

STEFANO CIACI

Transfinite geometric properties  
of the unit ball in Banach spaces





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of the unit ball in Banach spaces



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# Contents

<b>1</b>	<b>Introduction</b>	<b>11</b>
1.1	Background and motivation of the thesis . . . . .	11
1.2	Notation . . . . .	12
1.3	Summary of the thesis . . . . .	13
1.3.1	Chapter 2: Almost square Banach spaces . . . . .	13
1.3.2	Chapter 3: The symmetric strong diameter two property . . . . .	13
1.3.3	Chapter 4: The strong diameter two property . . . . .	14
1.3.4	Chapter 5: Ball-covering properties . . . . .	15
1.3.5	Overview . . . . .	15
1.4	Preliminaries . . . . .	16
1.4.1	Set theoretical topology . . . . .	16
1.4.2	Banach space theory . . . . .	17
<b>2</b>	<b>ASQ Banach spaces</b>	<b>25</b>
2.1	Definition and some examples . . . . .	26
2.1.1	When $\kappa$ is $\aleph_0$ . . . . .	27
2.1.2	When $\kappa$ is uncountable . . . . .	28
2.2	Stability results . . . . .	30
2.2.1	Direct sums . . . . .	30
2.2.2	Tensor products . . . . .	32
2.2.3	Ultraproducts . . . . .	33
2.3	Renorming results . . . . .	34
2.4	Locally almost square Banach lattices . . . . .	40
2.4.1	Introduction . . . . .	40
2.4.2	Positive LASQ and weakly null nets . . . . .	42
2.4.3	Positive LASQ and the diameter two property . . . . .	44
2.5	Open problems . . . . .	46
<b>3</b>	<b>The SSD2P</b>	<b>47</b>
3.1	Definition and some examples . . . . .	47
3.2	Stability results . . . . .	50

3.2.1	Direct sums . . . . .	51
3.2.2	Projective tensor products . . . . .	53
3.3	Transfinite versus finite . . . . .	54
3.3.1	The implication (ii) $\iff$ (iii) . . . . .	54
3.3.2	The implication (i) $\iff$ (ii) . . . . .	55
3.3.3	Final remarks . . . . .	56
3.4	Open problems . . . . .	58
<b>4</b>	<b>The SD2P</b>	<b>59</b>
4.1	Introduction . . . . .	60
4.1.1	Norm-additive sets . . . . .	60
4.1.2	Some important notation . . . . .	61
4.1.3	Intermezzo about octahedral norms . . . . .	63
4.1.4	Redefining the SD2P . . . . .	65
4.2	Definition and a first examples . . . . .	67
4.3	Duality . . . . .	69
4.3.1	When $\kappa$ is at most $\aleph_2$ . . . . .	71
4.4	Stability results . . . . .	72
4.4.1	Direct sums . . . . .	73
4.4.2	Tensor products . . . . .	78
4.5	More examples . . . . .	79
4.5.1	$C(K)$ spaces . . . . .	79
4.5.2	$L_1(\mu)$ spaces . . . . .	81
4.5.3	Lebesgue-Bochner spaces . . . . .	82
4.6	Open problems . . . . .	86
<b>5</b>	<b>Ball-covering properties</b>	<b>87</b>
5.1	Some important notation . . . . .	87
5.2	Definition and some examples . . . . .	89
5.3	Renorming results . . . . .	89
5.4	Stability results . . . . .	91
5.4.1	Absolute direct sums . . . . .	92
5.4.2	Infinite $\ell_\infty$ and $\ell_1$ sums . . . . .	93
5.5	More examples . . . . .	96
5.5.1	$\text{ROH}_{\aleph_1}$ is distinct from failing the $\alpha$ -BCP $_\kappa$ . . . . .	96
5.5.2	$\text{OH}_\kappa$ is distinct from failing the $\alpha$ -BCP $_{\aleph_1}$ . . . . .	97
	<b>Bibliography</b>	<b>99</b>
	<b>Kokkuvõte (Summary in Estonian)</b>	<b>106</b>

<i>CONTENTS</i>	7
<b>Curriculum Vitae</b>	<b>108</b>
<b>Elulookirjeldus</b>	<b>109</b>
<b>List of original publications</b>	<b>110</b>



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# Chapter 1

## Introduction

### 1.1 Background and motivation of the thesis

In an infinite-dimensional Banach space, one might encounter extreme geometric phenomena in which every slice or every relatively weakly open subset of its unit ball has a diameter of two. This happens, for instance, in infinite-dimensional uniform algebras [NW], Banach spaces with the Dugavet property [Shv], infinite-dimensional  $C^*$ -algebras [BLR2], and nonreflexive  $M$ -embedded spaces [Lop]. This can never happen in a reflexive Banach space (e.g, in a finite-dimensional space) because these spaces have the Radon-Nikodým property, which allows for the existence of slices with arbitrarily small diameters.

A common theory of diameter two properties began from the paper [ALN1], where T. A. Abrahamsen, V. Lima, and O. Nygaard systematically surveyed previous research and introduced three versions of the diameter two properties. A Banach space is said to have the *local diameter two property* (respectively, *diameter two property*, *strong diameter two property*) if every slice (respectively, every nonempty relatively weakly open subset, every finite convex combination of slices) of the unit ball has diameter two. Dual characterizations in terms of octahedral norms for these diameter two properties were obtained in the subsequent papers [BLR3] and [HLP1].

From there on, various strengthenings of diameter two properties and their relatives have emerged. For example, a Banach space is said to have the *symmetric strong diameter two property* if, given a finite number of slices of the unit ball, there exists a direction such that all these slices contain a line segment of length almost 2 in this direction [ANP]. Or, a Banach space is said to be *almost square* if for every finite set  $x_1, \dots, x_n$  in the unit sphere there is a unit vector  $y$  such that  $\|x_i - y\|$  is almost one for every  $1 \leq i \leq n$

[ALL].

A common property for all of these notions above is that they are *finitely defined*; that is, the definitions use that for every finite number of elements in a Banach space or its dual, there is some special element in the space or the dual. Such geometric properties were formalized by J. D. Hardtke as test families in [Har2], where he used it to show that if the property in question is stable under certain finite absolute sums, then it is also stable under the formation of corresponding Köthe-Bochner spaces.

Surprisingly, many classical Banach spaces enjoy even a transfinite analogue of the diameter two properties or their relatives. For example, the dual of every Banach space with the Daugavet property is octahedral with respect to separable subspaces [KSSW].

The main aim of the thesis is to systematically study transfinite extensions of diameter two properties, almost squareness, and octahedral norms. For some properties like almost squareness and octahedral norms, the idea of extending the notion to an infinite setting is straightforward. For properties like the strong diameter two property, this requires equivalently redefining the original property, so it could be extended to uncountable cardinals too. In general, the transfinite case exhibits significantly distinct behavior and is technically more complex. Hence, it gives new fruitful results and examples that complement the existing theory of spaces with the diameter two properties.

## 1.2 Notation

Given a real Banach space  $X$ , we call  $B_X$  and  $S_X$  its unit ball and sphere, respectively. We call  $X^*$  its topological dual and the remaining notation used is standard. Moreover, we call  $\mathcal{P}_\kappa(A)$  the set of all subsets of  $A$  with cardinality less than  $\kappa$ . For any undefined notation or missing definition, the reader is referred to [AB] for Banach space theory and [Jec] for set theory. It should be noted that some transfinite properties might be denoted differently from how they were originally introduced in the literature. Specifically, certain properties  $P_\kappa$ , which depend on a cardinal  $\kappa$ , were initially defined as:

$$P_\kappa : \iff (\forall A \in \mathcal{P}_{\kappa^+}(X)) \sigma(A, X),$$

where  $\sigma$  represents a particular statement. In this paper, however, we aim to standardize the notation. Hence, we will define  $P_\kappa$  consistently as:

$$P_\kappa : \iff (\forall A \in \mathcal{P}_\kappa(X)) \sigma(A, X).$$

This approach ensures uniformity across discussions involving transfinite properties.

## 1.3 Summary of the thesis

### 1.3.1 Chapter 2: Almost square Banach spaces

A Banach space  $X$  is said to be *almost square* (ASQ) whenever, for any finite-dimensional subspace  $Y \subset X$  and  $\varepsilon > 0$ , there exists  $x \in S_X$  such that the inequality

$$\|y + rx\| \leq (1 + \varepsilon) \max\{\|y\|, |r|\}$$

holds for all real numbers  $r$  and all  $y \in Y$  [ALL]. In this chapter, we investigate transfinite ASQ Banach spaces, providing examples, analyzing their geometrical characteristics, and studying their stability under direct sums, tensor products, and ultraproducts. We also explore the relationship between the presence of isomorphic copies of  $c_0(\kappa)$  and the existence of equivalent transfinite ASQ norms. In particular, the following result is the highlight of the chapter.

**Theorem.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. If  $X$  is  $\text{SQ}_\kappa$ , then it contains an isomorphic copy of  $c_0(\kappa)$ . Suppose additionally that  $\kappa$  has uncountable cofinality and  $d(X) = \kappa$ , then  $X$  admits an equivalent  $\text{SQ}_{\text{cf}(\kappa)}$  norm whenever it contains an isomorphic copy of  $c_0(\kappa)$ .*

Additionally, we conclude the chapter with some remarks on locally almost square (LASQ) Banach lattices in which we prove the following result.

**Theorem.** *If  $X$  is a positive LASQ Banach lattice, then, for every  $x \in S_X$ , there exists a weakly null net  $(y_\alpha)$  in  $S_X$  satisfying  $\lim \|x \pm y_\alpha\| = 1$ .*

### 1.3.2 Chapter 3: The symmetric strong diameter two property

A Banach space  $X$  has the *symmetric strong diameter two property* (SSD2P) if and only if for every  $n \in \mathbb{N}$ , for all slices  $S_1, \dots, S_n$  of  $B_X$ , and  $\varepsilon > 0$ , there are  $x_1, \dots, x_n, y \in B_X$  such that

(a)  $\|y\| \geq 1 - \varepsilon$ ,

(b)  $\|x_i \pm y\| \leq 1$ ,

(c)  $x_i \in S_i$

hold for  $1 \leq i \leq n$  [ANP]. In this chapter, we examine transfinite analogues of the SSD2P. We provide numerous examples, assess the stability of these properties when subjected to direct sums, and under projective tensor products, while contrasting these transfinite properties with their classical counterparts. It is known that the SSD2P admits the following equivalent characterization [HLLN, Theorem 2.1].

**Theorem.** *A Banach space  $X$  has the SSD2P if and only if, given  $n \in \mathbb{N}$ , non-empty relatively weakly open sets  $U_1, \dots, U_n$  in  $B_X$ , and  $\varepsilon > 0$ , there exist  $x_1, \dots, x_n, y \in B_X$  such that  $\|y\| \geq 1 - \varepsilon$ ,  $x_i \pm y \in B_X$  and  $x_i \in U_i$  for all  $1 \leq i \leq n$ .*

We will demonstrate that the transfinite case exhibits significantly distinct behavior. The central role in substantiating this claim will be served by the following result.

**Theorem.** *Let  $\{X_n : n \in \mathbb{N}\}$  be a family of Banach spaces. If, given any countable family of slices  $\mathcal{S}$  in  $B_{c_0(\mathbb{N}, X_n)}$  and  $\varepsilon > 0$ , there exist a set  $B$  and  $y$  in  $B_{c_0(\mathbb{N}, X_n)}$  such that  $\|y\| \geq 1 - \varepsilon$ ,  $B \pm y \subset B_{c_0(\mathbb{N}, X_n)}$  and  $B$  visits every slice in  $\mathcal{S}$ , then at least one of the  $X_n$ 's is not uniformly convex.*

### 1.3.3 Chapter 4: The strong diameter two property

A Banach space  $X$  enjoys the *strong diameter two property* (SD2P) if and only if every convex combination of slices of the unit ball has diameter two [ALN1]. This chapter is devoted to the transfinite analogues of the SD2P. It is clear that the concept of convex combination of slices poorly extends to uncountable cardinals, hence the first part of the chapter is dedicated to equivalently redefine the SD2P.

**Theorem.** *A Banach space  $X$  enjoys the SD2P if and only if, for every finite set  $A$  in  $S_{X^*}$  and  $\lambda \in (0, 1)$ , there are  $B \subset S_X$  and  $x^* \in S_{X^*}$  such that  $B$   $\lambda$ -norms  $A$  and  $x^*(x) \geq \lambda$  holds for all  $x \in B$ .*

Through this equivalent definition, we extend the SD2P to its transfinite analogue and investigate its dualities. Among others, we will prove the following result.

**Theorem.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. Then  $X$  has the SD2P $_{\kappa}$  if and only if  $X^*$  is OH $_{\kappa}$ .*

Here, by OH $_{\kappa}$ , we mean the transfinite extension of the notion of an octahedral norm defined in an unpublished paper by Godefroy and Maurey

in order to characterize those Banach spaces that contain isomorphic copies of  $\ell_1$ . For the reader's convenience, let us recall that a Banach space  $X$  is called *octahedral* (OH) if, for every finite-dimensional subspace  $Y \subset X$  and  $\varepsilon > 0$ , there exists  $x \in S_X$  such that

$$\|y + rx\| \geq (1 - \varepsilon)(\|y\| + |r|)$$

holds for every  $y \in Y$  and  $r \in \mathbb{R}$ . The chapter continues by investigating the stability of the transfinite SD2P under direct sums and projective tensor products. Eventually, the chapter ends by providing numerous examples, focusing in particular on continuous function spaces and  $L_1$  spaces.

### 1.3.4 Chapter 5: Ball-covering properties

This last chapter aims to extend the characterization of OH Banach spaces via the containment of  $\ell_1$  to the transfinite setting. In order to do so, we investigate transfinite extensions of the ball-covering property introduced by Cheng [Che], which is a close relative to the concept of an OH norm. For the reader's convenience, let us recall that a Banach space is said to enjoy the *ball-covering property* (BCP) if its unit sphere can be covered by countably many open balls that miss the origin. The main result of this chapter is the following.

**Theorem.** *Let  $X$  be a Banach space,  $\kappa$  an infinite cardinal, and  $\alpha \in [-1, 0)$ . If  $X$  fails the  $\alpha$ -BCP $_{\kappa}$ , then it contains an isomorphic copy of  $\ell_1(\kappa)$ . On the other hand, if  $X$  contains an isomorphic copy of  $\ell_1(\kappa)$ , then  $X$  admits an equivalent norm which fails the  $\alpha$ -BCP $_{\text{cf}(\kappa)}$ , for every  $\alpha \in (-1, 1)$ .*

We end the chapter by investigating the transfinite BCP's behaviour under direct sums, and we provide numerous examples.

### 1.3.5 Overview

Let's summarize the implications of the main properties mentioned above, which will also clarify the organization and structure of this thesis.

$$\begin{array}{ccc}
X \text{ is ASQ} & & \\
\Downarrow & & \\
X \text{ has the SSD2P} & & \\
\Downarrow & & \\
X \text{ has the SD2P} & \iff & X^* \text{ is OH}
\end{array}$$

$$(\forall \alpha \in (-1, 1)) X \text{ fails the } \alpha\text{-BCP}_{\aleph_0} \implies X \text{ is OH}$$

## 1.4 Preliminaries

In this section, we introduce preliminary concepts that will be used throughout this thesis.

### 1.4.1 Set theoretical topology

#### Some cardinal functions

Given a topological space  $X$ , the symbol  $d(X)$  denotes the *density character* of  $X$ , that is

$$d(X) := \min\{|\mathcal{D}| : \mathcal{D} \subset X \text{ is dense}\} + \aleph_0.$$

On the other hand,  $c(X)$  denotes the *cellularity* of  $X$ , that is

$$c(X) := \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a cellular family in } X\} + \aleph_0,$$

where a family  $\mathcal{C}$  is said to be a *cellular family* if it consists of pairwise disjoint open sets in  $X$ . It is well-known that  $d(X) \leq c(X)$ . We refer the reader to [Hod] for more details about these cardinal functions and others.

#### Stone's representation theorem for Boolean algebras

Let  $B$  be a Boolean algebra. We can associate to  $B$  a compact totally disconnected Hausdorff space, the *Stone space*  $\mathcal{S}(B)$ , defined as the set of all the ultrafilters on  $(B, \subset)$  endowed with the topology generated by the sets of the form

$$\{\mathcal{U} \in \mathcal{S}(B) : b \in \mathcal{U}\},$$

where  $b \in B$ . Stone's representation theorem for Boolean algebras states that every Boolean algebra  $B$  is isomorphic to the Boolean algebra of clopen sets of the Stone space  $\mathcal{S}(B)$ ; and furthermore, every compact totally disconnected

Hausdorff space  $X$  is homeomorphic to the Stone space belonging to the Boolean algebra of clopen sets of  $X$ . These assignments are functorial, and we obtain a category-theoretic duality between the category of Boolean algebras (with homomorphisms as morphisms) and the category of compact totally disconnected Hausdorff spaces (with continuous maps as morphisms).

If  $X$  is a discrete topological space, then its Stone-Čech compactification  $\beta X$  coincides with the Stone space of the complete Boolean algebra of all subsets of  $X$ .

## 1.4.2 Banach space theory

### Slices

Let  $X$  be a Banach space. By a *slice* of a set  $A \subset X$ , we mean the intersection of the set  $A$  with some half-space, i.e. a set of the form

$$\{x \in A : x^*(x) > \alpha\},$$

where  $x^* \in X^*$  and  $\alpha > 0$ .

### Convex combinations and series

Let  $A_1, \dots, A_n$  be finitely many sets in a Banach space  $X$ . A convex combination of  $A_1, \dots, A_n$  is a set of the form

$$\sum_{i=1}^n \lambda_i A_i,$$

where  $\lambda_1, \dots, \lambda_n$  are non-negative reals satisfying the equation

$$\sum_{i=1}^n \lambda_i = 1.$$

Similarly, a convex series of a countable family of sets  $A_i$ 's in  $X$  is a set of the form

$$\sum_{i=1}^{\infty} \lambda_i A_i,$$

where the  $\lambda_i$ 's are non-negative reals satisfying the equation

$$\sum_{i=1}^{\infty} \lambda_i = 1.$$

### The Daugavet property

A Banach space  $X$  is said to have the *Daugavet Property* if for every rank-1 operator  $T : X \rightarrow X$ , the following equation holds:

$$\|I + T\| = 1 + \|T\|,$$

where  $I$  denotes the identity operator. For example, if  $K$  is a compact topological space without isolated points, then  $C(K)$  exhibits the Daugavet property. This property is also present in the spaces  $L_1(\mu)$  and  $L_\infty(\mu)$  for any non-atomic measure  $\mu$ . A detailed overview on the Daugavet property can be found in [Wer].

### Absolute normalized sums

A norm  $N$  on  $\mathbb{R}^2$  is called *absolute* [BD] if

$$N(a, b) = N(|a|, |b|)$$

holds for all  $(a, b) \in \mathbb{R}^2$ , and *normalized* if

$$N(0, 1) = N(1, 0) = 1.$$

For example, the  $\ell_p$  norms on  $\mathbb{R}^2$  are absolute and normalized for  $1 \leq p \leq \infty$ . Additionally, if  $N$  is absolute, then  $N(a, b) \leq N(c, d)$ , whenever  $|a| \leq |c|$  and  $|b| \leq |d|$ . Now, given two Banach spaces  $X$  and  $Y$ , and an absolute normalized norm  $N$  on  $\mathbb{R}^2$ , we define  $X \oplus_N Y$  to be the product space  $X \times Y$  endowed with the norm

$$\|(x, y)\|_N := N(\|x\|, \|y\|).$$

The natural duality

$$(X \oplus_N Y)^* = X^* \oplus_{N^*} Y^*,$$

where

$$N^*(u, v) := \max_{N(x, y) \leq 1} (|xu| + |yv|)$$

is immediate to prove. Additionally notice that  $N^*$  defines an absolute normalized norm too.

Eventually, we let  $\pi_X$  and  $i_X$  denote the natural projection from  $X \times Y$  onto  $X$  and the natural embedding from  $X$  into  $X \times Y$ , respectively.

### Direct sums via (transfinite) sequence spaces

Let  $\{X_\alpha : \alpha \in \mathcal{A}\}$  be a family of Banach spaces and  $E$  a Banach space consisting of functions from  $\mathcal{A}$  into  $\mathbb{R}$ . We let

$$E(\mathcal{A}, X_\alpha) := \left\{ x : \mathcal{A} \rightarrow \prod X_\alpha : (\forall \alpha \in \mathcal{A}) x(\alpha) \in X_\alpha \text{ and } \|x\| < \infty \right\}$$

denote a Banach space with the norm

$$\|x\| := \|\alpha \in \mathcal{A} \mapsto \|x(\alpha)\|_{X_\alpha} \in \mathbb{R}\|_E.$$

### Tensor products

Given two Banach spaces  $X$  and  $Y$ , let  $\mathcal{B}(X \times Y)$  be the space of all bilinear forms from  $X \times Y$  into  $\mathbb{R}$ . The *tensor product*  $X \otimes Y$  is the subspace of the algebraic dual of  $\mathcal{B}(X \times Y)$  spanned by the elements  $x \otimes y$ , where

$$x \otimes y(B) := B(x, y)$$

for every  $B \in \mathcal{B}(X \times Y)$ . The *projective norm* is defined by

$$\|u\|_\pi := \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\},$$

and the *projective tensor product*  $X \hat{\otimes}_\pi Y$  is the closure of  $X \otimes Y$  endowed with the projective norm. It is known that the dual of  $X \hat{\otimes}_\pi Y$  is isometrically isomorphic to both  $\mathcal{B}(X \times Y)$  and to  $\mathcal{L}(X, Y^*)$ , and that the unit sphere of  $X \hat{\otimes}_\pi Y$  coincides with the closed convex hull of  $S_X \otimes S_Y$ . On the other hand, the *injective norm* is defined by

$$\|u\|_\varepsilon := \sup \left\{ \left| \sum_{i=1}^n \varphi(x_i) \psi(y_i) \right| : \varphi \in B_{X^*}, \psi \in B_{Y^*} \text{ and } u = \sum_{i=1}^n x_i \otimes y_i \right\},$$

and the *injective tensor product*  $X \hat{\otimes}_\varepsilon Y$  is the closure of  $X \otimes Y$  endowed with the injective norm. It is known that  $X \hat{\otimes}_\varepsilon Y$  isometrically embeds both in  $\mathcal{L}(X^*, Y)$  and in  $\mathcal{L}(Y^*, X)$ . We refer the reader to [Rya] for a comprehensive introduction to tensor products of Banach spaces.

### Ultraproducts

Fix a family of Banach spaces  $\{X_\alpha : \alpha \in \mathcal{A}\}$  and a non-principal ultrafilter  $\mathcal{U}$  in  $\mathcal{A}$ . Let  $c_0^\mathcal{U}(\mathcal{A}, X_\alpha)$  be the set consisting of all the elements  $x \in \ell_\infty(\mathcal{A}, X_\alpha)$

satisfying  $\lim_{\mathcal{U}} \|x(\alpha)\| = 0$ . The *ultraproduct of the  $X_\alpha$ 's with respect to  $\mathcal{U}$*  is the quotient space

$$\prod_{\mathcal{U}} X_{(\cdot)} := \ell_\infty(\mathcal{A}, X_\alpha) / c_0^{\mathcal{U}}(\mathcal{A}, X_\alpha)$$

We naturally identify the bounded functions  $x \in \ell_\infty(\mathcal{A}, X_\alpha)$  with the transfinite sequences  $(x(\alpha))_\alpha \in \mathcal{A}$ . So, the coset

$$[(x(\alpha))_\alpha] \in x + c_0^{\mathcal{U}}(\mathcal{A}, X_\alpha),$$

will be simply denoted by  $(x_\alpha)_{\mathcal{U}}$  if no confusion arises. It is straightforward to observe that the quotient norm over the ultraproduct can easily be computed via the equality

$$\|(x_\alpha)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_\alpha\|.$$

### The dual space of continuous function spaces

If  $K$  is a compact Hausdorff space, then the Banach space of continuous functions over  $K$  endowed with the supremum norm is  $C(K)$ . Additionally, the dual space  $C(K)^*$  is isometrically isomorphic to the space of regular signed Borel measures of bounded variation [AB, Theorem 14.14]. On the other hand, if  $L$  is a locally compact Hausdorff space, then,  $C_0(L)$  represents the Banach space consisting of the continuous functions over  $K$  which vanishes at infinity endowed with the supremum norm, i.e.

$$C_0(L) := \{f : K \rightarrow \mathbb{R} : (\forall \varepsilon > 0) (\exists K \subset L \text{ compact}) \\ (\forall x \in L \setminus K) |f(x)| < \varepsilon\}.$$

Moreover, thanks to the Riesz-Markov representation theorem, every continuous linear functional on  $C_0(L)$  admits a unique representation as a regular countably additive Borel measure on  $L$ .

### The dual space of $L_1$ spaces

Let  $(\Omega, \Sigma, \mu)$  be a measure space. The measure  $\mu$  is called *localizable* if the following conditions hold.

- (a) For every  $A \in \Sigma$  such that  $\mu(A) = \infty$  there exists  $B \subset A$  belonging to  $\Sigma$  such that  $0 < \mu(B) < \infty$ .
- (b) For every  $\mathcal{A} \subset \Sigma$  there exists  $B \in \Sigma$  such that  $\mu(A \setminus B) = 0$  for every  $A \in \mathcal{A}$  and, if  $C \in \Sigma$  satisfies  $\mu(A \setminus C) = 0$  for every  $A \in \mathcal{A}$ , then  $\mu(B \setminus C) = 0$ .

It is well-known that the standard duality  $L_1(\mu)^* = L_\infty(\mu)$  holds if and only if  $\mu$  is localizable [Fre1, Theorem 243G].

### Lebesgue-Bochner spaces

Given a measure space  $(\Omega, \Sigma, \mu)$ , a Banach space  $X$ , and  $1 \leq p < \infty$ , the *Lebesgue-Bochner spaces*  $L_p(\mu; X)$  is defined as the Kolmogorov quotient (by equality almost everywhere) of the space of Bochner-measurable functions  $f : \Omega \rightarrow X$  such that the norm

$$\|f\| := \left( \int_{\Omega} \|f\|^p d\mu \right)^{\frac{1}{p}}$$

is finite. Here we say that  $f$  is *Bochner-measurable* if  $f$  equals almost everywhere the limit of a sequence consisting of measurable countably-valued functions. The Lebesgue-Bochner space  $L_{\infty}(\mu; X)$  is built analogously, except for the fact that we consider the essential sup norm instead. It is clear that  $L_p(\mu; \mathbb{R}) = L_p(\mu)$ . It is known that the standard duality  $L_1(\mu; X)^* = L_{\infty}(\mu; X^*)$  holds if either  $\mu$  is decomposable (i.e. a measure that is a disjoint union of finite measures) and  $X^*$  is separable [Din, p. 282], or  $\mu$  is  $\sigma$ -finite and  $X^*$  has the Radon-Nikodym property with respect to  $\mu$  [DU, p. 98].

### Köthe-Bochner Function Spaces

Let  $(\Omega, \Sigma, \mu)$  be a complete  $\sigma$ -finite measure space. Let  $E$  be the Kolmogorov quotient (by equality almost everywhere) of a space of real-valued measurable functions on  $\Omega$ .  $E$  is called a *Köthe function space over*  $(\Omega, \Sigma, \mu)$  if the following conditions hold.

- (a) The characteristic function  $\chi_A$  belongs to  $E$  for every  $A \in \Sigma$  such that  $\mu(A) < \infty$ .
- (b) If  $f \in E$  and  $A \in \Sigma$  with  $\mu(A) < \infty$ , then  $f$  is integrable over  $A$ .
- (c) If  $g$  is measurable and  $f \in E$  is such that  $|g| \leq |f|$  almost everywhere, then  $g \in E$  and  $\|g\| \leq \|f\|$ .

Eventually, we call  $E(X)$  the Kolmogorov quotient (by equality almost everywhere) of the space of all Bochner-measurable functions  $f : \Omega \rightarrow X$  such that  $\|f(\cdot)\| \in E$ . The *Köthe-Bochner space* induced by  $E$  and  $X$  is the space  $E(X)$  endowed with the norm

$$\|f\| := \|\|f(\cdot)\|\|_E.$$

It is clear that  $L_p(\mu)(X) = L_p(\mu; X)$ . For more information on Köthe-Bochner spaces the reader is referred to [Lin].

### L- and M-embedded spaces

A Banach space  $X$  is called *L-embedded* if there exists a contractive projection  $P$  from  $X^{**}$  onto  $X$ , viewed as a subspace of  $X^{**}$  via the canonical embedding, and

$$X^{**} = X \oplus_1 \ker(P).$$

On the other hand, a Banach space  $X$  is called *M-embedded* if its dual space can be decomposed as

$$X^* = X^\perp \oplus_1 Z,$$

for some subspace  $Z \subset X^*$ .

### Banach lattices

A *lattice* is an algebraic structure  $(L, \vee, \wedge)$  consisting of a set  $L$  and two binary, commutative and associative operations  $\vee$  and  $\wedge$  on  $L$ , satisfying the equation

$$a \vee (a \wedge b) = a \wedge (a \vee b) = a$$

for each  $a, b \in L$ . A lattice generates a natural partial order via the relation defined by  $a \leq b$  if and only either  $a = a \wedge b$  or  $b = a \vee b$ . A *Banach lattice*  $X$  is a complete normed vector space with a lattice order  $\leq$ , such that, for every  $x, y \in X$ , the implication

$$|x| \leq |y| \implies \|x\| \leq \|y\|$$

holds, where the *absolute value*  $|\cdot|$  is defined by the equality  $|x| = x \vee -x$ . We call

$$X^+ := \{x \in X : x \geq 0\}$$

the *positive cone* of  $X$  and, given any set  $A \subset X$ , we will write  $A^+ := A \cap X^+$ . Eventually, let us recall that a Banach lattice  $X$  is said to be an *AM-space* whenever the norm commutes with the supremum on its positive cone, while  $X$  is called an *AL-space* whenever the norm is additive on its positive cone. For a comprehensive treatment of Banach lattices we refer the reader to [AB].

