DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
75

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Differential calculus $d^3 = 0$ on binary and ternary associative algebras

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Contents

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Introduction

During the last decades a spectacular development of noncommutative generalizations of differential geometry and Lie group theory has been achieved, and the respective new chapters of mathematical physics are known under the names of noncommutative geometry, quantum groups and quantum spaces. The correspondence between spaces and commutative algebras of functions determined on these spaces is well known in mathematics and in theoretical physics. This correspondence gives us the possibility to describe a geometric concept defined on a space in terms of the appropriate algebra of functions determined on this geometric space. For example, given a vector field on a smooth finite dimensional manifold, we may describe it as the derivation of the algebra of smooth functions determined on this manifold. The basic idea of noncommutative geometry is to replace this commutative algebra of functions by a noncommutative algebra, and this replacement leads us to noncommutative generalizations of geometries where a notion such as "a space of points" is not involved. A noncommutative generalization of geometry has its origin in quantum field theory and this generalization may be used to better understand the relations between classical field theory and its quantum counterpart.

A noncommutative generalization of a geometric space can be constructed by means of a q-deformation, where $q \in \mathbb{C}$ is a parameter of deformation. Given a plane with commuting coordinates x, y, we have the commutative algebra of polynomials with complex coefficients generated by variables x and y. This algebra can be considered as an algebra of polynomial functions on a plane. One can construct a noncommutative generalization of this plane replacing the classical commutation relation $x\,y = y\,x$ by its q-deformed analog $x\,y = q\;y\,x$, where q is a complex number satisfying $q \neq 1$. Then the algebra of polynomials over $\mathbb{C}$ generated by x, y which are subjected to the single commutation relation $x\,y = q\;y\,x$, where $q \in \mathbb{C}$, $q \neq 1$, is known as the algebra of polynomials on quantum plane. If we assume that q is a primitive Nth root of unity and the generators x and y obey the relations $x\,y = q\,y\,x,\ x^N = y^N = 1$ then the algebra of polynomials over $\mathbb{C}$ generated by x, y is called the algebra of polynomials on a reduced quantum plane at Nth root of unity. Let us note that the last-mentioned algebra is a particular case of a generalized Clifford algebra.

Development of methods of noncommutative differential geometry has increased the interest towards possible q-deformations of the known algebraic structures when q is a primitive Nth root of unity. Indeed, we may regard the factor (-1) which plays an important role in graded structures as the particular case of an Nth primitive root of unity when $N = 2$, and this suggests us a way to construct a noncommutative generalization of a given algebraic structure. Let us remind that a graded differential algebra is an unital associative graded algebra equipped with a differential d which is a linear operator of degree 1 satisfying the graded Leibniz rule $d(\omega\theta) = d(\omega)\,\theta + (-1)^{\deg(\omega)}\,\omega\,d(\theta)$, where ω, θ are elements of an algebra, $\deg(\omega)$ is the grading of an element ω, and $d^2 = 0$. This notion has a generalization called a graded q-differential algebra, where q is a primitive Nth root of unity, which was proposed and studied within the framework of noncommutative geometry [26, 30, 31]. The basic idea of this generalization is to replace the factor (-1) in the graded Leibniz rule, which may be viewed as a primitive square root of unity, by an Nth primitive root of unity, where $N \geq 2$. Since primitive Nth root of unity satisfies the relations $q^N = 1$ and $1 + q + q^2 + \ldots + q^{N-1} = 0$, the replacement $(-1) \longrightarrow q$ will lead to a consistent algebraic structure if we replace the condition $d^2 = 0$ in the definition of a graded differential algebra by a more general condition $d^N = 0$, where $N \geq 2$ [40]. The latter gives rise to notions such as N-differential, N-differential module, N-complex of modules, generalized homology of N-differential module [29].

It is well known that one can associate to each finite dimensional smooth manifold the graded differential algebra constructed by means of differential forms and exterior differential, and this algebra is a very powerful tool for studying the geometry and differential topology of manifold. Since we have the q-generalization of a graded differential algebra with N-differential satisfying $d^N = 0$, it is natural to put a question: given a space with commuting or noncommuting coordinates is it possible to construct a q-generalization of exterior calculus with exterior differential satisfying $d^N = 0$, i.e. to construct a graded q-differential algebra by means of functions and the differentials of coordinates of this space? The answer to this question in the case when a space is a smooth finite dimensional manifold was given in [3]. One of the purposes of the present thesis is to give an answer to the above posed question in the case of a noncommutative space. It is easy to show that the subspace of elements of grading zero of a graded q-differential algebra is the subalgebra of graded algebra. It is also easy to show that the subspace of elements of grading $k \geq 1$ of a graded q-differential algebra is the bimodule over the algebra of elements of grading zero. Since the differential of a graded q-differential algebra is a linear mapping of degree one, which maps the algebra of elements of grading zero to the bimodule of elements of grading one, and in this case it satisfies the Leibniz rule, we have the first order differential calculus associated to a given graded q-differential algebra. This means that if our goal is to construct an q-generalization of exterior calculus on a noncommutative space with differential d satisfying $d^3 = 0$ then we should begin our construction by choosing an appropriate first order differential calculus on a given noncommutative space and to extend it to the tensor algebra by the differential satisfying the q-Leibniz rule and condition $d^3 = 0$. If a first order differential calculus is a coordinate first order differential calculus then it induces the right partial derivatives with respect to coordinates of a noncommutative space. It can be shown that these derivatives satisfy the twisted Leibniz rule [12, 13, 14, 15, 17].

In Chapter 1 we give a detailed review of noncommutative geometry approach to a first order differential calculus on associative algebras and we also give a detailed description of the structure of a graded differential algebra with $d^2 = 0$. A first order differential calculus on an associative unital algebra $\mathcal{A}$ is $d : \mathcal{A} \longrightarrow \mathcal{M}$, where $\mathcal{M}$ is an $\mathcal{A}$-bimodule and a differential d satisfies the Leibniz rule. If an algebra $\mathcal{A}$ is generated by a finite set of variables and the structure of right $\mathcal{A}$-module of $\mathcal{M}$ is finite freely generated $\mathcal{A}$-module then we give the definitions of a coordinate and semi-coordinate first order differential calculus and show that a differential d induces the right partial derivatives on an algebra $\mathcal{A}$ which satisfy the twisted Leibniz rule. We give description of an universal first order differential calculus which is constructed with the help of tensor product of given associative algebra. We end this chapter with a detailed description of the structure of a graded differential algebra and universal differential envelope.

In Chapter 2 we study a generalization of a graded differential algebra which is called graded Q-differential algebra and this generalization was proposed and studied within the framework of noncommutative geometry in 1990s. We begin Chapter 2 with the definition of graded Q-differential algebra. A peculiar property of this generalization of graded differential algebra is that a differential satisfies $d^N = 0$, where N is any integer greater than or equal to 2. We consider the first non-trivial generalization, where Q is a primitive cubic root of unity and differential satisfies $d^3 = 0$. An important question which arises in applications of graded Q-differential algebras to noncommutative geometry and theoretical physics is how to construct a graded Q-differential algebra if we are given a first order differential calculus. In Chapter 2 we give a detailed description of construction of graded Q-differential algebra if we are given an algebra generated by finite set of variables which obey some relations. Our approach is based on tensor algebra and the construction of graded differential algebra for a given semi-coordinate first order differential calculus which is described in the previous chapter.

In Chapter 3 we apply the described before method to construction of an algebra of differential forms with exterior differential satisfying $d^3 = 0$ on the following noncommutative spaces: the anyonic line, the quadratic algebra, the q- and h-deformed quantum planes. We also find the conditions for the second order differentials generate the q- and h-deformed quantum planes respectively. We give an explicit description of the structure of algebra of differential forms with differential satisfying $d^3 = 0$ in the case of the mentioned above noncommutative spaces by finding all commutation relations between coordinates and their differentials. We show that in the case of one dimensional space the structure of an algebra of differential forms with differential satisfying $d^3 = 0$ is self-consistent, if this one dimensional space is the anyonic line [1]. We show that the first order differential calculus induced by our algebra of differential forms is well known first order differential calculus of fractional supersymmetry in dimension one [33].

In Chapter 4 we wish to stress the fact that the natural structure on which the $\mathbb{Z}_3$-grading takes its full meaning is a ternary algebra, which means a linear vector space over complex numbers on which a ternary composition law is defined. Although ternary laws can be modeled in ordinary algebras with an associative binary law by defining corresponding ternary ideals and dividing the algebra by the equivalence relations induced by these ideals, one may also introduce ternary composition laws for the entities which can not be derived from a binary law. We begin this chapter with a brief description of the structure of a ternary algebra giving the notions such as associative ternary algebra, structure constants of a ternary algebra, ternary algebra with involution and so on. Then we propose a notion of analog of module over a ternary algebra algebra which we call tri-module. We also define derivations of ternary algebra and construct several examples of ternary differential algebras. We propose a construction of universal envelope for associative ternary algebra and the universal differential calculus on a ternary algebra. Although we believe that these novel algebraic constructions might be pertinent for the description of quark fields and new models of elementary interactions in particle physics, we shall stress here mathematical rather than physical aspects, keeping hope that further developments and physical applications of ternary structures will follow soon.

Chapter 1

Noncommutative geometry approach to differential calculus

In this chapter our main concern is a concept of first order differential calculus (FODC) on associative unital algebra and the structure of a graded differential algebra. We give the definitions of basic notions related to first order differential calculus on associative algebra and give a detailed description of the structure of a graded differential algebra stressing the point that differential of a graded differential algebra induces the first order differential calculus on the subalgebra of elements of grading zero. In Section 1.1 we remind the definition of a first order differential calculus on an associative algebra. The basic components of FODC are a bimodule over this algebra and a mapping d from an algebra to bimodule which satisfies the Leibniz rule. A mapping d is called a differential of FODC. In Section 1.1 we show how to construct the FODC on an algebra with finite number of generators which can be interpreted as coordinates in a noncommutative space. If the right module structure of bimodule of FODC is finite freely generate module over an algebra and the differentials of generators of an algebra form the basis for this module then the corresponding FODC is called a first order coordinate differential calculus, and differential d is called a coordinate differential. If the differentials of generators of an algebra generate a submodule of the right module of bimodule of FODC then the corresponding FODC will be referred to as semi-coordinate first order differential calculus.

We show that in the case of coordinate and semi-coordinate first order differential calculus the differential d induces the right partial derivatives with respect to generators of an algebra, and we also show that these partial derivatives satisfy a twisted Leibniz rule. In the next chapters we shall use these partial derivatives induced by a differential d in order to construct a generalization of graded differential algebra with differential satisfying $d^3 = 0$. In this section we also study the problem of existence and uniqueness of coordinate and semi-coordinate differentials for the given partial derivatives and any homomorphism which relates the left structure of bimodule to the right one. In our descriptions of both cases of FODC we follow the approach proposed in [12, 13, 14, 15, 17], where a coordinate calculus is thoroughly studied. Since in Chapter4 we will construct the FODC on a ternary algebra, in this chapter following [18] we give a description of universal FODC in the case of binary unital algebra.

We start Section 1.2 by reminding a reader the definition of a graded differential algebra whose differential d satisfies the Leibniz rule and the condition $d^2 = 0$. Given a graded differential algebra we can consider a first order differential calculus of this algebra which can be described as follows: it is well known that the subspace of elements of grading zero of a graded algebra is subalgebra of this graded algebra and the subspace of elements of grading one can be viewed as bimodule over the subalgebra of elements of grading zero, then, taking into account that a differential of graded differential algebra is a mapping of degree one, we have the first order differential calculus which consists of the algebra of elements of grading zero, bimodule of elements of grading one over this algebra and a differential. On the other hand, if we have a first order differential calculus on an associative algebra it is natural to try to construct a graded algebra in such a way that initial algebra will be a subalgebra of elements of grade 0 of this graded algebra and to extend the differential d to the entire graded algebra by assuming that $d^2 = 0$. In Section 1.2 we describe a universal method for constructing a graded differential algebra by means of factorization of tensor algebra if we are given a first order differential calculus on an associative algebra. The detailed description of this method in the case of coordinate and semi-coordinate calculus with right partial derivatives can be found in [12],[13]. Since a semi-coordinate FODC with right partial derivatives is a more general calculus than a coordinate one we shall use a semi-coordinate calculus in order to construct the graded differential algebra in the next chapter.

1.1 First order differential calculus with right partial derivatives

Let $\mathcal{A}$ be an associative unital algebra over the field $\mathbb{K}$ of characteristic 0, and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Let d be a differential on the algebra $\mathcal{A}$, i.e. d is a linear map $d : \mathcal{A} \to \mathcal{M}$ which satisfies the Leibniz rule

$$d(uv) = d(u)v + ud(v) \quad \forall\ u,\ v \in \mathcal{A}. \tag{1.1.1}$$

Definition 1.1.1. The triple $\{\mathcal{M},\ \mathcal{A},\ d\}$ is called a first order differential calculus (FODC) on an algebra $\mathcal{A}$ with values in $\mathcal{A}$-bimodule $\mathcal{M}$. The elements of $\mathcal{M}$ are called one-forms and the bimodule $\mathcal{M}$ is linear span of elements $u\, d(v)\, w$ with $u,\ v,\ w \in \mathcal{A}$.

From the Leibniz rule 1.1.1, it follows that for any $u,\ v,\ w \in \mathcal{A}$ we have

$$u\, d(v)\, w = d(uv)w - d(u)vw \quad \text{or} \quad u\, d(v)\, w = u\, d(vw) - uv\, d(w).$$

Hence $\mathcal{M} \doteq \mathcal{A} \cdot d(\mathcal{A}) \cdot \mathcal{A} = d(\mathcal{A}) \cdot \mathcal{A} = \mathcal{A} \cdot d(\mathcal{A})$. Two FODC $\{\mathcal{M}_{\mathrm{I}},\ \mathcal{A},\ d_{\mathrm{I}}\}$ and $\{\mathcal{M}_{\mathrm{J}},\ \mathcal{A},\ d_{\mathrm{J}}\}$ on an algebra $\mathcal{A}$ are said to be isomorphic if there exists a bijective linear mapping $\psi : \mathcal{M}_{\mathrm{I}} \to \mathcal{M}_{\mathrm{J}}$ such that $\psi(u\, d_{\mathrm{I}}(v)\, w) = u\, d_{\mathrm{J}}(v)\, w$ for $u,\ v,\ w \in \mathcal{A}$.

Let us suppose that $\mathcal{A}$ is an algebra generated by $\{x^1,\ x^2, \dots,\ x^n\}$ which are linearly but not necessarily algebraically independent. The generators $x^1,\ x^2, \dots,\ x^n$ may be interpreted as coordinates in space in the sense that the elements of an algebra $\mathcal{A}$ can be considered as the polynomial functions on this space.

Definition 1.1.2. A first order differential calculus $\{\mathcal{M},\ \mathcal{A},\ d\}$ over an algebra $\mathcal{A}$ with generators $\{x^1,\ x^2, \dots,\ x^n\}$ is said to be *a coordinate calculus* if the differentials of generators $\{dx^1,\ dx^2, \dots,\ dx^n\}$ form the basis for the right $\mathcal{A}$-module of bimodule $\mathcal{M}$, i.e. the right $\mathcal{A}$-module $\mathcal{M}$ is freely generated by the differentials $dx^i = d(x^i),\ i = 1, \dots,\ n.$ A differential $d : \mathcal{A} \to \mathcal{M}$ will be referred to as coordinate differential.

Definition 1.1.3. A first order differential calculus $\{\mathcal{M},\ \mathcal{A},\ d\}$ on an algebra $\mathcal{A}$ with a finite number of generators is said to be *a semi-coordinate calculus* if there are no generators of the algebra $\mathcal{A}$ such that the corresponding differentials freely generate the right $\mathcal{A}$-module $\mathcal{M}$.

1.1.1 The right partial derivatives in the case of coordinate calculus

Let $\{\mathcal{M},\ \mathcal{A},\ d\}$ be a coordinate FODC, where $\mathcal{A} = \mathbb{K} < x^1,\ x^2, \dots,\ x^n >$, i.e. $\mathcal{A}$ is a free associative unital algebra $\mathcal{A}$ generated by the variables $x^1,\ x^2,\ \dots,\ x^n$, and $\mathcal{M}$ be an $\mathcal{A}$-bimodule with free right structure generated by the differentials of generators $dx^1,\ dx^2, \dots,\ dx^n$. Let us consider left module structure of $\mathcal{A}$-bimodule $\mathcal{M}$ in more details. We can interpret the left multiplication $u\,dv$, where $u \in \mathcal{A}$ and $dv \in \mathcal{M}$, as an endomorphism of the right module $\mathcal{M}$. It is known that the ring of all endomorphisms $End(\mathcal{M}) : \mathcal{M} \to \mathcal{M}$ of rank n is isomorphic to the ring $Mat_n(\mathcal{A})$ of $(n \times n)$-matrices with entries from the algebra $\mathcal{A}$: $End(\mathcal{M}) \cong Mat_n(\mathcal{A})$. This isomorphism allows us to find such an algebra homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ that

$$udx^i = dx^k \xi(u)^i_k \qquad \forall u \in \mathcal{A}. \tag{1.1.2}$$

Throughout this thesis we will use the so-called Einstein summation convention $dx^k \xi(u)^i_k \equiv \sum_{k=1}^n dx^k \xi(u)^i_k$ over repeating lower and upper indices.

It is easy to check that the map ξ is a homomorphism. Indeed

$$dx^k \xi(uv)^i_k = (uv)dx^i = u(vdx^i) = udx^l \xi(v)^i_l = dx^k \xi(u)^l_k \xi(v)^i_l. \tag{1.1.3}$$

We would like to point out that the homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ defined in (1.1.2) relates the left module structure of $\mathcal{A}$-bimodule $\mathcal{M}$ with the free right one.

Since the elements dx^i form the basis for the free right module $\mathcal{M}$ the differential of an arbitrary element $u \in \mathcal{A}$ can be uniquely expressed as follows

$$d(u) = dx^k\, D_k(u). \tag{1.1.4}$$

Definition 1.1.4. The linear maps $D_k : \mathcal{A} \to \mathcal{A}$ defined by (1.1.4) are called partial derivatives of coordinate FODC $\{\mathcal{M},\ \mathcal{A},\ d\}$.

Obviously the partial derivatives of the coordinate FODC $\{\mathcal{M},\ \mathcal{A},\ d\}$ satisfy the property

$$D_k(x^i) = \delta^i_k, \tag{1.1.5}$$

where δ^i_k is the Kronecker delta. As it follows from the Leibniz rule (1.1.1) rewritten in the terms of the decomposition (1.1.4), partial derivatives D_k and the homomorphism ξ are connected by the relations

$$D_k(uv) = D_k(u)v + \xi(u)^j_k D_j(v), \qquad \forall\, u,\, v \in \mathcal{A}, \tag{1.1.6}$$

i.e. the partial derivatives D_k satisfy the twisted Leibniz rule. In other words, the property (1.1.6) expresses compatibility conditions between the partial derivatives D_k and the homomorphism ξ. Moreover, comparing the Leibniz rule (1.1.1) for the product $u\,x^i \in \mathcal{A}$ written in the following form

$$u\,dx^i = d(u\,x^i) - du\,x^i = dx^j(D_j(ux^i) - D_j(u)x^i)$$

with the commutation rule (1.1.2), we obtain that the mapping $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ can be completely defined by the partial derivatives D_k as follows

$$\xi(u)^j_k = D_k(ux^j) - D_k(u)x^j \qquad \forall\, u \in \mathcal{A}. \tag{1.1.7}$$

Let us show that the map ξ defined in (1.1.7) is a homomorphism of algebras. Indeed we can compute the differential of the product $uvx^i \in \mathcal{A}$ with the help of the Leibniz rule (1.1.1)

$$\begin{aligned} d(uvx^i) &= d(u)vx^i + ud(v)x^i + uvdx^i = dx^k D_k(u)vx^i + \\ &\quad dx^k\big(D_k(ux^j) - D_k(u)x^j\big)D_j(v)x^i + uvdx^i. \end{aligned}$$

Making use of the definition of partial derivatives 1.1.4 we can compute the differential of the same product in another way

$$d(uvx^i) = dx^k D_k(uvx^j).$$

Hence the term $uvdx^i$ can be expressed as follows

$$\begin{aligned} uvdx^i &= dx^k\big(D_k(uvx^i) - D_k(u)vx^i - D_k(ux^j)D_j(v)x^i + \\ &\quad D_k(u)x^j D_j(v)x^i\big). \end{aligned} \tag{1.1.8}$$

From the other hand applying the differential d to the product uvx^i we get

$$\begin{aligned} d(uvx^i) = d(u(vx^i)) = du\,vx^i + ud(vx^i) = dx^k\big(D_k(u)vx^i + \\ D_k(ux^j)D_j(vx^i) - D_k(u)x^j D_j(vx^i)\big). \end{aligned}$$

This gives the following relation for the partial derivatives D_k

$$D_k(uvx^i) = D_k(u)vx^i + D_k(ux^j)D_j(vx^i) - D_k(u)x^jD_j(vx^i). \qquad (1.1.9)$$

Substituting (1.1.9) into (1.1.8) we obtain

$$\begin{aligned} uvdx^i &= dx^k\big(D_k(u)vx^i + D_k(ux^j)D_j(vx^i) - D_k(u)x^jD_j(vx^i) - \\ &\quad -D_k(u)vx^i - D_k(ux^j)D_j(v)x^i + D_k(u)x^jD_j(v)x^i\big) \\ &= dx^k\big(D_k(ux^j) - D_k(u)x^j\big)\big(D_j(vx^i) - D_j(v)x^i\big) \\ &= dx^k\xi(u)^j_k\xi(v)^i_j, \end{aligned}$$

and this implies that ξ is a homomorphism of algebras. Hence we have shown that the differential of FODC induces the partial derivatives D_k which satisfy the property (1.1.5) and the twisted Leibniz rule (1.1.6).

Next we consider the problem of existence and uniqueness of coordinate differential for any given homomorphism ξ and partial derivatives D_k. We assume that the set of matrices $\xi^1, \ldots, \xi^n \in Mat_n(\mathcal{A})$ is fixed. Then we construct a homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ by extending the map $x^i \mapsto \xi^i$, $i = 1, \ldots, n$, to the whole algebra. Note that, since $\mathcal{A}$ is a free algebra generated by x^i such an extension always exists and is uniquely determined. We define the linear maps $D_k : \mathcal{A} \to \mathcal{A}$ on monomials in $x^1, \ldots, x^n$ by induction with respect to their degree as follows:

$$D_k(x^i) = \delta^i_k \quad \text{and} \quad D_k(x^i\, v) = \delta^i_k\, v + \xi^{ij}_k\, D_j(v), \qquad (1.1.10)$$

where $v \in \mathcal{A}$ is any monomial and $\xi^{ij}_k = \xi(x^i)^j_k$. Now we can prove by means of induction that the linear maps D_k satisfy the twisted Leibniz rule (1.1.6). It is easy to check that it holds for any monomial v of degree one. Let us assume that the formula (1.1.6) holds for an arbitrary element $zv \in \mathcal{A}$ of degree n. Then for the element $uv = x^izv$ of degree $n+1$ we have

$$\begin{aligned} D_k(uv) &= D_k(x^iz\,v) = (\delta^i_k\, z)v + (\xi^i)^j_k\, D_j(zv) \\ &= \big(\delta^i_k z + (\xi^i)^j_k\, D_j(z)\big)v + (\xi^i)^l_k\, \xi(z)^j_l\, D_j(v) = D_k(u)v + \xi(u)^j_k D_j(v), \end{aligned}$$

where we use the property $\xi(u\,v)^i_k = \xi(u)^j_k\, \xi(v)^i_j$ since ξ is an algebra homomorphism.

Let $\mathcal{M}_{\Delta_\xi} = \big(\sum_{i=1}^n dx^i\big)\,\mathcal{A}$. Let us show that the linear map $\Delta_\xi : \mathcal{A} \to \mathcal{M}_{\Delta_\xi}$ defined by

$$\Delta_\xi(v) = dx^k\, D_k(v) \qquad \forall\, v \in \mathcal{A}$$

is a coordinate differential, i.e. the linear map Δ_ξ satisfies the property $\Delta_\xi(x^i) = dx^i$ and the Leibniz rule (1.1.1). Indeed we have

$$\begin{aligned}
\Delta_\xi(x^i) &= dx^k D_k(x^i) = dx^k \delta^i_k = dx^i, \qquad \forall x^i \in \mathcal{A},\\
\Delta_\xi(u\,v) &= dx^k\, D_k(u\,v) = dx^k (D_k(u)\,v + \xi(u)^i_k\, D_i(v))\\
&= \Delta_\xi(u)v + u\,\Delta_\xi(v), \qquad \forall u,\ v \in \mathcal{A}.
\end{aligned}$$

Thus the existence of coordinate FODC is equivalent with the existence of the homomorphism (1.1.2) and the partial derivatives (1.1.4) which satisfy the relations (1.1.6). Uniqueness of the differential Δ_ξ follows from the inductive definition of partial derivatives D_k (1.1.10). The first formula of this definition allows us to calculate the partial derivatives on generators of $\mathcal{A}$ and the second one on products in terms of their factors. Therefore there exists no more than one collection of partial derivatives $D_k,\ k = 1, \ldots,\ n$, determined by (1.1.10).

1.1.2 The cover differential in the case of coordinate calculus

In this section we consider the construction of coordinate FODC on a non-free algebra generated by a set of variables which are subjected to some relations. Let $\overline{\mathcal{A}} = \mathcal{A}/I = \mathcal{F} < x^1, \ldots,\ x^n >/I$, where $I \subsetneq \mathcal{A}$ is a two-sided ideal generated by the set of elements $f_s(x^1, \ldots,\ x^n) \in \mathcal{A},\ s \in \mathbb{N}$. In order to simplify notations we will denote the elements $f_s(x^1, \ldots, x^n)$ generating two-sided improper ideal I by f_s .

