Tensor Product Rings and Morita Equivalence

Kristo Väljako

Universitas Tartuensis

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Basis article

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$$\langle p + p', q\rangle = \langle p, q\rangle + \langle p', q\rangle,$$
$$\langle p, q + q'\rangle = \langle p, q\rangle + \langle p, q'\rangle,$$
$$\langle rp, q\rangle = r\langle p, q\rangle,$$
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Definition 1

Tensor product of modules $Q \otimes_R^{\beta} P$ with multiplication $\star$ defined by

$$(q \otimes p) \star (q' \otimes p') := q \otimes \langle p, q'\rangle p'$$

is called a ***tensor product ring*** *defined by an (R,R)-bilinear mapping $\beta = \langle\,,\,\rangle$.*

Pseudo-surjectivity

For any ring R and a set $A \subseteq R$, we will denote by $\langle A \rangle_{\mathrm{s}}$ the subgroup generated by A in the additive group $(R, +)$.

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Definition 2

Let R be a ring and B a set. We call a mapping $f\colon B \longrightarrow R$ ***pseudo-surjective****, if $\langle \operatorname{Im} f \rangle_{\mathrm{s}} = R$, i.e. the additive subgroup of R generated by the set $\operatorname{Im} f$ is equal to R.*

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Let $\psi\colon\ P \otimes_S Q \longrightarrow A$ a homorphism of abelian groups. Denote $\hat{\psi} := \psi \circ \otimes$, i.e., for every $p \in P$ and $q \in Q$, we have

$$\hat{\psi}(p, q) = \psi(p \otimes q).$$

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If ${}_R P_S$ and ${}_S Q_R$ are (R, S)- and (S, R)-bimodules, respectively, then $\hat{\psi}$ is also (R, R)-bilinear.

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If ${}_R P_S$ and ${}_S Q_R$ are (R, S)- and (S, R)-bimodules, respectively, then $\hat{\psi}$ is also (R, R)-bilinear. If $\psi\colon P \otimes_R Q \longrightarrow A$ is surjective, then $\hat{\psi}$ is pseudo-surjective.

Morita equivalence

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Definition 3

A six-tuple $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, where R and S are rings and ${}_RP_S$, ${}_SQ_R$ are bimodules, is called a ***Morita context****, if*

$$\theta\colon\ {}_R(P \otimes_S Q)_R \longrightarrow {}_RR_R, \quad \phi\colon\ {}_S(Q \otimes_R P)_S \longrightarrow {}_SS_S$$

are bimodule homomorphisms such that

$$\theta(p \otimes q)p' = p\phi(q \otimes p'),$$
$$q\theta(p \otimes q') = \phi(q \otimes p)q'$$

for every $p, p' \in P$ and $q, q' \in Q$.

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We will call idempotent rings R and S **Morita equivalent**, if there exists a unitary surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$.

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Proposition 4

Let R be an idempotent ring and ${}_RP$, Q_R unitary R-modules. Then every pseudo-surjectively defined tensor product ring $Q \otimes_R P$ is idempotent.

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Proposition 4

Let R be an idempotent ring and ${}_RP$, Q_R unitary R-modules. Then every pseudo-surjectively defined tensor product ring $Q \otimes_R P$ is idempotent.

Theorem 5

Let R be an idempotent ring, ${}_RP$ and Q_R unitary R-modules and $\langle\, , \rangle\colon\ P \times Q \longrightarrow R$ a pseudo-surjective (R, R)-bilinear mapping. Then the tensor product ring $Q \otimes_R P$ defined by $\langle\, , \rangle$ is Morita equivalent to R.

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A ring R is called **firm**, if

$$\nu_R \colon R \otimes_R R \longrightarrow R, \qquad \sum_{k=1}^{k^*} r_k \otimes r'_k \mapsto \sum_{k=1}^{k^*} r_k r'_k$$

is an isomorphism.

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Corollary 6

Let R be an idempotent ring. The rings R and $R\otimes_R^{\hat{\nu}} R$ are Morita equivalent with a corresponding surjective unitary Morita context $(R, R\otimes_R^{\hat{\nu}} R, R, R, \nu, \mathrm{id}_{R\otimes R})$.

