\mu \in [0, \infty)$) murrulise diferentseerimise operaator ning

$$(M^\nu u)(t) = t^\nu u(t), \quad 0 < t \leq T, \ \nu \in \mathbb{R} = (-\infty, \infty), \ u \in C[0, T].$$

Väitekirja üheks peamiseks eesmärgiks on uurida võrrandite (6.1.1) ja (6.1.2) ühest lahenduvust ruumis $C^m[0,T]$, kus $m \in \mathbb{N}_0 = \{0, 1, \ldots\}$. Ruumi $C^m[0,T]$ moodustavad lõigul $[0,T]$ defineeritud funktsioonid u, mis on m korda pidevalt diferentseeruvad (kui $m = 0$, siis tähistame $C^0[0,T] = C[0,T]$). Meie metoodika peamine tööriist on südamlike Volterra integraaloperaatorite teooria [34, 35, 37].

Võrrandites (6.1.1) ja (6.1.2) esinevad murrulise diferentseerimise operaatorid D_0^μ ($\mu \geq 0$) on defineeritud kui Riemann-Liouville'i murrulise integraaloperaatori J^μ pöördoperaatorid hulgal $J^\mu C[0,T]$, st

$$D_0^\mu := (J^\mu)^{-1}, \quad \mu \geq 0.$$

Riemann-Liouville'i murruline integraaloperaator J^μ on defineeritud valemiga:

$$(J^\mu u)(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u(s) ds, \quad u \in C[0,T], \quad t > 0, \ \mu > 0; \quad J^0 = I,$$

kus Γ tähistab Euleri gammafunktsiooni ja I ühikkujutust.

Väitekiri koosneb viiest peatükist. Esimeses peatükis antakse lühike ülevaade tööst ning murruliste tuletiste ja hajuvate integraalide ajaloost. Teises peatükis tutvustatakse töös vajalikke tähistusi, mõisteid ja tulemusi.

Töö kolmas peatükk on pühendatud singulaarsetele võrranditele kujul (6.1.2), kus $f \in C^m[0,T]$, $\alpha, \alpha_k \in \mathbb{R}$ ja

$$m < \alpha \leq m+1, \quad \alpha > \alpha_k \geq 0, \quad b_k \in C^m[0,T], \quad k = 1,2,\ldots,l, \quad m \in \mathbb{N}_0. \tag{6.1.3}$$

Alustuseks uuritakse konstantsete kordajatega võrrandeid, nende jaoks tõestatakse teoreem 3.1.1. Peatüki põhitulemus on teoreem 3.1.2, mis annab võrrandite (6.1.2) ühese lahenduvuse kriteeriumid ruumis $C^m[0,T]$ eeldustel (6.1.3).

Neljandas peatükis on vaatluse all singulaarsed võrrandid kujul (6.1.1), kus $f \in C^m[0,T]$, $\beta, \beta_k \in \mathbb{R}$ ja

$$m < \beta < m+1, \quad \beta > \beta_k \geq 0, \quad a_k \in C^m[0,T], \quad k = 1,2,\ldots,l, \quad m \in \mathbb{N}_0. \tag{6.1.4}$$

Võrrandite (6.1.1) puhul vaadeldakse eraldi juhte kus $0 < \beta < 1$ ja $\beta > 1$. Juhu $0 < \beta < 1$ jaoks tõestatakse teoreemid 4.1.1 ja 4.1.3. Kui $\beta > 1$, siis on olukord keerulisem. Esiteks jätkatakse operaatorid D_0^β ja $D_0^{\beta_k}$ ning vastavad südamlikud Volterra integraaloperaatorid ruumi $C^m[0,T]$ ja siis tõestatakse teoreemid 4.1.2 ja 4.1.4 laiendatud operaatorite kohta. Käsitletud murrulist järku diferentseerimise operaatorid D_0^μ on tihedalt seotud praktikas enam kasutust leidvate Riemann-Liouville'i ja Caputo murruliste tuletistega. Kolmanda ja neljanda peatüki põhitulemused saab ümber formuleerida juhule, kui võrrandites on D_0^μ asemel kasutatud Riemann-Liouville'i või Caputo murrulisi tuletisi.

Hajuvaid integraale (st integraale, mis klassikalises mõttes hajuvad) ja neid sisaldavaid võrrandeid on uuritud juba aastakümneid. Üks tähtsamaid käsitusi hajuvate integraalide puhul on nende lõpliku osa (f.p.) defineerimine. Aastate jooksul on välja pakutud mitmeid erinevaid võimalusi hajuva integraali lõpliku osa (vastava f.p.-integraali) defineerimiseks (vt [15, 20]).

Väitekirja viimases peatükis defineeritakse logaritmilist tegurit omavate hajuvate integraalide

$$\int_0^T a(t) t^{-\lambda-1} (\ln t)^n dt, \tag{6.1.5}$$

kus λ on kompleksarv, mille reaalosa on mittenegatiive, $m, n \in \mathbb{N}_0$, $0 < \alpha \leq 1$ ja

$$a \in C^m[0,T], \quad \left| a^{(m)}(t) - a^{(m)}(0) \right| \leq ct^\alpha, \quad 0 \leq t \leq T, \ c > 0, \ c = \text{const.}$$

lõplik osa (vastav f.p.-integraal) ja tõestatakse muutujavahetuse eeskiri selliste f.p.-integraalide jaoks.

Enamik käesoleva töö põhitulemustest sisalduvad avaldatud teadusartiklites [23, 24] ning neid tulemusi on tutvustatud viiel teaduskonverentsil. Töö sisaldab ka tulemusi, mis ei ole veel avaldatud.

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"The 15th International Conference on Mathematical Modelling and Analysis" Druskininkais (2010),

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List of Publications

1. Lätt, K., φ- and β- topologies in sequence spaces. Acta Comment. Univ. Tartu. Math. 13 (2009), 43 – 49.
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3. Lätt, K., Pedas, A. and Vainikko, G., A smooth solution of a singular fractional differential equation. Z. Anal. Anwend. 34(2) (2015), 127 – 146.

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