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Plasticity of the unit ball of a strictly convex Banach space

Mathematics
Bachelor's thesis (9 ECTS)

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Tartu 2022

RANGELT KUMERA BANACHI RUUMI ÜHIKKERA PLASTILISUS

Bakalaureusetöö

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Lühikokkuvõte

Käesoleva lõputöö eesmärgiks on üksikasjalikult lahti kirjutada rangelt kumera Banachi ruumi plastilisuse tõestus, mille avaldasid B. Cascales, V. Kadets, J. Orihuela ja E. J. Wingler aastal 2016.

CERCS teaduseriala: P140 Jadad, Fourier analüüs, funktsionaalanalüüs.

Märksõnad: Mittelaiendav kujutus, plastiline ruum, rangelt kumer Banachi ruum.

PLASTICITY OF THE UNIT BALL OF A STRICTLY CONVEX BANACH SPACE

Bachelor's thesis

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Abstract

The aim of this thesis is to provide details for the proof of plasticity of the unit ball of a strictly convex Banach space presented by B. Cascales, V. Kadets, J. Orihuela, and E. J. Wingler in 2016.

CERCS research specialization: P140 Series, Fourier analysis, functional analysis.

Keywords: Nonexpansive map, plastic space, strictly convex Banach space.

Contents

Introduction

For a metric space X, a function $f : X \to X$ is called

- *nonexpansive*, if for each x and y in X we have
  $d\big(f(x), f(y)\big) \leq d(x, y)$;
- *noncontractive*, if for each x and y in X we have
  $d\big(f(x), f(y)\big) \geq d(x, y)$;
- an *isometry*, if for each x and y in X we have
  $d\big(f(x), f(y)\big) = d(x, y)$.

We call a metric space *plastic* if every nonexpansive bijection from this space onto itself is an isometry. Equivalently, a metric space is plastic if every noncontractive bijection from this space onto itself is an isometry. The first systematic study of this property was carried out by S. A. Naimpally, Z. Piotrowski, and E. J. Wingler in 2006 [5]. The central result concerning the property is that every totally bounded metric space is plastic. This result has been known for a while, the proof can be found even in a 1936 article by H. Freudenthal and W. Hurewicz [2]. The mentioned result is also a major source of motivation for studying the property.

In 2016, B. Cascales, V. Kadets, J. Orihuela, and E. J. Wingler proved the plasticity of the unit ball of a strictly convex Banach space and posed the question of whether unit balls of all Banach spaces are plastic [1]. The latter remains an open question as of today. However, numerous partial positive results have been obtained since then. So far, the plasticity of the unit ball has been proved for the following spaces:

- strictly convex spaces
  (B. Cascales, V. Kadets, J. Orihuela, and E. J. Wingler; 2016);
- the space ℓ_1
  (V. Kadets and O. Zavarzina; 2016);
- ℓ_1-sums of strictly convex spaces
  (V. Kadets and O. Zavarzina; 2018);

- spaces whose unit sphere is the union of all its finite-dimensional polyhedral extreme subsets
  (C. Angosto, V. Kadets, and O. Zavarzina; 2018);

- the space c
  (N. Leo; 2022);

- spaces $\ell_1 \oplus_2 \mathbb{R}$ and $X \oplus_\infty Y$ for strictly convex X and Y
  (R. Haller, N. Leo, and O. Zavarzina; 2022).

The thesis at hand is based upon the article concerning the plasticity of the unit ball of a strictly convex space, the article that started the whole series. The proof presented in this article is brief, missing many of the tedious details. We aim to fill in these details and provide all the necessary preliminaries for understanding the proof.

The thesis is divided into two major sections. The first section provides the reader with all the necessary definitions and results of convex analysis with complete proof. We define the notions of extreme point and strictly convex Banach space, prove the existence of directional derivative of a convex function and its sublinearity, apply the latter to the norm function, define the notion of smooth point and demonstrate its connection to the derivative of the norm.