Proposition 7

Let $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ *be a unitary surjective Morita context connecting idempotent rings* R *and* S*, and let* $Q \otimes_R^{\hat{\theta}} P$*,* $P \otimes_S^{\hat{\phi}} Q$ *be tensor product rings defined by the mappings* $\hat{\theta}$*,* $\hat{\phi}$*, respectively. Then the rings* R*,* S*,* $P \otimes_S^{\hat{\phi}} Q$ *and* $Q \otimes_R^{\hat{\theta}} P$ *are all Morita equivalent.*

Local injectivity and strict local isomorphisms

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Local injectivity and strict local isomorphisms

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Definition 8

We call a homomorphism $\tau\colon R \longrightarrow S$ *of rings* ***locally injective*** *if its restriction to any subring of the form* aRb*, where* $a \in Ra$ *and* $b \in bR$*, is injective.*

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A locally injective homomorphism of rings, which is also surjective, is called a ***strict local isomorphism****.*

Local injectivity and strict local isomorphisms

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Proposition 9

Let R be a ring, M_R be an R-module and $f\colon\ M_R \longrightarrow R_R$ a homomorphism of modules. If we define a multiplication on the abelian group M by

$$m \bullet m' := mf(m'), \qquad (m, m' \in M),$$

then we obtain a ring and f is a locally injective homomorphism of rings. If S is a right s-unital ring then all strict local isomorphisms $S \longrightarrow R$ can be obtained using this construction.

Theorem 10

Let R and S be rings that are connected by a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Consider the tensor product ring $P \otimes_S^{\hat{\phi}} Q$ defined by $\hat{\phi}$. Then $\theta\colon\ P \otimes_S^{\hat{\phi}} Q \longrightarrow R$ is a locally injective homomorphism of rings.

Theorem 10

Let R and S be rings that are connected by a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Consider the tensor product ring $P \otimes_S^{\hat{\phi}} Q$ defined by $\hat{\phi}$. Then $\theta\colon\ P \otimes_S^{\hat{\phi}} Q \longrightarrow R$ is a locally injective homomorphism of rings.

Corollary 11

Let R and S be two Morita equivalent idempotent rings. Then there exist pseudo-surjectively defined tensor product rings $Q \otimes_R P$, $P \otimes_S Q$ and strict local isomorphisms $Q \otimes_R P \longrightarrow S$ and $P \otimes_S Q \longrightarrow R$.

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Proposition 12

Let R and S be idempotent rings. If R is isomorphic to some pseudo-surjectively defined tensor product ring $P \otimes_S Q$, where P_S and ${}_SQ$ are unitary modules, then the rings R and S are Morita equivalent.

Adjoint endomorphisms

Adjoint endomorphisms

Definition 13

Module endomorphisms $f \in \mathrm{End}(_R P)$ *and* $g \in \mathrm{End}(Q_R)$ *are called* ***adjoint*** *(with respect to* $\beta = \langle\, ,\, \rangle$*) if, for every* $p \in P$ *and* $q \in Q$*, we have*

$$\langle f(p), q \rangle = \langle p, g(q) \rangle.$$

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Definition 13

Module endomorphisms $f \in \mathrm{End}({}_RP)$ *and* $g \in \mathrm{End}(Q_R)$ *are called* ***adjoint*** *(with respect to* $\beta = \langle\,,\,\rangle$*) if, for every* $p \in P$ *and* $q \in Q$*, we have*

$$\langle f(p), q\rangle = \langle p, g(q)\rangle.$$

Lemma 14

Let ${}_RP$ *and* Q_R *be* R*-modules and* $\beta = \langle\,,\,\rangle\colon\ P \times Q \longrightarrow R$ *an* (R,R)*-bilinear mapping. For any* $k^* \in \mathbb{N}$*,* $p_1, \ldots, p_{k^*} \in P$ *and* $q_1, \ldots, q_{k^*} \in Q$*, the mappings*

$$f := \sum_{k=1}^{k^*} \langle \_\_, q_k\rangle p_k\colon\ {}_RP \longrightarrow {}_RP \quad \text{and} \quad g := \sum_{k=1}^{k^*} q_k \langle p_k, \_\_\rangle\colon\ Q_R \longrightarrow Q_R$$

are adjoint endomorphisms.

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Denote

$$\Sigma^{\beta} := \left\{ \sum_{k=1}^{k^*} (\langle \_, q_k \rangle p_k, q_k \langle p_k, \_ \rangle) \;\middle|\; k^* \in \mathbb{N}; \forall k\colon \; p_k \in P, q_k \in Q \right\}.$$

Denote

$$\Sigma^\beta := \left\{ \sum_{k=1}^{k^*} (\langle \_, q_k \rangle p_k, q_k \langle p_k, \_ \rangle) \,\middle|\, k^* \in \mathbb{N}; \forall k\colon\ p_k \in P, q_k \in Q \right\}.$$

Theorem 15

Let R be a ring. Then, for every (R, R)-bilinear mapping $\beta = \langle\, , \rangle\colon\ {}_RP \times Q_R \longrightarrow R$, there exists a strict local isomorphism $Q \otimes_R^\beta P \longrightarrow \Sigma^\beta$ of rings.

