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Lifting bounded approximation properties from Banach spaces to their dual spaces

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Chapter 1

Introduction

1.1 Background

Recall that a Banach space has the *approximation property* if its identity operator can be approximated, uniformly on compact sets, by finite-rank operators. If in addition the finite-rank operators can be chosen with norms less than or equal to λ for some $1 \leq \lambda < \infty$, then the Banach space is said to have the *bounded approximation property*, more precisely, the λ-*bounded approximation property*. The 1-bounded approximation property is called the *metric approximation property*. From the definitions, it is clear that the metric approximation property implies the bounded approximation property, which in its turn implies the approximation property. In our brief survey below, we rely on [O3, Section 1].

Systematic studies of the approximation properties started mid 1950s when the terms of the approximation property and the metric approximation property were introduced by Grothendieck as *la condition d'approximation* and *la condition d'approximation métrique* (see [G, Chapter I, pp. 167, 178]), and deeply studied in his famous Memoir [G]. The bounded approximation property was also occasionally introduced in [G] as *la variante affaiblie de la propriété d'approximation métrique* (see [G, Chapter I, p. 182]). However, the origin of the bounded approximation property dates back to 1930s – the notion was essentially considered already in Banach's book [B, p. 237], in a more general setting of the bounded compact approximation property (where "finite-rank operators" are replaced by "compact operators" in the definition of bounded approximation property).

Grothendieck proved (see [G, "Proposition" 37, pp. 170–171]) that the question whether all Banach spaces have the approximation property is equivalent to the following two famous problems: the Mazur's Problem 153 in the *Scottish Book* and the approximation problem.

Problem 153 was posed on November 6, 1936, had a live goose as a prize (see, e.g., [Scot, p. 231]), and was as follows. Given a continuous function $f = f(s,t)$ defined on $[0,1] \times [0,1]$ and any number $\varepsilon > 0$; do there exist numbers $a_1, \dots, a_n$; $b_1, \dots, b_n$; $c_1, \dots, c_n$ such that

$$\left| f(s,t) - \sum_{k=1}^{n} a_k f(s, b_k) f(c_k, t) \right| < \varepsilon$$

for all $s, t \in [0,1]$?

According to Pełczyński [P, p. 68], Mazur knew that the positive answer to Problem 153 would imply the positive answer to the *approximation problem*: can all compact operators between arbitrary Banach spaces be approximated, in the norm topology of operators, by finite-rank operators? It was considered to be one of the central open problems of functional analysis.

It was not until 1970s, when the first example of a Banach space failing the approximation property was produced: in 1972, Enflo [En] discovered a separable reflexive Banach space, which does not have the approximation property. A year later, in 1973, Figiel and Johnson [FJ1] constructed the first counterexamples showing that the approximation property, the bounded approximation property, and the metric approximation property are, in general, different notions. However, it is not known whether these notions are different for dual spaces. One of the most famous open problems is as follows (see, e.g., [C, Problem 3.8]; for an overview see [O3, Section 3]). *Does the approximation property of a dual space imply the metric approximation property?*

While approaching the problem, some new versions of the bounded approximation properties have emerged in the recent years. Two of these are the *bounded approximation property for the pairs*, consisting of a Banach space and its closed subspace, introduced in the seminal paper [FJP] in 2011, and its extension – the *bounded nest approximation property* considered in [FJ3] in 2016.

1.2 Summary of the thesis

The aim of the thesis is to systematically study the bounded approximation property of pairs, the bounded nest approximation property, and even more general version of the bounded approximation properties from [LisO] – the bounded convex approximation property. The latter concept also includes the bounded positive approximation property of Banach lattices.

The thesis has been organized as follows.

Chapter 1 briefly introduces the historic background of the approximation properties, contains a summary of the thesis, and describes notation used in the thesis.

Chapter 2 contains an overview of the notions and results needed in the following chapters. These include polar sets, nests in Banach spaces, the link between the space of finite-rank operators and tensor products, and Grothendieck's descriptions of its dual space as integral operators. A preliminary knowledge of some of the most important locally convex topologies defined on the space of bounded linear operators is given. Some preliminaries are due to [OV2].

In Chapter 3, we consider and describe various bounded approximation properties and their duality versions. The characterization of the bounded approximation properties defined by arbitrary operator ideals from [O2] is extended to bounded convex approximation properties. We prove that the bounded approximation property of the pair $(X^*, Y^\perp)$, where X is a Banach space and Y is its closed subspace, implies the bounded duality approximation property of (X, Y). This result extends an important theorem from [J1] on classical approximation properties to the approximation properties of pairs. The chapter is based on [OT, OV1, V].

In Chapter 4, we establish versions of the principle of local reflexivity which respect nests of subspaces. We prove a rather far-reaching extension of the Ringrose theorem on nests. We also extend a duality result on approximation properties of pairs from [LisO] and its bounded version from Chapter 3 to the context of nest approximation properties. Criteria of the nest approximation properties from [FJ3] are applied to obtain criteria of the duality nest approximation properties in the spirit of Grothendieck. This chapter is based on [OV2].

In Chapter 5, we study the lifting of bounded convex approximation properties from a Banach space to its dual space in some special cases. We show that for such a lifting rather weak forms of the principle of local reflexivity and the extendable local reflexivity are sufficient. It is also shown that such a lifting is possible whenever the dual space already enjoys a weaker bounded convex approximation property. We also complement and extend some results from [GS2, O2]. This chapter relies on [OV1, OV2, V].

The main results of the thesis are contained in [OT, OV1, OV2, V].

1.3 Notation

Our notation is standard.

Let X and Y be Banach spaces, both real or both complex. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y and by $\mathcal{W}(X,Y)$, $\mathcal{K}(X,Y)$, and $\mathcal{F}(X,Y)$ its subspaces of weakly compact, compact and finite-rank operators, respectively. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$ and, similarly, $\mathcal{W}(X)$ for $\mathcal{W}(X,X)$, $\mathcal{K}(X)$ for $\mathcal{K}(X,X)$, and $\mathcal{F}(X)$ for $\mathcal{F}(X,X)$. The range of an operator $S : X \to Y$ is denoted by $\operatorname{ran} S := \{Sx : x \in X\}$. The identity operator on X is denoted by I_X. If A and B are subsets of $\mathcal{L}(X)$ and $\mathcal{L}(X^{**})$, respectively, then $\{S^{**} : S \in A\} \circ B := \{S^{**}T : \; S \in A, \; T \in B\}$.

A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \to X^{**}$. Occasionally, we write $x \in X^{**}$ for $x \in X$, rather than $j_X x \in X^{**}$. The closed unit ball and the unit sphere of X are denoted by B_X and S_X, respectively. For a subset M of X, its closure is denoted by $\overline{M}$, its linear span by $\operatorname{span} M$, and its convex hull by $\operatorname{conv} M$. If Z is a subspace of X, then $Z^{\perp}$ is its annihilator in the dual space X^*, i.e., $Z^{\perp} := \{x^* \in X^* : x^*(z) = 0 \;\; \forall z \in Z\}$. For a subspace W of X^*, we denote by $W_{\perp}$ its annihilator in X, i.e., $W_{\perp} := \{x \in X : w(x) = 0 \;\; \forall w \in W\}$.

We consider Banach lattices over $\mathbb{R}$. If X is a Banach lattice and A is a subset of $\mathcal{L}(X)$, then $A_+ := A \cap \mathcal{L}(X)_+$ denotes the set of positive operators belonging to A.

We assume that the reader is familiar with well-known basic notions and theorems from the theory of Banach spaces and topological vector spaces (such as dual pairs, the Hausdorff theorem, the Hahn–Banach theorem, the Goldstine theorem, etc.) and we shall often use these without proper references.

Chapter 2

Preliminaries

This chapter contains an overview of the notions and results needed in the following chapters. These include polar sets, nests in Banach spaces, the link between the space of finite-rank operators and tensor products, and Grothendieck's descriptions of its dual space as integral operators. A preliminary knowledge of some of the most important locally convex topologies defined on the space of bounded linear operators is given. Some results are due to [OV2].

2.1 Polar sets

Let $\langle X, Y\rangle$ be a duality and let M be a subset of X. Recall that the *polar* of M is the set

$$M^\circ := \{y \in Y : \mathcal{R}e\ \langle x, y\rangle \leq 1 \ \ \forall x \in M\},$$

where $\mathcal{R}e\ \langle x, y\rangle$ denotes the real part of $\langle x, y\rangle$. The polar of M° is a subset of X, called the *bipolar* of M, and is denoted by $M^{\circ\circ}$.

If X is a normed space, then for a subset $Z \subset X$, we have

$$Z^\circ = \{x^* \in X^* : \mathcal{R}e\, x^*(z) \leq 1 \ \ \forall z \in Z\}$$

with respect to $\langle X, X^*\rangle$ and also with respect to $\langle X^{**}, X^*\rangle$.

By the definition, the following facts hold:

(a) if $\mu \neq 0$ and $\mu M \subset N$, then $N^\circ \subset \mu^{-1} M^\circ$;

(b) if X is a normed space, then $B_X^\circ = B_{X^*}$ with respect to $\langle X, X^* \rangle$ and also with respect to $\langle X^{**}, X^* \rangle$.

Proposition 2.1. *Let X be a normed space. The following properties hold for the dual pair $\langle X, X^* \rangle$.*

1. *If Z is a subset of X, then $\left(\overline{Z}\right)^\circ = Z^\circ$.*
2. *If Z is a subspace of X, then $Z^\perp = Z^\circ$.*

Proof. 1. Since $Z \subset \overline{Z}$, we have $\left(\overline{Z}\right)^\circ \subset Z^\circ$. Let us show that $Z^\circ \subset \left(\overline{Z}\right)^\circ$. Let $x^* \in Z^\circ$ be arbitrary. Then $x^* \in \left(\overline{Z}\right)^\circ$ whenever $\mathcal{R}e\, x^*(x) \leq 1$ for all $x \in \overline{Z}$. But this is the case. Indeed, for any $x \in \overline{Z}$, there is $(z_n) \subset Z$ such that $\lim_n z_n = x$. Hence, we have

$$\mathcal{R}e\, x^*(x) = \mathcal{R}e\, x^*(\lim_n z_n) = \lim_n \mathcal{R}e\, x^*(z_n) \leq 1.$$

2. Clearly, $Z^\perp \subset Z^\circ$. For $Z^\circ \subset Z^\perp$, assume that $Z^\circ \not\subset Z^\perp$. Then there is $x^* \in X^*$ such that $\mathcal{R}e\, x^*(z) \leq 1$ for all $z \in Z$, but there exists $z_0 \in Z$ such that $|x^*(z_0)| = r > 1$. Indeed, clearly there is $x \in Z$ such that $|x^*(x)| = \alpha > 0$. Put $z_0 = (1 + 1/\alpha)x$. Then $z_0 \in Z$ (because Z is a subspace) and we have

$$|x^*(z_0)| = |x^*((1 + 1/\alpha)x)| = (1 + 1/\alpha)\, |x^*(x)| = \alpha + 1 > 1.$$

Now pick $\phi \in \mathbb{R}$ such that $x^*(z_0) = re^{i\phi}$. Notice that then $r = x^*(e^{-i\phi} z_0)$, $e^{-i\phi} z_0 \in Z$ and we have

$$\mathcal{R}e\, x^*(e^{-i\phi} z_0) = \mathcal{R}e\, r = r > 1,$$

which is a contradiction with $\mathcal{R}e\, x^*(z) \leq 1$ for all $z \in Z$. □

Remark 2.2. Notice that in the literature the *absolute polar* of M, i.e., the set $\{y \in Y : |\langle x, y \rangle| \leq 1 \ \ \forall x \in M\}$, is occasionally referred to as "the polar" of M. The notions are distinguished, e.g., in [SchW].

2.2 Nests in Banach spaces

This section is based on [OV2].

Let X be a Banach space. Recall that a family of subspaces $\mathcal{N}$ of X is a *nest* if it is linearly ordered by inclusion, i.e., if $Y, Z \in \mathcal{N}$, then $Y \subset Z$ or $Z \subset Y$. By the definition, the following holds.

Proposition 2.3. *Let X be a Banach space. A finite nest $\{Y_1, Y_2, \dots, Y_n\}$ of subspaces of X is nested, e.g., $Y_1 \subset Y_2 \subset \cdots \subset Y_n$.*

A nest $\mathcal{N}$ of closed subspaces of X is said to be *complete* if $\mathcal{N}$ contains $\{0\}$ and X, and $\mathcal{N}$ is closed under arbitrary intersections and closures of arbitrary unions (meaning that

$$\bigcap_{Y \in \mathcal{N}'} Y \in \mathcal{N} \quad \text{and} \quad \overline{\bigcup_{Y \in \mathcal{N}'} Y} \in \mathcal{N}$$

whenever $\mathcal{N}'$ is a non-empty subfamily of $\mathcal{N}$).

Example 2.4. By Proposition 2.3, every finite nest of closed subspaces of a Banach space containing $\{0\}$ and the whole space is a complete nest.

Let X and Y be Banach spaces. Let $\mathcal{G}$ be a nest of subspaces of X^* and let $\mathcal{N}_{\mathcal{G}} = \{V_G : G \in \mathcal{G}\}$ be a nest of subspaces of Y. We say that $\mathcal{N}_{\mathcal{G}}$ is *increasing* on $\mathcal{G}$ if $V_G \subset V_H$ and $V_G \neq V_H$ whenever $G \subset H$ and $G \neq H$ in $\mathcal{G}$. Notice that in this case, $\mathcal{G}$ is also increasing on $\mathcal{N}_{\mathcal{G}}$. Indeed, if $V_G \subset V_H$, $V_G \neq V_H$, then $G \neq H$. We have either $G \subset H$ or $H \subset G$. If $H \subset G$, then $V_H \subset V_G$, which is a contradiction with $V_G \subset V_H$, $V_G \neq V_H$. Therefore, $G \subset H$, $G \neq H$.

We shall use the following easy observation.

Lemma 2.5. *Let X be a Banach space. Let Y and Z be closed subspaces of X. If $S \in \mathcal{L}(X)$, then $S(Y) \subset Z$ if and only if $S^*(Z^\perp) \subset Y^\perp$.*

Proof. We shall include a proof for completeness. The "only if" part is clear. For the "if" part, assume that there exists $S \in \mathcal{L}(X)$ such that $S^*(Z^\perp) \subset Y^\perp$, but there is $y_0 \in Y$ such that $Sy_0 \notin Z$. Using the Hahn–Banach theorem, we can find $x^* \in X^*$ such that $x^*(Sy_0) = 1$ and $x^*(z) = 0$ for all $z \in Z$. Then $x^* \in Z^\perp$. Hence, $S^*x^* \in Y^\perp$ and we have

$$x^*(Sy_0) = (S^*x^*)(y_0) = 0,$$

which is a contradiction with $x^*(Sy_0) = 1$. □

Notice that in the case when $S = I_X$ in Lemma 2.5, it reduces to Lemma 2.6 below.

Lemma 2.6. *Let X be a Banach space. Let Y and Z be closed subspaces of X. Then $Y \subset Z$ if and only if $Z^\perp \subset Y^\perp$. Consequently, $Y = Z$ if and only if $Z^\perp = Y^\perp$.*

If $\mathcal{N}$ is a nest of subspaces of a Banach space X, then we shall denote

$$\mathcal{N}^{\perp} := \{Y^{\perp} : Y \in \mathcal{N}\}$$

and $\mathcal{N}^{\perp\perp} := (\mathcal{N}^{\perp})^{\perp}$.

The following simple result clarifies some interrelations between the nests $\mathcal{N}$, $\mathcal{N}^{\perp}$, and $\mathcal{N}^{\perp\perp}$.

Proposition 2.7. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X.*

1. *The following assertions are equivalent:*
   (a) *$\mathcal{N}^{\perp}$ is closed under arbitrary intersections,*
   (b) *$\mathcal{N}$ is closed under closures of arbitrary unions.*
2. *The following assertions are equivalent:*
   (a) *$\mathcal{N}^{\perp\perp}$ is closed under arbitrary intersections,*
   (b) *$\mathcal{N}^{\perp}$ is closed under closures of arbitrary unions,*
   (c) *$\mathcal{N}$ is closed under arbitrary intersections and*

$$\Big(\bigcap_{Y\in\mathcal{N}'} Y\Big)^{\perp\perp} = \bigcap_{Y\in\mathcal{N}'} Y^{\perp\perp} \text{ for every non-empty subfamily } \mathcal{N}' \text{ of } \mathcal{N}. \quad (*)$$

Proof. Let $\mathcal{N}'$ be an arbitrary non-empty subfamily of $\mathcal{N}$.

1. First notice that, since $\bigcup_{Y\in\mathcal{N}'} Y$ is a subspace of X and the polar of a union coincides with the intersection of the polars (see, e.g., [SchW, p. 126]), by Proposition 2.1, we have

$$\Big(\overline{\bigcup_{Y\in\mathcal{N}'} Y}\Big)^{\perp} = \Big(\bigcup_{Y\in\mathcal{N}'} Y\Big)^{\perp} = \bigcap_{Y\in\mathcal{N}'} Y^{\perp}.$$

Therefore, by Lemma 2.6, the statement

$$\text{“there exists } Z \in \mathcal{N} \text{ such that } Z^{\perp} = \bigcap_{Y\in\mathcal{N}'} Y^{\perp}\text{ ”}$$

is equivalent to

$$\text{“there exists } Z \in \mathcal{N} \text{ such that } Z = \overline{\bigcup_{Y\in\mathcal{N}'} Y}\text{ ”}.$$

Hence, (a) and (b) are equivalent.

2. The equivalence of (a) and (b) is clear from part 1. Also, (c) clearly implies (a).

To prove (b) $\Rightarrow$ (c), assume that (b) holds. Then there exists $Z \in \mathcal{N}$ such that

$$Z^{\perp} = \overline{\bigcup_{Y \in \mathcal{N}'} Y^{\perp}}.$$

Therefore,

$$\bigcap_{Y \in \mathcal{N}'} Y = \bigcap_{Y \in \mathcal{N}'} (Y^{\perp})_{\perp} = \Big(\bigcup_{Y \in \mathcal{N}'} Y^{\perp}\Big)_{\perp} = \Big(\overline{\bigcup_{Y \in \mathcal{N}'} Y^{\perp}}\Big)_{\perp} = Z$$

and

$$\bigcap_{Y \in \mathcal{N}'} Y^{\perp\perp} = \Big(\bigcup_{Y \in \mathcal{N}'} Y^{\perp}\Big)^{\perp} = \Big(\overline{\bigcup_{Y \in \mathcal{N}'} Y^{\perp}}\Big)^{\perp} = Z^{\perp\perp} = \Big(\bigcap_{Y \in \mathcal{N}'} Y\Big)^{\perp\perp}.$$

Hence, (c) holds. $\square$

Remark 2.8. Notice that a nest $\mathcal{N}$ of subspaces of a Banach space satisfies condition $(*)$ if and only if

$$\big(\bigcap_{Y \in \mathcal{N}'} Y\big)^{\perp} = \overline{\bigcup_{Y \in \mathcal{N}'} Y^{\perp}} \text{ for every non-empty subfamily } \mathcal{N}' \text{ of } \mathcal{N}.$$

Indeed, by the proof of part 2 of Proposition 2.7, the equality

$$\bigcap_{Y \in \mathcal{N}'} Y^{\perp\perp} = \Big(\overline{\bigcup_{Y \in \mathcal{N}'} Y^{\perp}}\Big)^{\perp},$$

holds for every non-empty subfamily $\mathcal{N}'$ of $\mathcal{N}$. Therefore, the "if" and "only if" parts are clear by taking $(\,\cdot\,)^{\perp}$ and $(\,\cdot\,)_{\perp}$, respectively.

We have the following immediate consequence of Proposition 2.7.

Corollary 2.9. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X. Then the nest $\mathcal{N}^{\perp}$ is complete if and only if $\mathcal{N}$ is complete and satisfies condition $(*)$.*

2.3 The dual space of the space of finite-rank operators

Let X and Y be Banach spaces.

Recall that, for a functional $x^* \in X^*$ and an element $y \in Y$, the operator $x^* \otimes y : X \to Y$ is defined by $(x^* \otimes y)x = x^*(x)y$, $x \in X$. Clearly, $x^* \otimes y$ has rank one if and only if $x^* \neq 0$ and $y \neq 0$.

It is well known that a mapping $S : X \to Y$ belongs to $\mathcal{F}(X, Y)$ if and only if S can be represented as a finite sum of rank one operators

$$S = \sum_{k=1}^{n} x_k^* \otimes y_k,$$

where $x_1^*, \dots, x_n^* \in X^*$ and $y_1, \dots, y_n \in Y$. In this case, the conjugate operator of S has the representation

$$S^* = \sum_{k=1}^{n} y_k \otimes x_k^*.$$

An operator $T \in \mathcal{L}(X, Y)$ is an *integral operator* if there exists a probability measure space (with the measure μ) and operators $a \in \mathcal{L}(X, L_\infty(\mu))$ and $b \in \mathcal{L}(L_1(\mu), Y^{**})$ such that $j_Y T = b j_1 a$, where $j_1 : L_\infty(\mu) \to L_1(\mu)$ is the identity embedding, meaning that the diagram

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "a"'] & Y \arrow[r, "j_Y"] & Y^{**} \\
L_\infty(\mu) \arrow[rr, "j_1"] & & L_1(\mu) \arrow[u, "b"']
\end{tikzcd}$$

commutes. The *integral norm* $\|T\|_{\mathcal{I}}$ of an integral operator $T \in \mathcal{L}(X, Y)$ is defined by the equality

$$\|T\|_{\mathcal{I}} = \inf \|a\| \, \|b\|,$$

where the infimum is taken over all possible factorizations of T as above. The space of all integral operators from X to Y equipped with their integral norms will be denoted by $\mathcal{I}(X, Y)$. It is straightforward to verify that $\mathcal{I}(X, Y)$ is a Banach space (see, e.g., [DiJT, Theorem 5.2]).

Notice that since we have

$$\|j_Y T\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \quad \forall T \in \mathcal{I}(X, Y)$$

(see, e.g., [Ry, Proposition 3.21] or [DiJT, Theorem 5.14]), there exists a natural isometric embedding $J : \mathcal{I}(X,Y) \to \mathcal{I}(X,Y^{**})$ defined by $JT = j_Y T$ for $T \in \mathcal{I}(X,Y)$.

Integral operators were introduced by Grothendieck in [G, Chapter I, Proposition 27, pp. 124–127] (see, e.g., [DiU, pp. 231–232] or [Ry, p. 58]) with the aim to describe the dual space of an injective tensor product. Without entering into the theory of tensor products, we may reformulate this very important Grothendieck's description in the following way (cf. [O4, pp. 202–203]).

Theorem 2.10 (Grothendieck)**.** *Let X and Y be Banach spaces. Then the dual space $(\mathcal{F}(X,Y), \|\cdot\|)^*$ is linearly isometric with $\mathcal{I}(Y,X^{**})$ under the duality*

$$\Big\langle \sum_{k=1}^{n} x_k^* \otimes y_k, T \Big\rangle = \sum_{k=1}^{n} (Ty_k)(x_k^*)$$

and also with $\mathcal{I}(X^*,Y^*)$ under the duality

$$\Big\langle \sum_{k=1}^{n} x_k^* \otimes y_k, T \Big\rangle = \sum_{k=1}^{n} (Tx_k^*)(y_k).$$

We shall express Theorem 2.10 by writing $(\mathcal{F}(X,Y))^* = \mathcal{I}(Y,X^{**})$ and $(\mathcal{F}(X,Y))^* = \mathcal{I}(X^*,Y^*)$.

The link with tensor products of Banach spaces is that, in fact, $\mathcal{F}(X,Y)$ is algebraically the same as the algebraic tensor product $X^* \otimes Y$, with the rank one operator $x^* \otimes y$ corresponding to the elementary tensor $x^* \otimes y$. The link with the *injective tensor norm* $\varepsilon = \|\cdot\|_\varepsilon$ is that, in fact, $\|S\| = \|\sum_{k=1}^n x_k^* \otimes y_k\|_\varepsilon$ for all $S \in \mathcal{F}(X,Y)$, $S = \sum_{k=1}^n x_k^* \otimes y_k$.

By the above mentioned description due to Grothendieck, the dual of the algebraic tensor product $X \otimes Y \subset \mathcal{F}(X^*,Y)$ (where $X \otimes Y$ is equipped with the usual operator norm) can be identified with $\mathcal{I}(X,Y^*)$, i.e., $(X \otimes Y)^* = \mathcal{I}(X,Y^*)$. This identification is realized via the duality

$$\Big\langle \sum_{k=1}^{n} x_k \otimes y_k, T \Big\rangle = \sum_{k=1}^{n} (Tx_k)(y_k).$$

Let us recall the definition of the *projective tensor norm* $\pi = \|\cdot\|_\pi$ using finite-rank operators (cf. [O4, p. 203]): if $T \in \mathcal{F}(X,Y)$, then $\|T\|_\pi$ is defined by

$$\|T\|_\pi = \inf\Big\{ \sum_{k=1}^{n} \|x_k^*\|\,\|y_k\| : T = \sum_{k=1}^{n} x_k^* \otimes y_k \Big\},$$

where the infimum is taken over all possible representations of T. A straightforward verification shows that $(\mathcal{F}(X,Y), \|\cdot\|_\pi)$ is a normed space (see, e.g., [Ry, Proposition 2.1]). In fact, $(\mathcal{F}(X,Y), \|\cdot\|_\pi) = X^* \otimes_\pi Y := (X^* \otimes Y, \|\cdot\|_\pi)$. The completion of $X^* \otimes_\pi Y$ is called the *projective tensor product* and denoted by $X^* \hat{\otimes}_\pi Y$. Its elements have the following useful representation.

Proposition 2.11 (see, e.g., [Ry, Proposition 2.8])**.** *For every $u \in X^* \hat{\otimes}_\pi Y$ and $\varepsilon > 0$, there exist bounded sequences $(x_k^*) \subset X^*$ and $(y_k) \subset Y$ such that the series $\sum_{k=1}^\infty x_k^* \otimes y_k$ converges to u and $\sum_{k=1}^\infty \|x_k^*\| \, \|y_k\| < \|u\|_\pi + \varepsilon$.*

The next well-known lemma shows that every element of projective tensor product defines a bounded linear operator. We shall include a simple proof for completeness.

Lemma 2.12. *Let X and Y be Banach spaces. For every $u \in X^* \hat{\otimes}_\pi Y$, with a representation*

$$u = \sum_{k=1}^\infty x_k^* \otimes y_k,$$

where $(x_k^*) \subset X^*$, $(y_k) \subset Y$, and $\sum_{k=1}^\infty \|x_k^*\| \, \|y_k\| < \infty$, there exists an operator $\tilde{u} \in \mathcal{L}(X,Y)$ with $\|\tilde{u}\| \le \|u\|_\pi$ such that

$$\tilde{u}x = \sum_{k=1}^\infty x_k^*(x) y_k, \quad x \in X.$$

Proof. Since, algebraically, $X^* \otimes Y = \mathcal{F}(X,Y)$, there exists a natural embedding $j : X^* \otimes_\pi Y \to \mathcal{L}(X,Y)$ defined by

$$(ju)x = \sum_{k=1}^n x_k^*(x) y_k, \ x \in X,$$

where $u = \sum_{k=1}^n x_k^* \otimes y_k \in X^* \otimes_\pi Y$. Notice that j is a bounded linear operator with $\|j\| = 1$. Indeed, clearly, j is linear. For $u = \sum_{k=1}^n x_k^* \otimes y_k \in X^* \otimes_\pi Y$, we have

$$\|(ju)x\| \le \left(\sum_{k=1}^n \|x_k^*\| \, \|y_k\| \right) \|x\| \quad \forall x \in X,$$

and, since this holds for any representation of u, it follows that

$$\|(ju)x\| \le \inf \left\{ \sum_{k=1}^n \|x_k^*\| \, \|y_k\| : u = \sum_{k=1}^n x_k^* \otimes y_k \right\} \|x\| = \|u\|_\pi \, \|x\| \quad \forall x \in X.$$

Hence, $\|ju\| \le \|u\|_\pi$ for all $u \in X^* \otimes_\pi Y$ and thus $\|j\| \le 1$. On the other hand, if $x^* \otimes y \in X^* \otimes_\pi Y$ is an elementary tensor, then

$$\begin{aligned}\|j(x^* \otimes y)\| &= \sup_{\|x\|\le 1} \|(j(x^* \otimes y))x\| = \sup_{\|x\|\le 1} |x^*(x)|\,\|y\| = \|x^*\|\,\|y\| \\ &= \|x^* \otimes y\|_\pi ,\end{aligned}$$

implying that

$$\|x^* \otimes y\|_\pi \le \|j\|\,\|x^* \otimes y\|_\pi .$$

Therefore, $\|j\| = 1$.

Now, since $j \in \mathcal{L}(X^* \otimes_\pi Y, \mathcal{L}(X,Y))$, it can be uniquely extended to a bounded linear operator from $X^*\hat{\otimes}_\pi Y$ to $\mathcal{L}(X,Y)$. After passing to the unique extension, we shall denote

$$\tilde{u} = ju, \quad u \in X^*\hat{\otimes}_\pi Y.$$

Then $\tilde{u} \in \mathcal{L}(X,Y)$ and $\|\tilde{u}\| \le \|u\|_\pi$ for all $u \in X^*\hat{\otimes}_\pi Y$.

