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Does absolute or conditional convergence of the integrals define fractional derivatives?

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Does absolute or conditional convergence of the integrals define fractional derivatives?

In this master's thesis we analyse the class of fractionally differentiable functions. This work is built on Gennadi Vainikko's recent paper "Which functions are fractionally differentiable?", that characterises the class of fractionally differentiable functions in terms of the pointwise convergence or equiconvergence of certain improper integrals containing these functions. The aim of this thesis is to present and analyse an example, which shows us that in order to obtain all fractionally differentiable functions, one may not replace the conditional convergence of certain integrals by their absolute convergence. Also some supporting lemmas are formulated and proved.

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Keywords: integrals, fractional derivatives, integral calculus

Integraalide mitteabsoluutsest koondumisest murrulise tuletise definitsioonis

Käesoleva magistritöö eesmärgiks on analüüsida murruliselt diferentseeruvate funktsioonide klassi. Kuigi murruliselt diferentseeruvate funktsioonide idee ja mõiste on juba pikka aega teada, on seesuguste funktsioonide klass põhjalikumalt uuritud ning määratletud alles hiljutises Gennadi Vainikko artiklis "Which functions are fractionally differentiable?", mis on ka aluseks käesolevale magistritööle. Töö eesmärgiks on esitada ja analüüsida ühte näidet, mis demonstreerib, et kui teatud integraalide puhul asendada tinglik koondumine nende absoluutse koondumisega, siis ei saa kätte kõiki murruliselt diferentseeruvaid funktsioone. Lisaks on sõnastatud ja tõestatud mõned abitulemused.

CERCS kood - P130. Funktsioonid, diferentsiaalvõrrandid.

Märksõnad: integraalid, murdtuletised, integraalarvutus

Contents

1 Introduction

Fractional calculus is a field of mathematics that investigates the properties of derivatives and integrals of non-integer orders [1]. This branch grows out from the traditional definitions of the calculus of derivative and integral operators in much the same way as a fractional exponent is an outgrowth of exponents with integer value [2].

The significance of this field of mathematics is well proposed by Nicholas Wheeler in his book [3]: "The fractional calculus is a source of analytical power, latently too valuable to be casually dismissed. It has demonstrable applicability to a rich assortment of pure and applied subject areas. But it is valuable not least because it invites - indeed, it frequently requires - one to think about old things in new ways, and to become more intimately familiar with the resources of the ordinary calculus. It opens doors."

The history of fractional calculus goes back for more than 300 years (it is almost the same time when classical calculus was established) when it was first mentioned in a letter from Leibniz to L'Hospital in 1695 [1]. In this letter the idea of semiderivative was suggested as L'Hospital asked the question to the meaning of $d^n y/dx^n$ if $n = 1/2$; that is "what if n is fractional?" Leibniz replied that "$d^{1/2}$ will be equal to $x\sqrt{dx : x}$" [2]. But fractional calculus was actually built on formal foundations by many great and famous mathematicians, such as Liouville, Grünwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel, and others - they all have proposed different original approaches (which are explained chronologically in [4]) [1].

Although fractional calculus has quite a long history, it was until 1974 when the first book on the topic was published. It was written by Oldham and Spanier and was devoted exclusively to the subject of fractional calculus [5]. Today there exists at least two international journals which are devoted almost entirely to the subject of fractional calculus [5]:

- Journal of Fractional Calculus
- Fractional Calculus and Applied Analysis.

The subject of fractional calculus has become more and more popular over the years, especially during past three decades. This is due to its various applications in many fields of science and also engineering.

The first application of fractional calculus was actually made by Abel about the tautochronous problem (see e.g. [2]). But there are also many other very interesting and useful applications such as using fractional calculus in modelling (e.g. speech signals, cardic tissue electrode interface, etc), image proccessing (for edge detection), studying the electric transmission lines, developing different control systems or scemes, etc. One can read more about specific applications from article [2].

The present thesis focuses on examining fractional derivatives and what defines them. Current work is built on the paper by Gennadi Vainikko [6], which characterises the class of fractionally differentiable functions in terms of the pointwise convergence or equiconvergence of certain improper integrals containing these functions.

The thesis consists of two main chapters. The first chapter is devoted to definitions and explanations. Here the author also formulates the main theorem from Gennadi Vainikko's work [6], that answers the question: "which functions are fractionally differentiable?" In addition, author presents and proves some lemmas that are needed to establish the main result of the thesis.

In the second chapter the author presents the main result of the thesis by constructing and analysing an example of a certain function. This is an example which shows us that to obtain all fractionally differentiable functions, one may not replace the conditional convergence of certain integrals by their absolute convergence.

2 Preliminaries

2.1 Problem setting

In this section we introduce necessary definitions and the problem setting of the thesis. The same definitions are used in Gennadi Vainikko's paper [6].

Consider the Riemann-Liouville integral operator $J^\alpha \colon C[0,T] \to C[0,T]$ of order $\alpha > 0$, $\alpha \in \mathbb{R}$, defined by

$$(J^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad 0 \le t \le T, \quad u \in C[0,T],$$

where Γ is the Euler gamma - function. In particular, $(J^1 u)(t) = \int_0^t u(s) ds$. For $\alpha = m \in \mathbb{N} = \{1, 2, \dots\}$, the range of operator J^m is given by (see e.g. [7])

$$J^m C[0,T] = \{v \in C^m[0,T] \colon v^{(k)}(0) = 0, k = 0, \dots, m-1\} =: C_0^m[0,T],$$

and J^m is invertible on it, i.e., $(J^m)^{-1} v = D_0^m v$, where $D_0^m \colon C_0^m[0,T] \to C[0,T]$ is the restriction of the operator $D^m = (d/dt)^m \colon C^m[0,T] \to C[0,T]$. Due to the semigroup property (see e.g. [5, 8])

$$J^\alpha J^\beta = J^\beta J^\alpha = J^{\alpha+\beta} \quad \text{for} \quad \alpha > 0, \ \beta > 0.$$

Note that J^α is invertible on its range $J^\alpha C[0,T]$ also for fractional (noninteger) $\alpha > 0$. Indeed, if $J^\alpha u = 0$ for some $u \in C[0,T]$ then taking $m \in \mathbb{N}, \quad m > \alpha$, we have

$$J^m u = J^{m-\alpha} J^\alpha u = 0, \quad u = 0.$$

The description of the range $J^\alpha C[0,T]$, $\alpha > 0$, is closely related to the description of the class of fractionally differentiable functions. Namely, one possible definition of the fractional differentiation operator of order $\alpha > 0$ is given by

$$D_0^\alpha v = (J^\alpha)^{-1} v, \quad v \in J^\alpha C[0,T].$$

This most natural definition is used e.g. in the Mathematical Encyclopedia [9].