In the second section, we demonstrate the results presented in the aforementioned paper. We study the properties of extreme points and the properties of nonexpansive bijections on the unit ball in general and in the strictly convex case. At the end, we demonstrate the main theorem of this thesis, which states that a nonexpansive bijection on the unit ball of a strictly convex Banach space is an isometry.

1 Preliminaries

In this section, we shall list the preliminaries. Given that the research paper that this thesis is based upon is rich in results from convex analysis, we will present all the essentials in the required depth.

From this point onwards, X stands for a real Banach space. We denote by S_X and B_X the unit sphere and the closed unit ball of X.

1.1 Extreme points and strictly convex spaces

Definition 1.1. Subset $A \subset X$ is called a *convex* set if, for any $x, y \in A$, the line segment $[x, y]$ connecting x and y is included in A, or more formally,

$$tx + (1-t)y \in A \text{ for all } t \in [0, 1].$$

Convex sets contain a subset of exceptional interest, namely the set of extreme points. Extreme points possess numerous distinctive properties, which we will rely on in future investigations.

Definition 1.2. Let $A \subset X$ be a convex set. A point $z \in A$ is called an *extreme point* of A if there does not exist $x, y \in A$ and $0 < t < 1$ such that $x \neq y$ and $z = tx + (1-t)y$, or equivalently, if for every $y \in X \backslash \{0\}$ either $z + y \notin A$ or $z - y \notin A$. We will denote the set of all extreme points of A by $\text{ext}(A)$.

Definition 1.3. A set $A \subset X$ is called *strictly convex*, if all of its boundary points are extreme points.

Definition 1.4. Banach space X is called *strictly convex*, if B_X is strictly convex set.

1.2 Convex functions on real Banach spaces

Let D be a nonempty open convex subset of X.

Definition 1.5. A function $f : D \to \mathbb{R}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

whenever $x, y \in D$ and $t \in (0,1)$.

Definition 1.6. A function $p : X \to \mathbb{R}$ is said to be *subadditive* if, for all $x, y \in X$, we have $p(x+y) \leq p(x) + p(y)$.

Definition 1.7. A function $p : X \to \mathbb{R}$ is said to be *positively homogeneous* if, for all $\alpha > 0$ and $x \in X$, we have $p(\alpha x) = \alpha p(x)$.

Note that a positively homogeneous function $p : X \to \mathbb{R}$ satisfies $p(0) = 0$; this follows from positive homogeneity:

$$p(2 \cdot 0) = 2p(0) \implies p(0) = 2p(0) \implies p(0) = 0.$$

Definition 1.8. A *sublinear functional* is a function $p : X \to \mathbb{R}$ that is subadditive and positively homogeneous.

Lemma 1.9 ([6, Lemma 1.2])**.** *Let $f : D \to \mathbb{R}$ be a convex function and let x be a point of D. Then for each $y \in X$ there exists a one-sided finite limit*

$$\lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t},$$

which we will call the directional derivative of f at x along y and denote by $df(x)(y)$. Moreover, for fixed x, the function $df(x) : X \to \mathbb{R}$, $y \mapsto df(x)(y)$ turns out to be a sublinear functional, which we will call the derivative of f at x.

Proof. Let $y \in X$. Note that, since D is open, $f(x+ty)$ is defined for sufficiently small $t > 0$. The difference quotient $\frac{f(x+ty)-f(x)}{t}$ is nonincreasing as $t \to 0^+$, and bounded below.

If $0 < t < s$ (and they are sufficiently small), then we can rewrite $x + ty = t/s(x+sy) + (1-t/s)x$, and then use the convexity of f to obtain:

$$\begin{aligned} \frac{f(x+ty) - f(x)}{t} &\leq \frac{t/sf(x+sy) + (1-t/s)f(x) - f(x)}{t} \\ &= \frac{f(x+sy) - f(x)}{s}, \end{aligned}$$

which proves the monotonicity. Applying this to $-y$ in place of y, we see that

$$-\frac{f(x-ty)-f(x)}{t}$$

is nondecreasing as $t \to 0^+$. Moreover, by convexity of f, for every sufficiently small $t > 0$, we have