Let $\pi : \mathcal{A} \to \overline{\mathcal{A}}$ be the natural projection defined by

$$\pi(x^i) = x^i + I \doteq \overline{x}^i,\ (i = 1, \ldots,\ n); \quad \pi(v) = v + I \doteq \overline{v}; \quad \pi(1) = 1.$$

Obviously $ker\,\pi = I$, and the elements $\overline{x}^1,\ \ldots,\ \overline{x}^n$ generate the algebra $\overline{\mathcal{A}}$. Our aim is to construct a coordinate FODC $\{\overline{\mathcal{A}},\ \overline{\mathcal{M}},\ d\}$, where $\overline{\mathcal{M}} = \overline{\mathcal{A}}\,(d(\overline{\mathcal{A}}))\,\overline{\mathcal{A}}$, with right partial derivatives induced by d. Since in the previous section we proved the existence and uniqueness of the FODC in the case of free algebra for any given homomorphism and right partial derivatives consistent with this homomorphism we can give the following definition

Definition 1.1.5. The differential $d : \mathcal{A} \to \mathcal{M}$ is called a cover differential with respect to the commutation rule (1.1.2).

As it is easily seen from the preceding considerations, the coordinate differential $d : \mathcal{A} \to \mathcal{M}$ can be uniquely defined by an algebra homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ and partial derivatives $D_k : \mathcal{A} \to \mathcal{A}$, which satisfy the relation (1.1.6). Now our aim is to describe the procedure for construction of the algebra homomorphism $\overline{\xi} : \overline{\mathcal{A}} \to Mat_n(\overline{\mathcal{A}})$ and the partial derivatives $\overline{D}_k : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$, which determine the differential $d : \overline{\mathcal{A}} \to \overline{\mathcal{M}}$. First of all, in order to make consistent the left structure of $\overline{\mathcal{M}}$ with the right one, we define the algebra homomorphism $\overline{\xi}$ by means of the algebra homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ using the well-known property $Mat_n(\mathcal{A}) = \mathcal{A} \otimes Mat_n(\mathbb{K})$. We recall that the algebra homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ was obtained as an extension of the map $x^i \mapsto \xi^i$ which is unique because the algebra $\mathcal{A}$ is freely generated by the elements $x^i,\ i = 1, \ldots,\ n$. Each matrix $\xi^1, \ldots,\ \xi^n$ can be represented as the sum of elements of the linear space $\mathcal{A} \otimes Mat_n(\mathbb{K})$. Now let us fix such n elements $\xi^i \in \mathcal{A} \otimes Mat_n(\mathbb{K})$ that $\pi(\xi^i) = \overline{\xi}^i \in \overline{\mathcal{A}} \otimes Mat_n(\mathbb{K})$ $i = 1,\ \ldots,\ n$.

Let $Id : Mat_n(\mathbb{K}) \to Mat_n(\mathbb{K})$ be the identity map. The map

$$\Pi = \pi \otimes Id : \mathcal{A} \otimes Mat_n(\mathbb{K}) \to \overline{\mathcal{A}} \otimes Mat_n(\mathbb{K})$$

is an epimorphism. The following diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\xi} & Mat_n(\mathcal{A}) & \cong & \mathcal{A} \otimes Mat_n(\mathbb{K}) \\ \pi\big\downarrow & & \big\downarrow\ \ \Pi & = & \pi \otimes Id\big\downarrow \\ \overline{\mathcal{A}} & \xrightarrow{\overline{\xi}} & Mat_n(\overline{\mathcal{A}}) & \cong & \overline{\mathcal{A}} \otimes Mat_n(\mathbb{K}) \end{array} \tag{1.1.11}$$

is commutative iff

$$\overline{\xi}(\pi(f_s)) = \Pi(\xi(f_s)) = 0 \qquad \forall\ f_s \in I.$$

As a consequence of this commutativity

$$ker\,\Pi = ker(\pi \otimes Id) = ker\pi \otimes Mat_n(\mathbb{K}) = I \otimes Mat_n(\mathbb{K}) \cong Mat_n(I),$$

that implies the obvious inclusion

$$\xi(f_s) \in ker\,\Pi \cong Mat_n(I). \tag{1.1.12}$$

In view of this last inclusion, the following definition becomes evident.

Definition 1.1.6. An ideal I of free algebra $\mathcal{A}$ is said to be ξ-invariant if $\xi(I) \subseteq Mat_n(I)$, where $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ is the algebra homomorphism.

In what follows we will assume that the ideal $I \subsetneq \mathcal{A}$ is ξ-invariant. Let $\overline{D}_k$ be uniquely defined partial derivatives on $\overline{\mathcal{A}}$. We will define partial derivatives D_k by means of the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{D_k} & \mathcal{A} \\ \pi \Big\downarrow & & \Big\downarrow \pi \\ \overline{\mathcal{A}} & \xrightarrow{\overline{D}_k} & \overline{\mathcal{A}} \end{array} \tag{1.1.13}$$

finding a condition which guarantees that this diagram is commutative. In order to formulate this condition we give the following definition:

Definition 1.1.7. An ideal I of a free algebra $\mathcal{A}$ is said to be ξ-stable with respect to 1.1.2 if $D_k(I) \subseteq I$.

Theorem 1.1.8. *The diagram (1.1.13) is commutative iff the ξ-invariant ideal $I \in \mathcal{A}$ is ξ-stable.*

Proof. Necessity. Assume that $\pi \circ D_k = \overline{D}_k \circ \pi$. Then we obtain that on the generators of $\mathcal{A}$

$$\pi \circ D_k(x^i) = \overline{D}_k \circ \pi(x^i) = \overline{D}_k(\pi(x^i)) = \overline{D}_k(\overline{x}^i) = \delta^i_k \qquad \forall x^i \in \mathcal{A}.$$

The connection relations between the partial derivatives $\overline{D}_k$ and the homomorphism $\overline{\xi}$ (1.1.6) hold true

$$\begin{aligned} \pi \circ D_k(u\,v) \quad = \quad & \pi(D_k(u\,v)) = \pi\big(D_k(u)v + \xi(u)^j_k\, D_j(v)\big) = \pi(D_k(u))\,\pi(v) + \\ & \pi(\xi(u)^j_k)\,\pi(D_j(v)) = \overline{D}_k(\pi(u))\,\pi(v) + \overline{\xi}(\pi(u))^j_k\,\overline{D}_j(\pi(v)) = \\ & \overline{D}_k(\overline{u})\,\overline{v} + \overline{\xi}(\overline{u})^j_k\,\overline{D}_j(\overline{v}) = \overline{D}_k(\overline{u}\,\overline{v}) \qquad \forall u,\ v \in \mathcal{A}, \end{aligned}$$

since $\pi \circ D_k(u\,v) = \overline{D}_k \circ \pi(u\,v)$. For the relations $f_s \in I$ we have

$$(\pi \circ D_k)\,(f_s) = (\overline{D}_k \circ \pi)\,(f_s) = \overline{D}_k(\pi(f_s)) = \overline{D}_k(0) = 0 \qquad \forall f_s \in I,$$

therefore $D_k(f_s) \in I$. It means that the ideal I is ξ-stable.

Sufficiency. If the ξ-invariant ideal I is ξ-stable, then for any two elements $u,\ v \in \mathcal{A}$ and $f_s \in I$ we have that

$$D_k(u\,f_s\,v) = D_k(u)\,f_s\,v + \xi(u)^j_k\,D_j(f_s)\,v + \xi(u)^j_k\,\xi(f_s)^l_j\,D_l(v).$$

Since the ideal I is ξ-invariant, i.e. $\xi(f_s)^l_j \in I$ for any $f_s \in I$, it is obvious that $D_k(u\, f_s\, v) \in I$. Therefore $D_k(I) \subseteq I$, and the diagram (1.1.13) becomes commutative. $\square$

In this way, the facts obtained above allow us to unify the two properties of the ideal I in the following

Definition 1.1.9. An ideal $I \subsetneq \mathcal{A}$ is said to be consistent with the homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ if that ideal I is ξ-invariant and ξ-stable.

Finally we can formulate the main result of study of the coordinate FODC on the quotient algebra $\overline{\mathcal{A}} = \mathcal{A}/I$ with values in the $\overline{\mathcal{A}}$-bimodule $\overline{\mathcal{M}}$ with the free right structure: the coordinate differential $\overline{d} : \overline{\mathcal{A}} \to \overline{\mathcal{M}}$ exists iff the ideal $I \subsetneq \mathcal{A}$ is consistent with the homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$.

1.1.3 The right partial derivatives in the case of semi-coordinate calculus

By semi-coordinate case we mean the more general than coordinate case in which the generators $\{\mu^1_1,\ \mu^2_1, \ldots,\ \mu^m_1\}$, $m \in \mathbb{N}$, of module M_μ are not necessary of the form $\mu^i_1 = dx^i$ for some set of algebra generators $\{x^1,\ x^2, \ldots,\ x^n\}$, $n \in \mathbb{N}$, n does not need to be equal to m. Here the subscript index 1 means that we deal with the one-form. Now we assume that the $\mathcal{A}$-bimodule $M_\mu = \mathcal{A}\,(d\mathcal{A})\,\mathcal{A}$ is generated as a free right module. As it was in the coordinate case, the property $M_\mu = (d\mathcal{A})\,\mathcal{A} = \mathcal{A}(d\mathcal{A})$ immediately follows from the Leibniz rule (1.1.1) exactly.

We need to consider the semi-coordinate case of FODC here separately, because we will construct the q-differential algebra with the differential $d^3 = 0$ starting from such FODC in Chapter 2 and 3. Here we point out which details in the construction of semi-coordinate FODC coincide, and which differ from details of the coordinate FODC. In the semi-coordinate case there will be no confusion, if we use the same letters, which we have used before in the coordinate case of FODC in order to designate the differential, partial derivatives, and algebra homomorphism connecting the left structure of bimodule with the right one.

Because the bimodule M_μ has the free right structure, the differential of any element $u \in \mathcal{A}$ has the unique decomposition

$$d(u) = \mu^k_1\, D_k(u), \qquad k = i, \ldots,\ m, \tag{1.1.14}$$

where the linear maps $D_k : \mathcal{A} \to \mathcal{A}$ are right partial derivatives as before. From the uniqueness of this decomposition (1.1.14) it follows that the semi-coordinate differential $d : \mathcal{A} \to M_\mu$ completely defines the partial derivatives $D_k : \mathcal{A} \to \mathcal{A}$. It is easy to show that $D_k(1) = 0$. Indeed, applying the Leibniz rule (1.1.1) to the product $1 \cdot u$, we obtain

$$d(1 \cdot u) = \mu_1^k \, D_k(1)u + 1\mu_1^k \, D_k(u). \tag{1.1.15}$$

Since $d(1 \cdot u) = du = \mu_1^k \, D_k(u)$, we conclude that $D_k(1) = 0$.

Let us denote $D_k(x^i) \doteq \sigma_k^i$. Now we assume that $\sigma_k^i \neq \delta_k^i$ as a distinction from the coordinate case where $\sigma_k^i = \delta_k^i$. Then, in our notation, we write the differentials of generators of $\mathcal{A}$ by the following term:

$$d(x^i) = \mu_1^k \, \sigma_k^i, \qquad i = 1, \ldots, \; n, \; k = 1, \ldots, \; m. \tag{1.1.16}$$

Let the left structure of $\mathcal{A}$-bimodule M_μ be connected with right one by the algebra homomorphism $\xi : \mathcal{A} \to \mathrm{M}at_{m \times n}(\mathcal{A})$ such that

$$u \, \mu_1^j = \mu_1^k \xi(u)_k^j \qquad \forall u \in \mathcal{A},$$

or on the generators of $\mathcal{A}$

$$x^i \, \mu_1^j = \mu_1^k \xi_k^{ij}, \tag{1.1.17}$$

where $\xi_k^{ij} \doteq \xi(x^i)_k^j$.

As in the coordinate case, immediately from the Leibniz rule (1.1.1) rewritten in partial derivatives

$$\begin{aligned} \mu_1^k \, D_k(uv) \;\; &= \;\; d(uv) = d(u)v + u \; d(v) = \mu_1^k D_k(u)v + u\mu_1^l D_l(v) = \\ & \quad\;\; \mu_1^k(D_k(u)v + \xi(u)_k^l D_l(v)) \end{aligned}$$

we obtain the same type of the connection relations between the homomorphism ξ and the partial derivatives D_k

$$D_k(uv) = D_k(u)v + \xi(u)_k^l D_l(v) \qquad \forall u, \; v \in \mathcal{A}. \tag{1.1.18}$$

The algebra homomorphism $\xi : \mathcal{A} \to Mat_{m \times n}(\mathcal{A})$, partial derivatives $D_k : \mathcal{A} \to \mathcal{A}$ and initial conditions (1.1.16) define a semi-coordinate differential d uniquely. Now we need not to go into all details of the proof of this assertion because it coincides with the proof of its analogue described above for the coordinate case, but here we have to replace the initial conditions $D_k(x^i) = \delta^i_k$ by its semi-coordinate analogue $D_k(x^i) = \sigma^i_k$.

In the semi-coordinate case we have to correct the definition of the cover differential. Let $\{\mathcal{A},\ M_\mu,\ d\}$ be the semi-coordinate FODC.

Definition 1.1.10. The differential $d : \mathcal{A} \to M_\mu$ is called a cover differential with respect to the algebra homomorphism $\xi : \mathcal{A} \to Mat_{m\times n}(\mathcal{A})$ and the initial conditions $D_k(x^i) = \sigma^i_k$.

As in the coordinate case, the existence problem of FODC on the quotient algebra $\overline{\mathcal{A}} = \mathcal{A}/I$ described above has the analogous conclusion: the quotient algebra $\overline{\mathcal{A}}$ has a differential $\overline{d}$ with the partial derivatives $\overline{D}_k : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ such that $\overline{D}_k(\overline{x}^i) = \overline{\sigma}^i_k \in \overline{\mathcal{A}}$ and consistent with the homomorphism $\overline{\xi} : \overline{\mathcal{A}} \to Mat_{m\times n}(\overline{\mathcal{A}})$ iff the ideal $I \in \mathcal{A}$ is ξ-consistent. Remark that the bar labels elements from $\overline{\mathcal{A}}$ and maps that act on $\overline{\mathcal{A}}$. The method used for the proof of this fact in coordinate case works as proof in the semi-coordinate case too.

1.1.4 Universal first order differential calculus

Let $\mathcal{A}$ be an associative unital algebra with the unit element denoted by $1_\mathcal{A}$, and multiplication denoted by $m : \mathcal{A} \otimes_\mathbb{K} \mathcal{A} \to \mathcal{A}$. Obviously the tensor product $\mathcal{A} \otimes_\mathbb{K} \mathcal{A}$ can be considered as $\mathcal{A}$-bimodule defined by

$$u(a \otimes b)v = (u\,a) \otimes (b\,v). \tag{1.1.19}$$

Let us consider the submodule $J_m = \ker m \subset \mathcal{A} \otimes_\mathbb{K} \mathcal{A}$. We define the linear map $\delta_\mathcal{A} : \mathcal{A} \to J_m$ by

$$\delta_\mathcal{A}(a) = a \otimes 1_\mathcal{A} - 1_\mathcal{A} \otimes a. \tag{1.1.20}$$

This linear map satisfies the Leibniz rule and hence it is the differential on an algebra $\mathcal{A}$. Indeed we have

$$\begin{aligned} \delta_\mathcal{A}(ab) &= ab \otimes 1_\mathcal{A} - 1_\mathcal{A} \otimes ab = a \otimes b - 1_\mathcal{A} \otimes ab + ab \otimes 1_\mathcal{A} - a \otimes b = \\ &\quad \delta_\mathcal{A}(a)b + a\delta_\mathcal{A}(b). \end{aligned}$$

Thus the triple $(\mathcal{A}, J_m, \delta_\mathcal{A})$ is the J_m-valued first order differential calculus on an algebra $\mathcal{A}$. The submodule J_m considered as right $\mathcal{A}$-module is generated by $\operatorname{Im}\delta_{\mathcal{A}}$. Indeed if $\sum_i \ a_i \otimes \ b_i$ is an element of J_m, then $\sum_i a_i b_i = 0$. Consequently any element of J_m can be written as

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1 - 1 \otimes a_i)\ b_i = \sum_i \delta_{\mathcal{A}}(a_i)\ b_i, \tag{1.1.21}$$

where $a_i \otimes b_i \in J_m$.

Proposition 1.1.11. If $(\mathcal{A}, {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}, d)$ is an ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$-valued first order differential calculus on an associative unital algebra $\mathcal{A}$ then there exists a unique bimodule homomorphism $\varphi : J_m \longrightarrow {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ such that $d = \varphi \circ \delta_{\mathcal{A}}$ and the following diagram

$$\begin{tikzcd} & J_m \arrow[d, "\varphi"] \\ \mathcal{A} \arrow[ur, "\delta_{\mathcal{A}}"] \arrow[r, "d"] & {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} \end{tikzcd}$$

is commutative.

Proof. Let the linear map d be defined by

$$da \doteq (\varphi \circ \delta_{\mathcal{A}})(a) = \varphi(\delta_{\mathcal{A}}(a)) = \varphi(a \otimes 1_{\mathcal{A}} - 1_{\mathcal{A}} \otimes a) \quad \forall a \in \mathcal{A}.$$

This linear map is the differential because it satisfies the Leibniz rule. Indeed, we have

$$\begin{aligned} d(ab) &= (\varphi \circ \delta_{\mathcal{A}})(ab) = \varphi(\delta_{\mathcal{A}}(ab)) = \varphi(\delta_{\mathcal{A}}(a)b + a\delta_{\mathcal{A}}(b)) \\ &= (\varphi \circ \delta_{\mathcal{A}}(a))b + a(\varphi\delta_{\mathcal{A}}(b)) = d(a)b + ad(b) \quad \forall\ a,\ b \in \mathcal{A}. \end{aligned}$$

Next we need to show that for the differential $d : \mathcal{A} \to \mathcal{M}$ there exists an bimodule homomoprhism $\varphi : J_m \to \mathcal{M}$ such that $d = \varphi \circ \delta_{\mathcal{A}}$ and φ is uniquely defined. Since $\operatorname{Im}\delta_{\mathcal{A}}$ generates the right $\mathcal{A}$-module J_m we can define the homomorphism φ by

$$\begin{aligned} \varphi\left(\sum_i a_i \otimes b_i\right) &= \varphi\left(\sum_i (a_i \otimes 1_{\mathcal{A}} - 1_{\mathcal{A}} \otimes a_i) b_i\right) \\ &= \varphi\left(\sum_i \delta_{\mathcal{A}}(a_i) b_i\right) = \sum_i \varphi\delta_{\mathcal{A}}(a_i) b_i = \sum_i d(a_i)\, b_i. \end{aligned}$$

For the left $\mathcal{A}$-module J_m we have

$$\varphi\left(\sum_i a_i a_i' \otimes b_i\right) = \varphi\left(\sum_i (a_i a_i' \otimes 1_{\mathcal{A}} - 1_{\mathcal{A}} \otimes a_i a_i')\, b_i\right)$$

$$
\begin{aligned}
&= \varphi\Big(\sum_i (\delta_{\mathcal{A}}(a_i a_i')b_i)\Big) = \varphi \sum_i \big(\delta_{\mathcal{A}}(a_i)\, a_i' + a_i\delta_{\mathcal{A}}(a_i')\big)b_i \\
&= \varphi \sum_i \big(\delta_{\mathcal{A}}(a_i)\, a_i' b_i + a_i\delta_{\mathcal{A}}(a_i')b_i\big) = \sum_i \big(\varphi(\delta_{\mathcal{A}}(a_i)\, a_i' b_i) + \varphi(a_i\delta_{\mathcal{A}}(a_i')b_i)\big) \\
&= \sum_i \varphi(a_i\delta_{\mathcal{A}}(a_i')b_i) = \sum_i a_i\varphi(\delta_{\mathcal{A}}(a_i')b_i) \\
&= \sum_i a_i(\varphi\circ\delta_{\mathcal{A}})(a_i')b_i = \sum_i a_i\, d(a_i')b_i.
\end{aligned}
$$

Hence $\varphi : J_m \longrightarrow {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is the homomorphism of bimodules.

Taking into account that the differential $\delta_{\mathcal{A}}$ is uniquely defined for another homomorphism $\psi : J_m \longrightarrow {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ which satisfies

$$
da = (\psi\circ\delta_{\mathcal{A}})\ (a) = \psi(a\otimes 1 - 1\otimes a),
$$

we have

$$
\begin{aligned}
&((\psi-\varphi)\circ\delta_{\mathcal{A}})(a) = (\psi-\varphi)(a\otimes 1 - 1\otimes a) = \\
&\psi(a\otimes 1 - 1\otimes a) - \varphi(a\otimes 1 - 1\otimes a) = da - da = 0, \quad \forall\ a\in\mathcal{A}.
\end{aligned}
$$

Obviously the differential $\delta_{\mathcal{A}} : \mathcal{A}\to J_m$ is a cover differential for $d : \mathcal{A}\to {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Furthermore if ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} = d(\mathcal{A})\mathcal{A} = \mathcal{A}\ d(\mathcal{A})$, then the homomorphism φ becomes an epimorphism, and in this case the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is equal to the bimodule J_m factorized by some $\mathcal{A}$-subbimodule. □

Remark 1.1.12. Proposition 1.1.11 shows that the J_m-valued first order differential calculus on an associative unital algebra $\mathcal{A}$ has a property of universality. Therefore the J_m-valued first order differential calculus on an associative unital algebra $\mathcal{A}$ is called the universal first order differential calculus.

1.2 Graded differential algebra with $d^2 = 0$

We begin this section with the definition of graded differential algebra.

Definition 1.2.1. Let $\Omega = \bigoplus_{n=0}^{\infty}\Omega^{(n)}$ be an $\mathbb{N}$-graded algebra equipped with a linear map $d : \Omega\to\Omega$ which satisfies the following conditions:

1. graded Leibniz rule

$$d(\omega\,\theta) = (d\omega)\,\theta + (-1)^n \omega\,(d\theta), \forall \omega \in \Omega^{(n)}, \theta \in \Omega, \tag{1.2.1}$$

2. d has degree one

$$d : \Omega^{(n)} \to \Omega^{(n+1)}, \tag{1.2.2}$$

3.

$$d^2 = 0. \tag{1.2.3}$$

Then Ω is called a graded differential algebra with differential d.

Definition 1.2.2. Let $\Omega, \widetilde{\Omega}$ be two graded differential algebras respectively with differentials $d, \widetilde{d}$. A homogeneous algebra homomorphism $\phi : \Omega \to \widetilde{\Omega}$ of degree zero is said to be a morphism of graded differential algebras if it commutes with differentials $d, \widetilde{d}$, i.e.

$$\phi\,(\Omega^{(n)}) \subseteq \widetilde{\Omega}^{(n)}, \quad \phi(d\omega) = \widetilde{d}(\phi\omega), \quad \forall\,\omega \in \Omega^{(n)}.$$

Let us consider the structure of a graded differential algebra Ω. It follows immediately from Definition 1.2.1 that the subspace $\Omega^{(0)}$ of elements of grading zero is the subalgebra of Ω and the subspace $\Omega^{(n)}$ of elements of grading n is $(\Omega^{(0)})$-bimodule. Hence the triple $\{\Omega^{(0)},\ \Omega^{(1)},\ d\}$ can be considered as the FODC on the algebra $\Omega^{(0)}$ with the values in $\Omega^{(1)}$.

Now we assume the following conditions to be satisfied

$$\Omega^{(1)} = \Omega^{(0)}(d\,\Omega^{(0)})\Omega^{(0)} = \Omega^{(0)}(d\,\Omega^{(0)}) = (d\,\Omega^{(0)})\Omega^{(0)}.$$

Then we have $d\,(\Omega^{(n)}) \subseteq \Omega^{(n+1)}$, $\forall\,n \in \mathbb{N}$. Consequently for any $n > 0$ the $(\Omega^{(0)})$-bimodule $\Omega^{(n+1)}$ is generated by the subspace $\Omega^{(n)}$ and the differential d. Therefore if we are given the first order differential calculus $d : \Omega^{(0)} \to \Omega^{(1)}$ with differential d satisfying Leibniz rule (not graded Leibniz rule!) we can construct a graded differential algebra generating $(\Omega^{(0)})$-bimodules $\Omega^{(n)}$ and extending the differential d to $(\Omega^{(0)})$-bimodules $\Omega^{(n)}$ by means of graded Leibniz rule. A possible way to construct a graded differential algebra can be divided into two steps:

1. we construct a first order differential calculus,

2. then we generate bimodules of elements of higher gradings and extend a differential of a first order differential calculus to these bimodules requiring a differential to satisfy the conditions (1.2.1),(1.2.2),(1.2.3).

The detailed description of the first step has been given in Section 1.1. Now let us assume that we are given a semi-coordinate FODC $\{\mathcal{A},\ M_\mu,\ d^1\}$, where $\mathcal{A}$ is the associative unital algebra generated by the set of variables $\{x^1,\dots,\ x^n\}$ which are subjected to the relations $\{f_s=0\}$, $s\in\mathbb{N}$, $d^1:\mathcal{A}\to M_\mu$ is a differential, M_μ is the $\mathcal{A}$-bimodule generated as the free right module by the set of elements $\{\mu_1^1,\mu_1^2,\dots,\ \mu_1^m\}$ such that

$$d(x^i)=\mu_1^k\,\sigma_k^i\quad\text{and}\quad d\mu_1^k=0 \tag{1.2.4}$$

where $\sigma_k^i=D_k(x^i)\in\mathcal{A},\ i=1,\dots,\ n,\ k=1,\dots,\ m;\ n,\ m\in\mathbb{N},\ n\neq m$, and

$$u\,\mu_1^j=\mu_1^k\,\xi(u)_k^j,\qquad\forall\,u\in\mathcal{A}. \tag{1.2.5}$$

In order to construct the graded differential algebra with $d^2=0$ we assume that we are given a semi-coordinate FODC. A reason why we begin with a semi-coordinate FODC is that this is a more general case than a coordinate one. If our purpose is to construct a graded differential algebra for a coordinate FODC which is a particular case of a semi-coordinate FODC we will impose two additional conditions on semi-coordinate FODC getting a coordinate one. First of them is $n=m$, i.e. the number of the generators of $\mathcal{A}$ is equal to the number of elements of a bases of M_μ, and the second is $d(x^i)=\mu_1^i$ or $\sigma_j^i=\delta_j^i$.