Hence, for

$$u = \sum_{k=1}^{\infty} x_k^* \otimes y_k,$$

where $(x_k^*) \subset X^*$, $(y_k) \subset Y$, and $\sum_{k=1}^{\infty} \|x_k^*\|\,\|y_k\| < \infty$, we have

$$\tilde{u}x = \sum_{k=1}^{\infty} x_k^*(x) y_k, \quad x \in X.$$

□

Finally, let us recall that the *trace functional* trace : $X^*\hat{\otimes}_\pi X \to \mathbb{K}$ is defined as follows: if $u \in X^*\hat{\otimes}_\pi Y$ has a representation

$$u = \sum_{k=1}^{\infty} x_k^* \otimes y_k,$$

where $(x_k^*) \subset X^*$, $(y_k) \subset Y$, and $\sum_{k=1}^{\infty} \|x_k^*\|\,\|y_k\| < \infty$, then

$$\operatorname{trace} u = \sum_{k=1}^{\infty} x_k^*(x_k).$$

2.4 Locally convex topologies on the space of bounded linear operators

Let X and Y be Banach spaces.

The *strong operator topology* (SOT or τ_s) on $\mathcal{L}(X,Y)$ is defined by the system of seminorms $\{p_x\}_{x\in X}$ with

$$p_x(S) = \|Sx\|\,, \quad S \in \mathcal{L}(X,Y).$$

The SOT on $X^* = \mathcal{L}(X,\mathbb{K})$ is called the *weak* topology*.

The *weak operator topology* (WOT or τ_w) on $\mathcal{L}(X,Y)$ is defined by the system of seminorms $\{p_{(y^*,x)}\}_{y^*\in Y^*, x\in X}$ with

$$p_{(y^*,x)}(S) = |y^*(Sx)|\,, \quad S \in \mathcal{L}(X,Y).$$

The *topology of uniform convergence on compact sets* (τ_c) on $\mathcal{L}(X,Y)$ is defined by the system of seminorms $\{p_K : K \text{ is a compact subset of } X\}$ with

$$p_K(S) = \sup_{x\in K} \|Sx\|\,, \quad S \in \mathcal{L}(X,Y).$$

Notice that the following statements are true in $\mathcal{L}(X,Y)$.

(a) A net (S_ν) converges to S in the SOT if and only if $S_\nu x \to_\nu Sx$ for all $x \in X$, i.e., $S_\nu \to_\nu S$ *pointwise*.

(b) A net (S_ν) converges to S in the WOT if and only if $y^*(S_\nu x) \to_\nu y^*(Sx)$ for all $x \in X$ and $y^* \in Y^*$.

If $\tau_{\|\cdot\|}$ denotes the norm topology on $\mathcal{L}(X,Y)$, then we have

$$\tau_w \subset \tau_s \subset \tau_c \subset \tau_{\|\cdot\|}.$$

Proposition 2.13. *Let $(S_\nu) \subset \mathcal{L}(X,Y)$ be a bounded net. Then (S_ν) converges to S in the strong operator topology if and only if (S_ν) converges to S in the topology of uniform convergence on compact sets.*

Proof. The "if" part is clear.

For the "only if", let $M > 0$ be such that $\|S_\nu\| \le M$ for all ν and assume that $S_\nu \to_\nu S$ in the SOT, i.e., pointwise. We may assume that $\|S\| \le M$. Let us show that $S_\nu \to_\nu S$ uniformly on compact subsets of X. Let $K \subset X$ be a compact set and let $\varepsilon > 0$. Put $\delta = \varepsilon/(2M+1)$. By the Hausdorff theorem, there exists a finite δ-net $\{x_1, \ldots, x_n\}$ for K. Then for any $x \in K$ there is $k \in \{1, \ldots, n\}$ such that $\|x - x_k\| \le \delta$. Notice that there exists ν_ε such that if $\nu \ge \nu_\varepsilon$, then

$$\|S_\nu x_k - Sx_k\| \le \delta, \quad k = 1, \ldots, n.$$

Indeed, since $S_\nu x_1 \to_\nu Sx_1$, we can find ν_1 such that if $\nu \ge \nu_1$, then $\|S_\nu x_1 - Sx_1\| \le \delta$. Also, since $S_\nu x_2 \to_\nu Sx_2$, we can find ν_2 such that $\nu_2 \ge \nu_1$ and if $\nu \ge \nu_2$, then $\|S_\nu x_2 - x_2\| \le \delta$. This way we can find ν_n such that $\nu_n \ge \nu_{n-1} \ge \cdots \ge \nu_2 \ge \nu_1$ and if $\nu \ge \nu_n$, then $\|S_\nu x_n - Sx_n\| \le \delta$. It suffices to take $\nu_\varepsilon = \nu_n$.

If $\nu \ge \nu_\varepsilon$, then for any $x \in K$, we have

$$\begin{aligned}\|S_\nu x - Sx\| &\le \|S_\nu\| \, \|x - x_k\| + \|S_\nu x_k - Sx_k\| + \|S\| \, \|x_k - x\| \\ &\le (2M+1)\delta = \varepsilon,\end{aligned}$$

as wished. □

Theorem 2.14 (see, e.g., [DuS, Theorem VI.1.4])**.** *The weak and the strong operator topologies on $\mathcal{L}(X, Y)$ yield the same dual space.*

Proposition 2.15. *Let $(S_\nu) \subset \mathcal{L}(X, Y)$ be a net converging to S in the weak operator topology. Then there exists a net (T_μ) consisting of convex combinations of the elements of (S_ν) such that (T_μ) converges to S in the strong operator topology.*

Proof. Let ν_0 be an arbitrary index. Denote $A := \operatorname{conv}\{S_\nu : \nu \ge \nu_0\}$. Then any $T \in A$ can be represented as

$$T = \sum_{k=1}^{n} \lambda_k S_{\nu_k}, \;\; n \in \mathbb{N}, \;\; \nu_k \ge \nu_0, \;\; \lambda_k \ge 0, \;\; \text{and} \;\; \sum_{k=1}^{n} \lambda_k = 1.$$

By assumption, $(S_\nu)_{\nu \ge \nu_0}$ also converges to S in the WOT on $\mathcal{L}(X, Y)$. Since

$$\{S_\nu : \nu \ge \nu_0\} \subset A \subset \overline{A}^{WOT},$$

we have $S \in \overline{A}^{WOT}$ (because $\overline{A}^{WOT}$ is a closed subset in the WOT on $\mathcal{L}(X, Y)$).

Recall that the closure of a convex set in a locally convex space is determined by the dual space (see, e.g., [Day, Corollary 5, p. 25]). Hence, by Theorem 2.14, $\overline{A}^{WOT} = \overline{A}^{SOT}$.

Hence, $S \in \overline{A}^{SOT}$ and thus there exists a net $(T_\mu) \subset A$ converging to S in the SOT on $\mathcal{L}(X,Y)$. $\square$

Recall that Grothendieck's characterization [G] (see, e.g., [LT1, Proposition 1.e.3]) states that, algebraically,

$$(\mathcal{L}(X), \tau_c)^* = X^* \hat{\otimes}_\pi X,$$

under the duality

$$\langle T, u \rangle = \sum_{k=1}^{\infty} x_k^*(Tx_k),$$

where $u = \sum_{k=1}^{\infty} x_k^* \otimes x_k \in X^* \hat{\otimes}_\pi X$ and $T \in \mathcal{L}(X)$.

Chapter 3

Bounded convex approximation properties

In this chapter, we consider and describe various bounded approximation properties and their duality versions. The characterization of the bounded approximation properties defined by arbitrary operator ideals from [O2] is extended to bounded convex approximation properties. We prove that the bounded approximation property of the pair $(X^*, Y^\perp)$, where X is a Banach space and Y is its closed subspace, implies the bounded duality approximation property of (X, Y). This result extends an important theorem from [J1] on classical approximation properties to the approximation properties of pairs. The chapter is based on [OT, OV1, V].

3.1 Definitions

We shall begin with some general definitions which provide unified approach to various approximation properties. Throughout the thesis "approximation property" is often abbreviated to "AP".

Let X be a Banach space and let A be an arbitrary subset of $\mathcal{L}(X)$. The space X has the *A-approximation property* if for every compact subset K of X and for every $\varepsilon > 0$, there exists $S \in A$ such that $\|Sx - x\| \leq \varepsilon$ for all $x \in K$. Let $1 \leq \lambda < \infty$. The space X has the *λ-bounded A-approximation property* if S can be chosen with $\|S\| \leq \lambda$ (meaning that X has the $(A \cap \lambda B_{\mathcal{L}(X)})$-AP). If $\lambda = 1$, then one speaks about the *metric A-approximation property*. The dual space X^* of X is said to have the *(λ-bounded) A-approximation property with conjugate operators* if X^* has the (λ-bounded) $\{S^* : S \in A\}$-AP.

It is convenient to extend the well-known notion of *λ-bounded duality approximation property* (due to [J1]; see, e.g., [C, p. 288] or [S, p. 314]) as follows. We say that X has the *(λ-bounded) duality A-approximation property* if for all compact subsets K of X and L of X^*, there exists an operator $S \in A$ (with $\|S\| \le \lambda$) such that $\|Sx - x\| \le \varepsilon$ for all $x \in K$ and $\|S^*x^* - x^*\| \le \varepsilon$ for all $x^* \in L$. If A is convex and contains 0, then the λ-bounded duality A-AP is equivalent to the λ-bounded A-AP with conjugate operators (see Proposition 3.11 in Section 3.4).

Remark 3.1. It follows from a theorem by Godefroy and Saphar [GS2, Theorem 1.5] (see [LisO, Theorem 5.2] for an alternative proof of this theorem) that if A is a convex subset of $\mathcal{K}(X)$ containing 0, then the duality A-AP of X is always metric whenever X^* or X^{**} has the Radon–Nikodým property.

The concept of A-APs has been studied since the early 1980s by Reinov, Grønbœk, Willis, and others (see, e.g., [BB1], [BB2], [Lis], [LMO] for references and recent results). In the case when the set A is convex and contains 0, we speak about *convex approximation properties*. The study of convex APs was launched in [LisO] (these were occasionally introduced already in [LMO]). The concept includes the following notions (together with their duality and respective bounded versions, which are defined in the standard way).

(1) The *approximation property* of a Banach space X (when $A = \mathcal{F}(X)$).

(2) The *approximation property of pairs* (X, Y), consisting of a Banach space X and a closed subspace Y of X (when $A = \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$).

(3) The *nest approximation property* or the AP of pairs $(X, \mathcal{N})$, consisting of a Banach space X and a nest $\mathcal{N}$ of closed subspaces of X (when $A = \{S \in \mathcal{F}(X) : S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$).

(4) The *positive approximation property* of a Banach lattice X (when A consists of all the positive finite-rank operators, i.e., $A = \mathcal{F}(X)_+$).

Clearly, the APs for X, (X, X), and $(X, \{0\})$ are all equivalent. In their turn, the APs of pairs (X, Y) are the nest APs, namely when $\mathcal{N} = \{Y\}$, or, equivalently, $\mathcal{N} = \{\{0\}, Y, X\}$.

The APs of pairs were introduced and studied by Figiel, Johnson, and Pełczyński in their seminal paper [FJP] (see also [FJ2]). The nest APs are useful extensions of the APs of pairs; these were considered by Figiel and Johnson in [FJ3]. Even versions of the APs of pairs in the context where the space of finite-rank operators $\mathcal{F}(X)$ is replaced by a linear subspace A of $\mathcal{L}(X)$ have been studied (see [CKZ, ChZ]); these are called the A-APs of pairs.

The positive APs date back to 1967 (under the name "order approximation property"; see [Bl] for references and recent results). Szankowski [Sz1] was first to construct a Banach lattice without the AP, and thus without the positive AP. It is still an open question whether the AP implies the positive AP (and similar question holds for the bounded versions of these properties). The last process on this subject was made by Nielsen in [N].

After Enflo's construction of the Banach space failing the AP, examples of Banach spaces without the AP inside classical spaces quickly followed. According to Grothendieck's "Proposition" 37 in [G], there exists a subspace of c_0 failing the AP (see, e.g., [PF] for an explicit construction). Also, the spaces ℓ_p, $p \neq 2$, have subspaces without the AP. The case of $2 < p < \infty$ was shown independently by Davie [Dav] and Figiel [F] in 1973 (see, e.g., [LT1, pp. 86–90]). Examples in the case when $1 \leq p < \infty$, $p \neq 2$, were constructed by Szankowski [Sz2] in 1978 (see, e.g., [LT2, pp. 107–111]). Relying on the concept of bounded APs of pairs, Figiel, Johnson, and Pełczyński showed that c_0 and ℓ_1 have closed subspaces which have the AP, but fail the bounded AP (see [FJP, Corollary 1.13]).

3.2 Reformulating bounded (duality) approximation properties

Well-known reformulations of the λ-bounded AP of Banach spaces (see, e.g., [C, Theorem 3.3] or [S, Theorem 18.1]) can be extended to the following conditions (a) $-$ (d) in Theorem 3.2 below, all equivalent to the λ-bounded A-AP; and also to the conditions (a)$-$(f) in Theorem 3.5, in the next section, all equivalent to the λ-bounded AP of pairs.

Theorem 3.2. *Let X be a Banach space and let A be a subset of $\mathcal{L}(X)$. Let $1 \leq \lambda < \infty$. Then the following properties are equivalent.*

(a) *For every compact subset K of X and for every $\varepsilon > 0$ there exists $S \in A$ with $\|S\| \leq \lambda$ such that $\|Sx - x\| \leq \varepsilon$ for all $x \in K$.*

(b) *There exists a net* $(S_\nu) \subset A$ *satisfying* $\|S_\nu\| \le \lambda$ *for all* ν *such that* $S_\nu \to_\nu I_X$ *uniformly on compact subsets of* X.

(c) *There exists a net* $(S_\nu) \subset A$ *satisfying* $\|S_\nu\| \le \lambda$ *for all* ν *such that* $S_\nu \to_\nu I_X$ *pointwise.*

(d) *For every finite subset* M *of* X *and for every* $\varepsilon > 0$ *there exists* $S \in A$ *with* $\|S\| \le \lambda$ *such that* $\|Sx - x\| \le \varepsilon$ *for all* $x \in M$.

(e) *For every finite-dimensional subspace* E *of* X *and for every* $\varepsilon > 0$ *there exists* $S \in A$ *with* $\|S\| \le \lambda$ *such that* $\|Sx - x\| \le \varepsilon \|x\|$ *for all* $x \in E$.

Proof. The proof uses standard arguments but we shall include a proof for completeness.

To prove (a) $\Rightarrow$ (b), consider the set of all couples $\nu = (K, \varepsilon)$, where $K \subset X$ is a compact subset and $\varepsilon > 0$. Let us define an ordering in the following way:

$$(K_1, \varepsilon_1) \le (K_2, \varepsilon_2) \ \Leftrightarrow\ K_1 \subset K_2 \text{ and } \varepsilon_1 \ge \varepsilon_2.$$

Let us prove that this is indeed an ordering of directed set.

If $\nu = (K, \varepsilon)$, then $K \subset K$ and $\varepsilon \ge \varepsilon$. Therefore, $\nu \le \nu$.

Let $\nu_1 = (K_1, \varepsilon_1)$, $\nu_2 = (K_2, \varepsilon_2)$, $\nu_3 = (K_3, \varepsilon_3)$ be such that $\nu_1 \le \nu_2$ and $\nu_2 \le \nu_3$. In this case $K_1 \subset K_2 \subset K_3$ and $\varepsilon_1 \ge \varepsilon_2 \ge \varepsilon_3$. Hence, $\nu_1 \le \nu_3$.

Let $\nu_1 = (K_1, \varepsilon_1)$, $\nu_2 = (K_2, \varepsilon_2)$. We show that there exists $\nu_3 = (K_3, \varepsilon_3)$ such that $\nu_1 \le \nu_3$ and $\nu_2 \le \nu_3$. Take $K_3 = K_1 \cup K_2$. Since the union of two compact sets is compact, K_3 is compact. Put $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$. Then for $\nu_3 := (K_3, \varepsilon_3)$, we have $\nu_1 \le \nu_3$ and $\nu_2 \le \nu_3$.

Define a net (S_ν) in such a way that for every index ν take the corresponding operator $S_\nu \in A$ from (a). Then $\|S_\nu\| \le \lambda$ for all ν and if $\nu = (K, \varepsilon)$, then $\|S_\nu x - x\| \le \varepsilon$ for all $x \in K$.

Let us show that $S_\nu \to_\nu I_X$ uniformly on compact subsets of X. For any compact subset K_0 of X and for any $\varepsilon_0 > 0$, take $\nu_0 = (K_0, \varepsilon_0)$. If $\nu = (K, \varepsilon) \ge \nu_0$, then $K_0 \subset K$ and

$$\|S_\nu x - x\| \le \varepsilon \le \varepsilon_0 \ \ \forall x \in K_0,$$

as wished.

The implication (b) $\Rightarrow$ (c) is clear, because every one-element set is compact.

To prove (c) $\Rightarrow$ (d), consider a finite subset $M := \{x_1, \dots, x_m\} \subset X$ and let $\varepsilon > 0$. Let (S_ν) be a net from (c). We can find ν_m such that if $\nu \ge \nu_m$, then $\|S_\nu x_k - x_k\| \le \varepsilon$, $k = 1, \dots, m$ (cf. the proof of Proposition 2.13). Put $S = S_{\nu_m}$. Then $\|S\| \le \lambda$ and $\|Sx - x\| \le \varepsilon$ for all $x \in M$.

To prove (d) $\Rightarrow$ (e), let $E \subset X$ be a finite-dimensional subspace with a basis $\{e_1, \dots, e_m\}$ and let $\varepsilon > 0$. Then any $x \in E$ has a representation

$$x = \sum_{k=1}^{m} \lambda_k e_k.$$

Since all norms are equivalent in a finite-dimensional normed space, there exists $c > 0$ such that

$$\sum_{k=1}^{m} |\lambda_k| \le c\,\|x\| \quad \forall x \in E.$$

Let $S \in A$ with $\|S\| \le \lambda$ be an operator from (d) such that

$$\|Se_k - e_k\| \le \varepsilon/c, \quad k = 1, ..., m.$$

Then for any $x \in E$, we also have

$$\begin{aligned} \|Sx - x\| &= \left\| S\left(\sum_{k=1}^{m} \lambda_k e_k\right) - \sum_{k=1}^{m} \lambda_k e_k \right\| = \left\| \sum_{k=1}^{m} \lambda_k (Se_k - e_k) \right\| \\ &\le \sum_{k=1}^{m} |\lambda_k|\,\|Se_k - e_k\| \le \sum_{k=1}^{m} |\lambda_k|\,\varepsilon/c \le \varepsilon\,\|x\|. \end{aligned}$$

To prove (e) $\Rightarrow$ (a), let $K \subset X$ be a compact subset and let $\varepsilon > 0$. According to the Hausdorff theorem there exists a finite $\varepsilon/(\lambda+2)$-net $M := \{x_1, \dots, x_m\}$ for K. Then for every $x \in K$ there exists $x_k \in M$ such that

$$\|x - x_k\| \le \varepsilon/(\lambda + 2).$$

Put $E = \operatorname{span} M$ and $\delta = \varepsilon/\max_k \|x_k\|\,(\lambda + 2)$. By (e), there is $S \in A$ with $\|S\| \le \lambda$ such that $\|Sx - x\| \le \delta\,\|x\|$ for all $x \in E$. Hence,

$$\|Sx_k - x_k\| \le \delta\,\|x_k\| \le \varepsilon/(\lambda + 2)$$

for $k = 1, \dots, m$. For any $x \in K$, we have

$$\begin{aligned} \|Sx - x\| &= \|Sx - Sx_k + Sx_k - x_k + x_k - x\| \\ &\le \|S\|\,\|x - x_k\| + \|Sx_k - x_k\| + \|x_k - x\| \\ &\le \lambda\varepsilon/(\lambda + 2) + 2\varepsilon/(\lambda + 2) = \varepsilon, \end{aligned}$$

as wished. $\square$

To reformulate the λ-bounded duality A-AP, one can apply Theorem 3.3 below.

Theorem 3.3. *Let X be a Banach space and let A be a subset of $\mathcal{L}(X)$. Let $1 \le \lambda < \infty$. Then the following properties are equivalent.*

(a) *For all compact subsets K of X and L of X^*, and for every $\varepsilon > 0$ there exists $S \in A$ with $\|S\| \le \lambda$ such that $\|Sx - x\| \le \varepsilon$ for all $x \in K$ and $\|S^*x^* - x^*\| \le \varepsilon$ for all $x^* \in L$.*

(b) *There exists a net $(S_\nu) \subset A$ satisfying $\|S_\nu\| \le \lambda$ for all ν such that $S_\nu \to_\nu I_X$ and $S_\nu^* \to_\nu I_{X^*}$ uniformly on compact subsets of X and X^*, respectively.*

(c) *There exists a net $(S_\nu) \subset A$ satisfying $\|S_\nu\| \le \lambda$ for all ν such that $S_\nu \to_\nu I_X$ and $S_\nu^* \to_\nu I_{X^*}$ pointwise.*

(d) *For all finite subsets M of X and N of X^*, and for every $\varepsilon > 0$ there exists $S \in A$ with $\|S\| \le \lambda$ such that $\|Sx - x\| \le \varepsilon$ for all $x \in M$ and $\|S^*x^* - x^*\| \le \varepsilon$ for all $x^* \in N$.*

(e) *For all finite-dimensional subspaces E and F of X and X^*, respectively, and for every $\varepsilon > 0$ there exists $S \in A$ with $\|S\| \le \lambda$ such that $\|Sx - x\| \le \varepsilon \|x\|$ for all $x \in E$ and $\|S^*x^* - x^*\| \le \varepsilon \|x^*\|$ for all $x^* \in F$.*

Proof. The proofs are similar to the proofs in Theorem 3.2. We shall point out only the main differences.

To prove (a) $\Rightarrow$ (b), it is sufficient to consider the set of all triples $\nu = (K, L, \varepsilon)$ where $\varepsilon > 0$, $K \subset X$ and $L \subset X^*$ are compact subsets. A net (S_ν) as in (b) can be constructed using the idea from the proof of the respective implication in Theorem 3.2.

The implication (b) $\Rightarrow$ (c) is clear.

To prove (c) $\Rightarrow$ (d), consider finite subsets $M := \{x_1, \ldots, x_m\} \subset X$ and $N := \{x_1^*, \ldots, x_n^*\} \subset X^*$ and let $\varepsilon > 0$. Let (S_ν) be a net from (c). We can find ν_m as in the proof of the corresponding implication in Theorem 3.2 and ν_n such that $\nu_n \ge \nu_m$ and if $\nu \ge \nu_n$, then $\|S_\nu^* x^* - x^*\| \le \varepsilon$ for all $x^* \in N$. Clearly, then also $\|S_\nu x - x\| \le \varepsilon$ for all $x \in M$ and $\nu \ge \nu_n$. It sufficies to put $S = S_{\nu_n}$.

To prove (d) $\Rightarrow$ (e), let $E \subset X$ and $F \subset X^*$ be finite-dimensional subspaces with bases $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$, respectively. Let $\varepsilon > 0$. Take $c > 0$

as in the proof of the corresponding implication in Theorem 3.2 and choose $d \geq c$ such that

$$\sum_{k=1}^{n} |\mu_k| \leq d \|x^*\| \quad \forall x^* = \sum_{k=1}^{n} \mu_k f_k \in F.$$

Let $S \in A$ with $\|S\| \leq \lambda$ be from (d) such that

$$\|Se_k - e_k\| \leq \varepsilon/d, \quad k = 1, ..., m,$$

and

$$\|S^* f_k - f_k\| \leq \varepsilon/d, \quad k = 1, ..., n.$$

It is straightforward to verify that $\|Sx - x\| \leq \varepsilon \|x\|$ for all $x \in E$ and $\|S^* x^* - x^*\| \leq \varepsilon \|x^*\|$ for all $x^* \in F$.

To prove (e) $\Rightarrow$ (a), using the Hausdorff theorem, choose finite $\varepsilon/(\lambda+2)$-nets M and N for given compact subsets $K \subset X$ and $L \subset X^*$, respectively. Put $E = \operatorname{span} M$, $F = \operatorname{span} N$, and $\delta = \varepsilon/(\max_{x \in M} \|x\| + \max_{x^* \in N} \|x^*\|)(\lambda + 2)$. Let $S \in A$ be a operator from (e) such that $\|S\| \leq \lambda$, $\|Sx - x\| \leq \delta \|x\|$ for all $x \in E$ and $\|S^* x^* - x^*\| \leq \delta \|x^*\|$ for all $x^* \in F$. Then $\|Sx - x\| \leq \varepsilon/(\lambda + 2)$ for all $x \in M$, and $\|S^* x^* - x^*\| \leq \varepsilon/(\lambda + 2)$ for all $x^* \in N$. It is easy to verify that S satisfies condition (a). $\square$

3.3 Reformulating bounded (duality) approximation properties of pairs

This section is based on [OT].

We begin with the following auxiliary Lemma 3.4. Its special case when X is a Banach space was applied in the proof of [FJP, Lemma 1.5]. However, even in this special case, we have not found its proof in the literature. Therefore, we include a proof for completeness.

Lemma 3.4. *Let X be a locally convex Hausdorff space. Let Y be a closed subspace and F a finite-dimensional subspace of X. Then there exists a continuous linear projection P on X such that* $\operatorname{ran} P = F$ *and $P(Y) \subset Y$.*

Proof. 1. Let us consider first a particular case, assuming that $F \cap Y = \{0\}$. Let $(x_k)_{k=1}^n$ be a basis of F. Denote $F_m := \operatorname{span}\{x_k : k \neq m\}$. Then $Y + F_m$

is a closed subspace of X (see, e.g., [Ru, Theorem 1.42]) and $x_m \notin Y + F_m$. Indeed, if $x_m \in Y + F_m$, then

$$x_m = y + \sum_{k=1,\ k\neq m}^{n} \lambda_k x_k,$$

where $y \in Y$. Therefore,

$$x_m - \sum_{k=1,\ k\neq m}^{n} \lambda_k x_k \in F \cap Y.$$

Since $F \cap Y = \{0\}$, we have $x_m = \sum_{k=1,\ k\neq m}^{n} \lambda_k x_k$, which is a contradiction with the linear independence of $x_1, \dots x_n$.

According to a separation theorem (see, e.g., [Ru, Theorem 3.5]), there exist continuous linear functionals f_m, $m = 1, ..., n$, such that $f_m(x_m) = 1$ and $f_m|_{Y+F_m} = 0$, in particular $f_m|_Y = 0$ and $f_k(x_m) = \delta_{km}$, $k, m = 1, ..., n$.

Define $P : X \to X$ by

$$Px = \sum_{k=1}^{n} f_k(x) x_k.$$

Then, clearly, $\operatorname{ran} P \subset F$. We also have $F \subset \operatorname{ran} P$ because

$$Px_m = \sum_{k=1}^{n} f_k(x_m) x_k = f_m(x_m) x_m = x_m, \quad m = 1, \dots, n.$$

Let us show that P is continuous. For that it is sufficient to show that for any $k \in \{1, \dots, n\}$, the rank one operator $f_k \otimes x_k$ is continuous. Fix $k \in \{1, \dots, n\}$. Let $\mathcal{P}$ be a family of semi-norms generating the topology of X. Recall that $f_k \otimes x_k$ is continuous if and only if for each continuous semi-norm q on X, there exists a finite subset $\mathcal{P}'$ of $\mathcal{P}$ and a number $c > 0$ such that $q((f_k \otimes x_k)x) \leq c \max\{p(x) : p \in \mathcal{P}'\}$ for all $x \in X$ (see, e.g., [SchW, p. 74]). Since f_k is continuous, there is a finite subset $\mathcal{P}_k$ of $\mathcal{P}$ and $c_k > 0$ such that $|f_k(x)| \leq c_k \max\{p(x) : p \in \mathcal{P}_k\}$ for all $x \in X$. Let q be any continuous semi-norm on X. Then for any $x \in X$, we have

$$q((f_k \otimes x_k)x) = q(f_k(x)x_k) = |f_k(x)|\, q(x_k) \leq c_k q(x_k) \max\{p(x) : p \in \mathcal{P}_k\}.$$

It suffices to put $\mathcal{P}' = \mathcal{P}_k$ and $c = c_k q(x_k)$.

Since $Px_k = x_k$, $k = 1, ..., n$, we have $P^2 = P$, i.e., P is a projection.

For any $y \in Y$, we also have

$$Py = \sum_{k=1}^{n} f_k(y)x_k = \sum_{k=1}^{n} 0x_k = 0,$$

implying that $P(Y) = \{0\} \subset Y$.

2. Let us show that the general case can be reduced to the particular case above.

We start by decomposing $F = (F \cap Y) \oplus G$. Then $(F \cap Y) \cap G = \{0\}$ and $G \cap Y = \{0\}$. Hence, by the above, there exist continuous linear projections Q and R on X such that $\operatorname{ran} Q = F \cap Y$, $Q(G) \subset G$, and $\operatorname{ran} R = G$, $R(Y) \subset Y$. Define $P = Q + R - RQ$. Then P is continuous and linear, and $P(Y) \subset Y$ (because $\operatorname{ran} Q, R(Y) \subset Y$).

Clearly, $\operatorname{ran} P \subset F$. Let us show that $F \subset \operatorname{ran} P$. Notice that every $f \in F$ has a unique representation $f = y + g$, where $y \in F \cap Y$ and $g \in G$. Since $Qy = y$ and $Rg = g$, we have

$$\begin{aligned} Pf &= (Q + R - RQ)(y + g) = y + Ry - Ry + Qg + g - RQg \\ &= y + QRg + g - RQRg. \end{aligned}$$

Notice also that $QR = 0$ (because $Q(G) \subset (F \cap Y) \cap G = \{0\}$), implying that

$$Pf = y + g = f.$$

Therefore, $\operatorname{ran} P = F$ and $P^2 = P$. Hence, P is a continuous linear projection on X, as desired. □

Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. The λ-bounded AP of the pair (X, Y) and its duality version may be equivalently expressed using Theorems 3.5 and 3.6 below. The original notion of λ-bounded AP of the pair (X, Y) in [FJP] was defined as condition (f) of Theorem 3.5. The λ-bounded duality AP of the pair (X, Y) was introduced in [OT] as property (c) of Theorem 3.6. Note that we preferred to define the notions through conditions (a).

