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Partial result on the plasticity of the unit ball of the ℓ_∞-sum of finitely many strictly convex Banach spaces

Mathematics

Bachelor's thesis (9 EAP)

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OSALINE TULEMUS SEOSES LÕPLIKU HULGA RANGELT KUMERATE BANACHI RUUMIDE ℓ_∞-SUMMA ÜHIKKERA PLASTILISUSEGA

Bakalaureusetöö

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Lühikokkuvõte

Bakalaureusetöös üldistatakse lemma, mis esineb Nikita Leo tõestuses kahe rangelt kumera Banachi ruumi ℓ_∞-summa ühikkera plastilisusest. Selle üldistuse abil tõestame lõpliku hulga rangelt kumerate Banachi ruumide ℓ_∞-summa ühikkera omaduse, mis on plastilisusest formaalselt nõrgem. Veel tõestatakse kaks üldist tulemust Banachi ruumi ühikkera plastilisuse kohta.

CERCS teaduseriala: P140 Read, Fourier analüüs, funktsionaalanalüüs.

Märksõnad: Funktsionaalanalüüs, Banachi ruumid, rangelt kumerad Banachi ruumid, plastilisus, mittelineaarsed operaatorid.

PARTIAL RESULT ON THE PLASTICITY OF THE UNIT BALL OF THE ℓ_∞-SUM OF FINITELY MANY STRICTLY CONVEX BANACH SPACES

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Abstract

This thesis presents a generalization of a lemma appearing in Nikita Leo's proof of the plasticity of the closed unit ball of the ℓ_∞-sum of two strictly convex Banach spaces. The generalization is then used to show a property related to but weaker than plasticity for the closed unit ball of the ℓ_∞-sum of finitely many strictly convex Banach spaces. Two additional results related to

the plasticity of closed balls in Banach spaces are also proved.
CERCS research specialization: P140 Series, Fourier analysis, functional analysis.
Keywords: Functional analysis, Banach spaces, strictly convex Banach spaces, plasticity, non-linear operators.

Contents

Introduction

The central aim of this thesis is to present a generalization of a key lemma found in Nikita Leo's proof of the plasticity of the closed unit ball of the ℓ_∞-sum of two strictly convex Banach spaces, where the generalization extends the lemma to the ℓ_∞-sum of any finite number of strictly convex Banach spaces. In addition, the generalized lemma is applied to prove that any 1-Lipschitz bijection from the closed unit ball of such a space to itself which maps extreme points to extreme points or the sphere into itself must be an isometry.

1 Preliminaries

1.1 Background

The notion of plasticity for metric spaces was introduced by Naimpally, Piotrowski, and Wingler in their 2006 article [8]. A metric space is said to be *EC-plastic* (or just *plastic*) when all 1-Lipschitz bijections from the space into itself are isometries.

In their 2016 article [2], Cascales, Kadets, Orihuela, and Wingler began an investigation of the following question.

Problem. Is the closed unit ball of every Banach space plastic?

In said article, this question was answered affirmatively for the special case of strictly convex Banach spaces. (Recall that a Banach space is strictly convex if its unit sphere contains no segments with distinct endpoints.) The general case, however, remains open.

All totally bounded metric spaces are known to be plastic, including the unit balls of finite-dimensional Banach spaces [8].

The unit ball is also known to be plastic in the following cases:

- spaces whose unit sphere is a union of finite-dimensional polyhedral extreme subsets (incl. all strictly convex Banach spaces) [1, 2],
- any ℓ_1-sum of strictly convex Banach spaces (incl. ℓ_1 itself) [5, 6],
- the ℓ_∞-sum of **two** strictly convex Banach spaces [4],
- $\ell_1 \oplus_2 \mathbb{R}$ [4],
- $C(K)$, where K is a compact metrizable space with finitely many accumulation points (incl. $c \cong C(\omega + 1)$, i.e., the space of convergent real sequences) [3, 7].

In [4], Nikita Leo showed that the ℓ_∞-sum of two strictly convex Banach spaces has a plastic unit ball. While the original proof does not directly apply to an arbitrary finite sum of strictly convex Banach spaces, a crucial step in the proof can be modified to suit this purpose. By generalizing this step, we can establish a similar but weaker property than plasticity, which only considers a specific class of 1-Lipschitz bijections.

1.2 Conventions and notation

We adopt the convention that $0 \in \mathbb{N}$. For $n \in \mathbb{N}$ we define $[n] \coloneqq \{\, i \in \mathbb{N} \mid i < n \,\}$. Throughout, $(X_i, i \in [n])$ shall be a family of strictly convex Banach spaces, and B_X, S_X shall denote the closed unit ball and unit sphere of the Banach space X respectively. Given a family as above, we identify B_{X_i} with B_i and S_{X_i} with S_i for legibility.

For a family of Banach spaces as above, $\bigoplus_{i \in [n]} X_i$ will denote their ∞-sum, with components in the natural ascending order. For such a sum, let $(\pi_i, i \in [n])$ be the projection operators onto each component.

For two points in the ball of a Banach space X, we say they're *opposed* when the distance between them is 2. In a strictly convex Banach space, this is equivalent to the two points summing to 0. For an ∞-sum of strictly convex Banach spaces, this means the two points are opposed along their projection to some component space.

For any map f from a metric space (M, d) to itself, we say that it is *non-expansive* when it is a 1-Lipschitz map, and that it is *non-contractive* when for all $x, y \in M$ we have $d(x, y) \leq d(f(x), f(y))$ (note that this is dual to the inequality $d(x, y) \geq d(f(x), f(y))$ defining 1-Lipschitz maps).

2 Main result

We shall prove the following theorem.