There is another way to obtain the coordinate FODC from the given semi-coordinate one. From the Leibniz rule (1.2.1) and commutation relations (1.2.5) it follows that $\mu^k\in d(\mathcal{A})\,\mathcal{A}=\sum_i dx^i\,\mathcal{A}$. Thus for suitable $a_i^k\in\mathcal{A}$ we have $\mu^k=\mu^s\sigma_s^i\,a_i^k$, where a matrix $||\sigma_s^i||$ is right invertible. If $||a_i^k||$ is the reciprocal matrix of $||\sigma_s^i||$ then the change of generators $y^k=x^i\,a_i^k$ yields

$$dy^k=d(x^i\,a_i^k)=d(x^i)\,a_i^k=\mu^k\sigma_k^i\,a_i^k=\mu^k,$$

which means that we have a coordinate FODC.

Our next aim is to construct a graded differential algebra making use of the given above scheme. In other words we will construct a universal differential envelope for a given FODC $\{\mathcal{A},\ M_\mu,\ d^1\}$. An universal differential envelope is a graded differential algebra with the following universal property: any other graded differential algebra with the same FODC $\{\mathcal{A},\ M_\mu,\ d^1\}$ can be obtained from it by factorization. We extend the semi-coordinate FODC to a graded differential algebra with the help of differential d requiring $d\mu_1^k = 0$. It means that the basis of M_μ generates formal differentials.

Now our aim is to give a detailed description of universal differential envelope. We begin with the tensor algebra

$$T_{\mathcal{A}}(M_\mu) = \underset{n\geq 0}{\oplus} M_\mu^{\otimes n},$$

where $M_\mu^{\otimes n} = M_\mu \otimes_{\mathcal{A}} M_\mu \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} M_\mu$. As usual $M_\mu^0 = \mathcal{A}, M_\mu^1 = M_\mu$. The tensor algebra $T_{\mathcal{A}}(M_\mu)$ is generated by the set of variables $x^i \in \mathcal{A}$ which obey the relations $f_s = 0$, elements of a basis $\mu_1^j \in \mathcal{M}$ and relations (1.2.5). Each subspace $M_\mu^{\otimes n} \subset T_{\mathcal{A}}(M_\mu)$ is the free bimodule spanned by the elements of the form $\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_n}$. Any element $\omega \in M_\mu^{\otimes n}$ can be uniquely expressed as follows

$$\omega = \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_n}\, u_{i_1 i_2 \ldots i_n}, \tag{1.2.6}$$

where $u_{i_1 i_2 \ldots i_n} \in \mathcal{A}$. We shall call this form canonical.

Let us define a linear map $d : M_\mu^{\otimes n} \to M_\mu^{\otimes (n+1)}$ of degree one by

$$\begin{aligned} d(\omega) &= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_n}\, u_{i_1 i_2 \ldots i_n}) = \\ &(-1)^n \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_n} \otimes \mu_1^k\, D_k(u_{i_1 i_2 \ldots i_n}), \end{aligned} \tag{1.2.7}$$

where D_k are the right partial derivatives induced by a differential d^1. Further on we will require the linear map d to coincide with the differential d^1 on the algebra $\mathcal{A}$, and keeping this in mind we will use the same letter for both maps.

It is easy to see that the tensor algebra $T_{\mathcal{A}}(M_\mu)$ equipped with the operator d is not a graded differential algebra because of two reasons. The first reason is that in general the operator d defined in (1.2.7) does not satisfy the graded Leibniz rule (1.2.1). It is worth mentioning that the operator d does satisfy the graded Leibniz rule only in the case of elements which can be represented in the canonical form. The second reason is that the operator d does not satisfy the relation $d^2 = 0$. Indeed differentiating twice we obtain

$$d^2(u) = d(du) = d(\mu_1^k\, D_k(u)) = d(\mu_1^k)\, D_k(u) - \ \mu_1^k \otimes \mu_1^l\, D_l D_k(u) = \\ - \ \mu_1^k \otimes \mu_1^l\, D_l D_k(u) \neq 0 \qquad \forall u \, \in \mathcal{A}.$$

Hence if our purpose is to construct a graded differential algebra with differential d satisfying the graded Leibniz rule (1.2.1) and the relation $d^2 = 0$ we have to consider not the whole tensor algebra $T_{\mathcal{A}}(M_\mu)$ but its quotient algebra.

Let $J_\xi \subset T_{\mathcal{A}}(M_\mu)$ be the two-sided ideal generated by the following elements of grading two

$$\mu_1^k \otimes \mu_1^l\, \xi(\sigma_k^i)_l^j + \mu_1^k \otimes \mu_1^l\, D_l(\xi_k^{ij}), \quad \mu_1^i \otimes \mu_1^k\, D_k(\sigma_j^i). \tag{1.2.8}$$

Theorem 1.2.3. *The pair $\{\Omega_{\mathcal{A}}(M_\mu),\ d\}$, where $\Omega_{\mathcal{A}}(M_\mu) \doteq T_{\mathcal{A}}(M_\mu)/J_\xi$ is the quotient algebra modulo J_ξ, is the graded differential algebra.*

Proof. In order to prove this theorem we will use the commutation relations

$$\begin{aligned} &u\,\mu_1^j = \mu_1^k\, \xi(u)_k^j, \\ &\mu_1^j \otimes \mu_1^l(\xi(D_j(u))_l^i + D_l(\xi(u)_j^i)) = 0, \\ &\mu_1^i \otimes \mu_1^j\, D_j D_i(u) = 0, \qquad \forall\, u \in \mathcal{A}, \end{aligned} \tag{1.2.9}$$

which hold in the quotient algebra $\Omega_{\mathcal{A}}(M_\mu)$. Indeed, if $u = x^i$ then the relations (1.2.9) are satisfied because of (1.2.8). Assume that the relations (1.2.9) hold for any element $v \in \mathcal{A}$ of grading k. We can show that these relations hold for the element $u = x^i\, v$ of grading $k+1$ as follows

$$\begin{aligned} du \otimes \mu_1^j &= d(x^i\, v) \otimes \mu_1^j = dx^i \otimes v\, \mu_1^j + x^i\, dv \otimes \mu_1^j \\ &= \mu_1^j \sigma_j^i\, v \otimes \mu_1^k + x^i\, \mu_1^k D_k(v) \otimes \mu_1^t \\ &= \mu_1^j \otimes \mu_1^s \xi(\sigma_j^i)_s^k \xi(v)_k^t) + \mu_1^j \xi_j^{ik} D_k(v) \otimes \mu_1^t \\ &= -\mu_1^j \otimes \mu_1^s D_s(\xi_j^{ik}) \xi(v)_k^t + \mu_1^j \otimes \xi_j^{ik} \mu_1^p \xi(D_k(v))_p^t) \\ &= -\mu_1^j \otimes \mu_1^s D_s(\xi_j^{ik}) \xi(v)_k^t - \mu_1^j \otimes \xi_j^{ik} \mu_1^p D_p(\xi(v))_k^t \\ &= -\mu_1^j \otimes (d(\xi_j^{ik})\, \xi(v)_k^t + \xi_j^{ik}\, d(\xi(v))_k^t) \\ &= -\mu_1^j \otimes d(\xi_j^{ik}\, \xi(v)_k^t) = -\mu_1^j \otimes d(\xi(u)_k^t). \end{aligned}$$

Next step in the proof is to show that d satisfies the graded Leibniz rule (1.2.1) and we will do it by induction. Let $v,\ v_j \in \mathcal{A}$, $\theta = \mu_1^j\, v_j$, $\eta = v_j\, \mu_1^j \in M_\mu$,

m-form $\omega \in M_\mu^{\otimes m}$ be canonical, that is $\omega = \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} u_{i_1 i_2 \ldots i_m}$. Differentiating the products $\omega\, v$, $\omega \otimes \theta$ and $\omega \otimes \eta$ we obtain

$$\begin{aligned}
d(\omega v) &= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} u_{i_1 i_2 \ldots i_m}\, v) \\
&= (-1)^m \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes d(u_{i_1 i_2 \ldots i_m}\, v) \\
&= (-1)^m \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes d(u_{i_1 i_2 \ldots i_m})\, v \\
&\quad + (-1)^m \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes u_{i_1 i_2 \ldots i_m}\, d(v) \\
&= d(\omega)\, v + (-1)^m \omega \otimes dv,
\end{aligned}$$

$$\begin{aligned}
d(\omega \otimes \theta) &= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} u_{i_1 i_2 \ldots i_m} \otimes \mu_1^j v_j) \\
&= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \mu_1^l\, \xi(u_{i_1 i_2 \ldots i_m})_l^j\, v_j) \\
&= (-1)^{m+1} \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \mu_1^l \otimes d\big(\xi(u_{i_1 i_2 \ldots i_m})_l^j\, v_j\big) \\
&= (-1)^{m+1} \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \mu_1^l \otimes \big(d(\xi(u_{i_1 i_2 \ldots i_m})_l^j)\, v_j \\
&\qquad\qquad + \xi(u_{i_1 i_2 \ldots i_m})_l^j\, d(v_j)\big) \\
&= (-1)^m \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes d(u_{i_1 i_2 \ldots i_m}) \otimes \mu_1^j v_j \\
&\quad + (-1)^{m+1} \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} u_{i_1 i_2 \ldots i_m} \otimes \mu_1^j \otimes d(v_j) \\
&= d(\omega) \otimes \theta + (-1)^m \omega \otimes d(\theta),
\end{aligned}$$

$$\begin{aligned}
d(\omega \otimes \eta) &= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} u_{i_1 i_2 \ldots i_m} \otimes v_j \mu_1^j) \\
&= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \mu_1^l \xi(u_{i_1 i_2 \ldots i_m}\, v_j)_l^j) \\
&= (-1)^{m+1} \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \mu_1^l \otimes d(\xi(u_{i_1 i_2 \ldots i_k})_l^r\, \xi(v_j)_r^j) \\
&= (-1)^{m+1} \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \mu_1^l \otimes \big(d(\xi(u_{i_1 i_2 \ldots i_m})_l^r)\, \xi(v_j)_r^j \\
&\qquad\qquad + (\xi(u_{i_1 i_2 \ldots i_k})_l^r)\, d(\xi(v_j)_r^j)\big) \\
&= (-1)^m \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} \otimes \big(d(u_{i_1 i_2 \ldots i_m}) \otimes \mu_1^r \xi_r^j(v_j) \\
&\qquad\qquad + u_{i_1 \ldots i_k}\, \mu_1^r \otimes d(\xi_r^j(v_j))\big)
\end{aligned}$$

$$\begin{aligned}
&= d(\mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_m} u_{i_1 i_2 \ldots i_m}) \otimes v_j \mu_1^j \\
&\qquad + (-1)^k \mu_1^{i_1} \otimes \mu_1^{i_2} \otimes \cdots \otimes \mu_1^{i_k} u_{i_1 i_2 \ldots i_m} \otimes d(v_j) \otimes \mu_1^j \\
&= d(\omega) \otimes \theta + (-1)^k \omega \otimes d(\theta).
\end{aligned}$$

Hence d satisfies the graded Leibniz rule (1.2.1).

It remains to show that the operator d satisfies the relation $d^2 = 0$. Clearly this relation will be proved if we show that

$$d^2(\mathcal{A}) \subset J_\xi, \tag{1.2.10}$$
$$d^2(J_\xi) \subseteq J_\xi, \tag{1.2.11}$$
$$d^2(T_{\mathcal{A}}(\mathcal{M})) \subseteq J_\xi. \tag{1.2.12}$$

We prove the inclusion (1.2.10) by induction. For any generator x^i we have $d^2(x^i) = 0$. Assuming $d^2(v) \in J_\xi$ to be true for any element $v \in \mathcal{A}$ of grading k we will prove that this inclusion is true for any element of grading $k+1$, i.e. $d^2(u) \in J_\xi$, where $u = x^i\, v \in \mathcal{A}$ is the element of grading $k+1$. Indeed

$$\begin{aligned} d^2(u) &= d(d(x^i\, v)) = d(\mu^i_1\, v + x^i\, dv) \\ &= d^2x^i\, v - \mu^i_1 \otimes dv + \mu^i_1 \otimes dv + \mu^i_1 \otimes d^2v = \mu^i_1 \otimes d^2v \in J_\xi. \end{aligned}$$

In order to prove the inclusion (1.2.11) it is sufficient to show that this inclusion is true in the case of generators of J_ξ. Indeed

$$d^2(\mu^k_1 \otimes \mu^l_1(\xi(\sigma^i_k)^j_l + D_l(\xi^{ij}_k))) = \mu^k_1 \otimes \mu^l_1\, d^2(\xi(\sigma^i_k)^j_l + D_l(\xi^{ij}_k)) \in J_\xi,$$

since $\xi(\sigma^i_k)^j_l + D_l(\xi^{ij}_k) \in \mathcal{A}$, and

$$\begin{aligned} d^2(\mu^i_1 \otimes \mu^k_1\, D_k(\sigma^i_j)) &= d(d\mu^i_1 \otimes \mu^k_1\, D_k(\sigma^i_j) - \mu^i_1 \otimes d\mu^k_1\, D_k(\sigma^i_j) \\ &\quad + \mu^i_1 \otimes \mu^k_1 d(D_k(\sigma^i_j))) = d(\mu^i_1 \otimes \mu^k_1 d(D_k(\sigma^i_j)) \\ &= d\mu^i_1 \otimes \mu^k_1 d(D_k(\sigma^i_j)) - \mu^i_1 \otimes d\mu^k_1 d(D_k(\sigma^i_j)) + \mu^i_1 \otimes \mu^k_1 d^2(D_k(\sigma^i_j)) \\ &= 0 \in J_\xi, \end{aligned}$$

since $D_l(\sigma^i_j) \in \mathcal{A}$. Therefore the inclusion (1.2.11) is true.

Finally it remains to prove the inclusion (1.2.12). Assuming $d^2\eta \in J_\xi$ to be true for any $(n-1)$-form $\eta \in \mathcal{M}^{\otimes(n-1)}$ for the n-form $\omega = \mu^i_1 \otimes \eta \in \mathcal{M}^n$ we have

$$d^2(\omega) = d(d(\mu^i_1 \otimes \eta)) = d(d\mu^i_1 \otimes \eta - \mu^i_1 \otimes d\eta) = d(0) = 0 \in J_\xi.$$

This ends the proof of the property $d^2 = 0$ on the quotient algebra $\Omega_{\mathcal{A}}(M_\mu)$. Thus the pair $\{\Omega_{\mathcal{A}}(M_\mu), d\}$ is the graded differential algebra with the differential satisfying the property $d^2 = 0$. □

Remark 1.2.4. It should be pointed out that the ideal J_ξ is minimal for the construction of the obtained graded differential algebra $\Omega_{\mathcal{A}}(M_\mu)$.

Chapter 2

Graded differential algebra with differential $d^3 = 0$

An $d^N = 0$, where $N \geq 2$, generalization of a concept of graded differential algebra has been proposed and studied in [30, 31]. The peculiar property of this generalization is the appearance of higher order differentials of variables.

A first nontrivial generalization of a graded differential algebra, where $d^3 = 0$, has been studied in [1, 4, 42, 43, 44, 52]. Later on similar construction was realized in [6] on super-spaces with one- and two-parametric quantum groups as symmetry groups. Here from the beginning we note that we shall distinguish the parameter of deformation of quantum plane and the parameter used in the Leibniz rule. Therefore we shall denote these parameters by the letters "q" and "Q" accordingly, and thus our notations will differ from the notations used by the authors of previously mentioned articles.

In this chapter we construct the graded Q-differential algebra with differential $d^3 = 0$ on an arbitrary complex associative unital algebra with n generators. That is, we extend FODC by differential d satisfying the property $d^3 = 0$ and the Q-Leibniz rule

$$d(\omega\theta) = (d\omega)\theta + Q^n\omega(d\theta),$$

where $\omega \in \Omega^{(n)}$, $\theta \in \Omega$ and the complex number Q is a primitive cubic root of unity [31].

Our inspiration is twofold. On the one hand, we have the approach of S. Woronowicz [65] in order to construct quantum de Rham complex, for an ordinary differential ($d^2 = 0$), starting from FODC (see [12, 13, 14, 15, 16]). In Section 2.1 we choose for prolongation the semi-coordinate FODC, and in Section 2.2 the coordinate FODC. Also in Section 2.2 we apply the additional restriction on the bimodule generated by the second order differentials, namely that this bimodule is free as right module. On the other hand, we generalize the method applied in [30, 31] for the universal differential calculus and N-ary differential, where the graded (universal) Q-differential algebra with differential $d^N = 0$ is based on the universal FODC with underlying algebra $\mathcal{A}$. These original results can be found in the papers [8, 9].

Further in Chapter 3 we use the framework elaborated here for realization of the graded Q-differential algebra with differential $d^3 = 0$ on associative algebra with n generators and quadratic relations, on quantum planes, and on algebra with one generator specifically is constructed (see also [7]).

Doing a differential calculus on flat space coordinate $\mathbb{R}^n$ we wish that the span of one-forms will be generated by the exterior derivative of coordinate dx^i. However, in the case of curved space-time these differentials are local quantities, and not convenient to describe the generators of one-forms. For example, on Lie group the bases of left invariant one-forms (Cartan forms) plays essential role. In the case of quantum groups, it turns out that the classical dimension of group differ from the dimension of bi-coinvariant bimodule of one-forms. The same might be true for some quantum space [58].

From now on, we shall use the notation $[n]_Q = 1 + Q + Q^2 + \cdots + Q^{n-1}$. In our case, $Q = e^{i\frac{2\pi}{3}}$, we have $[3]_Q = 1 + Q + Q^2 = 0$.

2.1 Semi-coordinate case of graded Q-differential algebra with differential $d^3 = 0$

At the beginning of this section we remind the definition of graded Q-differential algebra with differential $d^3 = 0$ proposed in [30, 31], where such graded (universal) Q-differential algebra with differential $d^N = 0$ has been constructed starting from the universal FODC corresponding to a given algebra $\mathcal{A}$.

Definition 2.1.1. A pair $\{\Omega, d\}$ is said to be a graded Q-differential algebra, where $Q = e^{\frac{2\pi i}{3}}$, if $\Omega = \oplus_{n\geq 0}\,\Omega^{(n)}$ is a graded algebra with $\Omega^0 = \mathcal{A}$, and

$d : \Omega \to \Omega,\ d = \oplus_{n\geq 1} d^n$ is a homogeneous linear map of grading one which satisfies $d^3 = 0$ and the Q-Leibniz rule

$$d(\omega\theta) = (d\omega)\theta + Q^n\omega(d\theta),\ \ \omega \in \Omega^{(n)},\ \theta \in \Omega. \tag{2.1.1}$$

Remark 2.1.2. A trivial example of graded differential algebra $\{\Omega,\, d\}$ with differential $d^3 = 0$ is the triple $\{\mathcal{A},\ \mathcal{M},\ d\}$, where $\mathcal{A} = \Omega^{(0)}$, $\mathcal{M} = \Omega^{(1)}$, $\Omega^{(n)} = 0$, $n > 1$, and differential d coincides with differential on the algebra $\mathcal{A}$.

Further on we will generalize the construction of graded Q-differential algebra with differential satisfying $d^3 = 0$ on the semi-coordinate case of FODG $\{\mathcal{A},\, M_\mu,\, d\}$ described in Section 1.2. Here we assume that $\mathcal{A}$ is a unital associative algebra with the generating set $\{x^1, \ldots, x^n\}$ and the set of relations $\{f_s = 0\}$, M_μ is a $\mathcal{A}$-bimodule with the free right structure with bases $\{\mu_1^1,\ \mu_1^2, \ldots,\ \mu_1^m\}$, the elements of $\mathcal{A}$ and M_μ satisfy the commutation relations

$$u\,\mu_1^j = \mu_1^k \xi(u)_k^j, \tag{2.1.2}$$

where $\xi : \mathcal{A} \to Mat_{m\times n}(\mathcal{A})$ is an algebra homomorphism, $j = 1,\ \ldots, n$, $k = 1,\ \ldots, m$.
From now on we assume that $d\mu_1^i \neq 0$. Before specification of such algebra on the case of semi-coordinate calculus with right partial derivatives, we will introduce elements $\mu_2^i,\ i = 1, \ldots, m$, such that $d\mu_1^i = \mu_2^i$ and $d\mu_2^i = 0$. We will refer to these elements μ_2^i as the second order differentials. Let $\mathcal{M}_\mu^{(2)}$ be a bimodule over $\mathcal{A}$ with the set of generators $\{\mu_2^1,\ \mu_2^2, \ldots,\ \mu_2^m\}$.
Further we define a graded $\mathcal{A}$-bimodule $\mathcal{E}_\mu = \mathcal{M}_\mu^{(1)} \oplus \mathcal{M}_\mu^{(2)}$, and construct the tensor algebra of $\mathcal{E}_\mu$ over the ring $\mathcal{A}$:

$$T_{\mathcal{A}}\mathcal{E}_\mu = \mathcal{A} \oplus \mathcal{E}_\mu \oplus {\mathcal{E}_\mu}^{\otimes 2} \oplus {\mathcal{E}_\mu}^{\otimes 3} \oplus \cdots = \bigoplus_{r\geq 0} {\mathcal{E}_\mu}^{\otimes r}$$

where ${\mathcal{E}_\mu}^{\otimes r}$ means $\mathcal{E}_\mu \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{E}_\mu,\ {\mathcal{E}_\mu}^0 = \mathcal{A}$.

After expanding and rearranging terms in $T_{\mathcal{A}}\mathcal{E}_\mu$, we obtain a $\mathbb{N}$-graded algebra

$$T_{\mathcal{A}}\mathcal{E}_\mu = \oplus_{r\geq 0} T_\mu^r, \tag{2.1.3}$$

with a unique graduation compatible with graduation of $\mathcal{E}_\mu$ (see [30]). Here

$$\begin{aligned} T^0_\mu &= \mathcal{A}, \\ T^1_\mu &= \mathcal{M}^{(1)}_\mu, \\ T^2_\mu &= \mathcal{M}^{(2)}_\mu \oplus \mathcal{M}^{(1)}_\mu \otimes \mathcal{M}^{(1)}_\mu, \\ T^r_\mu &= \bigoplus_{\substack{m\le r,\, a_i=1,2;\\ a_1+\cdots+a_m=r}} \mathcal{M}^{(a_1)}_\mu \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}^{(a_m)}_\mu \quad \forall\ r>2 \end{aligned} \tag{2.1.4}$$

the last consists all possible tensor products of the total grade equal to r. In the sequel, writing $\omega \otimes \theta$ for elements from $T_{\mathcal{A}}\mathcal{E}_\mu$, we mean tensor product over the algebra $\mathcal{A}$, that is $\omega \otimes_{\mathcal{A}} \theta \doteq \omega \otimes \theta$.
Let us define a commutation rule between the elements of the algebra $\mathcal{A}$ and the forms μ^j_2 and $\mu^k_1 \otimes \mu^l_1$ from the bimodule T^2_μ:

$$x^i\, \mu^j_2 = \mu^k_2 \xi^{ij}_k + \mu^k_1 \otimes \mu^l_1\, (QD_l(\xi^{ij}_k) - \xi(\sigma^i_k)^j_l). \tag{2.1.5}$$

We would like to emphasize now that the $\mathcal{A}$-bimodule T^2_μ inherits the free right structure from $\mathcal{M}^{(1)}_\mu$ and $\mathcal{M}^{(2)}_\mu$, and the introduced commutation rule (2.1.5) defines a connection between the left and right structures of T^2_μ.
Now the tensor algebra $T_{\mathcal{A}}\mathcal{E}_\mu$ satisfying the relations (2.1.5) is generated by the elements x^i, μ^j_1, μ^k_2 and the relations $f_s = 0$, (2.1.2), and (2.1.5). Due to the facts that the tensor product in (2.1.4) is taken over the algebra $\mathcal{A}$ and the relations (2.1.2), and that (2.1.5) holds in $T_{\mathcal{A}}\mathcal{E}_\mu$, any homogenous element of grade r from this algebra has the unique decomposition in the terms

$$\mu^{i_1}_{a_1} \otimes \cdots \otimes \mu^{i_l}_{a_l}\, v_{i_1\ldots i_l}, \tag{2.1.6}$$

where $l \le r, ;\ v_{i_1\ldots i_l} \in \mathcal{A}$ and the sum in (2.2.3) is over all possible choices of $a_1, \ldots a_l;\ a_k \in \{1,2\}$, $k = 1,\ \ldots,\ l$, such that $a_1 + \cdots + a_l = r$.

Let $d : T_{\mathcal{A}}\mathcal{E} \to T_{\mathcal{A}}\mathcal{E}$ be defined by

$$\begin{aligned} &d(\mu^{i_1}_{a_1} \otimes \cdots \otimes \mu^{i_l}_{a_l}\, v_{i_1\ldots i_l}) = \\ &\sum_{j=1}^{m} Q^{(a_1+\cdots+a_{j-1})} \mu^{i_1}_{a_1} \otimes \cdots \otimes \mu^{i_{j-1}}_{a_{j-1}} \otimes \mu^{i_j}_{a_j+1} \otimes \mu^{i_{j+1}}_{a_{j+1}} \cdots \otimes \mu^{i_l}_{a_l}\, v_{i_1\ldots i_l} + \\ &\qquad Q^{(a_1+\cdots+a_l)} \mu^{i_1}_{a_1} \otimes \cdots \otimes \mu^{i_l}_{a_l} \otimes \mu^s_1 D_s(v_{i_1\ldots i_l}), \end{aligned} \tag{2.1.7}$$

and the property $d^3x^i = 0$ be assumed. It is easy to see that d is a linear mapping of grading one.

