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Optimization of Parameters of a Portfolio Reinsurance in Non-Life Insurance

Actuarial and Financial Engineering
Master’s Thesis (30 ECTS)

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TARTU 2025

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Abstract

This thesis assesses the optimization of reinsurance parameters within a non-life insurance portfolio, focusing specifically on determining the optimal retention levels for two insurance products. The research aims to balance the dual objectives of risk reduction and profitability enhancement, which are crucial for the financial stability and growth of insurance companies.

From a broader perspective, it was noted that this trade-off would determine an optimal level of retention which can be applied to both products. To achieve this objective, it was employed two methodologies, utility optimization theory and value at risk. The first one helps determine how much should be kept from what is earned. In contrast, the second one identifies potential maximum loss. It was shown that an increased level of retention may bring about higher profits but at the same time expose the company to significant financial loss calling for a balanced approach in reinsurance.

CERCS research specialization: P160 Statistics, operations research, programming, financial and actuarial mathematics.

Keywords: non-life reinsurance, reinsurance, insurer's portfolio, optimization, retention rate, claims, premium.

Portfelli edasikindlustusparameetrite optimeerimine kahjukindlustuses

Magistritöö

Arzu Miriyeva

Lühikokkuvõte

See lõputöö hindab edasikindlustusparameetrite optimeerimist kahjukindlustuse kindlustusportfellis, keskendudes eeskätt kahe kindlustustoote optimaalse omavastutuse taseme määramisele. Uurimistöö eesmärk on leida tasakaal riski vähendamise ja kasumlikkuse suurendamise eesmärkide vahel, mis on kindlustusseltside finantsstabiilsuse ja kasvu jaoks olulised. Laiemalt vaadates määrab see tasakaal optimaalse omavastutuse taseme, mida saaks rakendada mõlema toote puhul. Selle eesmärgi saavutamiseks kasutas töö autor kahte metoodikat: kasulikusse (utility) optimeerimise teooriat ja riski all olevat väärtust (Value at Risk). Esimene aitab kindlaks määrata, kui palju peaks teenitust säilitama, samas kui teine identifitseerib võimaliku maksimaalse kahju. Uurimus näitas, et omavastutuse taseme tõus võib tuua kaasa suuremaid kasumeid, kuid samal ajal võib see oluliselt kasvatada ettevõtte finantskahju riski, mistõttu on edasikindlustuses vaja tasakaalustatud lähenemist.

CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.

Märksõnad: kahjukindlustus, edasikindlustus, kindlustusandja portfell, optimeerimine, omavastutusmäär, kindlustuspreemia.

Contents

Introduction

Optimizing reinsurance parameters is a difficult and important task for non-life insurers who have to manage risks effectively and remain profitable. The term “non-life insurance” refers to the coverage for properties, casualties, liabilities among other insurable interests. These kinds of events can cause huge financial losses on the part of the insurance companies thereby necessitating effective risk management within their operations. Reinsurance serves as a risk transfer mechanism where an insurer cedes some portion of risk to another insurer.

Nonetheless, this does not eliminate the challenge of determining what would be considered as the best options or rather features for the reinsurance cover taking into account that these are very technical matters. One key issue is how much of the risk should be retained by the primary insurer before passing it over to the reinsurer since this affects many other things too.

In particular, this research focuses on two risk products within a non-life insurance portfolio and how they can be optimized in a best way. The study is geared towards determining the most suitable retention levels that strike a balance between reducing risks and increasing profits. By doing this, it will be possible to come up with some guidelines that may help insurance companies make better reinsurance strategies so that they can be financially sound over a long period of time.

The research is important because there are trade-offs inherent in risk portfolio management by insurance companies. Keeping more risk may increase profitability since the insurer does not have to pay out as much premium to the reinsurer; however, keeping too much risk can lead to huge financial loss especially when there are major claims (Swiss Re Institute, 2019).

This study will look into different mathematical models used for evaluating reinsurance portfolios including statistical analysis techniques. It will explore various types of risks present in the portfolio, such as dependent and independent ones, and consider different reinsurance treaties like quota share and excess of loss reinsurance.

By utilizing the optimization techniques, the research is intended to detect the retention levels that are the most optimal for the insurer's financial performance while being within the boundary of a risk level that can be considered acceptable.

Besides mathematical modelling, the research will also make use of the simulation methods to experiment with and validate the optimization strategies proposed. Such simulations will encompass different market scenarios, for instance, varying severity and frequency of claims.

The reinsurance optimization research is expected to be the potential source of knowledge related to this field.

The structure of the thesis is planned in such a way that the reader gets guidance for the different stages of the research. The first chapter provides the reader with an understanding of non-life insurance and reinsurance, and the different kinds of reinsurance contracts and their purposes. In this chapter, the notion of the optimization problem in portfolio reinsurance is also introduced, thereby, preparing the ground for the following in-depth analysis. The second chapter is about the mathematical background, which is the foundation of the optimization methods used in the study. It includes utility theory and optimization of value at risk approaches. The third chapter concentrates on the deployment of these methods via simulation tactics, demonstrating the outcomes of various reinsurance scenarios and scrutinizing their influence on the insurer's financial performance. In the end, the thesis wraps up with a discussion of the implications of the findings, both for the theoretical developments in the field and for the practical implementation in the insurance industry.

The main objective of the present thesis is to bridge the gap between theoretical models and their practical application in the real world to the problem of reinsurance optimization.

CHAPTER 1. Basics of non-life insurance and Reinsurance

1.1 Concept of non-life insurance

There are always associated risks with investing money from financial instruments to properties like home. Depending on the nature of the investment, the risk can be huge and cannot be taken by single individuals. For example, you bought an expensive car, but after a month a terrible accident happened which made that car totally useless, or a newly bought apartment had a big fire. In these cases, the owner of the assets would stay with a huge amount of loan and without a car or home.

To protect individuals from such terrible events risk transformation methods like non-life insurance policies related to our example come to help. Insurance policies let people transfer a big amount of the risk to insurance companies in exchange for agreed premiums. Through the insurance companies, society can divide the risk between each other and diminish the effect of terrible accidents on single individuals.

But what if the amount of the risk is huge, for example, a couple of billions of euros which can be much more than the capital of the insurance company? In this case, insurance companies apply risk mitigation technics and reinsurance is the most known of them. Theoretically, reinsurance companies can be accepted as the insurers of insurers. Similar to normal insurance policies, reinsurance companies require an agreed-upon premiums in order to accept transferred risk. Even at the same time, to spread over the whole risk, simultaneously there can be reinsurance agreements with several reinsurance companies.

1.2 Concept of Reinsurance

A reinsurance treaty is a formal agreement where a cedant transfers specific risks to a reinsurer under pre-agreed conditions. This transfer occurs at the same time and can involve either a portion or the entire portfolio of the insurance company. The treaty provides the cedant with ongoing reinsurance capacity throughout the agreement's term, effectively acting as a "blank cheque." In this setup, the cedant is aware of the risks being transferred before the reinsurer, who is informed of the treaty’s progress through regular reports, usually on a quarterly or semi-annual basis. Due to this, the treaty is sometimes referred to as a "blind" treaty. These treaties are classified as proportional because the transfer of risk to the reinsurer is directly proportional to the features of the underlying insurance policies. This means that both the cedant and reinsurer share the insured risks in proportion, with premiums, claims, and expenses being divided according to the agreed-upon ratio in the treaty. There is also a differentiation between proportional facultative reinsurance and proportional treaty reinsurance. (Cogo, 2017)

1.2.1 The Need for Reinsurance

Reinsurance is important to the insurance sector since it offers insurers an avenue to manage and mitigate their risk exposures effectively. The mechanics of insurance companies transferring parts of their liabilities to the reinsurer protect the capital and guarantee financial stability in the event of sudden and substantial losses (Cummins & Weiss, 2014). This is, hence, considered a major transfer of risk that is key to solvency maintenance and the ability to pay claims.

Further, because of reinsurance, insurers can increase their capacity to underwrite and take any large or complex risk they cannot write themselves (Swiss Re, 2020). This increase in capacity promotes growth and allows for the coverage of high-value assets and large-scale projects. It also significantly contributes to smoothing an insurer's financial results, as it dampens the fluctuations from year to year inherent in any insurance business (Froot, 2001). Since reinsurance distributes such large claims over more than one period or among several reinsurers, it enables insurance companies to realize more predictable and stable financial outcomes.

Moreover, reinsurance provides access to specialized competence and new solutions that assist insurers in developing their risk management practices further and offering their clients more competitive and tailored coverage (Lloyd's of London, 2019). Another critical angle is related to regulatory compliance: reinsurance can help the insurer meet its capital requirements more efficiently, enabling it to use its resources effectively to support business growth (Cummins & Mahul, 2009). Ultimately, reinsurance provides a vital extra layer of protection against disasters—natural or caused by pandemics—by the absorption of most of the feasible claims under any event for an insurer (OECD, 2021).

In other words, reinsurance is very beneficial for insurance companies to manage risks, protect capital, increase underwriting capacity, and stabilize financial outcomes; even expertise, regulatory compliance, and CAT risk management would guarantee resilience and confidence in a somewhat unpredictable market can be acquired.

1.2.2 Types of Reinsurance Contracts

There are many varieties of reinsurance contracts that provide flexibility to the cedant and reinsurer in serving many needs and objectives. There are two major categories for reinsurance contracts: proportional and non-proportional, each of which may include several types of agreements to suit different risk-sharing arrangements and financial structures.

Proportional Reinsurance:

In proportional reinsurance, the reinsurer shares risks and premiums in a predetermined proportion with the cedant. The reinsurer agrees to accept a predetermined percentage of all the risks, receiving the same percentage of the premiums to pay the same percentage of claims. Common forms of proportional reinsurance agreements include:

Quota Share Reinsurance: In this type of agreement, the cedant cedes a fixed percentage of all policies to the reinsurer that falls within a defined portfolio. In turn, the reinsurer agrees to pay the same rate for all claims arising from the said policies. This is a very simple arrangement that provides homogeneous protection for the insurer across all business lines.

Surplus Share Reinsurance: This type of reinsurance applies only to policies that are in excess of a retention limit for which the cedant is charged. In this, the reinsurer covers the risks above the retention limit and proportionately receives premiums and claims. This type of contract will allow insurers to retain more of the smaller, less risky policies while getting coverage for larger risks.

Non-Proportional Reinsurance:

Non-proportional reinsurance is the form of reinsurance where the reinsurer takes the losses in excess of the target retention that the reinsured wants to retain for his pool of business. The insurer, therefore, does not share in a proportion of the risk and premium of every individual policy in the primary insurer's portfolio. The basic non-proportional reinsurance agreements are:

Excess of Loss Reinsurance: The agreement is designed to cover losses in excess of the retention or priority sum, which the intermediary is expected to bear as a contribution. Over and above this sum, the reinsurer will pay for other losses incurred up to a prefixed limit. Excess of loss reinsurance normally purchases protection against catastrophes or big, unforeseen claims that may badly distort the insurer's financial position.

Stop-Loss Reinsurance: This kind of contract protects the cedant from losses that go beyond a stipulated percentage of total premiums earned during a period, usually one year. The reinsurer provides coverage for all losses above this threshold, hence helping the insurer to stabilize its financial performance in such years when claims are unusually high.

Aggregate Excess of Loss Reinsurance: This kind of contract is quite similar to stop loss reinsurance because the coverage applies if the sum of losses incurred within a particular period exceeds a specified amount. It provides protection against the accumulation of small or medium-sized claims that may amass and thus become an actual financial burden on the insurer.

The different forms of reinsurance contracts have been designed to meet very different risk management needs, therefore allowing insurers to adjust their reinsurance strategy according to their risk profile, financial objectives, and regulatory requirements. Insurers should use the proper type of reinsurance contract in a bid to manage risk exposure effectively, stabilize financial results, and ensure viability in the long term against unpredictable events.

1.2.3 Insurance premium

As compensation for transferring the whole or part of the risk to the insurance company, policyholders should pay insurance premiums to insurers based on the conditions of the policy. There are several ways to calculate insurance premiums, which differ according to mathematical principles. The most common insurance premium prices are:

1. **Expectation principle.**

$$P = (1+\eta)E[X], \qquad \eta > 0.$$

   Where non-negative notation η is the premium loading factor while $E[X]$ is the expected total claims at time t.

2. **Variance principle.**

$$P = E[X] + \eta D[X], \qquad \eta > 0.$$

Where $D[X]$ denotes the variance of risk X.

3. **Standard deviation principle.**

$$P = E[X] + \eta\sqrt{D[X]}, \qquad \eta > 0.$$

4. **Covariance principle.**

$$P = E[X] + 2\eta\sqrt{D[X]} - \eta Cov(X, Y), \qquad \eta > 0,$$

where Y is the random non-negative variable.

5. **Mean value principle.**

$$P = \sqrt{E[X^2]} = \sqrt{E[X]^2 + D[X]}.$$

6. **Exponential principle.**

$$P = \frac{1}{\eta}\log E[\exp(\eta X)], \qquad \eta > 0.$$

1.2.4 Solvency II capital requirement and reinsurance's effect on it

A regulatory framework named Solvency II, which is implemented across Europe, sets very strict capital requirements for insurance companies in order to maintain their financial stability and protect policyholders. It seeks to ensure that insurers have sufficient capital to meet their liabilities and are able to absorb shocks arising from adverse events thereby minimizing the risk of insolvency. The capital requirement under Solvency II is determined through a risk-based approach, taking into account different factors such as market risks, underwriting risks, and operational risks (EIOPA, 2019).