By $\mathcal{H}^\alpha[0,T]$, $0<\alpha\le 1$, we mean the standard Hoelder space consisting of functions $v\in C[0,T]$ such that

$$\|v\|_{\mathcal{H}^\alpha} := \max_{0\le t\le T}|v(t)| + \sup_{0\le s<t\le T}\frac{|v(t)-v(s)|}{(t-s)^\alpha} < \infty,$$

and by $\mathcal{H}_0^\alpha[0,T]$, $0<\alpha<1$, we mean the closed (see e.g. [10]) subspace of $\mathcal{H}^\alpha[0,T]$ consisting of functions $v\in\mathcal{H}^\alpha[0,T]$ such that

$$\sup_{0\le s<t\le T, t-s\le\varepsilon}\frac{|v(t)-v(s)|}{(t-s)^\alpha} \xrightarrow{\varepsilon\to 0} 0.$$

Although the concept of fractionally differentiable functions is old, the class of all fractionally differentiable functions has not been described until the recent work [6]. Below we formulate the main result of [6].

Theorem 1. *For an $\alpha\in(0,1)$ and a function $v\in C[0,T]$, the following conditions are equivalent:*

(i) *$v\in J^\alpha C[0,T]$, i.e., the fractional derivative $D_0^\alpha v := (J^\alpha)^{-1}v\in C[0,T]$ exists;*

(ii) *a finite limit $\gamma_0 := \lim_{t\to 0} t^{-\alpha}v(t)$ exists, and the improper integrals*

$$w(t) := \int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds, \quad 0<t\le T, \tag{1}$$

equiconverge in the sense that

$$\lim_{\Theta\uparrow 0}\sup_{0<t\le T}\left|\int_0^1 (t-s)^{-\alpha-1}(v(t)-v(s))ds - \int_0^{\Theta t}(t-s)^{-\alpha-1}(v(t)-v(s))ds\right| = 0;$$

(iii) *a finite limit $\gamma_0 := \lim_{t\to 0} t^{-\alpha}v(t)$ exists; the Riemann improper integral (1) converges for any $t\in(0,T]$ and defines a function $w\in C(0,T]$ which has a finite limit as $t\to 0$ (hence $w\in C[0,T]$); moreover, there is a majorant function $W\in L^1(0,T)$ such that*

$$\left|\int_\Theta^{\Theta t}(t-s)^{-\alpha-1}(v(t)-v(s))ds\right| \le W(t) \text{ for } 0<t<T,\ 0<\Theta<1;$$

(iv) *v has the structure $v=\gamma_0 t^\alpha + v_0$, where γ_0 is a constant, $v_0\in\mathcal{H}_0^\alpha[0,T]$, $v(0)=0$ and the improper integral (1) converges for any $t\in(0,T]$ and defines a function $w\in C(0,T]$, which has a finite limit $w(0) := \lim_{t\to 0} w(t)$ (so $w\in C[0,T]$).*

(v) *v has the structure $v = \gamma_0 t^\alpha + v_0$ where γ_0 is a constant, $v_0 \in \mathcal{H}_0^\alpha[0,T]$, $v(0) = 0$, and the improper integral $\int_0^t (t-s)^{-\alpha-1}(v_0(t) - v_0(s))ds =: w_0(t)$ converges for any $t \in (0,T]$ and defines with $w_0(0) = 0$ a function $w_0 \in C(0,T]$.*

For $v \in J^\alpha C[0,T]$, it holds for $0 < t \leq T$ that

$$(D_0^\alpha v)(t) := ((J^\alpha)^{-1}v)(t) = \frac{1}{\Gamma(1-\alpha)}\left(t^{-\alpha}v(t) + \alpha \int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds\right);$$

$$(D_0^\alpha v)(0) := ((J^\alpha)^{-1}v)(0) = \Gamma(\alpha+1)\gamma_0.$$

In the present thesis we answer the following question: in Theorem 1 in parts (iii)-(v) can one replace the conditional convergence of the improper integrals (1) by their absolute convergence? The answer occurs to be "no": restricting ourselves to the absolute convergence of those integrals, we do not obtain all functions $v \in J^\alpha C[0,T]$. To prove this claim, we construct a function $v \in J^\alpha C[0,1]$, satisfying (iv) such that integrals (1) do not converge absolutely for all $t \in (0,T]$. Namely,

$$v(t) = t(1-t)^\alpha(\log(1-t))^{-1}\sin(\log(1-t)), \quad 0 < t < 1, \tag{2}$$

with $v(0) = v(1) = 0$ occurs to be such a function. To see the continuity of $v(t)$ at points $t = 0$ and $t = T = 1$, observe that

1)

$$\lim_{t\to 0} v(t) = \lim_{t\to 0}\frac{t(1-t)^\alpha \sin(\log(1-t))}{\log(1-t)} = \lim_{t\to 0} t(1-t)^\alpha \cdot \lim_{t\to 0}\frac{\sin(\log(1-t))}{\log(1-t)} = 0,$$

as it is well known that (see [12])

$$\lim_{t\to 0}\frac{\sin(t)}{t} = 1; \tag{3}$$

2)

$$\lim_{t\to 1} v(t) = \lim_{t\to 1}\frac{t(1-t)^\alpha \sin(\log(1-t))}{\log(1-t)} = \lim_{t\to 1} t(1-t)^\alpha \cdot \lim_{t\to 1}\frac{\sin(\log(1-t))}{\log(1-t)} = 0.$$

In the third chapter we formulate a proposition for analysing this example.

2.2 Lemmas supporting the main result

In this section we present and prove some lemmas that are needed to show the main result.

To prove Lemma 2, which we are going to formulate beneath, we need to present Leibniz theorem about the convergence of alternating series [12] and also mean value theorem for integrals [7].

Theorem. ***(Leibniz theorem)*** *An alternating series of the form*

$$\sum_{k=1}^{\infty}(-1)^{k-1}f_k, \quad \textit{where} \quad f_k \geq 0,$$

converges if the following two conditions are satisfied:

1) $f_k \geq f_{k+1}$ *for all* $k \geq N$, *where* N *is some natural number*

2) $\lim_{k\to\infty} f_k = 0$.

Theorem. ***(mean value theorem for integrals)*** *If* f *and* g *are integrable functions on the closed interval* $[a, b]$ *and* g *does not change the sign, then there exists a number* μ *suvh that*

$$m := \inf_{x\in[a,b]} f(x) \leq \mu \leq \sup_{x\in[a,b]} f(x) =: M$$

and

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx.$$

For the needs of the next section let us formulate and prove the following lemma.