$$2f(x) \leq f(x-ty) + f(x+ty),$$

and therefore

$$-\frac{f(x-ty)-f(x)}{t} \leq \frac{f(x+ty)-f(x)}{t},$$

which shows that the right-hand side is bounded below and the left-hand side is bounded above as $t \to 0^+$. Thus both limits

$$\lim_{t\to 0^+} \frac{f(x-ty)-f(x)}{t} \quad \text{and} \quad \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$$

exist and are finite; the left-hand one is $-df(x)(-y)$ and we have

$$-df(x)(-y) \leq df(x)(y).$$

It is a simple consequence that $df(x)$ is positively homogeneous: for all $\alpha > 0$ and $y \in X$, we have

$$\begin{aligned} df(x)(\alpha y) &= \lim_{t\to 0^+} \frac{f(x+t(\alpha y))-f(x)}{t} \\ &= \lim_{t\to 0^+} \alpha \frac{f(x+(t\alpha)y)-f(x)}{t\alpha} \\ &= \alpha df(x)(y). \end{aligned}$$

To see that $df(x)$ is subadditive, we use convexity of f again to obtain first that, for all (sufficiently small) $t > 0$ and $y_1, y_2 \in X$, we have

$$\frac{f(x+t(y_1+y_2))-f(x)}{t} \leq \frac{f(x+2ty_1)-f(x)}{2t} + \frac{f(x+2ty_2)-f(x)}{2t},$$

and then take the limits as $t \to 0^+$. $\square$

Definition 1.10. A function $f : D \to \mathbb{R}$ is said to be *Gateaux differentiable* at $x \in D$ provided that the two-sided limit

$$\lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$$

exists and is finite for each $x \in X$.

Clearly, a convex function $f : D \to \mathbb{R}$ is Gateaux differentiable at $x \in D$ if and only if $df(x)(-y) = -df(x)(y)$ for each $y \in X$. Moreover, it turns out that if a convex function $f : D \to \mathbb{R}$ is Gateaux differentiable at $x \in D$, then the function $df(x)$, the derivative of f at x, is a linear functional. This follows from the next lemma.

Lemma 1.11. *Let $p : X \to \mathbb{R}$ be a sublinear functional. Then $p(-x) = -p(x)$ for every $x \in X$ if and only if p is linear.*

Proof. It is clear that if a function p is linear, then $p(-x) = -p(x)$ for every $x \in X$. Now, let us show the converse. First, we demonstrate that $p(x_1 + x_2) = p(x_1) + p(x_2)$ for all $x_1, x_2 \in X$. From the sublinearity of p we get $p(x_1 + x_2) \leq p(x_1) + p(x_2)$, hence it suffices to show that $p(x_1) + p(x_2) \leq p(x_1 + x_2)$. Multiplying both sides of the latter inequality by -1, we obtain an equivalent inequality:

$$-p(x_1) - p(x_2) \geq -p(x_1 + x_2).$$

Since $-p(x_1) = p(-x_1)$, $-p(x_2) = p(-x_2)$, and $-p(x_1 + x_2) = p(-x_1 - x_2)$ by the assumption, the latter inequality can be rewritten as

$$p(-x_1) + p(-x_2) \geq p(-x_1 - x_2).$$

The latter is, in turn, a straight consequence of the subadditivity of p.

It remains to prove the homogeneity of p. Let $\alpha \in \mathbb{R}$ and $x \in X$ be arbitrary. Our aim is to show that $p(\alpha x) = \alpha p(x)$. For $\alpha \geq 0$, the equality follows from the positive homogeneity of p. Now, let us consider the case $\alpha < 0$. Rewrite $p(\alpha x)$ as $p(-|\alpha| x)$ and use the assumption to obtain $p(-|\alpha| x) = -p(|\alpha| x)$. Then, use the positive homogeneity of p to obtain $p(|\alpha| x) = |\alpha| p(x)$. We have now obtained the following chain of equalities:

$$p(\alpha x) = p(-|\alpha| x) = -p(|\alpha| x) = -|\alpha| p(x) = \alpha p(x).$$

We see now that $p(\alpha x) = \alpha p(x)$, as was needed. $\square$

To sum up, we have the following result.