But, as before in Section 1.2, the tensor algebra $T_{\mathcal{A}}\mathcal{E}$ equipped with the operator d defined above does not become a graded Q-differential algebra with differential $d^3 = 0$ by the same reasons: firstly, the simple calculation shows that

$$\begin{aligned}
d^3(u) &= d^2(\mu_1^j\, D_j(u)) = d(\mu_2^j\, D_j(u) + Q\mu_1^j \otimes \mu_1^k\, D_k(D_j(u))) \\
&= Q[2]_Q\, \mu_2^j \otimes \mu_1^k\, D_k(D_j(u)) + Q^2\mu_1^j \otimes \mu_2^k\, D_k(D_j(u)) \\
&+ \mu_1^j \otimes \mu_1^k \otimes \mu_1^l\, D_l(D_k(D_j(u)))
\end{aligned}$$

does not vanish in general and, secondly, the definition (2.1.7) agrees with the Leibniz rule (2.1.1) for the canonical elements only.

Our aim now is to construct a minimal graded ideal J_Q in $T_{\mathcal{A}}\mathcal{E}_\mu$ such that $dJ_Q \subset J_Q$, and that the corresponding quotient algebra

$$\Omega(\mathcal{A}, M_\mu) \doteq \frac{T_{\mathcal{A}}\mathcal{E}_\mu}{J_Q} = \bigoplus_{n\geq 0} \Omega^{(n)}(\mathcal{A}, M_\mu) \tag{2.1.8}$$

becomes an $\mathbb{N}$-graded Q-differential ternary algebra, i.e. the property $d^3 = 0$ as well as the Q-Leibniz rule (2.1.1) are recovered in $\Omega(\mathcal{A}, M_\mu)$. We call such ideal by d-compatible. Here grading is meant with respect to $T_{\mathcal{A}}\mathcal{E}$: $J_Q^r \subset T^r$. In construction of the ideal J_Q, we shall follow general technique developed in [12] as before. The basic relations are

$$x^i\mu_1^j - \mu_1^k\xi_k^{ij} = 0 \quad \text{and} \quad dx^i - \mu_1^k\sigma_k^i = 0. \tag{2.1.9}$$

Consecutive differentiation of LHS of the first and second relations in (2.1.9) by means of (2.1.1) leads us to the following homogeneous elements in $T_{\mathcal{A}}\mathcal{E}$

$$x^i\mu_2^j - \mu_2^k\xi_k^{ij} - \mu_1^k \otimes \mu_1^l\, (QD_l(\xi_k^{ij}) - \xi(\sigma_k^i)_l^j), \tag{2.1.10}$$

$$\begin{aligned}
(\mu_1^k \otimes \mu_2^l - Q\mu_2^k \otimes \mu_1^l)\, \big(D_l(\xi_k^{ij}) + \xi(\sigma_k^i)_l^j\big) - \\
\mu_1^k \otimes \mu_1^l \otimes \mu_1^s\, D_s\big(QD_l(\xi_k^{ij}) - \xi(\sigma_k^i)_l^j\big),
\end{aligned} \tag{2.1.11}$$

$$(\mu_1^k \otimes \mu_2^l - Q\mu_2^k \otimes \mu_1^l)\, D_l(\sigma_k^i) - Q\mu_1^k \otimes \mu_1^l \otimes \mu_1^s\, D_sD_l(\sigma_k^i), \tag{2.1.12}$$

which, in fact, are generating relations

$$x^i\mu_2^j - \mu_2^k\xi_k^{ij} = \mu_1^k \otimes \mu_1^l\, (QD_l(\xi_k^{ij}) - \xi(\sigma_k^i)_l^j), \tag{2.1.13}$$

$$(\mu_1^k \otimes \mu_2^l - Q\mu_2^k \otimes \mu_1^l)\big(D_l(\xi_k^{ij}) + \xi(\sigma_k^i)_l^j\big) = \\ \mu_1^k \otimes \mu_1^l \otimes \mu_1^s\, D_s\big(QD_l(\xi_k^{ij}) - \xi(\sigma_k^i)_l^j\big), \tag{2.1.14}$$

$$(\mu_1^k \otimes \mu_2^l - Q\mu_2^k \otimes \mu_1^l)\, D_l(\sigma_k^i) = Q\mu_1^k \otimes \mu_1^l \otimes \mu_1^s\, D_s D_l(\sigma_k^i) \tag{2.1.15}$$

in the graded differential algebra with $d^3 = 0$.
If we assume that the elements $\mu_2^k \otimes \mu_1^l$, $\mu_1^k \otimes \mu_2^l$, and $\mu_1^k \otimes \mu_1^l \otimes \mu_1^m$ in (2.1.14) and (2.1.15) are connected by the one type of relations, then, comparing the left hand sides of these relations, we obtain the following connections between the homomorphism ξ and partial derivatives D_l

$$D_l(\sigma_k^i) = D_l(\xi_k^{ij}) + \xi(\sigma_k^i)_l^j \quad \text{and} \quad D_l(\sigma_k^i) = Q^2\xi(\sigma_k^i)_l^j - D_l(\xi_k^{ij}) \tag{2.1.16}$$

in addition to the relations (1.1.7). Rewriting the last relations as

$$D_l(\sigma_k^i - \xi_k^{ij}) = \xi(\sigma_k^i)_l^j \quad \text{and} \quad Q\, D_l(\sigma_k^i + \xi_k^{ij}) = \xi(\sigma_k^i)_l^j, \tag{2.1.17}$$

we thus get the relations between homomorphism ξ and partial derivatives on the generators x^i:

$$\xi_k^{ij} = (Q - Q^2)\sigma_k^i, \quad j = 1, \ldots,\ n. \tag{2.1.18}$$

Moreover, taking into account (1.1.7), it is obvious that the partial derivatives D_l act on product of generators in $\mathcal{A}$ as follows:

$$D_l(x^i\, x^j) = \sigma_l^i(x^j + Q - Q^2). \tag{2.1.19}$$

Thus the generating relations (2.1.13), (2.1.14) and (2.1.15) can be rewritten as:

$$\begin{aligned} x^i\mu_2^j &= (Q^2 - Q)\mu_2^k\sigma_k^i 3Q^2\mu_1^k \otimes \mu_1^l D_l(\sigma_k^i), \\ \mu_1^k \otimes \mu_2^l\, D_l(\sigma_k^i) &= Q\mu_2^k \otimes \mu_1^l\, D_l(\sigma_k^i) + 3/2(Q^2-1)^{-1}\mu_1^k \otimes \mu_1^l \otimes \mu_1^s\, D_s\, D_l(\sigma_k^i), \\ \mu_1^k \otimes \mu_2^l\, D_l(\sigma_k^i) &= Q\mu_2^k \otimes \mu_1^l)\, D_l(\sigma_k^i) - Q\mu_1^k \otimes \mu_1^l \otimes \mu_1^s\, D_s D_l(\sigma_k^i) \end{aligned} \tag{2.1.20}$$

The easy control shows that, as expected in [31], $d^2 : \mathcal{A} \to \Omega^{(2)}(\mathcal{A}, \mathcal{M})$ satisfies the Q-Leibniz rule

$$d^2(uv) = d^2(u)v + [2]_Q\ du\ dv + ud^2v, \quad \forall\ u,\ v \in \mathcal{A},$$

on the bimodule $\Omega^{(2)}(\mathcal{A}, M_\mu) = (\mathcal{M}_\mu^{(1)} \otimes \mathcal{M}_\mu^{(1)} \oplus \mathcal{M}_\mu^{(2)})/J_Q^{(2)}$, where $J_Q^{(2)}$ denotes a sub-bimodule in T_Q^2 generated by relations (2.1.9) and (2.1.10).
We can now state the following

Theorem 2.1.3. *If*

$$\Omega(\mathcal{A}, M_\mu) = \frac{T_{\mathcal{A}}\mathcal{E}_\mu}{J_Q},$$

where $T_{\mathcal{A}}\mathcal{E}_\mu$ is the tensor algebra and the homogeneous ideal $J_Q \subset T_{\mathcal{A}}\mathcal{E}_\mu$ is generated by the elements (2.1.10), (2.1.12) and

$$x^i \mu_1^j - \mu_1^k \xi_k^{ij},\ dx^i - \mu_1^k \sigma_k^i,$$

then the operator $d : \Omega(\mathcal{A}, M_\mu) \to \Omega(\mathcal{A}, M_\mu)$ defined by (2.1.7) is the graded differential satisfying $d^3 = 0$, and the pair $\{\Omega(\mathcal{A}, M_\mu), d\}$ is the graded Q-differential algebra.

Here we only give the main ideas of the proof, because the method of proof coincides with the one that we use for the proof of Theorem 2.2.1 in Section2.2. Let us only remark here that, in order to prove Theorem 2.1.3, we need to show by induction on degree of elements in $T_{\mathcal{A}}\mathcal{E}_\mu$ that the following properties:

1. $d(J_Q) \subset J_Q$,
2. the Q-Leibniz rule

   $$d(\omega\theta) = (d\omega)\theta + Q^n \omega(d\theta) \quad (mod\ J_Q)$$

   is satisfied $\forall\ \omega \in T_\mu^n$ and $\theta \in T_{\mathcal{A}}\mathcal{E}_\mu$,
3. $d^3(T_{\mathcal{A}}\mathcal{E}_\mu)) \subseteq \mathcal{J}_{\mathcal{Q}}$

hold in the tensor algebra $T_{\mathcal{A}}\mathcal{E}_\mu$.

2.2 Coordinate case of graded Q-differential algebra with differential $d^3 = 0$

In this section we will construct a graded differential algebra with differential $d^3 = 0$ in the case of coordinate FODC $\{\mathcal{A},\ \mathcal{M}^{(1)},\ d^1\}$ with right partial derivatives described in detail in Section 1.1. But before such specification we impose the restrictive condition on the bimodule $T^2 = \mathcal{M}^{(2)} \bigoplus \mathcal{M}^{(1)} \bigotimes \mathcal{M}^{(1)}$, namely, we define the second order differential (SOD) as it follows. Here $\mathcal{M}_\mu^{(1)}$ and $\mathcal{M}^{(2)}$ are a free right modules over $\mathcal{A}$ spanned by the elements

$\{dx^1,\ dx^2, \ldots,\ dx^n\}$ and $\{d^2x^1,\ d^2x^2, \ldots,\ d^2x^n\}$ respectively. Let us assume that the structure of bimodule on it is set by means of the same homomorphism $\xi = (\xi^j_k) : \mathcal{A} \to \mathrm{M}at_n\mathcal{A}$ like in the case of bimodule $\mathcal{M}$, i.e.

$$u\, d^2x^j = d^2x^k \xi(u)^j_k \qquad \forall u \in \mathcal{A},\ j,\ k = 1, \ldots,\ n.$$

Define a linear mapping $\tilde{d}^2 : \mathcal{A} \to \mathcal{M}^{(2)}$, such that

$$\tilde{d}^2(x^i) = d^2x^i, \quad \tilde{d}^2(u) = d^2x^i D_i(u),$$

where D_i are the same right partial derivatives, which was given and described in the case of FODC above in Section 1.1. As a consequence, the Leibniz rule

$$\tilde{d}^2(uv) = \tilde{d}^2(u)v + u\tilde{d}^2(v) \qquad \forall\ u,\ v \in \mathcal{A},$$

is automatically satisfied. Let us remark that, because of the obvious correspondence $dx^i \leftrightarrows d^2x^i$, the bimodules $\mathcal{M}^{(1)}_\mu$ and $\mathcal{M}^{(2)}$ are isomorphic. We also assume that the elements d^2x^i have the grade 2, and refer to elements of $\mathcal{M}^{(2)}$ as 2-forms.

We call the triple $\{\mathcal{A},\ \mathcal{M}^{(2)},\ \tilde{d}^2\}$ the canonical second order differential (SOD) associated to a given FODC $\{\mathcal{A},\ \mathcal{M},\ d^1\}$ (see Section 2.1 for non-canonical SOD).

Let us construct the tensor algebra of $\mathcal{E}$ over the ring $\mathcal{A}$

$$T_\mathcal{A}\mathcal{E} = \mathcal{A} \oplus \mathcal{E} \oplus \mathcal{E}^{\otimes 2} \oplus \mathcal{E}^{\otimes 3} \oplus \cdots = \bigoplus_{r \geq 0} \mathcal{E}^{\otimes r},$$

where $\mathcal{E} = \mathcal{M}^{(1)} \oplus \mathcal{M}^{(2)}$ is a graded $\mathcal{A}$-bimodule ($\mathcal{M}^{(1)} \equiv \mathcal{M}$), $\mathcal{E}^{\otimes r}$ means $\mathcal{E} \otimes_\mathcal{A} \cdots \otimes_\mathcal{A} \mathcal{E}$ (r times), $\mathcal{E}^0 = \mathcal{A}$. Expanding and rearranging terms in $T_\mathcal{A}\mathcal{E}$ we obtain an $\mathbb{N}$-graded algebra

$$T_\mathcal{A}\mathcal{E} = \bigoplus_{r \geq 0} T^r, \tag{2.2.1}$$

with a unique graduation compatible with graduation of $\mathcal{E}$ (see [30]). Remark that $T^0 = \mathcal{A},\ T^1 = \mathcal{M} \equiv \mathcal{M}^{(1)},\ T^2 = \mathcal{M}^{(2)} \oplus \mathcal{M}^{(1)} \otimes_\mathcal{A} \mathcal{M}^{(1)}$, and for $r > 2$

$$T^r = \bigoplus_{\substack{m \leq r,\, a_i = 1,2; \\ a_1 + \cdots + a_m = r}} \mathcal{M}^{(a_1)} \otimes_\mathcal{A} \cdots \otimes_\mathcal{A} \mathcal{M}^{(a_m)} \tag{2.2.2}$$

consists all possible tensor products of the total grade equal to r.

Any homogenous element of grade n has unique decomposition in the form

$$d^{a_1}x^{i_1} \otimes \cdots \otimes d^{a_m}x^{i_m} r_{i_1 \ldots i_m} \tag{2.2.3}$$

where $m \leq r, ;\ r_{i_1 \ldots i_m} \in \mathcal{A}$ and the sum in (2.2.3) is over all possible choices of $a_1, \ldots a_m$, where $a \in \{1,\, 2\}$ such that $a_1 + \cdots + a_m = r$ since the tensor product in (2.2.2) is taken over the algebra $\mathcal{A}$.

Let us define a linear mapping of grading one $d : T_{\mathcal{A}}\mathcal{E} \to T_{\mathcal{A}}\mathcal{E}$ by the formula

$$\begin{aligned} &d(d^{a_1}x^{i_1} \otimes \cdots \otimes d^{a_m}x^{i_m} r_{i_1 \ldots i_m}) = \sum_{j=1}^{m} Q^{(a_1+\cdots+a_{j-1})} d^{a_1}x^{i_1} \otimes \ldots \\ &\otimes d^{a_{j-1}}x^{i_{j-1}} \otimes d^{a_j+1}x^{i_j} \otimes d^{a_{j+1}}x^{i_{j+1}} \cdots \otimes d^{a_m}x^{i_m} r_{i_1 \ldots i_m} + \\ &Q^{(a_1+\cdots+a_m)} d^{a_1}x^{i_1} \otimes \cdots \otimes d^{a_m}x^{i_m} \otimes dx^s D_s(r_{i_1 \ldots i_m}) \end{aligned} \tag{2.2.4}$$

assuming that the condition $d^3x^i = 0$ is fulfilled. One already sees that $d(x^i) = dx^i$, $d^2(x^i) = d(dx^i) = d^2x^i$ and $d^3x^i = d^2(dx^i) = d(d^2x^i) = 0$, which means that d is prolongation of d^1 and $\tilde{d}^2$. Nevertheless, $\tilde{d}^2$ and d^2 are not identical, since

$$d^2u = d(dx^i D_i(u)) = \tilde{d}^2u + Q dx^i \otimes dx^j D_j(D_i(u))\,.$$

Similarly, simple calculation shows that

$$\begin{aligned} d^3u &= d(\tilde{d}^2u + Q dx^i \otimes dx^j D_j(D_i(u))) = Q[2]_Q\, d^2x^i \otimes dx^j D_j(D_i(u)) \\ &\quad + Q^2 dx^i \otimes d^2x^j D_j(D_i(u)) + dx^i \otimes dx^j \otimes dx^k D_k(D_j(D_i(u))) \end{aligned}$$

does not vanish in general. It should be also noticed that the definition (2.2.4) agrees with the Leibniz rule (2.1.1). However, for elements, which are not written in a canonical form (2.2.3), the Leibniz rule does not hold in general. Therefore the tensor algebra $T_{\mathcal{A}}\mathcal{E}$ equipped with the operator d, as defined above, is not a graded differential algebra with differential $d^3 = 0$.

We construct now a d-compatible ideal $I_Q = \bigoplus_{r \geq 0} I_Q^{(r)}$ in $T_{\mathcal{A}}\mathcal{E}$. To this end, we shall follow general technique developed in [12] as before in Section 2.1. The basic relations are

$$x^i dx^j - dx^k \xi_k^{ij} = 0 \quad \text{and} \quad x^i d^2x^j - d^2x^k \xi_k^{ij} = 0, \tag{2.2.5}$$

where $\xi_k^{ij} \doteq (\xi(x^i))_k^j \in \mathcal{A}$. By the consecutive differentiation of RHS (2.2.5) by means of (2.1.1), we obtain the following set of homogeneous elements in $T_{\mathcal{A}}\mathcal{E}$

$$dx^i \otimes dx^j - Qdx^k \otimes d\xi_k^{ij}, \tag{2.2.6}$$
$$dx^i \otimes d^2x^j - Q^2d^2x^k \otimes d\xi_k^{ij}, \tag{2.2.7}$$
$$d^2x^i \otimes dx^j + (1-Q)d^2x^k \otimes d\xi_k^{ij} - Q^2dx^k \otimes d^2\xi_k^{ij}, \tag{2.2.8}$$
$$d^3\xi_k^{ij}, \tag{2.2.9}$$
$$d^2x^i \otimes d^2x^j - Qd^2x^k \otimes d^2\xi_k^{ij}. \tag{2.2.10}$$

which, in fact, are generating relations for I_Q. Here

$$\begin{aligned}
d\xi_k^{ij} &= dx^l D_l(\xi_k^{ij});\\
d^2\xi_k^{ij} &= d^2x^l D_l(\xi_k^{ij}) + Qdx^l \otimes dx^m D_m(D_l(\xi_k^{ij}));\\
d^3\xi_k^{ij} &= Q[2]_Q\, d^2x^l \otimes dx^m D_m(D_l(\xi_k^{ij})) + Q^2dx^l \otimes d^2x^m D_m(D_l(\xi_k^{ij})) +\\
&\qquad\qquad dx^l \otimes dx^m \otimes dx^p D_p(D_m(D_l(\xi_k^{ij}))).
\end{aligned}$$

It means, that the graded differential algebra with $d^3 = 0$ have to be equipped by the following relations:

$$dx^i \otimes dx^j = Qdx^k \otimes d\xi_k^{ij}, \tag{2.2.11}$$
$$dx^i \otimes d^2x^j = Q^2d^2x^k \otimes d\xi_k^{ij}, \tag{2.2.12}$$
$$d^2x^i \otimes dx^j + (1-Q)d^2x^k \otimes d\xi_k^{ij} = Q^2dx^k \otimes d^2\xi_k^{ij}, \tag{2.2.13}$$
$$d^3\xi_k^{ij} = 0, \tag{2.2.14}$$
$$d^2x^i \otimes d^2x^j = Qd^2x^k \otimes d^2\xi_k^{ij}. \tag{2.2.15}$$

For example, the bimodule $\Omega^{(2)}(\mathcal{A}, \mathcal{M})$ has the structure

$$\Omega^{(2)}(\mathcal{A}, \mathcal{M}) = \frac{\mathcal{M} \otimes \mathcal{M} \oplus \mathcal{M}^{(2)}}{I_Q^{(2)}}$$

where $I_Q^{(2)}$ denotes a sub-bimodule in T^2 generated by relations (2.2.6). Now $d^2 : \mathcal{A} \to \Omega^{(2)}(\mathcal{A}, \mathcal{M})$ satisfies the Q-Leibniz rule

$$d^2(uv) = d^2(u)v + [2]_Q\, du\, dv + ud^2v, \quad \forall\ u,\ v \in \mathcal{A},$$

as expected [31].

Let us compare the relations (2.1.13)-(2.1.15) from Section 2.1 with the relations (2.2.11)-(2.2.15). It is evident that now we do not need the additional condition on the homomorphism ξ and partial derivatives D_k.

The main result of our study of the graded differential algebra with $d^3 = 0$ and SOD starting from the coordinate FODC is established by the following

Theorem 2.2.1. *Let $\Omega(\mathcal{A}, \mathcal{M})$ be the quotient algebra*

$$\Omega(\mathcal{A}, \mathcal{M}) \doteq \frac{T_{\mathcal{A}}\mathcal{E}}{I_Q} = \bigoplus_{n \geq 0} \frac{T^n}{I_Q^n} \doteq \bigoplus_{n \geq 0} \Omega^{(n)}(\mathcal{A}, \mathcal{M}), \tag{2.2.16}$$

where the homogeneous ideal $I_Q \subset T_{\mathcal{A}}\mathcal{E}$ is generated by the set of elements

$$x^i \mu_1^j - \mu_1^k \xi_k^{ij} \quad and \quad dx^i - \mu_1^k \sigma_k^i,$$

(2.2.6)-(2.2.10), and the operator $d : \Omega(\mathcal{A}, \mathcal{M}) \to \Omega(\mathcal{A}, \mathcal{M})$ is defined by the formula (2.2.4). Then the pair $\{\Omega(\mathcal{A}, \mathcal{M}), d\}$ is an $\mathbb{N}$-graded differential algebra with differential $d^3 = 0$, that is the properties $d^3 = 0$ and Q-Leibniz rule

$$d(\omega\theta) = (d\omega)\,\theta + Q^n \omega\,(d\theta), \;\; where\ \omega \in \Omega^n(\mathcal{A},\, \mathcal{M}), \;\; \theta \in \Omega(\mathcal{A}, \mathcal{M}),$$

are satisfied.

In order to prove this theorem, we need in additional results, which we formulate in the following

Proposition 2.2.2. *Let the ideal $I_Q \subset T_{\mathcal{A}}\mathcal{E}$ be generated by the elements $x^i dx^j - dx^k \xi_k^{ij}$, $x^i d^2x^j - d^2x^k \xi_k^{ij}$, and (2.2.6)-(2.2.10). Then, in the quotient algebra $\Omega(\mathcal{A}, \mathcal{M})$ (2.2.16) equipped with the operator d (2.2.4), the following relations*

$$dv \otimes dx^j = Q dx^k \otimes d\xi(v)_k^j, \tag{2.2.17}$$
$$dv \otimes d^2x^j = Q^2 d^2x^k \otimes d\xi(v)_k^j, \tag{2.2.18}$$
$$d^2v \otimes dx^j = (Q-1) d^2x^k \otimes d\xi(v)_k^j + Q^2 dx^k \otimes d^2\xi(v)_k^j, \tag{2.2.19}$$
$$d^3\xi(v)_k^j = 0, \tag{2.2.20}$$
$$d^2v \otimes d^2x^j = Q d^2x^k \otimes d^2\xi(v)_k^j \tag{2.2.21}$$

are true for all $v \in \mathcal{A}$.