### Lipschitz-free spaces

Recall that, given metric spaces  $(M, d_M)$  and  $(N, d_N)$ , a function  $f : M \rightarrow N$  is a *Lipschitz function* if there exists an  $L \geq 0$  such that, for all  $x, y \in M$  with  $x \neq y$ , one has

$$d_N(f(x), f(y)) \leq L d_M(x, y).$$

The smallest such  $L$  is called the *Lipschitz constant* of  $f$ . Let  $M$  be a *pointed metric space*, that is, a metric space with a fixed point  $0$ . The space of Lipschitz functions  $Lip_0(M)$  is the Banach space of all Lipschitz functions  $f : M \rightarrow \mathbb{R}$  with  $f(0) = 0$  equipped with the norm

$$\|f\| := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in M, x \neq y \right\},$$

i.e.,  $\|f\|$  is the Lipschitz constant of  $f$ . It is known that  $Lip_0(M)$  is a dual space whose predual, the *Lipschitz-free space*  $\mathcal{F}(M)$ , can be defined as the closed linear span of the functionals  $\delta_x$ 's in  $Lip_0(M)^*$  where

$$\delta_x(f) := f(x)$$

for every  $f \in Lip_0(M)$  and  $x \in M$ .



# Chapter 2

## Almost square Banach spaces

A Banach space  $X$  is said to be *almost square* (ASQ) whenever, for any finite-dimensional subspace  $Y \subset X$  and  $\varepsilon > 0$ , there exists  $x \in S_X$  such that the inequality

$$\|y + rx\| \leq (1 + \varepsilon) \max\{\|y\|, |r|\}$$

holds for all real numbers  $r$  and  $y \in Y$ .

Using a compactness argument, one can show that finding such  $x$  only for a given finite set  $y_1, \dots, y_n \in S_X$  suffices [ALL, Theorem 2.4]. Moreover, a standard argument shows that it is only necessary to require the inequality  $\|y_i + x\| \leq 1 + \varepsilon$  to hold for all  $1 \leq i \leq n$ . The concept of an ASQ Banach space was initially introduced in [ALL] and has since been the subject of extensive research. One noteworthy class of ASQ Banach spaces comprises non-reflexive M-embedded spaces [ALL, Corollary 4.3]. Other notable examples of ASQ Banach spaces include the  $c_0$  sum of a countable family consisting of arbitrary Banach spaces [ALL, Example 3.1], Gurariï spaces [ALLN, Proposition 3.2] and Banach spaces of the form  $C_0(X)$ , provided that  $X$  is an infinite locally-compact Hausdorff space that is not compact [AMR1, Proposition 2.1]. Here are two key results in the theory of ASQ Banach spaces:

- A Banach space contains an isomorphic copy of  $c_0$  if and only if it possesses an equivalent ASQ norm [BLR4, Corollary 2.4].
- A dual Banach space contains an isomorphic copy of  $c_0$  if and only if it possesses an equivalent dual ASQ norm [AHT, Theorem 3.8].

In this chapter, we investigate transfinite ASQ Banach spaces. We provide numerous examples, analyze the geometrical characteristics of these spaces, and study their stability under direct sums, tensor products, and ultraproducts. Additionally, we explore the relationship between the containment of

isomorphic copies of  $c_0(\kappa)$  and the existence of equivalent transfinite ASQ norms. Eventually, we end this chapter with some remarks and observations about locally almost square Banach lattices. This chapter is based on [ACLLR] and [Cia1].

## 2.1 Definition and some examples

**Definition 2.1.** Let  $X$  be a Banach space and  $\kappa$  a cardinal.

- (a) We say that  $X$  is *almost square with respect to  $\kappa$*  ( $\text{ASQ}_\kappa$ ) if, for every  $A \in \mathcal{P}_\kappa(S_X)$  and  $\varepsilon > 0$ , there exists  $y \in S_X$  satisfying

$$\|x \pm y\| \leq 1 + \varepsilon$$

for all  $x \in A$ .

- (b) We say that  $X$  is *square with respect to  $\kappa$*  ( $\text{SQ}_\kappa$ ) if, for every  $A \in \mathcal{P}_\kappa(S_X)$ , there exists  $y \in S_X$  satisfying

$$\|x \pm y\| \leq 1$$

for all  $x \in A$ .

Notice that, whenever  $\kappa$  is infinite, it suffices to require that  $\|x+y\| \leq 1+\varepsilon$  holds. Moreover, a standard argument shows that, in the SQ case, we can equivalently require  $\|x \pm y\| = 1$ . It is clear that being  $\text{ASQ}_{\aleph_0}$  naturally coincides with the classical definition of ASQ. An immediate application of [ALL, Lemma 2.2] shows the following result.

**Theorem 2.2.** *Let  $X$  be a Banach space and  $\kappa \geq \aleph_1$ . The following are equivalent:*

- (i)  $X$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ).
- (ii) For every subspace  $Y \subset X$ , with  $d(Y) < \kappa$ , and  $\varepsilon > 0$  (respectively,  $\varepsilon = 0$ ), there exists  $x \in S_X$  such that the inequality

$$\|y + rx\| \leq (1 + \varepsilon) \max\{\|y\|, |r|\}$$

holds for all  $y \in Y$  and  $r \in \mathbb{R}$ .

Let us proceed by providing some examples of transfinite ASQ Banach spaces.

### 2.1.1 When $\kappa$ is $\aleph_0$

**Example 2.3.** Given  $n \in \mathbb{N}$ , consider the Banach space  $X_n$  consisting of all the functions  $f \in C(S_{\mathbb{R}^n})$  satisfying  $f(x) = -f(-x)$  for all  $x \in S_{\mathbb{R}^n}$ . We claim at first that  $X_n$  is  $\text{SQ}_{n+1}$ . To this aim, fix  $f_1, \dots, f_n \in S_{X_n}$ , hence, by a Corollary of Borsuk-Ulam [AH, p. 485, Satz VIII], we can find  $x_0 \in S_{\mathbb{R}^n}$  such that all the functions  $f_i$ 's vanish at  $x_0$ . Now pick any function  $h \in S_{X_n}$  satisfying  $h(x_0) = 1$ , and define the function

$$g := (1 - \max |f_i|) \cdot h \in S_{X_n}.$$

Observe that, given  $1 \leq i \leq n$  and  $x \in S_{\mathbb{R}^n}$ , we obtain

$$\begin{aligned} |f_i(x) \pm g(x)| &\leq |f_i(x)| + |g(x)| \\ &\leq |f_i(x)| + 1 - \max |f_j(x)| \\ &\leq 1. \end{aligned}$$

This concludes the proof that  $X_n$  is  $\text{SQ}_{n+1}$ . Now, consider the Banach space  $c_0(\mathbb{N}, X_n)$  and let us show that it is  $\text{SQ}_{\aleph_0}$ . To this aim, fix  $x_1, \dots, x_m$  in its unit sphere. Since  $X_m$  is  $\text{SQ}_{m+1}$ , we can find  $y \in S_{X_m}$  satisfying

$$\|x_i(m) \pm y\| \leq 1$$

for every  $1 \leq i \leq m$ . This implies that

$$\|x_i \pm y \cdot e_m\| \leq 1$$

holds, as required.

**Example 2.4.** Fix a non-principal ultrafilter  $\mathcal{U}$  in  $\mathbb{N}$ . Given  $x \in \ell_\infty$ , define the equivalent norm

$$|x| := \max \left\{ \left| \lim_{\mathcal{U}}(x) \right|, \sup_{n \in \mathbb{N}} \left| \lim_{\mathcal{U}}(x) - x(n) \right| \right\}.$$

We claim that the Banach space  $X := (\ell_\infty, |\cdot|)$ , which has been proved in [BLR4] to be  $\text{ASQ}_{\aleph_0}$ , is actually  $\text{SQ}_{\aleph_0}$ . Fix  $x_1, \dots, x_n \in S_X$  and define, for all  $i \in \mathbb{N}$  and  $j = 1, \dots, n$ ,

$$A_{i,j} := \{m \in \mathbb{N} : |x_j(m) - \lim_{\mathcal{U}}(x_j)| < i^{-1}\} \in \mathcal{U}.$$

It is clear that

$$A_i := \bigcap A_{i,j} \in \mathcal{U},$$

moreover, since  $\mathcal{U}$  is non-principal, each  $A_i$  is infinite, thus we can find some  $f(i) \in A_i$  such that  $f(i) < f(i+1)$ . Since  $\emptyset \notin \mathcal{U}$ , either  $f(2\mathbb{N})$  or  $f(2\mathbb{N}+1)$  is not in  $\mathcal{U}$ , say, for example,  $f(2\mathbb{N}) \notin \mathcal{U}$  and define the formal series

$$y := \sum_{i \in 2\mathbb{N}} (1 - i^{-1}) \cdot e_{f(i)} \in \ell_\infty.$$

Notice that  $\lim_{\mathcal{U}}(y) = 0$ , hence the equality  $|y| = 1$  easily follows. Eventually, given  $1 \leq j \leq n$ , we obtain that

$$\begin{aligned} |x_j - y| &= \max \left\{ \left| \lim_{\mathcal{U}}(x_j) \right|, \sup_{i \in \mathbb{N}} \left| \lim_{\mathcal{U}}(x_j) - x_j(i) + y(i) \right| \right\} \\ &\leq \max \left\{ 1, \sup_{i \in 2\mathbb{N}} \left| \lim_{\mathcal{U}}(x_j) - x_j(f(i)) + 1 - i^{-1} \right|, \right. \\ &\quad \left. \sup_{i \in \mathbb{N} \setminus f(2\mathbb{N})} \left| \lim_{\mathcal{U}}(x_j) - x_j(i) \right| \right\} \\ &\leq 1. \end{aligned}$$

### 2.1.2 When $\kappa$ is uncountable

**Example 2.5.** Let  $\kappa \geq \aleph_1$  and  $\ell_\infty^c(\kappa)$  be the elements of  $\ell_\infty(\kappa)$  with countable support. If  $X$  is a subspace of  $\ell_\infty^c(\kappa)$  containing  $c_0(\kappa)$ , then  $X$  is  $\text{SQ}_\kappa$ .

*Proof.* Given  $A \in \mathcal{P}_\kappa(S_{\ell_\infty^c(\kappa)})$ , find  $\mu \in \kappa$  which doesn't belong to the support of any element belonging to  $A$ . It is clear that  $\|x \pm e_\mu\| = 1$  holds for every  $x \in A$ .  $\square$

In [Gur], spaces of (almost) universal disposition were introduced. Let us recall that, given a family of Banach spaces  $\mathcal{Y}$ , a Banach space  $X$  is *of almost universal disposition for  $\mathcal{Y}$*  if, for every pair  $S \subset T$  in  $\mathcal{Y}$  and  $\varepsilon > 0$ , any isometric embedding

$$S \rightarrow X$$

extends to an  $\varepsilon$ -isometric embedding

$$T \rightarrow X.$$

On the other hand, a Banach space  $X$  is *of universal disposition for  $\mathcal{Y}$*  if we additionally require the extension to be isometric. Gurariĭ spaces are the classical examples of spaces of almost universal disposition for the family of finite-dimensional Banach spaces.

**Example 2.6.** Let  $\kappa \geq \aleph_1$ . If  $X$  is of almost universal disposition (respectively, of universal disposition) for the family of Banach spaces with density character less than  $\kappa$ , then  $X$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ). Let us prove this statement for the  $\text{SQ}_\kappa$  case only. To this aim, fix a subspace  $Y \subset X$  with  $d(Y) < \kappa$ . The inclusion  $Y \rightarrow X$  extends to an isometric embedding

$$T : Y \oplus_\infty \mathbb{R} \rightarrow X.$$

The claim now follows by applying Theorem 2.2 for  $x = T(0, 1)$ .

**Theorem 2.7.** *Let  $X$  be a locally compact Hausdorff space and  $\kappa \geq \aleph_1$ . Then the following are equivalent.*

- (i)  $C_0(X)$  is  $\text{ASQ}_\kappa$ .
- (ii)  $C_0(X)$  is  $\text{SQ}_\kappa$ .
- (iii) If  $\mathcal{K}$  is a family consisting of less than  $\kappa$  many compact sets in  $X$ , then  $\bigcup \mathcal{K}$  is not dense in  $X$ .

*Remark 2.8.* Notice that property (iii), when  $\kappa = \aleph_1$ , is equivalent to requiring that  $X$  does not admit any dense  $\sigma$ -compact set.

*Proof.* (ii)  $\implies$  (i) is obvious. (i)  $\implies$  (iii). Let  $\mathcal{K}$  be a family consisting of less than  $\kappa$  many compact sets in  $X$ , and fix  $K \in \mathcal{K}$ . Since  $K$  is compact and  $X$  is locally compact, we can find a finite covering  $U_1^K, \dots, U_n^K$  for  $K$ , consisting of open relatively compact sets. Let

$$U_K := \bigcup U_i^K$$

and notice that  $U_K$  must differ from  $X$  because, otherwise, we would get that  $X$  is compact, which contradicts the fact that  $C_0(X)$  is  $\text{ASQ}$ . On the other hand,  $K$  and  $X \setminus U_K$  are disjoint closed non-empty sets, hence there exists a Urysohn function  $f_K : X \rightarrow [0, 1]$  satisfying  $f_K|_K = 1$  and  $f_K|_{X \setminus U_K} = 0$ . Notice that the support of  $f_K$  is contained in the relatively compact set  $U_K$ , hence  $f_K \in C_0(X)$ . Since  $C_0(X)$  is  $\text{ASQ}_\kappa$ , there exists some  $g \in S_{C_0(X)}$  satisfying

$$\|f_K \pm g\| \leq 3/2$$

for every  $K \in \mathcal{K}$ . By construction, it follows that  $|g(x)| \leq 1/2$  must hold for every  $x \in \bigcup \mathcal{K}$ , and, therefore, the non-empty open set

$$\{x \in X : |g(x)| > 1/2\}$$

is disjoint from  $\bigcup \mathcal{K}$ .

(iii)  $\implies$  (ii). Fix  $A \in \mathcal{P}_\kappa(S_{C_0(X)})$ . Given  $f \in A$  and  $n \in \mathbb{N}$ , there exists a compact set  $K_{f,n} \subset X$  so that  $|f(x)| < n^{-1}$  for all  $x \in K_{f,n}$ . Let  $\mathcal{K}$  be the family consisting of all such  $K_{f,n}$ 's and notice that  $|\mathcal{K}| < \kappa$ . By assumption, we can find a non-empty open set  $U$  which is disjoint from  $\bigcup \mathcal{K}$ , and, without loss of generality, we can assume it to be relatively compact. Let  $g : X \rightarrow [0, 1]$  be a Urysohn function satisfying  $\|g\| = 1$  and  $g|_{X \setminus U} = 0$ . Notice that the support of  $g$  is contained in the relatively compact set  $U$ , so  $g \in C_0(X)$ . By construction, we have that

$$\|f \pm g\| = 1$$

holds for all  $f \in A$ . □

## 2.2 Stability results

In this section, we provide more examples of transfinite ASQ Banach spaces by taking direct sums, tensor products, and ultraproducts.

### 2.2.1 Direct sums

In this subsection, we investigate the stability of transfinite ASQ under  $\ell_\infty$  and  $c_0$  sums, because they are known to be the only direct sums that preserve ASQ [HLLN, Theorem 3.1].

The following Theorem informally states that the  $\ell_\infty$  sum of a family of Banach spaces is  $\text{ASQ}_\kappa$  whenever some of the spaces belonging to the family are arbitrarily close to being  $\text{ASQ}_\kappa$ .

**Theorem 2.9.** *Let  $\{X_\alpha : \alpha \in \mathcal{A}\}$  be a family of Banach spaces and  $\kappa$  an infinite cardinal. If, for every  $\varepsilon > 0$ , there exists  $\beta \in \mathcal{A}$  satisfying*

$$(\forall A \in \mathcal{P}_\kappa(S_{X_\beta})) (\exists y \in S_{X_\beta}) (\forall x \in A) \|x + y\| \leq 1 + \varepsilon, \quad (2.1)$$

*then  $\ell_\infty(\mathcal{A}, X_\alpha)$  is  $\text{ASQ}_\kappa$ . If we additionally require that  $|\mathcal{A}| < \text{cf}(\kappa)$  and  $\lambda^{|\mathcal{A}|} < \kappa$  holds for all  $\lambda < \kappa$ , then the converse holds too.*

*Proof.* Fix  $A \in \mathcal{P}_\kappa(S_{\ell_\infty(\mathcal{A}, X_\alpha)})$ ,  $\varepsilon > 0$  and find some  $\beta \in \mathcal{A}$  satisfying (2.1). Let  $y \in S_{X_\beta}$  satisfy  $\|x(\beta) + y\| \leq 1 + \varepsilon$  for all  $x \in A$ . It is then clear that

$$\|x + y \cdot e_\beta\| \leq 1 + \varepsilon$$

holds for all  $x \in A$ , that is,  $\ell_\infty(\mathcal{A}, X_\alpha)$  is  $\text{ASQ}_\kappa$ .

Suppose now that  $\ell_\infty(\mathcal{A}, X_\alpha)$  is  $\text{ASQ}_\kappa$  and that, by contradiction, there exists  $\varepsilon > 0$  so that, for every  $\alpha \in \mathcal{A}$ , the negation of (2.1) holds true, that

is, there exists  $A_\alpha \in \mathcal{P}_\kappa(S_{X_\alpha})$  such that, for all  $y \in S_{X_\alpha}$ , we can find  $x \in A_\alpha$  satisfying  $\|x + y\| > 1 + \varepsilon$ . Notice that, thanks to our assumptions,

$$\left| \prod A_\alpha \right| \leq (\sup |A_\alpha|)^{|\mathcal{A}|} < \kappa.$$

Since  $\ell_\infty(\mathcal{A}, X_\alpha)$  is  $\text{ASQ}_\kappa$ , then there exists  $y \in S_{\ell_\infty(\mathcal{A}, X_\alpha)}$  satisfying

$$\|x + y\| \leq 1 + \varepsilon/2$$

for every  $x \in \prod A_\alpha$ . Let  $\beta \in \mathcal{A}$  be such that  $\|y(\beta)\| > 1 - \varepsilon/2$ , then

$$\begin{aligned} 1 + \varepsilon/2 &\geq \|x + y\| \\ &\geq \|x(\beta) + y(\beta)\| \\ &\geq \left\| x(\beta) + \frac{y(\beta)}{\|y(\beta)\|} \right\| - \left\| y(\beta) - \frac{y(\beta)}{\|y(\beta)\|} \right\| \\ &\geq \left\| x(\beta) + \frac{y(\beta)}{\|y(\beta)\|} \right\| - \varepsilon/2, \end{aligned}$$

holds for all  $x$ , which leads to a contradiction, since

$$\left\| x(\beta) + \frac{y(\beta)}{\|y(\beta)\|} \right\| \leq 1 + \varepsilon$$

holds for all  $x \in \prod X_\alpha$ . □

**Corollary 2.10.** *Let  $X$  and  $Y$  be Banach spaces and  $\kappa$  an infinite cardinal. Then  $X \oplus_\infty Y$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ) if and only if either  $X$  or  $Y$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ).*

*Proof.* The  $\text{ASQ}_\kappa$  case is a direct consequence of Theorem 2.9. On the other hand, the  $\text{SQ}_\kappa$  case follows by noting that Theorem 2.9 holds also with  $\varepsilon = 0$  whenever  $\mathcal{A}$  is finite. □

**Theorem 2.11.** *Let  $\{X_\alpha : \alpha \in \mathcal{A}\}$  be an uncountable family of Banach spaces. The Banach space  $c_0(\mathcal{A}, X_\alpha)$  is  $\text{SQ}_{|\mathcal{A}|}$ .*

*Proof.* Given  $A \in \mathcal{P}_{|\mathcal{A}|}(S_{c_0(\mathcal{A}, X_\alpha)})$ , we only need to find some  $\beta \in \mathcal{A}$  which doesn't belong to the support of any  $x \in A$ , and, therefore, the equality  $\|x \pm y \cdot e_\beta\| = 1$  must hold for every  $x \in A$  and  $y \in B_{X_\beta}$ . □

### 2.2.2 Tensor products

In this subsection, we give examples of projective and injective tensor products of Banach spaces which are transfinite ASQ. We aim to extend the known stability results of regular ASQ spaces by taking tensor products coming from [LLR] and [Rue2].

**Theorem 2.12.** *Let  $\kappa \geq \aleph_1$ . If  $X$  and  $Y$  are  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ) Banach spaces, then  $X \hat{\otimes}_\pi Y$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ).*

*Proof.* Let us prove the  $\text{SQ}_\kappa$  case only. Let  $A \in \mathcal{P}_\kappa(S_{X \hat{\otimes}_\pi Y})$ . Since the unit sphere of  $X \hat{\otimes}_\pi Y$  is the closed convex hull of  $S_X \otimes S_Y$ , for every  $u \in A$  and  $n \in \mathbb{N}$ , we can find  $m_n \in \mathbb{N}$ ,  $\lambda_i^{u,n} \geq 0$ ,  $x_i^{u,n} \in S_X$  and  $y_i^{u,n} \in S_Y$  satisfying

$$\sum_{i=1}^{m_n} \lambda_i^{u,n} = 1$$

and

$$\left\| u - \sum_{i=1}^{m_n} \lambda_i^{u,n} x_i^{u,n} \otimes y_i^{u,n} \right\| \leq n^{-1}.$$

Notice that, of the set consisting of all the indices  $(i, u, n) \in \mathbb{N} \times A \times \mathbb{N}$  has cardinality less than  $\kappa$ , therefore we can find  $x \in S_X$  and  $y \in S_Y$  satisfying

$$\|x_i^{u,n} + x\|, \|y_i^{u,n} + y\| \leq 1$$

for all choices of  $(i, u, n)$ . Thanks to [Rue2, Lemma 2.2], we have that

$$\|x_i^{u,n} \otimes y_i^{u,n} + x \otimes y\| \leq 1$$

holds for all  $(i, u, n)$ . It is clear that

$$\begin{aligned} \|u + x \otimes y\| &\leq n^{-1} + \left\| \sum_{i=1}^{m_n} \lambda_i^{u,n} (x_i^{u,n} \otimes y_i^{u,n} + x \otimes y) \right\| \\ &\leq n^{-1} + \sum_{i=1}^{m_n} \lambda_i^{u,n} \|x_i^{u,n} \otimes y_i^{u,n} + x \otimes y\| \\ &\leq n^{-1} + 1 \end{aligned}$$

holds for all  $u \in A$  and  $n \in \mathbb{N}$ , hence  $\|u + x \otimes y\| \leq 1$  holds for all  $u \in A$ .  $\square$

*Remark 2.13.* Requiring only one component to be  $\text{ASQ}_\kappa$  is not enough to ensure the claim in Theorem 2.12. Indeed, the projective tensor product  $c_0(\kappa) \hat{\otimes}_\pi \ell_p$  fails to be ASQ [LLR, Theorem 3.8], even though  $c_0(\kappa)$  is  $\text{SQ}_\kappa$ , whenever  $2 < p < \infty$  and  $\kappa \geq \aleph_1$ .

We now proceed to analyze injective tensor products. To do so, we need to study when spaces of operators can be transfinite ASQ. Given two Banach spaces  $X$  and  $Y$ , we denote by

$$\mathcal{L}_\kappa(X, Y) := \{T \in \mathcal{L}(X, Y) : d(T(X)) < \kappa\}.$$

**Theorem 2.14.** *Let  $\kappa \geq \aleph_1$  and  $X, Y$  non-trivial Banach spaces. Suppose that  $X$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ).*

(a) *If  $Y^* \otimes X \subset H \subset \mathcal{L}_\kappa(Y, X)$  is a closed subspace, then  $H$  is  $\text{ASQ}_{\text{cf}(\kappa)}$  (respectively,  $\text{SQ}_{\text{cf}(\kappa)}$ ).*

(b) *If  $Y \otimes X \subset H \subset \mathcal{L}_\kappa(Y^*, X)$  is a closed subspace, then  $H$  is  $\text{ASQ}_{\text{cf}(\kappa)}$  (respectively,  $\text{SQ}_{\text{cf}(\kappa)}$ ).*

*Proof.* Let us prove (a) for the  $\text{SQ}_\kappa$  case only. Fix  $\mathcal{T} \in \mathcal{P}_{\text{cf}(\kappa)}(S_H)$ , set

$$\mathcal{T}(Y) := \bigcup_{T \in \mathcal{T}} T(Y),$$

and notice that  $d(\mathcal{T}(Y)) < \kappa$ . By assumption, there exists  $y \in S_X$  satisfying

$$\|x + ry\| \leq \max\{\|x\|, |r|\}$$

for all  $x \in \mathcal{T}(Y)$  and  $r \in \mathbb{R}$ . Fix any  $y^* \in S_{Y^*}$  and notice that  $y^* \otimes x \in S_H$ . Moreover, given any  $T \in \mathcal{T}$ , we have that

$$\begin{aligned} \|(T + y^* \otimes x)(y)\| &\leq \max\{\|T(y)\|, \|y^*(y)\|\} \\ &\leq 1 \end{aligned}$$

holds for all  $y \in S_Y$ . If we pass to the sup over  $y$ , we obtain that

$$\|T + y^* \otimes x\| \leq 1$$

holds for all  $T \in \mathcal{T}$ . □

**Corollary 2.15.** *Let  $\kappa \geq \aleph_1$  and  $X, Y$  be non-trivial Banach spaces. If  $X$  is  $\text{ASQ}_\kappa$  (respectively,  $\text{SQ}_\kappa$ ), then  $X \hat{\otimes}_\varepsilon Y$  is  $\text{ASQ}_{\text{cf}(\kappa)}$  (respectively,  $\text{SQ}_{\text{cf}(\kappa)}$ ).*

### 2.2.3 Ultraproducts

In this subsection, we show examples of ultraproducts of Banach spaces which are transfinite ASQ. We aim to extend the known results for regular ASQ spaces coming from [Har1].

**Theorem 2.16.** *Let  $\kappa$  be an infinite cardinal and  $\{X_\alpha : \alpha \in \mathcal{A}\}$  an infinite family of  $\text{ASQ}_\kappa$  Banach spaces. If  $\mathcal{U}$  is an  $\aleph_1$ -incomplete non-principal ultrafilter over  $\mathcal{A}$ , i.e. such that there exist countably many  $U_i$ 's in  $\mathcal{U}$  satisfying  $\bigcap U_i \notin \mathcal{U}$ , then  $\prod_{\mathcal{U}} X_{(\cdot)}$  is  $\text{SQ}_\kappa$ .*

*Proof.* Since  $\mathcal{U}$  is  $\aleph_1$ -incomplete, there exists a function  $f : \mathcal{A} \rightarrow (0, \infty)$  such that  $\lim_{\mathcal{U}} f(\alpha) = 0$ . Fix a set  $A \in \mathcal{P}_\kappa(S_{\prod_{\mathcal{U}} X_{(\cdot)}})$ . By assumption, there exists, for all  $\alpha \in \mathcal{A}$ ,  $y_\alpha \in S_{X_\alpha}$  satisfying

$$\|x(\alpha) + y_\alpha\| \leq 1 + f(\alpha)$$

for all  $x \in A$ . Observe that

$$\begin{aligned} \|x + (y_\alpha)_{\mathcal{U}}\| &= \lim_{\mathcal{U}} \|x_\alpha + y_\alpha\| \\ &\leq \lim_{\mathcal{U}} (1 + f(\alpha)) \\ &= 1 \end{aligned}$$

holds for all  $x \in A$ . □

*Remark 2.17.* The vice versa of Theorem 2.16 doesn't hold true, as a matter of fact the ultraproduct  $\prod_{\mathcal{U}} \ell_{(\cdot)}(\kappa)$ , where  $\mathcal{U}$  is a non-principal ultrafilter over the set

$$\mathbb{N}_{\geq 2} := \{n \in \mathbb{N} : n \geq 2\},$$

is  $\text{SQ}_\kappa$  even though it is ultraproduct of reflexive Banach spaces. The key observation required to prove this, is the fact that, given any  $A \in \mathcal{P}_\kappa(S_{\prod_{\mathcal{U}} \ell_{(\cdot)}(\kappa)})$ , one can always find some  $y_n \in S_{\ell_n(\kappa)}$  satisfying  $\|x(n) + y_n\| \leq 2^{1/n}$  for all  $x \in A$ , from which it follows that

$$\begin{aligned} \|x + (y_n)_{\mathcal{U}}\| &= \lim_{\mathcal{U}} \|x(n) + y_n\| \\ &\leq \lim_{\mathcal{U}} 2^{1/n} \\ &= 1. \end{aligned}$$

## 2.3 Renorming results

In this section, we aim to analyze the relationship between the containment of isomorphic copies of  $c_0(\kappa)$ , and the existence of transfinite ASQ equivalent norms. Observe that an easy transfinite induction argument, proves that if a Banach space admits an equivalent  $\text{SQ}_\kappa$  norm, then it must contain an isomorphic copy of  $c_0(\kappa)$ . The same cannot be said for equivalent  $\text{ASQ}_\kappa$  norms, as the following example shows.

**Example 2.18.** Let  $\kappa$  be an infinite cardinal. The Banach space  $X := c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$  is  $\text{ASQ}_\kappa$ , but it doesn't contain any isomorphic copy of  $c_0(\aleph_1)$ .