Lemma 2. *If* $f \in C[0,\infty)$, $f(x) \geq 0$, f *is monotonically decreasing for* $x \geq x_0 \geq 0$ *and* $f(x) \to 0$ *as* $x \to \infty$, *then improper integrals*

$$\int_0^{\infty} f(x)\sin(x)dx \quad \textit{and} \quad \int_0^{\infty} f(x)\cos(x)dx$$

converge, i.e. there exist finite limits

$$\int_0^{\infty} f(x)\sin(x)dx = \lim_{b\to\infty}\int_0^b f(x)\sin(x)dx$$

and

$$\int_0^\infty f(x)\cos(x)dx = \lim_{b\to\infty}\int_0^b f(x)\cos(x)dx.$$

Proof. Let us first look at integral $\int_0^\infty f(x)\sin(x)dx$.

We expand integral $\int_0^\infty f(x)\sin(x)dx$ to series

$$\begin{aligned}\int_0^\infty f(x)\sin(x)dx &= \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} f(x)\sin(x)dx = \sum_{k=0}^\infty f(x_k)\int_{k\pi}^{(k+1)\pi}\sin(x)dx \\ &= \sum_{k=0}^\infty 2\cdot(-1)^k f(x_k),\end{aligned}$$

with some $x_k \in [k\pi, (k+1)\pi]$. To get an equality in step 2 we use the mean value theorem for integrals [7].

As x increases for $x \geq x_0$, $f(x)$ is monotonically decreasing, so by absolute value, all given members are non-increasing and moreover, they converge to 0.

So if we look at limit

$$\lim_{b\to\infty}\int_0^b f(x)\sin(x)dx,$$

then for every b there exists n, such that $\pi\cdot n \leq b \leq \pi(n+1)$ and as $n\to\infty$,

$$\left|\int_0^b f(x)\sin(x)dx - \int_0^{\pi(n+1)} f(x)dx\right| \leq \int_{n\pi}^{\pi(n+1)} |f(x)|\,dx \leq |f(n\pi)|\,\pi \to 0.$$

Hence,

$$\lim_{b\to\infty} f(x)\sin(x)dx = \sum_{k=0}^\infty 2\cdot(-1)^k f(x_k).$$

This is an alternating series absolutely converging to zero by Leibniz test, so given series converges by Leibniz theorem, i.e. integral $\int_0^\infty f(x)\sin(x)dx$ is convergent as $x\to\infty$.

For $\int_0^\infty f(x)\cos(x)dx$ the proof is very similar, but the only difference is how we choose the upper and lower limits. So, for $\cos(x)$ we have

$$\begin{aligned}\int_0^\infty f(x)\cos(x)dx &= \sum_{k=0}^\infty \int_{-\frac{\pi}{2}+k\pi}^{\frac{\pi}{2}+k\pi} f(x)\cos(x)dx = \sum_{k=0}^\infty f(x_k)\int_{-\frac{\pi}{2}+k\pi}^{\frac{\pi}{2}+k\pi}\cos(x)dx \\ &= \sum_{k=0}^\infty 2\cdot(-1)^k f(x_k),\end{aligned}$$

with some $x_k \in [-\pi/2 + k\pi, \pi/2 + k\pi]$. The rest of this proof was done previously.

□

Finally, there is one more lemma we need to formulate and prove for the next chapter.

Lemma 3. *Inequality* $|t^\alpha - s^\alpha| \leq |t-s|^\alpha$, *for* $0 < \alpha < 1$ *and* $s, t \geq 0$, *holds.*

Proof. First let us look at situation where $s = 0$ and $t > 0$. Then we have $|t^\alpha| \leq |t|^\alpha = |t^\alpha|$. This statement is true and the claim of Lemma 3 holds.
As we look at situation where $t = 0$ and $s > 0$, we see that the statement is very similar as in previous step and the claim of Lemma 3 holds.
Consider the case with $t > s > 0$ and look at inequality $t^\alpha - s^\alpha \leq (t-s)^\alpha$. First let us set $t = k \cdot s$, where $k \geq 1$ and reduce the claim to

$$k^\alpha - 1 \leq (k-1)^\alpha \quad \text{for} \quad k \geq 1.$$

For $k = 1$ the claim of Lemma 3 is clear, so we can take $k > 1$ and look at inequality

$$k^\alpha - 1 < (k-1)^\alpha \quad \text{for} \quad k > 1.$$

Now let α be fixed and consider

$$f(k) = k^\alpha - 1 - (k-1)^\alpha$$

defined on $[1, \infty)$. Note that $f(1) = 0$ and

$$f'(k) = \alpha k^{\alpha-1} - \alpha(k-1)^{\alpha-1} = \alpha\left(\frac{1}{k^{1-\alpha}} - \frac{1}{(k-1)^{1-\alpha}}\right) < 0, \quad \text{for} \quad k > 1.$$

Since $0 < k-1 < k$, we have $(k-1)^{1-\alpha} < k^{1-\alpha}$. As $f'(k) < 0$ for $k > 1$, the function $f(k)$ is decreasing and since

$$\lim_{k \to 1} f(k) = f(1) = 0,$$

this implies that $f(k) < 0$ and $k^\alpha - 1 < (k-1)^\alpha$ for $k > 1$.

□

3 Main result

The main goal of the present work is to show that requiring the absolute convergence of the improper integrals (1) in conditions (iii), (iv) or (v) of Theorem 1, we do not obtain the whole image $J^\alpha C[0,T]$. Namely we show that for the function (2)

$$v(t) = t(1-t)^\alpha (\log(1-t))^{-1} \sin(\log(1-t)), \quad 0 < t < 1,$$

for $T = 1$, condition (iii) is fulfilled, but the integrals (1)

$$w(t) := \int_0^t (t-s)^{-\alpha-1}(v(t) - v(s))ds, \quad 0 < t \leq T,$$

do not converge absolutely for all $t \in (0,1]$, more precisely, for $t = 1$ the integral

$$\int_0^1 (t-s)^{-\alpha-1}(v(t) - v(s))ds$$

does not converge absolutely. Hence, the integral

$$\int_0^1 (1-s)^{-\alpha-1}(v(1) - v(s))ds, \text{ with}$$

$$v(0) = v(1) = 0,$$

does not converge absolutely.

3.1 Examining the example

For the main goal of this thesis we have presented an example (2) of function $v \in J^\alpha C[0,T]$, in previous section, such that integrals given by (1) do not converge absolutely for all $t \in [0,T]$. To show that, we formulate and prove next proposition.

Proposition 1. *Function* (2), *where* $0 < t < 1$ *and* $v(0) = v(1) = 0$, *satisfies condition* (iv) *in Theorem 1, with* $T = 1$ *and* $\gamma_0 = 0$, *hence* $v \in J^\alpha C[0,1]$, *but integral*

$$\int_0^1 (1-s)^{-\alpha-1} |v(1) - v(s)| \, ds = \int_0^1 s(1-s)^{-1} |\log(1-s)|^{-1} |\sin(\log(1-s))| \, ds \qquad (4)$$

diverges.