Reinsurance is crucial for meeting these capital requirements for insurers. Primary insurers can reduce the level of capital they must hold against such risks by transferring some part of this risk to reinsurers. This is because reinsurance spreads effectively an insurer's exposure to large or catastrophic losses hence reduction in the overall risk profile of the insurer. Therefore, when used as part of an insurer's risk management strategy, reinsurance lowers the amount of capital required under the Solvency II standard formula or internal models (Bermúdez et al., 2013).

In addition, Solvency II recognizes the impact of reinsurance on the risk profile of an insurer and permits capital charge reduction where effective reinsurance arrangements are in place. The effectiveness of reinsurance in reducing capital requirements is however dependent upon the reinsurer's ability to meet its obligations, hence the reason why insurers should assess their credit qualities (Eling & Schmeiser, 2010). High-quality reinsurance agreements often lead to significant capital relief that allows insurers to optimize their capital management and participate in growth opportunities.

However, the use of reinsurance is not without its challenges under Solvency II. Insurers must be careful when considering costs associated with reinsurance as well as its influence on overall risk management policies. In order to ensure that it genuinely contributes towards a reduction in risk exposure and capital requirements, insurers have to comply with elaborate documenting and justifying rules for using reinsurance (Cummins & Weiss, 2014). Also, Solvency II demands continuous monitoring and reporting for further investigation if these agreements continue to comply with regulatory standards and achieve their intended objective of releasing additional funds.

Reinsurance turns out as a worthy tool for the Solvency II governed insurers in managing risk and lowering capital needs. Financial stability is enhanced by the effective use of reinsurance by insurance companies in meeting regulatory requirements and preparing for long-term growth. Nevertheless, the achievement of this goal depends on the selection and handling of reinsurance partners carefully based on Solvency II stipulations with their demands.

1.2.5 The optimization problem of portfolio reinsurance

Insurance companies face a key challenge in portfolio reinsurance: finding the optimal spot between managing risk and boosting profits. This means figuring out the best way to reinsure that cuts down on the insurer's risk while pumping up its financial results. It's no easy task, given all the moving parts to consider. These include how much reinsurance costs, how it affects capital needs how much risk the insurer is willing to take on, and the unique traits of the portfolio up for reinsurance.

A key issue in making portfolio reinsurance better is handling the give-and-take between risk and return. Reinsurance can cut down the chance of big losses a lot by shifting some of what the insurer owes to a reinsurer. But this isn't free - the money paid for reinsurance lowers the insurer's earnings. So, the problem to solve is to find the optimal spot where what you pay for reinsurance makes sense because it lowers risk enough, which in turn boosts the insurer's overall return on equity (ROE) (Cummins & Weiss, 2009).

Another key part of the optimization issue is making sure capital is used well while following rules like Solvency II. Smart use of reinsurance can cut down the money insurers need to keep by shifting risk to groups that can handle it better. This drop in required funds can free up cash to invest elsewhere giving the insurer more financial options. The tricky part is setting up reinsurance deals to make the most of this capital use without cutting too much into the insurer's profits (Bai & Wierzbicki, 2009).

Making reinsurance better also means spreading out reinsurance deals to lower the risk of one party not paying. If you rely on one reinsurer, you can be in big trouble if they can't pay up. So, a good reinsurance plan often involves using several different reinsurers all with high credit scores. You need to manage this spread to make sure it protects you from risk but doesn't cost too much (Gatzert & Schmeiser 2012).

The complexity of the optimization problem often requires the use of quantitative models, like stochastic programming dynamic optimization, and other advanced math techniques. These models help insurers run simulations of various reinsurance scenarios and evaluate how they affect the overall portfolio. By looking at different layers of coverage, retention levels, and types of reinsurance, insurers can figure out the best structure to meet their risk and return goals (Li et al. 2015).

Market conditions also have a big impact on how to optimize portfolio reinsurance. Things like the availability of reinsurance capacity, market pricing cycles, and the economic environment can change the cost and effectiveness of reinsurance. This means insurers need to be flexible often revisiting their reinsurance strategies to make sure they stay optimized as conditions change (Venter 2014).

The optimization problem of portfolio reinsurance is a multifaceted challenge that requires balancing risk reduction with financial performance, ensuring regulatory compliance, and managing market dynamics. By leveraging sophisticated quantitative models and maintaining a flexible approach, insurers can develop reinsurance strategies that optimize their portfolios, enhance capital efficiency, and support long-term profitability.

CHAPTER 2. Mathematical background of optimization methods

This chapter explains the probability models that describe aggregate claims in the fixed time interval by the non-life insurance process. An insurance process can be a single contract, a portfolio of contracts, or in other words a business line which means a particular set of risk products or a single risk product. In this research, a portfolio of contracts is categorized by business lines, and the aggregate claims amount means the total amount of claims that occurred in the portfolio.

2.1Risk model

The sum of all claims incurred in the insurance policies in the portfolio is called the aggregate loss of the insurance portfolio. Two main approaches are used to model the aggregate claim or loss of the portfolio: the individual risk model and the collective risk model.

In this research, the collective risk model will be used to estimate claim amounts because of its advantages, including the opportunity to use some known statistical distributions and to utilize both claim frequency and severity distributions to calculate the portfolio's aggregate claim amount.

2.1.1 Collective risk model

The collective risk model, with its flexibility, describes the aggregate loss amounts or total claims through the known distribution, considering the frequency and severity of the claims. This flexibility enables an analyst to provide feedback on these two elements, empowering them in their analysis.

Consider the random variables S_1 and S_2, which define the aggregate amount of loss in a non-life insurance portfolio:

$$S_1 = \sum_{i=1}^{N_1} X_i,$$

$$S_2 = \sum_{i=1}^{N_2} Y_i.$$

N_1 and N_2 are i.i.d random variables denoting the claim frequencies for each portfolio, respectively. X_i and Y_i are i.i.d random variables denoting the claim severity for two different portfolios.

Also, it is assumed that the severity of the claim size X_i and Y_i does not depend on the claim frequency N_1 and N_2 respectively.

Thus, the distribution functions will be:

$$F_X(x) := P\{X_i \leq x\} \text{ – the distribution of individual loss } X_i,$$

$$F_Y(y) := P\{Y_i \leq y\} \text{ – the distribution of individual loss } Y_i.$$

2.1.2 Compound distribution

In probability theory, the probability distribution of the sum of two or more independent, identically distributed (i.i.d) random variables, given that the number of terms to be summed is a Poisson-distributed variable, is known as a compound Poisson distribution. The outcome distribution can be discrete or continuous.

Consider i.i.d random variables of the claim sizes X_i and Y_i following the same distribution, and the number of claims, N_1 and N_2 follow a Poisson distribution with parameters λ_1 and λ_2. The compound Poisson random variables S_1 and S_2 are given by:

$$S_1 = \sum_{i=1}^{N_1} X_i \text{, where } N_1 \sim Poisson(\lambda_1),$$

$$S_2 = \sum_{i=1}^{N_2} Y_i \text{, where } N_2 \sim Poisson(\lambda_2).$$

Consider a new random variable Z_i which follows the mixture distribution of random variables X_i and Y_i. Here, the random variable Z_i denotes the total claim amounts for the direct insurer's whole portfolio rather than claim amounts of individual risk processes X_i and Y_i. So, the aggregate amount of the loss S_P for the whole portfolio is as follows

$$S_P = \sum_{i=1}^{N} Z_i,$$

where the frequency of claims is $N = N_1 + N_2$ and the distribution function of aggregated claim size amount Z_i equals to (S. Klugman, H. Panjer, G. Willmot, 2012.)

$$F_Z(z) := P\{Z_i \le z\} = \frac{\lambda_1 F_X(x) + \lambda_2 F_Y(y)}{\lambda_1 + \lambda_2}.$$

Thus, the compound Poisson random variable S_P with the frequency rate $\lambda = \lambda_1 + \lambda_2$ will be:

$$S_P = \sum_{i=1}^{N} Z_i, \text{where } N \sim Poisson(\lambda).$$

2.1.3 Moment Generating Function (MGF)

To simplificate the calculations of the key properties in compound Poisson distribution, moment generating function can be used. In this research, moment generating function is used to solve the derivatives of complex functions and to calculate mainly the expected value and variance of the compound Poisson distribution.

The moment generating function (MGF) of any random variable S_P is:

$$M_{S_P}(t) = E[e^{tS_P}].$$

In the compound Poisson distribution case, the MGF of the variable S_P can be calculated based on the MGF of number of claims N and the MGF of individual claim size Z.

MGF of total claim numbers which follows Poisson distribution:

$$M_N(t) = E[e^{tN}] = e^{\lambda(e^t - 1)}.$$

MGF of random variable Z:

$$M_Z(t) = E[e^{tZ}].$$

MGF for $S_P \sim \text{Compound Poisson}(\lambda, Z)$ will be:

$$M_{S_P}(t) = M_N\big(\ln(M_Z(t)\big) = e^{\lambda(M_Z(t)-1)}.$$

Thus, the expectation and variance of the random variable S_P will be:

$$E[S_P] = M'_{S_P}(0) = \lambda(E[Z]).$$

$$Var(S_P) = M''_{S_P}(0) - M'_{S_P}(0)^2 = \lambda(E[Z^2] + \lambda(E[X]^2)) - (\lambda E[X])^2$$
$$= \lambda E[Z^2] + \lambda^2(E[X])^2 - (\lambda E[X])^2 = \lambda E[Z^2].$$

The MGF, $M_Z(t)$ for the mixture distribution of the log-normally distributed random variables, Z_i, can be expressed as the weighted sum of the MGFs for random variables X_i and Y_i since MGF for any log-normally distributed random variables does not have a standard form:

$$M_Z(t) = \frac{\lambda_1 M_X(t) + \lambda_2 M_Y(t)}{\lambda_1 + \lambda_2}.$$

The MGFs, $M_X(t)$ and $M_Y(t)$, do not have any closed forms, because the MGF for any log-normally distributed random variables is not defined, but its moments can be computed. Thus, the the first moment of random variable Z_i with a mixture of two log-normally distributed random variables X_i and Y_i can be written as follows:

$$E[Z_i] = \frac{\lambda_1 E[X] + \lambda_2 E[Y]}{\lambda_1 + \lambda_2}.$$

2.2 Utility Maximization Theory

The utility maximization approach, a key concept in economics, finds applications in the insurance field. It is used to determine the optimal insurance coverage that maximizes expected wealth. In recent years, it has been particularly useful in portfolio reinsurance optimization research (Bugalho de Moura, Centeno, 2022), (Irgens, Paulsen, 2004)

The utility approach states that there is a utility function $U(W)$ that represents the total wealth of the insurers, which can be derived from the initial capital W at the end of the period. In non-life insurance, the exponential utility function is generally used for decision-making processes. Thus, assume a direct non-life insurer activity with the initial capital W adopting exponential utility function $u(x)$:

$$u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x}), \qquad \alpha > 0.$$

Where α is the risk aversion coefficient of the direct insurer, for the simplicity of the further calculations, the constants can be ignored, so assume the utility function

$$u(x) = -e^{-\alpha x}, \ \alpha > 0. \qquad (2.0)$$

Thus, at the end of the one year, the total expected wealth of the direct insurer without risk mitigation like reinsurance will be:

$$E[W + P - S_P] = E\left[-e^{-\alpha(W+P-S_P)}\right].$$

Where W is considered initial capital, P denotes the total premium of insurers, which is the total compensation for accepting the risk, and S_P represents the total claims accrued throughout the period. Also, the direct insurer is considered risk-averse and follows optimal business directions to maximize its gains.

2.2.1 The Quota Share Reinsurance

In this part the utility maximization method which is described in part 2.2 is applied to the portfolio with the Quota Share Reinsurance.

Consider a direct insurer having reinsurance with the retention parameter β for its activities in two different lines of the business in the portfolio of products. Each business line has the same category of risk properties and follows the same loss distribution. The aggregate claim amount of the portfolio follows compound Poisson distribution as defined in the previous section of this chapter. Thus, should be divided proportionally into two parts consisting of the direct insurer's responsibility $S_{P_I} = \beta S_P$ ($S_{P_I} \sim CP(\lambda, F_I)$, where $I = \beta(\frac{\lambda_1 X + \lambda_2 Y}{\lambda_1 + \lambda_2})$ and reinsurer's part $S_{P_R} = (1-\beta)S_P$ ($S_{P_R} \sim CP(\lambda, F_R)$, where $R = (1-\beta)(\frac{\lambda_1 X + \lambda_2 Y}{\lambda_1 + \lambda_2})$.

So, the moment-generating function of the S_{P_R} will be:

$$M_{S_{P_R}}(t) = e^{\lambda(M_R(t)-1)} = e^{\lambda\left(M_Z\left((1-\beta)t\right)-1\right)}.$$

Assume the utility function of the insurer 2.0 in the following way:

$$u(x) = -e^{-\alpha(W+P-(1-\beta)P-\beta S_P)}, \qquad \alpha > 0, \qquad \beta \geq 0.$$

For simplicity consider the initial capital W equal to 0 Then it follows:

$$-e^{-\alpha(P-(1-\beta)P-\beta S_P)} = -e^{-\alpha(P\beta-\beta S_P)} = -e^{-\alpha\beta(P-S_P)}.$$

2.2.1.1 Expected value premium principle

If P represents the total premium, then it can be replaced with $(1+\eta)S_P$, where η is the direct insurer's premium loading factor. Also, it is assumed that the direct insurer predicts its premiums based on the expected value principle.

$$-e^{-\alpha\beta(P-S_P)} = -e^{-\alpha\beta((1+\eta)S_P-S_P)} = -e^{-\alpha\beta S_P\eta}.$$

The maximum of this function can be found by taking its first partial derivative based on parameter β, it is then finding the critical points.