*Proof.* Let's initially assume that  $\kappa \geq \aleph_1$ . Given  $A \in \mathcal{P}_\kappa(S_X)$  and  $\varepsilon > 0$ , for every  $n \in \mathbb{N}$ , we can find  $y_n$  so that

$$\|x(n) + y_n\| \leq 2^{1/n}$$

holds for all  $x \in A$ . Let  $n$  be such that  $2^{1/n} \leq 1 + \varepsilon$ , hence

$$\|x + y_n \cdot e_n\| \leq 1 + \varepsilon$$

holds for all  $x \in A$ . If  $\kappa = \aleph_0$ , then the proof can be easily adapted, with the difference that we will require

$$\|x(n) + y_n\| \leq (2 + \varepsilon)^{1/n}$$

instead. This proves that  $X$  is  $\text{ASQ}_\kappa$ . For the second half of the proof, observe that  $X^*$  is countable sum of reflexive Banach spaces, hence it is weakly compactly generated. Therefore,  $X^*$  cannot contain any isomorphic copy of  $\ell_1(\aleph_1)$ . Now assume by contradiction that  $X$  contains an isomorphic copy of  $c_0(\aleph_1)$ . This would imply that  $\ell_1(\aleph_1)$  is isomorphic to a quotient of  $X^*$ , but, since  $\ell_1(\aleph_1)$  has the lifting property, then it would imply that  $\ell_1(\aleph_1)$  is isomorphic to a subspace of  $X^*$ , which clearly leads to a contradiction.  $\square$

*Remark 2.19.* Notice that with a similar argument as in Example 2.18, one can show that  $\ell_\infty(\mathbb{N}, \ell_n(\kappa))$  is a bidual  $\text{ASQ}_\kappa$  Banach space. The relevance of this observation comes from the fact that whether there exists a dual  $\text{ASQ}$  Banach space was asked in [ALL] and it was affirmatively answered in [AHT].

Example 2.18 shows that being  $\text{ASQ}_\kappa$  doesn't imply the containment of isomorphic copies of  $c_0(\kappa)$ . This is true in general, but the situation is different if we restrict ourselves to the class of continuous function spaces.

**Theorem 2.20.** *Let  $K$  be a compact Hausdorff space and  $\kappa$  an infinite cardinal. If the Banach space  $C(K)$  admits an equivalent  $\text{ASQ}_\kappa$  norm, then it contains an isomorphic copy of  $c_0(\kappa)$ .*

*Proof.* Let  $X$  be  $C(K)$  endowed with an equivalent  $\text{ASQ}_\kappa$  norm  $|\cdot|$ , say

$$C^{-1}\|\cdot\| \leq |\cdot| \leq C\|\cdot\|.$$

Now, find  $p \in \mathbb{N}$  big enough and  $\varepsilon > 0$  small enough so that

$$p > 2C^2(1 + \varepsilon)^p.$$

Since  $X$  is  $\text{ASQ}_\kappa$ , then we can construct via transfinite induction a set  $\{f_\alpha : \alpha \in \kappa\} \subset S_X$  satisfying

$$|f + rf_\alpha| \leq (1 + \varepsilon) \max\{|f|, |r|\}$$

for all  $r \in \mathbb{R}$  and  $f$  belonging to the closed linear span of the set  $\{f_\beta : \beta \in \alpha\}$ . Up to considering  $-f_\alpha$  if necessary, we can assume that the open sets

$$V_\alpha := \{x \in K : f_\alpha(x) > (2C)^{-1}\}$$

are non-empty. Now, assume by contradiction that  $c_0(\kappa)$  does not isomorphically embed in  $C(K)$ , then, by [Ros, p. 227], we can conclude that  $K$  satisfies the  $\kappa$ -chain condition. Additionally, thanks to [Ros, p. 227], we can find infinitely many  $\alpha_n$ 's in  $\kappa$  such that there exists some  $x \in \bigcap V_{\alpha_n}$ . This implies that

$$\begin{aligned} (1 + \varepsilon)^p &> \left| \sum_{i=1}^p f_{\alpha_i} \right| \\ &\geq C^{-1} \left\| \sum_{i=1}^p f_{\alpha_i} \right\| \\ &\geq C^{-1} \sum_{i=1}^p f_{\alpha_i}(x) \\ &\geq (2C^2)^{-1} p \end{aligned}$$

which leads to a contradiction.  $\square$

Now we turn our attention to the opposite direction of the problem. Does a Banach space containing an isomorphic copy of  $c_0(\kappa)$  admit an equivalent  $\text{SQ}_\kappa$  norm? The answer to this question in full generality is unknown, but in the remainder of this section some partial results are provided.

**Theorem 2.21.** *If  $X$  is a dual Banach space containing an isomorphic copy of  $c_0$ , then  $X$  admits an equivalent  $\text{SQ}_{\aleph_0}$  norm.*

*Proof.* If  $X$  is a dual Banach space containing an isomorphic copy of  $c_0$ , then it contains an isomorphic copy of  $\ell_\infty$ . Moreover, because of its injectivity,  $\ell_\infty$  is complemented in  $X$ , say

$$X = Y \oplus \ell_\infty$$

for some subspace  $Y \subset X$ . Let  $|\cdot|$  be the  $\text{SQ}_{\aleph_0}$  norm defined in Example 2.4. Then, thanks to Corollary 2.10, we can conclude that the Banach space

$$Y \oplus_\infty (\ell_\infty, |\cdot|)$$

is  $\text{SQ}_{\aleph_0}$ .  $\square$

**Theorem 2.22.** *Let  $\lambda \geq \kappa \geq \aleph_1$ . If there exists a  $\kappa$ -complete ultrafilter  $\mathcal{U}$  in  $\lambda$ , then  $\ell_\infty(\lambda)$  admits an equivalent  $\text{SQ}_\kappa$  norm.*

*Proof.* Let  $|\cdot|$  be the transfinite analogous of the equivalent norm defined in Example 2.4. Let  $X$  be  $\ell_\infty(\lambda)$  endowed with the norm  $|\cdot|$  and fix  $A \in \mathcal{P}_\kappa(S_X)$ . Given  $n \in \mathbb{N}$  and  $x \in X$ , set

$$B_{n,x} := \left\{ \mu \in \lambda : |x(\mu) - \lim_{\mathcal{U}}(x)| < n^{-1} \right\} \in \mathcal{U}.$$

By assumption, we can find some

$$\mu \in \bigcap_{n,x} B_{n,x}.$$

It is then clear that  $|x + e_\mu| = 1$  holds for all  $x \in A$ . □

Theorem 2.22 is quite unsatisfactory because  $\lambda$  must be a large cardinal, at least the first measurable cardinal. Using a variation of this idea, by taking multiple ultrafilters instead of just a fixed one, we obtain another general result which says that, when  $X$  contains  $c_0(\kappa)$  and  $X/c_0(\kappa)$  is “small”, then  $X$  admits an equivalent  $\text{SQ}_\kappa$  norm.

**Theorem 2.23.** *Let  $\kappa$  be a cardinal with uncountable cofinality and  $X$  a Banach space with density character  $\kappa$ . If  $X$  contains an isomorphic copy of  $c_0(\kappa)$ , then  $X$  admits an equivalent  $\text{SQ}_{\text{cf}(\kappa)}$  norm.*

*Proof.* Without loss of generality, assume that the copy of  $c_0(\kappa)$  is isometric in  $X$ . Let  $Y \subset X$  be a subspace together with an isometric isomorphism

$$S : Y \rightarrow c_0(\kappa).$$

Extend  $S$  by Hahn-Banach to a norm-1 operator

$$T : X \rightarrow \ell_\infty(\kappa).$$

Now, we aim to define a suitable one-to-one mapping  $g : \kappa \rightarrow B_{X^*}$  such that all  $g(\alpha)$ 's vanish on  $Y$ . After doing so, we can define the equivalent norm

$$|x| := \max \left\{ \|x\|_{X/Y}, \sup_{\alpha \in \kappa} |T_\alpha(x) - g(\alpha)(x)| \right\}.$$

**First step:**  $|\cdot|$  defines an equivalent norm on  $X$ .

It is clear that  $|\cdot|$  is a norm and that  $|\cdot| \leq 2\|\cdot\|$ . Suppose by contradiction that we cannot obtain the opposite inequality with respect to any fixed constant,

then we can find a sequence  $(x_n)$  in  $S_X$  satisfying  $\lim |x_n| = 0$ . This implies that  $\lim \|x_n\|_{X/Y} = 0$ , so we can find a sequence  $(y_n)$  in  $Y$  such that

$$\lim \|x_n - y_n\| = 0.$$

We therefore can infer that  $\lim |x_n - y_n| = 0$ . Since  $\lim |x_n| = 0$ , it follows that  $\lim |y_n| = 0$ . On the other hand, we have that

$$\begin{aligned} |y_n| &= \sup |T_\alpha(y_n)| \\ &= \|y_n\|, \end{aligned}$$

hence  $\lim \|y_n\| = 0$ . This allows us to conclude that  $\lim \|x_n\| = 0$ , which is a contradiction.

**Second step:** If  $T_\beta(x) = g(\beta)(x)$ , then  $|x + tS^{-1}(e_\beta)| = \max\{|x|, |t|\}$ . Set  $u_\beta := S^{-1}(e_\beta)$  and notice that, since  $g(\beta)(u_\beta) = 0$ , we deduce that

$$|T_\beta(x + tu_\beta) - g(\beta)(x + tu_\beta)| = |t|.$$

Moreover,  $T_\alpha(u_\beta) = g_\alpha(u_\beta) = 0$ . Therefore,

$$\begin{aligned} |x + tu_\beta| &= \max \left\{ \|x\|_{X/Y}, |T_\beta(x + tu_\beta) - g(\beta)(x + tu_\beta)|, \right. \\ &\quad \left. \sup_{\alpha \in \kappa \setminus \{\beta\}} |T_\alpha(x) - g(\alpha)(x)| \right\} \\ &= \max\{|x|, |t|\}. \end{aligned}$$

**Third step:** If  $Z \subset X$  is a subspace with density character less than  $\kappa$ , then, for all  $\alpha \in \kappa$ , except for  $< \kappa$  many  $\alpha$ 's, there are functionals  $g_\alpha \in B_{X^*}$  that vanish on  $Y$  which coincide with  $T_\alpha$  over  $Z$ . Consider the continuous functions  $\phi : \beta\kappa \rightarrow B_{Z^*}$  defined by

$$\phi(\mathcal{U})(z) := \lim_{\mathcal{U}} T_\gamma(z),$$

where the ultralimit is taken with respect to  $\gamma$ , and the topology on  $B_{Z^*}$  is the weak\* topology. Notice that, given  $\alpha \in \kappa$ , if a non-principal ultrafilter  $\mathcal{U}_\alpha$  in  $\kappa$  satisfies  $\phi(\mathcal{U}_\alpha) = \phi(\alpha)$ , then

$$g_\alpha := \lim_{\mathcal{U}_\alpha} T_\gamma$$

satisfies the desired conditions. Hence, it is enough to show that for all but less than  $< \kappa$  many  $\alpha$ 's such an ultrafilter exists. Suppose by contradiction the opposite, then there is a set  $A \subset \kappa$  of cardinality  $\kappa$  so that  $\phi^{-1}\{\phi(\alpha)\}$

contains no non-principal ultrafilter for all  $\alpha \in A$ . In other words,  $\phi^{-1}\{\phi(\alpha)\}$  consists only of isolated points in  $\beta\kappa$ , but it is also compact by continuity. This implies that each set  $\phi^{-1}\{\phi(\alpha)\}$  is finite, thus  $\{\phi(\alpha) : \alpha \in A\}$  has cardinality  $\kappa$ . Let us now prove that each point  $\phi(\alpha)$  is isolated in the range  $\phi(\beta\kappa)$ . This leads a contradiction with the fact that  $B_{Z^*}$  has weight less than  $\kappa$ , since  $Z$  has density less than  $\kappa$ . Suppose then that  $\phi(\alpha)$  is not isolated in that range. Since  $\kappa$  is dense in  $\beta\kappa$ , then we have that  $\phi(\alpha)$  does not belong to the closure of the set

$$\{\phi(\beta) : \beta \in \kappa, \phi(\beta) \neq \phi(\alpha)\}.$$

Define the set

$$\mathcal{F} := \{B \subset \kappa : \exists W \text{ neighborhood of } \phi(\alpha) \text{ with } \phi^{-1}(W \setminus \{\phi(\alpha)\}) \subset B\}.$$

Notice that  $\mathcal{F}$  is a filter in  $\kappa$  that contains all cofinite sets, and satisfies

$$\phi(\alpha) = \bigcap_{B \in \mathcal{F}} \overline{\phi(B)}.$$

There is a non-principal ultrafilter  $\mathcal{U}$  in  $\kappa$  that contains  $\mathcal{F}$ , and it implies that

$$\begin{aligned} \phi(\mathcal{U}) &= \phi\left(\lim_{\mathcal{U}} \beta\right) \\ &= \lim_{\mathcal{U}} \phi(\beta) \\ &= \phi(\alpha). \end{aligned}$$

This contradicts the assumption that  $\phi^{-1}(\alpha)$  doesn't contain any non-principal ultrafilter.

**Fourth step:** Defining the map  $g$ .

Let  $\{X_\gamma : \gamma < \text{cf}(\kappa)\}$  be a family consisting of subspaces of  $X$  with density character less than  $\kappa$  such that every subspace of  $X$  with density character less than  $\text{cf}(\kappa)$  is contained in some  $X_\gamma$ . Thanks to the previous step, we can inductively select  $\alpha(\gamma) \in \kappa$  and  $g_\gamma \in B_{X^*}$  such that  $g_\gamma$  vanishes on  $Y$ ,  $g_\alpha$  coincides with  $T_{\alpha(\gamma)}$  over  $X_\gamma$  and  $\alpha(\gamma') \neq \alpha(\gamma)$  for all  $\gamma' < \gamma$ . Define

$$g(\alpha) := \begin{cases} g_{\alpha(\gamma)} & \text{if } \alpha = \alpha(\gamma) \text{ for some } \gamma < \text{cf}(\kappa), \\ 0 & \text{if } \alpha \notin \{\alpha(\gamma) : \gamma < \text{cf}(\kappa)\}. \end{cases}$$

**Fifth step:** Conclusion.

Fix a subspace  $Z \subset X$  with  $d(Z) < \text{cf}(\kappa)$  and find  $\gamma \in \text{cf}(\kappa)$  such that  $Z$  is contained in  $X_\gamma$ . By construction, we have that

$$T_{\alpha(\gamma)}(x) = g_{\alpha(\gamma)}(x)$$

holds for all  $x \in X_\gamma$ . By the second step, we can find  $y \in S_{(X,|\cdot|)}$  satisfying

$$|x + ty| \leq \max\{|x|, |t|\}$$

for every  $x \in X_\gamma$  and  $t \in \mathbb{R}$ . □

**Corollary 2.24.**  $\ell_\infty/c_0$  admits an equivalent  $\text{SQ}_{\text{cf}(2^{\aleph_0})}$  norm.

*Proof.*  $\ell_\infty/c_0$  contains a subspace isometric to  $c_0(2^{\aleph_0})$ , coming from an almost disjoint family of cardinality  $2^{\aleph_0}$ . □

## 2.4 Locally almost square Banach lattices

In this final section, we end with some remarks and observations about the local version of ASQ in the class of Banach lattices.

### 2.4.1 Introduction

**Definition 2.25.** Let  $X$  be a Banach space.

- (a)  $X$  is *locally almost square* (LASQ) if, for every  $x \in S_X$  and  $\varepsilon > 0$ , there exists  $y \in S_X$  such that

$$\|x \pm y\| \leq 1 + \varepsilon.$$

- (b)  $X$  is *weakly almost square* (WASQ) if, for every  $x \in S_X$ , there exists a weakly-null sequence  $(y_n)$  in  $S_X$  satisfying

$$\lim \|x \pm y_n\| = 1.$$

These properties were introduced in [ALL]. It is known that being ASQ implies being WASQ. On the other hand the fact that being WASQ implies being LASQ is trivial. It is known that  $L_1[0, 1]$  is WASQ [ALL, Example 2.4] but it is not ASQ, but the fact that being LASQ is not equivalent to being WASQ was proved in [KV]. In this section we aim to prove that being LASQ sometimes implies possessing a property which is very close to WASQ, namely, by substituting the sequence with a net. In order to do so, we have to restrict ourselves to the class of Banach lattices and we introduce a strengthening of being LASQ in which we additionally require the M-orthogonal element to be in the positive cone of the lattice.

**Definition 2.26.** Let  $X$  be a Banach lattice. We say that  $X$  is *positive locally almost square* (positive LASQ) whenever, given  $x \in S_X$  and  $\varepsilon > 0$ , there exists  $y \in S_X^+$  satisfying  $\|x \pm y\| \leq 1 + \varepsilon$ .

Notice that in Definition 2.26 it suffices to require that  $\|x + y\| \leq 1 + \varepsilon$  and  $x \in S_X^+$ . This is equivalent because, given  $x \in S_X$  and  $\varepsilon > 0$ , we can find  $y \in S_X^+$  such that  $\||x| + y\| \leq 1 + \varepsilon$  holds true. Since

$$|x \pm y| \leq |x| + y,$$

then

$$\|x \pm y\| \leq \||x| + y\| \leq 1 + \varepsilon.$$

Additionally, it is clear that being positive LASQ is more restrictive than being LASQ, nevertheless the two properties are different from each other, since  $L_1[0, 1]$  is LASQ, but it is also clear that no AL-space can be positive LASQ. The following lemma allows us to easily identify a class of positive LASQ Banach lattices.

**Lemma 2.27.** *Let  $X$  be an AM-space. If  $X$  is LASQ, then it is positive LASQ.*

*Proof.* Fix  $x \in S_X$  and  $\varepsilon > 0$ . Since  $X$  is LASQ, we can find  $y \in S_X$  satisfying  $\|x \pm y\| \leq 1 + \varepsilon$ . Observe that

$$\begin{aligned} |x + |y|| &= |x + (y \vee -y)| \\ &= |(x + y) \vee (x - y)| \\ &\leq |x + y| \vee |x - y|, \end{aligned}$$

therefore

$$\begin{aligned} \|x + |y|\| &\leq \||x + y| \vee |x - y|\| \\ &= \max\{\|x + y\|, \|x - y\|\} \\ &\leq 1 + \varepsilon. \end{aligned}$$

□

**Example 2.28.** The Banach space

$$X := \{f \in C[0, 1] : f(0) = -f(1)\}$$

is the classical example of a Banach space which is LASQ but not ASQ. It is clear that  $X$  embeds in  $C[0, 1]$  as a lattice, hence it is an AM-space. Therefore  $X$  is positive LASQ.

### 2.4.2 Positive LASQ and weakly null nets

**Theorem 2.29.** *If  $X$  is a positive LASQ Banach lattice, then, for every  $x \in S_X$ , there exists a weakly null net  $(y_\alpha)$  in  $S_X$  satisfying  $\lim \|x \pm y_\alpha\| = 1$ .*

*Proof.* Fix  $x \in S_X$  and let us temporarily fix some  $m \in \mathbb{N}$ . Find a sequence of strictly positive reals  $(\varepsilon_n^m)$  such that

$$\prod (1 + \varepsilon_n^m) \leq 1 + m^{-1}.$$

Construct recursively a sequence  $(x_n^m)$  in  $S_X^+$ . Define at first  $x_0^m := |x|$ . Now assume that we are given  $x_0^m, \dots, x_n^m$  and call

$$\hat{x}_n^m := \frac{\sum x_i^m}{\|\sum x_i^m\|} \in S_X^+,$$

and, since  $X$  is positive LASQ, find  $x_{n+1}^m \in S_X^+$  satisfying

$$\|\hat{x}_n^m \pm x_{n+1}^m\| \leq 1 + \varepsilon_n^m.$$

This concludes the construction of the sequence  $(x_n^m)$ . Since  $B_{X^{**}}$  is weak\* compact, then there is a subnet  $\{x_{n_\alpha}^m : \alpha \in \mathcal{A}_m\}$ , for some set of indices  $\mathcal{A}_m$ , of  $(x_n^m)$  which is weak\* convergent to some  $x_m^{**} \in X^{**}$ . Given  $\alpha, \beta \in \mathcal{A}_m$ , define

$$y_{\alpha, \beta}^m := x_{n_\alpha}^m - x_{n_\beta}^m$$

and observe that, thanks to [ALL, Lemma 2.2],

$$\begin{aligned} 1 &= \|x_0^m\| \\ &\leq \left\| \sum_{i=0}^{n+1} x_i^m \right\| \\ &= \left\| \left\| \sum_{i=0}^n x_i^m \right\| \cdot \hat{x}_n^m + x_{n+1}^m \right\| \\ &\leq (1 + \varepsilon_n^m) \left\| \sum_{i=0}^n x_i^m \right\| \\ &\leq \dots \\ &\leq \prod_{i=1}^n (1 + \varepsilon_i^m) \\ &\leq 1 + m^{-1} \end{aligned}$$

holds for all  $n \in \mathbb{N}$ . Since the  $x_n^m$ 's belongs to  $S_X^+$ , then, given any choice of  $\varepsilon_i$ 's in  $\{-1, 0, 1\}$ , we have that

$$\left| \sum \varepsilon_i x_i^m \right| \leq \sum x_i^m,$$

and, in particular,

$$\begin{aligned} \left\| \sum \varepsilon_i x_i^m \right\| &\leq \left\| \sum x_i^m \right\| \\ &\leq 1 + m^{-1}. \end{aligned}$$

From this observation, it follows that

$$\|x \pm y_{\alpha, \beta}^m\| \leq 1 + m^{-1}$$

and

$$\|y_{\alpha, \beta}^m\| \leq 1 + m^{-1}$$

hold for all  $\alpha, \beta \in \mathcal{A}_m$  such that  $n_\alpha \neq n_\beta$ . Additionally, it also implies that

$$\|x_{n_\alpha}^m \pm x_{n_\beta}^m\| \leq 1 + m^{-1}$$

must hold too. A standard argument now can be used to show that

$$\|x_{n_\alpha}^m \pm x_{n_\beta}^m\| \geq 1 - m^{-1}$$

and

$$\|x \pm y_{\alpha, \beta}^m\| \geq 1 - m^{-1}.$$

To sum up, we have thus far proved that

$$(\forall \alpha, \beta \in \mathcal{A}_m : n_\alpha \neq n_\beta) \left| \|x \pm y_{\alpha, \beta}^m\| - 1 \right| \leq m^{-1} \quad (2.2)$$

and

$$(\forall \alpha, \beta \in \mathcal{A}_m : n_\alpha \neq n_\beta) \left| \|y_{\alpha, \beta}^m\| - 1 \right| \leq m^{-1}. \quad (2.3)$$

We can now proceed to building the desired net. Let  $\mathcal{O}$  be the neighborhood filter around 0 in  $X$  endowed with the weak topology and let us choose, for every  $(m, U) \in \mathbb{N} \times \mathcal{O}$ , some  $y_{m, U} \in U$ , where  $y_{m, U} = y_{\alpha, \beta}^m$  for some  $\alpha, \beta \in \mathcal{A}_m$  satisfying  $n_\alpha \neq n_\beta$ . Notice that such  $\alpha$  and  $\beta$  can be found because, for all  $m \in \mathbb{N}$ ,  $y_{\alpha, \beta}^m = x_{n_\alpha}^m - x_{n_\beta}^m$ , where the net  $(x_{n_\gamma}^m)$  is weak\* convergent to  $x_m^{**}$  and the map  $\gamma \mapsto n_\gamma$  is both cofinal and increasing. Clearly the set  $\mathbb{N} \times \mathcal{O}$ , endowed with its natural preorder, is a directed set. It is easy to verify that the net

$$\{y_{m, U} : (m, U) \in \mathbb{N} \times \mathcal{O}\}$$

is weakly null and, thanks to (2.2) and (2.3), satisfies

$$\lim \|y_{m, U}\| = \lim \|x \pm y_{m, U}\| = 1.$$

Eventually, a normalization of the  $y_{m, U}$ 's provides the desired net.  $\square$

A small adaptation to the proof of Theorem 2.29 gives the following criterion.

**Theorem 2.30.** *Let  $X$  be a Banach lattice such that  $X^*$  is separable. If  $X$  is positive LASQ, then it is WASQ.*

*Proof.* Since  $X^*$  is separable, then  $B_{X^{**}}$  is metrizable, with respect to the weak\* topology. Hence, in the proof of Theorem 2.29, we can replace the subnet  $(x_{n_\alpha}^m)$  with a subsequence  $(x_{n_k}^m)$ , and define

$$y_k^m := x_{n_k}^m - x_{n_{k-1}}^m,$$

which is clearly weakly null. Let now  $(x_n^*)$  be a dense sequence in  $X^*$  and, for every  $m \in \mathbb{N}$ , find  $n_m \in \mathbb{N}$  satisfying  $|x_i^*(y_n^m)| \leq m^{-1}$  for all  $1 \leq i \leq m$  and  $n \geq n_m$ . The sequence  $(y_{n_m}^m)$  is weakly null and, thanks to (2.2) and (2.3), it satisfies

$$\lim \|y_{n_m}^m\| = \lim \|x \pm y_{n_m}^m\| = 1.$$

Again, a normalization argument concludes the proof.  $\square$

### 2.4.3 Positive LASQ and the diameter two property

In this last subsection, we employ Theorem 2.29 to prove that being positive LASQ implies the *diameter two property* (D2P), i.e. every relatively weakly open set of the unit ball has diameter two. From this, we are able to extend previously known results to provide new examples of spaces enjoying the D2P.

**Theorem 2.31.** *Let  $X$  be a Banach lattice. If  $X$  is positive LASQ, then  $X$  has the D2P.*

*Proof.* Let  $U$  be a non-empty relatively weakly open set in  $B_X$  and  $\varepsilon > 0$ . Notice that

$$U_\delta := \bigcap_{i=1}^n \{x \in B_X : x_i^*(x) \geq 1 - \delta\} \subset U$$

for some  $0 < \delta \leq \varepsilon$  and  $x_1^*, \dots, x_n^* \in S_{X^*}$ . Since  $X$  is LASQ, then it is infinite dimensional, hence we can find  $x \in U_{\delta/2} \cap S_X$  and construct the net  $(y_\alpha)$  as in Theorem 2.29 with respect to  $x$ . It is now clear that we can find  $\alpha$  satisfying  $|x_i^*(y_\alpha)| \leq \delta/2$  for all  $1 \leq i \leq n$  and such that both  $\|y_\alpha\|$  and  $\|x \pm y_\alpha\|$  belongs to the interval  $[1 - \delta/2, 1 + \delta/2]$ . Therefore

$$\begin{aligned} \|x + y_\alpha - (x - y_\alpha)\| &= 2\|y_\alpha\| \\ &\geq 2 - \delta \\ &\geq 2 - \varepsilon \end{aligned}$$

and

$$\begin{aligned} x_i^*(x \pm y_\alpha) &\geq 1 - \delta/2 - \delta/2 \\ &\geq 1 - \varepsilon \end{aligned}$$

holds for all  $1 \leq i \leq n$ , hence, up to a perturbation argument,  $x \pm y_\alpha \in U$ .  $\square$

Notice that the converse of Theorem 2.31 doesn't hold. In fact  $L_1[0, 1]$ , as already noted, is not positive LASQ, but it is WASQ and therefore has the D2P [Kub, Proposition 2.6].

**Corollary 2.32.** *Let  $(\Omega, \Sigma, \mu)$  be a complete,  $\sigma$ -finite measure space,  $E$  a Köthe function space over  $\Omega$  and  $X$  a Banach lattice. If  $X$  is positive LASQ, then the Köthe-Bochner space  $E(X)$  is positive LASQ and therefore has the D2P.*

*Proof.* Notice that  $E(X)$  is a Banach lattice in a natural way, so the claim is meaningful. The proof is just an adaptation of [Har3, Theorem 3.1]. So in the following we will point out only the small changes needed to make the proof work.

1. Find the sequence  $(z_n)$  in  $S_X^+$ .
2. Consider the set  $F'(x) := F(x) \cap S_Z^+$  instead of  $F(x)$ .
3. Define  $g(x, z)$  for  $x \in \Omega'$  and  $z \in S_Z^+$ .

With these choices the selection  $\tilde{f}$  is hence defined from  $\Omega'$  to  $S_Z^+$  and therefore  $h \in S_{E(X)}^+$ .  $\square$

The next result concerns ultrapowers. Let us recall that the ultrapower of a Banach lattice is still a Banach lattice if we take as positive cone the elements that admit representatives such that each coordinate is positive [JL, Page 55]. Keeping this in mind, an obvious adaptation of [Har1, Proposition 4.2] proves the following.

**Corollary 2.33.** *Let  $X$  be a Banach lattice and  $\mathcal{U}$  a non-principal ultrafilter on  $\mathbb{N}$ . If  $X$  is positive LASQ, then the ultrapower  $\prod_{\mathcal{U}} X$  is positive LASQ and therefore has the D2P.*

Before stating the last result, let us introduce some notation. Given  $y \in S_X$ , define  $n(X, y)$  as the largest non-negative real number  $r$  satisfying

$$r\|x\| \leq \sup\{|x^*(x)| : x^* \in S_X \text{ and } x^*(y) = 1\}.$$

**Corollary 2.34.** *Let  $X$  be a Banach lattice and  $Y$  an AL-space such that there is  $y^* \in S_{Y^*}$  satisfying  $n(Y^*, y^*) = 1$ . If  $X$  is positive ASQ, i.e., for every  $x_1, \dots, x_n \in S_X$  and  $\varepsilon > 0$ , there exists  $y \in S_X^+$  satisfying  $\|x_i \pm y\| \leq 1 + \varepsilon$  for  $1 \leq i \leq n$ , then  $X \hat{\otimes}_\pi Y$  is positive LASQ and therefore has the D2P.*

*Proof.* Since  $Y$  is an AL-space, then  $X \hat{\otimes}_\pi Y$  is a Banach lattice [Fre2, Theorem 2B]. The rest of the proof is an adaptation of [LLR, Proposition 2.11]. So let us point out the main differences.