Proof. In case of $v = x^i,\ i = 1, \ldots,\ n$, we have relations directly obtained from the generators of ideal I_Q (2.2.6) -(2.2.10). This enables us to do the proof of proposition by induction. Assuming (2.2.17)-(2.2.21) hold for $v_i \in \mathcal{A}$, we will prove their for $v = x^i\, v_i$. Combining the basic relations (2.2.5), our assumptions, and the definition of the operator d, we obtain the following equalities

$$\begin{aligned} dv \otimes dx^j &= d(x^i\, v_i) \otimes dx^j = dx^i\, v_i \otimes dx^j + x^i\, dv_i \otimes dx^j \\ &= dx^i \otimes dx^k\, \xi(v_i)^j_k + Q\, x^i\, dx^k \otimes d\xi(v_i)^j_k \\ &= Q\, dx^l \otimes d\xi^{ik}_l\, \xi(v_i)^j_k + Q\, dx^l \xi^{ik}_l \otimes d\xi(v_i)^j_k \\ &= Q\, dx^l \otimes d(\xi^{ik}_l \xi(v_i)^j_k) = Q\, dx^l \otimes d\xi(x^i\, v_i)^j_l = Q dx^l \otimes d\xi(v)^j_l, \end{aligned}$$

which prove the inclusion (2.2.17);

$$\begin{aligned} dv \otimes d^2x^j &= d(x^i\, v_i)\, d^2x^j = dx^i\, v_i \otimes d^2x^j + x^i\, dv_i \otimes d^2x^j \\ &= dx^i \otimes d^2x^k\, \xi(v_i)^j_k + Q^2\, x^i\, d^2x^k \otimes d\xi(v_i)^j_k \\ &= Q^2\, d^2x^l \otimes d\xi^{ik}_l\, \xi(v_i)^j_k + Q^2\, d^2x^l \otimes \xi^{ik}_l\, d\xi(v_i)^j_k \\ &= Q\, dx^l \otimes d(\xi^{ik}_l \xi(v_i)^j_k) = Q^2\, d^2x^l \otimes d\xi(x^i\, v_i)^j_l = Q^2 d^2x^l \otimes d\xi(v)^j_l, \end{aligned}$$

which prove the inclusion (2.2.18);

$$\begin{aligned} d^2(v) \otimes dx^j &= d^2(x^i\, v_i) \otimes dx^j = d^2x^i\, v_i \otimes dx^j + [2]_Q\, dx^i \otimes dv_i \otimes dx^j + \\ &\qquad\qquad\qquad\qquad\qquad x^i\, d^2v_i \otimes dx^j \\ &= d^2x^i \otimes dx^k\, \xi(v_i)^j_k + Q\, [2]_Q\, dx^i \otimes dx^k \otimes d\xi(v_i)^j_k + \\ &\qquad (Q-1)\, x^i d^2x^k \otimes d\xi(v_i)^j_k + Q^2\, x^i\, dx^k \otimes d^2\xi(v_i)^j_k \\ &= (Q-1)\, d^2x^l \otimes d\xi^{ik}_l\, \xi(v_i)^j_k + Q^2\, dx^l \otimes d^2\xi^{ik}_l\, \xi(v_i)^j_k + \\ &\quad Q^2 [2]_Q\, dx^l \otimes d\xi^{ik}_l \otimes d\xi(v_i)^j_k + (Q-1)\, d^2x^l \otimes \xi^{ik}_l\, d\xi(v_i)^j_k + \\ &\qquad\qquad\qquad\qquad Q^2\, dx^l \otimes \xi^{ik}_l d^2\xi(v_i)^j_k \\ &= (Q-1)\, d^2x^l \otimes \big(d\xi^{ik}_l\, \xi(v_i)^j_k + \xi^{ik}_l\, d\xi(v_i)^j_k\big) + \\ &\quad Q^2\, dx^l \otimes \big(d^2\xi^{ik}_l \otimes \xi(v_i)^j_k + [2]_Q\, d\xi^{ik}_l \otimes d\xi(v_i)^j_k + \xi^{ik}_l d^2\xi(v_i)^j_k\big) \\ &= (Q-1)\, d^2x^l \otimes d(\xi^{ik}_l\, \xi(v_i)^j_k) + Q^2\, dx^l \otimes d^2(\xi^{ik}_l\, \xi(v_i)^j_k) \\ &= (Q-1)\, d^2x^l \otimes d\xi(x^i\, v_i)^j_l + Q^2\, dx^l \otimes d^2\xi(x^i\, v_i)^j_l \\ &= (Q-1)\, d^2x^l \otimes d\xi(v)^j_l + Q^2\, dx^l \otimes d^2\xi(v)^j_l, \end{aligned}$$

which prove the inclusion (2.2.19);

$$d^3\big(\xi(v)^j_k\big) \;=\; d^3\big(\xi(x^i\, v_i)^j_k\big) = d^3\big(\xi^{ij}_l \xi(v_i)^l_k\big) = d^3\xi^{ij}_l\, \xi(v_i)^l_k + [3]_Q\, d(xi^{ij}_l) \otimes$$

$$d^2(\xi(v_i)^l_k) + [3]_Q\, d(\xi^{ij}_l) \otimes d^2(\xi(v_i)^l_k) + \xi^{ij}_l\, d^3(\xi(v_i)^l_k) \equiv 0,$$

which prove the relation (2.2.20);

$$\begin{aligned}
d^2(v) \otimes d^2x^j &= d^2(x^i\, v_i) \otimes d^2x^j = d^2x^i\, v_i \otimes d^2x^j + [2]_Q\, dx^i \otimes dv_i \otimes d^2x^j + \\
&\qquad\qquad x^i\, d^2v_i \otimes d^2x^j \\
&= d^2x^i \otimes d^2x^k\, \xi(v_i)^j_k + Q^2[2]_Q\, dx^i \otimes d^2x^k \otimes d\xi(v_i)^j_k + \\
&\qquad\qquad Q\, x^i\, d^2x^k \otimes d\xi(v_i)^j_k \\
&= Q\, d^2x^l \otimes d^2\xi^{ik}_l\, \xi(v_i)^j_k + Q\,[2]_Q\, d^2x^l \otimes d\xi^{ik}_l \otimes d\xi(v_i)^j_k + \\
&\qquad\qquad Q\, d^2x^l \otimes \xi^{ik}_l\, d^2\xi(v_i)^j_k \\
&= Q\, d^2x^l \otimes \big(d^2\xi^{ik}_l\, \xi(v_i)^j_k + [2]_Q\, d\xi^{ik}_l \otimes d\xi(v_i)^j_k + \xi^{ik}_l\, d^2\xi(v_i)^j_k\big) \\
&= Q\, d^2x^l \otimes d(\xi^{ik}_l \xi(v_i)^j_k) = Q\, d^2x^l \otimes d^2\xi(x^i\, v_i)^j_l \\
&= Q\, d^2x^l \otimes d^2\xi(v)^j_l
\end{aligned}$$

which prove the inclusion (2.2.21) and completes the proof of proposition. □

We begin the proof of Theorem 2.2.1.

Proof. In order to prove the theorem, we need to show the following properties of ideal I_Q in $T_{\mathcal{A}}\mathcal{E} = \bigoplus_{n\geq 0} T^n$:

1. $d(I_Q) \subset I_Q$ (i.e. the ideal I_Q is d-compatible);
2. the Q-Leibniz rule
$$d(\omega\theta) = (d\omega)\theta + Q^n\omega(d\theta) \quad (\mod I_Q) \qquad (2.2.22)$$
is satisfied $\forall\ \omega \in T^n$ and $\theta \in T_{\mathcal{A}}\mathcal{E}$;
3. $d^3(T_{\mathcal{A}}\mathcal{E})) \subseteq \mathcal{I}_Q$.

We begin by proving the property (2), which we do by induction on degree of the form $\theta \in T_{\mathcal{A}}\mathcal{E}$. We first show that the Q-Leibniz rule (2.2.22) holds for the elements: $v \in \mathcal{A}$, $\theta_1 = v_k\, dx^k \in \mathcal{M} \equiv T^1$, and $\theta_2 = v_k\, d^2x^k \in \mathcal{M}^{(2)}$, where $v_k \in \mathcal{A}$.

Let $\omega = d^{a_1}x^{i_1}\dots d^{a_m}x^{i_m}\, r_{i_1\dots i_m} \in T^n$, i.e. $a_1+\dots+a_m = n$. Using the definition of operator d (2.2.4) and the relations from proposition 2.2.2, we get

$$\begin{aligned}
d(\omega\, v) &= d(d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m}\, v) = \sum_{j=1}^{m} Q^{a_1+\dots+a_j} d^{a_1}x^{i_1}\otimes\dots\\
&\quad \otimes d^{a_{j-1}}x^{i_{j-1}}\otimes d^{a_j+1}x^{i_j}\otimes d^{a_{j+1}}x^{i_{j+1}}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m}\, v +\\
&\quad Q^{a_1+\dots+a_m}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes d(r_{i_1\dots i_m})\, v +\\
&\quad Q^{a_1+\dots+a_m}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m}\otimes d(v)\\
&= d(\omega)\, v + Q^n\,\omega\otimes dv,
\end{aligned}$$

$$\begin{aligned}
d(\omega\otimes\theta_1) &= d(d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m}\otimes v_k dx^k)\\
&= d\big(d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes dx^l \xi(r_{i_1\dots i_m})^k_s \xi(v_k)^s_l\big)\\
&= \sum_{j=1}^{m} Q^{a_1+\dots+a_j} d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_{j-1}}x^{i_{j-1}}\otimes d^{a_j+1}x^{i_j}\otimes d^{a_{j+1}}x^{i_{j+1}}\otimes\\
&\qquad\qquad \dots\otimes d^{a_m}x^{i_m}\otimes dx^l\xi(r_{i_1\dots i_m}\, v_k)^k_l +\\
&\quad Q^{a_1+\dots+a_m}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes d^2x^l\,\xi(r_{i_1\dots i_m}\, v_k)^k_l +\\
&\quad Q^{a_1+\dots+a_m+1}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes dx^l\otimes d(\xi(r_{i_1\dots i_m})^k_s)\xi(v_k)^s_l +\\
&\quad Q^{a_1+\dots+a_m+1}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes dx^l\otimes \xi(r_{i_1\dots i_m})^k_s d(\xi(v_k)^s_l)) +\\
&= \big(\sum_{j=1}^{m} Q^{a_1+\dots+a_j} d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_{j-1}}x^{i_{j-1}}\otimes d^{a_j+1}x^{i_j}\otimes\\
&\qquad\qquad d^{a_{j+1}}x^{i_{j+1}}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m} +\\
&\quad Q^{a_1+\dots+a_m}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes d(r_{i_1\dots i_m})\big)\otimes v_k dx^k +\\
&\quad Q^{a_1+\dots+a_m}\, d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m}\otimes\big(dv_k\otimes dx^k + v_k d^2x^k\big)\\
&= d(\omega)\otimes\theta_1 + Q^n\,\omega\otimes d(\theta_1);\\
d(\omega\otimes\theta_2) &= d(d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\, r_{i_1\dots i_m}\otimes v_k\, d^2x^k)
\end{aligned}$$

$$\begin{aligned}
&= d\big(d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_m}x^{i_m}\otimes d^2x^l\xi(r_{i_1\dots i_m})^k_s\xi(v_k)^s_l\big)\\
&= \sum_{j=1}^{m} Q^{a_1+\dots+a_j} d^{a_1}x^{i_1}\otimes\dots\otimes d^{a_{j-1}}x^{i_{j-1}}\otimes d^{a_j+1}x^{i_j}\otimes d^{a_{j+1}}x^{i_{j+1}}\otimes
\end{aligned}$$

$$
\begin{aligned}
& \qquad\qquad\qquad\qquad \cdots \otimes d^{a_m} x^{i_m} \otimes d^2 x^l \xi(r_{i_1 \ldots i_m}\, v_k)^k_l + \\
& Q^{a_1+\cdots+a_m+2}\, d^{a_1} x^{i_1} \otimes \cdots \otimes d^{a_m} x^{i_m} \otimes d^2 x^l \otimes d(\xi(r_{i_1\ldots i_m})^k_s) \xi(v_k)^s_l + \\
& Q^{a_1+\cdots+a_m+2}\, d^{a_1} x^{i_1} \otimes \cdots \otimes d^{a_m} x^{i_m} \otimes d^2 x^l \otimes \xi(r_{i_1\ldots i_m})^k_s d(\xi(v_k)^s_l) \\
= \; & \Big(\sum_{j=1}^{m} Q^{a_1+\cdots+a_j} d^{a_1} x^{i_1} \otimes \cdots \otimes d^{a_{j-1}} x^{i_{j-1}} \otimes d^{a_j+1} x^{i_j} \otimes d^{a_{j+1}} x^{i_{j+1}} \otimes \\
& \qquad\qquad\qquad\qquad\qquad \cdots \otimes d^{a_m} x^{i_m}\, r_{i_1\ldots i_m} + \\
& Q^{a_1+\cdots+a_m} \otimes d^{a_1} x^{i_1} \otimes \cdots \otimes d^{a_m} x^{i_m} r_{i_1\ldots i_m} \Big) \otimes v_k d^2 x^k + \\
& Q^{a_1+\cdots+a_m}\, d^{a_1} x^{i_1} \otimes \cdots \otimes d^{a_m} x^{i_m}\, r_{i_1\ldots i_m} \otimes dv_k \otimes d^2 x^k
\end{aligned}
$$

$$
= \; d(\omega) \otimes \theta_2 + Q^n\, \omega \otimes d(\theta_2).
$$

Assume the Q-Leibniz rule (2.2.22) to hold for the form $\rho \in T^l$, we will prove it for the forms $\theta = \rho \otimes dx^k\, r_k \in T^{l+1}$, $r_k \in \mathcal{A}$. We have

$$
\begin{aligned}
d(\omega \otimes \theta) &= d(\omega \otimes \rho \otimes dx^k\, r_k) \\
&= d(\omega \otimes \rho) \otimes dx^k\, r_k + Q^{n+l} \omega \otimes \rho \otimes d(dx^k\, r_k) \\
&= d(\omega) \otimes \rho \otimes dx^k\, r_k + Q^n \omega \otimes d(\rho) \otimes dx^k\, r_k + \\
&\qquad\qquad + Q^{n+l} \omega \otimes \rho \otimes d(dx^k\, r_k) \\
&= d(\omega) \otimes (\rho \otimes dx^k\, r_k) + Q^n \omega \otimes d(\rho \otimes dx^k\, r_k) \\
&= d(\omega) \otimes \theta + Q^n \omega \otimes d(\theta).
\end{aligned}
$$

Analogously for $\theta = \rho \otimes d^2 x^k\, r_k \in T^{l+2}$, we obtain

$$
\begin{aligned}
d(\omega \otimes \theta) &= d(\omega \otimes \rho \otimes d^2 x^k\, r_k) \\
&= d(\omega \otimes \rho) \otimes d^2 x^k\, r_k + Q^{n+l} \omega \otimes \rho \otimes d(d^2 x^k\, r_k) \\
&= d(\omega) \otimes \rho \otimes d^2 x^k\, r_k + Q^n \omega \otimes d(\rho) \otimes d^2 x^k\, r_k + \\
&\qquad\qquad + Q^{n+l} \omega \otimes \rho \otimes d(d^2 x^k\, r_k) \\
&= d(\omega) \otimes (\rho \otimes d^2 x^k\, r_k) + Q^n \omega \otimes d(\rho \otimes d^2 x^k\, r_k) \\
&= d(\omega) \otimes \theta + Q^n\, \omega \otimes d(\theta).
\end{aligned}
$$

The property (2) is proved.

Further, in order to prove the properties (1) and (3), we need in the following

Lemma 2.2.3. $d^3(\mathcal{A}) \subseteq I_Q$.

Proof. Since $d^3x^i = 0$, we can do the prove of the lemma by induction. Let $v = x^i\, v_i$, where $d^3(v_i) \in I_Q$ is assumed. Since the assertion (2) is true, the straightforward calculation shows that

$$\begin{aligned} d^3(v) &= d^3x^i\, v_i + [3]_Q\, d^2(x^i) \otimes dv_i + [3]_Q\, d(x^i) \otimes d^2v_i + x^i\, d^3v_i \\ &= x^i\, d^3v_i \;\in x^i I_Q \subseteq I_Q, \end{aligned}$$

which is the required inclusion. □

Next we prove the property (1).

Since the generators of I_Q (2.2.6)-(2.2.10) were obtained from the basic relations (2.2.5) by the consecutive differentiation, we need to check only that the differentials of two last generators (2.2.9) and (2.2.10) belong to the ideal I_Q. By direct calculation, we get

$$\begin{aligned} d(d^3\xi^{ij}_k) &= d\big(Q[2]_Q\, d^2x^l \otimes dx^m D_m D_l(\xi^{ij}_k) + \\ &\quad Q^2 dx^l \otimes d^2x^m D_m D_l(\xi^{ij}_k) + dx^l \otimes dx^m \otimes dx^p D_p\, D_m\, D_l(\xi^{ij}_k)\big) \\ &= [2]_Q\, d^2x^l \otimes d^2x^m D_m D_l(\xi^{ij}_k) + \\ &\quad Q\,[2]_Q\, d^2x^l \otimes dx^m \otimes dx^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad Q^2 d^2x^l \otimes d^2x^m D_m D_l(\xi^{ij}_k) + \\ &\quad Q^2 dx^l \otimes d^2x^m \otimes dx^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad d^2x^l \otimes dx^m \otimes dx^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad Q dx^l \otimes d^2x^m \otimes dx^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad Q^2 dx^l \otimes dx^m \otimes d^2x^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad dx^l \otimes dx^m \otimes dx^p D_p \otimes dx^s D_s\, D_m\, D_l(\xi^{ij}_k) \\ &= [3]_Q\, d^2x^l \otimes d^2x^m D_m D_l(\xi^{ij}_k) + \\ &\quad [3]_Q\, d^2x^l \otimes dx^m \otimes dx^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad Q[2]_Q\, dx^l \otimes d^2x^m \otimes dx^p D_p D_m D_l(\xi^{ij}_k) + \\ &\quad Q^2\, dx^l \otimes dx^m \otimes d^2x^p D_p\, D_m\, D_l(\xi^{ij}_k) + \\ &\quad dx^l \otimes dx^m \otimes dx^p \otimes dx^s D_s D_p\, D_m\, D_l(\xi^{ij}_k) \\ &= dx^l \otimes d^3(D_l(\xi^{ij}_k)). \end{aligned}$$

Recall that $D_l(\xi^{ij}_k) \in \mathcal{A}$. Lemma 2.2.3 implies the inclusion $d^3(D_l(\xi^{ij}_k)) \in I_Q$, from which it follows that $dx^l \otimes d^3(D_l(\xi^{ij}_k)) \in dx^l \otimes I_Q \subseteq I_Q$.

Analogously,

$$\begin{aligned} d(d^2x^i \otimes d^2x^j - Qd^2x^k \otimes d^2\xi_k^{ij}) = d^3x^i \otimes d^2x^j + Q^2d^2x^i \otimes d^3x^j - \\ Qd^3x^k \otimes d^2\xi_k^{ij} - d^2x^k \otimes d^3\xi_k^{ij} \in d^2x^k \otimes I_Q \subseteq I_Q, \end{aligned}$$

and hence the property (1) is proved.

In order to prove the property (3), it is sufficient to show that $d^3(T^n) \subseteq I_Q$. Since the case $n = 0$, i.e. $T^0 = \mathcal{A}$, is already proved (see the lemma 2.2.3), this enables us to prove the required inclusion by induction on degree of the forms from $T_{\mathcal{A}}\mathcal{E}$.

Let $\omega_1 = dx^k \otimes \theta_k$, where $\theta_k \in T^{n-1}$, and $\omega_2 = d^2x^k \otimes \rho_k$, where $\rho_k \in T^{n-2}$, and the property (3) be satisfied for the forms θ_k and ρ_k. Consequently applying the operator d to the forms ω_1 and ω_2, we obtain the following inclusions for ω_1

$$\begin{array}{rcl} d(\omega_1) & \in & d^2x^k\,\theta_k + Qdx^k\,d\theta_k + I_Q; \\ d^2(\omega_1) & \in & Q[2]_Q d^2x^k\,d\theta_k + Q^2dx^k\,d^2\theta_k + d(I_Q) + I_Q; \\ d^3(\omega_1) & \in & dx^k\,d^3\theta_k + d^2(I_Q) + d(I_Q) + I_Q \\ & \subseteq & d^2x^k\,I_Q + d^2(I_Q) + d(I_Q) + I_Q \subseteq I_Q; \end{array}$$

and for ω_2

$$\begin{array}{rcl} d(\omega_2) & \in & Q^2d^2x^k\,d\rho_k + I_Q; \\ d^2(\omega_2) & \in & Qd^2x^k\,d^2\rho_k + d(I_Q) + I_Q; \\ d^3(\omega_2) & \in & d^2x^k\,d^3\rho_k + d^2(I_Q) + d(I_Q) + I_Q \\ & \subseteq & d^2x^k\,I_Q + d^2(I_Q) + d(I_Q) + I_Q \subseteq I_Q. \end{array}$$

Thus the proof of theorem is completed. □

Chapter 3

Quadratic algebras and quantum plane with $d^3 = 0$

Since the discovery of quantum plane by Yu. V. Manin [53] and its possible applications for the description of deformed or more intricate than usual symmetries in mathematical physics by Wess and Zumino [64], an immense activity followed, especially during the past decade. Quite naturally, after the purely algebraic properties of those newly discovered spaces have been quite deeply investigated, and the related quantum groups and Hopf algebras analyzed and described, the study of analytic properties had followed. This is why the q-deformed algebras have become the next object of many excellent studies [21, 23, 49].

Parallely, novel ternary and $\mathbb{Z}_3$-graded algebraic structures have been introduced and investigated [49], then generalized to the $\mathbb{Z}_N$-graded case [26, 31, 32]. Thus an important class of $\mathbb{Z}_N$-graded differential algebraic structures has been investigated in an exhaustive manner.

It becomes natural now to combine these two novel and important structures – the q-deformed algebras and the graded Q-differential algebras – in order to see whether they can lead to further generalizations of many useful algebraic and analytic tools such as de Rham complexes, homology, Hecke and braided algebras, and the like. Our aim in this chapter is to show how the graded Q-differential algebra with the non-standard differential satisfying $d^3 = 0$ but with $d^2 \neq 0$ (the de Rham complex with $d^3 = 0$) can be realized on the associative unital algebra with quadratic relations.

In the first section, we construct the de Rham complex with the differential $d^3 = 0$ on the free algebra with one generator following to the general formalism elaborated in Chapter 2 before.

In the second section, using this general formalism, we construct the de Rham complex on an associative unital algebra generated by n generators and quadratic relations. In what follows, we shall use the parameter χ, where χ is a complex number, in the deformed Leibniz rule. We will mention separately the case when $\chi = Q(\doteq e^{\frac{2\pi i}{3}})$. We find the relations connecting the generators x^i with d^2x^j, dx^i with d^2x^j. Also we show that the second order differentials are connected by commutation relations in the case when the parameter $\chi \neq Q$.

In the classical de Rham complex with $d^2 = 0$, two quantum planes ($xy = qyx$ and $dx\,dy = pdy\,dx$) are parts of the de Rham complex, and these planes and all cross-commutation relations between x, y, dx, dy are compatible with the action of the group $GL_{p,q}(2)$ [53]. In the third section, we find the values of complex parameter χ from the commutation relations on the second order differentials d^2x and d^2y. For these values of χ we find that d^2x and d^2y generate bosonic quantum plane. This quantum plane is preserved by the action of quantum group $GL_q(2)$.

In the forth section, we construct the de Rham complex on Jordanian plane, i.e. on h-deformed quantum plane $[x, y] = -hy^2$. As in the previous section, we show that the generators d^2x and d^2y define the quantum plane, which is like the quantum plane $xy = qyx$. This quantum plane is preserved by the action of quantum group $GL_h(2)$.

Also as before, we use the notation $[k]_\chi = 1 + \chi + \chi^2 + \cdots + \chi^{k-1}$, and the property $[3]_Q = 0$ for $Q = e^{\frac{2\pi i}{3}}$. We shall denote by E the identity operator (or matrix) acting in linear space defined by the context.

3.1 Algebra of differential forms in dimension one

This section is based on the paper [2]. We construct a graded Q-differential algebra of differential forms with exterior differential d satisfying $d^3 = 0$ in dimension one. We study the structure of a bimodule of second order differentials and show that it is homogeneous in the case of the anyonic line. In this section, Q is a primitive cubic root of unity. Let $\mathcal{A}$ be a free unital associative $\mathbb{C}$-algebra generated by a variable x. If $\xi : \mathcal{A} \longrightarrow \mathcal{A}$ is a homomorphism of this algebra and $D : \mathcal{A} \longrightarrow \mathcal{A}$ is a linear map such that

$$D(x) = 1, \quad D(fg) = D(f)g + \xi(f)D(g), \quad \forall f, g \in \mathcal{A}, \tag{3.1.1}$$

then according to coordinate calculi the map

$$d : f \longrightarrow D(f)dx, \tag{3.1.2}$$

where dx is the first order differential of a variable x, is a coordinate differential, i.e. d is a linear map $d : \mathcal{A} \longrightarrow {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ satisfying the Leibniz rule

$$d(fg) = d(f)g + fd(g),$$

and ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is a free bimodule over $\mathcal{A}$ generated by dx with the right module structure defined by the commutation rule

$$dxf = \xi(f)dx. \tag{3.1.3}$$

If

$$f = \alpha_0 + \alpha_1 x^1 + \cdots + \alpha_p x^p, \quad \alpha_0, \alpha_1, \ldots, \alpha_p \in \mathbb{C}$$

is an element of $\mathcal{A}$ then the partial derivative $D : \mathcal{A} \longrightarrow \mathcal{A}$ can be written in explicit form

$$D(f) = \sum_{m \geqslant 1} \sum_{k=0}^{m-1} \alpha_m \xi^k(x) x^{m-k-1}. \tag{3.1.4}$$

In order to construct a generalization of exterior calculus with exterior differential d satisfying $d^2 \neq 0$ let us introduce the second order differential d^2x. Let $(dx)^k(d^2x)^m$ be monomials composed from first and second order differentials, where k, m are nonnegative integers. As usual we assume that $(dx)^0 = (d^2x)^0 = 1$ where 1 is the unit element of $\mathcal{A}$. Let $\Omega_\xi(\mathcal{A})$ be a free left module over the algebra $\mathcal{A}$ generated by the above introduced monomials. It is easy to see that $\mathcal{A} \subset \Omega_\xi(\mathcal{A})$, ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} \subset \Omega_\xi(\mathcal{A})$. The module $\Omega_\xi(\mathcal{A})$ becomes an unital associative algebra if we define a multiplication law on $\Omega_\xi(\mathcal{A})$ by the relations

$$dx\, x \;=\; \xi(x)dx, \quad d^2x\, f = \xi(f)\, d^2x + [D, \xi]_Q(f)\, (dx)^2, \tag{3.1.5}$$

$$(dx)^3 \;=\; 0, \quad d^2x\,dx = Q^2 dx\,d^2x, \tag{3.1.6}$$

where $[D,\xi]_Q(f) = D(\xi(f)) - Q\xi(D(f))$, and $f \in \mathcal{A}$.