Which gives:

$$\frac{\partial f(\beta)}{\partial\beta} = \alpha S_P\eta e^{-\alpha\beta S_P\eta}.$$

where the parameters $\alpha,\ S_P$ and η are always non zero-positive variables. At the same time due to exponential function properties e^x is always non-zero positive value as well for all real x. So the other part of the function, $e^{-\alpha\beta S_P\eta}$ always get non-zero positive value. Thus it can be concluded that $\frac{\partial f(\beta)}{\partial\beta}$ is always positive and never zero for all the values of parameter β $(0<\beta<1)$.

But from the another point of view, if the behavior of the function is evaluated as $\beta\to\infty$:

$$f(\beta) = -e^{-\alpha\beta S_P\eta} \to -e^{-\infty} = 0$$

From this point of solution it can be concluded that as the parameter β approaches its maximum value which is 1 in our case, the function $f(\beta)$ is getting close to 0, but never being equal to 0, since the parameter β is bounded by 0 and 1, and never can be equal to infinity. This result suggests that as much as the the parameter β getting close to 1, direct insurer's utility function is getting larger values. These reasonings confirm that in expected value premium principle there is no non-trivial solution for parameter β.

2.2.1.2 Variance premium principle

Also, consider an alternative approach by assuming a different premium calculation principle: the variance principle.

$$P_R = (1-\beta)E[S_P] + \eta(1-\beta)^2D[S_P] = (1-\beta)\lambda E[Z] + \eta(1-\beta)^2\lambda E[Z^2].$$

The expected value of the direct insurer's utility function is:

$$E\left[u\left(W+P-P_R-S_{P_I}\right)\right]=E\left[-e^{-\alpha\left(W+P-P_R-S_{P_I}\right)}\right]=-e^{-\alpha(W+P)}e^{\alpha P_R}E\left[e^{\alpha\beta S_P}\right].$$

Finally, the optimal value of the retention parameter $\beta(0\leq\beta\leq 1)$ should be found to maximize the equation above, which aims to maximize the direct insurer wealth. This can be found by finding the suitable value of the $\beta,$ which minimizes the insurer's expected payout:

$$e^{\alpha P_R}E\left[e^{\alpha\beta S_P}\right]=e^{\alpha P_R}M_{S_P}(\alpha\beta).$$

Additionally, $M_{S_P}\,(t)$ can be approximated with the help of Taylor expansion theory for small values of $t,$ around $t=0$ as follows.

$$\ln\left(M_{S_P}\,(\alpha\beta)\right)\approx\alpha\beta E[S_P]+\frac{1}{2}(\alpha\beta)^2Var[S_P].$$

If the logarithms of the previous equation are taken, then it can be written as:

$$h(\beta):=\alpha P_R+\ln\left(M_{S_P}(\alpha\beta)\right)=\alpha\left((1-\beta)E[S_P]+\eta(1-\beta)^2Var[S_P]\right)+\alpha\beta E[S_P]+\frac{1}{2}(\alpha\beta)^2Var[S_P]=\alpha\left((1-\beta)\lambda E[Z]+\eta(1-\beta)^2\lambda E[Z^2]\right)+\alpha\beta\lambda E[Z]+\frac{1}{2}(\alpha\beta)^2\lambda E[Z^2]. \tag{2.1}$$

We find such a β which minimizes the function $h(\beta)$ above. It is done by finding the derivative of equation 2.1 with respect to β:

$$\frac{\partial h(\beta)}{\partial\beta}=-\alpha\lambda E[Z]-2\eta\alpha(1-\beta)\lambda E[Z^2]+\alpha\lambda E[Z]+\alpha^2\beta\lambda E[Z^2]. \tag{2.2}$$

By setting the partial derivative to 0, the optimal retention rate β can be found:

$$-\alpha\lambda E[Z]-2\eta\alpha(1-\beta)\lambda E[Z^2]+\alpha\lambda E[Z]+\alpha^2\beta\lambda E[Z^2]=0$$

$$-2\eta\alpha(1-\beta)\lambda E[Z^2]+\alpha^2\beta\lambda E[Z^2]=0$$

Dividing through by $\alpha\lambda E[Z^2]$ and assuming $\alpha\lambda E[Z^2]\neq 0$:

$$-2\eta(1-\beta)+\alpha\beta=0$$

$$-2\eta+2\eta\beta=-\alpha\beta$$

$$\beta=\frac{2\eta}{2\eta+\alpha}. \tag{2.3}$$

Thus, it is visible from the result, that equation (2.2) only gets zero value if the retention rate is derived by formula (2.3) under the variance premium principle. According to this formula, the optimal value of beta depends on the value of parameters η and α. As the premium loading η increases, an insurer can retain a higher proportion of risk, since it is better compensated for potential losses. On the other hand, if the insurer's risk aversion rate α is stronger, it results in a lower retention rate and reduces total risk exposure by transferring more risk to the reinsurer. In section 2.2.1.1 optimal retention parameter was analyzed under the expected value premium principle. Similar to the previous analysis (Weng, 2009), there is no unique optimal retention rate under the performed method of optimization if the premium principle is the expected value, which means the insurer gets higher wealth as much as retention parameter β close to one. Only with the company-specific numerical methods based on the historical results, a suitable retention rate can be found.

In real life, while implementing these results into practice, also the expenses of direct insurers should be considered. By selecting optimal β, the insurer is able to minimize his expected losses but at the same time maximize his expected utility given the premiums and risks involved.

2.2.2 The Excess of Loss Reinsurance

Now consider the utility maximation approach for excess of loss reinsurance agreements for a non-life insurance portfolio consisting of two insurance products. And assume the policies are independent by nature which means claim sizes or numbers of the policies do not affect each other. The retention level is d in excess of the reinsurance aggreement which means in other words „deductible". Direct insurer's total agregated net reinsured claims amount S_I follows a compound Poisson distribution $S_I \sim CP(\lambda_1 + \lambda_2, F_Z)$, where $Z = \frac{\lambda_1(X-d)+\lambda_2(Y-d)}{\lambda_1+\lambda_2}$ and indicates the mixture of risk process for net reinsured insurance claim sizes of each policies', $X \text{ and } Y$ together in the portfolio of insurance products. $\lambda_1 + \lambda_2$ is the total number of claims in the portfolio during the cover period. Thus, the moment-generating function of S_I will be

$$M_{S_I}(t) = E[e^{tS_I}] = e^{\lambda(M_Z(t)-1)}.$$

It should be stressed that since the direct insurer's aggregated premium amount P_R depends on deductible d, then the MGF, $M_{S_I}(t)$ also depends on d.

P_R is the reinsurance premium with the safety loading rate θ. The expected value premium principle (Weng, 2009) was used to calculate it

$$P_R = (1+\theta)E[S_R] = (1+\theta)\lambda\left(\int_0^{\infty} z\, f_Z(z)dz - \int_0^{d} zf_Z(z)dz - d\big(1 - F_Z(d)\big)\right)$$
$$= (1+\theta)\lambda\int_d^{\infty} (z-d)\, f_Z(z)dz.$$

Thus, the expected value of the direct insurer's total wealth at the end of the period will be:

$$E[u(W+P-P_R-S_I)] = E\left[-e^{-\alpha(W+P-P_R-S_I)}\right] = -e^{-\alpha(W+P)}e^{\alpha P_R}E[e^{\alpha S_I}].$$

In order to maximize the direct insurer's wealth at the end of the period, the deductible amount should be chosen in such a way that the direct insurer would have the maximum expected wealth. Alternatively, to maximize the direct insurer's total expected financial results, the total expected out payments should be minimized based on the possible deductible level d. So, the direct insurer's expected payout can be expressed as

$$e^{\alpha P_R}E[e^{\alpha S_I}] = e^{\alpha P_R}M_{S_I}(\alpha, d) = e^{\alpha P_R}M_N\big(\ln\big(M_Z(\alpha, d)\big)\big) =$$
$$= e^{\alpha P_R}e^{\lambda(M_Z(\alpha,d)-1).} \qquad (2.4)$$

By taking the logarithms of each side of the equation (2.4), the maximizing problem can be stated as follows to find the optimal deductible level d:

$$h(d) = \alpha P_R + \lambda(M_Z(\alpha, d) - 1).$$

Applying more simplifications gives

$$M_Z(\alpha, d) = \int_0^{d} e^{\alpha z} f_Z(z)dz + e^{\alpha d}\big(1 - F_Z(d)\big).$$

Putting them together

$$h(d) = \alpha(1+\theta)\lambda\left(\int_0^\infty z\, f_Z(z)dz - \int_0^d z f_Z(z)dz - d\big(1 - F_Z(d)\big)\right)$$

$$+ \lambda[\int_0^d e^{\alpha z} f_Z(z)dz + e^{\alpha d}\big(1 - F_Z(d)\big)].$$

Thus, a derivative of the last function should be found to find the optimal deductible level d, which follows as

$$\frac{\partial h(d)}{\partial d} = \alpha\lambda\big(1 - F_Z(d)\big)\big(-(1+\theta) + e^{\alpha d}\big). \qquad (2.5)$$

The optimal retention level d will be the values that make the equation (2.5) equal zero, $\frac{\partial h(d)}{\partial d} = 0$. (Tverdostup, 2014) The analysis of the equation shows that $h'(d) = 0$ occurs in its critical points:

1. $1 - F_Z(d) = 0$, which can only happen when $F_Z(d) = 1$. It can occur when d reaches the upper bound of the distribution Z, meaning d represents the maximum posible value of the distribution Z which makes $P(Z \leq d) = 1$. So, the first solution is the upper maximum value of the distribution Z.
2. When $-(1+\theta) + \alpha e^{\alpha d} = 0$, where if the natural logarithm is applied to both sides of the equation, then the optimal retention rate is

$$\ln(1+\theta) = \alpha d$$

$$d = \frac{1}{\alpha}\ln(1+\theta). \qquad (2.6)$$

Visually the relationship between the variable parameters of equation (2.6) can be plotted in the following graph:

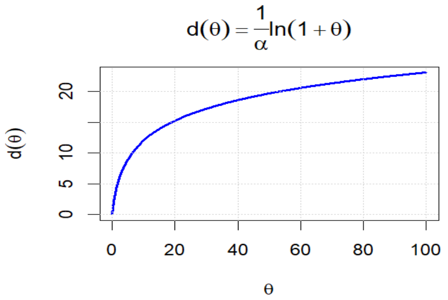


Figure 2.1: Visual representation of equation (2.6).

From Figure 2.1 it is obvious that the function is concave and increasing for the values of $\theta > 0$ in the x-axis.

2.3 VaR optimization approach

In this section, the optimal retention rate is analyzed for the same portfolio consisting of two products which was mentioned in section 2.2, but with a different optimization methodology. This optimization method also is used to find optimal measures for both quota share and excess of loss insurance types. The second optimization approach allows to make better comparisons and conclusions to determine the best solution for the direct insurer. One of the well-known risk assessment approach Value-at-Risk (VaR_α) is used as a second optimization approach in this section.

Recent developments in the financial industry have made risk management crucial. (Cerqueti, D'Ecclesia, Levantesi, 2021) Risk measurements, such as VaR (Denuit, Dhaene, Goovaerts, Kaas, 2005) have been a great example of risk assessment methods widely used in banking and insurance regulations to determine capital requirements. VaR is nowadays broadly accepted and considered the standard way of measuring risk. In simpler terms, it calculates the maximum expected loss at a specified confidence level (e.g., 95%) over a specified period.

VaR concept firstly was introduced by company J.P. Morgan in 1994 (J. P. Morgan, 1994) through its development of RiskMetrics methodology, which is a well-known risk management methodology. Even though assessment of the total risk of the portfolio was not a new concept at that time, J.P. Morgan was the first one to standardize this risk measure and make it accessible in the financial field.

In the broader form, VaR can be defined as:

Definition 1. "The maximum potential loss in value of a portfolio, over a given time period, for a given confidence level." (J. P. Morgan, 1994)

The VaR_α of a total aggregate non-negative claim amount S_P at a confidence level $1-\alpha$, where $0 < \alpha < 1$ is defined as

$$VaR_\alpha(S_P) = \inf\{v \geq 0 : P(S_P > v) \leq \alpha\}.$$

VaR can thus be defined as the smallest threshold v to which S_P has the greatest probability of exceeding. The $1-\alpha$ quantile of the random variable S_P indicates the maximum possible loss at the confidence level of $1-\alpha$. In the formulas above, inf stands for infimum and means the largest number that remains less than or equal to all other numbers in a set. In other words, it can be described as the highest value which is still less than or equal to any other given value in the set.

Crucial properties of VaR

The relevant Value-at-Risk, denoted by $VaR_\alpha(S_P)$ given in a risk S_P, and a probability level $p \in [0,1]$ is determined as

$$VaR_\alpha(S_P) = F_{S_P}^{-1}(1-\alpha) = S_{S_P}^{-1}(\alpha).$$

The right side of the equation is the inverse cumulative density function of r.v. S_P with the probability p $(0 \leq \mathrm{p} \leq 1)$. At the same time, the VaR_α measure equals the inverse cumulative survival function with its value in α $VaR_\alpha(S_P) = S_{S_P}^{-1}(\alpha)$, which is also known as the quantile function in survival analysis. Here, α is the risk coefficient for the insurer or, in other words, the risk rate accepted by the cedent.

If it is assumed that S_P is strictly decreasing function, then

$$Var_\alpha(S_P) = S_{S_P}^{-1}(\alpha) = m, P(S_P > m) \leq \alpha \leq P(S_P \geq m). \qquad (2.7)$$

And if

$$P(S_P > m) = \alpha, \qquad \text{then } Var_\alpha(S_P) = S_{S_P}^{-1}(\alpha) = m.$$

Before any reinsurance policies, the total risk exposure expected by the cedent is S_P. A reinsurance agreement can be beneficial for reducing risk exposures, considering the additional premiums that should be paid to the reinsurer company for bearing the transferred risk.