Theorem 3.5 (cf. [FJP, Lemma 1.5])**.** *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. Then the following properties of the pair (X, Y) are equivalent.*

(a) *For every compact subset K of X and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda$ such that $S(Y) \subset Y$ and $\|Sx - x\| \leq \varepsilon$ for all $x \in K$.*

(b) *There exists a net* $(S_\nu) \subset \mathcal{F}(X)$ *such that* $\|S_\nu\| \le \lambda$ *and* $S_\nu(Y) \subset Y$ *for all* ν, *and* $S_\nu \to_\nu I_X$ *uniformly on compact subsets of* X.

(c) *There exists a net* $(S_\nu) \subset \mathcal{F}(X)$ *such that* $\|S_\nu\| \le \lambda$ *and* $S_\nu(Y) \subset Y$ *for all* ν, *and* $S_\nu \to_\nu I_X$ *pointwise.*

(d) *For every finite subset* M *of* X *and for every* $\varepsilon > 0$ *there exists* $S \in \mathcal{F}(X)$ *with* $\|S\| \le \lambda$ *such that* $S(Y) \subset Y$ *and* $\|Sx - x\| \le \varepsilon$ *for all* $x \in M$.

(e) *For every finite-dimensional subspace* E *of* X *and for every* $\varepsilon > 0$ *there exists* $S \in \mathcal{F}(X)$ *with* $\|S\| \le \lambda$ *such that* $S(Y) \subset Y$ *and* $\|Sx - x\| \le \varepsilon \|x\|$ *for all* $x \in E$.

(f) *For every finite-dimensional subspace* E *of* X *and for every* $\varepsilon > 0$ *there exists* $S \in \mathcal{F}(X)$ *with* $\|S\| \le \lambda + \varepsilon$ *such that* $S(Y) \subset Y$ *and* $Sx = x$ *for all* $x \in E$.

Proof. Let $A = \{S \in \mathcal{F}(X) : \ S(Y) \subset Y\}$. The equivalence of (a) – (e) is immediate from Theorem 3.2.

The implication (e) $\Rightarrow$ (f) is proved in [FJP, Lemma 1.5]. We shall include a proof for completeness. Let $E \subset X$ be a finite-dimensional subspace and let $\varepsilon > 0$. Using Lemma 3.4, choose a projection $P \in \mathcal{F}(X)$ such that $\operatorname{ran} P = E$ and $P(Y) \subset Y$.

Let $\delta > 0$ be such that $\delta \|P\| < \varepsilon$. By (e), there is $T \in \mathcal{F}(X)$ with $\|T\| \le \lambda$ such that $T(Y) \subset Y$ and $\|Tx - x\| \le \delta \|x\|$ for all $x \in E$.

Put
$$S = P + T - TP = P + T(I_X - P).$$
Then, clearly, $S \in \mathcal{F}(X)$ and S is the identity on E. Also, $S(Y) \subset Y$ because $Py, Ty \in Y$ for all $y \in Y$.

Finally, let us observe that
$$\|S - T\| = \|TP - P\| = \sup_{x \in B_X} \|TPx - Px\| \le \sup_{x \in B_X} \delta \|Px\| = \delta \|P\| \le \varepsilon.$$

Hence, we have
$$\|S\| \le \|T\| + \|S - T\| \le \lambda + \varepsilon.$$

To prove (f) $\Rightarrow$ (e), let $E \subset X$ be a finite-dimensional subspace and let $\varepsilon > 0$. By (f), there is $T \in \mathcal{F}(X)$ with $\|T\| \le \lambda + \varepsilon$ such that $T(Y) \subset Y$ and $Tx = x$ for all $x \in E$.

Put $S = \lambda T/(\lambda + \varepsilon)$. Then $S \in \mathcal{F}(X)$, $S(Y) \subset Y$ and

$$\|S\| = \lambda \|T\| / (\lambda + \varepsilon) \leq \lambda.$$

Notice that $Sx = \lambda x/(\lambda + \varepsilon)$ for all $x \in E$. Hence, for any $x \in E$, we have

$$\|Sx - x\| = |\lambda/(\lambda + \varepsilon) - 1| \|x\| = \varepsilon/(\lambda + \varepsilon) \|x\| < \varepsilon \|x\|,$$

as wished. □

Theorem 3.6. *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. Then the following properties of the pair (X, Y) are equivalent.*

(a) *For all compact subsets K of X and L of X^*, and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda$ such that $S(Y) \subset Y$, $\|Sx - x\| \leq \varepsilon$ for all $x \in K$ and $\|S^*x^* - x^*\| \leq \varepsilon$ for all $x^* \in L$.*

(b) *There exists a net $(S_\nu) \subset \mathcal{F}(X)$ such that $\|S_\nu\| \leq \lambda$ and $S_\nu(Y) \subset Y$ for all ν, and $S_\nu \to_\nu I_X$ and $S_\nu^* \to_\nu I_{X^*}$ uniformly on compact subsets of X and X^*, respectively.*

(c) *There exists a net $(S_\nu) \subset \mathcal{F}(X)$ such that $\|S_\nu\| \leq \lambda$ and $S_\nu(Y) \subset Y$ for all ν, and $S_\nu \to_\nu I_X$ and $S_\nu^* \to_\nu I_{X^*}$ pointwise.*

(d) *For all finite subsets M of X and N of X^*, and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda$ such that $S(Y) \subset Y$, $\|Sx - x\| \leq \varepsilon$ for all $x \in M$ and $\|S^*x^* - x^*\| \leq \varepsilon$ for all $x^* \in N$.*

(e) *For all finite-dimensional subspaces E of X and F of X^*, and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda$ such that $S(Y) \subset Y$, $\|Sx - x\| \leq \varepsilon \|x\|$ for all $x \in E$ and $\|S^*x^* - x^*\| \leq \varepsilon \|x^*\|$ for all $x^* \in F$.*

(f) *For all finite-dimensional subspaces E of X and F of X^*, and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda + \varepsilon$ such that $S(Y) \subset Y$, $Sx = x$ for all $x \in E$ and $S^*x^* = x^*$ for all $x^* \in F$.*

Proof. The equivalence of (a) – (e) is immediate from Theorem 3.3 (in the special case when $A = \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$).

To prove (e) ⇒ (f), let $E \subset X$ and $F \subset X^*$ be finite-dimensional subspaces and let $\varepsilon > 0$. Using Lemma 3.4, choose a projection $P \in \mathcal{L}(X)$ such that $\operatorname{ran} P = E$ and $P(Y) \subset Y$. Look at X^* endowed with its weak* topology.

Notice that $Y^\perp$ is weak* closed. Indeed, let $f \in X^*$ and let $(f_\nu) \subset Y^\perp$ be a net converging to f in the weak* topology of X^*, i.e., $f_\nu \to_\nu f$ pointwise. Then, for any $y \in Y$, $f(y) = \lim_\nu f_\nu(y) = 0$. Hence, $f \in Y^\perp$.

Using Lemma 3.4 again, choose a weak*-to-weak* continuous linear projection R on X^* such that $\operatorname{ran} R = F$ and $R(Y^\perp) \subset Y^\perp$. Then there exists $Q \in \mathcal{L}(X)$ such that $R = Q^*$ (see, e.g., [M, Theorem 3.1.11]).

Since $Q^*(Y^\perp) \subset Y^\perp$, we have, by Lemma 2.5, $Q(Y) \subset Y$.

Let $\delta > 0$ satisfy

$$\delta(\|P\| + \|I_X - P\| \, \|Q\|) < \varepsilon.$$

According to (e), there is $T \in \mathcal{F}(X)$ with $\|T\| \le \lambda$ such that $T(Y) \subset Y$, $\|Tx - x\| \le \delta \|x\|$ for all $x \in E$ and $\|T^*x^* - x^*\| \le \delta \|x^*\|$ for all $x^* \in F$.

Applying a perturbation argument inspired by [OP, proof of Lemma 1.2], we denote

$$S := I_X + (I_X - Q)(T - I_X)(I_X - P),$$

i.e.,

$$S = T + P - TP + Q - QT + QTP - QP.$$

Then, clearly, $S \in \mathcal{F}(X)$. Since $Px = x$, i.e., $(I_X - P)x = 0$, for all $x \in E = \operatorname{ran} P$ and $Q^*x^* = x^*$, i.e., $(I_{X^*} - Q^*)x^* = 0$, for all $x^* \in F = \operatorname{ran} Q^*$, S is the identity on E and $S^* = I_{X^*} + (I_{X^*} - P^*)(T^* - I_{X^*})(I_{X^*} - Q^*)$ is the identity on F. Also, $S(Y) \subset Y$ because $Py,\ Ty,\ Qy \in Y$ for all $y \in Y$.

Finally, let us observe that

$$S = T + (I_X - T)P - Q(T - I_X)(I_X - P).$$

Let us also observe that

$$\|(I_X - T)P\| = \sup_{x \in B_X} \|TPx - Px\| \le \sup_{x \in B_X} \delta \|Px\| = \delta \|P\|$$

and

$$\begin{aligned} \|Q(T - I_X)(I_X - P)\| &\le \|I_X - P\| \, \|Q(T - I_X)\| \\ &= \|I_X - P\| \, \|(T^* - I_{X^*})Q^*\| \end{aligned}$$

with

$$\begin{aligned} \|(T^* - I_{X^*})Q^*\| &= \sup_{x^* \in B_{X^*}} \|T^*Q^*x^* - Q^*x^*\| \le \sup_{x^* \in B_{X^*}} \delta \|Q^*x^*\| \\ &= \delta \|Q^*\| = \delta \|Q\| . \end{aligned}$$

Hence, we have

$$\begin{aligned}\|S-T\| &\le \|(I_X-T)P\| + \|Q(T-I_X)(I_X-P)\| \\ &\le \delta(\|P\| + \|I_X-P\|\,\|Q\|) \le \varepsilon\end{aligned}$$

and

$$\|S\| \le \|T\| + \|S-T\| \le \lambda + \varepsilon.$$

To prove (f) $\Rightarrow$ (e), it is easy to verify that it suffices to take S as in the corresponding implication in Theorem 3.5. □

Remark 3.7. The λ-bounded duality AP of a Banach space X is usually defined as the special case of either property (c) or (e) in Theorem 3.6 when $Y = \{0\}$ (equivalently, $Y = X$). The equivalence of (e) and (f) in the context of the λ-bounded duality AP of a Banach space X has been established in [J1, Lemma 3] (see also, e.g., [S, Lemma 9.2]).

3.4 Describing bounded convex approximation properties

This section is based on [OV1, V].

Let X be a Banach space. In [O2, Theorem 2.1], Oja characterizes bounded A-APs, in the special case when A is a component of an arbitrary operator ideal (i.e., $A = \mathcal{A}(X,X)$, where $\mathcal{A}$ is an operator ideal), via elementary functionals defined on A. However, by the proof of this theorem, A could also be assumed to be a linear subspace of $\mathcal{L}(X)$ (see Theorem 3.8 below).

Recall that if A is a linear subspace of $\mathcal{L}(X)$, $x^{**} \in X^{**}$, and $x^* \in X^*$, then an *elementary functional* $x^* \otimes x^{**} : A \to \mathbb{K}$ is defined by the equality

$$(x^* \otimes x^{**})(T) = x^{**}(T^*x^*), \quad T \in A.$$

Clearly, $x^* \otimes x^{**} \in A^*$ and $\|x^* \otimes x^{**}\| \le \|x^*\|\,\|x^{**}\|$.

Theorem 3.8 (cf. [O2, Theorem 2.1])**.** *Let X be a Banach space and let A be a linear subspace of $\mathcal{L}(X)$. Let $1 \le \lambda < \infty$. Then*

(a) *X has the λ-bounded A-approximation property if and only if there exists $\Phi \in A^{**}$ such that $\|\Phi\| \le \lambda$ and*

$$\Phi(x^* \otimes j_X x) = x^*(x) \quad \forall x^* \in X^*,\ \forall x \in X;$$

(b) *X^* has the λ-bounded A-approximation property with conjugate operators if and only if there exists $\Phi \in A^{**}$ such that $\|\Phi\| \leq \lambda$ and*

$$\Phi(x^* \otimes x^{**}) = x^{**}(x^*) \ \ \forall x^* \in X^*, \ \forall x^{**} \in X^{**}.$$

The following Theorem 3.9 extends Theorem 3.8 to bounded convex APs. Its proof is inspired by the proof of [O2, Theorem 2.1]. In Theorem 3.9 we consider elementary functionals defined on $\mathcal{L}(X)$ and the duality $\langle(\mathcal{L}(X))^{**}, (\mathcal{L}(X))^*\rangle$.

Theorem 3.9. *Let X be a Banach space and let A be a bounded convex subset of $\mathcal{L}(X)$ containing 0. Then*

(a) *X has the A-approximation property if and only if there exists $\Phi \in A^{\circ\circ} \subset (\mathcal{L}(X))^{**}$ such that*

$$\Phi(x^* \otimes j_X x) = x^*(x) \ \ \forall x^* \in X^*, \ \forall x \in X;$$

(b) *X^* has the A-approximation property with conjugate operators if and only if there exists $\Phi \in A^{\circ\circ} \subset (\mathcal{L}(X))^{**}$ such that*

$$\Phi(x^* \otimes x^{**}) = x^{**}(x^*) \ \ \forall x^* \in X^*, \ \forall x^{**} \in X^{**}.$$

Proof. First, notice that $\overline{j_{\mathcal{L}(X)}A}^{w^*}$ is bounded whenever A is. In fact, $\overline{j_{\mathcal{L}(X)}A}^{w^*} \subset \lambda B_{(\mathcal{L}(X))^{**}}$ whenever $A \subset \lambda B_{\mathcal{L}(X)}$. Indeed, take $S \in \overline{j_{\mathcal{L}(X)}A}^{w^*}$, then there exists a net $(S_\nu) \subset j_{\mathcal{L}(X)}A \subset \lambda B_{(\mathcal{L}(X))^{**}}$ such that $S_\nu \to_\nu S$ in the weak* topology of $(\mathcal{L}(X))^{**}$, i.e., pointwise. Therefore, for any $f \in (\mathcal{L}(X))^*$, we have

$$|Sf| = \left|\lim_\nu S_\nu f\right| = \lim_\nu |S_\nu f| \leq \sup_\nu \|S_\nu\| \, \|f\| \leq \lambda \, \|f\| \, .$$

Hence, $\|S\| \leq \lambda$, i.e., $S \in \lambda B_{(\mathcal{L}(X))^{**}}$, as needed.

For the "only if" part, consider a net $(S_\nu) \subset A$. For (a) assume that $S_\nu \to_\nu I_X$ pointwise, and for (b) assume that $S_\nu^* \to_\nu I_{X^*}$ pointwise.

Since $\overline{j_{\mathcal{L}(X)}A}^{w^*}$ is bounded and weak* closed, by the Banach–Alaoglu theorem (see, e.g., [DuS, Corollary V.4.3]), it is weak* compact. We also have $(j_{\mathcal{L}(X)}S_\nu) \subset j_{\mathcal{L}(X)}A \subset \overline{j_{\mathcal{L}(X)}A}^{w^*}$. After passing to a weak* converging subnet, we may assume that there exists a weak* limit of (S_ν). Take Φ to be equal to that weak* limit. Then we have

$$\Phi f = \lim_\alpha f(S_\nu), \quad f \in (\mathcal{L}(X))^*,$$

$\Phi \in (\mathcal{L}(X))^{**}$, and $\Phi \in \overline{j_{\mathcal{L}(X)}A}^{w^*}$. By the bipolar theorem (see, e.g., [SchW, Theorem 1.5]),

$$A^{\circ\circ} = \overline{\operatorname{conv}(j_{\mathcal{L}(X)}A \cup \{0\})}^{w^*}.$$

Since A is convex and contains 0,

$$A^{\circ\circ} = \overline{j_{\mathcal{L}(X)}A}^{w^*}.$$

Therefore, $\Phi \in A^{\circ\circ}$.

In the case (a), we have

$$\begin{aligned}\Phi(x^* \otimes j_X x) &= \lim_\nu (x^* \otimes j_X x)(S_\nu) = \lim_\nu (j_X x)(S_\nu^* x^*) = \lim_\nu (S_\nu^* x^*)(x) \\ &= \lim_\nu x^*(S_\nu x) = x^*(\lim_\nu S_\nu x) = x^*(x),\end{aligned}$$

for all $x^* \in X^*$ and $x \in X$. In the case (b), we have

$$\begin{aligned}\Phi(x^* \otimes x^{**}) &= \lim_\nu (x^* \otimes x^{**})(S_\nu) = \lim_\nu x^{**}(S_\nu^* x^*) = x^{**}(\lim_\nu S_\nu^* x^*) \\ &= x^{**}(x^*),\end{aligned}$$

for all $x^* \in X^*$ and $x^{**} \in X^{**}$.

For the "if" part, let $\Phi \in A^{\circ\circ}$. Then, by the bipolar theorem, $\Phi \in \overline{j_{\mathcal{L}(X)}A}^{w^*}$. Hence, there exists a net $(S_\nu) \subset A$ such that $j_{\mathcal{L}(X)}S_\nu \to_\nu \Phi$ in the weak* topology of $(\mathcal{L}(X))^{**}$. In particular, for any $x^* \otimes x^{**} \in (\mathcal{L}(X))^*$,

$$\lim_\nu x^{**}(S_\nu^* x^*) = \lim_\nu (x^* \otimes x^{**})(S_\nu) = \Phi(x^* \otimes x^{**}),$$

and for any $x^* \otimes j_X x \in (\mathcal{L}(X))^*$,

$$\lim_\nu (S_\nu^* x^*)(x) = \lim_\nu (x^* \otimes j_X x)(S_\nu) = \Phi(x^* \otimes j_X x).$$

In case $\Phi(x^* \otimes j_X x) = x^*(x)$, as in (a), we have

$$\lim_\nu x^*(S_\nu x) = \lim_\nu (S_\nu^* x^*)(x) = \Phi(x^* \otimes j_X x) = x^*(x),$$

meaning that $S_\nu \to_\nu I_X$ in the WOT on $\mathcal{L}(X)$. By passing to convex combinations, we may assume that $S_\nu \to_\nu I_X$ in the SOT on $\mathcal{L}(X)$, i.e., pointwise.

Hence, by Theorem 3.2, X has the A-AP.

In case $\Phi(x^* \otimes x^{**}) = x^{**}(x^*)$, as in (b), we have

$$\lim_\alpha x^{**}(S_\nu^* x^*) = \Phi(x^* \otimes x^{**}) = x^{**}(x^*),$$

meaning that $S_\nu^* \to_\nu I_{X^*}$ in the WOT on $\mathcal{L}(X^*)$. By passing to convex combinations, we may assume that $S_\nu^* \to_\nu I_{X^*}$ pointwise. Hence, by Theorem 3.2, X^* has the $\{S^* : S \in A\}$-AP, i.e., X^* has the A-AP with conjugate operators. □

Remark 3.10. Notice that if A is a linear subspace of $\mathcal{L}(X)$ and $A \cap \lambda B_{\mathcal{L}(X)}$ is considered in the place of A in Theorem 3.9, then it reduces to Theorem 3.8. Indeed, in this case, by the Goldstine theorem, $j_{\mathcal{L}(X)}A$ is weak* dense in A^{**}. Thus, by the bipolar theorem, $A^{\circ\circ} = A^{**}$. We have $\Phi \in (A \cap \lambda B_{\mathcal{L}(X)})^{\circ\circ} \subset A^{\circ\circ} = A^{**}$. On the other hand, $(A \cap \lambda B_{\mathcal{L}(X)})^{\circ\circ} \subset \lambda(B_{\mathcal{L}(X)})^{\circ\circ}$ and $\lambda(B_{\mathcal{L}(X)})^{\circ\circ} = \lambda B_{(\mathcal{L}(X))^{**}}$. Therefore, $\|\Phi\| \le \lambda$.

Application of Theorem 3.9 is a lifting result for metric convex approximation properties from a Banach space, with the unique extension property, to its dual space (see Section 5.5).

In the case of bounded convex approximation properties, the following result is useful. For instance, it shows that "X^* has the A-AP with conjugate operators" in Theorem 3.9 (b) can be replaced with "X has the duality A-AP".

Proposition 3.11. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. Let $1 \le \lambda < \infty$. Then the following properties are equivalent.*

(a) *X has the λ-bounded duality A-approximation property.*

(b) *X^* has the λ-bounded A-approximation property with conjugate operators.*

(c) *There exists a net $(S_\nu) \subset A$ such that*

$$\limsup_\nu \|S_\nu\| \le \lambda$$

and

$$x^{**}(S_\nu^* x^*) \to_\nu x^{**}(x^*) \quad \forall x^* \in X^*,\ \forall x^{**} \in X^{**}.$$

In the proof of this proposition, we shall need the following simple lemma.

Lemma 3.12. *Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. Let $1 \le \lambda < \infty$. Then X has the λ-bounded duality A-approximation property if and only if X has the $(\lambda + \varepsilon)$-bounded duality A-approximation property for all $\varepsilon > 0$.*

Proof. The "only if" part is obvious and holds for any subset $A \subset \mathcal{L}(X)$.

For the "if" part, let us show that condition (d) of Theorem 3.3 holds. For that, let $M \subset X$ and $N \subset X^*$ be finite subsets. Let $\varepsilon > 0$. Put $\delta = \varepsilon/(\lambda + \max_{x \in M} \|x\| + \max_{x^* \in N} \|x^*\|)$. Then $\delta(\lambda + \|x\|) \leq \varepsilon$ for all $x \in M$ and $\delta(\lambda + \|x^*\|) \leq \varepsilon$ for all $x^* \in N$. By assumption (and Theorem 3.3 (d)), there is $T \in A$ with $\|T\| \leq \lambda + \delta$ such that $\|Tx - x\| \leq \delta$ for all $x \in M$ and $\|T^*x^* - x^*\| \leq \delta$ for all $x^* \in N$.

Put $S = (\lambda/(\lambda + \delta))T$. Then $S \in A$ because A is convex and contains 0. Clearly, $\|S\| \leq \lambda$. For any $x \in M$,

$$\begin{aligned}\|Sx - x\| &= \|(\lambda/(\lambda + \delta))Tx - x\| = \|\lambda(Tx - x) - \delta x\| / (\lambda + \delta) \\ &\leq \lambda \|Tx - x\| + \delta \|x\| \leq \delta(\lambda + \|x\|) \leq \varepsilon.\end{aligned}$$

Similarly, for $x^* \in N$, we have

$$\|S^*x^* - x^*\| = \|(\lambda/(\lambda + \delta))T^*x^* - x^*\| \leq \delta(\lambda + \|x^*\|) \leq \varepsilon,$$

as wished. □

Proof of Proposition 3.11. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious ((b) $\Rightarrow$ (c) holds thanks to Theorem 3.2 (c)). Notice that these hold for any subset $A \subset \mathcal{L}(X)$.

To prove (c) $\Rightarrow$ (a), we shall show that X has the $(\lambda + \varepsilon)$-bounded duality A-AP for all $\varepsilon > 0$. Then, by Lemma 3.12, condition (a) holds.

Let $(S_\nu) \subset A$ be a net from (c). Notice that

$$\limsup_\nu \|S_\nu\| \leq \lambda$$

is equivalent to the fact that for every $\varepsilon > 0$ there exists ν_ε such that if $\nu \geq \nu_\varepsilon$, then $\|S_\nu\| \leq \lambda + \varepsilon$. By assumption, $(S_\nu)_{\nu \geq \nu_\varepsilon}$ converges to I_{X^*} in the WOT on $\mathcal{L}(X^*)$. By passing to convex combinations, we may assume that $S_\nu^* \to_\nu I_{X^*}$ in the SOT on $\mathcal{L}(X^*)$, i.e. pointwise. In such a case, $S_\nu \to_\nu I_X$ in the WOT. By passing to convex combinations again, we may assume that $S_\nu \to_\nu I_X$ pointwise. By Theorem 3.3, X has the $(\lambda + \varepsilon)$-bounded duality A-AP.

Hence, X has the $(\lambda + \varepsilon)$-bounded duality A-AP for all $\varepsilon > 0$. □

Remark 3.13. Let A be a convex subset of $\mathcal{L}(X)$ containing 0. Consider the following statements.

(a) X has the duality A-approximation property.

(b) X^* has the A-approximation property with conjugate operators.

(c) There exists a net $(S_\nu) \subset A$ such that

$$x^{**}(S_\nu^* x^*) \to_\nu x^{**}(x^*) \quad \forall x^* \in X^*,\ \forall x^{**} \in X^{**}.$$

Notice that then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold (even when $A \subset \mathcal{L}(X)$ is just a subset). However, the implication (c) $\Rightarrow$ (a) does not hold in general. Indeed, in the particular case when $A = \mathcal{F}(X)$, condition (c) holds in every Banach space (see Proposition 3.14 below), but there exist Banach spaces without the AP and for these spaces the statement in (a) is false.

Proposition 3.14. *Let X be a Banach space. Then there exists a net $(S_\nu) \subset \mathcal{F}(X)$ such that*

$$x^{**}(S_\nu^* x^*) \to_\nu x^{**}(x^*) \quad \forall x^* \in X^*,\ \forall x^{**} \in X^{**}.$$

Proof. Consider a set of all couples $\nu = (F, \varepsilon)$, where $F \subset X^*$ is a finite-dimensional subspace and $\varepsilon > 0$, directed in the natural way. Fix $\nu = (F, \varepsilon)$. By the Auerbach lemma (see, e.g., [Pi, B.4.9]), there exists a projection $P_\nu \in \mathcal{F}(X^*)$ such that $\operatorname{ran} P_\nu = F$. Using the version of the principle of local reflexivity, according to which finite-rank operators on a dual space are "locally conjugate", (see, e.g., [OP, Theorem 2.5]), we can find $S_\nu \in \mathcal{F}(X)$ such that $\operatorname{ran} S_\nu^* = \operatorname{ran} P_\nu$ and $S_\nu^* x^* = P_\nu x^*$ for all $x^* \in F$. Notice that then $S_\nu^* \to_\nu I_{X^*}$ pointwise. Indeed, let $x^* \in X^*$ be arbitrary. For $\varepsilon_0 > 0$, put $\nu_0 = (\operatorname{span}\{x^*\}, \varepsilon_0)$. Now, if $\nu = (F, \varepsilon) \geq \nu_0$, then $x^* \in \operatorname{span}\{x^*\} \subset F$ and $\varepsilon \leq \varepsilon_0$. We have $S_\nu^* x^* = P_\nu x^* = x^*$, implying that $\|S_\nu^* x^* - x^*\| = 0 < \varepsilon_0$, as wished. □

3.5 Duality of bounded approximation properties of pairs

This section is based on [OT].

Let X be a Banach space. Let $1 \leq \lambda < \infty$. By the well-known Enflo–James–Lindenstrauss result (see, e.g., [LT1, p. 34]), the metric AP of X does not imply the AP of the dual space X^*. However, if X^* has the λ-bounded AP, then also X has the same property. This result is essentially due to Grothendieck (proved in [G, Chapter I, p. 180] for the metric AP), but its essence resides in the following important result due to Johnson [J1] (see, e.g., [C, Proposition 3.5] or [S, Proposition 9.8]).

Theorem 3.15 (Johnson)**.** *Let X be a Banach space. Let $1 \leq \lambda < \infty$. If X^* has the λ-bounded approximation property, then X^* has the λ-bounded approximation property with conjugate operators.*

Theorem 3.15 follows easily from the version of the principle of local reflexivity (see, e.g., [C, Proposition 3.5]), according to which finite-rank operators on a dual space are "locally conjugate" (see, e.g., [OP, Theorem 2.5]). We shall include a proof.

Proof. Let $F \subset X^*$ be a finite-dimensional subspace and let $\varepsilon > 0$. By Theorem 3.5 (f), it is sufficient to show that there is $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda + \varepsilon$ such that $S^* x^* = x^*$ for all $x^* \in F$. By assumption (and Theorem 3.5 (f)), there is $T \in \mathcal{F}(X^*)$ with $\|T\| \leq \lambda + \varepsilon/2$ such that $Tx^* = x^*$ for all $x^* \in F$. Then, by [OP, Theorem 2.5], there exists $S \in \mathcal{F}(X)$ such that $|\|S\| - \|T\|| < \varepsilon/2$ and $S^* x^* = Tx^*$ for all $x^* \in F$. Hence, $\|S\| < \|T\| + \varepsilon/2 \leq \lambda + \varepsilon$ and $S^* x^* = x^*$ for all $x^* \in F$, as wished. □

Remark 3.16. Already in the special case when $A = \mathcal{K}(X)$, the notions "A-AP with conjugate operators" and "A-AP" for the dual space differ. Grønbœk and Willis showed in [GW, Example 4.3] that there exists a Banach space with its dual space having the bounded $\mathcal{K}(X)$-AP, but failing the $\mathcal{K}(X)$-AP with conjugate operators. Therefore, in general, Theorem 3.15 cannot be extended to A-APs.

Consider the following duality conditions.

(a) (X, Y) has the λ-bounded AP.

(a*) $(X^*, Y^\perp)$ has the λ-bounded AP.

According to the discussion in the beginning of this section, the implication (a) $\Rightarrow$ (a*) does not hold in general. But the implication (a*) $\Rightarrow$ (a) holds for an arbitrary X in the particular case when $Y = \{0\}$ (or $Y = X$).

Proposition 3.17. *If X is a reflexive Banach space, then conditions* (a) *and* (a*) *are equivalent.*

Proof. We shall describe the λ-bounded APs of pairs using the criterion (c) in Theorem 3.5.