Analyzing the defining commutation relations (3.1.5), (3.1.6) of the algebra $\Omega_\xi(\mathcal{A})$ one can note that the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ of coordinate calculi is a submodule of $\Omega_\xi(\mathcal{A})$ and the relation (3.1.3) between its left and right structures follows from the commutation relation between the first order differential dx and a variable x. The second remark concerns the structure of the algebra $\Omega_\xi(\mathcal{A})$ with respect to the second order differential. The relations (3.1.5), (3.1.6) show that any power of the second order differential does not vanish. Hence the algebra $\Omega_\xi(\mathcal{A})$ is an infinite-dimensional vector space and an arbitrary element ω of this algebra can be written in the form

$$\omega = \sum_{m\geqslant 0}\sum_{k=0}^{2} f_{km}(dx)^k(d^2x)^m, \quad f_{km} \in \mathcal{A}. \tag{3.1.7}$$

We shall call elements of the algebra $\Omega_\xi(\mathcal{A})$ differential forms on one-dimensional space generated by a variable x. The algebra of differential forms $\Omega_\xi(\mathcal{A})$ becomes an $\mathbb{N}$-graded algebra if we assign grading zero to each element of the algebra $\mathcal{A}$ and grading $k+2m$ to monomial $(dx)^k(d^2x)^m$, i.e. we assume that a variable x has grading zero and the gradings of the differentials dx, d^2x are respectively 1, 2. Then the algebra of differential forms splits into the direct sum of its subspaces

$$\Omega_\xi(\mathcal{A}) = \bigoplus_{m=0}^{\infty} \Omega_\xi^m(\mathcal{A}),$$

where

$$\begin{aligned}
\Omega_\xi^0(\mathcal{A}) &= \mathcal{A}, \\
\Omega_\xi^{2k}(\mathcal{A}) &= \{f(d^2x)^k + h(dx)^2\,(d^2x)^{k-1} : f,h \in \mathcal{A}\}, \qquad (3.1.8)\\
&\qquad\qquad\qquad\qquad k = 1,2,\ldots \\
\Omega_\xi^{2k+1}(\mathcal{A}) &= \{f\,dx\,(d^2x)^k \;:\; f \in \mathcal{A}\},\; k = 0,1,\ldots. \qquad (3.1.9)
\end{aligned}$$

We now extend the differential (3.1.2) of the coordinate calculus to the whole algebra $\Omega_\xi(\mathcal{A})$ as follows:

$$d(\omega) \;=\; (D\,f - h)\,dx\,(d^2x)^k, \quad \omega \in \Omega_\xi^{2k}(\mathcal{A}),$$

$$d(\omega) \;=\; f\,(d^2x)^{k+1} + D\,f\,(dx)^2\,(d^2x)^k, \quad \omega \in \Omega_\xi^{2k+1}(\mathcal{A}).$$

We shall call the above defined differential an exterior differential on the algebra of differential forms $\Omega_\xi(\mathcal{A})$. It follows from the definition that exterior differential is an endomorphism of degree 1 of the algebra $\Omega_\xi(\mathcal{A})$, i.e. $d : \Omega_\xi^m(\mathcal{A}) \longrightarrow \Omega_\xi^{m+1}(\mathcal{A})$.

Proposition 3.1.1. The algebra of differential forms $\Omega_\xi(\mathcal{A})$ is a graded Q-differential algebra with respect to exterior differential d, i.e. for any two differential forms ω, θ the exterior differential d satisfies

$$d^3(\omega) \;=\; 0, \tag{3.1.10}$$

$$d(\omega\theta) \;=\; d(\omega)\theta + Q^{|\omega|}\omega d(\theta), \tag{3.1.11}$$

where $|\omega|$ is the grading of a form ω.

Proof. It follows from (3.1.8), (3.1.9) that any differential form ω can be decomposed into the sum of two forms ω_o, ω_e respectively of odd and even grading, where

$$\omega_o \;=\; \sum_{k\geqslant 0} g_k\,dx\,(d^2x)^k,$$

$$\omega_e \;=\; \sum_{k\geqslant 1}[f_k\,(d^2x)^k + h_{k-1}\,(dx)^2(d^2x)^{k-1}]. \tag{3.1.12}$$

From the definition of the exterior differential it also follows that a differential form of even grading (3.1.12) is d-closed ($d\omega_e = 0$) if it satisfies the following condition

$$Df_k = h_{k-1}. \tag{3.1.13}$$

Now it is easy to show that any form of odd grading is d^2-closed, i.e. $d^2\omega_o = 0$. Indeed applying the exterior differential d to form ω_o, one obtains the form of even grading

$$d\omega_o = \sum_{k\geqslant 0}[g_k\,(d^2x)^{k+1} + D\,g_k\,(dx)^2(d^2x)^k],$$

which is d-closed according to (3.1.13). Differentiating form (3.1.12) of even grading twice, one obtains the form

$$d^2\omega_e = \sum_{k\geqslant 0}[(Df_k - h_{k-1})(d^2x)^{k+2} + (D^2f_k - Dh_{k-1})(dx)^2(d^2x)^{k+1}],$$

which is d-closed. Thus the cube nilpotency (3.1.10) of the exterior differential is proved. The Q-Leibniz rule (3.1.11) can be verified by a direct calculation. $\square$

A homomorphism ξ plays a role of a parameter in the structure of the algebra of differential forms, and, choosing particular homomorphism, we can specify the structure of $\Omega_\xi(\mathcal{A})$. We remind that according to the definition the algebra of differential forms $\Omega_\xi(\mathcal{A})$ is a free left module over the algebra $\mathcal{A}$ generated by the monomials $(dx)^k\,(d^2x)^m$ with associative multiplication law determined by the relations (3.1.5), (3.1.6). Actually this algebra is generated by three generators x, dx, d^2x. Hence its structure will be more transparent if we define it by means of commutation relations imposed on the generators x, dx, d^2x. The only relation in (3.1.5), (3.1.6), which is not a commutation relation between generators, is the second relation in (3.1.5) containing an arbitrary element f of the algebra $\mathcal{A}$. The reason why it contains an arbitrary element f is that in contrast to the commutation relation $dx\,x = \xi(x)\,dx$ the right-hand side of this relation is not a homomorphism of the algebra $\mathcal{A}$ because of the non-homogeneous term $[D,\xi]_Q(f)$. Obviously imposing the condition

$$[D,\xi]_Q(f) = 0, \tag{3.1.14}$$

we can replace the second relation in (3.1.5) by the commutation relation

$$d^2x\,x = \xi(x)\,d^2x, \tag{3.1.15}$$

which has exactly the same form as the first one involving the first order differential.

Actually the choice for a homomorphism ξ is not very wide. Indeed every homomorphism ξ of the algebra $\mathcal{A}$ is determined by an element $h_\xi \in \mathcal{A}$ such that $\xi(x) = h_\xi$. From (3.1.1) it follows that if the derivative D determined by a homomorphism ξ should satisfy $D(x^m) \sim x^{m-1}$ then $\xi(x) = \alpha_\xi x$ where α_ξ is a complex number. The condition (3.1.14) can be solved with regard to a homomorphism ξ, and the following proposition describes a structure which is induced on one-dimensional space in this case.

Proposition 3.1.2. The condition (3.1.14) is satisfied if and only if $\xi(x) = Q\,x$. This solution leads to the Q-differential calculus on anyonic line with derivative

$$D(f)=\sum_{k\geqslant 1}\alpha_k\frac{x^{k-1}}{[k-1]_Q!},\quad f=\sum_{k\geqslant 0}\alpha_k\frac{x_k}{[k]_Q!}. \tag{3.1.16}$$

This means that one can consistently add the relation $x^3=0$ to the relations (3.1.5), (3.1.6).

Proof. Using the formula (3.1.4) one can find

$$\begin{aligned} D(\xi(f)) &= \alpha_\xi\sum_m\alpha_m\sum_{k=0}^{m-1}\xi^k(h)h^{m-k-1}, \\ Q\xi(D(f)) &= Q\sum_m\alpha_m\sum_{k=0}^{m-1}\xi^k(h)h^{m-k-1}. \end{aligned}$$

Thus

$$[D,\xi]_Q(f)=(\alpha_\xi-Q)\sum_m\alpha_m\sum_{k=0}^{m-1}\xi^k(h)h^{m-k-1}. \tag{3.1.17}$$

From the above formula it immediately follows that $\xi(x)=Q\,x$ or $\xi(f(x))=f(Qx)$. Putting $\xi(x)=Qx$ in (3.1.4), one obtains (3.1.16). It should be mentioned that Q-differential calculus determined by the derivative (3.1.16) is correctly defined at cubic root of unity on one-dimensional space generated by a variable x only in the case when $x^3=0$. It is easy to show that the relation $x^3=0$ can be consistently added to the relations (3.1.5), (3.1.6). Now the algebra of differential forms $\Omega_\xi(\mathcal{A})$ on anyonic line can be defined as an unital associative algebra generated by three generators x, dx, d^2x satisfying the following commutation relations:

$$\begin{aligned} x^3 &= 0, \\ dx\,x &= Qx\,dx, \\ d^2x\,x &= Qx\,d^2x, \\ (dx)^3 &= 0, \\ d^2x\,dx &= Q^2dx\,d^2x. \end{aligned}$$

□

3.2 General case of graded differential algebra with differential $d^3 = 0$ on quadratic algebra

Let $\mathcal{A}$ be an associative unital algebra generated by variables $x^1,\ x^2,\ldots,x^n$, which satisfy the following set of quadratic commutation relations:

$$x^i x^j = B^{ij}_{kl} x^k x^l, \tag{3.2.1}$$

where B is a matrix with complex number entries. We shall call such algebra the n-dimensional quantum space. Thus the elements of kind

$$x^i x^j - B^{ij}_{kl} x^k x^l \quad \in \ \mathcal{A} \tag{3.2.2}$$

generate an ideal $I \subset \mathcal{A}$.

Our aim now is, following to the general formalism elaborated in Chapter 2, to determine a FODC $\{\mathcal{A},\, \mathcal{M},\, d\}$ and construct a graded χ-differential algebra $\{\Omega,\, d\}$ on the quantum space $\mathcal{A}$ assuming that the χ-Leibniz rule

$$d(\omega\theta) = d\omega\ \theta + \chi^{\deg(\omega)}\omega\ d\theta, \tag{3.2.3}$$

where ω and θ are differential forms, $\deg(\omega)$ is the grade of the element ω, and χ is a complex number, and the condition $d^3 = 0$ take place.

In our construction the FODC coincides with Wess-Zumino type differential calculus on the quantum plane (3.2.1) [64]. Let the $\mathcal{A}$-bimodule $\mathcal{M}$ be generated as a free right $\mathcal{A}$-module by the differentials dx^j, and let the left $\mathcal{A}$-module structure on $\mathcal{M}$ be completely defined by the following relations:

$$x^i dx^j = C^{ij}_{kl} dx^k x^l, \qquad i,j = 1,\ 2,\ldots,n, \tag{3.2.4}$$

and C is a matrix with complex number entries, see e.g. Chapter 1 or [12, 13, 14, 15, 16] for more details.

We interpret the equations (3.2.4) as an algebra homomorphism $\xi : \mathcal{A} \to Mat_n(\mathcal{A})$ such that $\xi(x^i)^j_k \doteq \xi^{ij}_k = C^{ij}_{kl}\, x^l$ (see e. g. (1.1.2)).

The requirement that the ideal I has to be ξ-consistent (compare 1.1.9) leads us to the two consistency conditions involving matrices B and C:

$$(E_{12} - B_{12})(E_{12} + C_{12}) \ = \ 0, \tag{3.2.5}$$

$$B_{12}C_{23}C_{12} \;=\; C_{23}C_{12}B_{23}, \tag{3.2.6}$$

where E_{12} denotes the identity operators tensor product, $E_{12} = I_1 \otimes I_2$, operating in the tensor product of the space of 1-forms dx^i with the quantum space generated by x^k.

In fact, it is easily seen that the condition (3.2.5) is equivalent to the ξ-stability of the algebra ideal I (see e.g. (1.1.7)). Indeed,

$$\begin{aligned}0 = D_r(x^i x^j - B^{ij}_{kl} x^k x^l) = \delta^i_r\, x^j + C^{ij}_{rm}\, x^m - B^{ij}_{rl}\, x^l - B^{ij}_{kl} C^{kl}_{rm}\, x^m \\ = (\delta^i_k\, \delta^j_l\, - B^{ij}_{kl})(\delta^k_r\, \delta^l_m\, + C^{kl}_{rm})\, x^m = 0.\end{aligned}$$

It can be easily shown also that the condition (3.2.6) occurs from the requirement of ξ-invariance on the algebra ideal I (see e.g. (1.1.6)):

$$\begin{aligned}0 = (x^i\, x^j - B^{ij}_{kl} x^k\, x^l) dx^m = C^{jm}_{kl}\, C^{ik}_{st}\, dx^s\, x^t\, x^l - B^{ij}_{kl}\, C^{lm}_{pr}\, C^{kp}_{st}\, dx^s\, x^p\, x^r = \\ (C^{jm}_{kl}\, C^{ik}_{st}\, B^{tl}_{pr} - B^{ij}_{kl}\, C^{lm}_{pr}\, C^{kp}_{st}) dx^s\, x^p\, x^r = 0,\end{aligned}$$

what establishes the formula (3.2.6).

Further on, since we are now assuming $d^2 \neq 0$, we have to introduce the second order differentials and replace and eventually generalize the classical Leibniz rule. Let the differential d, by means of which we extend the FODC, be now supposed to satisfy the χ-Leibniz rule (3.2.3). In particular, for $\deg(\omega) = \deg(\theta) = 0$, i.e. for a first order calculus, one recovers the standard (undeformed) Leibniz rule.

The first differentiation of (3.2.4) gives rise to the relations between the generators x^i, the first and second order differentials dx^j, d^2x^k:

$$x^i d^2 x^j = C^{ij}_{kl} d^2 x^k x^l + (\chi C^{ij}_{kl} - \delta^i_k \delta^j_l) dx^k dx^l. \tag{3.2.7}$$

Next differentiation gives the commutation relations between differentials dx^k and d^2x^l only:

$$([2]_\chi\, \delta^i_k \delta^j_l - \chi^2 C^{ij}_{kl}) dx^k d^2 x^l = ([2]_\chi\, \chi C^{ij}_{kl} - \delta^i_k \delta^j_l) d^2 x^k dx^l. \tag{3.2.8}$$

Finally, we obtain extra relations between differentials d^2x^i:

$$[3]_\chi\, d^2 x^i d^2 x^j = [3]_\chi\, \chi^2 C^{ij}_{kl} d^2 x^k d^2 x^l. \tag{3.2.9}$$

When χ is not a primitive cubic root of unity, i.e. $\chi \neq Q \doteq e^{\frac{2\pi i}{3}}$, we arrive at the following relations

$$d^2x^i d^2x^j = \chi^2 C^{ij}_{kl} d^2x^k d^2x^l. \tag{3.2.10}$$

Therefore we refer to the case $\chi = Q$ as canonical one because there is no need to introduce new additional (exotic) relations (3.2.10) between the generators d^2x^i in this case (for more general cases, see [28, 30], Chapter 2).

The relations (3.2.1 - 3.2.10) define a universal graded differential algebra with differential $d^3 = 0$ on quadratic algebras: any other graded differential algebra with differential $d^3 = 0$ on (3.2.1) admitting the first order calculus (3.2.4) can be obtained from this one via a standard quotient construction. In particular, as in Section 2.1, we can introduce the quasi-SOD by the commutation relations between generators x^i and second order differentials d^2x^i

$$x^i d^2x^j = F^{ij}_{kl} d^2x^k x^l \tag{3.2.11}$$

where F is a matrix with complex number entries. In other words, we suppose that there is the $\mathcal{A}$-bimodule $\mathcal{M}^{(2)}$ generated by the elements d^2x^i as a free right $\mathcal{A}$-module and the left structure of $\mathcal{M}^{(2)}$ is connected with the right one by (3.2.11). In this case we can not say about SOD (see Section 2.1) because we do not define the differential $\widetilde{d} : \mathcal{A} \rightarrow \mathcal{M}^{(2)}$.

As a consequence of (3.2.11), the first and the second order differentials have to satisfy the following relations

$$dx^i d^2x^j = Q^2 F^{ij}_{kl} d^2x^k dx^l, \tag{3.2.12}$$
$$d^2x^i d^2x^j = Q^4 F^{ij}_{kl} d^2x^k d^2x^l. \tag{3.2.13}$$

This requirement allows us to introduce a left $\mathcal{A}$-module again, therefore also a bimodule structure on a right free $\mathcal{A}$-module generated by the second order differentials d^2x^j. Because of this, F should satisfy quadratic consistency conditions analogous to the condition (3.2.6):

$$B_{12}F_{23}F_{12} = F_{23}F_{12}B_{23}. \tag{3.2.14}$$

Substituting now (3.2.11) into (3.2.7), one finds

$$(F^{ij}_{kl} - C^{ij}_{kl}) d^2x^k x^l = (QC^{ij}_{kl} - \delta^i_k \delta^j_l) dx^k dx^l. \tag{3.2.15}$$

Differentiating the last relations and using (3.2.12), we obtain that the consistency condition takes on the form:

$$E - [2]_Q\, C + (Q[2]_Q\, E - C)Q\, F = 0. \tag{3.2.16}$$

Fortunately, the last equation reduces to a linear (cf. (3.2.5)) Wess-Zumino-like condition on the matrices C and F:

$$(E + C)(E - QF) = 0. \tag{3.2.17}$$

In this way, we got independent confirmation that only the canonical case $\chi = Q$ is interesting since it does not lead to the exotic relations (3.2.10) and (3.2.17).

Following the well known Wess-Zumino method, we can now resolve the consistency conditions (3.2.14) and (3.2.17). To this end let us assume that a Hecke R-matrix is given, i.e. the matrix R satisfying the braid relation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{3.2.18}$$

together with the second-order minimal polynomial condition

$$(R - \mu E)(R + \lambda E) = 0. \tag{3.2.19}$$

Rewriting it in the form

$$(E - \frac{1}{\mu}R)(E + \frac{1}{\lambda}R) = 0, \tag{3.2.20}$$

one immediately sees that $B = \frac{1}{\mu}R$, $C = \frac{1}{\lambda}R$ and $F = \frac{Q^2}{\mu}R$ are the solution of the consistency conditions (3.2.5), (3.2.6) and (3.2.14), (3.2.17). These can be further generalized for non-Hecke R-matrices (cf. [39]).

More generally, having matrices B and C as a solution of the consistency conditions (3.2.5, 3.2.6), one can set $F = Q^2 B$ provided that B commutes with C and the braid relation (3.2.14) is satisfied. In what follows, we shall provide few concrete examples at $d^3 = 0$ differential calculus.

3.3 q-deformed quantum plane with $d^3 = 0$

In this section, we shall realize the graded χ-differential algebra with $d^3 = 0$ constructed in Section 3.2 on the two-dimensional two-parametric quantum plane $\mathcal{A}_q(2)$ such that its generators x and y obey the relation

$$xy = qyx, \tag{3.3.1}$$

where q is a complex deformation parameter [36].

As it is well known, the quantum plane $\mathcal{A}_q(2)$ can be determined by the two-parametric R-matrix

$$B = \frac{1}{q}\widehat{R}_{pq}, \quad \text{where} \quad \widehat{R}_{pq} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-p^{-1} & qp^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{3.3.2}$$

when p is the complex deformation parameter different from q. Let us note at once that $\widehat{R}_{pq}$ satisfies the braid relations (3.2.6).

There are two different eigenvalues of $\widehat{R}_{pq}$: $\mu = q$ and $\lambda = -1/p$. Therefore we can rewrite the linear consistency condition (3.2.5) for the matrix $\widehat{R}_{pq}$ as it follows:

$$(E - \frac{1}{q}\widehat{R}_{pq})(E + p\widehat{R}_{pq}) = 0. \tag{3.3.3}$$

Moreover, due to the property

$$\widehat{R}^2_{pq} = \frac{q}{p}E + (q - \frac{1}{q})\widehat{R}_{pq}, \tag{3.3.4}$$

we obtain a new linear consistency condition

$$(E - \frac{1}{q}\widehat{R}_{pq})(E + \frac{1}{p}\widehat{R}^{-1}_{pq}) = 0 \tag{3.3.5}$$

from the condition (3.3.3).

It turns out that there are two infinite and non-equivalent families of covariant first order differential calculi on the quantum-plane $\mathcal{A}_q(2)$, which are characterized by means of two matrices

$$C_1 = p\widehat{R}_{pq} = \begin{pmatrix} pq & 0 & 0 & 0 \\ 0 & pq-1 & q & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & pq \end{pmatrix}$$

and

$$C_2 = \frac{1}{p}\widehat{R}_{pq} = \begin{pmatrix} (pq)^{-1} & 0 & 0 & 0 \\ 0 & 0 & p^{-1} & 0 \\ 0 & q^{-1} & (pq)^{-1}-1 & 0 \\ 0 & 0 & 0 & (pq)^{-1} \end{pmatrix},$$

which define the commutation relations (3.2.4)(cf. [57, 64]). As a matter of fact, the two matrices C_1 and C_2 are not really independent: one can be obtained from another if we substitute x by y and simultaneously q by q^{-1}, and p by p^{-1}. The matrices $\widehat{R}_{pq}$, C_1, and C_2 define a generalization of the first-order differential calculi obtained by Wess and Zumino [64] on the Manin plane [53]. In fact, we obtain a Wess-Zumino first order differential calculus if $p=1$ (cf. [60]). Thus further here we investigate only realization of graded differential algebra with $d^3=0$ for matrix C_1.

Next we write the explicit relations between $x,\ y$ and $dx,\ dy$ defined by the matrix C_1:

$$\begin{aligned} x\,dx &= pq\,dx\,x, \\ x\,dy &= (pq-1)\,dx\,y + q dy\,x, \\ y\,dx &= p\,dx\,y, \\ y\,dy &= pq\,dy\,y. \end{aligned} \tag{3.3.6}$$

Firstly we consider the realization for non-canonical case of parameter χ. Following to the general formalism elaborated above, write explicitly the relations (3.2.4), (3.2.8) and (3.2.10):

$$\begin{aligned} x\,d^2x &= pq\,dx^2\,x + (\chi pq-1)dx\,dx, \\ x\,d^2y &= (pq-1)d^2x\,y + (\chi pq-[2]_\chi)dx\,dy + \chi q\,dy\,dx, \\ y\,d^2x &= pd^2x\,y + \chi p\,dx\,dy - dy\,dx, \\ y\,d^2y &= pq\,d^2y\,y + (\chi pq-1)dy\,dy, \end{aligned} \tag{3.3.7}$$

$$
\begin{aligned}
dx\,d^2x &= \big((\chi pq-1)(1-\frac{1}{\chi^2pq})-\frac{1}{[2]_\chi}\big)d^2x\,dx,\\
dx\,d^2y &= \big((\chi pq-1)-\frac{\chi^2}{[2]_\chi}+\frac{[2]_\chi}{\chi}\big)d^2x\,dy+\big(\chi q+\frac{1}{\chi^2p}\big)d^2y\,dx,\\
dy\,d^2x &= \big(\chi p+\frac{1}{\chi^2q}\big)d^2x\,dy-\big(\frac{\chi pq-1}{\chi^2pq}+\frac{1+\chi[2]_\chi}{\chi[2]_\chi}\big)d^2y\,dx,\\
dy\,d^2y &= \big((\chi pq-1)(1-\frac{1}{\chi^2pq})-\frac{1}{[2]_\chi}\big)d^2y\,dy,
\end{aligned}
\tag{3.3.8}
$$

$$
\begin{aligned}
d^2x\,d^2x &= \chi^2pqd^2x\,d^2x,\\
d^2x\,d^2y &= \chi^2(pq-1)d^2x\,d^2y+\chi^2qd^2y\,d^2x,\\
d^2y\,d^2x &= \chi^2pd^2x\,d^2y\\
d^2y\,d^2y &= \chi^2pqd^2y\,d^2y.
\end{aligned}
\tag{3.3.9}
$$

Considering the obtained relations (3.3.8) as equations with respect to the parameter of differentiation χ, we get the set of values of parameter $\chi \in \{\pm\frac{1}{\sqrt{pq}}\}$.

If $\chi = \pm\frac{1}{\sqrt{pq}}$, then we obtain the commutation relations between the second order differentials without the parameter p:

$$d^2x\,d^2y = qd^2y\,d^2x.$$

It means that the generators d^2x and d^2y define the quantum plane, which is like the given quantum plane $xy = q\,yx$. Both quantum planes are preserved by the action of the quantum group $GL_q(2)$, determined by generators $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d}$ satisfying the commutation relations:

$$
\begin{aligned}
&\mathfrak{a}\mathfrak{b} = q\mathfrak{b}\mathfrak{a}, && \mathfrak{a}\mathfrak{c} = q\mathfrak{c}\mathfrak{a}, && \mathfrak{b}\mathfrak{c} = \mathfrak{c}\mathfrak{b},\\
&\mathfrak{c}\mathfrak{d} = q\mathfrak{d}\mathfrak{b}, && \mathfrak{b}\mathfrak{d} = q\mathfrak{d}\mathfrak{b}, && \mathfrak{a}\mathfrak{d}-\mathfrak{d}\mathfrak{a} = (q-q^{-1})\mathfrak{b}\mathfrak{c}.
\end{aligned}
$$

Further on we consider the case when $\chi = Q$. Firstly in order to write the explicit relations on generator $x,\ y,\ dx,\ dy,\ d^2x,\ d^2y$, we find the matrix F (3.2.11).

Due to the property (3.3.4), the condition (3.2.17) on the matrix $\widehat{R}_{pq}$ has two different forms:

$$(E+p\widehat{R})(E-\frac{Q^2}{q}\widehat{R}) = 0,$$

$$(E + p\widehat{R})(E - \frac{q}{Q}\widehat{R}^{-1}) = 0,$$

and therefore we can choice two matrix F for the quasi-SOD:

$$F_1 = \frac{Q^2}{q}\widehat{R}_{pq} = \begin{pmatrix} Q^2 & 0 & 0 & 0 \\ 0 & \frac{Q^2}{q}(1-\frac{p}{q}) & \frac{Q^2}{p} & 0 \\ 0 & \frac{Q^2}{q} & 0 & 0 \\ 0 & 0 & 0 & Q^2 \end{pmatrix}$$

and

$$F_2 = \frac{q}{Q}\widehat{R}_{pq} = \begin{pmatrix} \frac{1}{Q} & 0 & 0 & 0 \\ 0 & 0 & \frac{q}{Q} & 0 \\ 0 & \frac{p}{Q} & \frac{1}{Q}(1-pq) & 0 \\ 0 & 0 & 0 & \frac{1}{Q} \end{pmatrix},$$

As it was before for the matrices C_1 and C_2, the matrices F_1 and F_2 are not independent: substituting x by y, q by q^{-1}, and p by p^{-1}, we obtain F_2 from F_1. For this reason, we will prefer the case $F_1 = \frac{Q^2}{q}\widehat{R}_{pq}$ and consider here only this case.