Theorem 1. *Let* $Z \coloneqq \bigoplus_{i\in[n]} X_i$ *and let* $G\colon B_Z \to B_Z$ *be a non-contractive function. Then there is some permutation* $\sigma\colon [n] \to [n]$ *and a family of non-contractive functions* $g_i\colon S_i \to S_{\sigma(i)}$ *such that for all points* $x \in B_Z$ *and all* $i \in [n]$*, we have* $\pi_i x \in S_i \Rightarrow \pi_{\sigma(i)} G(x) = g_i(\pi_i x)$*.*

We start with the crucial lemma, whose central construction will reoccur in subsequent lemmas. We use the notation of Theorem 1.

Lemma 1. *Given* $2^n - 1$ *pairwise opposed points in* B_Z*, there is at most one point in* B_Z *that is opposed to all of them.*

As a natural consequence of this lemma, the maximal size of a set of pairwise opposed points in B_Z is 2^n.

Proof. The proof is by induction on n. The base case with $n = 1$ is clear by inspection (note here that n determines how many components Z has).

Consider an enumerated family $(x_i)_{i\in[2^n]}$ of pairwise opposed points in B_Z. We want to show that any point $q \in B_Z$ that opposes all points x_i with $i > 0$ is equal to x_0.

To do so, we first partition the family of points into two subsets A and B, where A is any maximal subset of the family satisfying the condition that for all $a, a' \in A$, the projections $\pi_0 a$ and $\pi_0 a'$ are not opposed, and B is simply the complement of A.

Let us note two things: that every pair of elements in B are also unopposed along π_0, and $|A| = |B| = 2^{n-1}$. From this, it follows that B is also maximal for same property defining A.

For the first, note that due to the maximality of A, any element in B has to oppose some element of A along the projection π_0 – otherwise A could be extended by that element. This implies that $\pi_0 B \subseteq -\pi_0 A$. But simultaneously changing the sign of of the π_0 component of two elements that are unopposed along that component will preserve this property, so simultaneously changing the sign of all the π_0 components of A will keep them pairwise unopposed. Since the π_0 components of B are a subset of the sign-changed components for A, they are also pairwise unopposed.

For the second, suppose that $|A| > 2^{n-1}$. Since no pair of points in A oppose each other along the projection π_0, but each pair is opposed along some projection, it follows that each pair in A is opposed along some projection π_i with $i > 0$. If we consider the space obtained by dropping the π_0 component, we get a space that is similar to Z but with one less component. However, this space now contains a family of more than 2^{n-1} pairwise opposed points, which contradicts the induction hypothesis for $n - 1$, since the maximum number of such points in a space with one less component is 2^{n-1}.

Now suppose $|A| < 2^{n-1}$. Then we have that B is a set with the same pairwise non-opposition property as A, and since A and B partition our family of points, we have $|B| > 2^{n-1}$. We can then apply the $|A| > 2^{n-1}$ argument to B to arrive at a contradiction. (Note that that argument did not require maximality from A.)

Since both branches of $|A| \neq 2^{n-1}$ gave a contradiction, we conclude that $|A| = 2^{n-1}$.

Moreover, the above argument also allows us to extract $\pi_0 A = -\pi_0 B$.

Assume without loss of generality that $x_0 \in A$. Suppose that q fails to oppose any member of B along the projection π_0. Then, by definition, q opposes all of them along the other projections. This induces a family of more than 2^{n-1} opposed points in the $n - 1$ case, which is impossible by the induction hypothesis. Thus, q must oppose some member of B along π_0, which in turn implies that q is equal to some member of A along π_0. Consequently, q fails to oppose any member of A along

π_0, so q must oppose each member of $A \setminus \{x_0\}$ along some other projection. This induces a family of $2^{n-1} - 1$ points in the $n-1$ case, so by the induction statement, there is at most one point in the induced space (corresponding to dropping the π_0 component) that opposes all of them. By assumption, this point must agree with q along all the projections π_i with $i > 0$. Since x_0 also opposes all of $A \setminus \{x_0\}$ along the projections other than π_0, we conclude that for all $i > 0$, $\pi_i q = \pi_i x_0$.

Therefore q agrees with x_0 along all but at most one component. To get full agreement with x_0, repeat the argument along one other component. This is always possible after the base case since $n \geq 2$, so there is at least one other component along which to run the argument. □

For the following lemmas, we take that $x_0 \in S_Z$ is some extreme point, and the family $(x_i)_{i \in [2^n]}$ is generated by this point and the negations along each component. We then define $(y_i)_{i \in [2^n]}$ by $y_i = G(x_i)$.

Lemma 2. *Let x_a, x_b differ in only one component by a sign flip. Then y_a, y_b do also.*

Proof. Let v be the point that agrees with x_a, x_b everywhere except for the one component where the two disagree, where we define v to be 0. Since v opposes all of (x_i) other than x_a, x_b, $G(v)$ opposes all of (y_i) other than possibly y_a, y_b.

Suppose first that y_a, y_b differ in more than one component. Let the indices of these components be i and j, respectively.

We define two partitions of (y_i) by the logic of Lemma 1: one as A_i, B_i along the component π_i and the other as A_j, B_j along the component π_j. We choose these subject to the constraints $y_a \in A_i, A_j$ and $y_b \in B_i, B_j$.

To see that this is possible, first take any partition along π_i, and call the components of it A, B. Without loss of generality, suppose that y_b is in the component B. Then, due to $-\pi_i A = \pi_i B$, there must be some $y_c \in A$ such that $\pi_i y_c = -\pi_i y_b$. Define

$A^p = \{y_a, y_c\}$. Note that y_a and y_c do not oppose each other along π_i (otherwise, we would have $\pi_i y_a = \pi_i y_b$, contradicting the definition of i), so we can extend A^p to a maximal set A' where no pair of elements is π_i-opposed. We cannot have $y_b \in A'$ since it opposes y_c along π_i, so y_b must belong to the complement B' of A'. We can now take $A_i = A'$, $B_i = B'$ and perform an analogous procedure along π_j.