Proof. Let us divide this proof into four parts. We will show that

1) $v \in \mathcal{H}_0^\alpha[0,1]$;
2) $w(t) := \int_0^t (t-s)^{-\alpha-1}(v(t) - v(s))ds$ converges for $\forall\, t \in (0,1]$;
3) $w \in C(0,1]$ and there exists $\lim_{t\to 0} w(t) =: w(0)$, which shows that $w \in C[0,1]$;
4) the convergence of the integral $w(1) = \int_0^1 (1-s)^{-\alpha-1}(v(1) - v(s))ds$ is non-absolute: (4) holds true.

Let us look at the function (2), $v(t) = t(1-t)^\alpha (\log(1-t))^{-1} \sin(\log(1-t))$, where $0 < t < 1$ and $v(0) = v(1) = 0$.

1) In the first part let us show that $v(t)$ is continuously differentiable on $[0,1)$.

   For the derivative v' we have the formula

$$v'(t) = \sum_{i=1}^{4} u_i(t), \qquad (5)$$

   where

$$u_1(t) = (1-t)^\alpha (\log(1-t))^{-1} \sin(\log(1-t));$$
$$u_2(t) = -\alpha \cdot t(1-t)^{\alpha-1} (\log(1-t))^{-1} \sin(\log(1-t));$$
$$u_3(t) = t(1-t)^{\alpha-1} (\log(1-t))^{-2} \sin(\log(1-t));$$
$$u_4(t) = -t(1-t)^{\alpha-1} (\log(1-t))^{-1} \cos(\log(1-t)).$$

Clearly $v' \in C(0,1)$. We want to show that $v \in C^1[0,1)$ and to do that, we look, what happens to each $u_i(t)$ separately, as $t \to 0$. As we know that (3) holds and using the change of variable $\log(1-t) = x$ for $x \to 0$, we get (here we use (3) in step 2)

$$\lim_{t\to 0} \frac{\sin(\log(1-t))}{\log(1-t)} = \lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{6}$$

Knowing this, let us analyse addend's of limits in (5).

(a) For the first addend we have

$$\lim_{t\to 0} u_1(t) = \lim_{t\to 0} \frac{(1-t)^\alpha \sin(\log(1-t))}{\log(1-t)} = \lim_{t\to 0}(1-t)^\alpha \cdot \lim_{t\to 0} \frac{\sin(\log(1-t))}{\log(1-t)} = 1.$$

(b) For the second addend we have

$$\begin{aligned}\lim_{t\to 0} u_2(t) &= \lim_{t\to 0} \frac{-\alpha t(1-t)^{\alpha-1} \sin(\log(1-t))}{\log(1-t)} \\ &= \lim_{t\to 0} -\alpha t(1-t)^{\alpha-1} \cdot \lim_{t\to 0} \frac{\sin(\log(1-t))}{\log(1-t)} = 0.\end{aligned}$$

(c) For the third addend we need to find

$$\begin{aligned}\lim_{t\to 0} u_3(t) &= \lim_{t\to 0} \frac{t(1-t)^{\alpha-1} \sin(\log(1-t))}{(\log(1-t))^2} \\ &= \lim_{t\to 0} \frac{\sin(\log(1-t))}{\log(1-t)} \cdot \lim_{t\to 0}(1-t)^{\alpha-1} \cdot \lim_{t\to 0} \frac{t}{\log(1-t)} = \lim_{t\to 0} \frac{t}{\log(1-t)}.\end{aligned}$$

Using L'Hospital rule we can continue

$$\lim_{t\to 0} u_3(t) = \lim_{t\to 0} \frac{t}{\log(1-t)} = \lim_{t\to 0} \frac{1}{-\frac{1}{1-t}} = \lim_{t\to 0}(t-1) = -1.$$

(d) For the fourth addend we use again L'Hospital rule in step 4 and also note that

$$\lim_{t\to 0}(1-t)^{\alpha-1} = 1 \quad \text{and} \quad \lim_{t\to 0}\cos(\log(1-t) = 1,$$

so

$$\begin{aligned}\lim_{t\to 0} u_4(t) &= \lim_{t\to 0}\frac{-t(1-t)^{\alpha-1}\cos(\log(1-t))}{\log(1-t)}\\ &= \lim_{t\to 0}\frac{-t}{\log(1-t)}\cdot\lim_{t\to 0}(1-t)^{\alpha-1}\cdot\lim_{t\to 0}\cos(\log(1-t)\\ &= \lim_{t\to 0}\frac{-t}{\log(1-t)} = \lim_{t\to 0}(1-t) = 1.\end{aligned}$$

Thus, we have shown that

$$v'(0) := \lim_{t\to 0} v'(t) = 1 + 0 - 1 + 1 = 1,$$

and hence, $v \in C^1[0,1)$.

Further observe that

$$\lim_{t\to 1}(1-t)^{1-\alpha}v'(t) = \lim_{t\to 1}\sum_{i=1}^{\infty}(1-t)^{1-\alpha}u_i(t) = 0. \tag{7}$$

To show (7), we need to analyse again every addend of this limit separately, as we did in the previous step.

(a) For the first addend we get

$$\lim_{t\to 1}(1-t)^{1-\alpha}u_1(t) = \lim_{t\to 1}\frac{(1-t)\sin(\log(1-t))}{\log(1-t)} = 0.$$

Note that the numerator of the last fraction is bounded and the denominator tends to infinity, therefore this limit is 0.

(b) For the second addend we have

$$\lim_{t\to 1}(1-t)^{1-\alpha}u_2(t) = \lim_{t\to 1}\frac{-\alpha t\sin(\log(1-t))}{\log(1-t)} = 0.$$

Again we note that the numerator of the last fraction is bounded and the denominator tends to infinity, therefore this limit is 0.

(c) For the third addend we have

$$\lim_{t\to 1}(1-t)^{1-\alpha}u_3(t) = \lim_{t\to 1}\frac{t\sin(\log(1-t))}{(\log(1-t))^2} = 0.$$

Here we also note that the numerator of the last fraction is bounded and the denominator tends to infinity, therefore this limit is 0.

(d) For the fourth addend we get

$$\lim_{t\to 1}(1-t)^{1-\alpha}u_4(t) = \lim_{t\to 1}\frac{-t\cos(\log(1-t))}{\log(1-t)} = 0.$$

Once again we note that the numerator of the last fraction is bounded and the denominator tends to infinity, therefore this limit is 0.