1. It suffices to consider the case  $u \in S_{X \hat{\otimes}_\pi Y}^+$ .
2. Since the convex envelop of  $S_X^+ \otimes S_Y^+$  is dense in  $S_{X \hat{\otimes}_\pi Y}^+$  [Fre2, 1E (ii)], we can find  $v = \sum_{i=1}^n \lambda_i x_i \otimes y_i$  such that  $\|u - v\| \leq \varepsilon$ ,  $x_i \in S_X^+$  and  $y_i \in S_Y^+$  for  $1 \leq i \leq n$ .
3. As  $X$  is positive ASQ, we can find  $x \in S_X^+$  such that  $\|x_i \pm x\| \leq 1 + \varepsilon$  holds for  $1 \leq i \leq n$ .
4. With these choices,  $z := x \otimes \sum_{i=1}^n \lambda_i y_i \in S_X^+$  thanks to [Fre2, 1E (ii)] and  $\|u \pm z\| \leq 1 + 2\varepsilon$ .

□

## 2.5 Open problems

In this final section, let us point out the main questions that remain unsolved.

*Problem 2.1.* Does an equivalent of Theorem 2.2 hold for the property  $\text{SQ}_{\aleph_0}$  too? That is, is being  $\text{SQ}_{\aleph_0}$  equivalent to the property that, given any finite-dimensional subspace, we can find an element which is M-orthogonal to the subspace?

*Problem 2.2.* Let  $\kappa$  be an infinite cardinal. If a Banach space contains an isomorphic copy of  $c_0(\kappa)$ , does it admit an equivalent  $\text{SQ}_\kappa$  norm?

*Problem 2.3.* Is the assumption of being positive LASQ together with the lattice structure necessary in Theorem 2.29? That is, is it true that if a Banach space is LASQ, then, for every  $x$  in its unit sphere, we can find a weakly null net  $(y_\alpha)$  in the unit sphere such that  $\lim \|x \pm y_\alpha\| = 1$ ?

# Chapter 3

## The symmetric strong diameter two property

A Banach space  $X$  has the *symmetric strong diameter two property* (SSD2P) if and only if, for every  $n \in \mathbb{N}$ , for all slices  $S_1, \dots, S_n$  of  $B_X$ , and  $\varepsilon > 0$ , there are  $x_1, \dots, x_n, y \in B_X$  such that

(a)  $\|y\| \geq 1 - \varepsilon$ ,

(b)  $\|x_i \pm y\| \leq 1$ ,

(c)  $x_i \in S_i$

for  $1 \leq i \leq n$ . The SSD2P was introduced in [ANP]. Known examples of Banach spaces enjoying the SSD2P include (see [HLLN]) ASQ Banach spaces, infinite-dimensional Lindenstrauss spaces, infinite-dimensional uniform algebras, Banach spaces with an infinite-dimensional centralizer, somewhat regular subspaces of  $C_0(X)$  spaces, where  $X$  is an infinite locally compact Hausdorff space and Müntz spaces.

In this chapter, we study transfinite analogues of the SSD2P. We investigate the stability of these properties under direct sums, and under projective tensor products. Moreover, we analyze the differences between these transfinite extensions against the classical definition. This chapter is based on [ACLLR] and [Cia2].

### 3.1 Definition and some examples

**Definition 3.1.** Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal.

- (a) We say that  $X$  has the *symmetric strong diameter two property with respect to  $\kappa$*  (SSD2P $_{\kappa}$ ) if, for every  $A \in \mathcal{P}_{\kappa}(S_{X^*})$  and  $\lambda \in (0, 1)$ , there are  $B \subset B_X$  and  $y \in B_X$  such that  $B$  is  $\lambda$ -norming  $A$ ,  $B \pm y \subset B_X$  and  $\|y\| \geq \lambda$ .
- (b) We say that  $X$  has the *norming and attaining symmetric strong diameter two property with respect to  $\kappa$*  (NASSD2P $_{\kappa}$ ) if, for every  $A \in \mathcal{P}_{\kappa}(S_{X^*})$ , there are  $B \subset S_X$  and  $y \in S_X$  such that  $B$  norms  $A$  and  $B \pm y \subset S_X$ .

It is clear that the SSD2P $_{\aleph_0}$  coincides with the classical definition of the SSD2P. Now, let us devote the rest of the section to provide some examples of spaces enjoying these transfinite extensions of the SSD2P.

**Example 3.2.** If a Banach space is ASQ $_{\kappa}$  (respectively, SQ $_{\kappa}$ ), then it possesses the SSD2P $_{\kappa}$  (respectively, NASSD2P $_{\kappa}$ ), whenever  $\kappa \geq \aleph_1$ .

*Proof.* Let us prove the SQ $_{\kappa}$  case only. Fix  $A \in \mathcal{P}_{\kappa}(S_{X^*})$  and find any set  $B \in \mathcal{P}_{\kappa}(S_X)$  that norms  $A$ . Since  $X$  is SQ $_{\kappa}$ , there exists  $y \in S_X$  such that  $B \pm y \subset S_X$ .  $\square$

**Theorem 3.3.** *Let  $L$  be a locally compact Hausdorff space. Then  $C_0(L)$  fails the SSD2P $_{d(L)^+}$ . On the other hand, if  $c(L) \geq \aleph_1$ , then  $C_0(L)$  enjoys the NASSD2P $_{c(L)}$ .*

*Proof.* Let  $\mathcal{D}$  be a dense set in  $L$  and consider the set of functionals

$$\{\delta_x : x \in \mathcal{D}\}.$$

Assume by contradiction that  $C_0(L)$  possesses the SSD2P $_{d(L)^+}$ , then we can find functions  $f_x$ 's and  $g$  in  $B_{C_0(L)}$  satisfying  $\|g\| \geq 2/3$ ,  $f_x(x) \geq 2/3$  and  $\|f_x \pm g\| \leq 1$  for every  $x \in \mathcal{D}$ . Since  $\mathcal{D}$  is dense, then we can find  $x \in \mathcal{D}$  such that  $|g(x)| > 1/3$ , which clearly leads to a contradiction.

For the second part of the claim, fix  $\mathcal{M} \in \mathcal{P}_{c(L)}(S_{C_0(L)^*})$ . Find a cellular family  $\mathcal{C}$  in  $L$  of size  $|\mathcal{M}| < |\mathcal{C}| \leq c(L)$  and, given  $\mu \in \mathcal{M}$  and  $m \in \mathbb{N}$ , define the set

$$\mathcal{C}_{\mu,m} := \{C \in \mathcal{C} : |\mu|(C) > m^{-1}\},$$

and notice that there are less than  $|\mathcal{C}|$  many such  $\mathcal{C}_{\mu,m}$ 's. Therefore, there must exist some  $C \in \mathcal{C}$  satisfying  $|\mu|(C) = 0$  for each  $\mu \in \mathcal{M}$ . Without loss of generality, we may even assume that  $|\mu|(\overline{C}) = 0$ , in fact, if that's not the case, we can replace  $C$  with some non-empty open set  $C'$  satisfying  $\overline{C'} \subset C$ . Now, find functions  $f_{\mu,m}$ 's in  $S_{C_0(L)}$  such that

$$\mu(f_{\mu,m}) \geq 1 - (3m)^{-1}.$$

Moreover, since the measures  $\mu$ 's are regular, we can find compact sets  $K_{\mu,m}$  in  $X \setminus \overline{C}$  satisfying

$$|\mu|(K_{\mu,m}) \geq 1 - (3m)^{-1}.$$

Construct Urysohn's functions  $g_{\mu,m}$ 's and  $h$  in  $S_{C_0(L)}$  satisfying  $g_{\mu,m}|_{K_{\mu,m}} = 1$ ,  $g_{\mu,m}|_{\overline{C}} = 0$ , and  $h|_{X \setminus C} = 0$ . Define

$$i_{\mu,m} := \frac{f_{\mu,m} \cdot g_{\mu,m}}{\|f_{\mu,m} \cdot g_{\mu,m}\|} \in S_{C_0(L)}$$

and notice that  $\|i_{\mu,m} \pm h\| = 1$ . Moreover, given any  $\mu \in \mathcal{M}$  and  $m \in \mathbb{N}$ , we have that

$$\begin{aligned} \mu(i_{\mu,m}) &\geq \int_L f_{\mu,m} g_{\mu,m} d\mu \\ &\geq \int_{K_{\mu,m}} f_{\mu,m} d\mu - (3m)^{-1} \\ &\geq \int_L f_{\mu,m} d\mu - 2(3m)^{-1} \\ &\geq 1 - m^{-1}. \end{aligned}$$

□

**Example 3.4.** Let us employ Theorem 3.3 to produce some examples of spaces possessing or failing the transfinite SSD2P.

- (a) Let  $L$  be a separable locally compact Hausdorff space, e.g.  $L = [0, 1]$ . Then  $C_0(L)$  fails the SSD2P $_{\aleph_1}$ .
- (b)  $C(\beta\mathbb{N})$  and  $C(\beta\mathbb{N} \setminus \mathbb{N})$  enjoy the NASSD2P $_{2^{\aleph_0}}$ , since, thanks to [Hod, 7.22], we have that

$$c(\beta\mathbb{N}) = c(\beta\mathbb{N} \setminus \mathbb{N}) = 2^{\aleph_0}.$$

- (c) Let  $B$  be a Boolean algebra and let  $\mathcal{S}(B)$  be the Stone space associated to  $B$ . It is clear that the set  $\{\{b\} : b \in B\}$  defines a cellular family in  $\mathcal{S}(B)$ . Now let us consider a regular positive Borel measure  $\mu$  over some locally compact Hausdorff space  $X$ . Define  $\mathfrak{B}_\mu$  the set of measurable sets modulo the negligible sets in  $X$ . It is known that  $L_\infty(\mu)$  is isometrically isomorphic to  $C(\mathcal{S}(\mathfrak{B}_\mu))$  (see e.g. pages 27–29 in [DLS]), therefore we conclude that  $L_\infty(\mu)$  enjoys the NASSD2P $_{|\mathfrak{B}_\mu|}$ , whenever  $|\mathfrak{B}_\mu| \geq \aleph_1$ . In particular, whenever  $\kappa \geq \aleph_1$  and  $\mu$  is the counting measure over  $\kappa$ ,  $|\mathfrak{B}_\mu| = \kappa$ , thus it follows that  $\ell_\infty(\kappa)$  enjoys the NASSD2P $_\kappa$ , but it fails the SSD2P $_{\kappa^+}$ , because  $d(\ell_\infty(\kappa)) = \kappa$ .

To conclude this section, let us provide a criterion to identify cellular families in particular classes of topological spaces, including Alexandrov-discrete spaces.

**Lemma 3.5.** *Let  $X$  be a  $T_{2\frac{1}{2}}$  space and  $\kappa$  an infinite cardinal. If there are  $\kappa$  many points in  $X$  such that every non-empty intersection of at most  $\kappa$  many neighborhoods is still a neighborhood, then  $c(X) \geq \kappa$ .*

*Remark 3.6.* Lemma 3.5, when  $\kappa = \aleph_0$ , can be stated by requiring that  $X$  contains infinitely many p-points, that is, points for which every  $G_\delta$  containing them is a neighborhood (see [Mil] for a detailed treatment of p-points)

*Proof.* Let  $A \subset X$  be a set of cardinality  $\kappa$  such that every non-empty intersection of at most  $\kappa$  many neighborhoods of  $x$  is still a neighborhood for every  $x \in A$ . Since  $X$  is  $T_{2\frac{1}{2}}$ , for every distinct  $x, y \in A$  we can find a closed neighborhood  $U_{x,y}$  of  $x$  which doesn't contain  $y$ . By assumption

$$U_x := \left( \bigcap_{y \in A \setminus \{x\}} U_{x,y}^\circ \right) \cap \left( \bigcap_{y \in A \setminus \{x\}} X \setminus U_{y,x} \right)$$

is an open neighborhood of  $x$ . Notice that, given distinct  $x, y \in A$  we have that

$$\begin{aligned} U_x \cap U_y &\subset U_{x,y}^\circ \cap (X \setminus U_{y,x}) \cap U_{y,x}^\circ \cap (X \setminus U_{x,y}) \\ &= \emptyset \end{aligned}$$

In other words,  $\{U_x : x \in A\}$  defines a cellular family of size  $\kappa$ . □

Notice that the assumption in Lemma 3.5 is far from being necessary. It is consistent with ZFC that  $\beta\mathbb{N} \setminus \mathbb{N}$  contains no p-points. Nevertheless, as already recalled in Example 3.4 (b),  $\beta\mathbb{N} \setminus \mathbb{N}$  admits a cellular family of cardinality  $2^{\aleph_0}$ .

## 3.2 Stability results

In this section, we investigate the stability of the transfinite SSD2P under direct sums and projective tensor products.

### 3.2.1 Direct sums

In this subsection, we investigate the stability of the transfinite SSD2P under  $\ell_\infty$  and  $c_0$  sums, because they are known to be the only direct sums that preserve the SSD2P [HLLN, Theorem 3.1]. Moreover, since  $c_0(\mathbb{N}, X_n)$  is always ASQ, and hence has the SSD2P, and, thanks to Theorem 2.11, we only need to focus, as far as  $c_0$  sums are concerned, to countable sums for the SSD2P $_\kappa$ , for  $\kappa \geq \aleph_1$ . The following Theorem, informally states that the Banach space  $c_0(\mathbb{N}, X_n)$  has the SSD2P $_\kappa$ , whenever, for every  $\lambda \in (0, 1)$ , there is some  $n$  such that  $X_n$  enjoys “ $\lambda$  – SSD2P $_\kappa$ ”.

**Theorem 3.7.** *Let  $\{X_n : n \in \mathbb{N}\}$  be a family of Banach spaces and  $\kappa \geq \aleph_1$ . If, for every  $\lambda \in (0, 1)$ , there exists  $n \in \mathbb{N}$  satisfying:*

$$\begin{aligned} & (\forall A \in \mathcal{P}_\kappa(S_{X_n^*})) (\exists B \subset B_{X_n}) (\exists y \in B_{X_n}) \\ & B \text{ is } \lambda\text{-norming } A, \|y\| \geq \lambda \text{ and } B \pm y \subset B_{X_n}, \end{aligned} \quad (3.1)$$

then  $c_0(\mathbb{N}, X_n)$  enjoys the SSD2P $_\kappa$ . If in addition  $\text{cf}(\kappa) \geq \aleph_1$ , then the vice-versa holds too.

*Proof.* Fix  $A \in \mathcal{P}_\kappa(S_{c_0(\mathbb{N}, X_n)})$  and  $\varepsilon > 0$ . Let  $\{x_\alpha^* : \alpha \in \mathcal{A}\}$  be an enumeration of  $A$  and find  $m \in \mathbb{N}$  such that (3.1) holds for  $\lambda = (1 - \varepsilon)^{\frac{1}{2}}$ . There are  $x_\alpha^m$ 's and  $y^m$  in  $B_{X_m}$  satisfying

- (a)  $\|y^m\| \geq 1 - \varepsilon$ ,
- (b)  $\|x_\alpha^m \pm y^m\| \leq 1$ ,
- (c)  $x_\alpha^*(m)(x_\alpha^m) \geq (1 - \varepsilon)^{\frac{1}{2}} \|x_\alpha^*(m)\|$

for every  $\alpha \in \mathcal{A}$ . Now, for each  $m \neq n \in \mathbb{N}$  and  $\alpha \in \mathcal{A}$ , find  $x_\alpha^n \in B_{X_n}$  satisfying

$$x_n^*(x_\alpha^n) \geq (1 - \varepsilon)^{\frac{1}{2}} \|x_\alpha^*(n)\|.$$

Since  $x_\alpha^* \in \ell_1(\mathbb{N}, X_n^*)$ , there exists some  $n_\alpha \in \mathbb{N}$  such that

$$\left\| \sum_{n=1}^{n_\alpha} x_\alpha^*(n) e_n \right\| \geq (1 - \varepsilon)^{\frac{1}{2}}.$$

Call

$$x_\alpha := \sum_{n=1}^{n_\alpha} x_\alpha^n e_n \in B_{c_0(\mathbb{N}, X_n)}$$

and  $y := y^m e_m \in B_{c_0(\mathbb{N}, X_n)}$ . Now, notice that

$$\begin{aligned} x_\alpha^*(x_\alpha) &= \sum_{n=1}^{n_\alpha} x_\alpha^*(n)(x_\alpha^n) \\ &\geq (1 - \varepsilon)^{\frac{1}{2}} \sum_{n=1}^{n_\alpha} \|x_\alpha^*(n)\| \\ &\geq 1 - \varepsilon, \end{aligned}$$

which means that the  $x_\alpha$ 's are  $(1 - \varepsilon)$ -norming  $A$ . On the other hand, we have that

$$\begin{aligned} \|y\| &= \|y^m\| \\ &\geq 1 - \varepsilon \end{aligned}$$

and  $\|x_\alpha \pm y\| \leq 1$  holds true by construction for every  $\alpha \in \mathcal{A}$ .

For the vice-versa, assume that  $\text{cf}(\kappa) \geq \aleph_1$  and fix  $\lambda \in (0, 1)$ . For each  $n \in \mathbb{N}$ , fix some  $A_n \in \mathcal{P}_\kappa(S_{X_n^*})$ , and consider the set

$$A := \{x^* \cdot e_n : n \in \mathbb{N} \text{ and } x^* \in A_n\}.$$

Notice that

$$\begin{aligned} |A| &\leq \aleph_0 \cdot \sup |A_n| \\ &< \kappa, \end{aligned}$$

hence, there exists a set  $B \subset B_{c_0(\mathbb{N}, X_n)}$  which is  $\lambda$ -norming  $A$ , and  $y \in B_{c_0(\mathbb{N}, X_n)}$  such that  $\|y\| \geq \lambda$  and  $B \pm y \subset B_{c_0(\mathbb{N}, X_n)}$ . Find  $n \in \mathbb{N}$  such that  $\|y(n)\| \geq \lambda$  and notice that the set

$$B_n := \{x(n) : x \in B\}$$

is  $\lambda$ -norming  $A_n$ . Eventually, observe that, given  $x \in B$ ,

$$\begin{aligned} \|x(n) \pm y(n)\| &\leq \|x \pm y\| \\ &\leq 1. \end{aligned}$$

□

The proof of Theorem 3.7 can be easily adjusted to  $\ell_\infty$  sums too. As a matter of fact, it is not needed to find  $n_\alpha$  and one can define the formal series

$$x_\alpha := \sum x_\alpha^n e_n \in B_{\ell_\infty(\mathbb{N}, X_n)}.$$

Additionally, since  $\ell_\infty$  doesn't impose requirements on the size of the support of its element, it is easy to extend the result to higher cardinals too. Hence, the following theorem is proved.

**Theorem 3.8.** *Let  $\{X_\alpha : \alpha \in \mathcal{A}\}$  be a family of Banach spaces and  $\kappa \geq \aleph_1$ . If, for every  $\lambda \in (0, 1)$ , there exists  $\alpha \in \mathcal{A}$  satisfying:*

$$\begin{aligned} & (\forall A \in \mathcal{P}_\kappa(S_{X_\alpha^*})) (\exists B \subset B_{X_\alpha}) (\exists y \in B_{X_\alpha}) \\ & B \text{ is } \lambda\text{-norming } A, \|y\| \geq \lambda \text{ and } B \pm y \subset B_{X_\alpha}, \end{aligned}$$

*then  $\ell_\infty(\mathcal{A}, X_\alpha)$  enjoys the SSD2P $_\kappa$ . If in addition  $\text{cf}(\kappa) > |\mathcal{A}|$ , then the vice-versa holds too.*

**Corollary 3.9.** *Let  $X$  and  $Y$  be Banach spaces, and  $\kappa \geq \aleph_1$ . Then  $X \oplus_\infty Y$  has the SSD2P $_\kappa$  if and only if either  $X$  or  $Y$  enjoys the SSD2P $_\kappa$ .*

### 3.2.2 Projective tensor products

It is known that the SSD2P is preserved by projective tensor products [Lan2, Theorem 2.2]. In this subsection, we aim to extend this result to the transfinite SSD2P. It should be noted that our proof is significantly different from the original one, as its argument relies on an equivalent characterization of the SSD2P which doesn't extend to its transfinite analogue. We will discuss this difference in behaviour in the next section.

**Theorem 3.10.** *Let  $X$  and  $Y$  be Banach spaces, and  $\kappa \geq \aleph_1$ . If  $X$  and  $Y$  possess the SSD2P $_\kappa$ , then  $X \hat{\otimes}_\pi Y$  enjoys the SSD2P $_\kappa$ .*

*Proof.* Fix  $\mathcal{B} \in \mathcal{P}_\kappa(S_{X \hat{\otimes}_\pi Y})$  and  $\varepsilon > 0$ . There exists  $x_B \otimes y_B \in S_X \otimes S_Y$  satisfying  $B(x_B \otimes y_B) \geq (1 - \varepsilon)^{\frac{1}{3}}$ , and define

$$B' := \frac{B(\cdot \otimes y_B)}{\|B(\cdot \otimes y_B)\|} \in S_{X^*}.$$

Since  $X$  possesses the SSD2P $_\kappa$ , we can find  $x'_B$ 's and  $x$  in  $B_X$  such that  $\|x\| \geq (1 - \varepsilon)^{\frac{1}{2}}$ ,  $x'_B \pm x \in B_X$  and  $B'(x'_B) \geq (1 - \varepsilon)^{\frac{1}{3}}$  hold for every  $B \in \mathcal{B}$ . Now, call

$$B'' := \frac{B(x'_B \otimes \cdot)}{\|B(x'_B \otimes \cdot)\|} \in S_{Y^*}.$$

Since  $Y$  possesses the SSD2P $_\kappa$ , we can find  $y''_B$ 's and  $y$  in  $B_Y$  such that  $\|y\| \geq (1 - \varepsilon)^{\frac{1}{2}}$ ,  $y''_B \pm y \in B_Y$  and  $B''(y''_B) \geq (1 - \varepsilon)^{\frac{1}{3}}$  hold for every  $B \in \mathcal{B}$ . Eventually, define

$$\begin{aligned} u_B & := x'_B \otimes y''_B \in B_{X \hat{\otimes}_\pi Y}, \\ v & := x \otimes y \in B_{X \hat{\otimes}_\pi Y}, \end{aligned}$$

and notice that  $\|v\| = \|x\|\|y\| \geq 1 - \varepsilon$ . Moreover, the fact that  $u_B \pm v \in B_{X \hat{\otimes}_{\pi} Y}$  follows from [Lan2, Lemma 2.1]. Finally, we claim that the set  $\{u_B : B \in \mathcal{B}\}$  is  $(1 - \varepsilon)$ -norming  $\mathcal{B}$ . To this aim, we only need to observe that

$$\begin{aligned} B(x'_B \otimes y''_B) &\geq (1 - \varepsilon)^{\frac{1}{3}} \|B(x'_B \otimes \cdot)\| \\ &\geq (1 - \varepsilon)^{\frac{1}{3}} B(x'_B, y_B) \\ &\geq (1 - \varepsilon)^{\frac{2}{3}} \|B(\cdot \otimes y_B)\| \\ &\geq (1 - \varepsilon)^{\frac{2}{3}} B(x_B \otimes y_B) \\ &\geq 1 - \varepsilon. \end{aligned}$$

□

### 3.3 Transfinite versus finite

The argument showing that the SSD2P is preserved by projective tensor products [Lan2, Theorem 2.2] relies on the following equivalent characterization of the SSD2P [HLLN, Theorem 2.1].

**Theorem 3.11.** *Let  $X$  be a Banach space. The following are equivalent:*

- (i)  $X$  has the SSD2P.
- (ii) Given  $n \in \mathbb{N}$ , non-empty relatively weakly open sets  $U_1, \dots, U_n$  in  $B_X$ , and  $\varepsilon > 0$ , there exist  $x_1, \dots, x_n, y \in B_X$  such that  $\|y\| \geq 1 - \varepsilon$ ,  $x_i \pm y \in B_X$  and  $x_i \in U_i$  for all  $1 \leq i \leq n$ .
- (iii) Given  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in S_X$ , there exist nets  $(y_\alpha^i)$  and  $(z_\alpha)$  in  $S_X$  such that  $\lim \|y_\alpha^i \pm z_\alpha\| = 1$  and, with respect to the weak topology on  $X$ ,  $\lim z_\alpha = 0$  and  $\lim y_\alpha^i = x_i$  hold for all  $1 \leq i \leq n$ .

In this section, we aim to show that Theorem 3.11 doesn't extend to higher cardinals.

#### 3.3.1 The implication (ii) $\iff$ (iii)

In Example 3.4 (c), we showed that  $\ell_\infty(\kappa)$  fails the SSD2P $_{\kappa^+}$ . Nevertheless, we claim that it satisfies the transfinite extension of (iii) from Theorem 3.11 in a very strong sense. In fact, let us fix  $x \in S_{\ell_\infty(\kappa)}$  and any ordinal  $\mu \in \kappa$ . Set

$$y_{x,\mu} := x - x(\mu)e_\mu \in B_{\ell_\infty(\kappa)}$$

and  $z_\mu := e_\mu \in S_{\ell_\infty(\kappa)}$ . It is then clear that  $y_{x,\mu} \pm z_\mu \in S_{\ell_\infty(\kappa)}$  and that, in the weak topology,  $\lim z_\mu = 0$  and  $\lim y_{x,\mu} = x$  hold for every  $x \in S_{\ell_\infty(\kappa)}$ . In other words,  $\ell_\infty(\kappa)$  satisfies condition (iii) from Theorem 3.11 with respect to set of cardinality  $|\ell_\infty(\kappa)| = 2^\kappa$ .

### 3.3.2 The implication (i) $\iff$ (ii)

Consider the Banach space  $c_0(\mathbb{N}_{\geq 2}, \ell_n(\kappa))$ , which we proved in Example 2.18 to be  $\text{ASQ}_\kappa$ , and hence it enjoys the  $\text{SSD2P}_\kappa$ . Nevertheless, condition (ii) from Theorem 3.11 majorly fails with respect to countable families consisting of non-empty relatively weakly open sets in the unit ball. This claim follows from the following result.

**Theorem 3.12.** *Let  $\{X_n : n \in \mathbb{N}\}$  be a family of Banach spaces. If, given any infinite family of slices  $\mathcal{S}$  in  $B_{c_0(\mathbb{N}, X_n)}$  and  $\varepsilon > 0$ , there exist a set  $B$  and  $y$  in  $B_{c_0(\mathbb{N}, X_n)}$  such that  $\|y\| \geq 1 - \varepsilon$ ,  $B \pm y \subset B_{c_0(\mathbb{N}, X_n)}$  and  $B$  visits every slice in  $\mathcal{S}$ , then at least one of the  $X_n$ 's is not uniformly convex.*

*Proof.* Let

$$A := \{x_n^* : n \in \mathbb{N}\} \in \mathcal{P}_{\aleph_1}(S_{\ell_1(\mathbb{N}, X_n^*)})$$

be a set consisting of functionals satisfying  $x_n^*(m) \neq 0$  and  $\|x_n^*(n)\| \geq \|x_n^*(m)\|$  for every  $n, m \in \mathbb{N}$ . Now, consider the family  $\mathcal{S}$  consisting of all the slices of the form

$$S_{n,m} := \{x \in B_{c_0(\mathbb{N}, X_n)} : x_n^*(x) \geq 1 - m^{-1}\|x_n^*(m)\|\},$$

and fix  $\varepsilon > 0$ . By assumption, we can find  $x_{n,m} \in S_{n,m}$  and  $y \in B_{c_0(\mathbb{N}, X_n)}$  such that  $\|y\| \geq 1 - \varepsilon$  and  $x_{n,m} \pm y \in B_{c_0(\mathbb{N}, X_n)}$  hold for every  $n, m \in \mathbb{N}$ . We can find  $p \in \mathbb{N}$  such that  $\|y(p)\| \geq 1 - \varepsilon$  and, since  $x_{p,m} \in S_{p,m}$ , then

$$\begin{aligned} 1 - m^{-1}\|x_p^*(m)\| &\leq x_p^*(x_{p,m}) \\ &\leq \sum_{n \neq p} \|x_p^*(n)\| + x_p^*(p)x_{p,m}(p) \\ &= 1 - \|x_p^*(p)\| + x_p^*(p)x_{p,m}(p), \end{aligned}$$

which implies that

$$x_p^*(p)x_{p,m}(p) \geq \|x_p^*(p)\| - m^{-1}\|x_p^*(m)\|,$$

hence

$$\begin{aligned} \|x_{p,m}(p)\| &\geq 1 - m^{-1} \frac{\|x_p^*(m)\|}{\|x_p^*(p)\|} \\ &\geq 1 - m^{-1}. \end{aligned}$$

Moreover, since  $x_{p,m} \pm y \in B_{c_0(\mathbb{N}, X_n)}$ , then

$$\|x_{p,m}(p) \pm y(p)\| \leq 1.$$

Finally, let us evaluate the modulus of convexity of  $X_p$ .

$$\begin{aligned} \delta_{X_p}(2 - 2\varepsilon) &:= \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_{X_p} \text{ and } \|x - y\| \geq 2 - 2\varepsilon \right\} \\ &\leq \inf_m \left( 1 - \frac{\|(x_{p,m}(p) + y(p)) + (x_{p,m}(p) - y(p))\|}{2} \right) \\ &= \inf_m (1 - \|x_{p,m}(p)\|) \\ &\leq \inf_m m^{-1} \\ &= 0, \end{aligned}$$

which implies that  $X_p$  is not uniformly convex.  $\square$

### 3.3.3 Final remarks

Despite the fact that the Theorem 3.11 fails in the transfinite case, it is possible to recover some analogue of the statement if we work with the NASSD2P $_{\kappa}$  instead.