Lemma 1.12. *A convex function $f : D \to \mathbb{R}$ is Gateaux differentiable at $x \in D$ if and only if the function $df(x)$ is linear.*

1.3 Derivative of the norm

Consider the norm function $\|\cdot\| : X \to \mathbb{R}$, $x \mapsto \|x\|$. It is an easy consequence that this function is convex:

$$\|tx + (1-t)y\| \leq \|tx\| + \|(1-t)y\| = |t|\|x\| + |1-t|\|y\| = t\|x\| + (1-t)\|y\|$$

for all $x, y \in X$ and $t \in (0,1)$. So by Lemma 1.9 the "right hand" directional derivative of the norm function $\|\cdot\|$ exists at any $x \in S_X$ along any $y \in X$, i.e., the one-sided limit

$$\lim_{t \to 0^+} \frac{1}{t}\big(\|x + ty\| - \|x\|\big)$$

exists and is finite for any $x \in S_X$ and $y \in X$. We are going to denote this limit by $x^*(y)$. Moreover, Lemma 1.9 says that, for a fixed $x \in S_X$, the function $x^* : X \to \mathbb{R}$, $y \mapsto x^*(y)$, is a sublinear functional on X. That is, for all $y \in X$ and $\alpha > 0$, we have

$$x^*(\alpha y) = \alpha x^*(y),$$

and, for all $y_1, y_2 \in X$, we have

$$x^*(y_1 + y_2) \leq x^*(y_1) + x^*(y_2).$$

If in the latter inequality we take $y_1 = y$ and $y_2 = -y$, we obtain a special case:

$$x^*(y) \geq -x^*(-y). \tag{1}$$

It turns out that the function x^* admits the following estimation: for all $x \in S_X$ and $y_1, y_2 \in X$, we have

$$|x^*(y_1) - x^*(y_2)| \leq \|y_1 - y_2\|. \tag{2}$$

To show this, we use the definition of directional derivative and the triangle inequality:

$$\begin{aligned} x^*(y_1) - x^*(y_2) &= \lim_{t\to 0^+} \frac{1}{t}\big(\|x+ty_1\| - \|x\|\big) - \lim_{t\to 0^+} \frac{1}{t}\big(\|x+ty_2\| - \|x\|\big) \\ &= \lim_{t\to 0^+} \frac{1}{t}\big(\|x+ty_1\| - \|x+ty_2\|\big) \\ &\le \lim_{t\to 0^+} \frac{1}{t}\|x+ty_1 - x - ty_2\| = \|y_1 - y_2\|. \end{aligned}$$

By exchanging the roles of y_1 and y_2, we obtain

$$x^*(y_2) - x^*(y_1) \le \|y_1 - y_2\|.$$

Together these two inequalities imply $|x^*(y_1) - x^*(y_2)| \le \|y_1 - y_2\|$.

Moreover, by taking $y_1 = y$ and $y_2 = 0$, we obtain a special case:

$$|x^*(y)| \le \|y\|. \tag{3}$$

The latter is valid for all $x \in S_X$ and $y \in X$.

We also notice that by definition we have $x^*(x) = 1$ for every $x \in S_X$:

$$x^*(x) = \big(\|x+tx\| - \|x\|\big) = \lim_{t\to 0^+} \frac{1}{t}\|x\|\big(|1+t| - 1\big) = \|x\|.$$

Since x^* is sublinear, we also get $x^*(\alpha x) = \alpha$ for every $\alpha > 0$.

The main theorem of this thesis makes use of smooth points. The following is the standard definition of a smooth point.

Definition 1.13. A point $x \in S_X$ is called *smooth* if there is a unique $f \in X^*$ such that $\|f\| = 1$ and $f(x) = 1$.