Now we need to check weather $v \in \mathcal{H}_0^\alpha[0,1]$. Since $v \in C^1[0,1)$, we have $v \in \mathcal{H}_0^\alpha[0,\theta]$ for all $\theta \in (0,1)$. So it remains to show that

$$0 \le s_n < t_n \quad \text{and} \quad s_n, t_n \to 1 \Rightarrow \frac{v(t_n)-v(s_n)}{(t_n-s_n)^\alpha} \to 0 \quad \text{as} \quad n \to \infty.$$

We estimate (for $s_n, t_n \to 1$):

$$\begin{aligned} |v(t_n)-v(s_n)| &= \left|\int_{s_n}^{t_n} v'(\tau)d\tau\right| = \left|\int_{s_n}^{t_n}(1-\tau)^{\alpha-1}(1-\tau)^{1-\alpha}v'(\tau)d\tau\right| \\ &\le \max_{s_n\le\tau\le t_n}(1-\tau)^{1-\alpha}\left|v'(\tau)\right|\int_{s_n}^{t_n}(1-\tau)^{\alpha-1}d\tau. \end{aligned}$$

Due to Lemma 3 we get

$$\int_{s_n}^{t_n}(1-\tau)^{\alpha-1}d\tau = \frac{1}{\alpha}\left((1-s_n)^\alpha-(1-t_n)^\alpha\right) \le \frac{1}{\alpha}(t_n-s_n)^\alpha.$$

So as $n\to\infty$ due to (7) we have

$$\frac{|v(t_n)-v(s_n)|}{(t_n-s_n)^\alpha} \le \frac{1}{\alpha}\max_{s_n\le\tau\le t_n}(1-\tau)^{1-\alpha}\left|v'(\tau)\right| \to 0.$$

Thus we have shown that $v \in \mathcal{H}_0^\alpha[0,1]$.

2) Next we need to show that (1) converges for all $t \in (0,1]$.

We know that $v \in C^1[0,1)\cap\mathcal{H}_0^\alpha[0,1]$ and

$$|v'(t)| \le c\frac{(1-t)^{\alpha-1}}{|\log(1-t)|},\ 0\le t<1. \tag{8}$$

For $0 < t < 1$ we integrate by parts

$$\int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds = \frac{1}{\alpha}(t-s)^{-\alpha}(v(t)-v(s)) \mid_{s=0}^{t} + \frac{1}{\alpha}\int_0^t (t-s)^{-\alpha}v'(s)ds$$

$$= -\frac{1}{\alpha}t^{-\alpha}v(t) + \frac{1}{\alpha}\int_0^t (t-s)^{-\alpha}v'(s)ds. \tag{9}$$

The last integral converges for $0 < t < 1$, since $v \in C^1[0,1)$. We know that $v \in \mathcal{H}_0^\alpha[0,1]$ and $v(0) = 0$, hence

$$t^{-\alpha}v(t) = \frac{v(t)-v(0)}{t^\alpha} \xrightarrow{t\to 0} 0,$$

therefore $t^{-\alpha}v(t)$ is continuous in $[0,1]$. Let now $t = 1$. We prove that the improper integral $\int_0^1 (1-s)^{-\alpha}v'(s)dx$ converges. According to (5) we can write this integral as the sum of four integrals. We next analyse these four integrals with $u_1(s)$, $u_2(s)$, $u_3(s)$ and $u_4(s)$ separately.

For the integral with $u_1(s)$ we have

$$\int_0^1 (1-s)^{-\alpha}u_1(s)ds = \int_0^1 (1-s)^{-\alpha}(1-s)^{\alpha}(\log(1-s))^{-1}\sin(\log(1-s))ds$$
$$= \int_0^1 (\log(1-s))^{-1}\sin(\log(1-s))ds.$$

We make change in variables $\log(1-s) = -x$ which implies $1 - s = e^{-x}$ and so $s = 1 - e^{-x}$ and $ds = e^{-x}dx$. Now while $s \to 0$, for x we have $x \to 0$ and from $s \to 1$ we get $x \to \infty$. So we get

$$\int_0^1 (1-s)^{-\alpha}u_1(s)ds = \int_0^\infty \frac{1}{-x}\sin(-x)e^{-x}dx = \int_0^\infty \frac{\sin(x)e^{-x}}{x}dx.$$

By Lemma 2 the last improper integral converges. Hence, also $\int_0^1 (1-s)^{-\alpha}u_1(s)ds$ converges.

For integral with $u_2(s)$ we get

$$\int_0^1 (1-s)^{-\alpha}u_2(s)ds = \int_0^1 (1-s)^{-\alpha}s(1-s)^{\alpha-1}(\log(1-s))^{-1}\sin(\log(1-s))ds$$
$$= \int_0^1 s(1-s)^{-1}(\log(1-s))^{-1}\sin(\log(1-s))ds.$$

Making the same change in variable as for $u_1(s)$ we get

$$\begin{aligned}\int_0^1 (1-s)^{-\alpha} u_2(s)ds &= \int_0^\infty (1-e^{-x})(e^{-x})^{-1}(-x)^{-1}\sin(-x)e^{-x}dx \\ &= \int_0^\infty e^x(1-e^{-x})\cdot\frac{1}{-x}\cdot\sin(-x)e^{-x}dx = \int_0^\infty \frac{1-e^{-x}}{x}\sin(x)dx.\end{aligned}$$

Again by Lemma 2 the last improper integral converges, hence $\int_0^1(1-s)^{-\alpha}u_2(s)ds$ converges.

For integral with $u_3(s)$ we get

$$\begin{aligned}\int_0^1 (1-s)^{-\alpha}u_3(s)ds &= \int_0^1 (1-s)^{-\alpha}s(1-s)^{\alpha-1}(\log(1-s))^{-2}\sin(\log(1-s))ds \\ &= \int_0^1 s(1-s)^{-1}(\log(1-s))^{-2}\sin(\log(1-s))ds.\end{aligned}$$

Making again the same change in variable as for $u_1(s)$ and $u_2(s)$ we get:

$$\begin{aligned}\int_0^1 (1-s)^{-\alpha}u_3(s)ds &= \int_0^\infty (1-e^{-x})e^x(-x)^{-2}\sin(-x)e^{-x}dx \\ &= \int_0^\infty -\frac{1-e^{-x}}{x^2}\sin(x)dx = -\int_0^\infty \frac{1-e^{-x}}{x}\cdot\frac{sin(x)}{x}dx.\end{aligned}$$

Again the last improper integral converges as x approaches to 0 by using (3) for $\frac{sin(x)}{x}$ and L'Hospital rule for $\frac{1-e^{-x}}{x}$. Hence $\int_0^1(1-s)^{-\alpha}u_3(s)ds$ converges.
For the integral with $u_4(s)$ we get

$$\begin{aligned}\int_0^1 (1-s)^{-\alpha}u_4(s)ds &= \int_0^1 (1-s)^{-\alpha}s(1-s)^{\alpha-1}(\log(1-s))^{-1}\cos(\log(1-s))ds \\ &= \int_0^1 s(1-s)^{-1}(\log(1-s))^{-1}\cos(\log(1-s))ds.\end{aligned}$$

Making the same change in variable as for $u_1(s)$ and $u_2(s)$ and $u_3(s)$ we get:

$$\int_0^\infty (1-e^{-x})e^x\cdot\frac{1}{-x}\cdot\cos(-x)e^{-x}dx = -\int_0^\infty \frac{1-e^{-x}}{x}\cos(x)dx.$$

By Lemma 2 the last improper integral converges, hence $\int_0^1(1-s)^{-\alpha}u_4(s)ds$ converges.