Let us now examine $G(v)$. Suppose that $G(v)$ opposes some member of A_i along π_i. In this case, $G(v)$ agrees with some member of B_i along π_i, so $G(v)$ must oppose all the members of $B_i \setminus \{y_b\}$ along the other projections, which implies that for $k \neq i$, we have $\pi_k G(v) = \pi_k y_b$. Therefore, we can conclude that $\pi_j G(v) = \pi_j y_b$, which implies that $G(v)$ must oppose all the members of $B_j \setminus \{y_b\}$ along the other projections. As a result, for all $k \neq j$, we have $\pi_k G(v) = \pi_k y_b$. But these together imply $G(v) = y_b = G(x_b)$, which contradicts the injectivity of G.

Consequently, $G(v)$ does not oppose any member of A_i along π_i, nor any member of B_i along the same. By reasoning analogous to the above, this implies that for $k \neq i$ we have $\pi_k G(v) = \pi_k y_a$ and $\pi_k G(v) = \pi_k y_b$. So we have that $\pi_j y_a = \pi_j y_b$, contradicting our assumption that j is an index where the two differ.

So y_a, y_b differ in at most one component. Since y_a, y_b oppose each other, they must differ in that component by a sign flip. □

This implies that any two y_a, y_b differ only by sign flips in various components. To see this, take x_a and x_b, and note that these differ only by sign flips. You can form a sequence of points in the family (x_i) such that the sequence starts at x_a, ends at x_b, and every two consecutive members differ by a sign flip. Then the sequence of their images under G in the family (y_i) will start at y_a, end at y_b, and every two consecutive members will differ by a sign flip.

The following is a specification of this sign flipping idea. What it expresses is that all G does with regard to the signs of components is permute them.

Lemma 3. *There is a permutation $\sigma\colon [n] \to [n]$ such that for any function*

$f\colon [n] \to \{-1, 1\}$ and vector $q \in B_Z$ satisfying $\forall i \in [n], \pi_i q = f(i)\pi_i x_0$, we have $\forall i \in [n], \pi_{\sigma(i)} G(q) = f(i)\pi_{\sigma(i)} y_0$.

Proof. For $i \in [n]$, let q_i be defined as equal to x_0 in all components except the π_i component, where we have $\pi_i q_i = -\pi_i x_0$.

By Lemma 2 we have that $G(q_i)$ and $y_0 = G(x_0)$ differ in exactly one component by a sign flip. Let the index of that component be denoted by $\sigma(i)$.

First we need to show that σ is a permutation. Since σ is a function from $[n]$ to $[n]$, this is equivalent to σ being injective. If $\sigma(i) = \sigma(j)$, then $G(q_i) = G(q_j)$, which by the injectivity of G implies $q_i = q_j$, and thus $i = j$. Therefore, σ is a permutation.

Now we will use induction on the number of components, denoted as k, that differ between q and x_0. It is clear that the cases $k \in \{0, 1\}$ are covered, with $k = 1$ being covered by the above construction of σ.

Assume that we have established the induction statement for at most k components, and let q be a vector which differs from x_0 in $k + 1$ components by sign flips. Let the first two components where q differs from x_0 have indices i, j (these exist and are distinct since we have $k + 1 \geq 2$ for the induction step). Define a_0 as equal to q, but equal to x_0 at i, j. Further define a_i (resp. a_j) as equal to q, except equal to x_0 at i (resp. j).

By the induction statement, $G(a_0)$ and $G(a_i)$ differ by a sign flip at $\sigma(j)$, while $G(a_0)$ and $G(a_j)$ differ by a sign flip at $\sigma(i)$. By Lemma 2, $G(q)$ differs from each of $G(a_i), G(a_j)$ by a single sign flip. Let $G(q)$ differ from $G(a_i)$ at index u and from $G(a_j)$ at index v.

The prior facts together imply that there are two paths for flipping signs to get from $G(a_0)$ to $G(q)$ that have to commute — flipping at indices $\sigma(j), u$ or flipping at indices $\sigma(i), v$. If $u = \sigma(j)$ then we would have $G(a_0) = G(q)$, which would imply $a_0 = q$. However, this is a contradiction since a_0 and q differ from x_0 by a different number of sign flips. Hence $u \neq \sigma(j)$, which implies that $G(q)$ and $G(a_0)$ differ at $\sigma(j)$ by a sign flip. Symmetrically, $G(q)$ and $G(a_0)$ differ at $\sigma(i)$ by a sign flip. Since $\sigma(i) \neq \sigma(j)$ and $G(q)$ can differ from $G(a_0)$ in at most two components, it differs from $G(a_0)$ in exactly those components.

Now let $f\colon [n] \to \{-1, 1\}$ be the function satisfying $\pi_u a_0 = f(u)\pi_u x_0$, and let f' be the analogous function for q. We clearly have $f(u) = f'(u)$ for $u \neq i, j$, and $f(u) = -f'(u) = 1$ for $u = i, j$.

By the induction assumption, we have $\pi_{\sigma(u)}G(a_0) = f(u)\pi_{\sigma(u)}y_0$. Moreover, from the above line of reasoning, we have

$$\pi_{\sigma(u)}G(q) = \pi_{\sigma(u)}G(a_0) = f(u)\pi_{\sigma(u)}y_0 = f'(u)\pi_{\sigma(u)}y_0$$

for $u \neq i, j$, and we also have

$$\pi_{\sigma(u)}G(q) = -\pi_{\sigma(u)}G(a_0) = -f(u)\pi_{\sigma(u)}y_0 = f'(u)\pi_{\sigma(u)}y_0$$

for $u = i, j$. In both cases, the last equality in the chain follows from the prior analysis of the relation between f and f'. Since we get $\pi_{\sigma(u)}G(q) = f'(u)\pi_{\sigma(u)}y_0$ in both cases and the cases are exhaustive, that equality holds for all $u \in [n]$, which is what we wanted. □

The previous lemma only establishes that there is some such permutation σ for every choice of extreme point x_0. However, it immediately follows that the permutation is the same for every x_i in the family. To establish that this permutation is also the same for all extreme points of B_Z, we require another lemma.