Dual mappings

Definition 16

*An (R,R)-bilinear mapping $\langle\,,\,\rangle\colon\ {}_RP \times Q_R \longrightarrow {}_RR_R$ is said to be a **dual mapping**, if*

1. *for every finite subset $Y \subseteq Q$, there exist $p_1, \ldots, p_{k^*} \in P$ and $q_1, \ldots, q_{k^*} \in Q$ such that for every $y \in Y$*

$$y = \sum_{k=1}^{k^*} q_k \langle p_k, y\rangle;$$

2. *for every finite subset $X \subseteq P$, there exist $p_1, \ldots, p_{h^*} \in P$ and $q_1, \ldots, q_{h^*} \in Q$ such that for every $x \in X$*

$$x = \sum_{h=1}^{h^*} \langle x, q_h\rangle p_h.$$

Example 17 (Dual mapping I)

Let V be a Euclidean space. It can be considered as a right or a left $\mathbb{R}$-module. The inner product of V is an $(\mathbb{R}, \mathbb{R})$-bilinear mapping $\langle\ ,\ \rangle\colon {}_{\mathbb{R}}V \times V_{\mathbb{R}} \longrightarrow \mathbb{R}$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for V. Then

$$x = \sum_{h=1}^{n} \langle x, e_h \rangle e_h,$$

for every $x \in V$. The inner product of any Euclidean space is a dual mapping.

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$$x = \sum_{h=1}^{n} \langle x, e_h\rangle e_h,$$

for every $x \in V$. The inner product of any Euclidean space is a dual mapping.

Example 18 (Dual mapping II)

Let R and S be s-unital rings that are connected by a unitary surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. The mappings

$$\hat{\theta} : P \times Q \longrightarrow R, \quad (p,q) \mapsto \theta(p \otimes q),$$
$$\hat{\phi} : Q \times P \longrightarrow S, \quad (q,p) \mapsto \phi(q \otimes p)$$

are dual mappings.

Proposition 19

Let R be a ring and $\beta = \langle\, , \,\rangle \colon {}_RP \times Q_R \longrightarrow {}_RR_R$ a pseudo-surjective dual mapping. Then R is idempotent and the rings R and Σ^β are Morita equivalent.

Proposition 19

Let R be a ring and $\beta = \langle\, , \,\rangle \colon {}_RP \times Q_R \longrightarrow {}_RR_R$ a pseudo-surjective dual mapping. Then R is idempotent and the rings R and Σ^β are Morita equivalent.

Proposition 20

If R is a ring and $\beta = \langle\, , \,\rangle \colon \; {}_RP \times Q_R \longrightarrow {}_RR_R$ is a dual mapping, then Σ^β is isomorphic to the subring

$$\Pi^\beta := \left\{ \sum_{k=1}^{k^*} q_k \langle p_k, \_\rangle \,\middle|\, k^* \in \mathbb{N};\ \forall k\colon \;\; q_k \in Q, p_k \in P \right\}$$

of the endomorphism ring $\mathrm{End}(Q_R)$.

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of the endomorphism ring $\mathrm{End}(Q_R)$.

Corollary 21

Let R be a ring and $\beta = \langle\, ,\, \rangle\colon {}_RP \times Q_R \longrightarrow {}_RR_R$ a pseudo-surjective dual mapping. Then R is idempotent and the rings R and Π^β are Morita equivalent.

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Proposition 22

Let R be a ring. If $\langle \, , \rangle \colon {}_RP \times Q_R \longrightarrow {}_RR_R$ is a dual (R, R)-bilinear mapping, then the tensor product ring $Q \otimes_R P$ defined by $\langle \, , \rangle$ is s-unital.

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Proposition 22

Let R be a ring. If $\langle\ ,\ \rangle\colon\ {}_RP \times Q_R \longrightarrow {}_RR_R$ is a dual (R,R)-bilinear mapping, then the tensor product ring $Q \otimes_R P$ defined by $\langle\ ,\ \rangle$ is s-unital.

Theorem 23

Let R be a ring and $\beta = \langle\ ,\ \rangle\colon\ {}_RP \times Q_R \longrightarrow {}_RR_R$ be a dual (R,R)-bilinear mapping. Then the tensor product ring $Q \otimes_R^\beta P$ is isomorphic to Σ^β and Π^β.

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Theorem 24

Let R and S be firm rings. Then R and S are Morita equivalent if and only if R is isomorphic to a pseudo-surjectively defined tensor product ring $P \otimes_S Q$.

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Theorem 24

Let R and S be firm rings. Then R and S are Morita equivalent if and only if R is isomorphic to a pseudo-surjectively defined tensor product ring $P \otimes_S Q$.

Theorem 25

Two s-unital rings R and S are Morita equivalent if and only if there exist R-modules ${}_RP$, Q_R, a dual (R, R)-bilinear pseudo-surjective mapping $\beta = \langle\, , \rangle \colon {}_RP \times Q_R \longrightarrow {}_RR_R$ and $S \cong \Pi^\beta$ as rings.

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Thank you for listening!