One of the important properties of VaR was shown by Dhaene et al. in 2002, states that

$$VaR_\alpha\big(g(S_P)\big) = g\big(VaR_\alpha(S_P)\big).$$

holds for any left-continuous and increasing function g and for any constant c,

$$VaR_\alpha(S_P + c) = VaR_\alpha(S_P) + c.$$

For a collective risk model

$$S_P = \sum_{i=1}^{N} Z_i, \quad i = 1,2,3, \dots, k.$$

where S_P is the total loss for the portfolio. N is the number of claims in occurrence i, and Z_i is the size of the loss events.

2.3.1 The Quota share reinsurance

Assume a direct insurer offers two individual policies X and Y. Here, β is the retention parameter of the quota share reinsurance agreement. Similarly to the previous assumptions, the aggregate claim of the portfolio follows a compound Poisson distribution. In simpler words, the insurer transfers $1 - \beta$ of its risk to the reinsurance company and bears β percent of the total claims. Intuitively, when β equals to 1, the insurer retains all the losses, and when β equals 0, the insurer transfers all the losses to the reinsurer to pay. Thus, similarly to the method utilized in section 2.2.1, the total aggregate claims amount should be divided proportionally into two parts consisting of the direct insurer's responsibility $S_{P_I} = \beta S_P$ ($S_{P_I} \sim CP(\lambda,\ F_I)$, where $I = \beta(\frac{\lambda_1 X + \lambda_2 Y}{\lambda_1 + \lambda_2})$ and reinsurer's part $S_{P_R} = (1 - \beta)S_P$ ($S_{P_R} \sim CP(\lambda,\ F_R)$, where $R = (1 - \beta)(\frac{\lambda_1 X + \lambda_2 Y}{\lambda_1 + \lambda_2})$).

In quota share reinsurance, the optimal retention level can be trivial or nontrivial, and the word trivial means the optimal retention rate β is either 1 or zero. In other words, in trivial cases, optimal reinsurance is defined as having full protection by transferring all losses to the reinsurer or not having a reinsurance contract at all where β equals 0. The goal of this chapter is to find a nontrivial optimal retention level under the VaR assumption, where β should be in the interval of (0,1). To find a nontrivial optimal retention rate, different insurance premium principles will be used.

So, the total risk exposure notated by T of the insurance portfolio will be equal to:

$$T = S_{P_I} + \pi\big((1 - \beta)S_P\big).$$

Then using the VaR's property of invariance

$$VaR_\alpha(T) = VaR_\alpha\left(S_{P_I}\right) + \pi\left((1-\beta)S_P\right).$$

In order to assess VaR_α the insurer's survival function, $S_{S_{P_I}}(\alpha)$, of the insurer's part of loss S_{P_I} is needed. The values at $S_{P_I} \geq 0$ will be

$$S_{S_{P_I}}(\alpha) = P\left(\beta S_P > s_{P_I}\right) = \begin{cases} S_{S_{P_I}}\left(\frac{S_{P_I}}{\beta}\right), & 0 \leq \beta < 1, \\ 0, & \beta = 1. \end{cases}$$

Thus, under the quota-share reinsurance agreement, the VaR_α of the total claims for total risk processes at the confidence level $1-\alpha$ will be

$$VaR_\alpha\left(S_{P_I}\right) = \beta S_{S_P}^{-1}(\alpha) + \pi\left((1-\beta)S_P\right).$$

In order to find the optimal retention rate β, the minimal VaR_α based on the retention rate, at $1-\alpha$ confidence level should be examined. Based on Chengguo Weng's findings the optimal non-trivial quota share reinsurance exists at a retention rate β, if and only if there is a constant β in the interval (0,1) which is the solution of the following equation.

$$\pi'\left((1-\beta)S_P\right) + S_{S_{P_I}}^{-1}(\alpha) = 0. \qquad (2.8)$$

According to Chengguo Weng's same research under the variance, covariance, and exponential premium principles, there is a nontrivial optimal quota share reinsurance retention rate where $\pi\left(cS_{P_I}\right)$ is strictly convex for any $\beta, 0 \leq \beta \leq 1$ which is the reason for nontriviality.

Meanwhile under expectation, standard deviation or mean value premium principles optimal quota share reinsurance retention rate is trivial if $\pi(0) = 0$ and property of positive homogeneity is relevant ($\pi(cZ) = c\pi(Z)$.

So

$$\beta^* = \begin{cases} 0, & when\ \pi(\beta) < S_{S_{P_I}}^{-1}(\alpha), \\ all\ numbers\ in\ (0,1), & when\ \pi(\beta) = S_{S_{P_I}}^{-1}(\alpha), \\ 1, & when\ \pi(\beta) > S_{S_{P_I}}^{-1}(\alpha). \end{cases}$$

where β^* indicates the optimal retention rate for the portfolio.

Variance premium principle

Under the variance premium principle, the total insurance premium will be

$$P = E[S_P] + \eta D[S_P], \qquad \eta > 0.$$

If the insurer enters a quota share reinsurance agreement, then due to the expected value and variance's properties, the reinsurer will get the premium equal to the following equations.

$$P_R = (1-\beta)E[S_P] + \eta(1-\beta)^2 D[S_P], \qquad \eta > 0.$$

The derivative can be applied to find optimal retention rates to solve the equation (2.8)

$$P_R{}' = \pi'\big((1-\beta)S_P\big) = -E[S_P] - 2\eta(1-\beta)D[S_P],$$

$$E[S_P] + 2\eta(1-\beta)D[S_P] - \ S_{S_P}^{-1}(\alpha) = 0$$

$$E[S_P] + 2\eta(1-\beta)D[S_P] = \ S_{S_P}^{-1}(\alpha)$$

$$\beta^* = 1 - \frac{S_{S_P}^{-1}(\alpha) - E[S_P]}{2\eta D[S_P]}.$$

Thus, in the last system of equations, the solution of optimal retention rates for quota share reinsurance agreement under the variance premium principle is obtained.

It is also important to note that this solution only applies to non-trivial optimal retention rates, which is only possible if

$$0 < \frac{S_{S_P}^{-1}(\alpha) - E[S_P]}{2\eta D[S_P]} < 1.$$

Simplifies to

$$E[S_P] < S_{S_P}^{-1}(\alpha) < E[S_P] + 2\eta D[S_P]. \qquad (2.9)$$

Thus, under the dependency of β^* on the expected value, variance, the confidence level α and insurance premium rate η, there is a non-trivial optimal retention rate for quota share reinsurance policy with variance premium principle considering the condition 2.9 is obtained.

The obtained formula suggests that the ideal retention rate β reduces with an increase in either the expected claims or the extreme loss quantile, which in turn makes the insurer take more risk in order to handle very high potential losses. On the other hand, it grows when there is greater claim volatility or if the loading factor is high because such factors increase the cost of reinsurance thus making it more expensive to transfer risk. This trade-off is seen in the relationship between risk retention and reinsurance being a cost-effective measure. In Chapter 3 the practical implementation of the obtained method is analyzed.

2.3.2. The excess of loss reinsurance

To find the optimal reinsurance retention rate under an excess-of-loss reinsurance agreement, the minimization of the VaR measure can be used as well. In this part of the research, Chai and Tai's method will be used to find the optimal retention level for an excess of loss reinsurance portfolio reinsurance policy.

Consider a non-life insurance portfolio comprising two policies involving risks X and Y. The insurer decides to use an excess loss reinsurance as a risk mitigation technique to reduce its total risk exposure. Meanwhile, $F_X(x) = P\{X \leq x\}$ and $F_Y(y) = P\{Y \leq y\}$ are cumulative distribution functions, $F_Z(z) = P\{Z \leq z\}$ is the cumulative distribution function of their mixture distribution, continuous and strictly increasing. $S_Z(z) = P\{Z > z\}$ is the survival probability function for random variables Z with non-negative expected values $E[Z]$.

Deductible d can be treated under two assumptions: Individually and aggregated. Reinsurance can be applied on individual risk level, which means during the cover period if any of the individual claims amount exceeds the deductible level, then the rest of the amount other than the deductible should be paid by the reinsurance company. But in an aggregated way, the reinsurance company only pays the excess of the deductible level if the total amount of the claims in the whole portfolio or each policy type exceeds the total deductible limit. The decision about these assumptions depends on various reasons, such as the field of insurance, risk awareness, and capital reserves of the insurer.

Consider the deductible rate for individual loss cases. Independent and randomly distributed claim size variables, S_{Z_I} and S_{Z_R}, are the individual losses paid by the insurer and reinsurer, respectively. So, related to the risk processes, S_{Z_I} and S_{Z_R} will be equal to:

$$S_{Z_I} = \begin{cases} S_Z, & S_Z \leq d, \\ d, & S_Z > d. \end{cases}$$

$$S_{Z_R} = \begin{cases} 0, & S_Z \leq d, \\ S_Z - d, & S_Z > d. \end{cases}$$

According to an excess of loss reinsurance policy, the reinsurer will pay part of the loss, which is more than d. By transferring the excess part of the deductible limits, the insurer sets the maximal limit of bearable loss for the company and gets itself safe from huge losses.

The total reinsurance premium is calculated with the expected value premium principle:

$$P_R = \pi(d) = (1 + \theta)E\left[S_{P_R}\right].$$

where the total expected loss for the reinsurer is

$$E\left[S_{P_R}\right] = \lambda_i(E[S_Z] - E[S_Z, d]).$$

Meanwhile, the total risk exposure of the insurer will be equal to the following:

$$R = S_{P_I} + (1 + \theta)E\left[S_{P_R}\right].$$

where the total expected loss for the direct insurer is

$$S_{P_I} = \min(Z, \max(d, Z - d)).$$

In exchange for sharing the loss, the insurer will have to forfeit part of the premiums it receives, which makes figuring out the best possible retention level. The insurer's remaining risk will drop as the retention margin drops, but the premium it must pay to the reinsurer in exchange will rise. Conversely, the insurer will take on more risk if it decides to lower the cost of the reinsurance premium.

As shown in the beginning of Chapter 2, the VaR of a non-negative random variable S_{P_I} and assuming $(1 + \theta)E\left[S_{P_I}\right]$ is a constant, at a confidence level $1 - \alpha$, $0 < \alpha < 1$ is defined as

$$VaR_\alpha\left(S_{P_I} + (1 + \theta)E\left[S_{P_R}\right]\right) = \inf\left\{S_{P_I} \geq 0 : P\left(S_{P_I} > s_{P_I}\right) \leq \alpha\right\}.$$

One benefit of the VaR risk measure is its simplicity. It is assumed that the likelihood of the risk surpassing a given value if the VaR corresponding to that risk is known, will not exceed α. The probability of risk tolerance can be understood as the parameter α in this case. Typically, α is selected to be a tiny value, less than 5%, in practice. This measure's drawback is that it doesn't reveal any information regarding the degree of exposure to danger above this threshold.

Thus, the optimal value of the d^* will be the values, where $VaR_\alpha\left(S_{P_I} + (1+\theta)E\left[S_{P_R}\right]\right)$ gets the minimum value.

$$VaR_\alpha\left(S_{P_I} + (1+\theta)E\left[S_{P_R}\right]\right) = \min\left\{VaR_\alpha\left(S_{P_I} + (1+\theta)E\left[S_{P_R}\right]\right)\right\}, \qquad d > 0.$$

To achieve the optimal level of retention denoted as d^*, VaR should be minimized by selecting appropriate values for d which will ensure that the probability of direct insurer's loss S_{P_I} exceeding d is properly taken care of.

To find the optimal retention level for reinsurance cover the following function should be solved, where

$$\frac{\partial}{\partial d} VaR_\alpha(S_{S_{P_I}}^{-1}(\alpha) + (1+\theta)E\left[S_{P_R}\right]) = 0.$$

Which the solution will be

$$\frac{\partial}{\partial d} S_{S_{P_I}}^{-1}(\alpha) = \frac{\partial}{\partial d} VaR_\alpha\left(S_{P_I}\right) = 1.$$

which it is assumed that as the deductible level increases by constant rate, 1 unit, VaR also increases by 1 unit.

Since the expected value of the reinsurer's loss, $E\left[S_{P_R}\right]$ depends on the deductible d, as d increases reinsurer's contribution on loss decreases. So, the likelihood of the loss amount will exceed the deductible amount $P\left(s_{P_R} > d\right)$, would decrease as d increases. Thus, the derivation for expectation can be done by applying the indicator function

$$\frac{\partial}{\partial d}\left((1+\theta)E\left[S_{P_R}\right]\right) = -(1+\theta)P(s_z > d),$$

Combining to

$$\frac{\partial}{\partial d}(S_{S_{P_I}}^{-1}(\alpha) + (1+\theta)E\left[S_{P_R}\right]) = 1 - (1+\theta)P(s_z > d) = 0$$

$$(1+\theta)P(s_z > d) = 1$$

$$P(s_z > d) = (1+\theta)^{-1}$$

where $S_{P_R}(d) = P(s_z > d)$ shows the probability that the claims will exceed the retention rate d at the optimal retention rate. The optimal retention rate d^* is observed when the reinsurer's safety loading factor θ equals to $(1 + \theta)^{-1}$.

To assess the final optimal retention rate d^*, the inverse survival function $S_{S_{P_R}}^{-1}$ should be provided by the probability of exceeding the deductible amount, which will return the optimal retention rate d^*.

Finally, the optimal rate marked with * is equal to

$$d^* = S_{S_Z}^{-1}((1 + \theta)^{-1}).$$

This establishes the optimal deductible rate that balances the retained risk and the reinsurance premium, ensuring the insurer's VaR is minimized.