Lemma 4. *Let $q \in B_Z$ and $i \in [n]$ be such that $\pi_i q = \pi_i x_0$. Then $\pi_{\sigma(i)}G(q) = \pi_{\sigma(i)}y_0$, where σ is the permutation we get for x_0 from Lemma 3.*

Proof. This is trivial for the $n = 1$ case, so we will assume $n > 1$.

Let $(x_j)_{j\in J}$ be the subfamily of $(x_j)_{j\in[2^n]}$ that satisfies the condition $\pi_i x_j = -\pi_i x_0$. We have that $G(q)$ opposes y_j for each $j \in J$. Note that $\pi_{\sigma(i)}y_j = -\pi_{\sigma(i)}y_0$ by

Lemma 3 for all $j \in J$, and $|J| = 2^{n-1}$. Suppose that $\pi_{\sigma(i)}G(q) \neq \pi_{\sigma(i)}y_0$. Then $G(q)$ would fail to oppose any y_j along $\pi_{\sigma(i)}$, so it would have to oppose them along the other components. Dropping the $\pi_{\sigma(i)}$ component, we would get a family of $2^{n-1}+1$ pairwise opposed points in an $(n-1)$-component space, which contradicts the corollary of Lemma 1. Therefore, we must have $\pi_{\sigma(i)}G(q) = \pi_{\sigma(i)}y_0$. □

Lemma 5. *Let x, x' be extreme points of B_Z and let σ, σ' be their respective induced permutations (according to Lemma 3). Then $\sigma = \sigma'$.*

Proof. It suffices to consider the case where x and x' differ only in the component π_i. If they differ in more components, we can chain applications of the one-component case as we turn x into x' one component at a time.

By Lemma 4, we have $\pi_{\sigma(j)}G(x) = \pi_{\sigma(j)}G(x')$ for all $j \neq i$. Similarly, we have $\pi_{\sigma'(j)}G(x) = \pi_{\sigma'(j)}G(x')$ for all $j \neq i$. If $\sigma(i) \neq \sigma'(i)$, then we would have that $G(x)$ and $G(x')$ agree on all components, in which case $x = x'$ contradicting our assumption. So we must have $\sigma(i) = \sigma'(i)$.

Next, we pick any $j \neq i$ and define q, q' by taking x, x', respectively, and flipping the sign of the j component. Thus, we get

$$\pi_{\sigma(j)}G(q') = \pi_{\sigma(j)}G(q) = -\pi_{\sigma(j)}G(x) = -\pi_{\sigma(j)}G(x'),$$

where the second equality comes from Lemma 3, and the first and third from $i \neq j$ and Lemma 4. Note however that by Lemma 3 we have that $\pi_q G(q') = \pi_q G(x')$ for all $q \neq \sigma'(j)$. Since $\pi_{\sigma(j)}G(q') = -\pi_{\sigma(j)}G(x')$ and $\pi_{\sigma(j)}G(x') \neq 0$, we have $\sigma(j) = \sigma'(j)$.

So we have $\sigma(i) = \sigma'(i)$ and $\forall j \neq i, \sigma(j) = \sigma'(j)$. Consequently we have $\sigma = \sigma'$. □

We can now prove Theorem 1.

Proof of Theorem 1. Let σ be the permutation induced by an extreme point of S_Z. By Lemma 5, σ is unique.

For each $i \in [n]$, we define the mapping $\iota_i \colon B_i \to B_Z$ as the standard inclusion satisfying $\pi_i \iota_i q = q$ and $\pi_j \iota_i q = 0$ for $j \neq i$. Then, we define $g_i \colon S_i \to S_{\sigma(i)}$ by $g_i(q) = \pi_{\sigma(i)} G(\iota_i q)$. To see that the codomain of g_i is correct, we use Lemma 4: note that $g_i(q)$ agrees with $\pi_{\sigma(i)} G(x)$ for any extreme point $x \in S_Z$ that satisfies $\pi_i x = q$, and that $G(x)$ is extreme (as a corollary of Lemma 1), so its components are all of norm 1.

Now suppose that $x \in B_Z$ and $\pi_i x \in S_i$. Fix any extreme v such that $\pi_i x = \pi_i v$. Then

$$\pi_i \iota_i \pi_i x = \pi_i x = \pi_i v,$$

so by Lemma 4, we have both $\pi_{\sigma(i)} G(x) = \pi_{\sigma(i)} G(v)$ and $\pi_{\sigma(i)} G(\iota_i \pi_i x) = \pi_{\sigma(i)} G(v)$. Therefore, we have

$$\pi_{\sigma(i)} G(x) = \pi_{\sigma(i)} G(\iota_i \pi_i x) = g_i(\pi_i x).$$

To complete the proof, we must show that g_i is a non-contraction.

Take $u, u' \in S_i$ and let x, x' be extreme points of B_Z such that $\pi_i x = u$ and $\pi_i x' = u'$, and which agree along all other components. Then

$$\begin{aligned} \left\| g_i(u) - g_i(u') \right\| &= \left\| \pi_{\sigma(i)} G(x) - \pi_{\sigma(i)} G(x') \right\| \\ &= \left\| G(x) - G(x') \right\| \\ &\geqslant \left\| x - x' \right\| \\ &= \left\| u - u' \right\|. \end{aligned}$$

The second inequality in the chain follows from Lemma 4: since x, x' differ only in i, then $G(x), G(x')$ differ only in $\sigma(i)$. $\square$

3 Main corollary

We will now apply the main theorem to prove a weaker version of plasticity for the closed unit ball of the ℓ_∞-sum of finitely many strictly convex Banach spaces. We continue to use the notation of Theorem 1, letting $Z \coloneqq \bigoplus_{i \in [n]} X_i$.