To prove (a) $\Rightarrow$ (a*), let $(S_\nu) \subset \mathcal{F}(X)$ be such that $\|S_\nu\| \leq \lambda$ and $S_\nu(Y) \subset Y$ for all ν, and $S_\nu \to_\nu I_X$ in the SOT on $\mathcal{L}(X)$. We have $S_\nu^{**} = S_\nu$ for all ν (because X is reflexive), implying that $S_\nu^{**} \to_\nu I_{X^{**}}$ in the SOT on $\mathcal{L}(X^{**})$. In that case $S_\nu^* \to_\nu I_{X^*}$ in the WOT on $\mathcal{L}(X^*)$. By passing to convex combinations, we may assume that $S_\nu^* \to_\nu I_X^*$ in the SOT on $\mathcal{L}(X^*)$, i.e., pointwise. Clearly, $(S_\nu^*) \subset \mathcal{F}(X^*)$ and $\|S_\nu^*\| = \|S_\nu\| \leq \lambda$ for all ν. Also for any ν, $S_\nu^*(Y^\perp) \subset Y^\perp$ (because $S_\nu(Y) \subset (Y)$). Therefore, $(X^*, Y^\perp)$ has the λ-bounded AP.

For (a*) $\Rightarrow$ (a), notice that if $(X^*, Y^\perp)$ has the λ-bounded AP, then, by the implication (a) $\Rightarrow$ (a*), $(X^{**}, Y^{\perp\perp})$ has the λ-bounded AP. Since X is reflexive, we have $X^{**} = X$ and $Y^{\perp\perp} = (Y^\perp)_\perp = Y$. Hence, (X, Y) has the λ-bounded AP. $\square$

Concerning Proposition 3.17, it may be added that, by [LisO, Corollary 5.3], the AP of a pair (X, Y) is always metric whenever X is reflexive.

It is natural to say that the *pair* $(X^*, Y^\perp)$ has the λ-*bounded approximation property with conjugate operators* if X^* has the λ-bounded $\{S \in \mathcal{F}(X) : S(Y) \subset Y\}$-AP with conjugate operators. The principal result of this section is as follows (see also Theorem 3.21).

Theorem 3.18. *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. The pair $(X^*, Y^\perp)$ has the λ-bounded approximation property if and only if $(X^*, Y^\perp)$ has the λ-bounded approximation property with conjugate operators.*

Theorem 3.18 extends Theorem 3.15 from X^* to $(X^*, Y^\perp)$, showing that the implication (a*) $\Rightarrow$ (a) holds in full generality.

Let X and Y be Banach spaces. The following Lemma 3.19 will concern some structure of Banach spaces of integral operators considered as dual spaces of the space $\mathcal{F}(X, Y)$ of finite-rank operators. We shall use Lemma 3.19 (or, more precisely, its Corollary 3.20) in the proof of the "only if" part of Theorem 3.18. It uses the canonical identifications $(\mathcal{F}(X))^* = \mathcal{I}(X^*, X^*)$ and $(\mathcal{F}(X^*))^* = \mathcal{I}(X^*, X^{***})$ from Theorem 2.10.

Lemma 3.19. *Let X be a Banach space and let Y be a closed subspace of X. Denote*

$$\mathcal{R} = \{R \in \mathcal{F}(X) : R(Y) \subset Y\}$$

and

$$\mathcal{S} = \{S \in \mathcal{F}(X^*) : S(Y^\perp) \subset Y^\perp\},$$

and consider $\mathcal{R}^\perp$ and $\mathcal{S}^\perp$ as subspaces of $\mathcal{I}(X^*, X^*)$ and $\mathcal{I}(X^*, X^{***})$, respectively. If $T \in \mathcal{I}(X^*, X^*)$ is such that $T \in \mathcal{R}^\perp$, then $j_{X^*}T \in \mathcal{S}^\perp$.

Proof. Let $T \in \mathcal{R}^\perp$, i.e., $\langle R, T\rangle = 0$ for every $R \in \mathcal{R}$. We have to show that $\langle S, j_{X^*}T\rangle = 0$ for every $S \in \mathcal{S}$. Suppose that $S \in \mathcal{S}$, i.e., $S \in \mathcal{F}(X^*)$ and $S(Y^\perp) \subset Y^\perp$. Consider a representation of S

$$S = \sum_{k=1}^{n} x_k^{**} \otimes x_k^*,$$

where $(x_k^*)_{i=k}^n \subset X^*$ and $(x_k^{**})_{k=1}^n \subset X^{**}$.

Since $Y^\perp$ is a linear subspace of X^*, $Y^\perp$ is algebraically complemented. This means that there is a linear subspace W of X^* such that $X^* = W \oplus Y^\perp$, i.e., $X^* = W + Y^\perp$ and $W \cap Y^\perp = \{0\}$. Thus, for every $x^* \in X^*$ there is a unique representation $x^* = w + y^\perp$, where $w \in W$ and $y^\perp \in Y^\perp$. Since $x_k^* = w_k + y_k^\perp$, where $w_k \in W$ and $y_k^\perp \in Y^\perp$, we have $S = S_1 + S_2$, where

$$S_1 = \sum_{k=1}^{n} x_k^{**} \otimes w_k \quad\text{and}\quad S_2 = \sum_{k=1}^{n} x_k^{**} \otimes y_k^\perp.$$

Note that $\operatorname{ran} S_1 \subset W$. Let $(\bar{w}_k)_{k=1}^m \subset W$ be an algebraic basis of $\operatorname{ran} S_1$. Then there is a system $(\bar{x}_k^{**})_{k=1}^m \subset X^{**}$ such that

$$S_1 = \sum_{k=1}^{m} \bar{x}_k^{**} \otimes \bar{w}_k$$

(see, e.g., [FHH$^+$, p. 203]). Let $x^* \in Y^\perp$ be arbitrary. Since $S(Y^\perp) \subset Y^\perp$,

$$Sx^* = S_1x^* + S_2x^* = \sum_{k=1}^{m} \bar{x}_k^{**}(x^*)\bar{w}_k + \sum_{k=1}^{n} x_k^{**}(x^*)y_k^\perp \in Y^\perp,$$

implying that

$$\sum_{k=1}^{m} \bar{x}_k^{**}(x^*)\bar{w}_k \in W \cap Y^\perp = \{0\}.$$

Since $\bar{w}_1, \ldots, \bar{w}_m$ are linearly independent, it follows that $\bar{x}_k^{**}(x^*) = 0$ for all $k = 1, \ldots, m$ and for all $x^* \in Y^\perp$, i.e., $(\bar{x}_i^{**})_{i=1}^k \subset Y^{\perp\perp}$.

Let us consider the canonical isometry $I : Y^{\perp\perp} \to Y^{**}$, defined by $(Iy^{\perp\perp})(y^*) = y^{\perp\perp}(x^*)$, where $y^* \in Y^*$, $y^{\perp\perp} \in Y^{\perp\perp}$, and $x^* \in X^*$ is an arbitrary extension of y^* (see, e.g., [M, Proposition 1.11.14]). Then $(I\bar{x}_k^{**})_{k=1}^m \subset Y^{**}$ and, by Theorem 2.10, we get

$$\begin{aligned}\langle S, j_{X^*}T\rangle &= \langle S_1, j_{X^*}T\rangle + \langle S_2, j_{X^*}T\rangle \\ &= \sum_{k=1}^m (j_{X^*}T)(\bar{w}_k)(\bar{x}_k^{**}) + \sum_{k=1}^n (j_{X^*}T)(y_k^\perp)(x_k^{**}) \\ &= \sum_{k=1}^m \bar{x}_k^{**}(T\bar{w}_k) + \sum_{k=1}^n x_k^{**}(Ty_k^\perp) \\ &= \sum_{k=1}^m (I\bar{x}_k^{**})(T\bar{w}_k|_Y) + \sum_{k=1}^n x_k^{**}(Ty_k^\perp).\end{aligned}$$

Denote $y_k^{**} := I\bar{x}_k^{**} \in Y^{**}$ and $y_k^* := T\bar{w}_k|_Y \in Y^*$, and choose elements $y_k \in Y$, $k = 1, ..., m$, such that $y_k^{**}(y_k^*) = y_k^*(y_k)$. Also choose $x_k \in X$, $k = 1, ..., n$, such that $x_k^{**}(Ty_k^\perp) = (Ty_k^\perp)(x_k)$. (Such elements exist. Indeed, let Z be a normed space, $z^* \in Z^*$, $z^{**} \in Z^{**}$, and denote $a := z^{**}(z^*)$. If $a = 0$, then $a = z^*(z)$ for $z = 0$. If $a \neq 0$, then there is $w \in Z$ such that $b := z^*(w) \neq 0$. Take $z = ab^{-1}w$. Then $a = ab^{-1}b = ab^{-1}z^*(w) = z^*(z)$.) Using elements y_k and x_k, define

$$R := \sum_{k=1}^m \bar{w}_k \otimes y_k + \sum_{k=1}^n y_k^\perp \otimes x_k \in \mathcal{F}(X).$$

Then $R(Y) \subset Y$, because for every $y \in Y$,

$$Ry = \sum_{k=1}^m \bar{w}_k(y)y_k + \sum_{k=1}^n y_k^\perp(y)x_k = \sum_{k=1}^m \bar{w}_k(y)y_k \in Y.$$

Hence, $R \in \mathcal{R}$ and therefore $\langle R, T\rangle = 0$. On the other hand,

$$\begin{aligned}\langle R, T\rangle &= \sum_{k=1}^m (T\bar{w}_k)(y_k) + \sum_{k=1}^n (Ty_k^\perp)(x_k) = \sum_{k=1}^m y_k^*(y_k) + \sum_{k=1}^n x_k^{**}(Ty_k^\perp) \\ &= \sum_{k=1}^m y_k^{**}(y_k^*) + \sum_{k=1}^n x_k^{**}(Ty_k^\perp) = \sum_{k=1}^m (I\bar{x}_k^{**})(T\bar{w}_k|_Y) + \sum_{k=1}^n x_k^{**}(Ty_k^\perp) \\ &= \langle S, j_{X^*}T\rangle.\end{aligned}$$

Hence $\langle S, j_{X^*}T\rangle = 0$, as desired. □

Recall from Section 2.3 that there exists a natural isometric embedding $J : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^{***})$ defined by $JT = j_{X^*}T$ for $T \in \mathcal{I}(X^*, X^*)$.

Corollary 3.20. *Let $\mathcal{R}$ and $\mathcal{S}$ be as in Lemma 3.19, and let*

$$\bar{J} : \mathcal{I}(X^*, X^*)/\mathcal{R}^\perp \to \mathcal{I}(X^*, X^{***})/\mathcal{S}^\perp$$

be defined by

$$\bar{J}(T + \mathcal{R}^\perp) = JT + \mathcal{S}^\perp,\ T \in \mathcal{I}(X^*, X^*),$$

where $J : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^{***})$ is the natural isometric embedding. Then $\bar{J}$ is a well-defined operator, $\|\bar{J}\| \le 1$, and $\bar{J}q_1 = q_2J$, where $q_1 : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^*)/\mathcal{R}^\perp$ and $q_2 : \mathcal{I}(X^*, X^{***}) \to \mathcal{I}(X^*, X^{***})/\mathcal{S}^\perp$ denote the quotient mappings.

Proof. Notice that since, by Lemma 3.19, $J(\mathcal{R}^\perp) \subset \mathcal{S}^\perp$, the definition of $\bar{J}$ is correct. Indeed, if $T_1, T_2 \in \mathcal{I}(X^*, X^*)$ are such that $T_1 + \mathcal{R}^\perp = T_2 + \mathcal{R}^\perp$, then $T_1 - T_2 \in \mathcal{R}^\perp$. It follows that $J(T_1 - T_2) \in \mathcal{S}^\perp$, i.e., $JT_1 + \mathcal{S}^\perp = JT_2 + \mathcal{S}^\perp$.

Clearly, $\bar{J}$ is linear. Also, we have $q_2J = \bar{J}q_1$. Indeed, for any $T \in \mathcal{I}(X^*, X^*)$,

$$(q_2J)(T) = q_2(JT) = JT + \mathcal{S}^\perp = \bar{J}(T + \mathcal{R}^\perp) = \bar{J}(q_1T) = (\bar{J}q_1)(T).$$

It follows that

$$\|\bar{J}\| = \|\bar{J}q_1\| = \|q_2J\| \le \|J\| = 1,$$

as wished. □

Proof of Theorem 3.18. The "if" part of Theorem 3.18 is clear because if $S(Y) \subset Y$, then $S^*(Y^\perp) \subset Y^\perp$.

To prove the "only if" part, let $\mathcal{R} = \{R \in \mathcal{F}(X) : R(Y) \subset Y\}$ and $\mathcal{S} = \{S \in \mathcal{F}(X^*) : S(Y^\perp) \subset Y^\perp\}$ be as in Lemma 3.19.

Assume that the pair $(X^*, Y^\perp)$ has the λ-bounded AP. This is the same as the λ-bounded $\mathcal{S}$-AP of X^*. According to Theorem 3.8 (a), there exists $\Phi \in \mathcal{S}^{**}$ such that $\|\Phi\| \le \lambda$ and

$$\Phi(x^{**} \otimes j_{X^*}x^*) = x^{**}(x^*)\ \ \forall x^* \in X^*,\ \forall x^{**} \in X^{**}.$$

Since $\mathcal{R} \subset \mathcal{F}(X)$ and $\mathcal{S} \subset \mathcal{F}(X^*)$, $\mathcal{R}^*$ and $\mathcal{S}^*$ are isometrically isomorphic to $(\mathcal{F}(X))^*/\mathcal{R}^\perp$ and $(\mathcal{F}(X^*))^*/\mathcal{S}^\perp$, respectively (see, e.g., [M, Theorem 1.10.16]). Hence, under the canonical identifications (see Theorem 2.10), $\mathcal{R}^*$ and $\mathcal{S}^*$ are isometrically isomorphic to $\mathcal{I}(X^*, X^*)/\mathcal{R}^\perp$ and $\mathcal{I}(X^*, X^{***})/\mathcal{S}^\perp$, respectively. Let $r : \mathcal{R}^* \to \mathcal{I}(X^*, X^*)/\mathcal{R}^\perp$ and $s : \mathcal{I}(X^*, X^{***})/\mathcal{S}^\perp \to \mathcal{S}^*$ denote the corresponding isometric isomorphisms.

Let $i_1 : \mathcal{R} \to \mathcal{F}(X)$ and $i_2 : \mathcal{S} \to \mathcal{F}(X^*)$ be the identity embeddings and let $q_1 : \mathcal{I}(X^*, X^*) \to \mathcal{I}(X^*, X^*)/\mathcal{R}^\perp$ and $q_2 : \mathcal{I}(X^*, X^{***}) \to \mathcal{I}(X^*, X^{***})/\mathcal{S}^\perp$

denote the quotient mappings (as in Corollary 3.20). Then, under canonical identifications (see Theorem 2.10), we have $i_1^* = r^{-1}q_1$ and $i_2^* = sq_2$.

Define $\Psi : \mathcal{R}^* \to \mathbb{K}$ by $\Psi = \Phi s\bar{J}r$, where $\bar{J}$ is the operator from Corollary 3.20. Then the diagrams

$$\begin{tikzcd}
\mathcal{R}^* \arrow[r, "\Psi"] \arrow[d, "r"] & \mathbb{K} & \mathcal{S}^* \arrow[l, "\Phi"'] \\
\mathcal{I}(X^*, X^*)/\mathcal{R}^\perp \arrow[rr, "\bar{J}"] & & \mathcal{I}(X^*, X^{***})/\mathcal{S}^\perp \arrow[u, "s"'] \\
\mathcal{I}(X^*, X^*) \arrow[u, "q_1"] \arrow[rr, "J"] & & \mathcal{I}(X^*, X^{***}) \arrow[u, "q_2"']
\end{tikzcd}$$

commute, $\Psi \in \mathcal{R}^{**}$ and $\|\Psi\| \le \|\Phi\| \le \lambda$.

Let $x^* \in X^*$ and $x^{**} \in X^{**}$. We shall show that $\Psi(x^* \otimes x^{**}) = x^{**}(x^*)$.

Denote $f := x^* \otimes x^{**} \in \mathcal{R}^*$. Then $f(R) = x^{**}(R^*x^*)$ for all $R \in \mathcal{R}$. Notice that if $\bar{f} := x^{**} \otimes x^* \in \mathcal{I}(X^*, X^*)$ is a rank one operator, then $\langle R, \bar{f}\rangle = x^{**}(R^*x^*)$ for all $R \in \mathcal{F}(X)$. Indeed, let $R = \sum_{k=1}^n x_k^* \otimes x_k \in \mathcal{F}(X)$ be arbitrary. Then $R^* = \sum_{k=1}^n x_k \otimes x_k^*$ and we have

$$\begin{aligned}
\langle R, \bar{f}\rangle &= \sum_{k=1}^n (\bar{f}x_k^*)(x_k) = \sum_{k=1}^n x^{**}(x_k^*)x^*(x_k) = x^{**}\left(\sum_{k=1}^n x^*(x_k)x_k^*\right) \\
&= x^{**}(R^*x^*).
\end{aligned}$$

Hence, $f(R) = \bar{f}(i_1R) = i_1^*\bar{f}(R)$ for all $R \in \mathcal{R}$, implying that $f = i_1^*\bar{f} = r^{-1}q_1\bar{f}$.

Now denote $g := x^{**} \otimes j_{X^*}x^* \in \mathcal{S}^*$ and consider the rank one operator $\bar{g} := x^{**} \otimes j_{X^*}x^* \in \mathcal{I}(X^*, X^{***})$. In this case, we have $g(S) = (S^*x^{**})(x^*) = x^{**}(Sx^*)$ for all $S \in \mathcal{S}$ and $\langle S, \bar{g}\rangle = x^{**}(Sx^*)$ for all $S \in \mathcal{F}(X^*)$. Hence, $g(S) = \bar{g}(i_2S) = i_2^*\bar{g}(S)$ for all $S \in \mathcal{S}$, implying that $g = i_2^*\bar{g} = sq_2\bar{g}$.

Therefore, since $\bar{g} = j_{X^*}\bar{f}$ and $\Phi(g) = x^{**}(x^*)$, we get

$$\begin{aligned}
\Psi(x^* \otimes x^{**}) &= \Psi(f) = \Psi(r^{-1}q_1\bar{f}) = \Phi(s\bar{J}rr^{-1}q_1\bar{f}) = \Phi(s\bar{J}q_1\bar{f}) \\
&= \Phi(sq_2J\bar{f}) = \Phi(sq_2j_{X^*}\bar{f}) = \Phi(sq_2\bar{g}) = \Phi(g) \\
&= x^{**}(x^*).
\end{aligned}$$

According to Theorem 3.8 (b), X^* has the λ-bounded $\mathcal{S}$-AP with conjugate operators, i.e., $(X^*, Y^\perp)$ has the λ-bounded AP with conjugate operators. $\square$

Using Proposition 3.11, we may reformulate Theorem 3.18 as follows.

Theorem 3.21. *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. The pair $(X^*, Y^\perp)$ has the λ-bounded approximation property if and only if the pair (X, Y) has the λ-bounded duality approximation property.*

Remark 3.22. The version of Theorem 3.21, stating that *the pair $(X^*, Y^\perp)$ has the AP if and only if the pair (X, Y) has the duality AP*, was essentially established in [LisO, Proposition 5.11]. The proof in [LisO] uses the principle of local reflexivity (see Theorem 4.1). Also the special case of Theorem 3.21, when Y is of finite codimension, can be proved using the principle of local reflexivity. We could not figure out how to use the principle of local reflexivity, even in its most sophisticated form (see [Beh1] or, e.g., [OP, Theorem 2.4]), to prove Theorem 3.18. In [O5, Corollary 4.2], an alternative proof was given to Theorem 3.18. The proof is based on a new version of the principle of local reflexivity, also established in [O5].

Remark 3.23. If X^* or X^{**} has the Radon–Nikodým property, then [LisO, Proposition 5.11] states the following: *if $(X^*, Y^\perp)$ has the AP, then (X, Y) has the metric duality AP* (cf. Remark 3.1).

Let us conclude this section with applications to the lifting of the λ-bounded AP from (X, Y) to $(X^*, Y^\perp)$ in some special cases.

In [FJP, Corollary 1.3], Figiel, Johnson, and Pełczyński proved that *if X is a Banach space, $q : X \to Z$ is a quotient map, and $\dim \ker q < \infty$, then the λ-bounded AP of X implies the same property of Z.* Note, that their proof (a straightforward one, which only uses condition (f) of Theorem 3.5) actually yields the following auxiliary result.

Lemma 3.24 (see the proof of [FJP, Corollary 1.3])**.** *Let X be a Banach space and let Y be a finite-dimensional subspace of X. Let $1 \leq \lambda < \infty$. If X has the λ-bounded approximation property, then also the pair (X, Y) has the λ-bounded approximation property.*

Proof. We shall present a proof for completeness. Let $F \subset X$ be a finite-dimensional subspace. Then $E := Y + F$ is also a finite-dimensional subspace of X. By assumption, the pair (X, X) has the λ-bounded AP. Hence, by Theorem 3.5 (f), for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ such that $\|S\| \leq$

$\lambda + \varepsilon$ and $Sx = x$ for all $x \in E$. Hence, we also have $S(Y) \subset Y$ (because $Y \subset E$). Therefore, by Theorem 3.5, the pair (X, Y) has the λ-bounded AP, as wished. $\square$

The equivalence of conditions (a) and (c) below was established in [FJP, Proposition 1.6]. We can complement this result as follows, providing also an alternative proof for the implication (a) $\Rightarrow$ (c).

Theorem 3.25 (cf. [FJP, Proposition 1.6]). *Let X be a Banach space and let $1 \le \lambda < \infty$. Then the following conditions are equivalent.*

(a) *The dual space X^* has the λ-bounded approximation property.*

(b) *The pair $(X^*, Y^\perp)$ has the λ-bounded approximation property for every finite-codimensional closed subspace Y of X.*

(c) *The pair (X, Y) has the λ-bounded approximation property for every finite-codimensional closed subspace Y of X.*

Proof. Since $Y^\perp$ is a finite-dimensional subspace of X^*, the implication (a) $\Rightarrow$ (b) is immediate from Lemma 3.24. The implication (b) $\Rightarrow$ (c) is clear from Theorem 3.21 because, obviously, the λ-bounded duality AP of the pair (X, Y) implies its λ-bounded AP. The implication (c) $\Rightarrow$ (a) is proved in [FJP, Proposition 1.6]. $\square$

Finally, let us mention that, as a by-product, we have the following slight complement to [FJP, Corollary 1.4 (i)], asserting that *if the dual space X^* has the λ-bounded AP, then all finite-codimensional closed subspaces Y of X and their dual spaces Y^* have the λ-bounded AP.*

Proposition 3.26. *Let X be a Banach space and let Y be a finite-codimensional closed subspace of X. Let $1 \le \lambda < \infty$. If X^* has the λ-bounded approximation property, then all spaces $Y^\perp$, $X^*/Y^\perp$, Y^*, $(X/Y)^*$, X, Y and X/Y have the λ-bounded approximation property.*

Proof. By [FJP, Corollary 1.2], the λ-bounded AP of a pair (Z, W) implies that Z, W and Z/W all have the same property. According to Theorem 3.25, both pairs $(X^*, Y^\perp)$ and (X, Y) have the λ-bounded AP. Hence, $Y^\perp$, $X^*/Y^\perp$, X, Y, and X/Y all have the λ-bounded AP. Since the dual spaces Y^* and $(X/Y)^*$ are naturally isometric to $X^*/Y^\perp$ and $Y^\perp$, respectively, Y^* and $(X/Y)^*$ also have the λ-bounded AP. $\square$

In general, the bounded AP cannot be lifted from a closed subspace Y of X to the pair (X, Y). Indeed, let X be a Banach space without the AP and Y a finite-dimensional subspace of X. Then Y has the metric AP (because $I_Y \in \mathcal{F}(Y)$), but (X, X) does not have the AP. Hence, the pair (X, Y) also cannot have the AP, in particular, it fails the metric AP.

In [FJP, Proposition 1.8], it was shown that in the special case when the closed subspace Y of X has finite codimension, then the λ-bounded AP of Y implies that (X, Y) has the 3λ-bounded AP.

Also, in general, the bounded AP cannot be lifted from the quotient space X/Y to (X, Y). Indeed, let X be a Banach space without the AP. Let $f \in X^*$ be any non-zero functional. Put $Y = \ker f$. Notice that then $\dim X/Y = 1$. Indeed, since $(X/Y)^*$ is isometrically isomorphic to $Y^\perp$ (see, e.g., [M, Theorem 1.10.17]) and $\operatorname{ran} f^* = (\ker f)^\perp$, we have $\dim(X/Y)^* = \dim(\ker f)^\perp = 1$. Hence, $\dim X/Y = 1$. It follows that X/Y has the metric AP. By the discussion above, the pair (X, Y) fails the metric AP.

The following question is still open (see [FJP, Problem 6.1]). *If X, Y, and X/Y all have the bounded AP, then does the pair (X, Y) have the bounded AP?*

Chapter 4

Principle of local reflexivity respecting nests of subspaces and the nest approximation properties

In this chapter, we establish versions of the principle of local reflexivity which respect nests of subspaces. We prove a rather far-reaching extension of the Ringrose theorem on nests. We also extend a duality result on approximation properties of pairs from [LisO] and its bounded version from Chapter 3 to the context of nest approximation properties. Criteria of the nest approximation properties from [FJ3] are applied to obtain criteria of the duality nest approximation properties in the spirit of Grothendieck. This chapter is based on [OV2].

4.1 The principle of local reflexivity

The *principle of local reflexivity* (PLR) states that the bidual X^{**} of a Banach space X and the space X itself are "locally almost the same". Theorem 4.1 below seems to be the most well-known and widely used version of the PLR.

Theorem 4.1 (PLR in [JRZ]). *Let X be a Banach space. For all finite-dimensional subspaces E of X^{**} and F of X^*, and for every $\varepsilon > 0$ there exists a one-to-one operator $T \in \mathcal{F}(E, X)$ such that $\|T\|, \|T^{-1}\| < 1 + \varepsilon$,*

$Tx = x$ for all $x \in X \cap E$, and $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$.

The PLR was discovered by Lindenstrauss and Rosenthal [LR] in 1969. It was improved by Johnson, Rosenthal, and Zippin [JRZ] in 1971. Since then, many new proofs, refinements, and generalizations of the PLR have been given in the literature (see, e.g., [Beh2] and [OP] for results and references). For instance, there is a version of the PLR for Banach lattices due to Conroy and Moore [CoM], and Bernau [Ber], revisited in [LisO]. Recently, with the aim to study the APs of pairs, a PLR respecting subspaces was established by Oja in [O5].

We shall propose some versions of the PLR that respect given nests of subspaces so that these could be applied to the nest APs; see Theorems 4.10, 4.11, and 4.13 in Section 4.4. These theorems will be based on Lemma 4.7, our main PLR lemma, which we shall prove in Section 4.3. The main PLR lemma, in its turn, will essentially use (through Lemma 4.8) a rather far-reaching extension of the Ringrose theorem (see Theorem 4.3), which is established in the following section.

4.2 Extension of the Ringrose theorem on nests

Let X be a Banach space. Let $\mathcal{N}$ be a nest of subspaces of X containing $\{0\}$. For $Y \in \mathcal{N}$, we define the *subspace* Y_- of X as follows. If $Y \neq \{0\}$, then

$$Y_- = \bigcup \{H \in \mathcal{N} : H \subset Y,\ H \neq Y\},$$

and if $Y = \{0\}$, then $Y_- = \{0\}$.

We start by recalling the Ringrose theorem on nests from the 1960s (see [Er, Lemma 2 (this is Lemma 3.3 in [Ri]) and Theorem 1]).

Theorem 4.2 (Ringrose)**.** *Let $\mathcal{N}$ be a complete nest of closed subspaces of a Hilbert space H. Denote $\mathcal{R} = \{R \in H \otimes H : R(G) \subset G\ \forall G \in \mathcal{N}\}$.*

(a) *Let $R = x \otimes y$ be a rank one operator. Then $R \in \mathcal{R}$ if and only if there is a subspace G in $\mathcal{N}$ such that $x \in (G_-)^\perp$, the orthogonal complement of G_-, and $y \in G$.*

(b) *Let $R \in H \otimes H$ be an operator of rank $n > 0$. If $R \in \mathcal{R}$, then R can be written as the sum of n operators of rank one, each belonging to $\mathcal{R}$.*

Theorem 4.2 has been extended from the Hilbert space case to more general settings of Banach spaces (see [FJ3] for related references) and even of topological vector spaces (see [Sp, Lemma 1 and Theorem 2]). To prove our main PLR lemma (Lemma 4.7), we need the following far-reaching extension of the Ringrose theorem.

Theorem 4.3. *Let X and Y be Banach spaces. Let $\mathcal{G}$ be a nest of subspaces of X^* containing $\{0\}$ and let $\mathcal{N}_\mathcal{G} = \{V_G : G \in \mathcal{G}\}$ be a nest of closed subspaces of Y containing Y. Assume that $\mathcal{N}_\mathcal{G}$ is closed under arbitrary intersections and $\mathcal{N}_\mathcal{G}$ is increasing on $\mathcal{G}$. Denote $\mathcal{R} = \{R \in X \otimes Y : R(G) \subset V_G \ \ \forall G \in \mathcal{G}\}$.*

(a) *Let $R = x \otimes y$ be a rank one operator. Then $R \in \mathcal{R}$ if and only if there is a subspace G in $\mathcal{G}$ such that $x \in (G_-)_\perp$ and $y \in V_G$.*

(b) *Let $R \in X \otimes Y$ be an operator of rank $n > 0$. If $R \in \mathcal{R}$, then R can be written as the sum of n operators of rank one, each belonging to $\mathcal{R}$.*

Proof. The proof will be modelled after the proof of [FJ3, Lemma 1].