In conclusion, we have shown that the improper integral $\int_0^1(1-s)^\alpha v'(s)ds$ converges, hence $\int_t^1(1-s)^\alpha v'(s)ds \to 0$ as $t\to 0$.

Now we also need to show that $\int_0^t (t-s)^{-\alpha} v'(s)ds \to \int_0^1 (1-s)^{-\alpha} v'(s)ds$ as $t \to 1$. We already have $\int_t^1 (1-s)^{-\alpha} v'(s)ds \to 0$ as $t \to 1$, thus it is sufficient to show, that

$$\left| \int_0^t [(t-s)^{-\alpha} - (1-s)^{-\alpha}] v'(s)ds \right| \to 0, \quad \text{as} \quad t \to 1.$$

For that it is sufficient to show that

$$\int_0^t [(t-s)^{-\alpha} - (1-s)^{-\alpha}]ds \max_{0 \le s \le t} |v'(s)| \to 0, \quad \text{as} \quad t \to 1. \tag{10}$$

First we look at integral $\int_0^t [(t-s)^{-\alpha} - (1-s)^{-\alpha}]ds$ and write

$$\begin{aligned} \int_0^t [(t-s)^{-\alpha} - (1-s)^{-\alpha}]ds &= \frac{1}{1-\alpha}[(1-s)^{1-\alpha} - (t-s)^{1-\alpha}]|_0^t \\ &= \frac{1}{1-\alpha}\left[(1-t)^{1-\alpha} - (t-t)^{1-\alpha} - (1-0)^{1-\alpha} + (t-0)^{1-\alpha}\right] \\ &= \frac{1}{1-\alpha}\left[(1-t)^{1-\alpha} - 1 + t^{1-\alpha}\right] \le \frac{1}{1-\alpha}(1-t)^{1-\alpha}, \end{aligned} \tag{11}$$

because $t^{1-\alpha} - 1 < 0$.

Now let's look at the factor $\max_{0 \le s \le t} |v'(s)|$ in (10). From the previous we know that (8) holds. If $s \le t$, then $(1-s) \ge (1-t)$, $\quad (1-s)^{\alpha-1} \le (1-t)^{\alpha-1}$ and when $s \le t$, then $|\log(1-s)| \ge |\log(1-t)|$. Therefore we get

$$\frac{(1-s)^{\alpha-1}}{|\log(1-s)|} \le \frac{(1-t)^{\alpha-1}}{|\log(1-t)|} \quad \text{for} \quad 0 < s \le t,$$

and (8) implies that

$$\max_{0 \le s \le t} |v'(s)| \le \frac{c(1-t)^{\alpha-1}}{|\log(1-t)|}.$$

Together with (11) we obtain (10) and have that $w \in C[0,1]$.

3) In this part we need to show that $w \in C[0,1]$ and for this we show that $w \in C(0,1]$ and $\exists \lim_{t \to 0} w(t) =: w(0) = 0$. We recall here that (see (9))

$$w(t) = \int_0^t (t-s)^{-\alpha-1}(v(t) - v(s))ds = -\frac{1}{\alpha} t^{-\alpha} v(t) + \frac{1}{\alpha} \int_0^t (t-s)^{1-\alpha} v'(s)ds.$$

From the part 2) of this proof it is clear that $w(t) \in C(0,1]$, so we only need to show that limit

$$w(0) = \lim_{t \to 0} w(t) = \lim_{t \to 0} \int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds$$

exists.

As we previously showed in the part 2) of this proof, the integral

$$\int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds$$

converges as $t \to 0$, moreover it converges to 0, and thus we know that this limit indeed exists and $w(0) = 0$.

4) It remains to show that for some t the integral $\int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds$, $0 < t \leq 1$ does not converge absolutely. It occurs that this holds true for $t = 1$.

For our example, since $v(1) = 0$, we can write

$$\int_0^1 (1-s)^{-\alpha-1} |v(1)-v(s)| \, ds = \int_0^1 (1-s)^{-\alpha-1} |v(s)| \, ds$$
$$= \int_0^1 s(1-s)^{-1} |\log(1-s)|^{-1} |\sin(\log(1-s))| \, ds.$$

The divergence of the last integral may be caused by the singularity of the integrand at the point $s = 1$. So we analyse the integral

$$I = \int_{1/2}^1 (1-s)^{-1} |\log(1-s)|^{-1} |\sin(\log(1-s))| \, ds.$$

As we make the change in variables $\tilde{s} = 1 - s$ and after that write again s instead of $\tilde{s}$, we have

$$I = \int_0^{1/2} s^{-1} |\log(s)|^{-1} |\sin(\log(s))| \, ds.$$

If we make a change in variable and say that $t = -\log(s)$, then $e^{-t} = s$. Since $0 \leq s \leq 1/2$, for t we have $-\log(1/2) = \log(2) \leq t \leq \infty$ and thus also $dt = -\frac{1}{s}ds$, we get that

$$I = -\int_{\log(2)}^{\infty} e^t \cdot t^{-1} |\sin(t)| \, e^{-t} dt = \int_{\log(2)}^{\infty} \frac{|\sin(t)|}{t} dt.$$

This is well known integral (see e.g. [7]) that diverges and so

$$\int_0^1 s(1-s)^{-1} \left|\log(1-s)\right|^{-1} \left|\sin(\log(1-s))\right| ds = \infty.$$

The proof of Proposition 1 is complete.

$\square$

3.2 Behaviour of function v(t)

Let us illustrate the behaviour of the function $v(t) = t(1-t)^{\alpha}(\log(1-t))^{-1}\sin(\log(1-t))$ as $0 < t < 1$. The function $v(t)$ oscillates between the "envelope" function

$$\bar{v}(t) = \pm t(1-t)^{\alpha}(\log(1-t))^{-1}.$$

As the graphs of $v(t)$ and $\bar{v}(t)$ depend on how we choose the parameter α, we are going to give α three different values between $0 < \alpha < 1$. So we choose α close to 0, between 0 and 1 and finally close to 1.