Theorem 2. *Let $F\colon B_Z \to B_Z$ be a 1-Lipschitz bijection. If F maps extreme points to extreme points, or $F(S_Z) \subseteq S_Z$, then F is an isometry.*

The proof of this is a direct adaptation of Leo's argument in [4].

Let F be as in the statement of Theorem 2. Define G to be its inverse function. Since G satisfies Theorem 1, we can define the permutation σ and the family of functions $g_i\colon S_i \to S_{\sigma(i)}$ as given by Theorem 1.

Lemma 6. *Each g_i is a bijection.*

Proof. We begin by showing that g_i is surjective. It is enough to show that $y \in g_i(S_i)$ for any $y \in S_{\sigma(i)}$.

We shall first consider the case where $F(S_Z) \subseteq S_Z$.

Construct the point $x \in S_Z$ such that $\pi_{\sigma(i)}x = y$ and $\pi_j x = 0$ for all $j \neq \sigma(i)$. Let $J = \{j \in [n] \mid \|\pi_j F(X)\| = 1\}$. From Theorem 1, we know that $\left\|\pi_{\sigma(j)}x\right\| = 1$ for all $j \in J$. It follows that $J \subseteq \{i\}$. $F(S_Z) \subseteq S_Z$ implies that J is nonempty, from which we have $J = \{i\}$. This allows us to use Theorem 1 to conclude that $\pi_{\sigma(i)}x = g_i(\pi_i F(x))$. Since $\pi_{\sigma(i)}x = y$, this implies that $y \in g_i(S_i)$.

We now consider the case where F maps extreme points to extreme points. Construct an extreme point $x \in B_Z$ such that $\pi_{\sigma(i)}x = y$. Since $F(x)$ is an extreme point, we have that $\pi_{\sigma(i)}x = g_i(\pi_i F(x))$ by Theorem 1. Since $\pi_{\sigma(i)}x = y$, we have that $y \in g_i(S_i)$.

Hence, g_i is surjective. Moreover, since g_i is a non-contraction by construction, it is injective. Therefore, g_i is a bijection. $\square$

Since each g_i is a bijection, we can define inverses $f_{\sigma(i)} = g_i^{-1}$.

Lemma 7. *$F(S_Z) \subseteq S_Z$ implies that F maps extreme points to extreme points.*

Proof. Fix any extreme point $x \in S_Z$. We aim to show that $F(x)$ is an extreme point also.

Define $y \in S_Z$ by $\pi_i y = f_{\sigma(i)}(\pi_{\sigma(i)} x)$. Since y is an extreme point, we have by Theorem 1 that

$$\pi_{\sigma(i)} G(y) = g_i(\pi_i y) = g_i(f_{\sigma(i)}(\pi_{\sigma(i)} x)) = \pi_{\sigma(i)} x,$$

where the last equality follows from the definition of $f_{\sigma(i)}$.

Since $G(y)$ and x are equal on all components, we have that $G(y) = x$, which means that $F(x) = y$. Since y is extreme by construction, this is sufficient. □

The previous lemma allows the remainder of the proof to proceed by considering only the case where F maps extreme points to extreme points.

Definition 1. We say that G is *homogeneous in k components* if, for all $J \subseteq [n]$ with $|J| \leq k$ and for all $x \in B_Z$ such that x has norm 1 on components $[n] \setminus J$, we have $\forall i \in [n], \pi_{\sigma(i)} G(x) = \|\pi_i x\| g_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right)$, where the expression on the right hand side is understood to be 0 whenever $\|\pi_i x\| = 0$. Analogously, we say that F is homogeneous in k components when, for the same J and x, we have $\forall i \in [n], \pi_{\sigma^{-1}(i)} F(x) = \|\pi_i x\| f_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right)$.

Lemma 8. *The function G is homogeneous in k components if and only if F is also.*

Proof. We will show that G being homogeneous in k components implies that F is also. The proof in the opposite direction is analogous.

Let us have $J \subseteq [n]$ with $|J| \leq k$ and $x \in B_Z$ such that x has norm 1 on components $[n] \setminus J$.

Fix any extreme point y such that $\pi_i x = \alpha_i \pi_i y$ for all $i \in [n]$, where α_i are non-negative scalars. (If $\|\pi_i x\| > 0$ then $\alpha_i = \|\pi_i x\|$, otherwise $\alpha_i = 0$ and $\pi_i y$ can be arbitrary in S_i.)

Define q by $\pi_i q = \alpha_{\sigma(i)} \pi_i F(y)$. First, let us confirm that $G(q) = x$.

By homogeneity, we have

$$\pi_{\sigma(i)} G(q) = \|\pi_i q\| g_i(\pi_i F(y)) = \alpha_{\sigma(i)} g_i(\pi_i F(y)).$$

Since $F(y)$ is an extreme point, we also have $\pi_{\sigma(i)} y = g_i(\pi_i F(y))$, hence

$$\pi_{\sigma(i)} G(q) = \alpha_{\sigma(i)} \pi_{\sigma(i)} y = \pi_{\sigma(i)} x.$$

Since $G(q)$ and x coincide on all components, we have $G(q) = x$, hence $q = F(x)$.

To prove that F is homogeneous in k components, we need to show that $\pi_{\sigma^{-1}(i)} F(x) = \|\pi_i x\| f_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right)$. Equivalently, we need to show that $\pi_{\sigma^{-1}(i)} q = \alpha_i f_i(\pi_i y)$. Since $\pi_i y = g_{\sigma^{-1}(i)}(\pi_{\sigma^{-1}(i)} F(y))$ and f_i is the inverse of $g_{\sigma^{-1}(i)}$, we have $f_i(\pi_i y) = \pi_{\sigma^{-1}(i)} F(y)$. Thus, it suffices to show that $\pi_{\sigma^{-1}(i)} q = \alpha_i \pi_{\sigma^{-1}(i)} F(y)$, which is equivalent to $\pi_i q = \alpha_{\sigma(i)} \pi_i F(y)$, the way we defined q earlier. Consequently we have that F is homogeneous in k components. □

Lemma 9. *The function G is homogeneous in n components.*

Proof. We proceed by induction. The base case is covered by Theorem 1.