CHAPTER 3. Investigation of optimization methods through simulation methods

A deterministic model, which allows to calculate the future events without any random elements, can only be obtained when the problem is relatively easy, all state variable values are certain, and there are no uncertainties in any of the input data to the optimization model. Nevertheless, such a deterministic model may fail to account for much of the complexity inherent in decision-making under uncertainty, which always exists in insurance fields. Stochastic variables are those that depend on uncertain inputs, which are described through probability density functions; in this case, one can determine how likely different outcomes will be given what has been decided about the future course of action.

An analytical approach or Monte Carlo optimization can be used to determine the best solution for a deterministic model. Monte Carlo optimization is a technique that involves evaluating the objective function for a large number of randomly generated feasible solutions and selecting the one with the best value.

It is easier to optimize deterministic models than stochastic ones. When the best outcome for a model depending on probability cannot be determined using analytical methods, one can resort to Monte Carlo simulation. This approach involves the generation of random numbers that are employed for characterizing the state variables or system parameters and also used in seeking the best possible solution (Mun, 2006).

Monte Carlo simulation refers to a model that shows the likelihood of different results occurring in a process where we cannot easily predict outcomes, because of the randomness involved. This is a method for assessing risk and uncertainty. These simulations are applicable across various areas such as investment, commerce, physics, and engineering among others.

To begin with, in Monte Carlo simulations, a probability distribution is employed for every variable that is intrinsically uncertain. The results are then recalculated many times, each time using a different set of random numbers taken from an estimated range, in order to account for uncertainty. This leads to multiple possible results that converge in accuracy with increased numbers of inputs. Stated differently, the various results create a bell-shaped curve or normal distribution with the middle representing the most likely result.

While there can be some differences depending on the situation, a Monte Carlo simulation follows these steps:

- Create a model. Establish an appropriate mathematical model or transfer function.
- Identify simulation variables. Select dependent variables and specify probability distribution functions for each independent random variable.
- Carry out repeated simulations. Iterate many times over by inputting the random variables into the model.
- Combine the outcomes and compute the mean, standard deviation as well as variance for evaluating the hypothesis.

Also, it should be noted that because of the high confidentiality of the insurance companies, mainly due to market competition, the real claim dataset will not be used in simulations. Instead, the claim data will be generated by the well-known mathematic random number generation technics based on the properties of particular statistic distributions.

3.1 Assumptions of the Monte Carlo Simulation of non-life insurance surplus process

In this chapter, the numerical examination of the obtained optimal retention parameters will be implemented. Firstly, consider an insurance company selling two non-insurance products which are in a similar risk group based on their historical data. For simplicity, the initial capital W of the company in the beginning of the period, $t = 0$ will be considered 0. The reinsurance cover period will be assumed 12 months which makes $t = 0, 1,2, \dots, 12$.

These assumptions are relative to all obtained optimal retention parameter approaches, if there are exemptions it will be stated in the appropriate sections.

All claim amounts follow the compound Poisson distribution with claim frequency parameters λ_1 and λ_2 which are for individual claim sizes X_i and Y_i respectively. It should also be mentioned that the non-homogeneous Poisson process $N(t)$ should be considered in simulations since in real life it is unlikely to get the same average number of claims at the end of each period. So, naturally depending on a lot of reasons like the current season of the year, the average rate for Poisson distribution should be varied. If the constant $\lambda_0 = 120$ is considered the historical average rate of a number of claims for both risk products, then on a monthly basis the average claim intensity can be modelled in the following way:

- Month 1-3: During the first quarter of the year assume that on average, historically the monthly number of claims equals approximately: $\lambda_1 + \lambda_2 = (1.2\lambda_0 + 5) + (\lambda_0 + 3t)$.
- Month 4 and 5: During this period there were relatively fewer claims were obtained in comparison to the beginning of the year, which the average can be modeled as: $\lambda_1 + \lambda_2 = (0.7\lambda_0 + 3) + (1.1\lambda_0 - 2)$.
- Month 6 and 8: During the summer period, because of the active lifestyle of people on average, the claim intensity will be relatively high: $\lambda_1 + \lambda_2 = (1.3\lambda_0 + t) + (1.2\lambda_0 - 5)$.
- Month 9 and 10: On average both risk processes follow the yearly average of the claim's intensity: $\lambda_1 + \lambda_2 = 2\lambda_0$.
- Month 11: In November the number of claims again slightly increases and follows $\lambda_1 + \lambda_2 = 1.2\lambda_0 + (\lambda_0 + t)$.
- Month 12: At the end of the year, claims intensity again follows the average yearly rate: $\lambda_1 + \lambda_2 = 2\lambda_0$.

Additionally, it should be stated that claim sizes do not depend on the arrival time of the claims and the interval between the occurrence time of the claims are also independent random variables.

The simulations and other analyses will be done for the portfolio for a one-year period which will be calculating the total surplus amount for the end of each month. Initial capital at the beginning of the year will be considered 0, and variable C_t donates the surplus values at $t = 0,1,\dots,12$. Thus, the surplus process of the non-life insurance portfolio can be given as

$$C_t = C_{t-1} + P_t - \sum_{i=1}^{N(t)} Z_i.$$

where it is applied in the case of not having any reinsurance policy. If the reinsurance was considered, then the value of the surplus equals to

$$C_t = C_{t-1} + P_{I_t} - \sum_{i=1}^{N(t)} Z_{I_i}.$$

which is the sum of the surplus value at the end of the previous period and the direct insurer's part of the premium amount P_{I_t} minus the insurer's part of the claim amounts $\sum_{i=1}^{N(t)} Z_{I_i}$.

The premium values are calculated based on the expected value $P = (1+\eta)E[Z]$ or variance premium principle $P = E[Z] + \eta D[Z]$, depending on the type of optimization method and type of the reinsurance agreement. The appropriate calculation method will be noted in each section.

The number of iterations for Monte Carlo Simulation is $\mathrm{j} = 1,2,..,\mathrm{k}$, which means the amount of the cumulative surplus value will be calculated $\mathrm{k} = 100000$ times for $t = 1,2,\dots,\ T = 12$. Thus, the simulation process for the portfolio can be shown as

$$C_t^j = C_{t-1}^j + P_t^j - \sum_i^{N(t)} Z_i^j. \tag{3.1}$$

Based on equation (3.1) the mean and standard deviation of the simulated values can be calculated as follows

$$\widehat{C_T} = \frac{1}{k}\sum_j^k C_t^j,$$

$$s(C_T) = \sqrt{E[(C_t^j - E[C_T])]^2}.$$

All the simulations are implemented in R-Studio software with the help of built-in packages.

3.2 Quota share reinsurance

Both utility and VaR_α optimization methods have been discussed in Chapter 2 for quota share reinsurance. It is obtained that the optimal retention rate for quota share reinsurance can be trivial depending on which premium calculation principle has been utilized. In this research, the expected value premium principle is used for utility maximation and indeed has been proven that the optimal reinsurance retention rate is 0, which also means not having a reinsurance agreement is the most optimal choice for the direct insurer. However, with the variance premium calculation method, there are the non-trivial optimal retention rates, and the formula for the optimal retention rate is obtained. In this chapter, the practical implementation of the obtained results will be shown.

If the direct insurer decides to take a quota share reinsurance agreement with a retention rate β, then the reinsurer will be responsible for $(1-\beta)$ part of the loss. Thus, the surplus process of direct insurer is equal to

$$C_t^{\beta j} = C_{t-1}^{\beta j} + P_{I_t}^{\beta j} - \sum\nolimits_i^{N(t)} \beta Z_i^j. \tag{3.2}$$

where $P_{I_t}^j = \beta P_t^j$ is the direct insurer's part of the premium, since in quota share reinsurance, both claims and premiums are divided proportionally between them in the same retention rate.

To analyze the accuracy of the obtained optimization methods, firstly the comparison of surplus values of the portfolio with the reinsured case in m different values of retention rates $\beta(\beta = (\beta_1, \beta_2, \ldots\ldots, \beta_m)$ and original surplus process without any reinsurance strategy for the whole cover period t will be done. Then, the behavior of the surplus processes $C_t^{\beta j}$ and C_t^j will be visually presented in the form of a graph, and the effect of the reinsurance agreement can be discussed.

Secondly, the ruin probabilities $P(\widehat{C_T^{\beta j} < 0})$ of the insurance company, which means the probability of the average of reinsured surplus trajectories to become less than 0 in the different rates of the retention levels should be calculated to compare the risk of insurer is taking. The ruin probabilities can be found by dividing the number of cases where the terminal surplus rate was smaller than 0, by the total number of simulations.

$$P\left(\widehat{C_T^{\beta j}} < 0\right) = \frac{1}{k}\sum\nolimits_{j=1}^{k} I_{(C_t^{\beta j}<0)}.$$

Additionally, in real life, sensitivity analysis and stress testing are done to measure the strength or durability of the insurance portfolio and reinsurance's effect on it. These are the common risk management techniques that insurance companies are implementing, to determine the total risk the company faces, depending on the global and internal economic processes; and increase in claim frequencies and sizes due to some catastrophic events. Finally, it would be worth checking the company's SCR requirements from regulatory bodies and measuring the reinsurance effect on it to see, as a risk mitigation technique how the obtained optimal reinsurance retention rate is effective.

All the mentioned comparisons, calculations, and visualizations will be done for both utility and VaR_{α} optimization approaches.

Now for quota-share reinsurance policy, consider a random variable Z_i $(i = 1,2, \dots .12)$ which is the mixture of two independent and identical log-normally distributed random variables. Log-normally distributed claim sizes X_i and Y_i, have mean values $\mu_1 = 0.8,\ \mu_2 = 1.5$ and standard deviation $\sigma_1 = 0.05,\ \sigma_2 = 0.01$ respectively. The expected value and variance formula for random variable Z at time t will be

$$E[Z] = \frac{\lambda_1 e^{\mu_1 + \frac{\sigma_1^2}{2}} + \lambda_2 e^{\mu_2 + \frac{\sigma_2^2}{2}}}{\lambda_1 + \lambda_2}, \tag{3.3}$$

$$Var[Z] = \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(\left(e^{\sigma_1^2} - 1\right)e^{2\mu_1 + \sigma_1^2}\right) + \frac{\lambda_2}{\lambda_1 + \lambda_2}\left(\left(e^{\sigma_2^2} - 1\right)e^{2\mu_2 + \sigma_2^2}\right)$$
$$+ \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{\lambda_2}{\lambda_1 + \lambda_2}\left(e^{\mu_1 + \frac{\sigma_1^2}{2}} - e^{\mu_2 + \frac{\sigma_2^2}{2}}\right)^2.$$

Also, consider risk aversion coefficient $\alpha = 0.01$.

3.2.1 Utility Maximization approach

Simulations are conducted considering the variance premium principle with $\eta = 0.01$. The following graphs, Figure 3.1 and Figure 3.2 show the various surplus processes under the different values of β. It is obvious that the insurer gets the highest surplus values when there is no reinsurance, but the surplus values are extremely volatile.

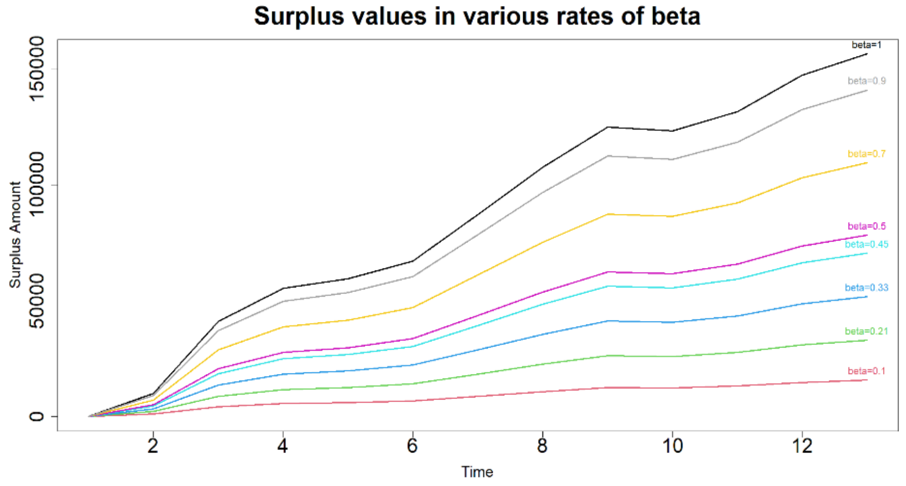


Figure 3.1. Random surplus processes under a positive scenario.

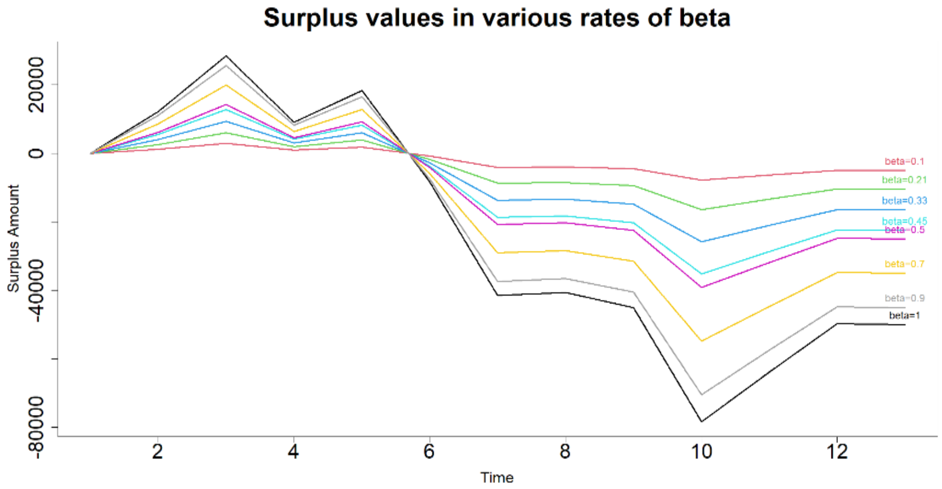
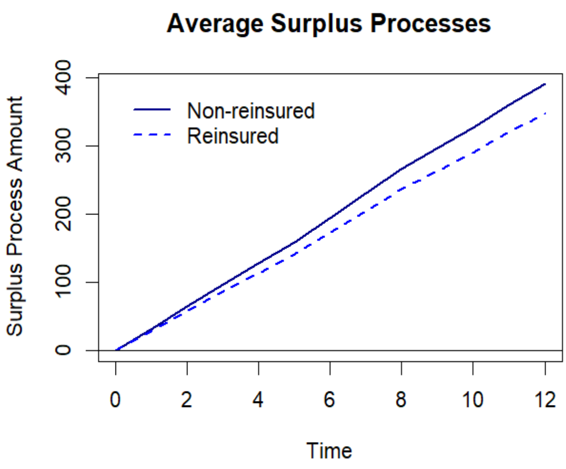


Figure 3.2 Random surplus processes under a negative scenario.