**Theorem 3.13.** *Let  $X$  be a Banach space and  $\kappa \geq \aleph_1$ . Consider the following statements:*

- (a)  $X$  enjoys the NASSD2P $_{\kappa}$ .
- (b) Given a family  $\mathcal{U}$  consisting of less than  $\kappa$  many relatively weakly open sets in  $B_X$ , a relatively weakly open neighborhood  $V$  of 0 in  $X$  and  $\varepsilon > 0$ , there are a set  $\{x_U : U \in \mathcal{U}\}$  and  $y \in V \cap S_X$  satisfying  $x_U \in U$  and  $x_U \pm y \in B_X$  for every  $U \in \mathcal{U}$ .
- (c) Given  $A \in \mathcal{P}_{\kappa}(S_X)$ , there are nets  $\{(y_{\alpha}^x) : x \in A\}$  and  $(z_{\alpha})$  in  $S_X$  satisfying  $\lim \|y_{\alpha}^x \pm z_{\alpha}\| = 1$  and, with respect to the weak topology,  $\lim z_{\alpha} = 0$  and  $\lim y_{\alpha}^x = x$  for every  $x \in A$ .

Then (a)  $\implies$  (b)  $\implies$  (c).

*Proof.* (a)  $\implies$  (b). Fix a family  $\mathcal{U}$  consisting of less than  $\kappa$  many relatively weakly open sets in  $B_X$ , a relatively weakly open neighborhood  $V$  of 0 in  $B_X$  and  $\varepsilon > 0$ . For every  $U \in \mathcal{U}$ , thanks to Bourgain's lemma, which states that every non empty relatively weakly open subset of the unit ball contains a convex combination of slices of the unit ball, we can find functionals

$x_{1,U}^*, \dots, x_{n_U,U}^* \in S_{X^*}$ ,  $\varepsilon_U > 0$  and convex coefficients  $r_{1,U}, \dots, r_{n_U,U}$  such that

$$\left\{ \sum_{i=1}^{n_U} r_i x_i : x_{i,U}^*(x_i) > 1 - \varepsilon_U \text{ for all } 1 \leq i \leq n_U \right\} \subset U.$$

Moreover, we can find  $x_{1,V}^*, \dots, x_{n_V,V}^* \in S_{X^*}$  and  $\varepsilon_V > 0$  satisfying

$$\{x \in B_X : |x_{i,V}^*(x)| \leq \varepsilon_V \text{ for all } 1 \leq i \leq n_V\} \subset V.$$

Since  $X$  has the NASSD2P $_{\kappa}$  and

$$|\{x_{i,U}^* : 1 \leq i \leq n_U \text{ and } U \in \mathcal{U} \cup \{V\}\}| < \kappa,$$

there exist

$$\{x_{i,U} : 1 \leq i \leq n_U \text{ and } U \in \mathcal{U} \cup \{V\}\}$$

and  $y$  in  $S_X$  satisfying  $x_{i,U}^*(x_{i,U}) \geq 1 - \varepsilon_U$  and  $x_{i,U} \pm y \in S_X$  for all  $1 \leq i \leq n_U$  and  $U \in \mathcal{U} \cup \{V\}$ . Now, given  $U \in \mathcal{U}$ , define

$$x_U := \sum_{i=1}^{n_U} r_i x_{i,U} \in B_X$$

and notice that  $x_U \in U$ . Moreover,

$$\begin{aligned} \|x_U \pm y\| &= \left\| \sum_{i=1}^{n_U} r_i (x_{i,U} \pm y) \right\| \\ &\leq \sum_{i=1}^{n_U} r_i \|x_{i,U} \pm y\| \\ &= 1. \end{aligned}$$

In order to conclude, it only remains to prove that  $y \in V$ . But this is clear because for every  $1 \leq i \leq n_V$  we have that

$$\begin{aligned} 1 &= \|x_{i,V} \pm y\| \\ &\geq x_{i,V}^*(x_{i,V} \pm y) \\ &\geq 1 - \varepsilon_V \pm x_{i,V}^*(y), \end{aligned}$$

which means that  $|x_{i,V}^*(y)| \leq \varepsilon_V$ , hence  $y \in V$ .

(ii)  $\implies$  (iii). Fix a set  $A \subset S_X$  of cardinality  $< \kappa$  and temporarily fix a weak neighborhood  $U$  of 0. Define

$$\mathcal{U} := \{(x + U) \cap B_X : x \in A\}$$

and find  $\{y_U^x : x \in A\} \subset B_X$  and  $z_U \in U \cap S_X$  satisfying  $y_U^x \in x + U$  and  $y_U^x \pm z_U \in B_X$  for all  $x \in A$ . Now semi-order the family of weakly open neighborhoods of 0 with respect to the inclusion and consider the nets  $(y_U^x)$  and  $(z_U)$ . It is clear that  $\lim y_U^x = x$ ,  $\lim z_U = 0$  and  $\lim \|z_U \pm y_U^x\| = 1$  holds for all  $x \in A$ . Moreover, up to a perturbation argument, we can assume that all  $y_U^x$ 's belong to  $S_X$ . Thus the claim is proved.  $\square$

The implication (c)  $\implies$  (b) from Theorem 3.13 fails. As already shown,  $\ell_\infty(\kappa)$  satisfies condition (iii) with respect to families of cardinalities less than  $2^\kappa$ , nevertheless it fails the SSD2P $_{\kappa^+}$ , which clearly is implied by (b) with respect to sets of cardinalities at most  $\kappa$ .

### 3.4 Open problems

In this final section, let us point out the main questions that remain unsolved.

*Problem 3.1.* [HLLN, Question 6.1] Does every Banach space with the SSD2P contain an isomorphic copy of  $c_0$ ?

Notice that the question remains open only for the case  $\kappa = \aleph_0$  thanks to Example 2.18.

*Problem 3.2.* Does the injective tensor product of two Banach space possesses the SSD2P $_\kappa$ , provided that one component enjoys the SSD2P $_\kappa$ ?

The answer to this question remains unknown even for  $\kappa = \aleph_0$  [Lan2, Question 3.2].

*Problem 3.3.* Are there sharper cardinal functions which can be used in the statement of Theorem 3.3? For example, given a locally compact Hausdorff space  $X$ , does  $C_0(X)$  fails the SSD2P $_{c(X)^+}$ ? Is it true that  $C_0(X)$  enjoys the NASSD2P $_{d(X)}$ , whenever  $d(X) \geq \aleph_1$ ?

*Problem 3.4.* Does the implication (b)  $\implies$  (a) hold in Theorem 3.13?

# Chapter 4

## The strong diameter two property

A Banach space  $X$  enjoys the *strong diameter two property* (SD2P) if and only if every convex combination of slices of the unit ball has diameter two. Notable examples of Banach spaces enjoying the SD2P include  $c_0$ ,  $\ell_\infty$ ,  $L_\infty[0, 1]$ ,  $C[0, 1]$  and  $L_1[0, 1]$ . One key result in the theory of the SD2P states that a Banach space possesses the SD2P if and only if its dual is octahedral [HLP1, Theorem 3.6].

Recall that Bourgain's lemma states that every non-empty relatively weakly open subset of the unit ball contains a convex combination of slices of the unit ball. The opposite implication doesn't hold true, as there are finite convex combinations of slices which fail to be relatively weakly open [GGMS, Remark IV.5]. Although in general the converse of Bourgain's lemma fails, it was proved in [GGMS, Remark IV.5] that it holds in the positive part of the unit sphere of  $L_1[0, 1]$  and it was shown in [AL, Theorem 2.3] that it also holds in the unit ball of  $C(K)$ , whenever  $K$  is a scattered compact space. The latter result was subsequently extended to spaces of the type  $C_0(K, X)$  in [ABHLP] and to certain  $C^*$ -algebras in [BF]. Recall that in an infinite-dimensional Banach space, every non-empty relatively weakly open subset of the unit ball intersects the unit sphere. Therefore, the requirement that every convex combination of slices of the unit ball intersects the unit sphere is a weaker property compared to the converse of Bourgain's lemma and implies the SD2P. In [AL, Section 3] it is wondered which Banach spaces satisfy this weaker condition. Motivated by this question, the property was studied under the name (P3) in [HKP], (CS) in [LMR] and *attaining strong diameter two property* (ASD2P) in [LM].

In this chapter, we study transfinite analogues of the SD2P. Additionally, the stability of these new properties under direct sums and tensor products

is also investigated. This chapter is based on [CLL1].

## 4.1 Introduction

It is clear that the SD2P poorly extends to higher cardinalities, therefore the aim of this section is to provide the tools that can allow us to equivalently re-define this property in a way that can easily be extended to higher cardinals. One could try to extend the SD2P by taking convex series of slices instead of finite convex combinations. This approach has two main limitations. The first one is that, obviously, this approach cannot extend to uncountable cardinals. The second issue is that, thanks to a perturbation argument, it is easy to show that taking convex series or finite convex combinations leads to the same property.

### 4.1.1 Norm-additive sets

In this subsection, we prove that a set in a Banach space behaves in an additive way with respect to the norm if and only if there exists an element in the dual that attains its norm on all the elements of the set.

**Lemma 4.1.** *Let  $X$  be a Banach space and  $x_1, \dots, x_n \in X$ . If the norm is additive with respect to  $x_1, \dots, x_n$ , i.e.  $\|\sum x_i\| = \sum \|x_i\|$ , then the following hold true.*

$$(a) \quad \|\sum r_i x_i\| = \sum r_i \|x_i\| \text{ for every } r_1, \dots, r_n \geq 0.$$

$$(b) \quad \|\sum r_i x_i\| \geq \sum r_i \|x_i\| \text{ for every } r_1, \dots, r_n \in \mathbb{R}.$$

*Proof.* (a) follows, for example, from [AA, Lemma 11.4] together with a simple inductive argument. On the other hand, (b) is a direct consequence of (a). In fact, notice that

$$\begin{aligned} \left\| \sum r_i x_i \right\| &\geq \left\| \sum_{r_i \geq 0} r_i x_i \right\| - \left\| \sum_{r_i < 0} r_i x_i \right\| \\ &\geq \sum_{r_i \geq 0} r_i \|x_i\| - \sum_{r_i < 0} |r_i| \|x_i\| \\ &= \sum r_i \|x_i\|. \end{aligned}$$

□

**Theorem 4.2.** *Let  $X$  be a Banach space and  $A \subset X$ . The following are equivalent.*

(i) There exists  $x^* \in X^* \setminus \{0\}$  such that  $x^*(x) = \|x^*\| \|x\|$  holds for every  $x \in A$ .

(ii)  $\|\sum x_i\| = \sum \|x_i\|$  holds for every choice of  $x_1, \dots, x_n \in A$ .

*Proof.* (i)  $\implies$  (ii) follows from the observation that

$$\begin{aligned} \left\| \sum x_i \right\| &\geq x^* \left( \sum x_i \right) \|x^*\|^{-1} \\ &= \sum \|x_i\| \end{aligned}$$

clearly holds for every  $x_1, \dots, x_n \in A$ .

(ii)  $\implies$  (i). Consider the functional  $x^*$  defined over the linear span of  $A$  by the formula

$$x^* \left( \sum r_i x_i \right) := \sum r_i \|x_i\|,$$

and let us show that  $x^*$  is well-defined. To this aim, let  $r_1, \dots, r_n, s_1, \dots, s_m \in \mathbb{R}$  and  $x_1, \dots, x_n, y_1, \dots, y_m \in A$  satisfy

$$\sum r_i x_i = \sum s_j y_j.$$

From Lemma 4.1 (b), it follows that

$$\begin{aligned} 0 &= \left\| \sum r_i x_i - \sum s_j y_j \right\| \\ &\geq \sum r_i \|x_i\| - \sum s_j \|y_j\|. \end{aligned}$$

Thanks to the symmetry of the inequality, we can conclude that the equality must hold, or, in other words,  $x^*$  is well-defined. The functional  $x^*$  is clearly linear and Lemma 4.1 implies that it is a norm-1 functional. The Hahn-Banach extension of  $x^*$  to  $X$  is the functional possessing the desired properties.  $\square$

### 4.1.2 Some important notation

In this subsection, we introduce some notation that will be used throughout this chapter. Let  $B \subset X$  and  $\lambda \in (0, 1]$ . We denote the set consisting of all  $\lambda$ -norm attaining functionals on  $B$  by

$$\text{NA}_\lambda(B) := \left\{ x^* \in X^* \setminus \{0\} : \sup_{x \in B \setminus \{0\}} \frac{x^*(x)}{\|x^*\| \|x\|} \geq \lambda \right\}.$$

Moreover, we will abuse the notation and write  $\text{NA}_\lambda(x)$  instead of writing  $\text{NA}_\lambda(\{x\})$ . Let us now list some useful properties of such sets.

**Lemma 4.3.** *Let  $X$  be a Banach space and  $\lambda, \mu \in (0, 1]$ . Additionally, let  $A, B \subset X$  and  $x, y \in X$ . The following hold true.*

(a) *If  $A \subset B$ , then  $\text{NA}_\lambda(A) \subset \text{NA}_\lambda(B)$ .*

(b)  *$x^* \in \text{NA}_\lambda(B)$  if and only if  $rx^* \in \text{NA}_\lambda(B)$  for every  $r \in (0, \infty)$ .*

(c) *Let  $f : B \rightarrow (0, \infty)$  be any function, then*

$$\text{NA}_\lambda(B) = \text{NA}_\lambda(\{f(x) \cdot x : x \in B\}).$$

(d) *If  $\text{NA}_\lambda(x) \cap \text{NA}_\mu(y) \neq \emptyset$ , then  $x \in \text{NA}_\lambda(\text{NA}_\mu(y))$ , which implies*

$$\|x + y\| \geq \lambda\|x\| + \mu\|y\|.$$

(e) *Let  $x_1, \dots, x_n \in S_X$  and  $\varepsilon \in [0, 1)$ . If  $\|\sum x_i\| \geq n - \varepsilon$ , then there exists  $x^* \in X^*$  such that*

$$x_i \in \text{NA}_{1-\varepsilon}(x^*) \subset \text{NA}_{1-\varepsilon}(\text{NA}_{1-\varepsilon}(x_j))$$

*holds for all  $1 \leq i, j \leq n$ .*

*Proof.* (a), (b) and (c) are obvious.

(d). The first implication is clear, so suppose that  $x$  belongs to the set  $\text{NA}_\lambda(\text{NA}_\mu(y))$ , and notice that

$$\begin{aligned} \|x + y\| &\geq \sup_{x^* \in \text{NA}_\mu(y) \cap S_{X^*}} x^*(x + y) \\ &\geq \lambda\|x\| + \mu\|y\|. \end{aligned}$$

(e). Fix  $x^* \in \text{NA}_1(\sum x_i)$ , then

$$\begin{aligned} n - 1 + x^*(x_j) &\geq x^*\left(\sum x_i\right) \\ &= \left\|\sum x_i\right\| \\ &\geq n - \varepsilon, \end{aligned}$$

hence  $x^*(x_j) \geq 1 - \varepsilon$  must hold for every  $1 \leq j \leq n$ . This, together with (a) implies the claim.  $\square$

Notice that (b) and (c) imply that we can restrict ourselves to norm-1 elements only when dealing with  $\text{NA}_\lambda(\cdot)$  sets.

### 4.1.3 Intermezzo about octahedral norms

A Banach space  $X$  is called *octahedral* (OH) if, for every finite-dimensional subspace  $Y$  of  $X$  and  $\varepsilon > 0$ , there exists  $y \in S_X$  such that

$$\|x + ry\| \geq (1 - \varepsilon)(\|x\| + |r|)$$

holds for every  $x \in Y$  and  $r \in \mathbb{R}$ . A compactness arguments shows that it suffices to consider finitely many elements belonging to the unit sphere of  $X$  instead of finite-dimensional subspaces [HLP1, Proposition 2.4]. Octahedral norms were introduced in an unpublished paper by Godefroy and Maurey (see [God] and [GM]). Notable examples of OH Banach spaces include  $\ell_1$ ,  $L_1[0, 1]$  and  $C[0, 1]$ . The most remarkable result in the theory of OH Banach spaces states that a Banach space contains an isomorphic copy of  $\ell_1$  if and only if it admits an equivalent OH norm [God, Theorem II.4]. In this subsection we briefly interrupt our analysis of diameter two properties and study transfinite extensions of OH norms. The aim of this subsection is not to give a complete overview of transfinite OH Banach spaces; its primary scope is to provide some tools that will be necessary in the rest of the chapter. We will come back to studying transfinite OH norms when discussing ball-covering properties in the next chapter.

**Definition 4.4.** Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal.

- (a) We say that  $X$  is *octahedral with respect to  $\kappa$*  ( $\text{OH}_\kappa$ ) if, for every  $A \in \mathcal{P}_\kappa(S_X)$  and  $\varepsilon > 0$ , there exists  $y \in S_X$  satisfying

$$\|x + y\| \geq 2 - \varepsilon$$

for every  $x \in A$ .

- (b) We say that  $X$  is *rigid octahedral with respect to  $\kappa$*  ( $\text{ROH}_\kappa$ ) if, for every  $A \in \mathcal{P}_\kappa(S_X)$ , there exists  $y \in S_X$  satisfying

$$\|x + y\| = 2$$

for every  $x \in A$ .

It is clear that being  $\text{OH}_{\aleph_0}$  coincides to being OH.

**Example 4.5.** If  $\kappa \geq \aleph_1$ , then  $\ell_1(\kappa)$  is  $\text{ROH}_\kappa$ .

*Proof.* Fixed  $A \in \mathcal{P}_\kappa(S_{\ell_1(\kappa)})$ , it suffices to find some  $\lambda \in \kappa$  not belonging to the support of any of the elements in  $A$  and consider  $y := e_\lambda$ .  $\square$

**Lemma 4.6.** *If a Banach space has the Daugavet property, then its dual is  $\text{ROH}_{\aleph_1}$ .*

*Proof.* A careful read of the proof of [KSSW, Lemma 2.12] implies the claim.  $\square$

**Example 4.7.** From Lemma 4.6, since  $L_1[0, 1]$  has the Daugavet property, it follows that  $L_\infty[0, 1]$  is  $\text{ROH}_{\aleph_1}$ .

**Theorem 4.8.** *Let  $M$  be a complete metric space. Then the following assertions are equivalent.*

- (i)  $Lip_0(M)$  has the Daugavet property.
- (ii)  $Lip_0(M)$  is  $\text{OH}_{\aleph_0}$ .
- (iii)  $Lip_0(M)$  is  $\text{ROH}_{\aleph_1}$ .
- (iv)  $M$  is a length space.

*Proof.* The equivalence (i)  $\iff$  (ii)  $\iff$  (iv) follows from [AM, Theorem 1.5], and (iii)  $\implies$  (ii) is clear from definition.

(i)  $\implies$  (iii). Assume that  $Lip_0(M)$  has the Daugavet property. Since the Daugavet property passes from the dual space to its predual, then the Lipschitz-free space  $\mathcal{F}(M)$  also has the Daugavet property. Lemma 4.6 implies the claim.  $\square$

We end this subsection by demonstrating that transfinite octahedrality can equivalently be described via subspaces of fixed density character.

**Theorem 4.9.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. Then  $X$  is  $\text{OH}_\kappa$  (respectively,  $\text{ROH}_\kappa$ ) if and only if for every subspace  $Y$  of  $X$  with density character less than  $\kappa$  and  $\varepsilon > 0$  (respectively,  $\varepsilon = 0$ ), there exists  $y \in S_X$  satisfying*

$$\|x + ry\| \geq (2 - \varepsilon)(\|x\| + |r|)$$

for every  $x \in Y$  and  $r \in \mathbb{R}$ .

*Proof.* We prove the  $\text{ROH}_\kappa$  case only. One direction is obvious, so suppose that  $X$  is  $\text{ROH}_\kappa$  and fix a subspace  $Y \subset X$  with  $d(Y) < \kappa$ . Let  $\mathcal{D} \in \mathcal{P}_\kappa(X)$  be dense, and call

$$\mathcal{D}' := \left\{ \frac{x}{\|x\|} : x \in (\mathcal{D} \cup (-\mathcal{D})) \setminus \{0\} \right\} \in \mathcal{P}_\kappa(S_X).$$

By hypothesis, there exists  $y \in S_X$  such that  $\|x + y\| = 2$  for every  $x \in \mathcal{D}'$ , and, therefore, Lemma 4.3 implies that

$$\mathcal{D}' \subset \text{NA}_1(\text{NA}_1(y)).$$

In other words, for every  $x \in \mathcal{D}'$ , we can find  $y_x^* \in S_{X^*}$  satisfying  $y_x^*(y) = 1$  and  $y_x^*(x) = \|x\|$ . Fix  $r \in \mathbb{R}$  and initially suppose that  $r \geq 0$ . Then, for every  $x \in A$ , we have that

$$\begin{aligned} \|x + ry\| &\geq y_x^*(x) + ry_x^*(y) \\ &= \|x\| + r. \end{aligned}$$

On the other hand, if  $r < 0$ , then, for every  $x \in A$ , we have that

$$\begin{aligned} \|x + ry\| &= \|-x - ry\| \\ &\geq y_{-x}^*(-x) - ry_{-x}^*(y) \\ &= \|x\| + |r|. \end{aligned}$$

□

#### 4.1.4 Redefining the SD2P

In this subsection, we equivalently redefine the SD2P in a way that can easily be extended to higher cardinalities.

**Theorem 4.10.** *Let  $X$  be a Banach space and  $\mathcal{S}$  a countable collection of slices of  $B_X$  satisfying  $\mathcal{S} = -\mathcal{S}$ , i.e. if  $S \in \mathcal{S}$ , then  $-S \in \mathcal{S}$ . Consider the following statements.*

- (i) *Every convex combination of slices belonging to  $\mathcal{S}$  has diameter two.*
- (ii) *Every convex combination of slices belonging to  $\mathcal{S}$  intersects  $B_X \setminus \lambda B_X$  for every  $\lambda \in (0, 1)$ .*
- (iii) *For every  $\lambda \in (0, 1)$ , there is  $A \subset B_X$  that visits every slice of  $\mathcal{S}$ , and there exists  $x^* \in S_{X^*}$  such that  $x^*(x) \geq \lambda$  holds for every  $x \in A$ .*

Then (i)  $\iff$  (ii)  $\iff$  (iii). If in addition  $\mathcal{S}$  is finite, then (ii)  $\implies$  (iii).

*Proof.* (i)  $\iff$  (ii) is a simple adaptation of [LMR, Theorem 3.1].

(iii)  $\implies$  (ii). Fix  $S_1, \dots, S_n \in \mathcal{S}$  and  $\lambda_1, \dots, \lambda_n \geq 0$  satisfying  $\sum \lambda_i = 1$ . Find a set  $\{x_1, \dots, x_n\} \subset S_X$  and  $x^* \in S_{X^*}$  such that  $x_i \in S_i$  and  $x^*(x_i) \geq \lambda$  for every  $1 \leq i \leq n$ . Then

$$\begin{aligned} \left\| \sum \lambda_i x_i \right\| &\geq x^* \left( \sum \lambda_i x_i \right) \\ &\geq \lambda. \end{aligned}$$

(ii)  $\implies$  (iii). Suppose that  $\mathcal{S} = \{S_1, \dots, S_n\}$  and fix  $\varepsilon > 0$ . By assumption, we can find  $x_1, \dots, x_n$  satisfying  $x_i \in S_i$  for  $1 \leq i \leq n$ , and

$$\left\| \sum n^{-1} x_i \right\| \geq 1 - \varepsilon n^{-1},$$

which implies  $\left\| \sum x_i \right\| \geq n - \varepsilon$ . Lemma 4.3 (e) implies that there exists  $x^* \in S_{X^*}$  such that  $x^*(x_i) \geq 1 - \varepsilon$  for  $1 \leq i \leq n$ .  $\square$

**Theorem 4.11.** *Let  $X$  be a Banach space and  $\mathcal{S}$  a finite (respectively, countable) collection of slices in  $B_X$  satisfying  $\mathcal{S} = -\mathcal{S}$ . The following are equivalent,*

- (i) *Every convex combination (respectively, series) of slices belonging to  $\mathcal{S}$  contains a closed line segment of length 2.*
- (ii) *Every convex combination (respectively, series) of slices belonging to  $\mathcal{S}$  intersects  $S_X$ .*
- (iii) *There is  $A \subset B_X$  that visits every slice of  $\mathcal{S}$ , and there exists  $x^* \in S_{X^*}$  such that  $x^*(x) = \|x\|$  holds for every  $x \in A$ .*

*Proof.* We will prove the claim when  $\mathcal{S}$  is countable only. (i)  $\iff$  (ii) is a simple adaptation of [LMR, Theorem 3.1].

(ii)  $\implies$  (iii). Let  $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ . For every  $i \in \mathbb{N}$ , find some  $x_i \in S_i \cap S_X$  such that

$$\left\| \sum 2^{-i} x_i \right\| = 1.$$

Theorem 4.2 implies the claim.

(iii)  $\implies$  (ii). Fix a convex series of slices  $\sum \lambda_i S_i$  in  $\mathcal{S}$ . By assumption, there are a sequence  $(x_n)$  in  $S_X$  and  $x^* \in S_{X^*}$  satisfying  $x_i \in S_i$  and  $x^*(x_i) = 1$  for every  $i \in \mathbb{N}$ . It is now clear that

$$\begin{aligned} \left\| \sum \lambda_i x_i \right\| &\geq x^* \left( \sum \lambda_i x_i \right) \\ &= 1. \end{aligned}$$

$\square$

Notice that, at the beginning of the section, we claimed that trying to extend the SD2P via convex series is not a valid approach. It has to be said that Theorem 4.11 shows that this approach could be taken if we consider the ASD2P. Nevertheless the limitation of extending these property to uncountable cardinals would still remain. We are finally ready to extend the SD2P to uncountable cardinals. For convenience sake, let us slightly change their definitions.

**Definition 4.12.** Let  $X$  be a Banach space and  $Y \subset X^*$  a subspace. We say that the pair  $(X, Y)$  has the *strong diameter two property* (SD2P) if every convex combination of slices in  $B_X$  defined by functionals in  $Y$  has diameter two. On the other hand, we say that the pair  $(X, Y)$  has the *attaining strong diameter two property* (ASD2P) if every convex combination of slices in  $B_X$  defined by functionals in  $Y$  intersects the unit sphere of  $X$ .

The original definitions for the SD2P are usually not given in terms of pairs, but only the pairs  $(X, X^*)$  and  $(X^*, X)$  are considered separately, nevertheless it is more convenient in the following to consider pairs in order to avoid repeating each statement twice. The following characterization is an immediate consequence of Theorems 4.10 and 4.11.

**Theorem 4.13.** *Let  $X$  be a Banach space and  $Y$  a subspace of  $X^*$ . Consider the following statements.*

(a)  $(X, Y)$  enjoys the SD2P.

(a') For every  $A \in \mathcal{P}_{\aleph_0}(S_Y)$  and  $\lambda \in (0, 1)$ , there are  $B \subset S_X$  and  $x^* \in S_{X^*}$  such that  $B$   $\lambda$ -norms  $A$  and  $x^*(x) \geq \lambda$  holds for all  $x \in B$ .

(b)  $(X, Y)$  enjoys the ASD2P.

(b') For every  $A \in \mathcal{P}_{\aleph_0}(S_Y)$  and  $\lambda \in (0, 1)$ , there are  $B \subset S_X$  and  $x^* \in S_{X^*}$  such that  $B$   $\lambda$ -norms  $A$  and  $x^*(x) = 1$  holds for all  $x \in B$ .

Then  $(a) \iff (a') \iff (b) \iff (b')$ .

## 4.2 Definition and a first examples

**Definition 4.14.** Let  $X$  be a Banach space,  $Y \subset X^*$  a subspace, and  $\kappa$  an infinite cardinal.

(a) We say that the pair  $(X, Y)$  enjoys the *strong diameter two property with respect to  $\kappa$*  (SD2P $_{\kappa}$ ) if, for every  $A \in \mathcal{P}_{\kappa}(S_Y)$  and  $\lambda \in (0, 1)$ , there are  $B \subset S_X$  and  $x^* \in S_{X^*}$  such that  $B$   $\lambda$ -norms  $A$  and  $x^*(x) \geq \lambda$  holds for all  $x \in B$ .

(b) We say that the pair  $(X, Y)$  enjoys the *norming and attaining strong diameter two property with respect to  $\kappa$*  (NASD2P $_{\kappa}$ ) if, for every  $A \in \mathcal{P}_{\kappa}(S_Y)$ , there are  $B \subset S_X$  and  $x^* \in S_{X^*}$  such that  $B$  norms  $A$  and  $x^*(x) = 1$  holds for all  $x \in B$ .

Additionally, we will say that  $X$  enjoys the SD2P $_{\kappa}$  whenever the pair  $(X, X^*)$  possesses it, and we will say that a dual Banach space  $X^*$  enjoys the weak\* SD2P $_{\kappa}$  whenever the pair  $(X^*, X)$  has it.

*Remark 4.15.* Let  $X$  be a Banach space and  $Y$  a subspace of  $X^*$ .

- (a)  $(X, Y)$  can enjoy at best the SD2P $_{|Y|}$ .
- (b) In Definition 4.14, it is equivalent to replace the set  $S_Y$  with  $Y \setminus \{0\}$ .
- (c) It is clear that the SD2P coincides with the SD2P $_{\aleph_0}$ , but, on the other hand, the ASD2P doesn't coincide with the NASD2P $_{\aleph_0}$ .

**Example 4.16.** If a Banach space  $X$  possesses the SSD2P $_{\kappa}$  (respectively, NASSD2P $_{\kappa}$ ), then it enjoys the SD2P $_{\kappa}$  (respectively, NASD2P $_{\kappa}$ ). This implication follows from a straightforward adaptation of [AL, Example 3.3].