All graphs are made using mathematical program MathCad15.

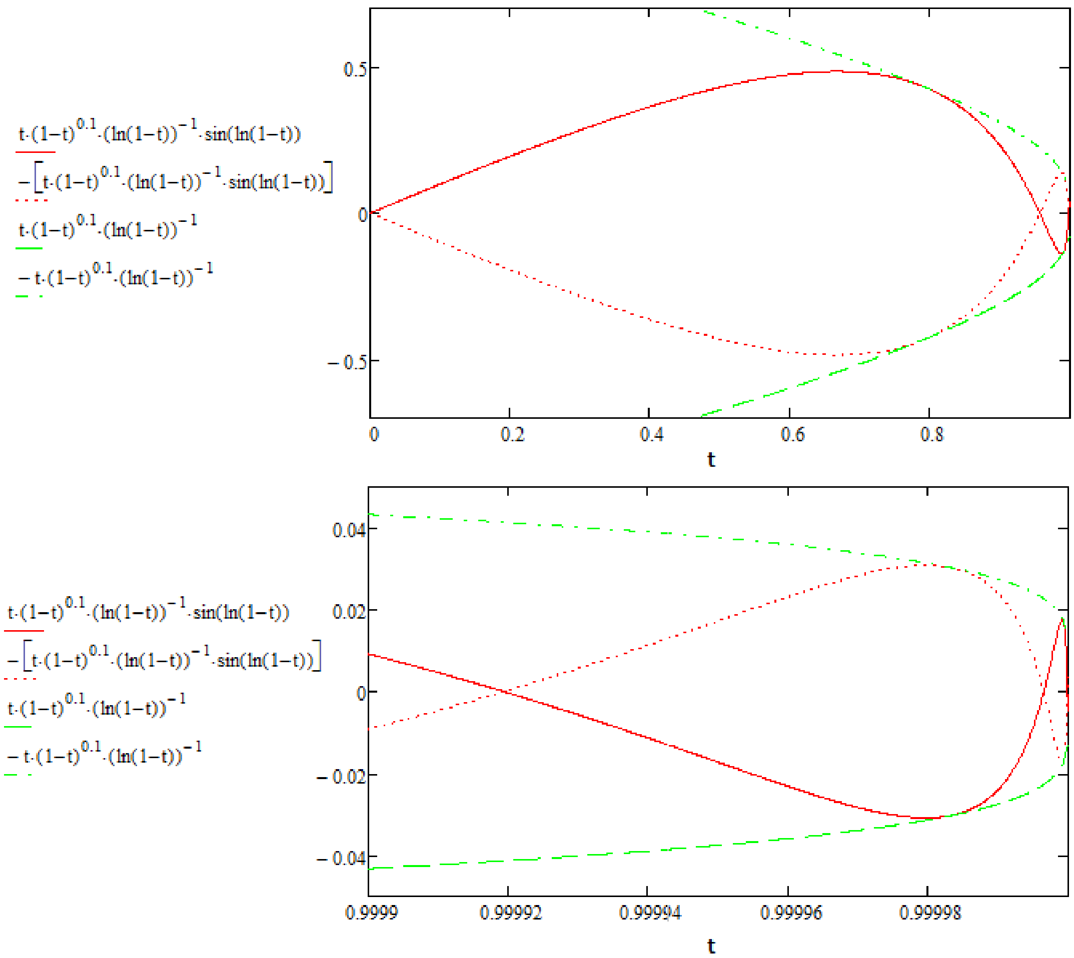

Figure 1: Graphs of $v(t)$ and $\bar{v}(t)$ for $\alpha = 0.1$

In Figure 1 we have shown the behaviour of functions $v(t)$ and $\bar{v}(t)$ as $\alpha = 0.1$. The red lines represent the function $v(t)$ and green lines represent the function for $\bar{v}(t)$. We can see that function $v(t)$ oscillates more rapidly as t gets close to point 1.

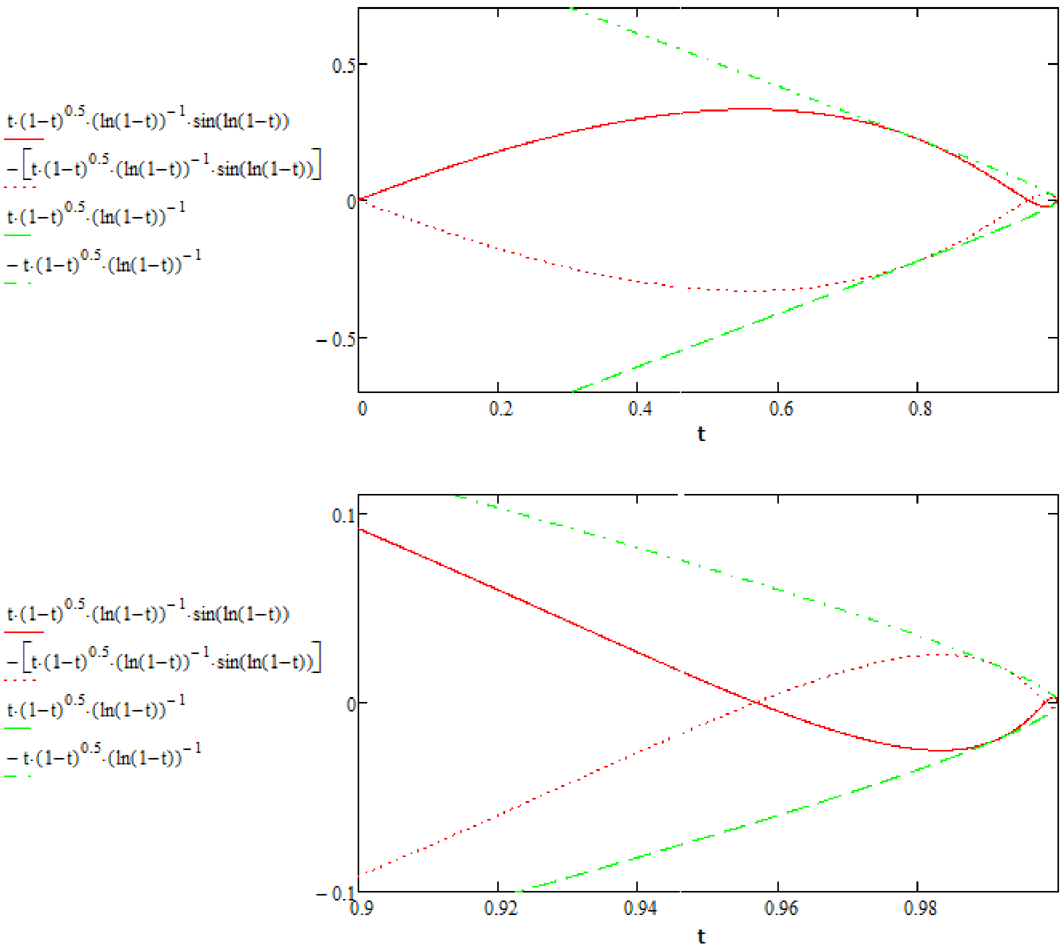

Figure 2: Graph of $v(t)$ and $\bar{v}(t)$ as $\alpha = 0.5$

In Figure 2 we have shown the behaviour of functions $v(t)$ and $\bar{v}(t)$ as $\alpha = 0.5$. The red lines represents again the function $v(t)$ and green lines represent the function $\bar{v}(t)$. And again we can see that function $v(t)$ oscillates more rapidly as it gets close to point 1.