The notion of a smooth point turns out to have a connection to the derivative of the norm. The following theorem establishes this relation.

Theorem 1.14 ([4, Theorem 5.4.17])**.** *A point $x \in S_X$ is smooth if and only if the norm of X is Gateaux differentiable at x.*

As was pointed out in Lemma 1.12, the assertion that the norm is Gateaux differentiable at x is equivalent to the assertion that x^* is a linear function. Therefore, we obtain the following corollary.

Corollary 1.15. *A point $x \in S_X$ is smooth if and only if the derivative of the norm at x is a linear function.*

The following result about smooth points is crucial to the proof of the main theorem of this thesis.

Theorem 1.16 (cf. [3, Theorem 20F])**.** *If X is a separable Banach space, then the set of smooth points is dense in S_X.*

2 Main results

In this section, we present and prove the main result of the thesis: the unit ball of a strictly convex Banach space is plastic. This theorem is due to B. Cascales, V. Kadets, J. Orihuela, and E. J. Wingler [1].

2.1 Properties of extreme points

Let us begin by showing some of the properties of extreme points. We use these properties later to study nonexpansive bijections.

For any $x \in S_X$ and $a \in (0,1)$, let

$$D(x,a) = aB_X \cap \big(x + (1-a)B_X\big).$$

Lemma 2.1 ([1, Lemma 2.1])**.** *For every $x \in S_X$ and $a \in (0,1)$, one has*

$$D(x,a) = a\Big\{x + y \in B_X : x - \frac{a}{1-a}\, y \in B_X\Big\}. \tag{4}$$

If $x \in S_X$ is an extreme point of B_X, then $D(x,a) = \{ax\}$ for every $a \in (0,1)$. If $x \in S_X$ is not an extreme point of B_X, then $D(x, \frac{1}{2})$ consists of more than one point.

Proof. Fix $x \in S_X$ and $a \in (0,1)$. We begin by showing the equality (4). We have to show that $z \in D(x,a)$ if and only if $z = a(x+y)$ with $x + y \in B_X$ and $x - \frac{a}{1-a}y \in B_X$. To prove this, consider an element $y \in X$. Clearly,

$$\begin{aligned}
a(x+y) \in D(x,a) &\iff a(x+y) \in aB_X \cap \big(x + (1-a)B_X\big) \\
&\iff \big(a(x+y) \in aB_X\big) \wedge \big(a(x+y) \in (x + (1-a)B_X)\big) \\
&\iff \big((x+y) \in B_X\big) \wedge \Big(\frac{a(x+y) - x}{1-a} \in B_X\Big) \\
&\iff \big(x + y \in B_X\big) \wedge \Big(-x + \frac{a}{1-a}y \in B_X\Big) \\
&\iff \big(x + y \in B_X\big) \wedge \big(x - \frac{a}{1-a}y \in B_X\big).
\end{aligned}$$

The equality (4) is now proved.

Assume that $x \in \text{ext}(B_X)$. We show that $D(x,a) = \{ax\}$. By the equality (4), every element in $D(x,a)$ is in the form of $a(x+y)$ with $x+y \in B_X$ and $x - \frac{a}{1-a}y \in B_X$. Let $y \in X$ be such that $x+y \in B_X$ and $x - \frac{a}{1-a}y \in B_X$. Since B_X is convex, the line segment $[x+y, x-\frac{a}{1-a}y]$ entirely lies in B_X. Notice that point x is contained in the interior of that line segment because

$$x = a(x+y) + (1-a)(x - \frac{a}{1-a}y).$$

However, by definition, an extreme point of B_X cannot be in the interior of a nontrivial line segment that lies in B_X; hence the line segment consists of just one point x, i.e., $y = 0$. Consequently, the set $D(x,a)$ is a singleton set consisting of ax only.