Suppose G is homogeneous in $k < n$ components. Let $J \subseteq [n]$ with $|J| = k + 1$, and let $x \in B_Z$ have norm 1 on components $[n] \setminus J$. We already know that for $i \in [n] \setminus J$, $\pi_{\sigma(i)} G(x) = g_i(\pi_i x)$ by the construction of g_i.

It thus suffices to show that for $i \in J$, we have $\pi_{\sigma(i)} G(x) = \|\pi_i x\| g_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right)$.

Fix any $i \in J$. Define y such that $\pi_k y = \pi_k G(x)$ on indices $k \neq \sigma(i)$, and $\pi_{\sigma(i)} y = \frac{\pi_{\sigma(i)} G(x)}{\left\|\pi_{\sigma(i)} G(x)\right\|}$, where we take $\pi_{\sigma(i)} y$ to be an arbitrary element of $S_{\sigma(i)}$ in the case $\pi_{\sigma(i)}G(x) = 0$. Note that y has norm 1 on k components, so we can apply the induction assumption to it. We also define z as equal to y, except at $\sigma(i)$, where we flip the sign of the component.

We have that

$$\|F(y) - x\| \leq \|y - G(x)\| = 1 - \left\|\pi_{\sigma(i)}G(x)\right\|.$$

Similarly, we have that

$$\|F(z) - x\| \leq \|z - G(x)\| = 1 + \left\|\pi_{\sigma(i)}G(x)\right\|.$$

By the homogeneity of F in k components, we have that $\pi_i F(y) = f_{\sigma(i)}(\pi_{\sigma(i)}y)$ and $\pi_i F(z) = f_{\sigma(i)}(\pi_{\sigma(i)}z)$.

Since $g_j(-x) = -g_j(x)$ holds for all $j \in [n]$, $x \in S_j$, we also have that $f_{\sigma(j)}(-x) = -f_{\sigma(j)}(x)$ holds for $j \in [n]$, $x \in S_{\sigma(j)}$. Since $\pi_{\sigma(i)}z = -\pi_{\sigma(i)}y$, we have that $f_{\sigma(i)}(\pi_{\sigma(i)}z) = -f_{\sigma(i)}(\pi_{\sigma(i)}y)$, from which $\pi_i F(z) = -\pi_i F(y)$.

Since $\|\pi_i F(y) - \pi_i x\| \leq \|F(y) - x\|$, we have that

$$\|\pi_i F(y) - \pi_i x\| \leq 1 - \left\|\pi_{\sigma(i)}G(x)\right\|.$$

Analogously we have that

$$\|-\pi_i F(y) - \pi_i x\| = \|\pi_i F(z) - \pi_i x\| \leq 1 + \left\|\pi_{\sigma(i)}G(x)\right\|.$$

This means that $\pi_i x$ lies in the intersection of the two balls $B(\pi_i F(y), 1 - \left\|\pi_{\sigma(i)}G(x)\right\|)$ and $B(-\pi_i F(y), 1 + \left\|\pi_{\sigma(i)}G(x)\right\|)$. The intersection of these is a convex set in X_i, and is contained in the sphere of each ball. Since X_i is a strictly convex space, this intersection can contain at most one point. Since $\left\|\pi_{\sigma(i)}G(x)\right\|\pi_i F(y)$ belongs to both balls, we must have that $\left\|\pi_{\sigma(i)}G(x)\right\|\pi_i F(y) = \pi_i x$. Since we had $\|\pi_i F(y)\| = 1$ by the homogeneity of F, this gives us $\|\pi_i x\| = \left\|\pi_{\sigma(i)}G(x)\right\|$.

If $\pi_i x = 0$, we are done, since this gives us $\pi_{\sigma(i)}G(x) = 0$. We shall proceed with the assumption that $\pi_i x \neq 0$.

In this case, we have that $\|\pi_i x\| f_{\sigma(i)}(\pi_{\sigma(i)} y) = \pi_i x$. Dividing both sides by $\|\pi_i x\|$ and applying g_i, we get $\pi_{\sigma(i)} y = g_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right)$. By the definition of y, this gives us

$$\frac{\pi_{\sigma(i)} G(x)}{\left\|\pi_{\sigma(i)} G(x)\right\|} = \frac{\pi_{\sigma(i)} G(x)}{\|\pi_i x\|} = g_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right),$$

from which $\pi_{\sigma(i)} G(x) = \|\pi_i x\| g_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right)$ is immediate. □

Lemma 10. *For all $x \in S_Z$ and $\alpha \in [-1, 1]$, we have $F(\alpha x) = \alpha F(x)$.*

Proof. By homogeneity, we have, for $\alpha \geq 0$, that

$$\pi_{\sigma^{-1}(i)} F(\alpha x) = \|\pi_i(\alpha x)\| f_i \left(\frac{\pi_i(\alpha x)}{\|\pi_i(\alpha x)\|} \right) = \alpha \|\pi_i x\| f_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right) = \alpha \pi_{\sigma^{-1}(i)} F(x).$$

In the case of $\alpha < 0$, it is sufficient to consider only $\alpha = -1$. Here we make use of the fact that $f_i(-x) = -f_i(x)$:

$$\begin{aligned} \pi_{\sigma^{-1}(i)} F(-x) &= \|\pi_i(-x)\| f_i \left(\frac{\pi_i(-x)}{\|\pi_i(-x)\|} \right) \\ &= \|\pi_i x\| \left(-f_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right) \right) \\ &= -\|\pi_i x\| f_i \left(\frac{\pi_i x}{\|\pi_i x\|} \right) \\ &= -\pi_{\sigma^{-1}(i)} F(x). \end{aligned}$$

□

Lemma 11. $F(S_Z) = S_Z$.