The following result states that the NASD2P $_{\kappa}$  can equivalently be defined via relatively weakly open sets instead of norming sets or slices.

**Lemma 4.17.** *Let  $X$  be a Banach space,  $\lambda \in (0, 1]$  and  $\kappa$  an infinite cardinal. The following are equivalent.*

- (i) *For every family  $\mathcal{S}$  consisting of slices of  $B_X$  of cardinality smaller than  $\kappa$ , there are  $A \subset S_X$  and  $x^* \in S_{X^*}$  such that  $A$  visits every slice of  $\mathcal{S}$  and  $x^*(x) \geq \lambda$  for every  $x \in A$ .*
- (ii) *For every family of non-empty relatively weakly open sets  $\mathcal{U}$  in  $B_X$  of cardinality smaller than  $\kappa$ , there are  $A \subset S_X$  and  $x^* \in S_{X^*}$  such that  $A$  visits every set of  $\mathcal{U}$  and  $x^*(x) \geq \lambda$  for every  $x \in A$ .*

*Proof.* (ii)  $\implies$  (i) is obvious since every slice is a non-empty relatively weakly open subset.

(i)  $\implies$  (ii). Fix  $\mathcal{U}$  as in the assumption. Thanks to Bourgain's lemma, for every  $U \in \mathcal{U}$ , we can find finite convex combination of slices

$$\sum_{i=1}^{n_U} \lambda_{i,U} S_{i,U} \subset U.$$

Hence, there is a set

$$\{x_{i,U} : 1 \leq i \leq n_U, U \in \mathcal{U}\} \subset S_X$$

such that  $x_{i,U} \in S_{i,U}$  and there is  $x^* \in S_{X^*}$  such that  $x^*(x_{i,U}) \geq \lambda$  for every  $1 \leq i \leq n_U$  and  $U \in \mathcal{U}$ . The set

$$\left\{ \sum_{i=1}^{n_U} \lambda_{i,U} x_{i,U} : U \in \mathcal{U} \right\}$$

visits every set of  $\mathcal{U}$  and

$$x^* \left( \sum_{i=1}^{n_U} \lambda_{i,U} x_{i,U} \right) \geq \lambda$$

holds for every  $U \in \mathcal{U}$ . □

We will postpone showing more examples since, in order to show interesting non-trivial examples, more theory concerning the structure of the transfinite SD2P needs to be developed.

### 4.3 Duality

The aim of this section is to show dual connections between the transfinite SD2P and other properties.

**Lemma 4.18.** *If  $X$  is a Banach space, then*

$$\bigcap_{\lambda \in (0,1)} \bigcup_{x^* \in X^*} \mathcal{P}(\text{NA}_\lambda(\text{NA}_\lambda(x^*)) \cap X^*) = \bigcap_{\lambda \in (0,1)} \bigcup_{x^* \in X^*} \mathcal{P}(\text{NA}_\lambda(\text{NA}_\lambda(x^*)) \cap X)$$

*Proof.* Let

$$A \in \bigcap_{\lambda \in (0,1)} \bigcup_{x^* \in X^*} \mathcal{P}(\text{NA}_\lambda(\text{NA}_\lambda(x^*)) \cap X^*).$$

For every  $\varepsilon > 0$ , there exists  $y^* \in S_{X^*}$  such that, for every  $x^* \in A$ , we can find  $x^{**} \in S_{X^{**}}$  satisfying

$$x^{**}(x^*) \geq 1 - \varepsilon/2$$

and

$$x^{**}(y^*) \geq 1 - \varepsilon/2.$$

By Goldstine's theorem, there is  $x \in S_X$  such that

$$|(x^{**} - x)x^*| < \varepsilon/2$$

and

$$|(x^{**} - x)y^*| < \varepsilon/2.$$

Therefore  $x^*(x) \geq 1 - \varepsilon$  and  $y^*(x) \geq 1 - \varepsilon$ , and the first inclusion is thus proved, since

$$A \in \mathcal{P}(\text{NA}_{1-\varepsilon}(\text{NA}_{1-\varepsilon}(y^*) \cap X)).$$

The opposite inclusion is obvious because

$$\text{NA}_\lambda(\text{NA}_\lambda(x^*) \cap X) \subset \text{NA}_\lambda(\text{NA}_\lambda(x^*)) \cap X^*$$

holds for every choice of  $x^*$  and  $\lambda$ . □

**Theorem 4.19.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. The following statements hold.*

- (a)  $X$  has the SD2P $_\kappa$  if and only if  $X^*$  is OH $_\kappa$ .
- (b)  $X^*$  has the weak\* SD2P $_\kappa$  if and only if for every  $A \in \mathcal{P}_\kappa(S_X)$  and  $\varepsilon > 0$ , there is  $x^{**} \in S_{X^{**}}$  such that

$$\|x + x^{**}\| \geq 2 - \varepsilon$$

holds true for all  $x \in A$ .

*Proof.* We will prove only (a). Notice that possessing the SD2P $_\kappa$  can be equivalently rephrased by requiring that

$$\mathcal{P}_\kappa(S_{X^*}) \subset \bigcap_{\lambda \in (0,1)} \bigcup_{x^* \in X^*} \mathcal{P}(\text{NA}_\lambda(\text{NA}_\lambda(x^*) \cap X)).$$

On the other hand, it's easy to verify that being OH $_\kappa$  for a dual Banach space  $X^*$  is equivalent to require

$$\mathcal{P}_\kappa(S_{X^*}) \subset \bigcap_{\lambda \in (0,1)} \bigcup_{x^* \in X^*} (\text{NA}_\lambda(\text{NA}_\lambda(x^*))).$$

Hence, Lemma 4.18 provides the desired conclusion. □

Let us now consider dual relations with ROH $_\kappa$  norms.

**Theorem 4.20.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. Consider the following statements.*

- (a)  $X$  is ROH $_\kappa$ .
- (b)  $X^*$  enjoys the weak\* NASD2P $_\kappa$ .

(c) For every  $A \in \mathcal{P}_\kappa(S_X)$ , there exists  $x^{**} \in S_{X^{**}}$  satisfying

$$\|x + x^{**}\| = 2$$

for all  $x \in A$ .

(d)  $X^*$  enjoys the weak\*  $\text{SD2P}_\kappa$ .

Then (a)  $\implies$  (b)  $\implies$  (c)  $\implies$  (d).

*Proof.* (a)  $\implies$  (b) follows from an argument similar to the one used in the proof of Theorem 4.19. (b)  $\implies$  (c). Fix  $A \in \mathcal{P}_\kappa(S_X)$  and find  $B \subset S_{X^*}$ , and  $x^{**} \in S_{X^{**}}$ , such that  $B$  norms  $A$ , and  $x^{**}(x^*) = 1$  for every  $x^* \in B$ . Hence, for each  $x \in A$ , we have that

$$\begin{aligned} \|x + x^{**}\| &\geq \sup_{x^* \in B} x^*(x) + x^{**}(x^*) \\ &= 2. \end{aligned}$$

Finally, (c)  $\implies$  (d) follows from Theorem 4.19.  $\square$

**Theorem 4.21.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. If  $X$  possesses the  $\text{NASD2P}_\kappa$ , then  $X^*$  is  $\text{ROH}_\kappa$ .*

*Proof.* The claim follows from an argument similar to the one used in the proof of Theorem 4.19.  $\square$

### 4.3.1 When $\kappa$ is at most $\aleph_2$

*Remark 4.22.* Let  $X$  be a Banach space and  $Y \subset X^*$  a subspace. It is clear that, if the pair  $(X, Y)$  enjoys the  $\text{NASD2P}_{\text{d}(Y)^+}$ , then it even possesses the  $\text{NASD2P}_{|Y|^+}$ .

**Theorem 4.23.** *If  $X$  is a Banach space with the Daugavet property, then  $X^*$  enjoys the weak\*  $\text{NASD2P}_{\aleph_1}$ . Moreover,*

(a) *if  $X$  is separable, then  $X^*$  has the weak\*  $\text{NASD2P}_{|X|^+}$ ;*

(b) *if  $X$  is non-separable, then for every subspace  $Y \subset X$  with  $\text{d}(Y) = \aleph_1$ , there exists  $x^{**} \in S_{X^{**}}$  such that*

$$\|x + \lambda x^{**}\| = \|x\| + |\lambda|$$

*holds for every  $x \in Y$  and  $\lambda \in \mathbb{R}$ . In particular  $X^*$  has the weak\*  $\text{SD2P}_{\aleph_2}$ .*

*Proof.* An attentive read of [AL, Example 2.4] shows the claim of the theorem. For the moreover part, (a) follows from Remark 4.22.

(b). Let  $Y \subset X$  be a subspace with  $d(Y) = \aleph_1$ . Thanks to [Abr, Remark 2.3], there exists an almost isometric ideal  $Y \subset Z \subset X$  with  $d(Z) = \aleph_1$ . By [ALN2, Proposition 3.8],  $Z$  has the Daugavet property. Now, [LR, Theorem 3.3] implies the existence of some  $x^{**} \in S_{Z^{**}}$  satisfying

$$\|z + \lambda x^{**}\| = \|z\| + |\lambda|$$

for every  $z \in Z$  and  $\lambda \in \mathbb{R}$ . Hence Theorem 4.19 implies the claim.  $\square$

**Theorem 4.24.** *If  $X$  is a Banach space, then the following assertions are equivalent.*

(i)  $X$  is  $\text{OH}_{\aleph_0}$ .

(ii) For every separable subspace  $Y \subset X$ , there exists  $x^{**} \in S_{X^{**}}$  satisfying

$$\|x + \lambda x^{**}\| = \|x\| + |\lambda|$$

for every  $x \in Y$  and  $\lambda \in \mathbb{R}$ .

(iii)  $X^*$  enjoys the weak\* SD2P $_{\aleph_0}$ .

(iv)  $X^*$  enjoys the weak\* SD2P $_{\aleph_1}$ .

*Proof.* (i)  $\implies$  (ii). Fix a separable subspace  $Y \subset X$ . Thanks to [Lan1, Proposition 3.36], we can find a separable OH subspace  $Y \subset Z \subset X$ . Hence, [GK, Lemma 9.1] guarantees the existence of some  $x^{**} \in S_{X^{**}}$  satisfying

$$\|z + \lambda x^{**}\| = \|z\| + |\lambda|$$

for every  $z \in Z$  and  $\lambda \in \mathbb{R}$ .

(i)  $\iff$  (iii) is [God, Remark II.5.2], (iv)  $\implies$  (iii) is obvious, and (ii)  $\implies$  (iv) follows from Theorem 4.19 (b).  $\square$

## 4.4 Stability results

In this section, we aim to extend the known examples of Banach spaces enjoying the transfinite SD2P by taking direct sums and tensor products.

### 4.4.1 Direct sums

**Definition 4.25.** We say that an absolute normalized norm  $N$  on  $\mathbb{R}^2$  has the *positive strong diameter two property* (positive SD2P) if, for every  $n \in \mathbb{N}$ , and any finite family of slices  $S_1, \dots, S_n$  in  $B_{(\mathbb{R}^2, N)}$  defined by positive functionals in  $(\mathbb{R}^2, N^*)$ , there are positive elements  $(a_i, b_i) \in S_i$  such that

$$N\left(\sum (a_i, b_i)\right) = \sum N(a_i, b_i).$$

The original definition of the positive SD2P [HLN] was given in terms of convex combination of slices, but Lemma 4.1 ensures that it is equivalent to Definition 4.25. It is known that an absolute normalized norm  $N$  on  $\mathbb{R}^2$  has the positive SD2P if and only if there is a positive element  $(a, b)$  such that

$$N(t(a, 1) + (1 - t)(b, 1)) = 1$$

holds for every  $t \in [0, 1]$ , hence it is clear that the  $\ell_1$  and the  $\ell_\infty$  norm both have the positive SD2P, but they are not the only such norms. The positive SD2P was introduced in order to characterize all the absolute normalized norms that preserve the SD2P [HLN, Theorem 3.5]. In addition, it was noted in [HKP, Proposition 2.2] that the same statement holds also for the ASD2P. In the following, we want to prove that both these statements can in fact be extended to infinite cardinals and arbitrary dual pairs.

**Lemma 4.26.** *Let  $N$  be an absolute normalized norm on  $\mathbb{R}^2$ . Then the following are equivalent.*

- (i)  $N$  has the positive SD2P.
- (ii) For every set  $A \subset S_{(\mathbb{R}^2, N)^*}$  consisting of positive functionals, there is a set  $B \subset S_{(\mathbb{R}^2, N)}$  consisting of positive elements and  $x^* \in S_{(\mathbb{R}^2, N^*)}$  such that  $B$  norms  $A$  and  $x^*(x) = 1$  for every  $x \in B$ .

*Proof.* (ii)  $\implies$  (i) follows from Theorem 4.2.

(i)  $\implies$  (ii). Suppose at first that  $A$  is countable, and let

$$\{x_n^* : n \in \mathbb{N}\}$$

be its enumeration. For every  $n \in \mathbb{N}$ , we can select positive elements  $(a_1^n, b_1^n), \dots, (a_n^n, b_n^n)$  satisfying

$$x_i^*(a_i^n, b_i^n) \geq 1 - n^{-1}$$

for  $1 \leq i \leq n$  and

$$N\left(\sum (a_i^n, b_i^n)\right) = \sum N(a_i^n, b_i^n).$$

We proceed with a diagonal argument.

Thanks to compactness, there is  $I_1 \subset \mathbb{N}$  such that

$$\lim_{i \in I_1} (a_1^i, b_1^i) = (a_1, b_1),$$

for some  $(a_1, b_1) \in \text{NA}_1(x_1^*)$ . Suppose we are given  $I_n$ , then we can find  $I_{n+1} \subset I_n$  and  $(a_{n+1}, b_{n+1}) \in \text{NA}_1(x_{n+1}^*)$  such that

$$\lim_{i \in I_{n+1}} (a_{n+1}^i, b_{n+1}^i) = (a_{n+1}, b_{n+1}).$$

Now, the sequence  $\{(a_n, b_n) : n \in \mathbb{N}\}$  norms  $A$  and, for every  $n \in \mathbb{N}$ , we have that

$$\begin{aligned} N \left( \sum_{j=1}^n (a_j, b_j) \right) &= \lim_{i \in I_n} N \left( \sum_{j=1}^n (a_j^i, b_j^i) \right) \\ &= \lim_{i \in I_n} \sum_{j=1}^n N(a_j^i, b_j^i) \\ &= \sum_{j=1}^n N(a_j, b_j). \end{aligned}$$

The desired conclusion now follows thanks to Theorem 4.2. If  $A$  is not countable, then a density argument will provide the claim.  $\square$

**Theorem 4.27.** *Let  $X_1$  and  $X_2$  be Banach spaces,  $Y_1 \subset X_1^*$  and  $Y_2 \subset X_2^*$  non-trivial subspaces,  $\kappa$  an infinite cardinal and  $N$  an absolute normalized norm on  $\mathbb{R}^2$ . If  $(X_1, Y_1)$  and  $(X_2, Y_2)$  enjoy the SD2P $_\kappa$  (respectively, NASD2P $_\kappa$ ), then  $N$  has the positive SD2P if and only if  $(X_1 \oplus_N X_2, Y_1 \oplus_{N^*} Y_2)$  possesses the SD2P $_\kappa$  (respectively, NASD2P $_\kappa$ ).*

*Proof.* We will prove the claim only for the NASD2P $_\kappa$ . Suppose that  $N$  has the positive SD2P and fix  $A \in \mathcal{P}_\kappa(S_{Y_1 \oplus_{N^*} Y_2})$ . Call

$$A_1 := \{y_1 / \|y_1\| : (y_1, y_2) \in A \text{ and } y_1 \neq 0\} \subset S_{Y_1}$$

and

$$A_2 := \{y_2 / \|y_2\| : (y_1, y_2) \in A \text{ and } y_2 \neq 0\} \subset S_{Y_2}.$$

Since the  $(X_i, Y_i)$ 's have the NASD2P $_\kappa$ , we can find  $B_i \subset S_{X_i}$  and  $x_i^* \in S_{X_i^*}$  such that  $B_i$  norms  $A_i$  and  $x_i^*(x) = 1$  for every  $x \in B_i$ , where  $i = 1, 2$ . Thanks to Lemma 4.26, we can find  $B \subset S_{(\mathbb{R}^2, N)}$  consisting of positive elements and  $(c, d) \in S_{(\mathbb{R}^2, N^*)}$  such that  $B$  norms the set

$$\{(\|y_1\|, \|y_2\|) : (y_1, y_2) \in A\}$$

and  $ac + bd = 1$  holds for every  $(a, b) \in B$ . Now, define

$$z^* := (cx_1^*, dx_2^*) \in S_{(X_1 \oplus_N X_2)^*}$$

and

$$B' := \{(ax_1, bx_2) : (a, b) \in B, x_1 \in B_1 \text{ and } x_2 \in B_2\} \subset S_{X_1 \oplus_N X_2}.$$

It is clear that, for every  $(ax_1, bx_2) \in B'$ , and

$$\begin{aligned} z^*(ax_1, bx_2) &= acx_1^*(x_1) + bdx_2^*(x_2) \\ &= ac + bd \\ &= 1. \end{aligned}$$

We claim that  $B'$  norms  $A$ . Fix  $(y_1, y_2) \in A$ , find  $x_i \in B_i$  and  $(a, b) \in B$  such that  $y_i(x_i) = \|y_i\|$  for  $i \in \{1, 2\}$  and

$$a\|y_1\| + b\|y_2\| = 1.$$

Thus

$$\begin{aligned} (y_1, y_2)(ax_1, bx_2) &= a\|y_1\| + b\|y_2\| \\ &= 1. \end{aligned}$$

Vice-versa assume that  $(X_1 \oplus_N X_2, Y_1 \oplus_{N^*} Y_2)$  has the the NASD2P $_{\kappa}$ , hence it enjoys the SD2P. It is easy to adapt the proof of [HLN, Theorem 3.5] to show that this implies that  $N$  has the positive SD2P.  $\square$

Let us now focus on  $\ell_1$  and  $\ell_\infty$  sums.

**Lemma 4.28.** *Let  $X_1$  and  $X_2$  be Banach spaces,  $Y_1 \subset X_1^*$  and  $Y_2 \subset X_2^*$  subspaces,  $\lambda \in (0, 1]$  and  $\varepsilon > 0$ . The following statements hold true.*

(a) *If  $A \subset S_{X_1}$  and  $B \subset S_{X_2}$ , then*

$$\text{NA}_\lambda(A) \oplus_1 \text{NA}_\lambda(B) \subset \text{NA}_\lambda(A \oplus_\infty B).$$

(b) *If  $A \subset S_{X_1 \oplus_\infty X_2}$ , then*

$$\text{NA}_{1-\varepsilon}(A) \cap (S_{Y_1} \oplus_1 S_{Y_2}) \subset \text{NA}_{1-2\varepsilon}(\pi_{X_1}(A)) \oplus_1 \text{NA}_{1-2\varepsilon}(\pi_{X_2}(A)).$$

(c) *If  $x^* \in X_1^* \setminus \{0\}$ , then*

$$(\text{NA}_\lambda(x^*) \cap S_{X_1}) \oplus_\infty B_{X_2} \subset \text{NA}_\lambda(x^* \oplus_1 0) \cap S_{X_1 \oplus_\infty X_2}.$$

(d) If  $x_1^* \oplus_1 x_2^* \in S_{X_1^* \oplus_1 X_2^*}$  and

$$A \subset \text{NA}_{1-\varepsilon}(x_1^* \oplus_1 x_2^*) \cap S_{X_1 \oplus_\infty X_2},$$

then either

$$\pi_{X_1}(A) \subset \text{NA}_{1-2\varepsilon}(x_1^*) \cap X_1$$

or

$$\pi_{X_2}(A) \subset \text{NA}_{1-2\varepsilon}(x_2^*) \cap X_2.$$

*Proof.* (a), (b) and (c) are clear. In order to prove (d), assume by contradiction that we can find  $x_1 \in \pi_{X_1}(A)$  and  $y_2 \in \pi_{X_2}(A)$  satisfying

$$x_1^*(x_1) < (1 - 2\varepsilon)\|x_1^*\|\|x_1\|$$

and

$$x_2^*(y_2) \leq (1 - 2\varepsilon)\|x_2^*\|\|y_2\|$$

(notice that this includes the case when  $x_2^* = 0$ ). Now find  $y_1$  and  $x_2$  so that  $(x_1, y_1)$  and  $(x_2, y_2)$  belong to  $A$ . Observe that

$$\begin{aligned} 2 - 2\varepsilon &\leq (x_1^*, x_2^*)((x_1, y_1) + (x_2, y_2)) \\ &< (1 - 2\varepsilon)(\|x_1^*\|\|x_1\| + \|x_2^*\|\|y_2\|) + x_1^*(x_2) + x_2^*(y_1) \\ &< 1 - 2\varepsilon + x_1^*(x_2) + x_2^*(y_1), \end{aligned}$$

therefore  $x_1^*(x_2) + x_2^*(y_1) > 1$ , which is a contradiction because  $(x_2, y_1)$  belongs to  $B_{X_1 \oplus_\infty X_2}$ .  $\square$

**Lemma 4.29.** *Let  $X_1$  and  $X_2$  be Banach spaces,  $Y_1 \subset X_1^*$  and  $Y_2 \subset X_2^*$  subspaces,  $\lambda, \mu \in (0, 1]$ , and  $\varepsilon, \delta > 0$ . The following statements hold true.*

(a) *If  $x^* \in X_1^* \setminus \{0\}$ , then*

$$\text{NA}_\mu(\text{NA}_\lambda(x^*) \cap X_1) \oplus_1 X_2^* \subset \text{NA}_\mu(\text{NA}_\lambda(x^* \oplus_1 0) \cap (X_1 \oplus_\infty X_2)).$$

(b) *Let  $A \subset S_{Y_1}$ ,  $B \subset S_{Y_2}$  and  $x_1^* \oplus_1 x_2^* \in S_{X_1^* \oplus_1 X_2^*}$ . If*

$$A \oplus_1 B \subset \text{NA}_{1-\delta}(\text{NA}_{1-\varepsilon}(x_1^* \oplus_1 x_2^*) \cap (X_1 \oplus_\infty X_2)),$$

then either

$$A \subset \text{NA}_{1-2\delta}(\text{NA}_{1-2\varepsilon}(x_1^*) \cap X_1)$$

or

$$B \subset \text{NA}_{1-2\delta}(\text{NA}_{1-2\varepsilon}(x_2^*) \cap X_2).$$

*Proof.* (a). Let us employ Lemma 4.28 (a) and (c), together with Lemma 4.3 (a). It follows that

$$\begin{aligned} \text{NA}_\mu(\text{NA}_\lambda(x^*) \cap X_1) \oplus_1 X_2^* &= \text{NA}_\mu(\text{NA}_\lambda(x^*) \cap S_{X_1}) \oplus_1 \text{NA}_\mu(S_{X_2}) \\ &\subset \text{NA}_\mu((\text{NA}_\lambda(x^*) \cap S_{X_1}) \oplus_\infty S_{X_2}) \\ &\subset \text{NA}_\mu(\text{NA}_\lambda(x^* \oplus_1 \mathbf{0}) \cap (X_1 \oplus_\infty X_2)). \end{aligned}$$

(b). If

$$C := \text{NA}_{1-\varepsilon}(x_1^* \oplus_1 x_2^*) \cap S_{X_1 \oplus_\infty X_2},$$

then, thanks to Lemma 4.28 (b), together with Lemma 4.3 (b) and (c), we conclude that

$$\begin{aligned} A \oplus_1 B &\subset \text{NA}_{1-\delta}(C) \cap (S_{Y_1} \oplus_1 S_{Y_2}) \\ &\subset \text{NA}_{1-2\delta}(\pi_{X_1}(C)) \oplus_1 \text{NA}_{1-2\delta}(\pi_{X_2}(C)), \end{aligned}$$

hence  $A \subset \text{NA}_{1-2\delta}(\pi_{X_1}(C))$  and  $B \subset \text{NA}_{1-2\delta}(\pi_{X_2}(C))$ . On the other hand, Lemma 4.28 (d), implies that either

$$\pi_{X_1}(C) \subset \text{NA}_{1-2\varepsilon}(x_1^*) \cap X_1$$

or

$$\pi_{X_2}(C) \subset \text{NA}_{1-2\varepsilon}(x_2^*) \cap X_2.$$

The desired conclusion now follows from Lemma 4.3 (a).  $\square$

**Theorem 4.30.** *Let  $X_1$  and  $X_2$  be Banach spaces,  $Y_1 \subset X_1^*$  and  $Y_2 \subset X_2^*$  subspaces, and  $\kappa$  an infinite cardinal. The following are equivalent.*

(i) *Either  $(X_1, Y_1)$  or  $(X_2, Y_2)$  enjoys the SD2P $_\kappa$  (respectively, NASD2P $_\kappa$ ).*

(ii)  *$(X_1 \oplus_\infty X_2, Y_1 \oplus_1 Y_2)$  enjoys the SD2P $_\kappa$  (respectively, NASD2P $_\kappa$ ).*

*Proof.* (i)  $\implies$  (ii) follows from Theorem 4.29 (a), while (ii)  $\implies$  (i) follows from Theorem 4.29 (b).  $\square$

**Theorem 4.31.** *Let  $\{X_\alpha : \alpha \in \mathcal{A}\}$  be a family of Banach spaces,  $Y_\alpha \subset X_\alpha^*$  subspaces, and  $\kappa$  an infinite cardinal. The following are equivalent.*

(i)  *$(X_\alpha, Y_\alpha)$  enjoys the SD2P $_\kappa$  (respectively, NASD2P $_\kappa$ ) for every  $\alpha \in \mathcal{A}$ .*

(ii)  *$(\ell_1(\mathcal{A}, X_\alpha), \ell_\infty(\mathcal{A}, X_\alpha^*))$  enjoys the SD2P $_\kappa$  (respectively, NASD2P $_\kappa$ ).*

*Proof.* We prove the NASD2P $_{\kappa}$  case only.

(i)  $\implies$  (ii). Fix  $A \in \mathcal{P}_{\kappa}(S_{\ell_{\infty}(\mathcal{A}, X_{\alpha}^*)})$ . For every  $\alpha \in \mathcal{A}$ , find  $C_{\alpha} \subset S_{X_{\alpha}}$  and  $x_{\alpha}^* \in S_{X_{\alpha}^*}$  such that  $C_{\alpha}$  norms  $\pi_{Y_{\alpha}}(A)$  and  $x_{\alpha}^*(x) = 1$  for each  $x \in C_{\alpha}$ . Set

$$C := \bigcup i_{X_{\alpha}}(C_{\alpha}) \subset S_{\ell_1(\mathcal{A}, X_{\alpha})}$$

and define  $x^* \in S_{\ell_{\infty}(\mathcal{A}, X_{\alpha}^*)}$  by  $x^*(\alpha) = x_{\alpha}^*$ . It is clear by construction that  $x^*(x) = 1$  for every  $x \in C$ . We now claim that  $C$  norms  $A$ . To this aim fix  $y^* \in A$  and, for every  $n \in \mathbb{N}$ , pick some  $\alpha_n \in \mathcal{A}$  such that

$$\|y^*(\alpha_n)\| \geq 1 - n^{-1},$$

and find  $x_{\alpha_n} \in C_{\alpha_n}$  satisfying

$$y^*(\alpha_n)x_{\alpha_n} \geq (1 - n^{-1})\|y^*(\alpha_n)\|.$$

We conclude that

$$\begin{aligned} y^*(i_{X_{\alpha_n}} x_{\alpha_n}) &= y^*(\alpha_n)(x_{\alpha_n}) \\ &\geq (1 - n^{-1})^2. \end{aligned}$$

By passing to the supremum over  $n$ , the claim follows.

(ii)  $\implies$  (i). It suffices to show that  $(X_1, Y_1)$  possesses the NASD2P $_{\kappa}$  provided that  $(X_1 \oplus_1 X_2, Y_1 \oplus_{\infty} Y_2)$  enjoys the NASD2P $_{\kappa}$ . To this purpose, fix  $A \in \mathcal{P}_{\kappa}(S_{Y_1})$  and find  $x_1^* \oplus_{\infty} x_2^* \in S_{X_1^* \oplus_{\infty} X_2^*}$  such that, if we define

$$C := \text{NA}_1(x_1^* \oplus_{\infty} x_2^*) \cap (X_1 \oplus_1 X_2),$$

then  $A \oplus_{\infty} \{0\} \subset \text{NA}_1(C)$ . It follows that  $A \subset \text{NA}_1(\pi_{X_1}(C))$  and, on the other hand,  $\pi_{X_1}(C) \cap S_{X_1} \subset \text{NA}_1(x_1^*)$ , due to Lemma 4.28 (b). We conclude, thanks to Lemma 4.3 (a), that  $A \subset \text{NA}_1(\text{NA}_1(x_1^*))$ .  $\square$

## 4.4.2 Tensor products

It has been proved that the SD2P is preserved by projective tensor products in [BLR1, Corollary 3.6] and [LMR, Proposition 3.6]. In this subsection, we aim to extend this result to the transfinite SD2P. Let us begin by investigating the norming and attaining case.