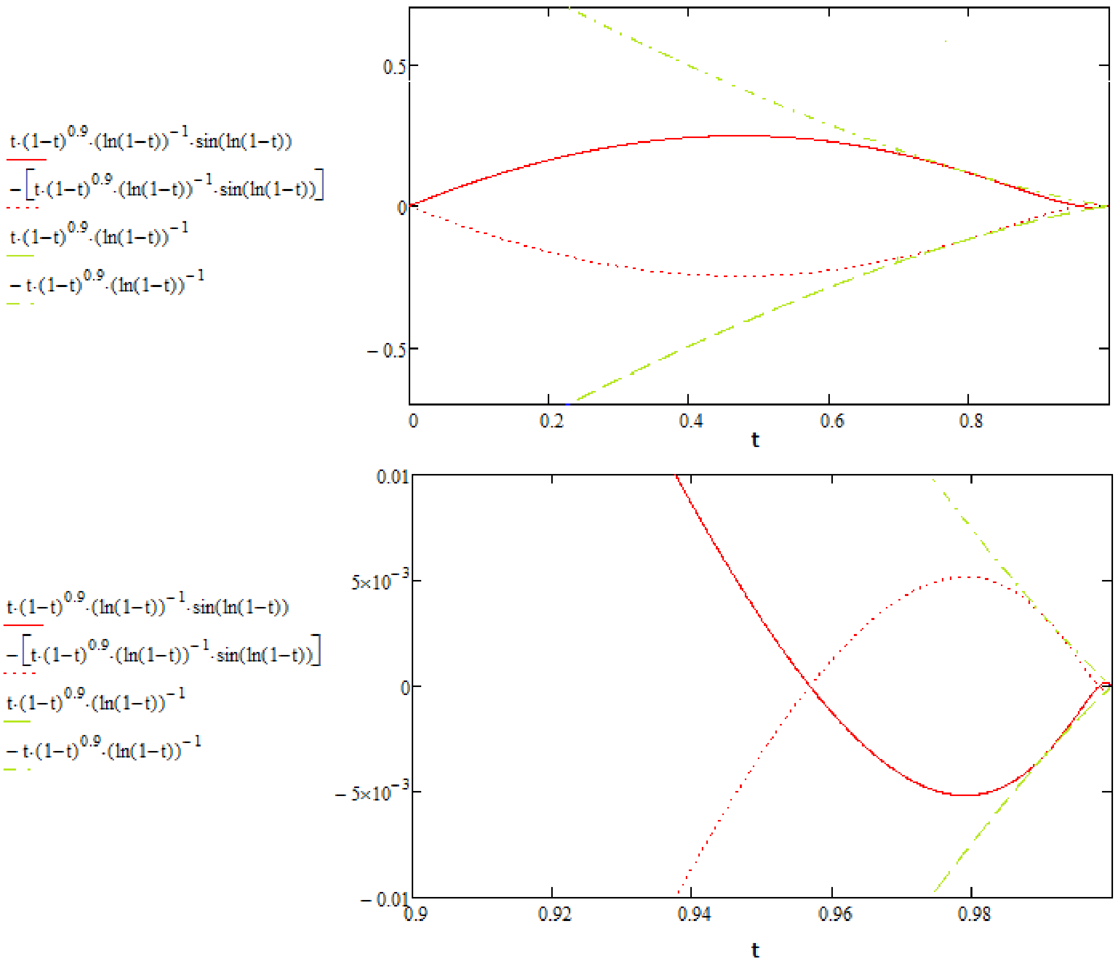

Figure 3: Graphs of $v(t)$ and $\bar{v}(t)$ for $\alpha = 0.9$

In Figure 3 we have shown the behaviour of functions $v(t)$ and $\bar{v}(t)$ as $\alpha = 0.9$. As in previous two figures, the red lines represent the function $v(t)$ and green lines represent $\bar{v}(t)$. Here we see as well that function $v(t)$ oscillates more as it gets close to point 1.

References

[1] Tomaš Kisela, *Fractional differential equations and their applications*, Diploma thesis, 2008, https://www.researchgate.net/publication/249993249_Fractional_Differential_Equations_and_Their_Applications.

[2] Mehdi Dalir, Majid Bashour, *Applications of Fractional Calculus*, Applied Mathematical Sciences, Vol. 4, No. 21, 2010, http://www.m-hikari.com/ams/ams-2010/ams-21-24-2010/bashourAMS21-24-2010.pdf.

[3] Nicholas Wheeler *Construction and physical application of the fractional calculus*, Reed College Physics Department, February, 1997.

[4] Keith B. Oldham, Jerome Spanier, *The Fractional Calculus*, London: Academic Press, 1974.

[5] Anatoly A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, *Theory and applications of fractional differential equations*, North Holland mathematics studies 204, 2006.

[6] Gennadi Vainikko, *Which functions are fractionally differentiable?*, Zeitschrift für Analysis und ihre Anwendungen, to appear.

[7] Gunnar Kangro, *Matemaatiline analüüs I*, Tallinn, Valgus, 1978.

[8] Kai Diethelm, *The analysis of fractional Differential Equations. An Application-oriented Exposition Using Differential Operators of Caputo Type*, Lect. Notes Math. 2004, Springer 2010.

[9] *Mathematical Encyclopedia*, Vol. 1-5 (in Russian). Moscow: Soviet Encyclopedia, 1977-1985.

[10] Gennadi Vainikko, *First kind cordinal Volterra integral equations 2.* Numer. Funct. Anal. Optim. 35 (2014), 1607-1637.

[11] Arvet Pedas, Gennadi Vainikko, *Harilikud diferentsiaalvõrrandid*, Tartu Ülikooli Kirjastus, 2009.

[12] Leiki Loone, Virge Soomer, *Matemaatilise analüüsi algkursus*, Tartu Ülikooli Kirjastus 2009.

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