Proof. Let us first show that for all $x \in S_Z$ we have $F(x) \in S_Z$.

Let $\|\pi_i x\| = 1$. Then by the homogeneity of F in n components, we have $\pi_{\sigma^{-1}(i)} F(x) = f_i(\pi_i x)$. Since $f_i(\pi_i x) \in S_{\sigma^{-1}(i)}$, we have that $\|F(x)\| = 1$.

Now it suffices to show that for $x \in S_Z$, we have $G(x) \in S_Z$. This follows immediately from the agreement Lemma 4 and the fact that the image of an extreme point under G is extreme. $\square$

It is now sufficient to apply the following result.

Lemma 12 ([2, Lemma 2.5])**.** *Let X be a Banach space and $F\colon B_X \to B_X$ be a* 1*-Lipschitz bijection. If $F(S_X) = S_X$ and $F(\alpha x) = \alpha F(x)$ for all $x \in S_X$ and $\alpha \in [-1, 1]$, then F is an isometry.*

By combining Lemmas 10, 11, and 12, the proof of Theorem 2 is immediate.

4 Additional results

In this section, we present some notable theorems that we encountered while investigating the topic. These theorems do not meaningfully relate to the main result of this document, and we drop the assumption of strict convexity here.

Theorem 3. *Suppose that there are Banach spaces X and Y, and a non-expansive bijection $F\colon B_X \to B_Y$ such that F is not an isometry. Then there is a Banach space Z and a non-expansive bijection $F'\colon B_Z \to B_Z$ such that F' is not an isometry.*

The inspiration for this theorem was drawn from the work of Olesia Zavarzina in [9]. Unfortunately, we are not aware if this theorem is already known as folklore. Since it has not yet been published, we present it here.

Proof. Let C_i be a Banach space for each $i \in \mathbb{Z}$, such that $C_i = X$ for $i < 0$ and $C_i = Y$ for $i \geqslant 0$. Take $Z \coloneqq \bigoplus_{i=-\infty}^{\infty} C_i$ with the ∞-norm. Define $F'\colon B_Z \to B_Z$ by $\pi_i F'(z) = \pi_{i-1} z$ for $i \neq 0$ and $\pi_0 F'(z) = F(\pi_{-1} z)$.

It is straightforward to verify that the codomain of F' is correct and that F' is a bijection. It remains to be shown that F' is non-expansive and not an isometry.

To show that F' is not an isometry, take any $x, x' \in B_X$ such that $\|F(x) - F(x')\| < \|x - x'\|$. Then take $z, z' \in B_Z$ to be the natural inclusions of x, x' into C_{-1}, so $\pi_{-1} z = x$ and $\pi_i z = 0$ for $i \neq -1$, and similarly for z'.

Then we have that $\pi_0 F'(z) = F(x)$ and $\pi_i F'(z) = 0$ for $i \neq 0$, and z' is analogous. So we have

$$\|F'(z) - F'(z')\| = \|F(x) - F(x')\| < \|x - x'\| = \|z - z'\|.$$

Thus, F' is not an isometry.

Now we will show that F' is non-expansive by using the following calculations for arbitrary $u, v \in B_Z$:

$$\begin{aligned}\big\|F'(u) - F'(v)\big\| &= \sup_{i\in\mathbb{Z}} \big\|\pi_i F'(u) - \pi_i F'(v)\big\| \\ &= \max\{\big\|\pi_0 F'(u) - \pi_0 F'(v)\big\|, \sup_{i\neq 0} \big\|\pi_i F'(u) - \pi_i F'(v)\big\|\} \\ &= \max\{\|F(\pi_{-1}u) - F(\pi_{-1}v)\|, \sup_{i\neq 0} \|\pi_{i-1}u - \pi_{i-1}v\|)\} \\ &\leqslant \max\{\|\pi_{-1}u - \pi_{-1}v\|, \sup_{i\neq -1} \|\pi_i u - \pi_i v\|\} \\ &= \sup_{i\in\mathbb{Z}} \|\pi_i u - \pi_i v\| \\ &= \|u - v\|.\end{aligned}$$

□

We now proceed to prove that from a homeomorphic 1-Lipschitz non-isometry from the closed unit ball of a Banach space to itself, we may extract a function with the same properties for the ball of a separable subspace of the original Banach space.

Lemma 13. *Let X be a Banach space and A be a subset of X that is closed under scaling by rationals. Then $\overline{A \cap B_X} = \overline{A} \cap B_X$.*

Proof. Since $\overline{A} \cap B_X$ is closed and $A \cap B_X \subseteq \overline{A} \cap B_X$, we have $\overline{A \cap B_X} \subseteq \overline{A} \cap B_X$.

To show the opposite inclusion, take any $a \in \overline{A} \cap B_X$, and choose a sequence $a_i \in A$ that converges to a. If $\|a\| < 1$, then $\|a_i\| < 1$ for all sufficiently large i, and therefore $a_i \in A \cap B_X$ for all sufficiently large i, so $a \in \overline{A \cap B_X}$.

On the other hand, if $\|a\| = 1$, then choose a sequence of rationals q_i such that $0 < q_i \leq \dfrac{1}{\|a_i\|}$ and $q_i \xrightarrow[i\to\infty]{} 1$. Such a sequence exists since $\|a_i\| \xrightarrow[i\to\infty]{} 1$. Then we have $q_i a_i \in A \cap B_X$ for each i and $q_i a_i \xrightarrow[i\to\infty]{} a$, so $a \in \overline{A \cap B_X}$.