Let us write the explicit relations (3.2.11, 3.2.12) for the graded differential algebra with $d^3 = 0$ and FODC (3.3.6) on the quantum plane (3.3.1):

$$\begin{aligned} x\,d^2x &= Q^2 d^2x\,x, \\ x\,d^2y &= Q(1-\frac{1}{pq})d^2x\,y + \frac{Q^2}{p}d^2y\,x, \\ y\,d^2x &= \frac{Q^2}{q}d^2x\,y, \\ y\,d^2y &= Q^2 d^2y\,y; \end{aligned} \tag{3.3.10}$$

$$\begin{aligned} dx\,d^2x &= Qd^2x\,dx, \\ dx\,d^2y &= Q(1-\frac{1}{pq})d^2x\,dy + \frac{Q}{p}d^2y\,dx, \\ dy\,d^2x &= \frac{Q}{q}d^2x\,dy, \\ dy\,d^2y &= Qd^2y\,dy, \end{aligned} \tag{3.3.11}$$

the additional relation (3.2.15):

$$
\begin{aligned}
d^2x\,x &= \frac{Qpq-1}{Q^2-pq}dx\,dx,\\
d^2x\,y &= \frac{Qpq}{Q^2-pq}dx\,dy - \frac{q}{Q^2-pq}dy\,dx\\
d^2x\,y &= -\frac{p}{Q^2-pq}dx\,dy + \frac{[2]_Q pq-1}{Q^2-pq}q\,dy\,dx,\\
d^2y\,y &= \frac{Qpq-1}{Q^2-pq}dy\,dy,
\end{aligned}
\tag{3.3.12}
$$

and the commutation relations between the second order differentials (3.2.13)

$$
d^2x\,d^2y = q\,d^2y\,d^2x. \tag{3.3.13}
$$

The quantum plane generated by the second order differential in this case coincides with the one in the case $\chi = \pm\frac{1}{\sqrt{pq}}$ described before and is preserved by the action of quantum group $GL_q(2)$ (3.3.10) as the given quantum plane (3.3.1).

The relations (3.3.12) allows us to make choice of value $pq \neq Q^2$ or $pq = Q^2$.

If $pq \neq Q^2$, the relations (3.3.12) take the following form:

$$
\begin{aligned}
d^2x\,x &= [2]_Q dx\,dx,\\
(1+\frac{1}{pq})d^2x\,y - \frac{1}{p}d^2y\,x &= -dx\,dy,\\
(Q^2-pq)d^2x\,y &= Qpq\,dx\,dy - q\,dy\,dx,\\
d^2y\,y &= -Qdy\,dy,
\end{aligned}
\tag{3.3.14}
$$

and their differentiation leads us to the relations (3.3.11).

Otherwise, if $pq = Q^2$, we obtain that the first order differentials generate the quantum plane:

$$
dx\,dy = q\,dy\,dx, \tag{3.3.15}
$$

which is like to the quantum planes (3.3.1) and (3.3.13), and also is preserved by the group $GL_q(2)$ (3.3.10).

3.4 h-deformed quantum plane with $d^3 = 0$

Let $\mathcal{A}_h(2)$ be two-dimensional quantum plane with coordinates x and y, which satisfy the condition

$$[x, y] = -hy^2. \tag{3.4.1}$$

This h-deformed quantum plane can be determined also by the following R-matrix:

$$\widehat{R}_h = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 0 & 1 & -h \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.4.2}$$

As the matrix $\widehat{R}_h$ has two eigenvalues 1 and -1, the second-order minimal polynomial condition

$$(\widehat{R}_h - E)(\widehat{R}_h + E) = 0,$$

rewritten in the form

$$(E - \widehat{R}_h)(E + \widehat{R}_h) = 0,$$

gives us the solution $C = \widehat{R}_h$ of the linear condition (3.2.5). Then the condition (3.2.6) is automatically satisfied.

In this section, as well as before, following to the general framework elaborated in Section 3.2, we shall construct the graded differential algebra with the χ-Leibniz rule (3.2.3) and the differential $d^3 = 0$ on the two-dimensional h-deformed quantum plane (3.4.1) for the non-canonical parameter χ and for the case of quasi-SOD with the canonical parameter χ assuming that FODC for both cases is defined by the following relations between generators $x,\ y$ and first order differentials $dx,\ dy$:

$$\begin{aligned} x\,dx + hx\,dy - hy\,dx &= dx\,x - h^2 dy\,y, \\ x\,dy + y\,dx &= dx\,y + dy\,x, \\ y\,dx &= dx\,y + hdy\,y, \\ y\,dy &= dy\,y. \end{aligned} \tag{3.4.3}$$

Next we consider the non-canonical case of parameter χ and write the commutation relations between the generators $x,\ y,\ dx,\ dy,\ d^2x,\ d^2y$ (3.2.7), (3.2.8) and (3.2.10) explicitly:

$$\begin{aligned}
x\,d^2x - hx\,d^2y + hy\,d^2x &= d^2x\,x - hd^2y\,y - (1-\chi)dx\,dx - hdx\,dy \\
&\quad + hdy\,dx + \chi h^2 dy\,dy, \\
x\,d^2y + y\,d^2x &= d^2x\,y + d^2y\,x - (1-\chi)dx\,dy \\
&\quad - (1-\chi)dy\,dx, \\
y\,d^2x &= d^2x\,y + hd^2y\,y + \chi dx\,dy - dy\,dx \\
&\quad + \chi hdy\,dy, \\
y\,d^2y &= d^2y\,y - (1-\chi)dy\,dy;
\end{aligned} \tag{3.4.4}$$

$$\begin{aligned}
&([2]_\chi - \chi^2)dx\,d^2x - \chi^2 hdx\,d^2y + \chi^2 hdy\,d^2x - \chi^2h^2dy\,d^2y \\
&\qquad = ([2]_\chi\chi - 1)d^2x\,dx + [2]_\chi\chi hd^2x\,dy - [2]_\chi\chi hd^2y\,dx + [2]_\chi\chi h^2d^2y\,dy, \\
&[2]_\chi dx\,d^2y] - \chi^2dy\,d^2x + \chi^2h\,dy\,d^2y \\
&\qquad = -d^2x\,dy + [2]_\chi\chi d^2y\,dx - [2]_\chi\chi h^2d^2y\,dy; \\
&([2]_\chi - \chi^2)dx\,d^2y + ([2]_\chi - \chi^2)dy\,d^2x \\
&\qquad = ([2]_\chi\chi - 1)d^2x\,dy + ([2]_\chi\chi - 1)d^2y\,dx; \\
&([2]_\chi - \chi^2)dy\,d^2y = ([2]_\chi\chi - 1)d^2y\,dy;
\end{aligned} \tag{3.4.5}$$

$$\begin{aligned}
(1-\chi^2)d^2x\,d^2x &= -(1-\chi^2)hd^2y\,d^2x - (1-\chi^2)h^2\chi^2d^2y\,d^2y, \\
d^2x\,d^2y &= \chi^2d^2y\,d^2x - \chi^2hd^2y\,d^2y, \\
(1-\chi^2)d^2x\,d^2y &= -(1-\chi^2)d^2y\,d^2x, \\
d^2y\,d^2y &= \chi^2d^2y\,d^2y.
\end{aligned} \tag{3.4.6}$$

Considering the last relations (3.4.6) as equations with respect to the parameter of differentiation χ, we get that $\chi \in \{\pm 1\}$.

If $\chi = \pm 1$, then the second order differentials d^2x, d^2y generate the h-deformed quantum plane

$$d^2x,\, d^2y - d^2y\,d^2x = -h(d^2y)^2, \tag{3.4.7}$$

which is like to the plane $\mathcal{A}_h(2)$ (3.4.1). This quantum plane is preserved by action of the quantum group $GL_h(2)$ generated by the elements α, β, γ, δ

satisfying the following relations:

$$\alpha\gamma - \gamma\alpha = -h\gamma^2, \qquad \alpha\delta = \gamma\beta - h\gamma\delta,$$
$$\beta\delta - \delta\beta = -h\gamma^2, \qquad \delta\alpha = \beta\gamma + h\delta\gamma.$$

All obtained relations (3.4.3 - 3.4.5) with the relation (3.4.7) define the de Rham complex with differential $d^3 = 0$ on the h-deformed quantum plane (3.4.1) for $\chi = \pm 1$.

If we deal with the canonical parameter $\chi = Q$, then the last relations (3.4.6) disappear, and the graded differential algebra with $d^3 = 0$ on the h-deformed quantum plane (3.4.1) is defined by the relations (3.4.3 - 3.4.4) and the relations

$$\begin{aligned} dx\, d^2x - dx\, d^2y &= -d^2x\, dx - Qhd^2y\, dx - Qh^2d^2y\, dy, \\ dx\, d^2y] + dy\, d^2x &= Qd^2x\, dy + Qd^2y\, dx, \\ dy\, d^2y &= Qd^2y\, dy \end{aligned} \tag{3.4.8}$$

instead of (3.4.5).

In the end of this section we construct the graded differential algebra with differential $d^3 = 0$ on the h-deformed quantum plane by assumption that $\chi = Q$, FODC is defined by the relations (3.4.3), and quasi-SOD is given by the matrix $F = \widehat{R}_h$ (3.4.2):

$$\begin{aligned} x\, d^2x - hx\, d^2y + hy\, d^2x &= Q^2d^2x\, x + Q^2hd^2y\, y, \\ x\, d^2y + y\, d^2x &= Q^2d^2x\, y + Q^2d^2y\, x, \\ y\, d^2x &= Q^2d^2x\, y + Q^2hd^2y\, y, \\ y\, dy &= Q^2d^2y\, y. \end{aligned} \tag{3.4.9}$$

Explicit relations (3.4.5-3.4.6) take the form:

$$\begin{aligned} dx\, d^2x + hdx\, d^2y - hdy\, d^2x &= Q^2d^2x\, dx + Q^2h^2d^2y\, dy, \\ dx\, d^2y + dy\, d^2x &= Q^2d^2x\, dy + Q^2d^2y\, dx, \\ dy\, d^2x &= Q^2d^2x\, dy + Q^2hd^2y\, dy, \\ dy\, d^2y &= Q^2d^2y\, dy; \end{aligned} \tag{3.4.10}$$

$$
\begin{aligned}
d^2x\,d^2x + hd^2x\,d^2y - hd^2y\,d^2x &= Qd^2x\,d^2x - Qh^2d^2y\,d^2y,\\
d^2x\,d^2y &= Qd^2y\,d^2x - Qhd^2y\,d^2y,\\
d^2y\,d^2x &= Qd^2x\,d^2y + Qhd^2y\,d^2y,\\
d^2y\,d^2y &= Qd^2y\,d^2y.
\end{aligned}
\tag{3.4.11}
$$

The additional condition (3.2.15), taking into account that now $F = C$, is:

$$
\begin{aligned}
dx\,dx + Qhdy\,dx - Qhdx\,dy &= Qdx\,dx + Q^2hd^2y\,dy,\\
dx\,dy &= Qdy\,dx - Qhdy\,dy,\\
dy\,dx &= Qdx\,dy + Qhdy\,dy,\\
dy\,dy &= Qdy\,dy.
\end{aligned}
\tag{3.4.12}
$$

Then, from the arrays of relations (3.4.11) and (3.4.12), we obtain the following conditions on the first and second order differentials:

$$
\begin{aligned}
(dx)^2 = (dy)^2 = dx\,dy = dy\,dx = 0,\\
(d^2x)^2 = (d^2y)^2 = d^2x\,d^2y = d^2y\,d^2x = 0.
\end{aligned}
\tag{3.4.13}
$$

It means that the graded differential algebra with the differential $d^3 = 0$ on the h-deformed quantum plane (3.4.1) is defined by the relations (3.4.3), (3.4.9), and (3.4.10).

Chapter 4

Universal differential calculus on ternary algebras

We start introducing some notation and conventions. Throughout this chapter, we shall work in the category of vector spaces over a field $\mathbb{K}$, which in our case, for simplicity, is assumed to be the field of real or complex numbers. This means that all objects considered here are linear spaces, all mappings are $\mathbb{K}$–linear mappings, the tensor product $\otimes$ is a shortcut for $\otimes_{\mathbb{K}}$. Algebras will be generically denoted by $\mathcal{A}$, and the modules will be denoted by $\mathcal{M}$.

In this chapter we are interested in *ternary* algebras, i.e. linear spaces over $\mathbb{K}$ endowed with a trilinear associative composition law. More general structures of this type, called *n-ary algebras* have been studied elsewhere ([11, 20, 24, 30, 35, 41, 45, 56, 62]), and it has been shown that many familiar notions from the theory of usual (i.e. "binary") algebras, such as nilpotency, solvability, simplicity algebras etc., can be quite naturally generalized to the n-linear case.

Our attention will be focused on particular properties of *ternary* algebras, including the relations existing between general ternary algebras or ternary algebras of particular types, and trivial ternary algebras induced by the associative law in ordinary algebras, which then play a role similar to the role played by associative algebras with respect to the classical non-associative Lie algebras. Next, we shall define the analog of *modules* over ordinary algebras, which will be called *tri-modules* in the present case.

Finally, we shall define the *derivations* of ternary algebras and show in several examples how such differential ternary algebras can be realized. General construction of the universal envelope for associative ternary algebras and the universal differential calculus will also be presented. This chapter is based on the paper [9].

4.1 Ternary algebras and tri-modules

4.1.1 Associative ternary algebras

By *ternary* (associative) algebra $(\mathcal{A}, [\ ])$ we mean a linear space $\mathcal{A}$ (over a field $\mathbb{K}$) equipped with a linear map $[\ ] : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ called a (ternary) multiplication (or product), which satisfies the following strong associativity condition :

$$[[abc]de] = [a[bcd]e] = [ab[cde]]$$

Weaker versions of ternary associativity, when only one of the above identities is satisfied, can be called *left* (respectively, *right* or *central*) associativity.

We look at associative ternary algebras as a natural generalization of binary one: if $(\mathcal{A}, \cdot)$ is the usual (binary, associative) algebra then an induced ternary multiplication can be, of course, defined by $[abc] = (a \cdot b) \cdot c = a \cdot (b \cdot c)$. In what follows, such ternary algebras will be called trivial; from now on we shall study exclusively non-trivial ternary algebras. It is known that unital ternary algebras are trivial. Later on we shall show that any finitely generated ternary algebra is a ternary subalgebra of some trivial ternary algebra, which is a ternary generalization of Ado's theorem for finite-dimensional Lie algebras.

As we have already mentioned, many notions known in the binary case can be directly generalized to the ternary case. For example, the notion of ternary $\star$–algebra is defined by $[abc]^* = [c^* b^* a^*]$, where the star operation $* : \mathcal{A} \to \mathcal{A}$ is, as it should be, (anti-) linear anti-involution which means $(a^*)^* = a$ and $(ab)^* = b^* a^*$. By the way, the very concept of involution can be generalized so that it becomes adapted to ternary structures. A ternary involution should satisfy $((a^*)^*)^* = a$, as an example, we can introduce the operation $*$ such that $[abc] = [b^* c^* a^*]$. In some applications, an important role is played by ternary algebras with a different associativity law:

$$[[abc]de] = [a[dcb]e] = [ab[cde]]. \tag{4.1.1}$$

Such associativity is sometimes called "type B-associativity" or "2nd kind" [19]. In the case of ternary $\star$–algebras both types of associativity are related to each other. Assuming that $(\mathcal{A}, [\ ], *)$ is a ternary $\star$–algebra, one can introduce another ternary multiplication $[\ ]_*$ such that

$$[abc]_* \stackrel{def}{=} [ab^*c], \quad \forall\ a,\ b,\ c \in \mathcal{A}.$$

The algebra $(\mathcal{A}, [\ ]_*, *)$ becomes an associative ternary $\star$–algebra of B-type. The converse statement is also true: any ternary $\star$–algebra of B-type gives rise to a standard ternary $\star$–algebra. Observe that in the case of algebras over the field of complex numbers one has to assume anti-linearity of ternary multiplication in the middle factor instead of linearity.

Example 4.1.1. Any Hilbert or symmetric scalar product (i.e. metric) vector space $\mathcal{H}$ bears a canonical structure of ternary algebra of B-type with ternary multiplication defined as follows:

$$\{a\, b\, c\} =< a,\, b >\ c$$

induced by scalar multiplication $<, >$ in $\mathcal{H}$. It is not a ternary $\star$–algebra.

In the finite-dimensional case, when we replace the Hilbert space with a metric vector space, in a given basis $\{e_k\}$, $k; m = 1, 2 \dots N$ we can define a non-degenerate metric $g_{ik} =< e_i, e_k >$. Then the Clifford algebra generated by the elements C_i satisfying

$$C_i C_j + C_j C_i = 2 g_{ij} \mathbf{1}$$

provides an appropriate associative algebra which can serve as a representation of the ternary product $\{e_i e_j e_k\} =< e_i, e_j > e_k$ as follows:

$$\{C_i\, C_j\, C_k\} = \frac{1}{2}\left(C_i C_j + C_j C_i\right) C_k$$

Obviously, there are two other possible choices of ternary product in a metric (or Hilbert) space, corresponding to cyclic permutations of three factors:

$$\{a\, b\, c\}' =< b,\, c >\ a, \quad \{a\, b\, c\}" =< c, a > b$$

In a finite-dimensional case, one may define the most general ternary product of this type as a linear combination of these three, i.e.

$$\{e_i\, e_j\, e_k\} = \sum_{l,m,n} M^{lmn}_{ijk} < e_l,\, e_m > \, e_n = \rho^n{}_{ijk}\, e_n \tag{4.1.2}$$

with tensors M^{lmn}_{ijk} symmetric in first two upper indices l, m. The four-index tensor $\rho^n{}_{ijk}$ plays the role of structure constants of our ternary algebra. It is easy to prove that it is impossible to impose strong associativity on such a product, because the set of equations it would imply on the coefficients of the tensor M^{lmn}_{ijk} is strongly over-determined (see ([62]) for example).

As in the usual algebraic case, we can impose particular symmetries on the ternary product, defining a new product displaying a representation property with respect to the permutations of its three lower indices, e.g. by requiring the total symmetry:

$$\{e_i\, e_j\, e_k\}_{sym} = \{e_i\, e_j\, e_k\} + \{e_j\, e_k\, e_i\} + \{e_k\, e_i\, e_j\} =$$

$$< e_i, e_j > \, e_k + < e_j, e_k > \, e_i + < e_k, e_i > \, e_j. \tag{4.1.3}$$

Other choices are possible; for example, a Z_3-generalization of the commutator in associative binary algebras, which generates non-associative Lie algebras, can be introduced as follows:

$$\{e_i\, e_j\, e_k\}_q = \{e_i\, e_j\, e_k\} + q\,\{e_j\, e_k\, e_i\} + q^2\{e_k\, e_i\, e_j\} \tag{4.1.4}$$

with q one of the primitive third roots of unity, $q = e^{\frac{2i\pi}{3}}$, satisfying $q^3 = 1$ and $q + q^2 + q^3 = 0$.

This algebra is particularly simple in dimension two, when there are only two basis vectors. Then the "ternary structure constants" $\rho^i{}_{jkm}$ are as follows:

$$\rho^i{}_{111} = \rho^i{}_{222} = 0;$$

$$\rho^1{}_{221} = q\,\rho^1{}_{212} = q^2\,\rho^1{}_{122} = 1; \quad \rho^2{}_{112} = q\,\rho^2{}_{121} = q^2\,\rho^2{}_{211} = 1$$

$$\rho^2{}_{221} = q\,\rho^2{}_{212} = q^2\,\rho^2{}_{122} = 0; \quad \rho^1{}_{112} = q\,\rho^1{}_{121} = q^2\,\rho^1{}_{211} = 0 \tag{4.1.5}$$

Besides their particular symmetry, the coefficients $\rho^i{}_{jkm}$ possess another interesting property akin to the representation property of antisymmetric structure constants of usual Lie algebras. Let us introduce the following ternary composition law for these coefficients, which can be also named "cubic matrices", with regard to their lower indices:

$$(\rho^i * \rho^j * \rho^k)_{prs} = \sum_{nmt} \rho^i_{npm}\, \rho^j_{mrt}\, \rho^k_{tsn} \tag{4.1.6}$$

Then, introducing the same Z_3-skew-symmetric product as

$$\{\rho^i\, \rho^j\, \rho^k\}_q = (\rho^i * \rho^j * \rho^k) + q\,(\rho^j * \rho^k * \rho^i) + q^2\,(\rho^k * \rho^i * \rho^j), \tag{4.1.7}$$

we can easily check that

$$\{\rho^i\, \rho^j\, \rho^k\}_q = \sum_m \rho^{ijk}_m\, \rho^m \tag{4.1.8}$$

which provides us with a faithful representation of our ternary algebra.

As in the usual case, we are interested to know whether such an algebra can be also represented by certain combinations of ternary products in an ordinary (i.e. binary) associative algebra, playing the role of an enveloping algebra. It is easy to see that the answer is positive. In the above example, the two generators of non-associative ternary algebra with Z_3-skew-symmetric product can be represented by any two Pauli matrices multiplied by the factor $i/2$. One can check that the matrices $\tau_i = \frac{i}{2}\,\sigma_i\,, \quad i = 1, 2$ satisfy

$$q^2 \tau_i \tau_j \tau_k + \tau_j \tau_k \tau_i + q \tau_k \tau_i \tau_j = \sum_m \rho^m_{ijk} \tau_m \tag{4.1.9}$$

4.1.2 Universal envelope of ternary algebra

In order to construct an universal envelope of ternary algebra one should consider the structure of ternary algebras in more detail. In the classical (binary) case, algebras defined by generators and relations between them play important role in concrete applications. It suffices to mention that the well-known Grassmann and Clifford algebras can be defined in this way. Let us briefly summarize this approach (see e.g. [18]). First, we recall that the tensor algebra

$$TV = \oplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \ldots$$

of a given vector space V is a free algebra with n-generators, where $n = dim\, V$. Thus for any subset $S \subset TV$ one can construct a two-sided ideal J_S generated by S and the quotient algebra

$$\mathcal{A}_S = TV/J_S \ .$$

Here V is called a space of generators, S is a set of generating relations. Conversely, by the well-known theorem (see e.g. [18]) any (unital) algebra with n- generators can be obtained in this way. Notice that a non-unital free algebra can be defined as

$$T'V = \oplus_{k=1}^{\infty} V^{\otimes k} = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots \ .$$

Much in the same way, for any vector space V one can construct a free ternary algebra generated by V. To this aim we define

$$T^{\mathrm{odd}}V = \oplus_{k=0}^{\infty} V^{\otimes (2k+1)} = V \oplus V^{\otimes 3} \oplus V^{\otimes 5} \oplus \dots \ . \tag{4.1.10}$$

as a ternary algebra with a ternary multiplication:

$$[uvw]_{\otimes} = u \otimes v \otimes w, \quad \forall\ u,\ v,\ w \in T^{\mathrm{odd}}V \ . \tag{4.1.11}$$

Observe that $T^{\mathrm{odd}}V$ is not a trivial ternary algebra, however it is a ternary subalgebra in the trivial ternary algebra TV (as well as in $T'V$).

$T^{\mathrm{odd}}V$ plays the role of free ternary algebra in the following way. Let $(\mathcal{A}, [\,])$ be a ternary algebra, V any vector space. Then for any linear map $\varphi : V \to \mathcal{A}$, there exists its unique lift $\tilde{\varphi} : T^{\mathrm{odd}}V \to \mathcal{A}$, which is a homomorphism of ternary algebras, such that $\varphi = \tilde{\varphi} \circ \mu$, which means that the following diagram

$$\begin{tikzcd} & T^{\mathrm{odd}}V \arrow[d, "\tilde{\varphi}"] \\ V \arrow[ur, hook, "\mu"] \arrow[r, "\varphi"] & \mathcal{A} \end{tikzcd}$$

is commutative. Here μ denotes the canonical embedding $\mu : V \hookrightarrow T^{\mathrm{odd}}V$.

In particular, if φ is an embedding and $\tilde{\varphi}$ is an epimorphism, then

$$\mathcal{A} \cong T^{\mathrm{odd}}V/Ker(\tilde{\varphi})$$

i.e. the algebra $\mathcal{A}$ becomes isomorphic to the quotient algebra $T^{\mathrm{odd}}V/Ker(\tilde{\varphi})$ Quite obviously, K$er\tilde{\varphi}$ is a ternary ideal in $T^{\mathrm{odd}}V$. The simplest example is provided tautologically by the fact that

$$\mathcal{A} \cong T^{\mathrm{odd}}\mathcal{A}/gen < a \otimes b \otimes c - [abc] >,$$

where $gen < a \otimes b \otimes c - [abc] >$ denotes a ternary ideal generated by elements $\{a \otimes b \otimes c - [abc] : \ a, \ b, \ c \in \mathcal{A}\}$.