(a) For the "only if" part, assume that $x \otimes y \in \mathcal{R}$. Denote

$$V_G := \bigcap_{H \in \mathcal{G}} \{V_H : y \in V_H\} \in \mathcal{N}_\mathcal{G}.$$

(Such a $G \in \mathcal{G}$ exists because $\mathcal{N}_\mathcal{G}$ is closed under intersections.) Clearly, $y \in V_G$.

If $G = \{0\}$, then $G_- = \{0\}$ and $(G_-)_\perp = X$. Hence, $x \in (G_-)_\perp$. Let $G \neq \{0\}$. If $x \notin (G_-)_\perp$, then there is $H \in \mathcal{G}$ such that $H \subset G$, $H \neq G$, together with $h \in H$ such that $h(x) \neq 0$. Therefore, as $\mathcal{N}_\mathcal{G}$ is increasing, we have $V_H \subset V_G$ and $V_G \neq V_H$. Since $x \otimes y \in \mathcal{R}$, $(x \otimes y)h = h(x)y \in V_H$, implying that $y \in V_H$, because $h(x) \neq 0$. Hence, $V_G \subset V_H$, which is a contradiction with $V_H \subset V_G$ and $V_G \neq V_H$. Therefore, $x \in (G_-)_\perp$.

For the "if" part, assume that there exists $G \in \mathcal{G}$ such that $x \in (G_-)_\perp$ and $y \in V_G$. Let us show that then $x \otimes y \in \mathcal{R}$, i.e., $(x \otimes y)(H) \subset V_H$ for every $H \in \mathcal{G}$. Let $H \in \mathcal{G}$. If $G \subset H$, then $y \in V_H$. Hence, $(x \otimes y)h = h(x)y \in V_H$ for every $h \in H$. If $H \subset G$, $H \neq G$, then $H \subset G_-$ and, as $x \in (G_-)_\perp$, we have $h(x) = 0$ for every $h \in H$.

(b) We shall proceed by induction on the rank of operators. The rank one case is clear. Suppose that the claim is true for operators of rank $n - 1 \geq 1$.

Let $R \in \mathcal{R}$ and let $\dim \operatorname{ran} R = n$. Denote

$$F_H := S_{\operatorname{ran} R} \cap V_H, \quad H \in \mathcal{G},$$

and

$$V_G := \bigcap_{H\in\mathcal{G}} \{V_H : \ F_H \neq \emptyset\}.$$

(Such a $G \in \mathcal{G}$ exists, because $\mathcal{N}_\mathcal{G}$ is closed under intersections.) Let us show that $F_G \neq \emptyset$. Observe that

$$F_G = \bigcap\{F_H : F_H \neq \emptyset, \ H \in \mathcal{G}\}.$$

Observe also that the family $\{F_H : F_H \neq \emptyset, H \in \mathcal{G}\}$ of closed subsets of $S_{\operatorname{ran} R}$ has the finite intersection property, because, by Proposition 2.3, its finite subfamilies are nested. Therefore, since $S_{\operatorname{ran} R}$ is compact, $F_G \neq \emptyset$.

Let $y_1 \in F_G$. Extend $\{y_1\}$ to a basis $\{y_1, \ldots, y_n\}$ for $\operatorname{ran} R$. Then there exist $x_1, \ldots, x_n \in X$ such that

$$R = \sum_{k=1}^{n} x_k \otimes y_k.$$

Let us show that $x_1 \in (G_-)_\perp$. If $G = \{0\}$, then $(G_-)_\perp = X$ and $x_1 \in (G_-)_\perp$. Let $G \neq \{0\}$. If $x_1 \notin (G_-)_\perp$, then there is $H \in \mathcal{G}$ such that $H \subset G$, $H \neq G$, together with $h \in H$ such that $h(x_1) \neq 0$. We have $V_H \subset V_G$, $V_H \neq V_G$, because $\mathcal{N}_\mathcal{G}$ is increasing. On the other hand, as $R \in \mathcal{R}$, $Rh \in V_H$. And since $h(x_1) \neq 0$, by the linear independence of $y_1, \ldots, y_n$, we have

$$Rh = \sum_{k=1}^{n} h(x_k) y_k \neq 0.$$

Hence, $F_H \neq \emptyset$ implying that $V_G \subset V_H$, which is a contradiction with $V_H \subset V_G$ and $V_G \neq V_H$. Therefore, $x_1 \in (G_-)_\perp$.

Since $x_1 \in (G_-)_\perp$ and $y_1 \in V_G$, by (a), $x_1 \otimes y_1 \in \mathcal{R}$. Therefore, we see that the rank $n-1$ operator

$$\sum_{k=2}^{n} x_k \otimes y_k = R - x_1 \otimes y_1 \in \mathcal{R}.$$

By the induction hypothesis, $x_k \otimes y_k \in \mathcal{R}$ also for all $k = 2, \ldots, n$, as wished. □

Corollary 4.4. *Let X and Y be Banach spaces. Let $\mathcal{G}$ be a nest of subspaces of X containing $\{0\}$ and let $\mathcal{N}_\mathcal{G} = \{V_G : G \in \mathcal{G}\}$ be a nest of closed subspaces of Y as in Theorem 4.3. Denote $\mathcal{R} = \{R \in \mathcal{F}(X, Y) : R(G) \subset V_G \ \ \forall G \in \mathcal{G}\}$.*

(a) *Let $R = x^* \otimes y$ be a rank one operator. Then $R \in \mathcal{R}$ if and only if there is a subspace G in $\mathcal{G}$ such that $x^* \in (G_-)^\perp$ and $y \in V_G$.*

(b) *Let $R \in \mathcal{F}(X, Y)$ be an operator of rank $n > 0$. If $R \in \mathcal{R}$, then R can be written as the sum of n operators of rank one, each belonging to $\mathcal{R}$.*

Proof. Recall that, algebraically $\mathcal{F}(X, Y) = X^* \otimes Y$. Hence, we have $\mathcal{R} = \{R \in X^* \otimes Y : R(G) \subset V_G \ \ \forall G \in \mathcal{G}\}$. Notice that if we consider $\mathcal{G}$ as a nest of subspaces of $X^{**} = (X^*)^*$, then the assumptions of Theorem 4.3 are satisfied. We also have $(G_-)_\perp = \{x^* \in X^* : x^*(x) = 0 \ \ \forall x \in G_-\} = (G_-)^\perp \subset X^*$ for any G in $\mathcal{G}$. The assertions (a) and (b) immediately follow from Theorem 4.3. $\square$

Corollary 4.5 below is a well-known Banach space version of the Ringrose Theorem 4.2. It is immediate from Corollary 4.4.

Corollary 4.5 (cf. [FJ3, Lemma 1])**.** *Let X be a Banach space. Let $\mathcal{N}$ be a nest of closed subspaces of X containing $\{0\}$ and X. Assume that $\mathcal{N}$ is closed under arbitrary intersections. Denote $\mathcal{R} = \{R \in \mathcal{F}(X) : R(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$.*

(a) *Let $R = x^* \otimes x$ be a rank one operator. Then $R \in \mathcal{R}$ if and only if there is a subspace Y in $\mathcal{N}$ such that $x^* \in (Y_-)^\perp$ and $x \in Y$.*

(b) *Let $R \in \mathcal{F}(X)$ be an operator of rank $n > 0$. If $R \in \mathcal{R}$, then R can be written as the sum of n operators of rank one, each belonging to $\mathcal{R}$.*

If we specify Corollary 4.5 for the nests containing only one non-trivial element, then we have the following result.

Corollary 4.6 (see [FJ2, Lemma 2.1])**.** *Let X be a Banach space and Y a closed subspace of X. Denote $\mathcal{R} = \{R \in \mathcal{F}(X) : \ R(Y) \subset Y\}$.*

(a) *Let $R = x^* \otimes x$ be a rank one operator. Then $R \in \mathcal{R}$ if and only if either $x^* \in Y^\perp$ or $x \in Y$.*

(b) *Let $R \in \mathcal{F}(X)$ be an operator of rank $n > 0$. If $R \in \mathcal{R}$, then R can be written as the sum of n operators of rank one, each belonging to $\mathcal{R}$.*

Proof. Denote $\mathcal{N} := \{\{0\}, Y, X\}$. Then the nest $\mathcal{N}$ clearly satisfies the assumptions of Corollary 4.5 and $\mathcal{R} = \{R \in \mathcal{F}(X) : R(Z) \subset Z \ \ \forall Z \in \mathcal{N}\}$.

(a) For the "only if" part, let $R = x^* \otimes x$ be a rank one operator such that $R \in \mathcal{R}$. By Corollary 4.5,

(1) $x^* \in (\{0\}_-)^\perp = \{0\}^\perp = X^*$ and $x \in \{0\}$,

or

(2) $x^* \in (Y_-)^\perp = \{0\}^\perp = X^*$ and $x \in Y$,

or

(3) $x^* \in (X_-)^\perp = Y^\perp$ and $x \in X$.

Condition (1) cannot hold because R has rank one. Therefore, we have either $x^* \in Y^\perp$ or $x \in Y$, as wished.

For the "if" part, let $R = x^* \otimes x$ be a rank one operator such that either $x^* \in Y^\perp$ or $x \in Y$. First, let us consider the case when $x^* \in Y^\perp$ and $x \in X$ is arbitrary. Then $x^* \in (X_-)^\perp$ (cf. condition (3)). Since $X \in \mathcal{N}$, it follows from Corollary 4.5 (a) that $R \in \mathcal{R}$. Now, let us consider the case when $x \in Y$ and $x^* \in X^*$ is arbitrary. Then we have $x^* \in (Y_-)^\perp$ (cf. condition (2)) and thus, by Corollary 4.5 (a), $R \in \mathcal{R}$. The assertion (b) immediately follows from Corollary 4.5 (b). □

4.3 The Main PLR Lemma

Let $\mathcal{N}$ be a nest of closed subspaces of a Banach space X. Recall the following condition (from Proposition 2.7):

$$\Big(\bigcap_{Y\in\mathcal{N}'} Y\Big)^{\perp\perp} = \bigcap_{Y\in\mathcal{N}'} Y^{\perp\perp} \text{ for every non-empty subfamily } \mathcal{N}' \text{ of } \mathcal{N}. \qquad (*)$$

Lemma 4.7 (Main PLR Lemma)**.** *Let X and Y be Banach spaces. Let $\mathcal{G}$ be a nest of subspaces of X^* containing $\{0\}$ and let $\mathcal{N}_\mathcal{G} = \{V_G : G \in \mathcal{G}\}$ be a nest of closed subspaces of Y containing Y. Assume that $\mathcal{N}_\mathcal{G}$ is closed under arbitrary intersections, $\mathcal{N}_\mathcal{G}$ is increasing on $\mathcal{G}$, and $\mathcal{N}_\mathcal{G}$ satisfies condition $(*)$. Let $T \in X \otimes Y^{**}$ be such that $T(G) \subset V_G^{\perp\perp}$ for all $G \in \mathcal{G}$. Then there exists a net $(T_\nu) \subset X \otimes Y$ satisfying $T_\nu(G) \subset V_G$ for all ν and for all $G \in \mathcal{G}$ such that*

1° $\|T_\nu\| \to_\nu \|T\|$,

2° *$T_\nu^* y^* \to_\nu T^* y^*$ for all $y^* \in Y^*$,*

3° *$T_\nu x^* \to_\nu T x^*$ for all those $x^* \in X^*$ for which $Tx^* \in Y$.*

The proof of Lemma 4.7 will use Lemma 4.8 below (this is an extension of Lemma 3.19 for nests), which in its turn uses Theorem 4.3, the extension of the Ringrose theorem.

Lemma 4.8 will use the canonical identifications $(X \otimes Y)^* = \mathcal{I}(X, Y^*)$ and $(X \otimes Y^{**})^* = \mathcal{I}(X, Y^{***})$ due to Grothendieck (see Theorem 2.10).

Lemma 4.8. *Let X and Y be Banach spaces. Let $\mathcal{G}$ and $\mathcal{N}_\mathcal{G} = \{V_G : G \in \mathcal{G}\}$ be nests of subspaces of X^* and of closed subspaces of Y, respectively, as in Lemma 4.7. Denote*

$$\mathcal{R} = \{R \in X \otimes Y : R(G) \subset V_G \quad \forall G \in \mathcal{G}\}$$

and

$$\mathcal{S} = \{S \in X \otimes Y^{**} : S(G) \subset V_G^{\perp\perp} \quad \forall G \in \mathcal{G}\},$$

and consider $\mathcal{R}^\perp$ and $\mathcal{S}^\perp$ as subspaces of $\mathcal{I}(X, Y^*)$ and $\mathcal{I}(X, Y^{***})$, respectively. If $T \in \mathcal{I}(X, Y^*)$ is such that $T \in \mathcal{R}^\perp$, then $j_{Y^*}T \in \mathcal{S}^\perp$.

Proof. We start by showing that Theorem 4.3 is applicable to the pair of nests $\mathcal{G}$ and $\mathcal{N}_\mathcal{G}^{\perp\perp} = \{V_G^{\perp\perp} : G \in \mathcal{G}\}$. Since $\mathcal{N}_\mathcal{G}$ is a nest that contains Y and is increasing on $\mathcal{G}$, by Lemma 2.6, $\mathcal{N}_\mathcal{G}^{\perp\perp}$ is a nest of closed subspaces of Y^{**} that contains Y^{**} and is increasing on $\mathcal{G}$. By Proposition 2.7, $\mathcal{N}_\mathcal{G}^{\perp\perp}$ is also closed under arbitrary intersections.

Let $T \in \mathcal{R}^\perp$, i.e., $\langle R, T\rangle = 0$ for every $R \in \mathcal{R}$. We have to show that $\langle S, j_{Y^*}T\rangle = 0$ for every $S \in \mathcal{S}$. By Theorem 4.3 (b), it is enough to show that $\langle x \otimes y^{**}, j_{Y^*}T\rangle = 0$ for every $x \otimes y^{**} \in \mathcal{S}$.

Let $x \otimes y^{**} \in \mathcal{S}$ be arbitrary. Then, by Theorem 4.3 (a), there is a subspace G in $\mathcal{G}$ such that $x \in (G_-)_\perp$ and $y^{**} \in V_G^{\perp\perp}$. Consider the canonical isometry $I : V_G^{\perp\perp} \to V_G^{**}$ defined by $(Iv^{\perp\perp})(v^*) = v^{\perp\perp}(y^*)$, where $v^{\perp\perp} \in V_G^{\perp\perp}$ and $y^* \in Y^*$ is an arbitrary extension of $v^* \in V_G^*$. Then $Iy^{**} \in V_G^{**}$ and we have $y^{**}(Tx) = (Iy^{**})(Tx|_{V_G})$.

Choose $v \in V_G$ such that $(Iy^{**})(Tx|_{V_G}) = (Tx|_{V_G})(v)$. (Such an element v exists. Indeed, let Z be a normed space, $z^* \in Z$, $z^{**} \in Z^{**}$, and denote $a := z^{**}(z^*)$. If $a = 0$, then $a = z^*(v)$ for $v = 0$. If $a \neq 0$, then there is $w \in Z$ such that $b := z^*(w) \neq 0$. Take $v = ab^{-1}w$; then $a = z^*(v)$.)

Put $R = x \otimes v$. Then, by Theorem 4.3 (a), $R \in \mathcal{R}$ and therefore $\langle R, T\rangle = 0$. On the other hand,

$$\langle R, T\rangle = (Tx)(v) = (Iy^{**})(Tx|_{V_G}) = y^{**}(Tx) = \langle x \otimes y^{**}, j_{Y^*}T\rangle.$$

Hence, $\langle x \otimes y^{**}, j_{Y^*}T\rangle = 0$, as desired. $\square$

Let Y be a closed subspace of a Banach space X. In the special case when $\mathcal{G} = \mathcal{N}_{\mathcal{G}} := \{\{0\}, Y, X\}$ in Lemma 4.8, it reduces to the following.

Corollary 4.9. *Let X be a Banach space and let Y be a closed subspace of X. Denote*

$$\mathcal{R} = \{R \in \mathcal{F}(X) : R(Y) \subset Y\}$$

and

$$\mathcal{S} = \{S \in \mathcal{F}(X, X^{**}) : S(Y) \subset Y^{\perp\perp}\},$$

and consider $\mathcal{R}^{\perp}$ and $\mathcal{S}^{\perp}$ as subspaces of $\mathcal{I}(X^*, X^*)$ and $\mathcal{I}(X^*, X^{***})$, respectively. If $T \in \mathcal{I}(X^*, X^*)$ is such that $T \in \mathcal{R}^{\perp}$, then $j_{X^*}T \in \mathcal{S}^{\perp}$.

Proof. Notice that since algebraically $\mathcal{F}(X) = X^* \otimes X$ and $\mathcal{F}(X, X^{**}) = X^* \otimes X^{**}$, we have $\mathcal{R} = \{R \in X^* \otimes X : R(G) \subset V_G \ \forall G \in \mathcal{G}\}$ and $\mathcal{S} = \{S \in X^* \otimes X^{**} : S(G) \subset V_G^{\perp\perp} \ \forall G \in \mathcal{G}\}$, where $\mathcal{G} = \mathcal{N}_{\mathcal{G}}(= \{V_G : G \in \mathcal{G}\}) := \{\{0\}, Y, X\}$ (because $\{0\}^{\perp\perp} = \{0\}$ and $X^{\perp\perp} = X^{**}$). Clearly, $\mathcal{N}_{\mathcal{G}}^{\perp}$ is a complete nest (because it is finite and contains $\{0\}$ $(= X^{\perp})$ and X^* $(= \{0\}^{\perp})$). Hence, by Corollary 2.9, $\mathcal{N}_{\mathcal{G}}$ satisfies $(*)$. If we consider $\mathcal{G}$ as a nest of subspaces of X^{**}, then the assumptions of Lemma 4.8 are satisfied. The assertion is immediate from Lemma 4.8. □

Proof of Lemma 4.7. Thanks to Lemma 4.8, the proof is close to part (a) of the proof of [O5, Lemma 2.1]. We shall include a proof for completeness.

Let $J : \mathcal{I}(X, Y^*) \to \mathcal{I}(X, Y^{***})$ be the natural embedding defined by $JA = j_{Y^*}A$ for $A \in \mathcal{I}(X, Y^*)$ (see Section 2.3). Let $\mathcal{R}$ and $\mathcal{S}$ be as in Lemma 4.8. Then, by Lemma 4.8, $J(\mathcal{R}^{\perp}) \subset \mathcal{S}^{\perp}$. This implies that the operator

$$\bar{J} : \mathcal{I}(X, Y^*)/\mathcal{R}^{\perp} \to \mathcal{I}(X, Y^{***})/\mathcal{S}^{\perp},$$

given by

$$\bar{J}(A + \mathcal{R}^{\perp}) = JA + \mathcal{S}^{\perp}, \quad A \in \mathcal{I}(X, Y^*),$$

is well defined. Moreover, (similarly to the proof of Corollary 3.20) it is easy to verify that $\|\bar{J}\| \le 1$, and $\bar{J}q_1 = q_2 J$, where $q_1 : \mathcal{I}(X, Y^*) \to \mathcal{I}(X, Y^*)/\mathcal{R}^{\perp}$ and $q_2 : \mathcal{I}(X, Y^{***}) \to \mathcal{I}(X, Y^{***})/\mathcal{S}^{\perp}$ denote the quotient mappings.

Since $\mathcal{R}$ and $\mathcal{S}$ are linear subspaces of $X \otimes Y$ and $X \otimes Y^{**}$, respectively, the duals $\mathcal{R}^*$ and $\mathcal{S}^*$ are canonically isometrically isomorphic to $(X \otimes Y)^*/\mathcal{R}^{\perp}$ and $(X \otimes Y^{**})^*/\mathcal{S}^{\perp}$, respectively. Hence, under the canonical identifications, $\mathcal{R}^*$ and $\mathcal{S}^*$ are isometrically isomorphic to $\mathcal{I}(X, Y^*)/\mathcal{R}^{\perp}$ and $\mathcal{I}(X, Y^{***})/\mathcal{S}^{\perp}$, respectively. Let $r : \mathcal{R}^* \to \mathcal{I}(X, Y^*)/\mathcal{R}^{\perp}$ and $s : \mathcal{I}(X, Y^{***})/\mathcal{S}^{\perp} \to \mathcal{S}^*$ denote the corresponding isometries.

Define $\Phi : \mathcal{R}^* \to \mathcal{S}^*$ by $\Phi = s\bar{J}r$. Then the diagrams

$$\begin{tikzcd}
\mathcal{R}^* \arrow[r, "\Phi"] \arrow[d, "r"'] & \mathcal{S}^* \\
\mathcal{I}(X,Y^*)/\mathcal{R}^\perp \arrow[r, "\bar{J}"] & \mathcal{I}(X,Y^{***})/\mathcal{S}^\perp \arrow[u, "s"'] \\
\mathcal{I}(X,Y^*) \arrow[r, "J"] \arrow[u, "q_1"] & \mathcal{I}(X,Y^{***}) \arrow[u, "q_2"']
\end{tikzcd}$$

commute and, clearly, $\|\Phi\| \leq 1$.

Let T be the given operator. Then $T \in \mathcal{S} \subset \mathcal{S}^{**}$ and $\Phi^*(T) \in \|T\|\, B_{\mathcal{R}^{**}}$. By the Goldstine theorem, there exists a net $(T_\nu) \subset \|T\|\, B_{\mathcal{R}}$ converging weak* to $\Phi^*(T)$. Therefore, $(T_\nu) \subset X \otimes Y$ is a net such that, for every ν, we have $\|T_\nu\| \leq \|T\|$ and $T_\nu(G) \subset V_G$ for all $G \in \mathcal{G}$. Moreover, if for every $x^* \in X^*$ and $y^* \in Y^*$, we consider $x^* \otimes y^* \in \mathcal{R}^*$ and $x^* \otimes j_{Y^*}y^* \in \mathcal{S}^*$, then

$$\begin{aligned}
x^*(T_\nu^* y^*) &= \langle T_\nu, x^* \otimes y^* \rangle \to_\nu \langle x^* \otimes y^*, \Phi^*(T) \rangle = \langle T, \Phi(x^* \otimes y^*) \rangle \\
&= \langle T, x^* \otimes j_{Y^*} y^* \rangle = x^*(T^* y^*).
\end{aligned}$$

This means that $T_\nu^* \to T^*|_{Y^*}$ in the WOT on $\mathcal{L}(Y^*, X)$.

For $1^\circ - 3^\circ$, we shall use a convex combination argument as in the proof of [OP, Lemma 1.1]. After passing to convex combinations, we may assume that $T_\nu^* \to T^*|_{Y^*}$ in the SOT on $\mathcal{L}(Y^*, X)$ (see Proposition 2.15), i.e., $T_\nu^* y^* \to_\nu T^* y^*$ for all $y^* \in Y^*$. Hence, T_ν converges to T in the WOT on $\mathcal{L}(T^{-1}(Y), Y)$. After passing to new convex combinations, we may assume that $T_\nu x^* \to_\nu T x^*$ for all $x^* \in T^{-1}(Y)$. Therefore, we have 2° and 3°. Since $\limsup_\nu \|T_\nu\| \leq \|T\|$ and, by 2°,

$$\|T^* y^*\| = \lim_\nu \|T_\nu^* y^*\| \leq \liminf_\nu \|T_\nu^*\|\, \|y^*\| \quad \forall y^* \in Y^*,$$

we have $\|T\| \leq \liminf_\nu \|T_\nu\|$, implying 1°. $\square$

4.4 Versions of the principle of local reflexivity respecting nests of subspaces

It was proved in [O5] by Oja (see [O5, Theorems 1.3 and 3.1]) that finite-rank operators between dual spaces are "locally conjugate" and respect a given pair of subspaces. Theorems 4.10 and 4.11 below are versions of this result for a pair of nests of subspaces.

Theorem 4.10. *Let X and Y be Banach spaces. Let $\mathcal{U}$ be a nest of closed subspaces of X containing $\{0\}$ and let $\mathcal{N}_{\mathcal{U}} = \{V_U : U \in \mathcal{U}\}$ be a nest of closed subspaces of Y containing Y. Assume that $\mathcal{N}_{\mathcal{U}}$ is closed under arbitrary intersections, $\mathcal{N}_{\mathcal{U}}$ is increasing on $\mathcal{U}$, and $\mathcal{N}_{\mathcal{U}}$ satisfies condition $(*)$. Let $S \in \mathcal{F}(Y^*, X^*)$ satisfy $S(V_U^{\perp}) \subset U^{\perp}$ for all $U \in \mathcal{U}$. Then there exists a net $(T_\nu) \subset \mathcal{F}(X, Y)$ satisfying $T_\nu(U) \subset V_U$ for all ν and for all $U \in \mathcal{U}$ such that*

1° $\|T_\nu\| \to_\nu \|S\|$,

2° $T_\nu^* y^* \to_\nu Sy^*$ *for all* $y^* \in Y^*$,

3° $T_\nu^{**} x^{**} \to_\nu S^* x^{**}$ *for all those* $x^{**} \in X^{**}$ *for which* $S^* x^{**} \in Y$.

Proof. Since $\mathcal{F}(Y^*, X^*)$ is algebraically the same as $Y^{**} \otimes X^*$, we have $S \in Y^{**} \otimes X^*$ and thus $S^* \in X^* \otimes Y^{**}$. We also have $S^*(U^{\perp\perp}) \subset V_U^{\perp\perp}$ for all $U \in \mathcal{U}$.

We shall apply Lemma 4.7 to S^*. For that take $\mathcal{G} = \mathcal{U}^{\perp\perp} := \{U^{\perp\perp} : U \in \mathcal{U}\}$. Then $\mathcal{G}$ is a nest of closed subspaces of X^{**} and $\{0\} = \{0\}^{\perp\perp}$ is in $\mathcal{G}$. Take $\mathcal{N}_{\mathcal{G}} = \mathcal{N}_{\mathcal{U}}$. By Lemma 2.6, there is a bijective correspondence between $\mathcal{U}$ and $\mathcal{U}^{\perp\perp}$, where U corresponds to $U^{\perp\perp}$. Therefore, we may assume that $\mathcal{N}_{\mathcal{U}}$ is indexed by $\mathcal{U}^{\perp\perp}$, i.e.,

$$\mathcal{N}_{\mathcal{G}} = \{V_U : U^{\perp\perp} \in \mathcal{G}\}.$$

Then $\mathcal{N}_{\mathcal{G}}$ is a nest of closed subspaces of Y containing Y, $\mathcal{N}_{\mathcal{G}}$ is closed under arbitrary intersections, and $\mathcal{N}_{\mathcal{G}}$ satisfies condition $(*)$. Also, $\mathcal{N}_{\mathcal{G}}$ is increasing on $\mathcal{G}$.

Lemma 4.7 produces a net $(S_\nu) \subset X^* \otimes Y$ which, considered as a net in $\mathcal{F}(X^{**}, Y)$, satisfies the following: $S_\nu(U^{\perp\perp}) \subset V_U$ for all ν and for all $U \in \mathcal{U}$, $\|S_\nu\| \to_\nu \|S^*\| = \|S\|$, $S_\nu^* y^* \to_\nu S^{**} y^* = Sy^*$ for all $y^* \in Y^*$, and $S_\nu x^{**} \to_\nu S^* x^{**}$ for all those $x^{**} \in X^{**}$ for which $S^* x^{**} \in Y$. On the other hand, $X^* \otimes Y = \mathcal{F}(X, Y)$, so that if we take $T_\nu = S_\nu|_X \in \mathcal{F}(X, Y)$, then $T_\nu^{**} = S_\nu$ for all ν. Therefore, 1° – 3° hold and $T_\nu(U) \subset V_U$ for all ν and for all $U \in \mathcal{U}$. □

Theorem 4.11. *Let X and Y be Banach spaces. Let $\mathcal{U}$ and $\mathcal{N}_{\mathcal{U}} = \{V_U : U \in \mathcal{U}\}$ be nests of closed subspaces of X and Y, respectively, as in Theorem 4.10. Let $S \in \mathcal{F}(Y^*, X^*)$ satisfy $S(V_U^{\perp}) \subset U^{\perp}$ for all $U \in \mathcal{U}$. If K and L are* *compact subsets of X^{**} and Y^*, respectively, and $\varepsilon > 0$, then there exists an operator $T \in \mathcal{F}(X,Y)$ satisfying $T(U) \subset V_U$ for all $U \in \mathcal{U}$ such that*

1° $|\|T\| - \|S\|| < \varepsilon$,

2° $\|T^*y^* - Sy^*\| < \varepsilon$ *for all* $y^* \in L$,

3° $\|T^{**}x^{**} - S^*x^{**}\| < \varepsilon$ *for all those* $x^{**} \in K$ *for which* $S^*x^{**} \in Y$.

Proof. Let (T_ν) be the net from Theorem 4.10. Since $\|T_\nu\| \to_\nu \|S\|$, we can choose ν_0 such that $|\|T_\nu\| - \|S\|| < \varepsilon$ whenever $\nu \geq \nu_0$. Hence, the net $(T_\nu)_{\nu \geq \nu_0}$ is bounded. Since the pointwise convergence of bounded nets of operators is uniform on compact sets (see Proposition 2.13), we have $(T_\nu^*)_{\nu \geq \nu_0}$ and $(T_\nu^{**})_{\nu \geq \nu_0}$ converging uniformly on L and K, respectively. Therefore, we can find $\nu \geq \nu_0$ such that $1° - 3°$ hold for $T := T_\nu$. □

Remark 4.12. If the nests in Theorems 4.10 and 4.11 contain only one non-trivial element (i.e., $\mathcal{U} = \{\{0\}, U, X\}$ and $\mathcal{N}_\mathcal{U} = \{\{0\}, V, Y\}$, where U and V are closed subspaces of X and Y, respectively), then these special cases of Theorems 4.10 and 4.11 are contained in [O5, Theorem 3.1] and in [O5, Theorem 1.3], respectively.