Having exhausted all cases, we have shown that $\overline{A} \cap B_X \subseteq \overline{A \cap B_X}$, as desired. □

Theorem 4. *Suppose X is a Banach space and $F\colon B_X \to B_X$ is a non-expansive homeomorphism that is not an isometry. Then X has a separable closed sub-*

space Y such that $F(B_Y) = B_Y$ and $G \coloneqq F\big|_{B_Y} : B_Y \to B_Y$ is a non-expansive homeomorphism that is not an isometry.

Proof. Let $x, x' \in B_X$ be points such that $\|F(x) - F(x')\| < \|x - x'\|$.

Let $S \subseteq X$ be the intersection of all subsets of X that are closed under addition, rational scaling, F, F^{-1} (where either is defined), and which contain both x and x'. This is well-defined since X is one such subset. Also, S satisfies all of those properties itself.

Since S is closed under addition and rational scaling, we have that $\overline{S}$ is closed under addition and real scaling, and since it is also a closed set, it must be a closed subspace of X. By the previous lemma, we have that $\overline{S \cap B_X} = \overline{S} \cap B_X = B_{\overline{S}}$. We claim that $\overline{S}$ is the subspace we are looking for.

First, note that S is countable, since it coincides with a set of finite expressions generated by the points x and x', addition, rational scaling, and applications of F', F^{-1} when either are defined. This implies that $\overline{S}$ is separable.

Since F is continuous and S is closed under F, we have that $F(\overline{S \cap B_X}) \subseteq \overline{F(S \cap B_X)} \subseteq \overline{S \cap B_X}$, so $F(B_{\overline{S}}) \subseteq B_{\overline{S}}$. Analogously, since F^{-1} is continuous and S is closed under it, we have that $F^{-1}(B_{\overline{S}}) \subseteq B_{\overline{S}}$, which implies $B_{\overline{S}} \subseteq F(B_{\overline{S}})$. Since we have inclusions going both ways, we must have $F(B_{\overline{S}}) = B_{\overline{S}}$.

From this, it is immediate that F restricted to $B_{\overline{S}}$ is a non-expansive homeomorphism from the set to itself. Moreover, since $x, x' \in B_{\overline{S}}$, we also have that F cannot be an isometry on $B_{\overline{S}}$. □

Conclusion

In this thesis, we showed that Nikita Leo's result in [4] regarding the plasticity of the closed unit ball of the ℓ_∞-sum of two strictly convex Banach spaces partially generalizes to the case of the sum of an arbitrary finite number of strictly convex Banach spaces. We generalized a key lemma in Nikita's proof and adapted the remainder to show that in the case of such spaces, any 1-Lipschitz bijection from the closed unit ball of the space to itself which maps extreme points to extreme points or the sphere into itself must be an isometry. Some additional results with simpler proofs were also collected, in the hope that they may prove useful for subsequent investigation.

Whether there exists a path from our results to a full generalization is presently unclear. Further investigation may make use of these results in order to prove plasticity in special cases — we believe the case of a finite ℓ_∞-sum of ℓ_2-spaces may be tractable using our results. Alternatively, they may be used to guide the construction of potential counterexamples, should any exist.

Furthermore, we believe that the results proved herein may directly generalize to the case of non-contractive maps between the closed unit balls of distinct Banach spaces, in the style of [9]. However, we have not yet verified this in detail.

References

[1] Carlos Angosto, Vladimir Kadets, and Olesia Zavarzina. "Non-expansive bijections, uniformities and polyhedral faces". In: *Journal of Mathematical Analysis and Applications* 471.1 (2019), pp. 38–52. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2018.10.058. URL: https://www.sciencedirect.com/science/article/pii/S0022247X18308874.

[2] Bernardo Cascales, Vladimir Kadets, Jose Orihuela, and Eric Wingler. "Plasticity of the unit ball of a strictly convex Banach space". In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas* 110 (Aug. 2016), pp. 723–727. DOI: 10.1007/s13398-015-0261-3.

[3] Micheline Fakhoury. *Plasticity of the unit ball of some $C(K)$ spaces*. 2022. DOI: 10.48550/ARXIV.2208.07967. URL: https://arxiv.org/abs/2208.07967.

[4] Rainis Haller, Nikita Leo, and Olesia Zavarzina. "Two new examples of Banach spaces with a plastic unit ball". In: *Acta et Commentationes Universitatis Tartuensis de Mathematica* 26.1 (2022), pp. 89–101. DOI: https://doi.org/10.12697/ACUTM.2022.26.07. URL: https://ojs.utlib.ee/index.php/ACUTM/article/view/ACUTM.2022.26.07.

[5] Vladimir Kadets and Olesia Zavarzina. "Nonexpansive bijections to the unit ball of the ℓ_1-sum of strictly convex Banach spaces". In: *Bulletin of the Australian Mathematical Society* 97.2 (2018), pp. 285–292. DOI: 10.1017/S0004972717001150.

[6] Vladimir Kadets and Olesia Zavarzina. "Plasticity of the unit ball of ℓ_1". In: *Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics* 83 (Sept. 2016), pp. 4–9.

DOI: 10.26565/2221-5646-2016-83-01. URL: https://periodicals.karazin.ua/mech_math/article/view/8069.

[7] Nikita Leo. “Plasticity of the unit ball of c and c_0”. In: *Journal of Mathematical Analysis and Applications* 507.1 (2022), p. 125718. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2021.125718. URL: https://www.sciencedirect.com/science/article/pii/S0022247X21007976.

[8] Somashekhar Amrith Naimpally, Zbigniew Piotrowski, and Eric Wingler. “Plasticity in metric spaces”. In: *Journal of Mathematical Analysis and Applications* 313.1 (2006), pp. 38–48. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2005.04.070. URL: https://www.sciencedirect.com/science/article/pii/S0022247X05004117.

[9] Olesia Zavarzina. “Non-expansive bijections between unit balls of Banach spaces”. In: *Annals of Functional Analysis* 9 (Apr. 2017). DOI: 10.1215/20088752-2017-0050.

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