**Theorem 4.32.** *Let  $X$  and  $Y$  be Banach spaces and  $\kappa$  and infinite cardinal. If  $X$  and  $Y$  enjoy the NASD2P $_{\kappa}$ , then  $X \hat{\otimes}_{\pi} Y$  enjoys the NASD2P $_{\kappa}$ .*

*Proof.* Let  $\mathcal{B} \in \mathcal{P}_\kappa(S_{(X \hat{\otimes}_\pi Y)^*})$  and choose elements  $u_B \otimes v_B \in S_X \otimes S_Y$  such that

$$B(u_B^n, v_B^n) \geq (1 - n^{-1})^{\frac{1}{3}}$$

for every  $B \in \mathcal{B}$  and  $n \in \mathbb{N}$ . Since  $X$  enjoys the NASD2P $_\kappa$ , there exist  $x_B^n$ 's belonging to  $S_X$  and  $x^* \in S_{X^*}$  satisfying

$$B(x_B^n, v_B^n) \geq (1 - n^{-1})^{\frac{1}{3}} \|B(\cdot, v_B^n)\|$$

and  $x^*(x_B^n) = 1$  for every  $B \in \mathcal{B}$  and  $n \in \mathbb{N}$ . Since  $Y$  possesses the NASD2P $_\kappa$ , there exist  $y_B^n$ 's belonging to  $S_Y$  and  $y^* \in S_{Y^*}$  satisfying

$$B(x_B^n, y_B^n) \geq (1 - n^{-1})^{\frac{1}{3}} \|B(x_B^n, \cdot)\|$$

and  $y^*(y_B^n) = 1$  for each  $B \in \mathcal{B}$  and  $n \in \mathbb{N}$ . Now set  $z_B^n := x_B^n \otimes y_B^n$  and  $z^* := x^* \otimes y^*$ . It is clear that  $z^*(z_B^n) = 1$  and that the  $z_B^n$ 's norm  $\mathcal{B}$ , as

$$\begin{aligned} B(x_B^n, y_B^n) &> (1 - n^{-1})^{\frac{1}{3}} \|B(x_B^n, \cdot)\| \\ &\geq (1 - n^{-1})^{\frac{1}{3}} B(x_B^n, v_B^n) \\ &\geq (1 - n^{-1})^{\frac{2}{3}} \|B(\cdot, v_B^n)\| \\ &\geq (1 - n^{-1})^{\frac{2}{3}} B(u_B^n, v_B^n) \\ &\geq 1 - n^{-1}. \end{aligned}$$

□

**Theorem 4.33.** *Let  $X$  and  $Y$  be Banach spaces, and  $\kappa$  an infinite cardinal. If  $X$  and  $Y$  enjoy the SD2P $_\kappa$ , then  $X \hat{\otimes}_\pi Y$  enjoys the SD2P $_\kappa$ .*

*Proof.* The proof follows from a straightforward extension of [HLP2, Theorem 2.2], together with Theorem 4.19. □

## 4.5 More examples

We are finally ready to provide interesting examples of Banach spaces enjoying the transfinite SD2P.

### 4.5.1 $C(K)$ spaces

**Theorem 4.34.** *Let  $K$  be a compact Hausdorff space. If  $|K| \geq \aleph_1$ , then  $C(K)$  possesses the NASD2P $_{|K|}$ .*

*Proof.* Fix  $\mathcal{M} \in \mathcal{P}_{|K|}(S_{C(K)^*})$ . Since each measure  $\mu \in \mathcal{M}$  has bounded variation, it must have at most countably many atoms, hence we can find  $x \in K$  which is not an atom for any  $\mu \in \mathcal{M}$ . Now, fix  $\mu \in \mathcal{M}$  and  $\varepsilon > 0$ . Since  $\mu(\{x\}) = 0$  and  $\mu$  is regular, then we can find an open neighborhood  $U_{\mu,\varepsilon}$  of  $x$  satisfying

$$|\mu|(U_{\mu,\varepsilon}) < \varepsilon/3.$$

Find  $f_{\mu,\varepsilon} \in S_{C(K)}$  such that

$$\mu(f_{\mu,\varepsilon}) \geq 1 - \varepsilon/3$$

and consider the Urysohn's functions  $g_{\mu,\varepsilon} : K \rightarrow [0, 1]$  satisfying  $g_{\mu,\varepsilon}(x) = 0$  and  $g_{\mu,\varepsilon}|_{K \setminus U_{\mu,\varepsilon}} = 1$ . Set

$$h_{\mu,\varepsilon} = 1 + g_{\mu,\varepsilon}(f_{\mu,\varepsilon} - 1) \in S_{C(K)}$$

and notice that

$$\begin{aligned} \mu(h_{\mu,\varepsilon}) &= \int_{K \setminus U_{\mu,\varepsilon}} f_{\mu,\varepsilon} d\mu + \int_{U_{\mu,\varepsilon}} h_{\mu,\varepsilon} d\mu \\ &\geq \mu(f_{\mu,\varepsilon}) - 2|\mu|(U_{\mu,\varepsilon}) \\ &\geq 1 - \varepsilon. \end{aligned}$$

In other words, the  $h_{\mu,\varepsilon}$ 's norm  $\mathcal{M}$ . Moreover, it is clear that  $\delta_x(h_{\mu,\varepsilon}) = 1$  holds true for every  $\mu \in \mathcal{M}$  and  $\varepsilon > 0$ .  $\square$

**Example 4.35.** Let us employ Theorem 4.34 to produce new examples of Banach spaces possessing the transfinite SD2P.

- (a)  $C[0, 1]$  enjoys the NASD2P $_{2^{\aleph_0}}$ .
- (b) Let  $B$  be an infinite Boolean algebra and let  $\mathcal{S}(B)$  be the Stone space associated to  $B$ . Now let us consider a regular positive Borel measure  $\mu$  over some locally compact Hausdorff space  $X$ . Define  $\mathfrak{B}_\mu$  the set of measurable sets modulo the negligible sets in  $X$ . It is known that  $L_\infty(\mu)$  is isometrically isomorphic to  $C(\mathcal{S}(\mathfrak{B}_\mu))$  (see e.g. pages 27–29 in [DLS]), therefore we conclude that  $L_\infty(\mu)$  enjoys the NASD2P $_{|\mathcal{S}(\mathfrak{B}_\mu)|}$ . In particular, the Banach spaces  $\ell_\infty$  and  $L_\infty[0, 1]$  enjoy the NASD2P $_{2^{2^{\aleph_0}}}$  thanks to [Jec, Theorem 7.6].

Theorem 4.34 can be used to fully describe  $C(K)$  spaces when  $\kappa = \aleph_1$ .

**Corollary 4.36.** *Let  $K$  be a compact Hausdorff space. The following are equivalent.*

- (i)  $|K| \geq \aleph_1$ .
- (ii)  $C(K)$  enjoys the  $\text{SD2P}_{\aleph_1}$ .
- (iii)  $C(K)$  enjoys the  $\text{NASD2P}_{\aleph_1}$ .
- (iv)  $C(K)^*$  is non-separable.
- (v)  $C(K)^*$  is  $\text{OH}_{\aleph_1}$ .
- (vi)  $C(K)^*$  is  $\text{ROH}_{\aleph_1}$ .

*Proof.* (iv)  $\iff$  (vi) is [GLM, Corollary 20], (ii)  $\iff$  (v) is Theorem 4.19 (a) and (i)  $\implies$  (iii) follows from Theorem 4.34. (iii)  $\implies$  (ii) and (vi)  $\implies$  (v)  $\implies$  (iv) are obvious. Notice that (iv)  $\implies$  (i) since, if  $|K| < \omega_1$ , then  $C(K)^*$  is the closed linear span of the set  $\{\delta_x : x \in K\}$  (see e.g. the discussion after Example 1.10 in [ADM] noting that it only requires  $\delta_x$  to be a measure for every  $x \in K$ ).  $\square$

### 4.5.2 $L_1(\mu)$ spaces

**Theorem 4.37.** *Let  $X$  be an  $L$ -embedded space and  $\kappa$  an infinite cardinal. Then  $X$  enjoys the  $\text{SD2P}_{\kappa}$  (respectively,  $\text{NASD2P}_{\kappa}$ ) if and only if  $X^{**}$  enjoys the weak\*  $\text{SD2P}_{\kappa}$  (respectively, weak\*  $\text{NASD2P}_{\kappa}$ ).*

*Proof.* Let us prove the case for the  $\text{NASD2P}_{\kappa}$  case only. If  $X$  has the  $\text{NASD2P}_{\kappa}$ , then clearly  $X^{**}$  has the weak\*  $\text{NASD2P}_{\kappa}$ . Conversely, let

$$X^{**} = X \oplus_1 Z$$

for some subspace  $Z \subset X^{**}$ . It is clear that

$$X^* = X^* \oplus_{\infty} \{0\},$$

therefore Theorem 4.31 implies that, if  $(X \oplus_1 Z, X^* \oplus_{\infty} \{0\})$  enjoys the  $\text{NASD2P}_{\kappa}$ , then  $(X, X^*)$  satisfies the  $\text{NASD2P}_{\kappa}$ .  $\square$

**Corollary 4.38.** *Let  $\mu$  be a localizable measure. The following statements are equivalent.*

- (i)  $\mu$  is atomless.
- (ii)  $L_1(\mu)$  has the  $\text{SD2P}_{\aleph_0}$ .
- (iii)  $L_1(\mu)$  has the  $\text{NASD2P}_{\aleph_1}$ .

(iv)  $B_{L_1(\mu)}$  has no strongly exposed points.

(v)  $L_1(\mu)$  has the Daugavet property.

(vi)  $L_\infty(\mu)$  is  $\text{OH}_{\aleph_0}$ .

(vii)  $L_\infty(\mu)$  is  $\text{ROH}_{\aleph_1}$ .

(viii)  $L_\infty(\mu)$  has the Daugavet property.

*Proof.* (i)  $\iff$  (iv)  $\iff$  (v) was proved in [BM] and (i)  $\iff$  (viii) is known (see e.g. [Wer]). (viii)  $\implies$  (iii) follows from combining the fact that  $L_1(\mu)$  is an  $L$ -embedded space together with Theorems 4.23 and 4.37. (iii)  $\implies$  (ii) and (ii)  $\implies$  (iv) are obvious, (ii)  $\iff$  (vi) follows from Theorem 4.18 and (iii)  $\implies$  (vii) is shown by Theorem 4.21. Eventually, (vii)  $\implies$  (vi) is obvious.  $\square$

*Remark 4.39.* The examples of separable Banach spaces that have been shown to enjoy the  $\text{NASD2P}_{\aleph_1}$  so far actually enjoy the Daugavet property. Nevertheless, they still are separate properties. In fact, fix some separable Daugavet space  $X$  that also has the  $\text{NASD2P}_{\aleph_1}$  (for example,  $X = C[0, 1]$ ) and let  $N$  be an absolute normalized norm with the positive SD2P which differs from the  $\ell_1$  and  $\ell_\infty$  norm, then  $X \oplus_N X$  has the  $\text{NASD2P}_{\aleph_1}$  by Theorem 4.27. On the other hand,  $X \oplus_N X$  is separable and cannot have the Daugavet property [BKSW, Corollary 5.4].

In [ALN1, Theorem 4.4] it was pointed out that every Daugavet space has the SD2P. In the following we will show that even more is true.

**Theorem 4.40.** *Let  $X$  be a Banach space. If  $X$  has the Daugavet property, then  $X$  has the  $\text{SD2P}_{\aleph_1}$ .*

*Proof.* If  $X$  has the Daugavet property, then, by Lemma 4.6, the dual  $X^*$  is  $\text{ROH}_{\aleph_1}$ . By Theorem 4.18, we conclude that  $X$  has the  $\text{SD2P}_{\aleph_1}$ .  $\square$

### 4.5.3 Lebesgue-Bochner spaces

In general it is not known when Lebesgue-Bochner spaces  $L_1(\mu; X)$  are  $L$ -embedded (see [HWW, IV.5] for some partial results). Therefore, more attention is needed to study under which conditions Lebesgue-Bochner spaces enjoy the transfinite SD2P.

**Lemma 4.41.** *Let  $\mu$  be an atomless measure. If  $\{B_i : i \in \mathbb{N}\}$  is a countable family consisting of non-negligible sets, then there exists a family  $\{E_i : i \in \mathbb{N}\}$  consisting of non-negligible pairwise disjoint sets satisfying  $E_i \subset B_i$  for every  $i \in \mathbb{N}$ .*

*Proof.* For every  $i > 1$  fix a set  $C_i^1 \subset B_i$  such that

$$0 < \mu(C_i^1) \leq 2^{-i} \mu(B_1)$$

(the existence of such  $C_i^1$ 's is ensured by [AB, Theorem 10.52]) and set

$$E_1 := B_1 \setminus \left( \bigcup_{i=2}^{\infty} C_i^1 \right).$$

Note that

$$\begin{aligned} \mu(E_1) &\geq \mu(B_1) - \left( \sum_{i=2}^{\infty} 2^{-i} \right) \mu(B_1) \\ &> 0. \end{aligned}$$

We now have  $E_1 \subset B_1$  and  $C_i^1 \subset B_i$ , all non-negligible, such that  $E_1$  is disjoint from  $\bigcup_{i=2}^{\infty} C_i^1$ . With an induction argument, we repeat the construction on

$$\{C_i^{j-1}\}_{i=j+1}^{\infty}$$

to get  $E_j \subset C_j^{j-1}$  disjoint from all  $C_i^j \subset C_j^{j-1}$ , where  $i > j$ . Clearly, the  $E_i$ 's satisfy the claim.  $\square$

**Lemma 4.42.** *Let  $X$  be a Banach space and  $\mu$  an atomless measure. If  $(f_i)_i \subset S_{L_{\infty}(\mu; X)}$ , then there exists a family  $\{E_i : i \in \mathbb{N}\} \subset \Sigma$  consisting of non-negligible pairwise disjoint sets such that  $(f_i \chi_{E_i})_i \subset S_{L_{\infty}(\mu; X)}$ .*

*Proof.* For all  $i, j \in \mathbb{N}$  find non-negligible  $B_{i,j} \in \Sigma$  such that  $\|f_i(t)\| > 1 - j^{-1}$  for almost all  $t \in B_{i,j}$ . By Lemma 4.41, we can assume that these sets are pairwise disjoint. Set

$$E_i := \bigcup_{j \in \mathbb{N}} B_{i,j}.$$

$\square$

**Theorem 4.43.** *Let  $\mu$  be a measure and  $X$  a Banach space. Suppose that either  $\mu$  is decomposable and  $X^*$  is separable or  $\mu$  is  $\sigma$ -finite and  $X^*$  has the Radon-Nikodym property with respect to  $\mu$ .*

- (a) *If  $\mu$  is atomless, then  $L_1(\mu; X)$  has the NASD2P $_{\aleph_1}$ .*
- (b) *If  $\mu$  has atoms, then  $L_1(\mu; X)$  enjoys the SD2P $_{\kappa}$  (respectively, NASD2P $_{\kappa}$ ) if and only if  $X$  possesses the SD2P $_{\kappa}$  (respectively, NASD2P $_{\kappa}$ ), whenever  $\kappa \in \{\aleph_0, \aleph_1\}$ .*

*Proof.* (a) Let  $(f_i)_i \subset S_{L_\infty(\mu; X)}$ . Thanks to Lemma 4.42, there exists a pairwise disjoint family

$$\{E_{i,j} : i, j \in \mathbb{N}\} \subset \Sigma$$

consisting of non-negligible pairwise disjoint sets such that the  $f_i \chi_{E_{i,j}}$ 's belong to  $S_{L_\infty(\mu; X)}$ . For every  $i, j \in \mathbb{N}$  find some  $g_{i,j} \in S_{L_1(\mu; X)}$  such that

$$(f_i \chi_{E_{i,j}})(g_{i,j}) \geq 1 - j^{-1}.$$

It is clear that the set

$$\{g_{i,j} \chi_{E_{i,j}} : i, j \in \mathbb{N}\}$$

satisfies condition (ii) of Lemma 4.1 and it norms the  $f_i$ 's since

$$\begin{aligned} f_i(g_{i,j} \chi_{E_{i,j}}) &= (f_i \chi_{E_{i,j}})(g_{i,j}) \\ &\geq 1 - j^{-1}. \end{aligned}$$

(b). If  $\mu$  has atoms, then

$$L_1(\mu; X) = L_1(\nu; X) \oplus_1 \ell_1(\{\eta : \eta \text{ is an atom of } \mu\}; X),$$

where  $\nu$  is atomless, thus the claim follows from (a) and Theorem 4.31.  $\square$

*Remark 4.44.* Notice that if  $X^*$  is separable, then  $X$  cannot have the SD2P $_{\aleph_1}$  due to Theorem 4.19. On the other hand, if  $X^*$  has the Radon-Nikodym property with respect to some  $\sigma$ -finite measure  $\mu$ , then  $X$  can still have the NASD2P $_{\aleph_1}$ . In fact  $c_0(\aleph_1)$  is SQ $_{\aleph_1}$  and  $\ell_1(\aleph_1)$  has the Radon-Nikodym property.

We now investigate when the dual of the Lebesgue-Bochner space  $L_1(\mu; X)$  enjoys the weak\* SD2P $_{\aleph_1}$  or the weak\* NASD2P $_{\aleph_1}$ .

**Lemma 4.45.** *If  $\{X_\alpha : \alpha \in \mathcal{A}\}$  is an infinite family of non-trivial Banach spaces, then  $\ell_1(\mathcal{A}, X_\alpha)^*$  has the weak\* NASD2P $_{|\ell_1(\mathcal{A}, X_\alpha)|}$ .*

*Proof.* Find  $y^* \in S_{\ell_\infty(\mathcal{A}, X_\alpha^*)}$  such that  $\|y^*(\alpha)\| = 1$  for every  $\alpha \in \mathcal{A}$  and set

$$A := (y^* + c_{00}(\mathcal{A}, X_\alpha^*)) \cap S_{\ell_\infty(\mathcal{A}, X_\alpha^*)}.$$

We claim that  $A$  norms  $\ell_1(\mathcal{A}, X_\alpha)$ .

Fix  $x \in S_{\ell_1(\mathcal{A}, X_\alpha)}$  and  $\varepsilon > 0$ . Find a finite set  $I$  such that

$$\sum_{\alpha \notin I} \|x(\alpha)\| < \varepsilon/2.$$

Pick  $x^* \in S_{\ell_\infty(\mathcal{A}, X_\alpha^*)}$  such that  $x^*(x) = 1$  and define  $\tilde{x}^* \in A$  by

$$\tilde{x}^*(\alpha) := \begin{cases} x^*(\alpha) & \text{if } \alpha \in I, \\ y^*(\alpha) & \text{if } \alpha \notin I. \end{cases}$$

Therefore,

$$\begin{aligned} \tilde{x}^*(x) &= \sum_{\alpha \in \mathcal{A}} \tilde{x}^*(\alpha)(x(\alpha)) \\ &\geq \sum_{\alpha \in I} x^*(\alpha)(x(\alpha)) - \varepsilon/2 \\ &\geq x^*(x) - \varepsilon \\ &= 1 - \varepsilon, \end{aligned}$$

hence the claim follows. In order to conclude, we only need to find  $x^{**} \in S_{X^{**}}$  such that  $x^{**}(x^*) = 1$  for every  $x^* \in A$ . For every  $x_1^*, \dots, x_n^* \in A$ , there is  $\alpha$  such that  $x_i^*(\alpha) = y^*(\alpha)$  for all  $i \in \{1, \dots, n\}$ . Thus,

$$\begin{aligned} \left\| \sum_{i=1}^n x_i^* \right\| &\geq \left\| \sum_{i=1}^n y^*(\alpha) \right\| \\ &= n \end{aligned}$$

and the conclusion follows from Lemma 4.1.  $\square$

**Theorem 4.46.** *Let  $\mu$  be a measure and  $X$  a Banach space. Then  $L_1(\mu; X)^*$  enjoys the weak\* SD2P $_\kappa$  (respectively, weak\* NASD2P $_\kappa$ ) if and only if  $\mu$  is not purely atomic with finitely many atoms or  $X^*$  enjoys the weak\* SD2P $_\kappa$  (respectively, weak\* NASD2P $_\kappa$ ), whenever  $\kappa \in \{\aleph_0, \aleph_1\}$ .*

*Proof.*

- (a) If  $\mu$  is atomless, then  $L_1(\mu; X)$  has the Daugavet property thanks to [Wer, page 81]. Therefore, Theorem 4.23 implies that  $L_1(\mu; X)^*$  has the weak\* NASD2P $_{\aleph_1}$ .
- (b) If  $\mu$  is purely atomic and has infinitely many atoms, then  $L_1(\mu; X)^*$  has the weak\* NASD2P $_{\aleph_1}$ , thanks to Lemma 4.45.
- (c) If  $\mu$  has atoms but is not purely atomic, then  $L_1(\mu; X) = L_1(\mu_1; X) \oplus_1 L_1(\mu_2; X)$ , where  $\mu_1$  is atomless, therefore, thanks to Theorem 4.30, we conclude that  $L_1(\mu; X)^*$  has the weak\* NASD2P $_{\aleph_1}$ .
- (d) If  $\mu$  is purely atomic and has finitely many atoms, then

$$L_1(\mu; X) = X \oplus_1 \dots \oplus_1 X,$$

and Theorem 4.30 implies the desired conclusion.  $\square$

## 4.6 Open problems

In this final section, let us point out the main questions that remain unsolved.

*Problem 4.1.* [CLL1, Question 3.7] If a dual Banach space  $X^*$  is  $\text{ROH}_\kappa$ , does this imply that  $X$  enjoys the  $\text{NASD2P}_\kappa$ ?

*Problem 4.2.* [CLL1, Question 6.10] If  $X$  is a Banach space with the Daugavet property, then does  $X$  have the ASD2P (or even the  $\text{NASD2P}_{\aleph_1}$ )?

The answer to this question is positive if either

- (a)  $X$  is an  $L$ -embedded space of density character  $\leq \aleph_1$ . Indeed, if  $X$  is  $L$ -embedded, has density character  $\leq \aleph_1$ , and has the Daugavet property, then  $X^*$  also has the Daugavet property (see [Rue1, Theorem 3.4] for the separable case and [LR, Theorem 4.1] for the case of cardinality  $\aleph_1$ ). Now, by Theorem 4.23, the bidual  $X^{**}$  has the weak\*  $\text{NASD2P}_{\aleph_1}$ , hence Theorem 4.37 shows that  $X$  has the  $\text{NASD2P}_{\aleph_1}$ .
- (b)  $X = C(K)$  is separable for some  $K$  compact Hausdorff. Since  $X$  has the Daugavet property, then  $K$  does not have any isolated points [Wer, Example (a)]. Now, by Corollary 4.36,  $X$  has the  $\text{NASD2P}_{\aleph_1}$  if and only if  $K$  is uncountable. If, by contradiction, we would assume that  $K$  was countable, then we would get that

$$K = \omega^\alpha + n$$

for some countable ordinal  $\alpha$  and  $n \in \mathbb{N}$  by Sierpinski–Mazurkiewicz theorem, which clearly has isolated points. Therefore,  $X$  must have the  $\text{NASD2P}_{\aleph_1}$ .

# Chapter 5

## Ball-covering properties

In [God, p. 12], it was noted that being OH is equivalent to the fact that every finite covering of the unit ball, via closed (equivalently, open) balls, has at least one member covering itself the unit ball. This observation allows us to connect OH norms to ball-coverings. Recall that a Banach space enjoys the *ball-covering property* (BCP) if its unit sphere can be covered by countably many open balls not containing the origin. It is clear that every separable Banach space possesses the BCP, but the condition is not necessary, as  $\ell_\infty$  has the BCP.

In [GLM], the BCP was extended to the  $\alpha$ -BCP, where  $\alpha \in [-1, 1)$ . A Banach space  $X$  is said to have the  $\alpha$ -BCP, where  $\alpha \in [0, 1)$ , if its unit sphere can be covered by countably many open balls that do not intersect  $\alpha B_X$ , while, for  $\alpha \in [-1, 0)$ , the balls are required not to contain  $\alpha B_X$ . Therefore the BCP coincides with the 0-BCP.

The aim of this chapter is to try to extend the well-known characterization of OH Banach spaces in terms of containment of  $\ell_1$  to the transfinite case. In order to do so, we will consider and study the failure of a transfinite version of the BCP instead, which is a property stronger than  $\text{OH}_\kappa$ , but weaker than  $\text{ROH}_\kappa$ . This chapter is based on [CLL2].

### 5.1 Some important notation

In this section, we introduce some notation that will be used throughout this chapter. Given a Banach space  $X$ ,  $A \subset X$ ,  $\kappa$  a cardinal, and  $\alpha, \beta \in [-1, 1]$ , define

$$C(\alpha, \beta, A) := \{x \in X : \exists a \in A (\|x - a\| < \alpha\|x\| + \beta\|a\|)\},$$

and

$$C(\alpha, \beta, \kappa) := \{B \subset C(\alpha, \beta, A) : \text{for some } A \in \mathcal{P}_\kappa(X)\}.$$

Notice that this notation well encapsulates the concept of covering a set with open balls, because a set  $B$  belongs to  $C(\alpha, \beta, \kappa)$  if and only if there exists a set  $A \in \mathcal{P}_\kappa(X)$  such that

$$B \subset \bigcup_{x \in A} B(x, \alpha + \beta\|x\|),$$

where by  $B(x, r)$  we mean the ball centered in  $x$  of radius  $r$ . Let us now list some useful properties of these sets.

**Lemma 5.1.** *Let  $X$  be a Banach space,  $A, B \subset X$ , and  $\alpha, \beta \in [-1, 1]$ . Then*

- (a)  $C(\alpha, \beta, A) = C(\alpha, \beta, \overline{A})$ .
- (b) If  $B \subset C(\alpha, \beta, A)$ , then  $tB \subset C(\alpha, \beta, tA)$  holds true for each  $t > 0$ .
- (c) If  $S_X \subset C(\alpha, \beta, A)$ , then  $X \setminus \{0\} = C(\alpha, \beta, \mathbb{R}^+A)$ .
- (d) Let  $f : A \rightarrow [1, \infty)$ . If  $B \subset C(\alpha, \beta, A)$ , then

$$B \subset C(\alpha, \beta, \{f(x) \cdot x : x \in A\}).$$

*Proof.* The proof is immediate. □

**Lemma 5.2.** *Let  $X$  be a Banach space,  $\kappa$  an infinite cardinal, and  $\alpha, \beta \in [-1, 1]$ . The following are equivalent:*

- (i)  $S_X \notin C(\alpha, \beta, \kappa)$ .
- (ii)  $X \setminus \{0\} \notin C(\alpha, \beta, \kappa)$ ,
- (iii) Given a subspace  $Y \subset X$  of density character less than  $\kappa$ , there exists  $x \in S_X$  such that

$$\|rx + y\| \geq \alpha|r| + \beta\|y\|$$

holds for every  $r \in \mathbb{R}$  and  $y \in Y$ .

*Proof.* (i)  $\implies$  (iii) follows from applying Lemma 5.1 (a). (iii)  $\implies$  (ii) is trivial, and (ii)  $\implies$  (i) is a consequence of Lemma 5.1 (a) and (c). □

## 5.2 Definition and some examples

**Definition 5.3.** Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. Then  $X$  enjoys the  $\alpha$ -ball covering property with respect to  $\kappa$  ( $\alpha$ -BCP $_{\kappa}$ ), for  $\alpha \in [0, 1)$ , if  $S_X$  can be covered by less than  $\kappa$  many open balls that do not intersect  $\alpha B_X$ , while, for  $\alpha \in [-1, 0)$ , the open balls are required not to contain  $\alpha B_X$ .

In this notation, the usual BCP corresponds to the 0-BCP $_{\aleph_1}$ . Moreover, it is clear that a Banach space  $X$  possesses the  $\alpha$ -BCP $_{\kappa}$  if, and only if  $S_X \in C(-\alpha, 1, \kappa)$ . Eventually, observe that being ROH $_{\kappa}$  is equivalent to requiring that  $S_X \notin C(1, 1, \kappa)$ , and that being OH $_{\kappa}$  is equivalent to the condition  $S_X \notin C(\alpha, \alpha, \kappa)$ , for every  $\alpha \in (0, 1)$ . In other words, ROH $_{\kappa}$  is equivalent to the failure of the  $(-1)$ -BCP $_{\kappa}$  and the failure of the  $\alpha$ -BCP $_{\kappa}$ , for every  $\alpha \in (-1, 1)$ , implies OH $_{\kappa}$ .

We will postpone showing more examples, since, in order to show interesting examples, more theory of the structure of the transfinite BCP needs to be developed.

## 5.3 Renorming results

In this section, we aim to analyze the relationship between the containment of isomorphic copies of the Banach space  $\ell_1(\kappa)$ , and the existence of equivalent norms which fail the transfinite BCP.

**Theorem 5.4.** *Let  $X$  be a Banach space,  $\kappa$  an infinite cardinal, and  $\alpha \in [-1, 0)$ . If  $X$  fails the  $\alpha$ -BCP $_{\kappa}$ , then it contains an isomorphic copy of  $\ell_1(\kappa)$ .*

*Proof.* We follow the ideas outlined in [GLM, Proposition 23]. Notice that it suffices to show that we can find a subset  $A \subset S_X$  of cardinality  $\kappa$  such that

$$\left\| \sum_{i=1}^n r_i x_i \right\| \geq -\alpha \sum_{i=1}^n |r_i|$$

holds for every  $x_1, \dots, x_n \in A$  and  $r_1, \dots, r_n \in \mathbb{R}$ . Let  $\mathcal{O}$  be the partially ordered set consisting of all sets, of any given cardinality, in  $S_X$  satisfying this property. Any singleton belongs to  $\mathcal{O}$ , hence it is non-empty. Moreover, any chain  $\mathcal{C}$  in  $\mathcal{O}$  has upper bound  $\bigcup \mathcal{C} \in \mathcal{O}$ . Therefore Zorn's Lemma applies, thus we can find a maximal element  $A \in \mathcal{O}$ . Observe that the cardinality of  $A$  must be at least  $\kappa$ , otherwise, since  $X$  fails the  $\alpha$ -BCP $_{\kappa}$ , we can find  $x \in S_X$  such that  $\{x\} \cup A \in \mathcal{O}$ , thus violating the maximality of  $A$ .  $\square$

We now proceed to investigate the opposite direction of the renorming. The cornerstone of our approach is the norm construction technique from [KSW, Section 4].