In [O5, Theorem 1.2], it was proved that finite-rank operators between bidual spaces are "locally biconjugate" and respect a given pair of subspaces. A version of this result for a given pair of nests of subspaces is as follows. Its special case, when the nests contain only one non-trivial element, is contained in [O5, Theorem 1.2].

Theorem 4.13. *Let X and Y be Banach spaces. Let $\mathcal{U}$ be a nest of closed subspaces of X containing $\{0\}$ and let $\mathcal{N}_\mathcal{U} = \{V_U : U \in \mathcal{U}\}$ be a nest of closed subspaces of Y containing Y. Assume that $\mathcal{N}_\mathcal{U}$ is increasing on $\mathcal{U}$, $\mathcal{N}_\mathcal{U}^{\perp\perp}$ is closed under arbitrary intersections, and $\mathcal{U}^{\perp\perp}$ is closed under closures of arbitrary unions. Let $S \in \mathcal{F}(X^{**}, Y^{**})$ be such that $S(U^{\perp\perp}) \subset V_U^{\perp\perp}$ for all $U \in \mathcal{U}$. If K and L are compact subsets of X^{**} and Y^*, respectively, and $\varepsilon > 0$, then there exists an operator $T \in \mathcal{F}(X,Y)$ satisfying $T(U) \subset V_U$ for all $U \in \mathcal{U}$ such that*

1° $|\|T\| - \|S\|| < \varepsilon$,

2° $|x^{**}(T^*y^*) - (Sx^{**})(y^*)| < \varepsilon$ *for all* $x^{**} \in K$ *and* $y^* \in L$.

Proof. To prove the theorem, we shall apply Theorem 4.11 twice.

First of all, notice that there is clearly no loss of generality to assume that $K \subset B_{X^{**}}$ and $L \subset B_{Y^*}$. Indeed, assume that Theorem 4.13 holds for all compact subsets $K_0 \subset B_{X^{**}}$ and $L_0 \subset B_{Y^*}$, and for all $\varepsilon_0 > 0$. Let us show that then the assertion of Theorem 4.13 holds for any compact subsets $K \subset X^{**}$ and $L \subset Y^*$, and for any $\varepsilon \in (0,1)$. Put $\varepsilon_0 = \varepsilon/(1 + \max_{x^{**} \in K} \|x^{**}\| + \max_{y^* \in L} \|y^*\|)$. Then $\varepsilon_0 < 1$, $\varepsilon_0 \|x\| < 1$ for all $x \in K$, and $\varepsilon_0 \|y^*\| < 1$ for all $y^* \in L$. There exists an operator $T \in \mathcal{F}(X,Y)$ satisfying $T(U) \subset V_U$ for all $U \in \mathcal{U}$ such that $|\|T\| - \|S\|| < \varepsilon_0^3$ and $|x^{**}(T^*y^*) - (Sx^{**})(y^*)| < \varepsilon_0^3$ for all $x^{**} \in \varepsilon_0 K \subset B_K$ and $y^* \in \varepsilon_0 L \subset B_L$. Clearly, 1° holds. Also, for any $x^{**} \in K$ and $y^* \in L$, we have

$$\varepsilon_0^2 \left|x^{**}(T^*y^*) - (Sx^{**})(y^*)\right| = \left|\varepsilon_0 x^{**}(T^*(\varepsilon_0 y^*)) - (S(\varepsilon_0 x^{**}))(\varepsilon_0 y^*)\right| < \varepsilon_0^3,$$

implying that

$$|x^{**}(T^*y^*) - (Sx^{**})(y^*)| < \varepsilon_0 < \varepsilon.$$

Hence, 2° holds.

We start by applying Theorem 4.11 to $S \in \mathcal{L}((X^*)^*, (Y^*)^*)$. Notice that $\mathcal{N}_{\mathcal{U}}^{\perp}$ and $\mathcal{U}^{\perp}$ are nests of closed subspaces of Y^* and X^*, respectively. Let us show that we can take $\mathcal{N}_{\mathcal{U}}^{\perp}$ and $\mathcal{U}^{\perp}$ in the roles of $\mathcal{U}$ and $\mathcal{N}_{\mathcal{U}}$ of Theorem 4.11, respectively.

We have $\{0\} = Y^{\perp} \in \mathcal{N}_{\mathcal{U}}^{\perp}$ and $X^* = \{0\}^{\perp} \in \mathcal{U}^{\perp}$. Since $\mathcal{N}_{\mathcal{U}}$ is increasing on $\mathcal{U}$, $\mathcal{U}$ is increasing on $\mathcal{N}_{\mathcal{U}}$. But this, via Lemma 2.6, means that $\mathcal{U}^{\perp}$ is increasing on $\mathcal{N}_{\mathcal{U}}^{\perp}$. Since $\mathcal{U}^{\perp\perp}$ is closed under closures of arbitrary unions, by Proposition 2.7, $\mathcal{U}^{\perp}$ is closed under arbitrary intersections and satisfies condition $(*)$.

An application of Theorem 4.11 to S yields $R \in \mathcal{F}(Y^*, X^*)$ such that $R(V_U^{\perp}) \subset U^{\perp}$ for all $U \in \mathcal{U}$, $|\|R\| - \|S\|| < \varepsilon/2$ and $\|R^*x^{**} - Sx^{**}\| < \varepsilon/2$ for all $x^{**} \in K$.

Notice that $\mathcal{U}$ and $\mathcal{N}_{\mathcal{U}}$ together with $R \in \mathcal{F}(Y^*, X^*)$ satisfy the assumptions of Theorem 4.11. Indeed, since $\mathcal{N}_{\mathcal{U}}^{\perp\perp}$ is closed under arbitrary intersections, by Proposition 2.7, $\mathcal{N}_{\mathcal{U}}$ is closed under arbitrary intersections and satisfies condition $(*)$. Now an application of Theorem 4.11 to R yields $T \in \mathcal{F}(X,Y)$ such that $T(U) \subset V_U$ for all $U \in \mathcal{U}$, $|\|T\| - \|R\|| < \varepsilon/2$ and $\|T^*y^* - Ry^*\| < \varepsilon/2$ for all $y^* \in L$.

Hence, 1° and 2° hold. □

Remark 4.14. The assertions of Theorems 4.10 and 4.11 hold whenever $\mathcal{N}_{\mathcal{U}} = \{V_U : U \in \mathcal{U}\}$ is a nest of closed subspaces of Y, which is increasing on $\mathcal{U}$ and such that the nest $\mathcal{N}_{\mathcal{U}}^{\perp}$ is complete. If, moreover, $\mathcal{U}^{\perp\perp}$ is complete, then also the assertion of Theorem 4.13 holds. These conditions are satisfied in the special case when $\mathcal{U}$ and $\mathcal{N}_{\mathcal{U}}$ are finite nests, having the same number of elements and both containing $\{0\}$ and the whole space (see also Corollary 2.9).

4.5 Duality of nest approximation properties

The following result extends Johnson's Theorem 3.15 (see Section 3.5) from APs to nest APs. Notice that in the special case when $\mathcal{N} = \{Y\}$, where Y is a closed subspace of a Banach space X, Theorem 4.15 below reduces to Theorem 3.21.

Theorem 4.15. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X satisfying condition $(*)$. Let $1 \le \lambda < \infty$. Then $(X, \mathcal{N})$ has the λ-bounded duality approximation property if and only if $(X^*, \mathcal{N}^\perp)$ has the λ-bounded approximation property.*

Proof. The "only if" part is clear, because if $S(Y) \subset Y$, then $S^*(Y^\perp) \subset Y^\perp$.

For the "if" part, consider the set of all couples $\nu = (L, \varepsilon)$, where L is a finite subset of X^* and $\varepsilon > 0$, directed in the natural way. Since the pair $(X^*, \mathcal{N}^\perp)$ has the λ-bounded AP, there exists a net $(T_\nu) \subset \mathcal{F}(X^*)$ with $\|T_\nu\| \le \lambda$ for all ν such that $T_\nu(Y^\perp) \subset Y^\perp$ for all ν and for all $Y \in \mathcal{N}$, and

$$\|T_\nu x^* - x^*\| \le \varepsilon/2 \ \ \forall x^* \in L.$$

By enlarging $\mathcal{N}$, if necessary, we clearly may assume that $\mathcal{N}$ contains $\{0\}$ and X. We also may assume that $\mathcal{N}$ is closed under intersections. Indeed, let $\mathcal{N}'$ be a non-empty subfamily of $\mathcal{N}$. Then, using condition $(*)$ twice, we have

$$\begin{aligned} T_\nu^*\Bigg(\Big(\bigcap_{Y\in\mathcal{N}'} Y\Big)^{\perp\perp}\Bigg) &= T_\nu^*\Big(\bigcap_{Y\in\mathcal{N}'} Y^{\perp\perp}\Big) \subset \bigcap_{Y\in\mathcal{N}'} T_\nu^*(Y^{\perp\perp}) \subset \bigcap_{Y\in\mathcal{N}'} Y^{\perp\perp} \\ &= \Big(\bigcap_{Y\in\mathcal{N}'} Y\Big)^{\perp\perp}. \end{aligned}$$

Hence, by Lemma 2.5,

$$T_\nu\Bigg(\Big(\bigcap_{Y\in\mathcal{N}'} Y\Big)^{\perp}\Bigg) \subset \Big(\bigcap_{Y\in\mathcal{N}'} Y\Big)^{\perp}.$$

Now, for every operator $T_\nu \in \mathcal{F}(X^*)$, by Theorem 4.11, there is an operator $S_\nu \in \mathcal{F}(X)$ such that $S_\nu(Y) \subset Y$ for all $Y \in \mathcal{N}$, $\|S_\nu\| \leq \lambda + \varepsilon$, implying that

$$\limsup_\nu \|S_\nu\| \leq \lambda,$$

and

$$\|S_\nu^* x^* - T_\nu x^*\| < \varepsilon/2 \quad \forall x^* \in L.$$

Hence,

$$\|S_\nu^* x^* - x^*\| \leq \|S_\nu^* x^* - T_\nu x^*\| + \|T_\nu x^* - x^*\| < \varepsilon \quad \forall x^* \in L,$$

implying that $S_\nu^* x^* \to_\nu x^*$ for all $x^* \in X^*$. Indeed, let $x^* \in X^*$. For a given $\varepsilon_0 > 0$ take $\nu_0 = (\{x^*\}, \varepsilon_0)$. If $\nu = (L, \varepsilon) \geq \nu_0$, then $x^* \in L$ and $\varepsilon \leq \varepsilon_0$, and we have

$$\|S_\nu^* x^* - x^*\| < \varepsilon \leq \varepsilon_0,$$

as needed. Therefore,

$$x^{**}(S_\nu^* x^*) \to_\nu x^{**}(x^*) \quad \forall x^* \in X^*, \ \forall x^{**} \in X^{**}.$$

By Proposition 3.11, X has the λ-bounded duality $\{S \in \mathcal{F}(X) : S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$-AP, i.e., the pair $(X, \mathcal{N})$ has the λ-bounded duality AP. $\square$

Next we shall also extend [LisO, Proposition 5.11] from APs of pairs to nest APs (see Theorem 4.16). In the special case when $\mathcal{N} = \{Y\}$, where Y is a closed subspace of X, Theorem 4.16 below reduces to [LisO, Proposition 5.11], providing an alternative proof to it. The original proof of [LisO, Proposition 5.11] relied on the classical PLR and was quite technical. Let us mention that we cannot figure out how the classical PLR could be used to prove Theorem 4.16. Our proof uses the canonical identification $(\mathcal{L}(X), \tau_c)^* = X^* \hat{\otimes}_\pi X$, due to Grothendieck [G], where τ_c denotes the (locally convex) topology on $\mathcal{L}(X)$ of uniform convergence on compact subsets of X (see Section 2.4).

Theorem 4.16. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X satisfying condition $(*)$. Then $(X, \mathcal{N})$ has the duality approximation property if and only if $(X^*, \mathcal{N}^\perp)$ has the approximation property.*

Proof. Similarly to Theorem 4.15, the "only if" part is clear.

For the "if" part, consider the set of all couples $\nu = (L, \varepsilon)$, where L is a compact subset of X^* and $\varepsilon > 0$, directed in the natural way. As in the proof of Theorem 4.15, using Theorem 4.11, we can find a net $(S_\nu) \subset \mathcal{F}(X)$ such that $S_\nu(Y) \subset Y$ for all ν and for all $Y \in \mathcal{N}$, and $S_\nu^* \to_\nu I_{X^*}$ in the topology τ_c. Indeed, since the pair $(X^*, \mathcal{N}^\perp)$ has the AP, there exists a net $(T_\nu) \subset \mathcal{F}(X^*)$ such that $T_\nu(Y^\perp) \subset Y^\perp$ for all ν and for all $Y \in \mathcal{N}$, and

$$\|T_\nu x^* - x^*\| \le \varepsilon/2 \ \ \forall x^* \in L.$$

Enlarging $\mathcal{N}$, if necessary, we may assume (as in the proof of Theorem 4.15) that $\mathcal{N}$ and T_ν satisfy the assumptions of Theorem 4.11. Hence, for $T_\nu \in \mathcal{F}(X^*)$, by Theorem 4.11, there exists $S_\nu \in \mathcal{F}(X)$ such that $S_\nu(Y) \subset Y$ for all $Y \in \mathcal{N}$, and

$$\|S_\nu^* x^* - T_\nu x^*\| \le \varepsilon/2 \ \ \forall x^* \in L.$$

Now, for any compact subset L_0 of X^* and any $\varepsilon_0 > 0$, take $\nu_0 = (L_0, \varepsilon_0)$. If $\nu = (L, \varepsilon) \ge \nu_0$, then $L_0 \subset L$ and $\varepsilon \le \varepsilon_0$, and we have

$$\|S_\nu^* x^* - x^*\| \le \|S_\nu^* x^* - T_\nu x^*\| + \|T_\nu x^* - x^*\| < \varepsilon \le \varepsilon_0 \ \ \forall x^* \in L_0.$$

Therefore, $S_\nu^* \to_\nu I_{X^*}$ in the topology τ_c, as wished.

Hence, $S_\nu^* \to_\nu I_{X^*} = I_X^*$ in the weak topology of the locally convex space $(\mathcal{L}(X^*), \tau_c)$, meaning that

$$\langle S_\nu^* - I_X^*, v\rangle = \sum_{k=1}^\infty x_k^{**}((S_\nu^* - I_X^*)x_k^*) \to_\nu 0$$

for all $v = \sum_{k=1}^\infty x_k^{**} \otimes x_k^* \in X^{**} \hat{\otimes}_\pi X^*$ $(= (\mathcal{L}(X^*), \tau_c)^*)$. But then

$$\langle S_\nu - I_X, u\rangle = \sum_{k=1}^\infty x_k^*((S_\nu - I_X)x_k) \to_\nu 0$$

for all $u = \sum_{k=1}^\infty x_k^* \otimes x_k \in X^* \hat{\otimes}_\pi X$, meaning that $S_\nu \to_\nu I_X$ in the weak topology of $(\mathcal{L}(X), \tau_c)$. After passing to convex combinations, we may assume that $S_\nu \to_\nu I_X$ in the topology τ_c (see, e.g., [Day, Theorem 2, p. 46]). Hence, the pair $(X, \mathcal{N})$ has the duality AP. $\square$

In view of Corollary 2.9, from Theorems 4.15 and 4.16, we immediately get the following.

Corollary 4.17. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X. If the nest $\mathcal{N}^\perp$ is complete, then the assertions of Theorems 4.15 and 4.16 hold.*

Let us also spell out the corresponding result for finite nests (cf. Remark 4.14).

Corollary 4.18. *Let X be a Banach space and let $\mathcal{N}$ be a finite nest of closed subspaces of X. Then the assertions of Theorems 4.15 and 4.16 hold.*

Remark 4.19. If X^* or X^{**} has the Radon–Nikodým property, then Theorems 4.15 and 4.16, and Corollaries 4.17 and 4.18 state that *the pair $(X, \mathcal{N})$ has the metric duality AP if and only if the pair $(X^*, \mathcal{N}^\perp)$ has the AP* (cf. Remark 3.1).

4.6 Criteria of the nest approximation properties in the spirit of Grothendieck

Let X be a Banach space. Recall the *"condition de biunivocité"*, a criterion of the AP, established by Grothendieck in his memoir [G, Chapter I, p. 165].

(AP). *If $u \in X^* \hat{\otimes}_\pi X$ is such that $\tilde{u}(X) = \{0\}$, then* $\operatorname{trace} u = 0$.

By Grothendieck [G] (see, e.g., [Ry, p. 74] or [LT1, p. 32]), a Banach space X has the AP if and only if condition (AP) holds.

The main result of the recent paper by Figiel and Johnson [FJ3, Theorem 2.1] provides a criterion of the nest AP. The authors call it "the dual version of the statement that $(X, \mathcal{N})$ has the AP" (see [FJ3, p. 569]), and it refines the dual form of the AP for the pair (X, Y), where Y is a closed subspace of X, from [FJ2, Theorem 2.2]. The Figiel–Johnson theorems are formulated in terms of nuclear operators under the hypothesis that X has the AP. The proof of [FJ3, Theorem 2.1] (using also Corollary 4.5) shows the following result. We shall include a proof for completeness.

Theorem 4.20 (cf. [FJ3, Theorem 2.1])**.** *Let X be a Banach space. Let $\mathcal{N}$ be a nest of closed subspaces of a Banach space X containing $\{0\}$ and X. Assume that $\mathcal{N}$ is closed under arbitrary intersections. Then $(X, \mathcal{N})$ has the approximation property if and only if the following condition holds.*

($\mathrm{AP}_{\mathcal{N}}$). *If $u \in X^* \hat{\otimes}_\pi X$ is such that $\tilde{u}(Y) \subset \overline{Y_-}$ for all $Y \in \mathcal{N}$, then* $\operatorname{trace} u = 0$.

Proof. For the "only if" part, assume that the condition ($\mathrm{AP}_{\mathcal{N}}$) does not hold. Then there exists $u \in X^* \hat{\otimes}_\pi X$,

$$u = \sum_{k=1}^{\infty} x_k^* \otimes x_k$$

where $(x_k^*) \subset X^*$, $(x_k) \subset X$, and $\sum_{k=1}^{\infty} \|x_k^*\| \, \|x_k\| < \infty$ such that $\tilde{u}(Y) \subset \overline{Y_-}$ for all $Y \in \mathcal{N}$, but

$$\operatorname{trace} u = \sum_{k=1}^{\infty} x_k^*(x_k) = \alpha \neq 0.$$

Since, algebraically, $(\mathcal{L}(X), \tau_c)^* = X^* \hat{\otimes}_\pi X$ (see Section 2.4), we have

$$\langle S, u \rangle = \sum_{k=1}^{\infty} x_k^*(S x_k), \;\; S \in \mathcal{L}(X).$$

Hence,

$$\langle I_X, u \rangle = \operatorname{trace} u = \alpha.$$

By assumption, there exists a net $(S_\nu) \subset \mathcal{F}(X)$ such that $S_\nu(Y) \subset Y$ for all ν and for all $Y \in \mathcal{N}$, and $S_\nu \to_\nu I_X$ in the topology τ_c. Hence, $S_\nu \to_\nu I_X$ in the weak topology of the locally convex space $(\mathcal{L}(X), \tau_c)$, meaning that

$$\langle S_\nu - I_X, u \rangle \to_\nu 0.$$

Let us show that $\langle S_\nu, u \rangle = 0$ for every ν. By Corollary 4.5, any S_ν can be written as

$$S_\nu = \sum_{k=1}^{m} y_k^* \otimes y_k,$$

where m is a rank of S_ν, and for each $k \in \{1, \ldots, m\}$ there is $Y \in \mathcal{N}$ such that $y_k^* \in (Y_-)^\perp$ and $y_k \in Y$. In particular, $\tilde{u} y_k \in \overline{Y_-}$ and, by Proposition 2.1, $y_k^* \in (\overline{Y_-})^\perp$. It follows that for any $k \in \{1, \ldots, m\}$,

$$\langle y_k^* \otimes y_k, u \rangle = \sum_{n=1}^{\infty} x_n^*(y_k^* \otimes y_k(x_n)) = y_k^*\left(\sum_{n=1}^{\infty} x_n^*(y_k) x_n\right) = y_k^*(\tilde{u} y_k) = 0.$$

Therefore,

$$\langle S_\nu, u \rangle = \sum_{k=1}^{m} \langle y_k^* \otimes y_k, u \rangle = 0,$$

implying that

$$\langle S_\nu - I_X, u \rangle = \langle S_\nu, u \rangle - \langle I_X, u \rangle \to_\nu -\alpha \neq 0,$$

which is a contradiction with

$$\langle S_\nu - I_X, u\rangle \to_\nu 0.$$

For the "if" part, assume that $(X, \mathcal{N})$ does not have AP. Then

$$I_X \notin \overline{\{R \in \mathcal{F}(X) : R(Y) \subset Y \quad \forall Y \in \mathcal{N}\}}^{\tau_c}.$$

According to a separation theorem, there exists $u \in (\mathcal{L}(X), \tau_c)^* = X^* \hat{\otimes}_\pi X$, such that $\langle I_X, u\rangle = \operatorname{trace} u = 1$ and $\langle R, u\rangle = 0$ for every $R \in \mathcal{F}(X)$ for which $R(Y) \subset Y$ for all $Y \in \mathcal{N}$. In particular, by Corollary 4.5, for any $Y \in \mathcal{N}$,

$$\langle y^* \otimes y, u\rangle = 0 \quad \forall y^* \in (Y_-)^\perp, \ \forall y \in Y.$$

Let us show that $\tilde{u}(Y) \subset \overline{Y_-}$ for all $Y \in \mathcal{N}$. Consider a representation of u

$$u = \sum_{k=1}^{\infty} x_k^* \otimes x_k,$$

where $(x_k^*) \subset X^*$, $(x_k) \subset X$, and $\sum_{k=1}^{\infty} \|x_k^*\| \, \|x_k\| < \infty$. For any $Y \in \mathcal{N}$, we have

$$y^*(\tilde{u}y) = y^*\left(\sum_{k=1}^{\infty} x_k^*(y)x_k\right) = \langle y^* \otimes y, u\rangle = 0 \quad \forall y \in Y, \quad \forall y^* \in (Y_-)^\perp.$$

Hence, $\tilde{u}y \in ((Y_-)^\perp)_\perp = \overline{Y_-}$ for all $y \in Y$ and $Y \in \mathcal{N}$, as wished.

By condition ($\mathrm{AP}_\mathcal{N}$), $\operatorname{trace} u = 0$, which is a contradiction with $\operatorname{trace} u = 1$. □

Clearly, in the special case when $\mathcal{N} = \{\{0\}, X\}$, condition ($\mathrm{AP}_\mathcal{N}$) reduces to (AP), because $X_- = \{0\}$.

Let us spell out Theorem 4.20 for the finite nests.

Corollary 4.21 (cf. [FJ3, Corollary 1])**.** *Let X be a Banach space. Let $\mathcal{N} = \{Y_1, \ldots, Y_n\}$ be an n-element nest of closed subspaces of X containing $\{0\}$ and X. Assume that $\{0\} = Y_1 \subset Y_2 \subset \cdots \subset Y_n = X$. Then $(X, \mathcal{N})$ has the approximation property if and only if the following condition holds.*
If $u \in X^* \hat{\otimes}_\pi X$ is such that $\tilde{u}(Y_{m+1}) \subset Y_m$ for $m = 1, \ldots, n-1$, then $\operatorname{trace} u = 0$.

Proof. This is an immediate corollary because $\overline{(Y_{m+1})_-} = \overline{Y_m} = Y_m$ for $m = 1, \ldots, n-1$ and $\overline{(Y_1)_-} = \{0\} = Y_1$. □

Remark 4.22. In the special case when $\mathcal{N} = \{\{0\}, Y, X\}$, where Y is a closed subspace of X, Corollary 4.21 reduces to [FJ2, Theorem 2.2], stating that *the AP of the pair* (X, Y) *is equivalent to the condition: if* $u \in X^* \hat{\otimes}_\pi X$ *is such that* $\tilde{u}(X) \subset Y$ *and* $\tilde{u}(Y) = \{0\}$, *then* $\operatorname{trace} u = 0$.

Our goal is to present a criterion of the *duality* nest APs in the same spirit. We shall rely on Theorems 4.16 and 4.20.

Let $\mathcal{N}$ be a nest of subspaces of X containing X. For $Y \in \mathcal{N}$, we define the *subspace* Y_+ of X, the "dual version" of Y_-, as follows. If $Y \neq X$, then

$$Y_+ = \bigcap \{H \in \mathcal{N} : Y \subset H,\ Y \neq H\},$$

and if $Y = X$, then $Y_+ = X$.

Theorem 4.23. *Let* X *be a Banach space. Let* $\mathcal{N}$ *be a nest of closed subspaces of* X *containing* $\{0\}$ *and* X. *Assume that* $\mathcal{N}$ *is closed under closures of arbitrary unions and satisfies condition* $(*)$. *Then* $(X, \mathcal{N})$ *has the duality approximation property if and only if the following condition holds.*
If $u \in X^{**} \hat{\otimes}_\pi X^*$ *is such that* $\tilde{u}(Y^\perp) \subset (Y_+)^\perp$ *for all* $Y \in \mathcal{N}$, *then* $\operatorname{trace} u = 0$.

Proof. Since $\mathcal{N}$ satisfies condition $(*)$, by Theorem 4.16, the pair $(X, \mathcal{N})$ has the duality AP if and only if the pair $(X^*, \mathcal{N}^\perp)$ has the AP. By Proposition 2.7, $\mathcal{N}^\perp$ is closed under arbitrary intersections because $\mathcal{N}$ is closed under closures of arbitrary unions. Also $\mathcal{N}^\perp$ contains $\{0\}(= X^\perp)$ and $X^*(= \{0\}^\perp)$. Hence, from Theorem 4.20, we see that the AP of $(X^*, \mathcal{N}^\perp)$ is in its turn equivalent to the condition in Theorem 4.23 whenever $\overline{(Y^\perp)_-} = (Y_+)^\perp$ for all $Y \in \mathcal{N}$. But this is indeed the case. If $Y \neq X$, then $\mathcal{N}' := \{H \in \mathcal{N} :\ Y \subset H,\ Y \neq H\}$ is a non-empty subfamily of $\mathcal{N}$ (because $\mathcal{N}$ contains X). Since $\mathcal{N}$ satisfies condition $(*)$, by Remark 2.8, we have

$$\overline{(Y^\perp)_-} = \overline{\bigcup_{H \in \mathcal{N}'} H^\perp} = \Big(\bigcap_{H \in \mathcal{N}'} H\Big)^\perp = (Y_+)^\perp.$$

If $Y = X$, then $\overline{(Y^\perp)_-} = \{0\} = X^\perp = (Y_+)^\perp$. □

For finite nests, the corresponding criterion of the duality nest AP is as follows.

Corollary 4.24. *Let* X *be a Banach space. Let* $\mathcal{N} = \{Y_1, \ldots, Y_n\}$ *be an* n*-element nest of closed subspaces of* X *containing* $\{0\}$ *and* X. *Assume that* $\{0\} = Y_1 \subset Y_2 \subset \cdots \subset Y_n = X$. *Then* $(X, \mathcal{N})$ *has the duality approximation*

property if and only if the following condition holds.
If $u \in X^{**}\hat{\otimes}_\pi X^*$ *is such that* $\tilde{u}(Y_m^\perp) \subset Y_{m+1}^\perp$ *for* $m = 1, \ldots, n-1$*, then* $\operatorname{trace} u = 0$.

Proof. This is an immediate corollary because $(Y_m)_+ = Y_{m+1}$ for $m = 1, \ldots, n-1$ and $(Y_n)_+ = X = Y_n$. $\square$

Taking $\mathcal{N} = \{\{0\}, Y, X\}$, where Y is a closed subspace of X, the following criterion of the duality AP for pairs is immediate from Corollary 4.24.

Corollary 4.25. *Let* X *be a Banach space and let* Y *be a closed subspace of* X*. Then* (X, Y) *has the duality approximation property if and only if the following condition holds.*
If $u \in X^{**}\hat{\otimes}_\pi X^*$ *is such that* $\tilde{u}(X^*) \subset Y^\perp$ *and* $\tilde{u}(Y^\perp) = \{0\}$*, then* $\operatorname{trace} u = 0$.

Chapter 5

Lifting bounded convex approximation properties from Banach spaces to their dual spaces

In this chapter, we study the lifting of bounded convex approximation properties from a Banach space to its dual space in some special cases. We show that for such a lifting rather weak forms of the principle of local reflexivity and the extendable local reflexivity are sufficient. It is also shown that such a lifting is possible whenever the dual space already enjoys a weaker bounded convex approximation property. We also complement and extend some results from [GS2, O2]. This chapter relies on [OV1, OV2, V].

5.1 Lifting of the bounded convex approximation property and the related local reflexivity

Let X be a Banach space. Starting from the seminal paper [JRZ] by Johnson, Rosenthal, and Zippin, cases when bounded APs can be lifted from X to its dual space X^* have been studied, for instance, in [FJ1, GS2, J2, KW, LisO, O1, O2]. By an important result, due to Johnson and Oikhberg [JO], such a lifting is possible when X is extendably locally reflexive (see Theorem 5.2).

Definition 5.1. A Banach space X is λ-*extendably locally reflexive* (ELR) if for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $T \in \mathcal{L}(X^{**})$ such that $T(E) \subset X$, $\|T\| \leq \lambda + \varepsilon$, and $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$.