Figures 3.1 and 3.2 clearly demonstrate that the insurer gets the maximum surplus amount if $\beta = 1$, but at the same time, under the worst-case scenario the loss amount is the highest and the slope is smoother when $\beta \to 0$, meanwhile the surplus amount moves more aggressively when $\beta \to 1$, because the insurer tends to bear all risk.



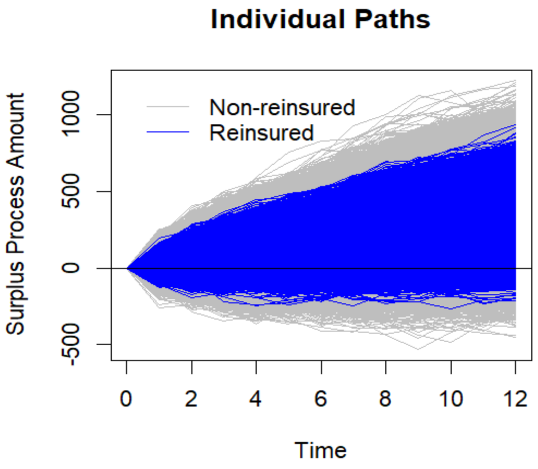


Figure 3.3. Average of Surplus processes $\widehat{C_T^{\beta J}}, \widehat{C_T^{J}}$.   *Figure 3.4. Individual trajectories*$C_t^{\beta^* j}, C_t^{\beta^* j}$.

In Figure 3.3 the reinsured surplus is slightly less than the non-reinsured surplus. The patterns of the individual trajectories from 100000 simulations can be seen in Figure 3.4, which shows, that with a reinsurance agreement, the insurer company would have more stable capital without huge ups and downs.

Another thing that should be considered is the value of the external non-controllable risks which can cause huge losses like hurricanes, earthquakes, or global pandemics. So, it makes sense to give up on a higher surplus to transfer such risks to the reinsurance company, since throughout history such high-risk events always existed.

	Non-reinsured	$\beta = 0.3$	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta^* = 0.67$	$\beta = 0.7$	$\beta = 0.9$
$\widehat{C^{\beta j}}$	390.33	199.2	249.96	292.90	328.02	347.10	355.33	386.48
$S(C^{\beta j})$	197.36	59.21	78.94	98.68	118.42	131.57	138.15	177.62

Table 3.1. Mean and Standard deviation of trajectories in $\beta = (\beta_1, \beta_2, \ldots\ldots, \beta_m)$.

From Table 3.1 it can be commented that the mean value of the trajectories of the simulations is an increasing function of parameter β which gives the maximum amount of mean and standard deviation when $\beta = 1$ or when there is no reinsurance agreement. The reason of this pattern is due to the nature of proportionality. Meanwhile, the optimal rate for the direct insurer alone is $\beta = 0.67$ with a given risk aversion level and premium loading factor.

Additionally, the ruin probabilities of the company are as follows:

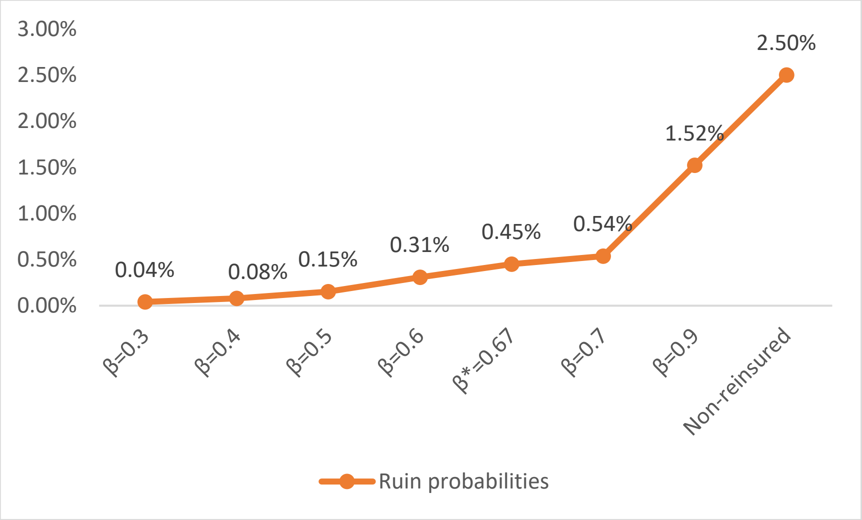


Figure 3.5. The pattern of the probability of ruin with respect to β.

This means the probability of getting negative values for the surplus process is increasing as the retention rate β approaches to 1. The insurer gets the lowest ruin probability in the lower values of β, but at the same time insurer sacrifices more surplus amount. Also, it is visible that in a retention rate above 0.7, the increase of the ruin probability is getting faster, which shows the importance of reinsurance. The optimal retention rate helps to set the retention rate within the risk appetite of the insurer, which depends on further analysis and is determined by the company.

The effect of other parameters like insurance premium loading factor η should be analyzed by comparing surplus amount in different values of η. In the next table, the effect of other parameters is demonstrated.

η	0.01	0.10	0.15	0.20	0.25	0.30	0.5
β^*	0.67	0.85	0.91	0.95	0.96	0.97	0.98
$\widehat{C^{\beta j}}$	346.64	1147.76	1937.33	3899.20	5856.58	7812.65	9768.18
$S(C^{\beta j})$	132.10	169.84	180.13	188.71	191.75	193.31	194.26

Table 3.2. The impact of premium loading factor η over Mean and Standard deviation of trajectories in optimal retention rates.

In Table 3.2 as expected, the mean value and standard deviation of the insurer’s surplus increases if the insurer sets the higher rate of premium loading factor. Since the optimal reinsurance retention rate formula depends directly on the premium factor, it is also affected as it is visible from the table above. Naturally, the results suggest that as more as the insurer gets a premium, the probability of ruin decreases and there is less need for reinsurance cover, considering the same level of risk aversion rate.

3.2.2 VaR Optimization Approach

Under the VaR optimization approach, there is a possibility to find the optimal retention rate of quota share reinsurance. Firstly, under the VaR optimization approach, consider the direct insurer’s parts of claims is retention rate β, and the reinsurer will be responsible for $1-\beta$ part of claim amounts. Premiums are also considered in the same retention rates with the variance premium calculation principle. Also, it is possible to find non-trivial optimal retention rate β under several premium principles like semi-variance and exponential, which are out of scope in this research. Consider the same parameters used in section 3.2.1.

According to the VaR approach, the non-trivial quota share reinsurance retention rate only exists if and only if the following equation is true.

$$E[S_P] < S_{S_P}^{-1}(\alpha) < E[S_P] + 2\eta D[S_P]. \tag{3.4}$$

Thus, based on the assumptions above the optimal retention rate for quota share reinsured portfolio will be equal to

$$\beta^* = 1 - \frac{S_{S_P}^{-1}(\alpha) - E[S_P]}{2\eta D[S_P]}.$$

Considering the above-mentioned parameters firstly, let's check if condition 3.4 holds to see if there is a non-trivial retention rate of risk process S_P:

$$E[S_P] < S_{S_P}^{-1}(\alpha) < E[S_P] + 2\eta D[S_P] \rightarrow \; 877.12 < \; 927.23 < 942.28.$$

So, the equation 3.4 is true with the defined parameters. Now the optimal non-trivial retention rate can be found.

Thus, considering the above-mentioned assumptions, the Average of Surplus processes $\widehat{C_t^J}, \widehat{C_t^{\beta^* J}}$ and Individual trajectories $C_t^{\beta^* j}$, $C_t^{\beta^* j}$ of direct insurer in non-reinsured and reinsured cases respectively, will be as shown in Figure 3.6 and Figure 3.7.

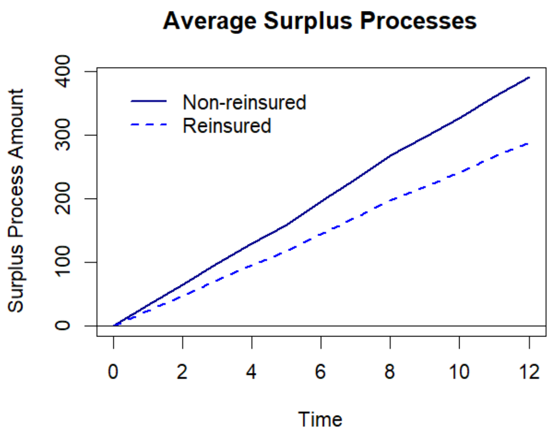


Figure 3.6. Average of Surplus processes $\widehat{C_t^J}, \widehat{C_t^{\beta^* J}}$.

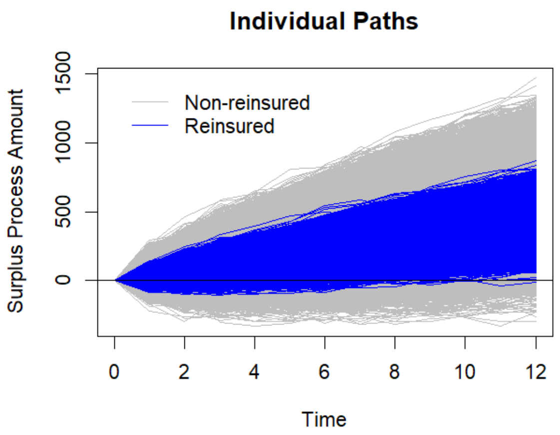


Figure 3.7. Individual trajectories $C_t^{\beta^* j}, C_t^{\beta^* j}$

Based on Figures 3.6 and 3.7 it is clear to see that the average surplus processes and individual paths of the trajectories of the direct insurer are with considerably different values. This is due to the proportionality of quota share reinsurance. As observed before, in a proportional reinsurance agreement: quota share reinsurance has triviality over the behavior of surplus amounts which means the direct insurer gets the highest surplus value in the non-reinsured case. Meanwhile, from the point of the value at the risk point there is an optimal retention rate which minimizes the value of faced total risk exposure VaR within the α rate of confidence interval. In the following table the pattern of the VaR_α in different values of retention rates β^* can be observed.

In Table 3.3 it is easy to see that the minimum values of VaR_α for both individual risk processes Z, and for the whole portfolio S_P are obtained at the optimal rate of the retention rate β^*. Thus, it is visible that there is indeed an optimal rate of retention rate of quota share reinsurance, from the value at risk perspective, and quota share reinsurance indeed can be beneficial to decrease the value of risk exposure to have more stable surplus amounts. As a risk mitigation technique quota share reinsurance can be more useful in real-life insurance processes since usually insurance business has a large amount of risk exposure. So, even though the optimal quota share decreased the value of risk exposure in a small amount in the example above, in real life the difference can be significant.

η	β^*	$VaR_\alpha(S_P^\beta)$	$VaR_\alpha(S_P)$	$P(\widehat{C_T^{\beta^*}} < 0)$	$P(\widehat{C_T} < 0)$	$\widehat{C_T^{\beta^*}}$	$S(\widehat{C_T^{\beta^*}})$
0.010	0.234	10895.64	11126.87	0.00028	0.0249	161.66	46.11
0.011	0.304	10916.66	11126.87	0.00011	0.0151	221.83	60.11
0.012	0.303	10934.18	11126.87	0.00003	0.00932	278.22	71.82
0.013	0.411	10949.00	11126.87	0.00001	0.00569	331.73	81.62
0.014	0.453	10961.71	11126.87	0.00003	0.00283	383.79	89.64
0.015	0.490	10972.72	11126.87	0.00000	0.00155	433.20	96.69
0.016	0.521	10982.35	11126.87	0.00001	0.0009	482.57	103.11

Table 3.3 The effect of the insurance premium loading factor η over retention levels β^*, terminal surplus $\widehat{C_T^{\beta^*}}$, and ruin probability $P(\widehat{C_T^{\beta^*}} < 0)$ at $\alpha = 0.01$.

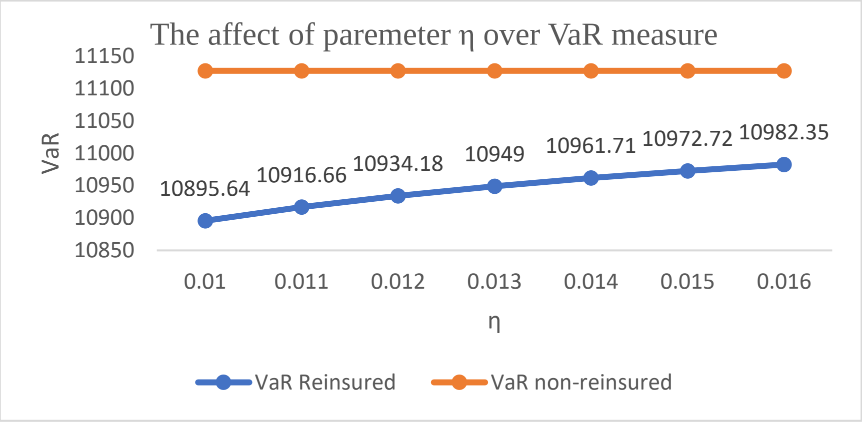


Figure 3.8. Pattern of the Value at Risk with respect to η.