Any n-nary algebra can be embedded into a binary one [20]. Here we are particulary interested in the ternary case [19]. For given ternary algebra $\mathcal{A}$ one can defined a $\mathbb{Z}_2$–graded vector space

$$\mathcal{U}_{\mathcal{A}} = \mathcal{A}_1 \oplus \mathcal{A}_0,$$

where $\mathcal{A}_1 = \mathcal{A}$ is an odd part. The even subspace $\mathcal{A}_0$ of $\mathcal{U}_{\mathcal{A}}$ is assumed to be the quotient vector space

$$\mathcal{A}_0 = (\mathcal{A} \otimes \mathcal{A})/span < [xyz] \otimes w - x \otimes [yzw] >$$

where $span < [xyz] \otimes w - x \otimes [yzw] >$ denotes a vector subspace of $\mathcal{A} \otimes \mathcal{A}$ span by elements $\{[xyz] \otimes w - x \otimes [yzw] : x, y, z, w \in \mathcal{A}\}$. Let $a \circledast b$ denote the equivalence class of the element $a \otimes b \in \mathcal{A} \otimes \mathcal{A}$. Now we are in a position to define the multiplication $\overline{\circledast}$ between elements from $\mathcal{U}_{\mathcal{A}}$ by the following:

$$a \overline{\circledast} b \stackrel{def}{=} a \circledast b;$$

$$(a \circledast b) \overline{\circledast} c = a \overline{\circledast} (b \circledast c) \stackrel{def}{=} [abc];$$

$$(a \circledast b) \overline{\circledast} (c \circledast d) \stackrel{def}{=} [abc] \circledast d = a \circledast [bcd] = a \overline{\circledast} ((b \circledast c) \overline{\circledast} d) = (a \overline{\circledast} (b \circledast c)) \overline{\circledast} d \ .$$

In this way, we have obtained a $\mathbb{Z}_2$-graded algebra, since

$$\mathcal{A}_i \overline{\circledast} \mathcal{A}_j \subset \mathcal{A}_{i+j(mod 2)}.$$

It is easy to see that this binary, nonunital algebra is associative. The initial ternary algebra $\mathcal{A} \equiv \mathcal{A}_1$ becomes a ternary subalgebra in the trivial ternary algebra $\mathcal{U}_{\mathcal{A}}$. Of course, $\mathcal{A}_0$ is a (binary) algebra which is also a subalgebra of $\mathcal{U}_{\mathcal{A}}$, and $\mathcal{A}$ becomes a $\mathcal{A}_0$–bimodule. Further on, to simplify the notation, we shall use the same symbol in order to denote the equivalence class $a \circledast b \in \mathcal{A}_0$ corresponding to the elements $a, \ b \in \mathcal{A}$, and for the multiplication $\circledast$ in $\mathcal{U}_{\mathcal{A}}$.

In other words, any ternary algebra can be extended to a binary one, i.e. a ternary algebra is a ternary subalgebra in some associative binary algebra. Furthermore, if $\phi : \mathcal{A} \to \mathcal{B}$ is a ternary homomorphism of $\mathcal{A}$ into an associative algebra $\mathcal{B}$, i.e. $\mathcal{A}$ is a ternary subalgebra in a binary algebra $\mathcal{B}$, then there exists one and only one (binary) algebra homomorphism $\tilde{\phi} : \mathcal{U}_{\mathcal{A}} \to \mathcal{B}$ such that $\phi = \tilde{\phi} \circ \iota$, where ι is the canonical embedding of $\mathcal{A}$ into $\mathcal{U}_{\mathcal{A}}$, i.e. the following diagram:

$$\begin{tikzcd} & \mathcal{U}_{\mathcal{A}} \arrow[d, "\tilde{\phi}"] \\ \mathcal{A} \arrow[ur, hook, "\iota"] \arrow[r, "\phi"'] & \mathcal{B} \end{tikzcd}$$

is commutative, and this universal property characterizes $(\iota\,, \mathcal{U}_{\mathcal{A}})$ up to an isomorphism. For example, $\mathcal{U}_{T^{\mathrm{odd}}V} = T'V$, i.e.

$$T'V = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus V^{\otimes 4} \oplus \ldots\ = T^{\mathrm{odd}}V \oplus T^{\mathrm{even}}V,$$

is the enveloping algebra, in which the ternary algebra $(T^{\mathrm{odd}}V, [\ ]_{\otimes})$ can be embedded.

4.1.3 Tri-modules over ternary algebras

The concept of tri-module is a particular case of the concept of module over an algebra over an operad defined in [37]. In the more general context of n-ary algebras it was then considered in [38]. Here, a structure of tri-module over a ternary algebra $\mathcal{A}$ is simply defined on a vector space $\mathcal{M}$ by the following three linear mappings called

$$\begin{array}{cc} \text{left} & [\ ]_L : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}, \\ \text{right} & [\ ]_R : \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{M}, \\ \text{and central} & [\ ]_C : \mathcal{A} \otimes \mathcal{M} \otimes \mathcal{A} \to \mathcal{M} \end{array}$$

multiplication, respectively (see also [19, 54]). They are assumed to satisfy the following compatibility conditions

$$[ab[cdm]_L]_L = [[abc]dm]_L = [a[bcd]m]_L, \tag{4.1.12}$$
$$[[mab]_R cd]_R = [ma[bcd]]_R = [m[abc]d]_R, \tag{4.1.13}$$
$$[a[b[cmx]_C y]_C z]_C = [[abc]m[xyz]]_C, \tag{4.1.14}$$
$$[a[bcm]_L d]_C = [ab[cmd]_C]_L = [[abc]md]_C, \tag{4.1.15}$$
$$[a[mbc]_R d]_C = [[amb]_C cd]_R = [am[bcd]]_C, \tag{4.1.16}$$

$$[[abm]_L cd]_R = [ab[mcd]_R]_L = [a[bmc]_C d]_C, \quad (4.1.17)$$
$$\forall\ a,\ b,\ c,\ d,\ x,\ y,\ z \in \mathcal{A},\ m \in \mathcal{M}$$

In the case of tri-module $\mathcal{M}$ over an algebra $\mathcal{A}$ of type B the conditions (4.1.12), (4.1.13), (4.1.17) remain unchanged while (4.1.14), (4.1.15), (4.1.16) have to be replaced correspondingly by

$$[a[b[cmx]_C y]_C z]_C = [[ayc]m[xbz]]_C, \quad (4.1.18)$$
$$[a[cbm]_L d]_C = [[amb]_C cd]_R, \quad (4.1.19)$$
$$[a[mbc]_R d]_C = [ab[cmd]_C]_L, \quad (4.1.20)$$
$$\forall\ a,\ b,\ c,\ d,\ x,\ y,\ z \in \mathcal{A},\ m \in \mathcal{M}.$$

Much in the same way as binary algebra is a trivial ternary algebra, the notion of tri-module generalizes the notion of bimodule. More exactly, one has (have to see paper!)

Remark 4.1.2. Let $\mathcal{A}$ be a (binary) algebra and $\mathcal{M}$ a bimodule over it. Thus defining $[abm]_L = a\cdot(b\cdot m) = (a\cdot b)\cdot m$, $[amb]_C = a\cdot(m\cdot b) = (a\cdot m)\cdot b$ and $[mab]_R = m\cdot(a\cdot b) = (m\cdot a)\cdot b$ we see that $\mathcal{M}$ becomes a tri-module over the same algebra considered as a trivial ternary algebra.

Analogously, we can obtain an enveloping module $\mathcal{U}_\mathcal{M}$ over the enveloping algebra $\mathcal{U}_\mathcal{A}$ of ternary module $\mathcal{M}$ over ternary algebra $\mathcal{A}$. Let us denote by $\mathcal{U}_\mathcal{M}$ a $\mathbb{Z}_2$-graded vector space

$$\mathcal{U}_\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_0, \quad (4.1.21)$$

where the odd part $\mathcal{M}_1 \equiv \mathcal{M}$. The even part is defined as a quotient vector space

$$\mathcal{M}_0 = (\mathcal{A}\otimes\mathcal{M}\oplus\mathcal{M}\otimes\mathcal{A})/lin < S >,$$

where S is a set of elements in $\mathcal{A}\otimes\mathcal{M}\oplus\mathcal{M}\otimes\mathcal{A}$ $(a,\ b,\ c, \in \mathcal{A},\ m\ \in \mathcal{M}.)$

$$[abc]\otimes m - a\otimes[bcm]_L,$$
$$[abm]_L\otimes c - a\otimes[bmc]_C,$$
$$[amb]_C\otimes c - a\otimes[mbc]_R,$$
$$[mab]_R\otimes c - m\otimes[abc],$$

which generate the subspace $lin < S >$.

As previously, denote by $a \circledast m$ or $m \circledast a$ the corresponding equivalence classes, elements of $\mathcal{M}_0$. Define left and right multiplication $\overline{\circledast}$ between elements from $\mathcal{U}_{\mathcal{A}}$ and those from $\mathcal{U}_{\mathcal{M}}$ in the following way:

$$a \overline{\circledast} m \stackrel{def}{=} a \circledast m;$$

$$m \overline{\circledast} a \stackrel{def}{=} m \circledast a;$$

$$(a \circledast b) \overline{\circledast} m = a \overline{\circledast} (b \circledast m) \stackrel{def}{=} [abm]_L;$$

$$m \overline{\circledast} (c \circledast d) = (m \circledast c) \overline{\circledast} d \stackrel{def}{=} [mcd]_R;$$

$$a \overline{\circledast} (m \circledast b) \stackrel{def}{=} [amb]_C; \qquad (a \circledast m) \overline{\circledast} b \stackrel{def}{=} [amb]_C;$$

$\forall\ a,\ b,\ c,\ d,\ e,\ f\ \in \mathcal{A},\ m\ \in \mathcal{M}$.

One can check the following properties of the action of algebra $\mathcal{A}$ on module $\mathcal{M}$

$$[abc] \circledast m = a \circledast [bcm]_L;$$

$$m \circledast [bcd] = [mbc]_R \circledast d;$$

$$[abc] \circledast (m \circledast d) = [ab[cmd]_C]_L;$$

$$(a \circledast m) \circledast [bcd] = [[amb]_C cd]_R;$$

$$(a \circledast [bcd]) \circledast m = ([abc] \circledast d) \circledast m = [abc] \circledast (d \circledast m) = [ab[cdm]_L]_L;$$

$$m \circledast ([cde] \circledast f) = m \circledast (c \circledast [def]) = (m \circledast c) \circledast [def] = [[mcd]_R ef]_R;$$

$$(a \circledast [bcd]) \circledast (m \circledast e) = ([abc] \circledast d) \circledast (m \circledast e) = [abc] \circledast [dme]_C;$$

$$(a \circledast m) \circledast ([bcd] \circledast e) = (a \circledast m) \circledast (b \circledast [cde]) = [[amb]_C \circledast [cde];$$

Thus $\mathcal{U}_{\mathcal{M}}$ becomes a $\mathbb{Z}_2$-graded bimodule over $\mathcal{U}_{\mathcal{A}}$ sinces

$$\mathcal{A}_i \overline{\circledast} \mathcal{M}_j \subseteq \mathcal{M}_{i+j(mod2)}, \quad \mathcal{M}_j \overline{\circledast} \mathcal{A}_i \subseteq \mathcal{M}_{i+j(mod2)}, \quad i,\ j \in \{0,1\}.$$

In particular, $\mathcal{M} \equiv \mathcal{M}_1$ and $\mathcal{M}_0$ are $\mathcal{A}_0$-bimodules.

Further on, we shall use the same symbol $\circledast$ to denote the equivalence class in $\mathcal{U}_{\mathcal{M}}$, its bimodule structure and for the multiplication in $\mathcal{U}_{\mathcal{A}}$.

Let us stress again that any bimodule over a (binary) algebra becomes automatically a trimodule over the same algebra considered as a trivial ternary algebra.

What we have shown above is that any trimodule is a sub-trimodule of some universal bimodule $\mathcal{U}_{\mathcal{M}}$ over $\mathcal{U}_{\mathcal{A}}$. Conversely, if $\mathcal{N}$ is a $\mathbb{Z}_2$–graded bimodule over $\mathcal{U}_{\mathcal{A}}$, then its odd part $\mathcal{N}_1$ is a trimodule over $\mathcal{A}$.

4.2 Universal differentiation of ternary algebra

A first order differential calculus (differential calculus in short) of ternary algebra $\mathcal{A}$ is a linear map from ternary algebra a into tri-module over it, i.e. $d : \mathcal{A} \to \mathcal{M}$, such that a ternary analog of the Leibniz rule takes place:

$$d([f\,g\,h]) = [df\,g\,h]_R + [f\,dg\,h]_C + [f\,g\,dh]_L, \quad \forall f, g, h \in \mathcal{A}. \qquad (4.2.1)$$

In particular, if $\mathcal{M} = \mathcal{A}$, then we shall call so defined differential *ternary derivation* of $\mathcal{A}$. An interesting example is provided by

Example 4.2.1. Ternary derivative in Hilbert (or metric) vector space.
As we already noticed, any Hilbert space $(\mathcal{H}, <,>)$ inherits a canonical ternary 2nd type associative structure given by $\{a\,b\,c\} =< a, b > c$.
For a linear operator being a ternary derivation $D : \mathcal{H} \to \mathcal{H}$ one calculates:

$$D\{a\,b\,c\} = \{Da\,b\,c\} + \{a\,Db\,c\} + \{a\,b\,Dc\},$$

Now, taking into account that $D\{a\,b\,c\} =< a, b > Dc = \{a\,b\,Dc\}$ it implies

$$< Da, b >= - < a, Db > \Rightarrow\ D^+ = -D, \ \ i.e.\ \ (iD)^+ = iD$$

i.e, that ternary derivations are in one-to-one correspondence with hermitian operators in $\mathcal{H}$. This makes possible a link with Quantum Mechanics, especially the version introduced by Nambu ([55]).

Let us refer again to the classical (binary) case. First order differential calculus from an algebra into bimodule can be automatically interpreted as a ternary differential calculus from trivial ternary algebra into a trivial tri-module over it. It can be easily seen from

$$d(fgh) = d((fg)h) = d(fg)\,h + fg\,d(h) = df\,gh + f\,dg\,h + fg\,dh\ .$$

The converse statement is, in general, not true. A ternary Leibniz rule for differential calculus from an algebra into bimodule does not necessarily imply, in case of non–unital algebras, the existence of a standard (binary) Leibniz rule. In particular, the set of ternary derivations of non-unital algebra should be an extension of the set of standard (binary) derivations.

Let $(\mathcal{A}, \mathcal{M}, d)$ be our ternary differential calculus from ternary algebra into tri-module. By the Leibniz rule

$$\tilde{d}(a \circledast b) = (da) \circledast b + a \circledast (db)$$

it can be uniquely extended to a 0-degree differential $\tilde{d} : \mathcal{U}_{\mathcal{A}} \to \mathcal{U}_{\mathcal{M}}$, in a way which ensures commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "d"] \arrow[d, "\iota"] & \mathcal{M} \arrow[d, "\iota'"] \\ \mathcal{U}_{\mathcal{A}} \arrow[r, "\tilde{d}"] & \mathcal{U}_{\mathcal{M}} \end{tikzcd}$$

Conversely, any 0-degree first order differential calculus from $\mathcal{U}_{\mathcal{A}}$ into $\mathcal{U}_{\mathcal{M}}$, such that $\tilde{d}\mid_{\mathcal{A}} \subset \mathcal{M}$ gives rise to ternary $\mathcal{M}$-valued differential calculus on $\mathcal{A}$.

The universal first order differential calculus on non-unital algebras is well describe in [24, 25, 41]. Let us recall this construction shortly. Determine a vector space $\Omega^1_u(\tilde{\mathcal{A}}) = \hat{\mathcal{A}} \oplus \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$, where $\hat{\mathcal{A}}$ is a non-unital (binary) algebra. Any element from $\Omega^1_u(\tilde{\mathcal{A}})$ can be written in the form: $(a, b \otimes c),\ where\ a,\ b,\ c\ \in \hat{\mathcal{A}}$. Define left and right multiplications by elements from $\hat{\mathcal{A}}$:

$$\begin{aligned} x(a, b \otimes c) &= (0, x \otimes a + xb \otimes c), \\ (a, b \otimes c)y &= (ay, -a \otimes y + b \otimes cy - bc \otimes y). \end{aligned}$$

In this way, $\Omega^1_u(\tilde{\mathcal{A}})$ becomes a $\hat{\mathcal{A}}$-bimodule since

$$(x(a, b \otimes c))y = x((a, b \otimes c)y).$$

Let $D : \hat{\mathcal{A}} \to \hat{\mathcal{A}} \oplus \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$, $Da = (a, 0), \quad \forall\, a \in \hat{\mathcal{A}}$ be a canonical embedding. Because it satisfies the Leibniz rule:

$$D(a)b + aD(b) = (a, 0)b + a(b, 0) = (ab, -a \otimes b) + (0, a \otimes b) = (ab, 0) = D(ab),$$

D is a differential. We shall call it the universal differential for a given algebra $\hat{\mathcal{A}}$.

For unital algebras, there exists an alternative construction of $\Omega^1_u(\tilde{\mathcal{A}})$ as a kernel of multiplication map [22] (see also [18]). Since our ternary algebras have no unit element, we can not use such construction here.

Our aim is to provide an analogous construction in the case of ternary algebra. From a $\mathbb{Z}_2$-graded $\mathcal{U}_{\mathcal{A}}$-bimodule $\Omega^1_u(\mathcal{U}_{\mathcal{A}}) = \mathcal{U}_{\mathcal{A}} \oplus \mathcal{U}_{\mathcal{A}} \otimes \mathcal{U}_{\mathcal{A}}$, let us extract its odd subspace $\mathcal{A} \oplus \mathcal{A}_0 \otimes \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A}_0$ with elements

$$(a, \beta \otimes b, c \otimes \gamma),\ \forall\ a,\ b,\ c,\ \in \mathcal{A},\ \beta,\ \gamma \in \mathcal{A}_0.$$

We shall denote it as $\Omega^1_T(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A}_0 \otimes \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A}_0$. As we already know from our previous considerations, $\Omega^1_T(\mathcal{A})$ is a tri-module over $\mathcal{A}$. Thus we have defined the left, central and right ternary multiplications

$$[xy(a, \beta \otimes b, c \otimes \gamma)]_L = (0, (x \circledast y) \otimes a + (x \circledast [y\beta]) \otimes b, [xyc] \otimes \gamma);$$

$$[x(a, \beta \otimes b, c \otimes \gamma)y]_C = (0, -(x \circledast a) \otimes y - ([x\beta] \circledast \gamma) \otimes y + (x \circledast c) \otimes [\gamma y] -$$

$$(x \circledast [c\gamma]) \otimes y, x \otimes (a \circledast y) + [x\beta] \otimes (b \circledast y)); \tag{4.2.2}$$

$$[(a, \beta \otimes b, c \otimes \gamma)xy]_R =$$

$$([axy], \beta \otimes [bxy], -a \otimes (x \circledast y) - [\beta b] \otimes (x \circledast y) + c \otimes ([\gamma x] \circledast y) - [c\gamma] \otimes (x \circledast y)).$$

The canonical embedding $D : \mathcal{A} \to \Omega^1_T \mathcal{A}$:

$$D(a) = (a, 0, 0), \quad \forall\ a \in \mathcal{A}. \tag{4.2.3}$$

defines a ternary differential (4.2.1). In fact, one has

$$[(a, 0, 0)bc]_R + [a(b, 0, 0)c]_C + [ab(c, 0, 0)]_L =$$
$$([abc], 0, -a \otimes (b \circledast c)) + (0, -(a \circledast b) \otimes c, a \otimes (b \circledast c)) + (0, (a \circledast b) \otimes c, 0) =$$
$$([abc], 0, 0).$$

This ternary differential calculus is universal because for any trimodule E and any ternary E-valued differential calculus $d : \mathcal{A} \to E$, there exists one and only one covering trimodule homomorphism $\tilde{\varphi}_d$ such that $d = \tilde{\varphi}_d \circ D$, i.e. the following diagram

$$\begin{tikzcd} & \Omega^1_T(\mathcal{A}) \arrow[d, "\tilde{\varphi}_d"] \\ \mathcal{A} \arrow[ur, hook, "D"] \arrow[r, "d"] & E \end{tikzcd}$$

is commutative. Moreover, if the trimodule E is spanned by the elements $d\mathcal{A}$, $[\mathcal{A}\ d\mathcal{A}\ \mathcal{A}]_C$ and $[d\mathcal{A}\ \mathcal{A}\ \mathcal{A}]_R$, then $\tilde{\varphi}_d$ is an epimorphism and $E = \Omega^1_T(\mathcal{A})/Ker(\tilde{\varphi}_d)$.

In this way, the problem of classification of all first order differential calculi over $\mathcal{A}$ can be translated into the problem of classification of all subtrimodules in $\Omega^1_T(\mathcal{A})$. Remember that $\Omega^1_T(\mathcal{A})$ is an odd part of $\Omega^1_u(\mathcal{U}_\mathcal{A})$ and our ternary differential (4.2.3) is, in fact, a restriction of the universal differential.

As it is well known [24, 25, 41] the bimodule $\Omega^1_u(\mathcal{U}_\mathcal{A})$ extends, by means of the graded Leibniz rule, to the universal graded differential algebra with $d^2 = 0$. This leads to higher order differential calculi. Another universal extension with $d^N = 0$, still for the case of binary (unital) algebras, has been considered in [27, 30]. These universal extensions have been provided by means of the $q-$Lebniz rule, for q being primitive $N-$ degree root of the unity, i.e. $q = e^{\frac{2\pi i}{N}}$ (see also [40] in this context). However, the so-called $N-$ary case ($d^N = 0$) seems also to be specially well adopted for $N-$ary algebras. Various constructions of higher order differentials for ternary algebras will be a subject of our future investigation.

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Kokkuvõte

Diferentsiaalarvutus $d^3 = 0$ binaarsetel ja ternaarsetel assotsiatiivsetel algebratel

Antud väitekiri on pühendatud mittekommutatiivse geomeetria raames tekkinud diferentsiaalarvutusele diferentsiaaliga d, mis rahuldab tingimust $d^3 = 0$. Antud diferentsiaalarvutus tugineb gradueeritud Q-diferentsiaalalgebra mõistele, kus Q on kuupjuur ühest. Gradueeritud Q-diferentsiaalalgebrat võib vaadelda gradueeritud diferentsiaalalgebra üldistusena, kusjuures gradueeritud diferentsiaalalgebrad mängivad tähtsat rolli nii kaasaegses diferentsiaalgeomeetrias kui ka väljateooriates.

Esimeses peatükis on antud esimest järku diferentsiaalarvutuse assotsiatiivsetel algebratel detailne kirjeldus. Esimest järku diferentsiaalarvutus assotsiatiivsel ühikuga algebral on kolmik $(\mathcal{A}, \mathcal{M}, d)$, kus $\mathcal{A}$ on assotsiatiivne ühikuga algebra, $\mathcal{M}$ on $\mathcal{A}$-bimoodul ja d on diferentsiaal, mis rahuldab Leibnizi valemit. Esimeses peatükis vaadeldakse koordinaatdiferentsiaalarvutust, mis tekib juhul, kui algebra $\mathcal{A}$ on lõplikult tekitatud algebra ja $\mathcal{A}$-bimoodul $\mathcal{M}$ kui parempoolne $\mathcal{A}$-moodul on vaba. Näidatakse, et koordinaatdiferentsiaalarvutuse korral diferentsiaal d tekitab parempoolsed osatuletised algebral $\mathcal{A}$, mis rahuldavad teatud homomorfismiga deformeeritud Leibnizi valemit. Antakse universaalse esimest järku diferentsiaalarvutuse ja gradueeritud diferentsiaalalgebra struktuuri detailne kirjeldus. Teises peatükis uuritakse gradueeritud diferentsiaalalgebra üldistust, mis tekkis mittekommutatiivse geomeetria raames 1990-ndatel aastatel. Seda üldistust nimetatakse gradueeritud q-diferentsiaalalgebraks ja teises peatükis antakse selle üldistuse definitsioon. Gradueeritud q-diferentsiaalalgebra diferentsiaal d rahuldab tingimust $d^N = 0$, kus $N \geq 2$. Teises peatükis on vaadeldud esimest mittetriviaalset üldistust, kus q on kuupjuur ühest ja diferentsiaal rahuldab $d^3 = 0$. Gradueeritud Q-diferentsiaalalgebra rakendustes

mittekommutatiivses geomeetrias ja teoreetilises füüsikas on tähtis probleem, kuidas konstrueerida gradueeritud q-diferentsiaalalgebra, kui on antud esimest järku diferentsiaalarvutus assotsiatiivsel algebral. Teises peatükis näidatakse, kuidas konstrueerida gradueeritud Q-diferentsiaalalgebrat, kui on antud esimest järku diferentsiaalarvutus algebral, mis on lõplikult tekitatud teatuid seoseid rahuldavate moodustajate poolt. Kolmandas peatükis rakendatakse teises peatükis välja töötatud meetodit diferentsiaalvormide algebra analoogi konstrueerimiseks järgmistel ruumidel: anioonilisel (anyonic) ühedimensionaalsel ruumil, ruutalgebral, h-deformeeritud kvanttasandil, q-deformeeritud kvanttasandil. On antud diferentsiaalvormide algebra analoogi detailne kirjeldus, kusjuures on leitud kõik kommutatsiooniseosed koordinaatide ja nende diferentsiaalide vahel. Ühedimensionaalse ruumi korral on näidatud, et diferentsiaalvormide algebra struktuuri erinevad komponendid on kooskõlas, kui ühedimensionaalne ruum on aniooniline. On näidatud, et anioonilise ruumi korral diferentsiaalvormide algebra analoog sisaldab esimest järku diferentsiaalarvutust, mida kasutatakse murdsupersümmeetrilistes teooriates.
Viimases peatükis vaadeldakse diferentsiaalarvutust ternaarsetel algebratel. On antud ternaarse algebra struktuuri ja sellega seotud mõistete (ternaarne assotsiatiivsus, involutsioon, struktuurikonstandid) lühikirjeldus. On defineeritud $\mathcal{A}$-mooduli mõiste üle ternaarse algebra. Moodulit üle ternaarse algebra nimetatakse trimooduliks. Tuuakse sisse ternaarse algebra derivatsiooni mõiste ja uuritakse selliste derivatsioonide ruumi struktuuri. Konstrueeritakse ternaarseid diferentsiaalalgebraid ja universaalne diferentsiaalarvutus ternaarsel algebral.

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