The ELR was discovered by Rosenthal and studied by Johnson, Oikhberg, and Rosenthal in [JO] and [OR]. The next theorem is proved in [JO, Theorem 3.1 (1)]; for its quantized version, see [OR, Theorem 3.13].

Theorem 5.2 (Johnson–Oikhberg)**.** *If a Banach space X is λ-extendably locally reflexive and has the μ-bounded approximation property, then X^* has the $\lambda\mu$-bounded approximation property.*

The proof in [JO] relies on the PLR (see [O2, Corollary 3.13] for an alternative proof which does not use the PLR). The method of the proof in [JO] seems to suggest that the PLR for Banach lattices and the PLR respecting subspaces could be used for the lifting of the positive bounded AP and of the bounded AP of pairs, respectively. Since these APs are special cases of the convex AP, a question arises about a *unified approach* to lifting results in the framework of convex APs.

We introduce the following general forms of the ELR and the PLR. In Theorem 5.6, we shall see that these rather weak forms of the ELR and the PLR are sufficient for the lifting of different bounded APs from Banach spaces to their dual spaces.

Definition 5.3. Let X be a Banach space and let C be a subset of $\mathcal{L}(X^{**})$. Let $1 \leq \lambda < \infty$. We say that X is λ-*extendably locally reflexive of type* C if for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $T \in C$ with $\|T\| \leq \lambda + \varepsilon$ such that $T(E) \subset X$ and

$$|x^*(Tx^{**}) - x^{**}(x^*)| \leq \varepsilon \ \ \forall x^{**} \in S_E, \ \forall x^* \in S_F.$$

Notice that if C and D are subsets of $\mathcal{L}(X^{**})$ such that $C \subset D$ and X is λ-ELR of type C, then X is λ-ELR of type D. The λ-ELR of a Banach space X clearly implies the λ-ELR of type $\mathcal{L}(X^{**})$. More examples will be presented in the following sections.

Definition 5.4. Let X be a Banach space, let A and B be subsets of $\mathcal{L}(X)$ and $\mathcal{L}(X^{**})$, respectively. We say that the *principle of local reflexivity of type* $B \to A$ holds in X if for all $T \in B$, for all finite-dimensional subspaces $E \subset$

X^{**} and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $S \in A$ with $\|S\| \leq \|T\| + \varepsilon$ such that

$$|(Tx^{**})(x^*) - x^{**}(S^*x^*)| \leq \varepsilon \ \ \forall x^{**} \in S_E,\ \forall x^* \in S_F.$$

The PLR of type $B \to A$ means that the operators on X^{**} of "type B" are "locally" of "type A" on X.

Examples 5.5. The following assertions are true.

(1) By the PLR (see [JRZ, the proof of Theorem 3.3] or, e.g., [OP, Theorem 2.5]), in every Banach space X, the PLR of type $\mathcal{F}(X^{**}) \to \mathcal{F}(X)$ holds.

(2) Let X be a Banach space and let Y be a closed subspace of X. By the PLR respecting subspaces (see [O5, Theorem 1.2]), the PLR of type $\{T \in \mathcal{F}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\} \to \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$ holds in X.

(3) Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X such that the nest $\mathcal{N}^{\perp\perp}$ is complete. By Theorem 4.13, the PLR of type $\{T \in \mathcal{F}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\ \forall Y \in \mathcal{N}\} \to \{S \in \mathcal{F}(X) : S(Y) \subset Y\ \forall Y \in \mathcal{N}\}$ holds in X.

(4) In every Banach lattice X, the PLR of type $\mathcal{F}(X^{**})_+ \to \mathcal{F}(X)_+$ holds (see Corollary 5.20).

(5) Let A be a subset of $\mathcal{L}(X)$. Trivially, in every Banach space X, the PLR of type $\{S^{**} : S \in A\} \to A$ holds (even for $\varepsilon = 0$).

Theorem 5.6. *Let X be a Banach space. Let A be a convex subset of $\mathcal{L}(X)$ containing 0. Let B and C be subsets of $\mathcal{L}(X^{**})$ such that $\{S^{**} : S \in A\} \circ C \subset B$. Let $1 \leq \lambda, \mu < \infty$. Assume that the principle of local reflexivity of type $B \to A$ holds in X. If X is λ-extendably locally reflexive of type C and has the μ-bounded A-approximation property, then X has the $\lambda\mu$-bounded duality A-approximation property.*

Proof. By Proposition 3.11 (c), it suffices to construct a net $(R_\nu) \subset A$ such that $\limsup_\nu \|R_\nu\| \leq \lambda\mu$ and

$$x^{**}(R_\nu^* x^*) \to_\nu x^{**}(x^*) \ \ \forall x^* \in X^*,\ \forall x^{**} \in X^{**}.$$

Consider the set of all $\nu = (E, F, \varepsilon)$, where $E \subset X^{**}$ and $F \subset X^*$ are finite-dimensional subspaces and $\varepsilon > 0$, directed in the natural way. Since

X is λ-ELR of type C, for every ν, there exists an operator $T_\nu \in C$ with $\|T_\nu\| \leq \lambda + \varepsilon$ such that $T_\nu(E) \subset X$ and

$$|x^*(T_\nu x^{**}) - x^{**}(x^*)| \leq \varepsilon \ \ \forall x^{**} \in S_E, \ \forall x^* \in S_F.$$

The set $T_\nu(S_E) \subset X$ is compact because S_E is compact. Since X has the μ-bounded A-approximation property, there exists $S_\nu \in A$ with $\|S_\nu\| \leq \mu$ such that

$$\|S_\nu T_\nu x^{**} - T_\nu x^{**}\| \leq \varepsilon \ \ \forall x^{**} \in S_E.$$

Hence, for any $x^{**} \in S_E$,

$$\begin{aligned}\|S_\nu^{**} T_\nu x^{**} - T_\nu x^{**}\| &= \|S_\nu^{**} - I_{X^{**}}\| \, \|T_\nu x^{**}\| = \|S_\nu - I_X\| \, \|T_\nu x^{**}\| \\ &= \|S_\nu T_\nu x^{**} - T_\nu x^{**}\| \leq \varepsilon.\end{aligned}$$

We have $S_\nu^{**} T_\nu \in \{S^{**} : S \in A\} \circ C \subset B$. By the PLR of type $B \to A$, there exists $R_\nu \in A$ with

$$\|R_\nu\| \leq \|S_\nu^{**} T_\nu\| + \varepsilon \leq \mu(\lambda + \varepsilon) + \varepsilon,$$

implying that

$$\limsup_\nu \|R_\nu\| \leq \lambda\mu,$$

and

$$|(S_\nu^{**} T_\nu x^{**})(x^*) - x^{**}(R_\nu^* x^*)| \leq \varepsilon \ \ \forall x^{**} \in S_E, \ \forall x^* \in S_F.$$

For any $x^{**} \in S_E$ and $x^* \in S_F$, we have

$$\begin{aligned}|x^{**}(R_\nu^* x^*) - x^{**}(x^*)| &\leq |x^{**}(R_\nu^* x^*) - (S_\nu^{**} T_\nu x^{**})(x^*)| \\ &\quad + |(S_\nu^{**} T_\nu x^{**})(x^*) - (T_\nu x^{**})(x^*)| \\ &\quad + |(x^*)(T_\nu x^{**}) - x^{**}(x^*)| \\ &\leq 3\varepsilon.\end{aligned}$$

Let us show that

$$\lim_\nu x^{**}(R_\nu^* x^*) = x^{**}(x^*) \ \ \forall x^* \in X^*, \ \forall x^{**} \in X^{**}.$$

Clearly, it is sufficient to show the equality for the elements of unit spheres. Let $x^* \in S_{X^*}$ and $x^{**} \in S_{X^{**}}$ be arbitrary elements. For $\varepsilon_0 > 0$, take $\nu_0 = (\mathrm{span}\{x^{**}\}, \mathrm{span}\{x^*\}, \varepsilon_0/3)$. If $\nu = (E, F, \varepsilon) \geq \nu_0$, then $x^{**} \in \mathrm{span}\{x^{**}\} \subset E$, $x^* \in \mathrm{span}\{x^*\} \subset F$, $\varepsilon \leq \varepsilon_0/3$, and we have

$$|x^{**}(R_\nu^* x^*) - x^{**}(x^*)| \leq 3\varepsilon \leq \varepsilon_0,$$

as needed. $\square$

From Theorem 5.6 and Example 5.5 (5), we have the following immediate corollary that will be applied in lifting results Theorem 5.28 and Corollary 5.29 in Section 5.3.

Corollary 5.7. *Let X be a Banach space. Let A be a convex subset of $\mathcal{L}(X)$ containing 0 and let C be a subset of $\mathcal{L}(X^{**})$ such that $\{S^{**} : S \in A\} \circ C \subset \{S^{**} : S \in A\}$. Let $1 \leq \lambda, \mu < \infty$. If X is λ-extendably locally reflexive of type C and has the μ-bounded A-approximation property, then X has the $\lambda\mu$-bounded duality A-approximation property.*

Remark 5.8. If X^* or X^{**} has the Radon–Nikodým property, then Corollary 5.7 asserts that X has the metric duality A-AP (cf. Remark 3.1).

5.2 Applications of the lifting theorem

Let X be a Banach space. Taking $A = \mathcal{F}(X)$, $B = \mathcal{F}(X^{**})$, and $C = \mathcal{L}(X^{**})$, and noticing that "X has the $\lambda\mu$-bounded duality AP" trivially implies that "X^* has the $\lambda\mu$-bounded AP", the following result, hence also Theorem 5.2, is immediate from Example 5.5 (1) and Theorem 5.6.

Corollary 5.9 (cf. Theorem 5.2)**.** *Let X be a Banach space. Let $1 \leq \lambda, \mu < \infty$. If X is λ-extendably locally reflexive of type $\mathcal{L}(X^{**})$ and has the μ-bounded approximation property, then X has the $\lambda\mu$-bounded duality approximation property.*

Remark 5.10. The proof of Theorem 5.2 in [JO] uses the PLR, which states that X^{**} and X are "locally almost the same". Our proof uses the version of the PLR stating that $\mathcal{F}(X^{**})$ and $\mathcal{F}(X)$ are "locally almost the same". This appropriately chosen version of the PLR considerably shortens and eases the proof of Theorem 5.2

By Theorem 5.6, even stronger versions of Corollary 5.9 hold (see Theorems 5.12 and 5.15 below). These versions are expressed using APs of pairs and APs of nests, respectively. It seems natural to define the ELR also for pairs as follows.

Definition 5.11. Let X be a Banach space and Y a closed subspace of X. Let $1 \leq \lambda < \infty$. We say that the *pair* (X, Y) is λ-*extendably locally reflexive* if X is λ-extendably locally reflexive of type $\{T \in \mathcal{L}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\}$.

In the special case of $(X, \{0\})$, the following Theorem 5.12 coincides with Corollary 5.9.

Theorem 5.12. *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \le \lambda, \mu < \infty$. If the pair (X, Y) is λ-extendably locally reflexive and has the μ-bounded approximation property, then the pair (X, Y) has the $\lambda\mu$-bounded duality approximation property.*

Proof. Let $A = \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$, $B = \{T \in \mathcal{F}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\}$, and $C = \{T \in \mathcal{L}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\}$. Then $\{S^{**} : S \in A\} \circ C \subset B$, because if $S(Y) \subset Y$ for $S \in \mathcal{L}(X)$, then clearly $S^*(Y^\perp) \subset Y^\perp$, hence also $S^{**}(Y^{\perp\perp}) \subset Y^{\perp\perp}$. Therefore, the claim is immediate from Example 5.5 (2) and Theorem 5.6. □

Remark 5.13. If, in Theorem 5.12, X^* or X^{**} has the Radon–Nikodým property, then the pair (X, Y) has the metric duality AP (cf. Remark 3.1).

It is convenient to define the ELR also for pairs $(X, \mathcal{N})$ extending Definition 5.11 as follows.

Definition 5.14. Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X. Let $1 \le \lambda < \infty$. We say that the pair $(X, \mathcal{N})$ is λ-*extendably locally reflexive* if X is λ-ELR of type $\{T \in \mathcal{L}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\ \forall Y \in \mathcal{N}\}$.

Taking $A = \{S \in \mathcal{F}(X) : S(Y) \subset Y\ \forall Y \in \mathcal{N}\}$, $B = \{T \in \mathcal{F}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\ \forall Y \in \mathcal{N}\}$ and $C = \{T \in \mathcal{L}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\ \forall Y \in \mathcal{N}\}$, Example 5.5 (3) and Theorem 5.6 immediately yield the following lifting result for bounded nest APs.

Theorem 5.15. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X such that the nest $\mathcal{N}^{\perp\perp}$ is complete. Let $1 \le \lambda, \mu < \infty$. If the pair $(X, \mathcal{N})$ is λ-extendably locally reflexive and has the μ-bounded approximation property, then the pair $(X, \mathcal{N})$ has the $\lambda\mu$-bounded duality approximation property.*

For finite nests, Theorem 5.15 reads as follows.

Corollary 5.16. *Let X be a Banach space and let $\mathcal{N}$ be a finite nest of closed subspaces of X. Then the assertion of Theorem 5.15 holds.*

Proof. Notice that $(X, \mathcal{N})$ is λ-ELR if and only if $(X, \mathcal{N} \cup \{0\} \cup X)$ is λ-ELR. Also notice that the bounded AP and its duality version of $(X, \mathcal{N})$ are equivalent to the bounded AP and its duality version of $(X, \mathcal{N} \cup \{0\} \cup X)$, respectively. Therefore, the assertion is immediate from Theorem 5.15. □

Remark 5.17. Theorem 5.15 and Corollary 5.16 are nest versions of Theorem 5.12. In the special case when $\mathcal{N} = \{Y\}$, where Y is a closed subspace of X, Corollary 5.16 reduces to Theorem 5.12.

Remark 5.18. If, in Theorem 5.15 (or Corollary 5.16), X^* or X^{**} has the Radon–Nikodým property, then $(X, \mathcal{N})$ has the metric duality AP (cf. Remark 3.1).

In [LisO, Theorem 5.6], it was established that positive finite-rank operators between dual Banach lattices are "locally conjugate". From this theorem, the following result about "locally biconjugate" operators can be obtained.

Theorem 5.19. *Let X and Y be Banach lattices. If $T \in \mathcal{F}(X^{**}, Y^{**})_+$, $E \subset X^{**}$ and $F \subset Y^*$ are finite-dimensional subspaces, and $\varepsilon > 0$, then there exists $S \in \mathcal{F}(X, Y)_+$ with $\|S\| \le \|T\| + \varepsilon$ such that*

$$|(Tx^{**})(y^*) - x^{**}(S^*y^*)| \le \varepsilon \quad \forall x^{**} \in S_E,\ \forall y^* \in S_F.$$

Proof. Let E and F be finite-dimensional subspaces of X^{**} and Y^*, respectively, and $T \in \mathcal{F}(X^{**}, Y^{**})_+$. Let $\varepsilon > 0$. Take $\delta > 0$ such that $\delta(2 + \delta) \le \varepsilon/(1 + \|T\|)$.

We shall apply [LisO, Theorem 5.6 and Remark 5.3] twice. On the first application, there exists $R \in \mathcal{F}(Y^*, X^*)_+$ such that $\|R\| \le (1 + \delta)\|T\|$, and $\|R^*x^{**} - Tx^{**}\| \le \delta\|x^{**}\|$ for all $x^{**} \in E$. On the second application, we have an operator $S \in \mathcal{F}(X, Y)_+$ such that $\|S\| \le (1+\delta)\|R\|$, and $\|S^*y^* - Ry^*\| \le \delta\|y^*\|$ for all $y^* \in F$. Hence,

$$\|S\| \le (1 + \delta)^2\|T\| = \|T\| + \delta(2 + \delta)\|T\| \le \|T\| + \varepsilon.$$

For any $x^{**} \in S_E$ and $y^* \in S_F$, we have

$$\begin{aligned} |(Tx^{**})(y^*) - x^{**}(S^*y^*)| &\le |(Tx^{**})(y^*) - (R^*x^{**})(y^*)| \\ &\quad + |(R^*x^{**})(y^*) - x^{**}(S^*y^*)| \\ &= |(R^*x^{**})(y^*) - (Tx^{**})(y^*)| \\ &\quad + |x^{**}(Ry^*) - x^{**}(S^*y^*)| \\ &\le \delta\|x^{**}\|\|y^*\| + \|x^{**}\|\delta\|y^*\| = 2\delta \\ &< \delta(2 + \delta) \le \varepsilon, \end{aligned}$$

as desired. □

We shall need the immediate consequence.

Corollary 5.20. *In every Banach lattice X, the principle of local reflexivity of type $\mathcal{F}(X^{**})_+ \to \mathcal{F}(X)_+$ holds.*

It is convenient to introduce a positive version of the ELR in Banach lattices as follows.

Definition 5.21. Let X be a Banach lattice. Let $1 \leq \lambda < \infty$. We say that X is *positively λ-extendably locally reflexive* if it is λ-extendably locally reflexive of type $\mathcal{L}(X^{**})_+$.

It is natural to call the λ-bounded duality $\mathcal{F}(X)_+$-AP of a Banach lattice X the *λ-bounded duality positive approximation property.* This property trivially implies the λ-bounded positive AP both for X and its dual lattice X^*. Thus, taking $A = \mathcal{F}(X)_+$, $B = \mathcal{F}(X^{**})_+$, and $C = \mathcal{L}(X^{**})_+$, we immediately get from Corollary 5.20 and Theorem 5.6 the following version of Theorem 5.2 (or Corollary 5.9) for positive APs.

Theorem 5.22. *Let X be a Banach lattice. Let $1 \leq \lambda, \mu < \infty$. If X is positively λ-extendably locally reflexive and has the μ-bounded positive approximation property, then X has the $\lambda\mu$-bounded duality positive approximation property.*

Remark 5.23. If, in Theorem 5.22, X^* or X^{**} has the Radon–Nikodým property, then X has the metric duality positive approximation property (cf. Remark 3.1).

Remark 5.24. It is well known that abstract L-spaces and M-spaces have the metric positive AP. Since the dual of an M-space is an L-space and vice versa, by Theorem 5.27 below, abstract L-spaces and M-spaces are positively 1-ELR. On the other hand, Pełczyński's universal space U for unconditional bases, considered as a Banach lattice, has the metric positive AP, but U^* fails the AP (see, e.g., [LisO, Remark 3.2]); hence, by Theorem 5.22, U is not positively ELR. We do not know any example of a non-reflexive Banach lattice without the (metric positive) AP which is positively ELR. Neither do we know of a Banach lattice which is ELR, but not positively ELR.

5.3 Extendable local reflexivity implied by bounded convex approximation properties

The lifting Theorem 5.2 by Johnson and Oikhberg has a strong converse, due to Rosenthal (see [JO, Theorem 3.1 (2)]).

Theorem 5.25 (Rosenthal)**.** *Let X be a Banach space. Let $1 \le \lambda < \infty$. If X^* has the λ-bounded approximation property, then X is λ-extendably locally reflexive.*

Recall that, by Johnson's Theorem 3.15 (see Section 3.5), the assumption "X^* has the λ-bounded AP" is equivalent to "X^* has the λ-bounded AP with conjugate operators". Keeping this in mind, we shall see that Rosenthal's Theorem 5.25 can be extended as follows, providing a general converse to Theorem 5.6.

Proposition 5.26. *Let X be a Banach space. Let A be a subset of $\mathcal{W}(X)$. Let $1 \le \lambda < \infty$. If X^* has the λ-bounded A-approximation property with conjugate operators, then X is λ-extendably locally reflexive of type $\{S^{**} : S \in A\}$.*

Proof. Let $F \subset X^*$ be a finite-dimensional subspace and let $\varepsilon > 0$. Since S_F is compact and X^* has the λ-bounded $\{S^* : S \in A\}$-AP, there exists $S \in A$ with $\|S\| \le \lambda$ such that

$$\|S^*x^* - x^*\| \le \varepsilon \ \ \forall x^* \in S_F.$$

Then $S^{**} \in \{S^{**} : S \in A\}$ and $\|S^{**}\| \le \lambda$. Since $S \in \mathcal{W}(X)$, we have that $S^{**}(X^{**}) \subset X$ (see, e.g., [DuS, Theorem VI.4.2]). For any $x^{**} \in S_{X^{**}}$ and $x^* \in S_F$, we have

$$|x^*(S^{**}x^{**}) - x^{**}(x^*)| = |x^{**}(S^*x^*) - x^{**}(x^*)| \le \|x^{**}\| \, \|S^*x^* - x^*\| \le \varepsilon.$$

Hence, for any finite-dimensional subspace E of X^{**}, the conditions of the λ-ELR of type $\{S^{**} : S \in A\}$ for X are satisfied. □

In the case of Banach lattices, we have the following version of Rosenthal's Theorem 5.25.

Theorem 5.27. *Let X be a Banach lattice. Let $1 \le \lambda < \infty$. If the dual lattice X^* has the λ-bounded positive approximation property, then X is positively λ-extendably locally reflexive.*

Proof. Since X^* has the λ-bounded positive AP, it has the λ-bounded positive AP with conjugate operators (see [LisO, Proposition 5.7]). Let $A = \mathcal{F}(X)_+$. Then $A \subset \mathcal{W}(X)$ and $\{S^{**} : S \in A\} \subset \mathcal{F}(X^{**})_+ \subset \mathcal{L}(X^{**})_+$. The claim is immediate from Proposition 5.26. □

The following result shows that the lifting of convex APs from a Banach space to its dual space is possible whenever the dual space already enjoys a weaker AP.

Theorem 5.28. *Let X be a Banach space. Let A be a convex subset of $\mathcal{L}(X)$ containing 0 and let B be a subset of $\mathcal{W}(X)$ such that $A \circ B \subset A$. Let $1 \leq \lambda, \mu < \infty$. If X^* has the λ-bounded B-approximation property with conjugate operators and X has the μ-bounded A-approximation property, then X has the $\lambda\mu$-bounded duality A-approximation property.*

Proof. The result is immediate from Proposition 5.26 and Corollary 5.7. Indeed, it follows from Proposition 5.26 that X is λ-ELR of type $\{T^{**} : T \in B\}$. Notice that $\{S^{**} : S \in A\} \circ \{T^{**} : T \in B\} \subset \{S^{**} : S \in A\}$ (because $A \circ B \subset A$). Now, since X is λ-ELR of type $\{T^{**} : T \in B\}$ and has the μ-bounded A-AP, by Corollary 5.7, X has the $\lambda\mu$-bounded duality A-AP. $\square$

If, in Theorem 5.28, $A \subset \mathcal{K}(X)$ and X^* or X^{**} has the Radon–Nikodým property, then X has the metric duality A-AP (cf. Remark 3.1).

Let us spell out an immediate general application of Theorem 5.28 to (positive) approximation properties of pairs.

Corollary 5.29. *Let X be a Banach space and Y a closed subspace of X. Let $\mathcal{A}$ be an operator ideal. Denote $A = \{S \in \mathcal{A}(X) : S(Y) \subset Y\}$ and $B = \{T \in \mathcal{W}(X) : T(Y) \subset Y\}$. Let $1 \leq \lambda, \mu < \infty$. Then the assertion of Theorem 5.28 holds. In the special case when X is a Banach lattice, A and B may be replaced by $A_+ := A \cap \mathcal{L}(X)_+$ and $B_+ := B \cap \mathcal{L}(X)_+$.*

The classical cases when Corollary 5.29 applies are $A = \mathcal{F}(X)$ and $A = \mathcal{K}(X)$. For instance, it follows that the dual lattice X^* has the bounded (metric if X^* has the Radon–Nikodým property) positive AP whenever X has the bounded positive AP and X^* has the bounded positive weakly compact AP with conjugate operators.

Let $\mathcal{A}$ be an operator ideal. It is natural to consider weaker versions of nest APs by replacing "$\mathcal{F}(X)$" in the definitions of the nest APs with "$\mathcal{A}(X)$", the component of $\mathcal{A}$. We say that the *pair* $(X, \mathcal{N})$ has the (λ-*bounded*) $\mathcal{A}$-*approximation property* if X has the (λ-bounded) $\{S \in \mathcal{A}(X) : S(Y) \subset Y \;\; \forall Y \in \mathcal{N}\}$-AP. The ($\lambda$-*bounded*) *duality* $\mathcal{A}$-*AP* of a pair $(X, \mathcal{N})$ is defined in a standard way, as the (λ-bounded) duality $\{S \in \mathcal{A}(X) : S(Y) \subset Y \;\; \forall Y \in \mathcal{N}\}$-AP of X. Clearly, the nest $\mathcal{A}$-APs coincide with the nest APs whenever $\mathcal{A} = \mathcal{F}$.

Proposition 5.26, specialized to nest $\mathcal{A}$-APs, yields the following.

Proposition 5.30. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X. Let $\mathcal{A} \subset \mathcal{W}$ be an operator ideal. Let $1 \le \lambda < \infty$. If the pair $(X, \mathcal{N})$ has the λ-bounded duality $\mathcal{A}$-approximation property, then X is λ-extendably locally reflexive of type $\{S^{**} : S \in \mathcal{A}(X),\ S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$.*

Let us spell out the special case of Proposition 5.30 for the nest APs, because this provides a strong converse to Theorem 5.15. The "moreover" part below follows from Theorem 4.15.

Theorem 5.31. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X. Let $1 \le \lambda < \infty$. If the pair $(X, \mathcal{N})$ has the λ-bounded duality approximation property, then X is λ-extendably locally reflexive of type $\{S^{**} : S \in \mathcal{F}(X),\ S(Y) \subset Y\ \forall Y \in \mathcal{N}\}$; hence, the pair $(X, \mathcal{N})$ is λ-extendably locally reflexive.*
Moreover, if the nest $\mathcal{N}$ satisfies condition $(*)$, then the claim holds under the hypothesis that the pair $(X^*, \mathcal{N}^{\perp})$ has the λ-bounded approximation property.

The next Theorem 5.32 complements Theorem 5.15, showing that the lifting of bounded nest APs from a Banach space to its dual space is possible whenever the dual space already enjoys a weaker nest AP. Important cases when Theorem 5.32 applies are $\mathcal{A} = \mathcal{F}$ and $\mathcal{A} = \mathcal{K}$. For instance, it follows that $(X, \mathcal{N})$ has the bounded duality AP whenever $(X, \mathcal{N})$ has the bounded AP and the bounded duality weakly compact AP.

Theorem 5.32. *Let X be a Banach space and let $\mathcal{N}$ be a nest of closed subspaces of X. Let $\mathcal{A}$ be an operator ideal. Let $1 \le \lambda, \mu < \infty$. If the pair $(X, \mathcal{N})$ has the λ-bounded duality $\mathcal{W}$-approximation property and the μ-bounded $\mathcal{A}$-approximation property, then the pair $(X, \mathcal{N})$ has the $\lambda\mu$-bounded duality $\mathcal{A}$-approximation property.*

Proof. Since $(X, \mathcal{N})$ has the λ-bounded duality $\mathcal{W}$-AP, by Proposition 5.30, X is λ-ELR of type $C := \{S^{**} : S \in \mathcal{W}(X),\ S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$. Clearly, the PLR of type $B := \{S^{**} : S \in \mathcal{A}(X),\ S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\} \to \{S \in \mathcal{A}(X) : S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\} =: A$ holds in X (cf. Example 5.5 (5)).

Moreover, we have the inclusion $\{S^{**}T : S \in A,\ T \in C\} \subset B$. Indeed, let $S \in A$ and $T \in C$. Then there is $R \in \mathcal{W}(X)$ such that $R(Y) \subset Y$ for all $Y \in \mathcal{N}$ and $R^{**} = T$. Hence, $SR \in \mathcal{A}(X)$ (because $\mathcal{A}$ is an operator ideal) and $(SR)(Y) \subset S(Y) \subset Y$ for all $Y \in \mathcal{N}$, implying that $S^{**}T = S^{**}R^{**} = (SR)^{**} \in B$.

An immediate application of Theorem 5.6 concludes the proof. $\square$

Remark 5.33. In the special case when $\mathcal{N} = \{Y\}$, where Y is a closed subspace of X, Theorem 5.32 is contained in Corollary 5.29.

5.4 The strong extendable local reflexivity

Let X be a Banach space. If one adds in the definition of the λ-ELR (see Definition 5.1) the requirement that the operator $T \in \mathcal{L}(X^{**})$ also satisfies $T^*(X^*) \subset X^*$, then one obtains the notion of the *strong λ-extendable local reflexivity*. The strong λ-ELR was introduced and studied in [O2]. Among others, Rosenthal's Theorem 5.25 was strengthened and extended in [O2, Theorem 3.4] as follows.

Theorem 5.34 (Oja)**.** *Let $\mathcal{A}$ be an operator ideal and let X be a Banach space. Let $1 \leq \lambda < \infty$. If $\mathcal{A}(X) \subset \mathcal{W}(X)$ and X^* has the λ-bounded $\mathcal{A}(X)$-approximation property with conjugate operators, then X is strongly λ-extendably locally reflexive.*

It was also observed in [O2, Proposition 3.2] that Rosenthal's Theorem 5.25 fails already for the bounded compact AP, i.e., $\mathcal{F}(X)$ cannot be replaced by $\mathcal{K}(X)$ in Theorem 5.25. This also means that the assumption "X has the λ-bounded $\mathcal{A}(X)$-AP *with conjugate operators*" is essential in Theorem 5.34. A "strong" example of this phenomenon was presented in [O2, Theorem 3.6]: there exists a strongly 1-ELR Banach space X with a monotone shrinking basis such that:

(a) its even duals $X^{**}, X^{****}, \ldots$ are strongly 1-ELR, have the metric compact AP, but do not have the bounded weakly compact AP with conjugate operators;

(b) its odd duals $X^*, X^{***}, \ldots$ are not ELR, but have the metric compact AP with conjugate operators.