The impact of the changes in the value of the direct insurer's premium loading factor can be evaluated in Table 3.3. There is a straight relationship between the values of optimal retention rate β^* and the value of premium loading factor η. But in contrast, as the parameter η increases, the value of total risk exposure VaR_α also increases. This can be explained as, while the value of the parameter η is growing, the amount of the direct insurer's premium and as a result the price of the reinsurance agreement also grows, which makes bigger the value of risk exposure. The decreasing pattern of optimal retention rate β^* is due to the proportionality of quota share insurance retention rate which is trivial in most of the cases. Additionally, the ruin probabilities of having a negative surplus amount have a decreasing pattern with the decreasing level of speed. The reason behind this is the magnitude of growth in the amount of premiums is higher than the magnitude of the change in VaR_α. Expected values and standard deviation of the surplus amounts both are growing as parameter η increases. However, the degree of the change in standard deviation is relatively smaller than a change in expected value which can be the stabilizing effect of optimal quota share reinsurance retention rate.

Overall, practically the effectivity of optimal retention rate for quota share reinsurance has been proved. So, it can be concluded that there is indeed an optimal retention rate for quota share reinsurance which is decreasing the total amount of risk exposure.

3.3 Excess of loss reinsurance

The reinsurer premium for excess of loss reinsurance agreement is calculated based on the expected value principle, which means the premium the reinsurer gets depends on the amount of the loss that will be covered by the reinsurer. Unlike proportional reinsurance, for the excess of reinsurance, the reinsurance premium rate is different than the rate of the covered loss, and the rate should be agreed upon by the participation of both parties.

$$P_R = (\lambda_1 + \lambda_2)(1 + \theta)(E[Z] - E[Z, d]).$$

Similarly to quota share reinsurance, the simulations of the insurer's surplus process will be discussed in this section. Optimization was obtained only considering the benefit of the direct insurer, so there is no guarantee that reinsurer companies will agree with the obtained deductible amount.

The number of Monte Carlo simulations is $k = 100000$ for any point of time $t = 1,2, \ldots\ldots, 12$, and the assumptions are the same as described before, at the beginning of the 3rd chapter. Thus, the surplus process of the direct insurer can be written as follows

$$C_t^d = C_{t-1}^d + P_t^d - \textstyle\sum_{t=1}^{n_t} \min(Z_t, d),$$

where P_t^d is the net of the reinsurance which means the reinsurer's portion already has been deducted. The direct insurer's premium is calculated as follows:

$$P_t^d = P_t - P_R = (\lambda_1 + \lambda_2)(1 + \eta)E[Z] - (\lambda_1 + \lambda_2)(1 + \theta)(E[Z] - E[Z, d]).$$

where the value of $E[Z]$ should be calculated according to formula (3.3).

In order to find the expected value of claim sizes $E[Z, d]$ under excess of loss reinsurance policy the cumulative distribution functions are necessary to find $P(Z \leq d)$ and $P(Z > d)$. Since the single claim size Z comes from the mixture of the two log-normal distributions, there is no direct formula for evaluating these probabilities. So, only for this step, the individual deductible levels are derived from the obtained formula for optimal level then the expected value of each individual log-normal distribution is calculated. Then, the probabilities $P(Z \leq d)$ and $P(Z > d)$ are determined based on the individual deductible levels and finally, the expected value of the claims under excess of reinsurance policy was calculated by taking the proportion of each obtained individual expected value:

$$E[X, d] = e^{\mu_1 + \frac{\sigma_1^2}{2}} P(Z \leq d) + d_1 P(Z > d),$$

$$E[Y, d] = e^{\mu_2 + \frac{\sigma_2^2}{2}} P(Z \leq d) + d_2 P(Z > d),$$

$$E[Z, d] = \frac{\lambda_1}{\lambda_1 + \lambda_2} E[X, d] + \frac{\lambda_2}{\lambda_1 + \lambda_2} E[Y, d].$$

The mean of the reinsured and non-reinsured surplus processes respectively will be

$$\widehat{C^d} = \frac{1}{k} \sum_{t=1}^{k} \min(Z_t, d),$$

$$\hat{C} = \frac{1}{k}\sum_{t=1}^{k} (Z_t).$$

Deductible level d is considered for the single claim amount in the portfolio, which means if the value of any claims passes the deductible level, the reinsurer will cover the excess of the incurred loss unlimitedly.

Lastly, the probability of ruin for direct insurer will be calculated based on the surplus trajectories.

$$P\left(\widehat{C^d} < 0\right) = \frac{1}{k}\sum_{t=1}^{k} I_{\{R_t^{dj} < 0\}} \ \forall \ t = (1,2,\ldots,12).$$

The ruin probability will be calculated with different rates of deductible level d, then will be compared with each other to see if indeed retention level affects the company's ruin probability.

All the mentioned analyses will be done for both utility and VaR maximization.

3.3.1 Utility maximization

For the excess of loss portfolio reinsurance agreement, the following optimal level of retention d has been obtained:

$$d = \frac{1}{\alpha}\ln(1+\theta)\,, \alpha > 0 \ and \ 0 \leq \theta \leq 1.$$

where the optimal retention rate depends on the premium loading factor θ of the reinsurer and the risk coefficient, α of the direct insurer. The behavior of the function can be commented in a way that as parameter θ closer to 0, the logarithmic part of the function $\ln(1+\theta)$ would get approximately equal to θ which would make $d \approx \frac{\theta}{\alpha}$. Additionally, as θ increases, $\ln(1+\theta)$ grows more slowly which means function d would increase at a decreasing rate which is also visible from the graph of the function. An increase in the value of α leads to a decrease in d. This means that with higher sensitivity (or risk aversion), the influence of θ decreases; therefore, a larger α will lead to a smaller d.

Similarly to the quota share reinsurance, the simulations were done in R studio.

Now for excess of loss portfolio reinsurance policy, consider a random variable Z_i $(i = 1,2,\ldots.12)$ which is the mixture of two independent and identical log-normally distributed random variables. Log-normally distributed claim sizes X_i and Y_i, have mean values $\mu_1 = 7.5$, $\mu_2 = 6$ and standard deviation $\sigma_1 = 1$, $\sigma_2 = 1$ respectively.

Firstly, the trajectories are obtained at $\alpha = 0.01$ and $\eta = 0.05$, but later, for comparison purposes to see the effect of risk coefficient rate and premium loading factor, the calculations were repeated in different levels of parameter α and η.

In the graphs below, the visual observation of direct insurer's average original and reinsured surplus processes $\widehat{C_t}$ and $\widehat{C_t^d}$ and individually simulated trajectories $\widehat{C_t^J}$, $\widehat{C_t^{dJ}}$ at time $t = 1,2,\ldots,12$ and constant $\theta = 0.05$ is described:

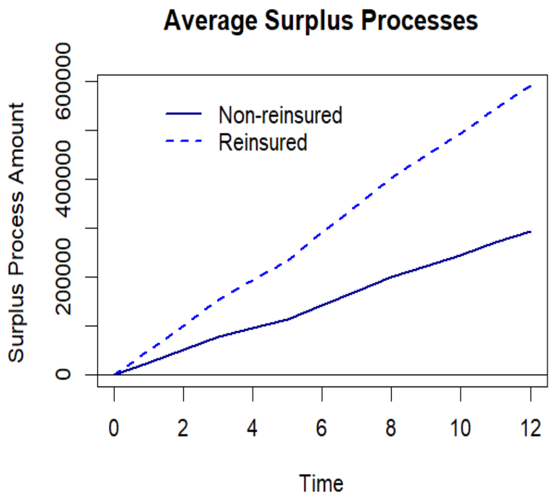


Figure 3.9. Average of Surplus processes $\widehat{C_t^J}$, $\widehat{C_t^{dJ}}$.

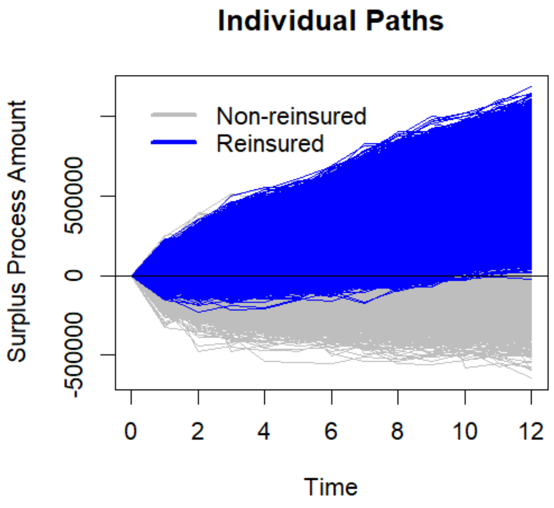


Figure 3.10. Individual trajectories $\widehat{C_t^J}$, $\widehat{C_t^{dJ}}$.

In Figure 3.9, there is a significant difference between Non-reinsured and Reinsured surplus values at the end of each period. While reinsured surplus values are improving faster, there is a slighter slope of difference for non-reinsured surplus values. The difference in the results suggests that with defined parameters, the surplus process is very volatile, and the probability of massive claim events is higher which increases the loss of bigger amounts. It is visible that the reinsurance agreement significantly decreased the higher losses due to the limit in maximum payable claim amount for the insurer. At the same time, reinsured surplus processes in the excess of loss reinsurance are less likely to depend on the company's original surplus, which in this case surplus value mostly depends on retention level d. Thus, the excess of loss reinsurance most likely would decrease the magnitude of losses significantly, would make the surplus processes less volatile, and would make the direct insurer's solvency much better.

To analyze it better, the impact of the changes in parameters θ and α and their influences on the value of ruin probability of the company are collected in the following tables:

θ	d^*	$P(\widehat{C_T^d} < 0)$	$\frac{P(\widehat{C_{\theta_i}^d} < 0)}{P(\widehat{C_{\theta_{i-1}}^d} < 0)}$	$\widehat{C_T^d}$	$S(C_T^d)$
0.03	5428.80	0.00	-	823015.8	117154.6
0.05	8960.84	0.0003	-	588645.3	145592.5
0.07	12426.25	0.00208	6.93	474389.1	161960.1
0.08	14134.73	0.00511	2.45	439288.6	167676.0
0.09	15827.47	0.009	1.76	413333.3	172587.6
0.1	17504.75	0.0144	1.6	392619.0	175548.6
0.2	33485.33	0.05202	3.61	316525.4	192791.7

Table 3.4. The effect of the reinsurance premium loading factor θ over retention levels, terminal surplus $\widehat{C_T^d}$, and ruin probability $P(\widehat{C_T^d} < 0)$ at $\alpha = 0.01$.

From Table 3.4 it is visible that if the reinsurer's loading factor θ increases, then both the threshold d and the cost of the reinsurance treaty (or rather the premium of the reinsurer) increase. Nevertheless, the cedent's retained premium decreases with θ. It means that the expected value of terminal values of the portfolio's surplus process $\widehat{C_T^d}$ decreases as θ increases. This is not surprising given that for a higher threshold level (where only a small proportion of risks are transferred), the premium paid to the reinsurer for cover will rise significantly on account of an added loading factor θ. As such, there will be an increase in the standard deviation of the ultimate values of the terminal surplus, due to a larger maximum liability which is imposed by the reinsured leading to wider swings in paid claims. At the same time, the probability of the portfolio's terminal surplus values being negative also significantly increases even with small changes in the value of retention level d. So, ruin probabilities are very sensitive to changes in the reinsurer's premium loading factor θ, and intuitively with the lower responsibilities there is a smaller chance that the insurer's surplus can be a negative value and vice-versa.

In the following table below, Table 3.5 the impact of the changes in direct insurer's risk aversion rate α has been analyzed. Unlike the behavior of parameter θ, the small increases in the value of α result in having smaller retention levels d, which also positively affects the company's readability to high-risk situations. As risk aversion parameter α increases, both the probability of having a negative surplus and the change in the value of ruin probability decreases. A positive pattern was also observed in the amount of the standard deviation of the surplus process by having a less volatile portfolio surplus. Meanwhile, there is an increasing pattern in the expected value of surplus $\widehat{C_T^d}$ until a particular value of the parameter α (when $\alpha = 0.05$ in this example), and then the expected value starts decreasing, which means in order to operate more securely, the direct insurer transfers the higher value of premiums with the less value of retained risk. The decision of the parameter α depends on the various risk factors like economic, global, liquidity of the company, etc. Individual analysis should be done to decide the optimal risk aversion parameter for each individual company.

α	d^*	$P(\widehat{C_T^d} < 0)$	$\frac{P(\widehat{C_{\alpha_\iota}^d} < 0)}{P(\widehat{C_{\alpha_{\iota-1}}^d} < 0)}$	$\widehat{C_T^d}$	$S(C_T^d)$
0.01	8960.844	0.00002	-	588514.8	146564.2
0.02	4480.422	0.00	0.00	899601.5	105725.1
0.03	2986.948	0.00	0.00	1065518.2	82110.89
0.05	1792.169	0.00	0.00	1150000.0	56398.49
0.1	896.084	0.00	0.00	976484.3	31302.1
0.2	448.042	0.00	0.00	635251.0	16485.77
0.3	298.695	0.00	0.00	454224.0	11177.25

Table 3.5. The effect of the direct insurer's risk aversion rate α *over retention levels, terminal surplus* $\widehat{C_T^d}$*, and ruin probability* $P(\widehat{C_T^d} < 0)$ *at* $\theta = 0.05$.