**Lemma 5.5.** *Let  $(X, \|\cdot\|)$  be a Banach space and  $Y \subset X$  a subspace. If  $p$  is a seminorm on  $X$  dominated by  $\|\cdot\|$  and equivalent to it in  $Y$ , then*

$$|\cdot| := p(\cdot) + \|\cdot + Y\|_{X/Y}$$

*defines an equivalent norm on  $X$ . Moreover, if  $\kappa$  is an infinite cardinal,  $\alpha \in [-1, 1)$ , and, for every  $A \in \mathcal{P}_\kappa(X)$ , there exists  $y \in Y \setminus \{0\}$  satisfying*

$$p(y - x) \geq -\alpha p(y) + p(x)$$

*for every  $x \in A$ , then  $(X, |\cdot|)$  fails the  $\alpha$ -BCP $_\kappa$ .*

*Proof.* Assume  $p \leq C\|\cdot\|$  on  $X$  and  $p \geq c\|\cdot\|$  on  $Y$ , for some  $c, C > 0$ . On the one hand, it is clear that  $|\cdot| \leq (C + 1)\|\cdot\|$ , on the other, we claim that  $|\cdot| \geq \tilde{c}\|\cdot\|$ , where  $\tilde{c} := (1 + c + C)^{-1}$ .

To this aim, fix  $x \in X$  and, without loss of generality, assume that

$$\|x + Y\| < \tilde{c}\|x\|,$$

otherwise the claim becomes trivial to prove. There exists  $y \in Y$  such that  $\|x - y\| < \tilde{c}\|x\|$ , hence

$$\begin{aligned} \|y\| &\geq \|x\| - \|x - y\| \\ &> (1 - \tilde{c})\|x\|. \end{aligned}$$

Therefore

$$\begin{aligned} |x| &\geq p(x) \\ &\geq p(y) - p(y - x) \\ &\geq c\|y\| - C\|x - y\| \\ &> c(1 - \tilde{c})\|x\| - C\tilde{c}\|x\| \\ &= \tilde{c}\|x\|. \end{aligned}$$

Thus the claim is proved. For the moreover part, fix  $A \in \mathcal{P}_\kappa(X)$ . There is  $y \in Y \setminus \{0\}$  satisfying

$$p(y - x) \geq -\alpha p(y) + p(x)$$

for every  $x \in A$ . Thus

$$\begin{aligned} |y - x| &= p(y - x) + \|x + Y\|_{X/Y} \\ &\geq -\alpha p(y) + |x| \\ &= -\alpha|y| + |x|, \end{aligned}$$

which means that  $X \setminus \{0\} \notin C(\alpha, 1, \kappa)$ .  $\square$

**Theorem 5.6.** *Let  $X$  be a Banach space and  $\kappa$  an infinite cardinal. If  $X$  contains an isomorphic copy of  $\ell_1(\kappa)$ , then  $X$  admits an equivalent norm  $|\cdot|$  such that  $(X, |\cdot|)$  fails the  $\alpha$ -BCP<sub>cf( $\kappa$ )}</sub>, for every  $\alpha \in (-1, 1)$ .*

*Proof.* Up to renorming, we can assume that  $X$  contains a subspace  $Y$  isometrically isomorphic to  $\ell_1(\kappa)$ . Let the set  $\{e_\eta : \eta \in \kappa\} \subset S_Y$  be isometrically equivalent to the canonical basis of  $\ell_1(\kappa)$ . Let  $\mathcal{S}$  be the family of all seminorms on  $X$  dominated by the norm  $\|\cdot\|$  which coincide with  $\|\cdot\|$  on  $Y$ . It is clear that  $\mathcal{S}$  is non-empty, as  $\|\cdot\|$  belongs to it. Therefore  $p := \inf \mathcal{S}$  is a well-defined seminorm belonging to  $\mathcal{S}$ . By applying the claim in the proof of [BK, Theorem 1.3] to the ultrafilters in  $\kappa$ , we obtain that

$$\lim p(x + e_\eta) = p(x) + 1$$

holds for every  $x \in X$ . Now, let us fix  $\alpha \in (-1, 1)$  and  $A \in \mathcal{P}_{\text{cf}(\kappa)}(X)$ . Given  $x \in A$ , we can find  $\eta_x < \kappa$  so that

$$p(x + e_\eta) \geq p(x) - \alpha$$

holds for every  $\eta_x \leq \eta < \kappa$ . Find  $\theta < \kappa$  greater than all  $\eta_x$ 's, hence it is clear that

$$p(x + e_\theta) \geq p(x) - \alpha$$

holds true for all  $x \in A$ . Eventually, Lemma 5.5 provides the desired norm.  $\square$

It has to be mentioned that Theorem 5.6 has been improved in [AMR2] by Avilés, Martínez-Cervantes, and Rueda Zoca, where the authors replaced the failure of the  $\alpha$ -BCP<sub>cf( $\kappa$ )}</sub> with the stronger condition ROH $_\kappa$ . On the other hand, Theorem 5.4 cannot be improved by requiring only OH $_\kappa$ , as implied by Example 2.18

## 5.4 Stability results

In this section, we investigate the stability of possessing and failing the transfinite BCP under direct sums.

### 5.4.1 Absolute direct sums

In [GLM, Proposition 8] it was shown that the  $\alpha$ -BCP is preserved by  $p$ -sums, and it is easy to adapt their proof to the transfinite case. At first, we aim to investigate the opposite direction of this result. Notice that the following results can be applied to  $\text{OH}_\kappa$  norms too.

**Theorem 5.7.** *Let  $X$  and  $Y$  be Banach spaces,  $N$  an absolute normalized norm on  $\mathbb{R}^2$ ,  $\alpha \in [-1, 0]$  and  $\beta \in [-1, 1]$ . If  $S_{X \oplus_N Y} \subset C(\alpha, \beta, A)$ , for some set  $A$ , then  $S_X \subset C(\alpha, \beta, \pi_X(A))$ . Moreover, if  $N$  is the  $\ell_1$  norm, then the statement also holds for  $\alpha \in (0, 1)$ .*

*Proof.* Fix  $x \in S_X$  and consider the element  $(x, 0) \in S_{X \oplus_N Y}$ . By assumption there is  $(a_X, a_Y) \in A$  such that

$$N(\|x - a_X\|, \|a_Y\|) < \alpha + \beta N(\|a_X\|, \|a_Y\|).$$

Assume by contradiction that

$$\|x - a_X\| \geq \alpha + \beta \|a_X\|.$$

Therefore,

$$\begin{aligned} \alpha + \beta N(\|a_X\|, \|a_Y\|) &> N(\|x - a_X\|, \|a_Y\|) \\ &\geq N(\alpha + \beta \|a_X\|, \|a_Y\|) \\ &\geq -N(\alpha, 0) + N(\beta \|a_X\|, \|a_Y\|) \\ &= -|\alpha| + N(\beta \|a_X\|, \|a_Y\|). \end{aligned}$$

Since  $\alpha \in [-1, 0]$  we have that  $-|\alpha| = \alpha$ . In addition notice that

$$N(\beta \|a_X\|, \|a_Y\|) \geq \beta N(\|a_X\|, \|a_Y\|),$$

which leads to a contradiction. For the moreover part, if  $N$  is the  $\ell_1$  norm on  $\mathbb{R}^2$ , we only need to adjust the previous inequality to

$$\begin{aligned} \alpha + \beta(\|a_X\| + \|a_Y\|) &> \|x - a_X\| + \|a_Y\| \\ &\geq \alpha + \beta \|a_X\| + \|a_Y\| \end{aligned}$$

to get a contradiction. □

**Corollary 5.8.** *Let  $X$  and  $Y$  be Banach spaces,  $N$  an absolute normalized norm on  $\mathbb{R}^2$ ,  $\kappa$  an infinite cardinal and  $\alpha \in [0, 1)$ . If  $X \oplus_N Y$  has the  $\alpha$ -BCP $_\kappa$ , then  $X$  and  $Y$  have the  $\alpha$ -BCP $_\kappa$ .*

### 5.4.2 Infinite $\ell_\infty$ and $\ell_1$ sums

We now turn our attention to the behaviour of the  $\alpha$ -BCP $_\kappa$  for infinite direct sums of families of Banach spaces. Firstly, recall that for a family  $\{X_n : n \in \mathbb{N}\}$  of Banach spaces and  $1 \leq p \leq \infty$ , the Banach space  $\ell_p(\mathbb{N}, X_n)$  has the BCP if and only if each  $X_n$  has the BCP (see [LZ, Corollary 2.7 and Theorem 2.8]). Our goal in the remaining part of this section is to investigate the  $\alpha$ -BCP $_\kappa$  in infinite  $\ell_1$  and  $\ell_\infty$ -sums in more detail.

**Theorem 5.9.** *Let  $\{X_\eta : \eta \in \mathcal{A}\}$  be a family of Banach spaces and for each  $\eta \in \mathcal{A}$  let  $A_\eta$  be a subset of  $X_\eta$ .*

(a) *If  $\alpha \in [0, 1]$  and  $S_{X_\eta} \subset C(\alpha, 1, A_\eta)$  for all  $\eta \in \mathcal{A}$ , then*

$$S_{\ell_\infty(\mathcal{A}, X_\eta)} \subset C\left(\alpha, 1, \mathbb{R}^+ \bigcup i_{X_\eta}(A_\eta)\right).$$

(b) *If  $\alpha \in (-1, 0)$  and  $S_{X_\eta} \subset C(\alpha, 1, A_\eta)$  for all  $\eta \in \mathcal{A}$ , then*

$$S_{\ell_\infty(\mathcal{A}, X_\eta)} \subset C\left(\alpha + \varepsilon, 1, \mathbb{R}^+ \bigcup i_{X_\eta}(A_\eta)\right)$$

*for every  $\varepsilon > 0$ .*

*Proof.* Assume that  $S_{X_\eta} \subset C(\alpha, 1, A_\eta)$  for all  $\eta$ . Then, by Lemma 5.1 (c), one has that  $X_\eta \setminus \{0\} = C(\alpha, 1, \mathbb{R}^+ A_\eta)$ . Fix  $x \in S_{\ell_\infty(\mathcal{A}, X_\eta)}$ .

(a) If  $\alpha \in [0, 1]$ , then find some  $\eta$  such that  $\|x(\eta)\| > 0$ .

(b) If  $\alpha \in (-1, 0)$ , then fix  $\varepsilon \in (0, -\alpha)$  and find  $\eta$  such that  $\|x(\eta)\| \geq 1 + \varepsilon/\alpha$ .

By our assumption there is some  $a \in \mathbb{R}^+ A_\eta \setminus \{0\}$  such that

$$\|x(\eta) - a\| < \alpha \|x(\eta)\| + \|a\|,$$

Hence,

$$\|x(\eta) - ta\| < \alpha \|x(\eta)\| + \|ta\|$$

for any  $t \geq \max\{2/\|a\|, 1\}$  by Lemma 5.1 (d). Notice that with this choice of  $t$  we have that  $\|ta\| \geq 2$ , therefore

$$\|x - t\tilde{a}\| = \|x(\eta) - ta\|.$$

We deduce that

$$\begin{aligned} \|x - t\tilde{a}\| &= \|x(\eta) - ta\| \\ &< \alpha \|x(\eta)\| + \|ta\| \\ &= \alpha \|x(\eta)\| + \|ti_{X_\eta}(a)\|. \end{aligned}$$

(a) If  $\alpha \in [0, 1]$ , then  $\alpha\|x(\eta)\| \leq \alpha$ . This proves that

$$S_{\ell_\infty(\mathcal{A}, X_\eta)} \subset C\left(\alpha, 1, \mathbb{R}^+ \bigcup i_{X_\eta}(A_\eta)\right).$$

(b) If  $\alpha \in (-1, 0)$ , then

$$\begin{aligned} \alpha\|x(\eta)\| &\leq \alpha(1 + \varepsilon/\alpha) \\ &= \alpha + \varepsilon. \end{aligned}$$

This shows that

$$S_{\ell_\infty(\mathcal{A}, X_\eta)} \subset C\left(\alpha + \varepsilon, 1, \mathbb{R}^+ \bigcup i_{X_\eta}(A_\eta)\right).$$

□

**Corollary 5.10.** *Let  $\kappa$  be an infinite cardinal and  $\{X_\eta : \eta \in \kappa\}$  a family of Banach spaces.*

- (a) *If  $\alpha \in [-1, 0]$  and each  $X_\eta$  satisfies the  $\alpha$ -BCP $_\kappa$ , then  $\ell_\infty(\kappa, X_\eta)$  has the  $\alpha$ -BCP $_\kappa$ .*
- (b) *If  $\alpha \in (0, 1)$  and each  $X_\eta$  satisfies the  $\alpha$ -BCP $_\kappa$ , then, for every  $\varepsilon > 0$ ,  $\ell_\infty(\kappa, X_\eta)$  has the  $(\alpha - \varepsilon)$ -BCP $_\kappa$ .*

Notice that the converse of Corollary 5.10, for  $\alpha \in [0, 1)$ , is given by Corollary 5.8.

**Theorem 5.11.** *Let  $\{X_n : n \in \mathbb{N}\}$  be a family of Banach spaces, for each  $n$  let  $A_n$  be a subset of  $X_n$  and let  $\alpha, \beta \in [-1, 1]$ .*

(a) *If  $X_n \setminus \{0\} = C(\alpha, \beta, A_n)$  for all  $n \in \mathbb{N}$ , then*

$$\ell_1(\mathbb{N}, X_n) \setminus \{0\} = C(\alpha + \varepsilon, \beta, c_{00}(\mathbb{N}, A_n))$$

*for all  $\varepsilon > 0$ .*

(b) *If each  $A_n$  is a subspace and there exists some  $c > 0$  such that  $S_{X_n} \subset C(\alpha, \beta, cB_{A_n})$  for all  $n \in \mathbb{N}$ , then*

$$S_{\ell_1(\mathbb{N}, X_n)} \subset C(\alpha, \beta, cB_{\ell_1(\mathbb{N}, A_n)}).$$

*Proof.* (a). Fix  $x \in \ell_1(\mathbb{N}, X_n) \setminus \{0\}$  and  $\varepsilon > 0$ . Find some  $n \in \mathbb{N}$  such that

$$\sum_{j=n+1}^{\infty} \|x(j)\| \leq \varepsilon \|x\|.$$

There are  $a_1 \in A_1, \dots, a_n \in A_n$  such that

$$\|x(j) - a_j\| < \alpha \|x(j)\| + \beta \|a_j\|$$

for every  $j \leq n$  that satisfies  $x(j) \neq 0$ . Define  $a \in c_{00}(\mathbb{N}, A_n)$  by

$$a(j) := \begin{cases} a_j & \text{if } j \leq n \text{ and } x(j) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \|x - a\| &= \sum_{j=1}^n \|x(j) - a_j\| + \sum_{j=n+1}^{\infty} \|x(j)\| \\ &< \sum_{j=1}^n (\alpha \|x(j)\| + \beta \|a_j\|) + \varepsilon \|x\| \\ &= (\alpha + \varepsilon) \|x\| + \beta \|a\| - \alpha \sum_{j=n+1}^{\infty} \|x(j)\|. \end{aligned}$$

If  $\alpha \geq 0$ , then

$$\|x - a\| < (\alpha + \varepsilon) \|x\| + \beta \|a\|.$$

If  $\alpha < 0$ , then

$$\begin{aligned} -\alpha \sum_{j=n+1}^{\infty} \|x(j)\| &\leq -\alpha \varepsilon \|x\| \\ &\leq \varepsilon \|x\|, \end{aligned}$$

hence

$$\|x - a\| < (\alpha + 2\varepsilon) \|x\| + \beta \|a\|.$$

(b). Fix  $x \in S_{\ell_1(\mathbb{N}, X_n)}$ . For all  $n \in \mathbb{N}$  with  $x(n) \neq 0$ , we can find  $y_n \in c\|x(n)\| \cdot B_{A_n}$  satisfying

$$\|x(n) - y_n\| < \alpha \|x(n)\| + \beta \|y_n\|,$$

whose existence is guaranteed by Lemma 5.1 (b). Define  $y \in cB_{\ell_1(\mathbb{N}, A_n)}$  by

$$y(n) := \begin{cases} y_n & \text{if } n \in \mathbb{N} \text{ is such that } x(n) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\|x - y\| < \alpha\|x\| + \beta\|y\|.$$

□

We end this section by spelling out two useful consequences of the aforementioned results.

**Corollary 5.12.** *Let  $\{X_n : n \in \mathbb{N}\}$  be a family of Banach spaces,  $\kappa$  an infinite cardinal and  $\alpha, \beta \in [-1, 1]$ .*

(a) *If there is  $m \in \mathbb{N}$  such that  $S_{X_m} \notin C(\alpha, \beta, \kappa)$ , then*

$$S_{\ell_1(\mathbb{N}, X_n)} \notin C(\alpha, \beta, \kappa).$$

(b) *If  $S_{\ell_1(\mathbb{N}, X_n)} \notin C(\alpha, \beta, \kappa)$ , then, for every  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that  $S_{X_n} \notin C(\alpha - \varepsilon, \beta, \kappa)$ .*

*Proof.* (a) follows by Theorem 5.7. On the other hand, (b) is an easy consequence of Theorem 5.11 (a). □

## 5.5 More examples

### 5.5.1 $\text{ROH}_{\aleph_1}$ is distinct from failing the $\alpha$ -BCP $_{\kappa}$

**Lemma 5.13.** *Let  $\kappa$  be an infinite cardinal and  $\alpha \in (-1, 0)$ . Let  $X_{\alpha, \kappa}$  be  $\ell_1(\kappa)$  endowed with the equivalent norm*

$$\|\cdot\|_{\alpha} := -\alpha\|\cdot\|_1 + (1 + \alpha)\|\cdot\|_{\infty}.$$

*Then  $X_{\alpha, \kappa}$  fails the  $\alpha$ -BCP $_{\kappa}$ , but  $S_{X_{\alpha, \kappa}} \subset C(1, 1, \{\mp e_1\})$ .*

*Proof.* Fix a set  $A \in \mathcal{P}_{\kappa}(X_{\alpha, \kappa})$  and find

$$\theta \in \kappa \setminus \bigcup_{a \in A} \text{supp}(a).$$

Then for all  $a \in A$

$$\begin{aligned} \|a - e_\theta\| &= -\alpha(\|a\|_1 + 1) + (1 + \alpha) \max\{\|a\|_\infty, 1\} \\ &\geq -\alpha(\|a\|_1 + 1) + (1 + \alpha)\|a\|_\infty \\ &= \|a\| - \alpha. \end{aligned}$$

This proves that  $X_{\alpha,\kappa}$  fails the  $\alpha$ -BCP $_\kappa$ .

Finally, we show that  $S_{X_{\alpha,\kappa}} \subset C(1, 1, \{\mp e_1\})$ . Let  $x \in S_{X_{\alpha,\kappa}}$ . If  $x(1) = 0$ , then  $\|x + e_1\|_\infty = 1$ . Otherwise, say  $x(1) < 0$ , note that  $\|x + e_1\|_1 < 2$  (if  $x(1) > 0$ , then use  $-e_1$  instead). In either case we conclude that for all  $x \in X_{\alpha,\kappa}$  there exists  $e \in \{\mp e_1\}$  such that  $\|x - e\|_\alpha < 2$ .  $\square$

**Theorem 5.14.** *For every infinite cardinal  $\kappa$  there exists a Banach space  $X$  which fails the  $\alpha$ -BCP $_\kappa$  for every  $\alpha \in (-1, 1)$ , but which fails to be ROH $_{\aleph_1}$*

*Proof.* Let  $\kappa$  be an infinite cardinal. Fix a sequence  $(\alpha_n) \subset (-1, 0)$  such that  $\inf_n \alpha_n = -1$ . For every  $n \in \mathbb{N}$  define a Banach space  $X_n := X_{\alpha_n, \kappa}$  as in Lemma 5.13 and set  $X := \ell_1(\mathbb{N}, X_n)$ . Now, Corollary 5.12 (a) implies that  $X$  fails the  $\alpha$ -BCP $_\kappa$  for every  $\alpha \in (-1, 1)$ . Moreover, by Theorem 5.11 (b),  $X$  fails to be ROH $_{\aleph_1}$ . In fact, if  $A_n := \text{span}\{\mp e_1\}$ , then  $S_{X_n} \subset C(1, 1, B_{A_n})$ , hence  $S_X \subset C(1, 1, B_{\ell_1(\mathbb{N}, A_n)})$ , therefore  $S_X \in C(1, 1, \aleph_1)$  thanks to Lemma 5.1 (a).  $\square$

### 5.5.2 OH $_\kappa$ is distinct from failing the $\alpha$ -BCP $_{\aleph_1}$

We end this section by proving the existence of a Banach space which is OH $_\kappa$ , but that has the  $\alpha$ -BCP $_{\aleph_1}$  for all  $\alpha \in [-1, 0)$ .

**Lemma 5.15.** *For all  $\alpha \in (1/2, 1)$  and infinite cardinal  $\kappa$  there exists a Banach space  $X_{\alpha,\kappa}$  that has the  $\beta$ -BCP $_{\aleph_1}$  for all  $\beta \in [-1, 0)$  and such that  $S_{X_{\alpha,\kappa}} \notin C(\alpha, \alpha, \kappa)$ .*

*Proof.* Set  $p := (1 + \log_2(\alpha))^{-1} \in (1, \infty)$  and  $X_{\alpha,\kappa} := \ell_p(\kappa)$ . Fix a set  $A \in \mathcal{P}_\kappa(X_{\alpha,\kappa})$ . Find some

$$\theta \in \kappa \setminus \bigcup_{a \in A} \text{supp}(a).$$

For all  $a \in A$ , by Jensen's inequality, we have that

$$\begin{aligned} \|e_\theta - a\|_p &= \left(1 + \sum |a(\eta)|^p\right)^{\frac{1}{p}} \\ &\geq \frac{1}{2}2^{\frac{1}{p}} + \frac{1}{2} \left(2 \sum |a(\eta)|^p\right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}-1}(1 + \|a\|_p) \\ &= \alpha(1 + \|a\|_p). \end{aligned}$$

This proves that  $S_{X_{\alpha,\kappa}} \notin C(\alpha, \alpha, \kappa)$ . On the other hand  $X_{\alpha,\kappa}$  is reflexive, hence it does not contain  $\ell_1$ . Therefore, by Theorem 5.4,  $X_{\alpha,\kappa}$  has the  $\beta$ -BCP $_{\aleph_1}$  for all  $\beta \in [-1, 0)$ .  $\square$

**Theorem 5.16.** *For every infinite cardinal  $\kappa$  there exists a Banach space which is  $\text{OH}_\kappa$ , but that has the  $\alpha$ -BCP $_{\aleph_1}$  for all  $\alpha \in [-1, 0)$ .*

*Proof.* Fix a sequence  $(\alpha_n) \subset (1/2, 1)$  such that  $\sup_n \alpha_n = 1$  and for all  $n \in \mathbb{N}$  pick a Banach space  $X_n := X_{\alpha_n, \kappa}$  as in Lemma 5.15. Now, Corollary 5.12 (b) implies that  $X := \ell_1(\mathbb{N}, X_n)$  has the  $\alpha$ -BCP $_{\aleph_1}$  for all  $\alpha \in [-1, 0)$ . Moreover, Corollary 5.12 (a) shows that  $X$  is  $\text{OH}_\kappa$ .  $\square$

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# Banachi ruumi ühikera transfiniitsed geomeetrilised omadused

## Kokkuvõte

Lõpmatumõõtmelises Banachi ruumis võib kohata äärmuslikke geomeetrilisi nähtusi, kus iga ühikera viilu või iga suhteliselt nõrgalt lahtise alamhulga diameeter on kaks. Selline nähtus leiab aset näiteks lõpmatumõõtmelises ühtlases algebras [NW], Daugaveti omadusega Banachi ruumis [Shv], lõpmatumõõtmelises  $C^*$ -algebras [BLR2] ja mitterefleksiivses Banachi ruumis, mis on  $M$ -ideaal oma teises kaasruumis [Lop]. Refleksiivses Banachi ruumis (nt lõplikumõõtmelises ruumis) sellist omadust olla ei saa, kuna neil ruumidel on Radon–Nikodými omadus ehk nende ruumide ühikkerades leidub kui tahes väikese läbimõõduga viile.

Diameeter-2 omaduste teooria sai alguse artiklist [ALN1], kus T. A. Abrahamsen, V. Lima ja O. Nygaard uurisid süstemaatiliselt varasemaid tulemusi ja tutvustasid diameeter-2 omaduse jaoks kolm erinevat versiooni. On teada, et Banachi ruumil on tugev diameeter-2 omadus parajasti siis, kui tema kaasruumi norm on oktaeedriline. Sarnaseid duaalseid kirjeldusi omavad ka ülejäänud kaks diameeter-2 omadust [BLR3] ja [HLP1]. Sealt edasi on tekkinud erinevad diameeter-2 omaduste tugevdused ja nendega seotud mõisted, näiteks sümmeetriline tugev diameeter-2 omadust [ANP] ja peaaegu ruudu omadus Banachi ruumis [ALL].

Kõigi nende ülaltoodud mõistete ühine omadus on see, et need on nii öelda *lõplikult määratletud*; see tähendab, et definitsioonid kasutavad seda, et mis tahes lõpliku arvu elementide jaoks Banachi ruumis või tema kaasruumis leidub mingi eriline element selles samas ruumis või tema kaasruumis, mis annab vastava omaduse. Sellised geomeetrilised omadused üldistas J. D. Hardtke testperekonna nime alla artiklis [Har2], kus ta kasutas seda näitamaks, et kui vastav omadus on teatud lõplike absoluutsete summade korral stabiilne, siis see on stabiilne ka vastavate Köthe-Bochneri ruumide moodustamisel.

Üllataval kombel on paljudel klassikalistel Banachi ruumidel isegi diameeter-2 omaduste või nendega seotud omaduste transfiniitne analoog. Näiteks, mis tahes Daugaveti ruumi kaasruum pole lihtsalt oktaeedriline, vaid on isegi loenduvalt oktaeedriline [KSSW].

Käesoleva väitekirja põhieesmärk on süstemaatiliselt uurida diameeter-2 omaduste, peaaegu ruudu omaduse ja oktaeedriliste normide transfiniitseid

analooge. Mõne omaduse puhul, nagu peaaegu ruudu omadus ja oktaeedrilised normid, on definitsioonide laiendamine lihtne. Teisalt, näiteks tugeva diameeter-2 omaduse puhul nõuab see algele defintisioonile uue samaväärsse kirjelduse leidmist, et seda saaks laiendada ka lõpmatule arvule viiludele. Üldiselt käituvad transfiniitsed analoogid olemasolevatest diameeter-2 omadustest erinevalt ja on tehniliselt keerulisemad. Seega annab see suund uusi viljakaid tulemusi ja näiteid, mis oluliselt täiendavad olemasolevat diameeter-2 omadusega ruumide teooriat.

Väitekirjaga koosneb viiest peatükist. Esimene peatükk sisaldab ülevaadet töö põhilistest mõistetest ja nende tekkimise ajaloost. Samuti kirjeldatakse väitekirja ülesehitust ja töös kasutatavaid tähistusi.

Teises peatükis tuuakse sisse transfiniitse peaaegu ruudu omadusega Banachi ruumi mõiste ja selgitatakse tema erinevust klassikalisest peaaegu ruudu omaduse mõistest. Peatüki põhitulemusena uuritakse, kuidas on seotud ekvivalentse lõpmatu peaaegu ruudu omadusega normi olemasolu ning Banachi ruumi  $c_0(\kappa)$  isomorfne sisaldumine selles Banachi ruumis. See peatükk põhineb artiklidel [ACLLR] ja [Cia1].

Kolmandas peatükis vaadeldakse sümmeetrilise tugeva diameeter-2 omaduse transfiniitset analoogi. Osutub, et vastav lõpmatu analoog erineb oluliselt klassikalisest omadusest. Põhitulemusena leitakse, millal on ruumil  $C_0(L)$ , kus  $L$  on lokaalselt kompaktne Hausdorffi ruum, transfiniitne sümmeetriline tugev diameeter-2 omadus. Samuti uuritakse selle omaduse päranduvust nii absoluutsetele summadele kui ka projektiivsele tensorsorkorrutisele. See peatükk tugineb artiklitele [ACLLR] ja [Cia2].

Neljandas peatükis leitakse esmalt klassikalise tugeva diameeter-2 omaduse ekvivalentne definitsioon, mida saaks laiendada ka lõpmatule juhule. Seejärel tõestatakse, et sarnaselt klassikalise duaalsusega tugeva diameeter-2 omadusega ruumi ja tema kaasruumi oktaeedrilisuse vahel, kehtib see ka transfiniitsete analoogide korral. Siin peatükis tuuakse mitmeid uusi näiteid Banachi ruumidest, millel on lõpmatu tugev diameeter-2 omadus. See peatükk põhineb artiklil [CLL1].

Viiendas peatükis tuuakse esmalt sisse lõpmatu arvuga kerade katmise omadus ja selgitatakse selle vahekorra oktaeedriliste normidega. Peatüki põhitulemusena uuritakse, kuidas on seotud ekvivalentse transfiniitse oktaeedrilise normi olemasolu ning Banachi ruumi  $\ell_1(\kappa)$  isomorfne sisaldumine selles Banachi ruumis. See peatükk tugineb artiklile [CLL2].

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# List of original publications

1. A. Avilés, S. Ciaci, J. Langemets, A. Lissitsin, and A. RuedaŽoca, *Transfinite almost square Banach spaces*, *Studia Mathematica*, **271**(1), 39–63 (2023).
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