Finally, assume that $x \in S_X \backslash \text{ext}(B_X)$. Since x is not an extreme point of B_X, by definition, there is an element $y \in X \backslash \{0\}$ such that $x+y \in B_X$ and $x - y \in B_X$. Then both the points $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$ belong to B_X, and therefore also to $D(x, \frac{1}{2})$ by the equality (4). Consequently, the set $D(x, \frac{1}{2})$ is not a singleton set $\square$

Let us consider another useful collection of sets. For any $x \in S_X$, let

$$D_1(x) = (x + B_X) \cap (-x + B_X).$$

Now we show a virtue of a statement similar to the previous one.

Lemma 2.2 ([1, Lemma 2.2])**.** *For every $x \in S_X$, one has*

$$D_1(x) = -x + \{x + y \in B_X : x - y \in B_X\}. \tag{5}$$

If $x \in S_X$ is an extreme point of B_X, then $D_1(x) = \{0\}$.

Proof. Let $x \in S_X$. The equality (5) is proved by the following equivalences:

$$\begin{aligned} y \in D_1(x) &\iff y \in (x + B_X) \cap (-x + B_X) \\ &\iff \big(y \in (x + B_X)\big) \wedge \big(y \in (-x + B_X)\big) \\ &\iff \big(y - x \in B_X\big) \wedge \big(x + y \in B_X\big) \\ &\iff \big(x - y \in B_X\big) \wedge \big(x + y \in B_X\big). \end{aligned}$$

Suppose now that $x \in \mathrm{ext}(B_X)$. By the definition of an extreme point, there is no such element $y \in X \backslash \{0\}$ satisfying both $x + y \in B_X$ and $x - y \in B_X$. Therefore

$$\{x + y \in B_X : x - y \in B_X\} = \{x\},$$

and it follows that $D_1(x) = -x + \{x\} = \{0\}$. $\square$

2.2 Properties of nonexpansive bijections

The following theorem lists some main properties of a nonexpansive bijection on the unit ball of a Banach space.

Theorem 2.3 ([1, Theorem 2.3])**.** *Let $F : B_X \to B_X$ be a nonexpansive bijection. Then the following hold:*

(1) *$F(0) = 0$;*

(2) *$F^{-1}(S_X) \subset S_X$;*

(3) *$F(D(x, a)) \subset D(F(x), a)$ for all $x \in F^{-1}(S_X)$ and $a \in (0, 1)$;*

(4) *if $F(x)$ is an extreme point of B_X, then $F(ax) = aF(x)$ for all $a \in (0, 1)$;*

(5) *if $F(x)$ is an extreme point of B_X, then x is an extreme point of B_X;*

(6) *if $F(x)$ is an extreme point of B_X, then $F(-x) = -F(x)$.*

Moreover, if X is strictly convex, then

(i) *F maps the sphere S_X bijectively onto itself;*

(ii) *$F(ax) = aF(x)$ for all $x \in S_X$ and $a \in (0, 1)$;*

(iii) *$F(-x) = -F(x)$ for all $x \in S_X$.*

Proof. (1) Let us show that 0 cannot be mapped to any other point other than 0. Suppose that $F(0) \neq 0$ and let $\alpha = \|F(0)\|$. Then $\alpha > 0$. By the bijectivity of F, there exists a point $x \in B_X$ such that $F(x) = -\frac{1}{\alpha}F(0)$. Then

$$\begin{aligned}\|F(0) - F(x)\| &= \|F(0) + \frac{1}{\alpha}F(0)\| = \|F(0)\| \big|1 + \frac{1}{\alpha}\big| \\ &= \alpha\big(1 + \frac{1}{\alpha}\big) = \alpha + 1 > 1 \geq \|0 - x\|.\end{aligned}$$

The obtained inequality $\|F(0) - F(x)\| > \|0 - x\|$ clearly contradicts the fact that F is nonexpansive.

(2) We are required to check that a point from the interior of B_X cannot be mapped to S_X. Notice that any point x from the interior of B_X satisfies $\|x\| < 1$. From (1) and from nonexpansiveness of F we deduce that

$$\|F(x)\| = \|F(0) - F(x)\| \leq \|0 - x\| < 1,$$

which means that no point x from the interior of B_X can be mapped to S_X. Thus $F^{-1}(S_X) \subset S_X$.