The following graphs show the behavior of the expected value and standard deviation due to changes in parameters α and θ respectively:

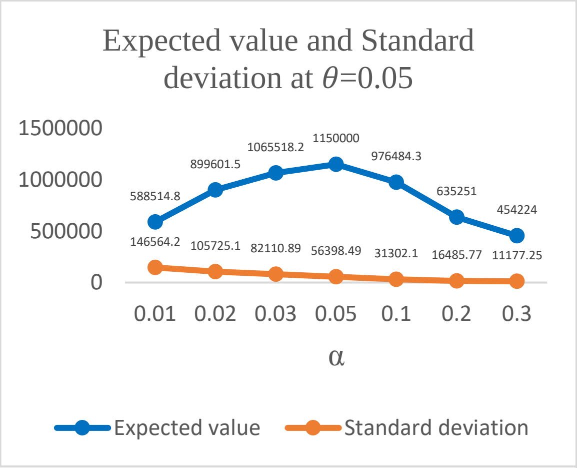


Figure 3.11.The pattern of expected v. and s.d over (α)

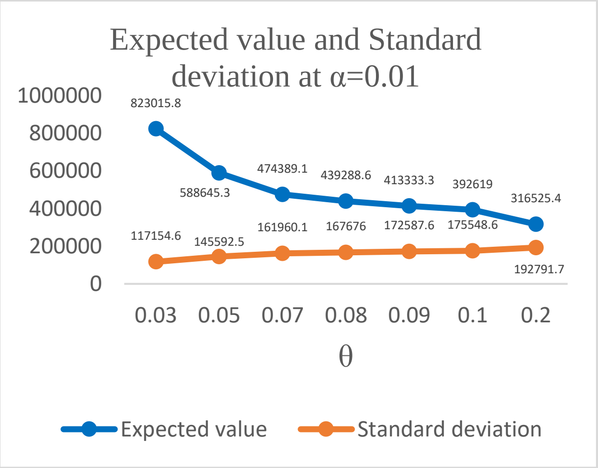


Figure 3.12.The pattern of expected v. and s.d over (θ)

The next graph shows the behavior of the change in ruin probabilities with respect to the reinsurer's premium loading parameter, θ :

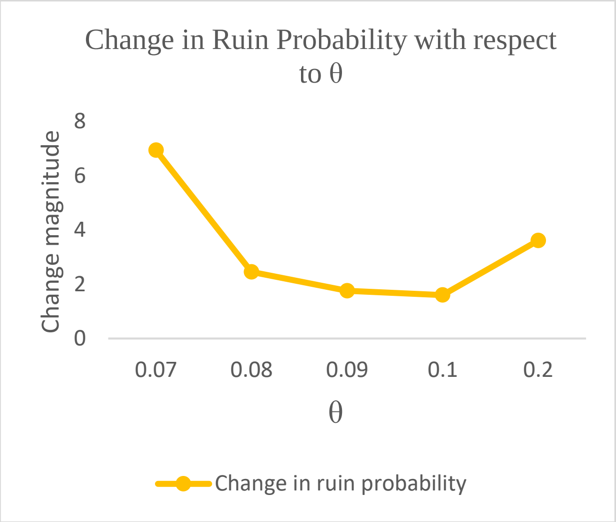


Figure 3.13. The pattern of ruin probability w.r.t. θ.

Also, it is worth mentioning that in real businesses there are more expenses of the direct insurer including administration, acquisition, and other expenses which can have a big impact on the company's surplus process. For simplicity, the effects of these expenses were ignored, so they are out of this study. At the same time, the obtained optimal retention rate cannot be accepted by the reinsurer, so real-life discussions and offers should be considered.

3.3.2 VaR Optimization Approach

The optimal retention level d^* under the VaR_α optimization approach is determined as follows:

$$d^* = S_{S_Z}^{-1}((1+\theta)^{-1}).$$

where the optimal retention level d^* depends on the quantile function of the insurer's portfolio. However, according to Chengguo Weng (Weng, 2009), it is only possible to get a unique non-trivial optimal retention level when the following assumptions are true

$$S_Z^{-1}(\alpha) \geq d^* + (1+\theta)\int_{d^*}^{\infty} S_Z(z)dz.$$

for optimal excess of loss retention levels.

Consider all the assumptions about parameters are the same as stated in part 3.3.1, except the direct insurer risk aversion rate is $\alpha = 0.005$.

Thus, considering the above-mentioned assumptions are in force then, the Average Surplus processes $\widehat{C_t^d}, \widehat{C_t^{d^*J}}$, and Individual trajectories $C_t^{d^*j}, C_t^{d^*j}$ of the direct insurer in non-reinsured and reinsured cases respectively, will be as in Figures 3.14 and 3.15.

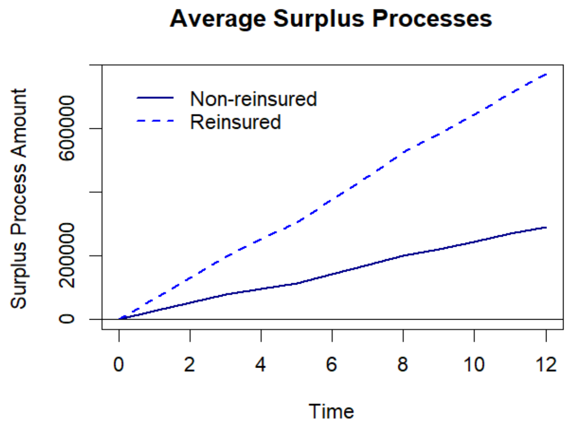


Figure 3.14. Average of Surplus processes $\widehat{C_t^J}$, $\widehat{C_t^{dJ}}$

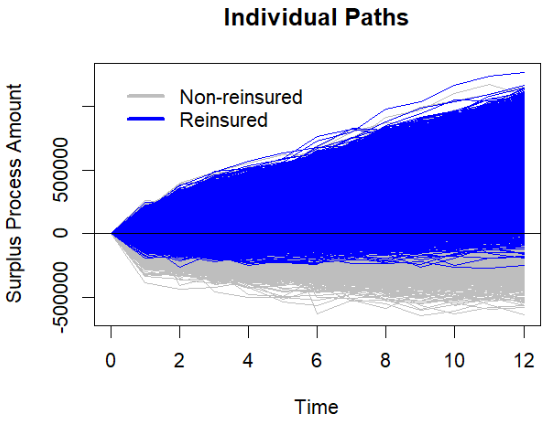


Figure 3.15. Individual trajectories $\widehat{C_t^J}$, $\widehat{C_t^{dJ}}$

The gap between the Non-reinsured and Reinsured average Surplus process is considerably large, the direct insurer reaches a higher surplus at the end of the year due to the existence of an additional cover – reinsurance agreement. Unlike the quota-share reinsurance agreement in excess of reinsurance, the direct insurer gets a higher surplus value than the non-reinsured case of the portfolio in the stated examples. However, in real life the difference in the parametrization should be considered since the agreement also should be profitable for the reinsurer, otherwise, there wouldn't exist such business. Thus, with the real historical data from insurance companies, the output of the obtained models can be analyzed in a better way. In Figure 3.14, it is clearly seen that the reinsurance agreement significantly decreases the ruin probability for the direct insurer. Also, in the positive surplus processes, the reinsured individual trajectories overlap the non-reinsured ones which support the results from the previous graph.

From the following table, it is clear that the direct insurer's minimal risk exposure VaR_α for the whole portfolio is obtained in the optimal level of excess of loss retention level d^*. This means the optimization approach from value at risk point of view is indeed working.

In the following table and graph, the sensitivity of the optimal retention rates and other measures over the reinsurance premium loading factor θ can be analyzed.

θ	d^*	$VaR_\alpha(S_P^d)$	$VaR_\alpha(S_P)$	$P(\widehat{C_T^{d^*}} < 0)$	$P(\widehat{C_T} < 0)$	$\widehat{C_T^{d^*}}$	$S(C_T^{d^*})$
0.005	14665.15	3501396.00	3879013.00	0.00562	0.07753	438561.00	168916.10
0.008	12425.4	2971243.00	3879013.00	0.00155	0.07753	482686.80	161672.30
0.010	11450.11	2741006.00	3879013.00	0.00085	0.07753	508824.90	158492.20
0.012	10693.43	2562746.00	3879013.00	0.00045	0.07753	532462.30	155132.80
0.015	9814.905	2356317.00	3879013.00	0.00007	0.07753	562620.90	150570.00
0.06	9570.344	2298977.00	3879013.00	0.00009	0.07753	571982.80	149899.60
0.02	8756.86	2108734.00	3879013.00	0.00000	0.07638	607246.7	144798.3

Table 3.6. The effect of the reinsurance premium loading factor θ over retention levels d^*, terminal surplus $\widehat{C_T^d}$, and ruin probability $P(\widehat{C_T^{d^*}} < 0)$ at $\alpha = 0.005$.

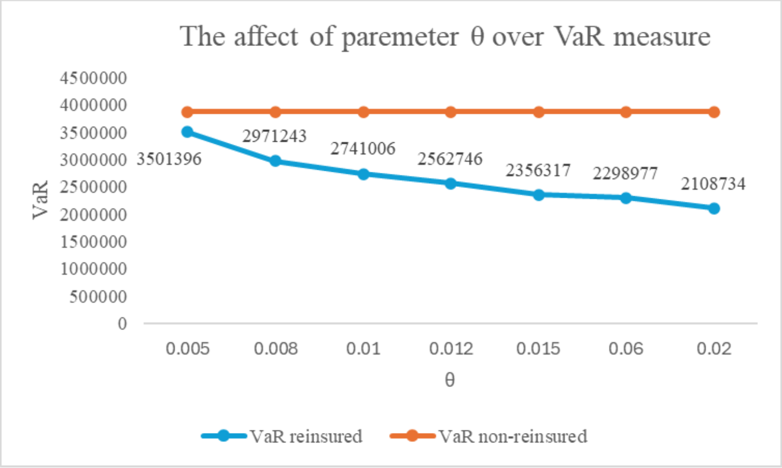


Figure 3.16. The pattern of the Value at Risk with respect to θ.

The main dependence of the optimal retention levels under the VaR_α is the reinsurer's safety loading factor θ. As the value of parameter θ increases, there is a perfect decreasing tendency of all measures, other than the expected value of the surplus. Since the reinsurer safety loading factor is the price of the reinsurance treaty, it is normal to experience lesser amounts of deductible parameter d^* in case of higher rates of parameter θ. Logically, if the direct insurer pays more premiums to the reinsurer, the reinsurer also must be responsible for a bigger part of the occurred losses. As the reinsurer covers more claims the value of standard deviation decreases and the value of the expected value increases. The results for the ruin probabilities suggest that the reinsurance policy can mitigate the solvency risk even to zero value if the reinsurer company gets enough payments. But once again, the application of these results to real business can be a bit tricky, since it depends on the many analytical measures and interests of both parties.

Overall, from a value risk perspective, it can be concluded that the excess of loss reinsurance treaty can indeed decrease the direct insurer's risk exposure. Depending on the characteristics of the business area and global economic or financial process which can increase the amount of the losses dramatically, some part of the extra surplus amount should be given up protecting the business. To do it, excess of loss reinsurance agreement in the optimal level of retention level from the direct insurer's side can be a beneficial risk mitigating technique.

CONCLUSIONS

In this research, the complex problem of optimizing reinsurance parameters for an insurance portfolio consisting of two risk products has been investigated. The research has focused specifically on determining the optimal retention levels that balance risk reduction with profitability, using various mathematical models and simulation techniques.

One of the key findings is the triviality of the optimal reinsurance retention rates which mostly depends on the premium calculation method used. Especially, this is the case for quota share reinsurance. But by using correct premium calculation and optimization approach this problem can be solved. It has been found out that under expected values premium calculation the optimal quota share retention is trivial, but under the variance principle is it possible to find a non-trivial optimal retention rate? In further research, other premium calculation methods can be checked to determine the triviality of the optimal retention rates.

In this research bivariate non-life insurance portfolio has been examined, in the future optimal parameters of reinsurance can be determined for a portfolio consisting of three or more products which is a usual case in the real insurance business.

One of the key conclusions of this research is that the optimization of reinsurance retention levels is highly dependent on the trade-off between risk and return. The results from the simulation models demonstrate that while higher retention levels can lead to greater profitability, they also increase the risk of significant financial losses in the event of large claims. Conversely, lower retention levels reduce risk but may result in excessive reinsurance costs, thereby diminishing overall profitability. This trade-off highlights the importance of carefully balancing these factors to achieve an optimal reinsurance strategy.

The study also confirms the effectiveness of using both utility maximization and value-at-risk approaches in optimizing reinsurance parameters.

The simulations conducted in this study also provided valuable insights into the practical application of reinsurance optimization. By modeling different reinsurance scenarios, including quota share and excess of loss reinsurance, the research was able to identify optimal retention levels that minimize the insurer's risk exposure while maintaining an acceptable level of profitability. The results indicated that in many cases, a combination of reinsurance types might be the most effective strategy.

In conclusion, this thesis has demonstrated that the optimization of reinsurance retention parameters is a critical aspect of non-life insurance management, requiring a careful balance between risk and profitability. The research has provided a framework for determining optimal retention levels, incorporating both theoretical models and practical simulations. Future research can build on this work by exploring the optimization of reinsurance strategies in portfolios with more complex structures or in different regulatory environments especially including the impacts of Solvency capital requirement.

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