Inspired by the idea of the proof of Theorem 5.34 (see [O2, Lemma 3.3]), we shall extend Theorem 5.34 to *convex APs of pairs* as follows (see also Theorem 5.38).

Theorem 5.35. *Let X be a Banach space and Y a closed subspace of X. Let A be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let $1 \leq \lambda < \infty$. If X^* has the λ-bounded $\{S \in A : S(Y) \subset Y\}$-approximation property with conjugate operators, then for every finite-dimensional subspace $F \subset X^*$ and*

for every $\varepsilon > 0$, there exists $S \in A$ with $S(Y) \subset Y$ such that $\|S\| \le \lambda + \varepsilon$ and $S^*x^* = x^*$ for all $x^* \in F$.
Moreover, if $A \subset \mathcal{W}(X)$, then the operator $T := S^{**}$ has the following properties: $T(X^{**}) \subset X$, $T(Y^{\perp\perp}) \subset Y$, $\|T\| \le \lambda + \varepsilon$, $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in X^{**}$ and $x^* \in F$, and $T^*(X^*) \subset X^*$.

Proof. Let $F \subset X^*$ be a finite-dimensional subspace and let $\varepsilon > 0$. Look at X^* endowed with its weak* topology and, using Lemma 3.4, choose a weak*-to-weak* continuous linear projection R on X^* such that $\operatorname{ran} R = F$ and $R(Y^\perp) \subset Y^\perp$. Then there exists $Q \in \mathcal{F}(X)$ such that $R = Q^*$. Hence, $F = \operatorname{ran} Q^*$ and, by Lemma 2.5, $Q(Y) \subset Y$.

Since X^* has the λ-bounded $\{S^* : S \in A,\ S(Y) \subset Y\}$-AP, by condition (e) of Theorem 3.2, there exists $P \in A$ with $P(Y) \subset Y$ and $\|P\| \le \lambda$ such that $\|P^*x^* - x^*\| \le (\varepsilon/\|Q\|)\,\|x^*\|$ for all $x^* \in F$.

Put $S = P + Q(I_X - P)$. Then, clearly, $S \in A$ (because $\mathcal{F}(X) \subset A$) and $S(Y) \subset Y$. Let us observe that

$$\begin{aligned}\|(I_{X^*} - P^*)Q^*\| &= \sup_{x^* \in B_{X^*}} \|P^*(Q^*x^*) - Q^*x^*\| \le \sup_{x^* \in B_{X^*}} (\varepsilon/\|Q\|)\,\|Q^*x^*\| \\ &= (\varepsilon/\|Q\|)\,\|Q^*\| = \varepsilon.\end{aligned}$$

Hence, we have

$$\|S\| = \|S^*\| = \|P^* + (I_{X^*} - P^*)Q^*\| \le \|P^*\| + \|(I_{X^*} - P^*)Q^*\| \le \lambda + \varepsilon.$$

Let us also observe that

$$S^* = I_{X^*} + (I_{X^*} - P^*)(Q^* - I_{X^*}).$$

Hence, clearly, S^* is identity on $F = \operatorname{ran} Q^*$.

Assume now that $A \subset \mathcal{W}(X)$. Then $S \in \mathcal{W}(X)$ and $S^* \in \mathcal{W}(X^*)$. Therefore, $T := S^{**} \in \mathcal{W}(X^{**}, X)$ and $T^* \in \mathcal{W}(X^{***}, X^*)$ (see, e.g., [DuS, Theorem VI.4.2]). Moreover, since $S(Y) \subset Y$, we get that $T(Y^{\perp\perp}) \subset X \cap Y^{\perp\perp}$. Notice that $X \cap Y^{\perp\perp} \subset Y$. Indeed, assume that there is x_0 such that $x_0 \in X \cap Y^{\perp\perp}$, but $x_0 \notin Y$. Using the Hahn–Banach theorem, choose $x^* \in X^*$ such that $x^*(x_0) = 1$ and $x^*(y) = 0$ for all $y \in Y$. Hence, $x^* \in Y^\perp$. Since $x_0 \in Y^{\perp\perp}$, it follows that $x^*(x_0) = 0$, which is a contradiction with $x^*(x_0) = 1$. Therefore, $X \cap Y^{\perp\perp} \subset Y$.

We also have

$$x^*(Tx^{**}) = x^{**}(S^*x^*) = x^{**}(x^*) \quad \forall x^{**} \in X^{**},\ \forall x^* \in F.$$

$\square$

Remark 5.36. Theorem 5.35 without its "moreover" part also follows from [CKZ, Proposition 2.3], the equivalent conditions to the bounded A-APs of pairs. In [CKZ], the observation was made by extending the proof of [FJP, Lemma 1.5]. The converse of the statement in Theorem 5.35 also holds and is standard to deduce (see, e.g., the proof of Theorem 3.5 (f) $\Rightarrow$ (e)).

It is appropriate to extend the strong ELR to pairs as follows.

Definition 5.37. Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. We say that the *pair* (X, Y) is *strongly λ-extendably locally reflexive* if for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $T \in \mathcal{L}(X^{**})$ such that $T(E) \subset X$, $T(Y^{\perp\perp}) \subset Y$, $\|T\| \leq \lambda + \varepsilon$, $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$, and $T^*(X^*) \subset X^*$.

Let A be a linear subspace of $\mathcal{L}(X)$. It is natural to say that the *pair* $(X^*, Y^\perp)$ has the λ-bounded *A-approximation property with conjugate operators* if X^* has the λ-bounded $\{S \in A : \; S(Y) \subset Y\}$-AP with conjugate operators. Thus, the "moreover" part of Theorem 5.35 may be reformulated as follows.

Theorem 5.38. *Let X be a Banach space and Y a closed subspace of X. Let A be a linear subspace of $\mathcal{W}(X)$ containing $\mathcal{F}(X)$. Let $1 \leq \lambda < \infty$. If the pair $(X^*, Y^\perp)$ has the λ-bounded A-approximation property with conjugate operators, then the pair (X, Y) is strongly λ-extendably locally reflexive.*

Theorem 5.38 contains Theorem 5.34 as the special case when $Y = \{0\}$ and A is the component of an arbitrary operator ideal, since the strong ELR of X coincides with the strong ELR of the pair $(X, \{0\})$.

Recall that by Theorem 3.18, the λ-bounded AP of the pair $(X^*, Y^\perp)$ implies that it has the λ-bounded AP with conjugate operators. Therefore, taking $A = \mathcal{F}(X)$, we immediately get from Theorem 5.38 the following version of Rosenthal's Theorem 5.25 for pairs.

Corollary 5.39. *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leq \lambda < \infty$. If the pair $(X^*, Y^\perp)$ has the λ-bounded approximation property, then the pair (X, Y) is strongly λ-extendably locally reflexive.*

Remark 5.40. Theorem 5.25 is a special case of Corollary 5.39 when $Y = \{0\}$. Notice that the proof of our result, unlike the proof of Theorem 4.1 in [JO], does not use any kind of PLR.

5.5 The unique extension property

Finally, let us mention the important case of lifting the metric APs from a Banach space X to its dual space X^* which was discovered by Godefroy and Saphar in [GS2]. This is the case when X has the unique extension property (UEP), a useful concept studied by Godefroy and Saphar in [GS1] (using the term "X is uniquely decomposed") and [GS2]. Recall that a Banach space X has the *unique extension property* (UEP) if the only operator $T \in \mathcal{L}(X^{**})$ such that $\|T\| \leq 1$ and $T|_X = I_X$ is the identity operator on X^{**}, i.e., $T = I_{X^{**}}$.

For instance (see [GS2]), the following Banach spaces have the unique extension property: Hahn–Banach smooth spaces, in particular, spaces which are M-ideals in their biduals (for example, closed subspaces of c_0); spaces with a Fréchet-differentiable norm; separable polyhedral Lindenstrauss spaces; spaces of compact operators $\mathcal{K}(X, Y)$ for reflexive Banach spaces X and Y.

By [GS2, Theorem 2.2], the UEP permits to lift the metric A-AP from X to X^* in the special case when $A = \mathcal{F}(X)$ or $A = \mathcal{K}(X)$. In [O2, Corollary 2.5], the result was extended to components of an arbitrary operator ideal. However, the proof in [O2] holds for any linear subspace of $\mathcal{L}(X)$ (see Proposition 5.41 below) and we shall show that, thanks to Theorem 3.9, it can be modified even for convex subsets of $\mathcal{L}(X)$ (see Proposition 5.43 below).

Proposition 5.41 (cf. [O2, Corollary 2.5])**.** *Let X be a Banach space having the unique extension property. Let A be a linear subspace of $\mathcal{L}(X)$. If X has the metric A-approximation property, then X^* has the metric A-approximation property with conjugate operators.*

Proof. The proof is essentially the same as in [O2, Corollary 2.5]. We present it for completeness. Since X has the metric A-AP, by Theorem 3.8 (a), there exists $\Phi \in A^{**}$ such that $\|\Phi\| \leq 1$ and

$$\Phi(x^* \otimes j_X x) = x^*(x) \ \ \forall x^* \in X^*, \ \forall x \in X.$$

Define $T \in \mathcal{L}(X^{**})$ by

$$(Tx^{**})(x^*) = \Phi(x^* \otimes x^{**}), \quad x^{**} \in X^{**}, \ x^* \in X^*.$$

Then clearly $\|T\| \leq 1$, and $T|_X = I_X$ because

$$(Tx)(x^*) = \Phi(x^* \otimes j_X x) = x^*(x) \ \ \forall x \in X, \ \forall x^* \in X^*.$$

By the UEP, $T = I_{X^{**}}$. Hence,

$$\Phi(x^* \otimes x^{**}) = (I_{X^{**}}x^{**})(x^*) = x^{**}(x^*) \ \ \forall x^* \in X^*, \ \forall x^{**} \in X^{**},$$

meaning that X^* has the metric A-AP with conjugate operators (see Theorem 3.8 (b)). □

Since the metric AP of the pair (X, Y) is precisely the metric A-AP of X, where $A = \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$, by Proposition 5.41, X^* has the metric A-AP with conjugate operators. But the latter is, by Proposition 3.11, equivalent to the metric duality A-AP of X, which in its turn is precisely the metric duality AP of the pair (X, Y). Thus, looking also at Lemma 2.5 (or Theorem 3.21), we have obtained the following lifting result.

Theorem 5.42. *Let X be a Banach space having the unique extension property and let Y be a closed subspace of X. If the pair (X, Y) has the metric approximation property, then the pair (X, Y) has the metric duality approximation property; hence, the pair $(X^*, Y^\perp)$ has the metric approximation property.*

However, Proposition 5.41 is not applicable in the case of the metric positive AP or its version for pairs. The use of the idea of the proof of [O2, Corollary 2.5] and Theorem 3.9 yield the following general lifting result for the metric convex APs.

Proposition 5.43. *Let X be a Banach space having the unique extension property. Let A be a convex subset of $\mathcal{L}(X)$ containing 0. If X has the metric A-approximation property, then X^* has the metric A-approximation property with conjugate operators.*

Proof. The proof is similar to the proof of Proposition 5.41.

Let $\Phi \in (A \cap B_{\mathcal{L}(X)})^{\circ\circ} \subset (\mathcal{L}(X))^{**}$ be from Theorem 3.9 (a). Then $\|\Phi\| \leq 1$. Indeed, we have $(B_{\mathcal{L}(X)})^\circ \subset (A \cap B_{\mathcal{L}(X)})^\circ$, and thus, $(A \cap B_{\mathcal{L}(X)})^{\circ\circ} \subset (B_{\mathcal{L}(X)})^{\circ\circ}$, where $(B_{\mathcal{L}(X)})^{\circ\circ} = B_{(\mathcal{L}(X))^{**}}$. As in the proof of Proposition 5.41, one can verify that Φ satisfies Theorem 3.9 (b), which completes the proof. □

Let us spell out an immediate application of Proposition 5.43 to the positive approximation properties of pairs.

Corollary 5.44. *Let X be a Banach lattice having the unique extension proerty and let Y be a closed subspace of X. Let $\mathcal{A}$ be an operator ideal. Denote $A = \{S \in \mathcal{A}(X, X)_+ : S(Y) \subset Y\}$. If X has the metric A-approximation property, then the dual lattice X^* has the metric A-approximation property with conjugate operators.*

The classical cases where Corollary 5.44 applies are $\mathcal{A} = \mathcal{F}$, $\mathcal{A} = \mathcal{K}$, and $\mathcal{A} = \mathcal{W}$. For instance, taking $\mathcal{A} = \mathcal{F}$ and $Y = \{0\}$, we have the following version of Corollary 5.44 for positive approximation properties.

Corollary 5.45. *Let X be a Banach lattice. If X has the unique extension property and the metric positive approximation property, then the dual lattice X^* has the metric positive approximation property with conjugate operators.*

Remark 5.46. If X is a Banach lattice, then the metric positive approximation property and the metric positive approximation property with conjugate operators coincide for the dual lattice X^* (see [LisO, Proposition 5.7]).

We also have the following immediate result from Propositions 5.26 and 5.43.

Proposition 5.47. *Let X be a Banach space having the unique extension property. Let A be a convex subset of $\mathcal{W}(X)$ containing 0. If X has the metric A-approximation property, then X is 1-extendably locally reflexive of type $\{S^{**} : \ S \in A\}$.*

Recall that if A is a convex subset of $\mathcal{L}(X)$ containing 0 (as in Proposition 5.43), then "X^* has the metric A-AP with conjugate operators" is equivalent to "X has the metric duality A-AP" (see Proposition 3.11). Therefore, we immediately have the following Theorem 5.48 for metric nest APs from Propositions 5.43 and 5.47.

Theorem 5.48. *Let X be a Banach space having the unique extension property and let $\mathcal{N}$ be nest of closed subspaces of X. Let $\mathcal{A}$ be an operator ideal. If $(X, \mathcal{N})$ has the metric $\mathcal{A}$-approximation property, then $(X, \mathcal{N})$ has the metric duality $\mathcal{A}$-approximation property.*
Moreover, if $\mathcal{A} \subset \mathcal{W}$, then X is 1-extendably locally reflexive of type $\{S^{**} : S \in \mathcal{A}(X),\ S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$.

Remark 5.49. In the special case, when $\mathcal{A} = \mathcal{F}$ and $\mathcal{N} = \{Y\}$, where Y is a closed subspace of X, Theorem 5.48 reduces to Theorem 5.42, a lifting theorem of the metric APs for pairs.

Applying Theorem 5.48 to $\mathcal{A} = \mathcal{F}$ yields the following metric nest AP version.

Corollary 5.50. *Let X be a Banach space having the unique extension property and let $\mathcal{N}$ be a nest of closed subspaces of X. If the pair $(X, \mathcal{N})$ has the metric approximation property, then $(X, \mathcal{N})$ is 1-extendably locally reflexive.*

Proof. By Theorem 5.48, X is 1-ELR of type $\{S^{**} : S \in \mathcal{F}(X),\ S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$. Since

$$\begin{gathered}\{S^{**} : S \in \mathcal{F}(X),\ S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\} \subset \\ \{T \in \mathcal{F}(X^{**}) :\ T(Y^{\perp\perp}) \subset Y^{\perp\perp} \ \ \forall Y \in \mathcal{N}\} \subset \\ \{T \in \mathcal{L}(X^{**}) :\ T(Y^{\perp\perp}) \subset Y^{\perp\perp} \ \ \forall Y \in \mathcal{N}\},\end{gathered}$$

we see that X is 1-ELR of type $\{T \in \mathcal{L}(X^{**}) :\ T(Y^{\perp\perp}) \subset Y^{\perp\perp} \ \ \forall Y \in \mathcal{N}\}$, i.e., $(X, \mathcal{N})$ is 1-ELR. □

In general, the UEP does not guarantee lifting of the λ-bounded AP from a Banach space to its dual space, at least when $\lambda \geq 6$. Indeed, let X_{JS} be the closed subspace of c_0 constructed by Johnson and Schechtman (see [JO, Corollary JS]). Then X_{JS} has the UEP (all closed subspaces of c_0 have, as was already mentioned) and X_{JS} has the 6-bounded AP (see [Z]), but X_{JS}^* does not have the AP, in particular, it does not have the λ-bounded AP for any $\lambda \geq 1$.

We conclude with the following rather surprising result, showing that the UEP permits to lift the λ-bounded convex APs from a Banach space to its dual space, $1 \leq \lambda < \infty$, whenever the space already enjoys a weaker convex AP.

Theorem 5.51. *Let X be a Banach space having the unique extension property. Let A and B be a convex subsets of $\mathcal{L}(X)$ and $\mathcal{W}(X)$, respectively, such that $A \circ B \subset A$ and both contain 0. Let $1 \leq \lambda < \infty$. If X has the metric B-approximation property and the λ-bounded A-approximation property, then X has the λ-bounded duality A-approximation property.*

Proof. By Proposition 5.43, the dual space X^* has the metric B-AP with conjugate operators. It follows from Theorem 5.28 that X has the λ-bounded duality A-AP. □

Let $\mathcal{A}$ be an operator ideal. Taking $A = \{S \in \mathcal{A}(X) : S(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$ and $B = \{T \in \mathcal{W}(X) : T(Y) \subset Y \ \ \forall Y \in \mathcal{N}\}$, we get from Theorem 5.51 the following version for nest APs.

Theorem 5.52. *Let X be a Banach space having the unique extension property and let $\mathcal{N}$ be a nest of closed subspaces of X. Let $\mathcal{A}$ be an operator ideal and let $1 \leq \lambda < \infty$. If the pair $(X, \mathcal{N})$ has the metric $\mathcal{W}$-approximation property and the λ-bounded $\mathcal{A}$-approximation property, then the pair $(X, \mathcal{N})$ has the λ-bounded duality $\mathcal{A}$-approximation property.*

For instance, it follows from Theorem 5.52 that if X has the UEP, then the pair $(X, \mathcal{N})$ has the bounded duality AP whenever $(X, \mathcal{N})$ has the bounded AP and the metric weakly compact AP.

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Tõkestatud aproksimatsiooniomaduste ülekandumine Banachi ruumide kaasruumidesse

Kokkuvõte

Aproksimatsiooniomadusi on uuritud alates 1930. aastatest. Tõkestatud aproksimatsiooniomadust vaadeldi esmakordselt juba Banach raamatus [B] (seda küll üldisemas, kompaktse aproksimatsiooniomaduse kontekstis). Süstemaatilised ja aktiivsed uuringud algasid 1955. aastal, mil Grothendieck [G] aproksimatsiooniomaduse ja meetrilise aproksimatsiooniomaduse mõisted kasutusele võttis. Aproksimatsiooniomadustega on seotud mitmeid siiani lahendamata probleeme. Kuulsaim neist on järgmine: kas Banachi ruumi kaasruumi puhul on aproksimatsiooniomaduse ning meetrilise aproksimatsiooniomaduse mõisted erinevad? Seoses antud probleemi võimalike lahendusteede uurimisega on võetud kasutusele mitmeid uusi tõkestatud aproksimatsiooniomaduse versioone. Aastal 2011 toodi artiklis [FJP] sisse tõkestatud aproksimatsiooniomaduse versioon paaride jaoks, mis koosnevad Banachi ruumist ja tema kinnisest alamruumist. Erijuhul, kui alamruumiks on nullalamruum (või terve ruum), ühtib antud versioon klassikalise tõkestatud aproksimatsiooniomaduse mõistega. Hiljuti vaadeldi artiklis [FJ3] selle omaduse üldistust – Banachi ruumi kinniste alamruumide ahela tõkestatud aproksimatsiooniomadust.

Käesoleva väitekirja põhieesmärk on süstemaatiliselt uurida paaride ja ahela tõkestatud aproksimatsiooniomadusi ning nende üldisemat versiooni artiklist [LisO] – tõkestatud kumerat aproksimatsiooniomadust. Viimane hõlmab erijuhul ka Banachi võrede positiivse aproksimatsiooniomaduse mõiste.

Käesolev väitekiri koosneb viiest peatükist. Väitekirja esimene sissejuhatav peatükk sisaldab aproksimatsiooniomaduste ajaloolise tausta tutvustust, väitekirja kokkuvõtet ning väitekirjas kasutatavate tähistuste kirjeldust.

Teises peatükis tuuakse ära töö ülejäänud osade jaoks vajalikud mõisted ja tulemused. Vaatluse all on polaarid, ahelad Banachi ruumides, lõplikumõõtmeliste operaatorite ruumi seos tensorkorrutistega ning selle ruumi kaasruumi kirjeldus integraalsete operaatorite kaudu, mis on antud Grothendiecki [G] poolt. Tutvustatakse ka pidevate lineaarsete operaatorite ruumi olulisemaid lokaalselt kumeraid topoloogiaid. Tulemused Banachi ruumi alamruumide ahelate kohta pärinevad artiklist [OV2].

Väitekirja kolmandas peatükis defineeritakse üldised aproksimatsiooniomadusega seotud mõisted, nende hulgas ka artiklitest [FJP, FJ3, LisO] pärinevad aproksimatsiooniomaduste versioonid. Kirjeldatakse tõkestatud aproksimatsiooniomaduste erinevaid versioone ning nende duaalseid vorme. Seejuures tõestatakse paaride tõkestatud (duaalse) aproksimatsiooniomadusega viis samaväärset tingimust, millest neli tõestatakse üldisemas tõkestatud (duaalse) A-aproksimatsiooniomaduse kontekstis, kus A on pidevate lineaarsete operaatorite ruumi alamhulk. Töötatakse välja järgmine tõkestatud kumera aproksimatsiooniomaduse kriteerium, mis laiendab artiklis [O2] saadud operaatorideaali poolt defineeritud tõkestatud aproksimatsiooniomaduse kirjeldust.

Teoreem 3.9. *Olgu X Banach ruum ja olgu A ruumi $\mathcal{L}(X)$ tõkestatud kumer alamhulk, mis sisaldab nulloperaatorit. Siis*

(a) *ruumil X on A-aproksimatsiooniomadus parajasti siis, kui leidub $\Phi \in A^{\circ\circ} \subset (\mathcal{L}(X))^{**}$ nii, et*

$$\Phi(x^* \otimes j_X x) = x^*(x) \quad \forall x^* \in X^*, \ \forall x \in X;$$

(b) *kaasruumil X^* on kaasoperaatoritega A-aproksimatsiooniomadus parajasti siis, kui leidub $\Phi \in A^{\circ\circ} \subset (\mathcal{L}(X))^{**}$ nii, et*

$$\Phi(x^* \otimes x^{**}) = x^{**}(x^*) \quad \forall x^* \in X^*, \ \forall x^{**} \in X^{**}.$$

Lisaks tõestatakse, et paari, mis koosneb Banachi ruumi kaasruumist ja kinnise alamruumi annulaatorist, tõkestatud aproksimatsiooniomadus toob endaga kaasa lähtepaari vastava omaduse. See tulemus üldistab järgmist Johnsoni [J1] klassikalise aproksimatsiooniomaduse kohta käivat olulist teoreemi kaasruumilt X^* paarile $(X^*, Y^\perp)$.

Teoreem 3.15 (Johnson)**.** *Olgu X Banachi ruum. Olgu $1 \leq \lambda < \infty$. Kui kaasruumil X^* on λ-tõkestatud aproksimatsiooniomadus, siis tal on kaasoperaatoritega λ-tõkestatud aproksimatsiooniomadus.*

See peatükk tugineb artiklitele [OT, OV1, V].

Neljandas peatükis töötatakse välja lokaalse refleksiivsuse printsiibi versioonid, mis on kooskõlas Banachi ruumide kinniste alamruumide ahelatega. Muuhulgas üldistatakse 1960. aastatest pärinev Ringrose'i ahelateoreem Hilberti ruumi ja selle kinniste alamruumide täieliku ahela juhult allolevale kahe Banachi ruumi ning kahe ahela juhule.

Teoreem 4.3. *Olgu X ja Y Banachi ruumid. Olgu $\mathcal{G}$ kaasruumi X^* selline alamruumide ahel, et $\{0\} \in \mathcal{G}$, ning olgu $\mathcal{N}_\mathcal{G} = \{V_G : \ G \in \mathcal{G}\}$ ruumi Y selline kinniste alamruumide ahel, et $Y \in \mathcal{N}_\mathcal{G}$. Eeldame, et $\mathcal{N}_\mathcal{G}$ on kinnine ühisosade moodustamise suhtes ja kasvav ahelal $\mathcal{G}$. Tähistame $\mathcal{R} = \{R \in X \otimes Y : R(G) \subset V_G \ \ \forall G \in \mathcal{G}\}$.*

(a) *Olgu $R = x \otimes y$ ühemõõtmeline operaator. Siis $R \in \mathcal{R}$ parajasti siis, kui leidub alamruum $G \in \mathcal{G}$ nii, et $x \in (G_-)_\perp$ ja $y \in V_G$.*

(b) *Olgu $R \in X \otimes Y$ n-mõõtmeline operaator, kus $n > 0$. Kui $R \in \mathcal{R}$, siis R on avaldatav summana n-st ühemõõtmelisest operaatorist, mis kuuluvad hulka $\mathcal{R}$.*

Uute lokaalse relfeksiivsuse printsiibi versioonide rakendusena laiendatakse paari aproksimatsiooniomaduse ning selle tõkestatud versiooni duaalsustulemused (mis on vastavalt tõestatud artiklis [LisO] ning väitekirja kolmandas peatükis) vastavalt ahela aproksimatsiooniomaduse ning selle tõkestatud versiooni juhule. Seejuures üldistatakse Johnsoni [J1] teoreem 3.15 kaasruumilt X^* paarile $(X^*, \mathcal{N}^\perp)$, kus $\mathcal{N}^\perp$ on teatud ahel, mis koosneb ruumi X kinniste alamruumide annulaatoritest. Näidatakse, et ahela aproksimatsiooniomaduste kirjeldused artiklist [FJ3] on rakendatavad analoogiliste Gorthendiecki [G] tüüpi kriteeriumite saamiseks duaalsete omaduste jaoks. See peatükk põhineb artiklil [OV2].

Väitekirja viiendas peatükis vaadeldakse, millistel tingimustel saab tõkestatud kumerat aproksimatsiooniomadust üle kanda Banachi ruumilt tema kaasruumile. Näidatakse, et seda saab teha kahel põhilisel juhul. Esiteks eeldusel, et lähteruum rahuldab laiendatava lokaalse refleksiivsuse ning lokaalse refleksiivsuse printsiibi teatavaid nõrgendatud vorme. Teiseks eeldusel, et kaasruumil on olemas tõkestatud kumera aproksimatsiooniomaduse nõrgem versioon.

Teoreem 5.28. *Olgu X Banachi ruum. Olgu A ruumi $\mathcal{L}(X)$ kumer alamhulk, mis sisaldab nulloperaatorit ning B ruumi $\mathcal{W}(X)$ selline alamhulk, et $A \circ B \subset A$. Olgu $1 \leq \lambda, \mu < \infty$. Kui kaasruumil X^* on kaasoperaatoritega λ-tõkestatud B-aproksimatsiooniomadus ja ruumil X on μ-tõkestatud A-aproksimatsiooniomadus, siis on ruumil X $\lambda\mu$-tõkestatud duaalne A-aproksimatsiooniomadus.*

Need tulemused annavad üldise meetodi erisuguste tõkestatud aproksimatsiooniomaduste ülekandmiseks lähteruumilt kaasruumile ning üldistavad ja parendavad teadaolevaid tulemusi klassikalise tõkestatud aproksimatsiooniomaduse kohta. Esimesest üldisest tulemusest järeldatakse muuhulgas

Johnson–Oikhbergi [JO] tõkestatud aproksimatsiooniomaduse kohta käiv teoreem, mida seejärel laiendatakse paari ja ahela tõkestatud aproksimatsiooniomaduste juhule. Väitekirja kolmandast peatükist pärineva tõkestatud kumera aproksimatsiooniomaduse kriteeriumi, teoreemi 4.3, rakendusena tõestatakse järgmine üldistus Godefroy–Saphari teoreemile meetrilise aproksimatsiooniomaduse ülekandumiseks Banachi ruumi kaasruumile, mis rakendub ka Banachi võredes.

Lause 5.43. *Olgu X Banachi ruum, millel on ühese jätkamise omadus. Olgu A ruumi $\mathcal{L}(X)$ kumer alamhulk, mis sisaldab nulloperaatorit. Kui ruumil X on meetriline A-aproksimatsiooniomadus, siis on kaasruumil X^* kaasoperaatoritega meetriline A-aproksimatsiooniomadus.*

Teatavasti üldjuhul ei võimalda ühese jätkamise omadus tõkestatud aproksimatsiooniomadust üle kanda Banachi ruumilt tema kaasruumile, kuid lausest 5.43 ning teoreemist 5.28 selgub, et see on võimalik juhul, kui lähteruumil on olemas meetrilise aproksimatsiooniomaduse nõrgem versioon.

Teoreem 5.51. *Olgu X Banachi ruum, millel on ühese jätkamise omadus. Olgu A ja B vastavalt ruumide $\mathcal{L}(X)$ ja $\mathcal{W}(X)$ sellised kumerad alamhulgad, et $A \circ B \subset A$ ning mõlemad sisaldavad nulloperaatorit. Olgu $1 \leq \lambda < \infty$. Kui ruumil X on meetriline B-aproksimatsiooniomadus ja λ-tõkestatud A-aproksimatsiooniomadus, siis on ruumil X λ-tõkestatud duaalne A-aproksimatsiooniomadus.*

Antud peatükis on mõningad tulemused tõestatud paaride ja ahela aproksimatsioonomaduste üldisemates kontekstides, milleks on paaride kumer aproksimatsiooniomadus ning operaatorideaali poolt defineeritud ahela aproksimatsiooniomadus. See peatükk põhineb artiklitel [OV1, OV2, V].

Väitekirja olulisemad tulemused on ilmunud/ilmumas artiklites [OT, OV1, OV2, V].

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