(3) Let $x \in F^{-1}(S_X)$ and $a \in (0, 1)$. We need to show that $F(D(x, a)) \subset D(F(x), a)$. Fix $z \in D(x, a)$. By definition, $\|z\| \leq a$ and $\|x - z\| \leq 1 - a$. Since F is nonexpansive, $\|z\| = \|z - 0\| \geq \|F(z) - F(0)\|$ and $\|x - z\| \geq F(x) - F(z)\|$, and since $F(0) = 0$ by (1), we conclude that $\|F(z)\| \leq a$ and $\|F(x) - F(z)\| \leq 1 - a$, that is, $F(z) \in D(F(x), a)$ by definition.

(4) Fix $a \in (0, 1)$. Assume that $F(x) \in \mathrm{ext}(B_X)$. We need to show that $F(ax) = aF(x)$. By the assumption, Lemma 2.1 implies that $D(F(x), a) = \{aF(x)\}$. On the other hand, for every $y \in S_X$ by definition we clearly have that $ay \in D(y, a)$, and therefore also $aF(x) \in D(F(x), a)$. Thus $F(ax) = aF(x)$.

(5) Assume that $F(x) \in \mathrm{ext}(B_X)$. We need to show that $x \in \mathrm{ext}(B_X)$ where $x \in S_X$ is the unique preimage of the element $F(x)$. By Lemma 2.1 it suffices to show that $D(x, \frac{1}{2})$ is a singleton set. From (3) it immediately follows that $F(D(x, \frac{1}{2})) \subset D(F(x), \frac{1}{2})$. Since $D(F(x), \frac{1}{2})$ is a singleton set by our assumption and Lemma 2.1, the injectivity of F implies that $D(x, \frac{1}{2})$ may consist of only one point.

(6) Assume that $F(x) \in \mathrm{ext}(B_X)$. We need to show that $F(-x) = -F(x)$ where $x \in S_X$ is the unique preimage of the element $F(x)$. To show this, we will use the set $D_1(F(x)$, for which we know by our assumption and Lemma 2.2 that it is a singleton set, $D_1(F(x) = \{0\}$. By the surjectivity of F there is a $y \in S_X$ such that $F(y) = -F(x)$. Clearly, $\|x - y\| \leq 2$, i.e., $\frac{1}{2}\|x - y\| \leq 1$. Consider the point $z = \frac{1}{2}(x + y) \in B_X$. Since F is nonexpansive and $\|x - z\| = \frac{1}{2}\|x - y\| \leq 1$ and $\|y - z\| = \frac{1}{2}\|x - y\| \leq 1$, we have

$$\|F(x) - F(z)\| \leq \|x - z\| \leq 1$$

and

$$\|F(y) - F(z)\| \leq \|y - z\| \leq 1.$$

Thus $F(x) - F(z) \in B_X$ and $F(z) + F(x) = F(z) - F(y) \in B_X$. From this we obtain that $F(z) \in D_1(F(x))$ because, by Lemma 2.2 again,

$$\begin{aligned} F(z) &= -F(x) + \big(F(x) + F(z)\big) \\ &\in -F(x) + \{F(x) + u \in B_X : F(x) - u \in B_X\} = D_1(F(x)). \end{aligned}$$

Hence $F(z) = 0$, i.e., $z = 0$ by (1), which means that $y = -x$, and it follows that $F(-x) = -F(x)$.

We assume in the following that X is strictly convex, thus $\operatorname{ext}(B_X) = S_X$.

(i) We argue by contradiction. Suppose, contrary to our claim, that $F(S_X) \neq S_X$. Let $y \in S_X$ be such that $\|F(y)\| < 1$. Clearly, $F(y) \neq 0$ by (1). By surjectivity of F there exists $x \in X$ such that $F(x) = \frac{1}{\|F(y)\|}F(y)$. Notice that such an x is in S_X by (2). Since $\|F(x)\| = 1$, we have $F(x) \in \operatorname{ext}(B_X)$ by our assumption. Hence, by (4), $F(ax) = aF(x)$ for every $a \in (0, 1)$, and it follows that

$$F(\|F(y)\|x) = \|F(y)\|F(x) = \|F(y)\|\frac{1}{\|F(y)\|}F(y) = F(y).$$

This contradicts the injectivity of F since $\|F(y)\|x \neq y$, because $\|y\| = 1$, whereas $\|\|F(y)\|x\| = \|F(y)\| < 1$.

Since $\operatorname{ext}(B_X) = S_X$, assertions (ii) and (iii) are direct consequences of (4) and (6), respectively. □

2.3 Main theorem

Lemma 2.4 ([1, Lemma 2.4]). *Let $F : B_X \to B_X$ be a bijective nonexpansive mapping that satisfies (i), (ii), and (iii) of Theorem 2.3, and suppose that for some $x \in S_X$ and $y \in B_X$ we have $x^*(-y) = -x^*(y)$. Then $(F(x))^*(F(y)) = x^*(y)$.*

Proof. If $y = 0$, then $F(y) = 0$ by item (1) of Theorem 2.3, and the equality $(F(x))^*(F(y)) = x^*(y)$ follows immediately from the definition.

Assume now that $y \neq 0$. Let us show that $(F(x))^*(-F(y)) \leq x^*(-y)$. According to the definition,

$$(F(x))^*(-F(y)) = \lim_{t\to 0^+} \frac{1}{t}\big(\|F(x) - tF(y)\| - \|F(x)\|\big).$$

By (i) we have $\|F(x)\| = \|x\|$ and by (ii) we have $tF(y) = F(ty)$ for every $t \in (0,1)$. Thus

$$\begin{aligned}(F(x))^*(-F(y)) &= \lim_{t\to 0^+} \frac{1}{t}\big(\|F(x) - tF(y)\| - \|F(x)\|\big)\\ &= \lim_{t\to 0^+} \frac{1}{t}\big(\|F(x) - F(ty)\| - \|x\|\big)\\ &\le \lim_{t\to 0^+} \frac{1}{t}\big(\|x - ty\| - \|x\|\big) = x^*(-y).\end{aligned}$$

The last inequality follows directly from the fact that F is a nonexpansive mapping.

Repeating the same argument for $-y$ in the place of y, we also get

$$(F(x))^*(-F(-y)) \le x^*(y).$$

Note that the items (ii) and (iii) ensure that $F(-y) = -F(y)$. Hence the latter inequality takes the form

$$(F(x))^*(F(y)) \le x^*(y).$$

Therefore, it is left to show that $(F(x))^*(F(y)) \ge x^*(y)$. For this we use (1), the inequality $(F(x))^*(-F(y)) \le x^*(-y)$ obtained in the beginning of the proof at hand, and the assumption $x^*(-y) = -x^*(y)$:

$$(F(x))^*(F(y)) \ge -(F(x))^*(-F(y)) \ge -x^*(-y) = x^*(y).$$

The inequalities $(F(x))^*(F(y)) \le x^*(y)$ and $(F(x))^*(F(y)) \ge x^*(y)$ yield the equality we needed. □

Lemma 2.5 ([1, Lemma 2.5]). *Let $F : B_X \to B_X$ be a bijective nonexpansive mapping that satisfies (i), (ii), and (iii) of Theorem 2.3. Then F is an isometry.*

Proof. Since we know that F is nonexpansive, it suffices to prove that for arbitrary $y_1, y_2 \in B_X$ we have $\|y_1 - y_2\| \le \|F(y_1) - F(y_2)\|$. If $y_1 = y_2$, the conclusion is immediate, so we can assume $y_1 \neq y_2$. Fix arbitrary $y_1, y_2 \in B_X$. Let $E = \operatorname{span}\{y_1, y_2\}$. Define W to be the set of smooth points of S_E. By the Mazur density theorem (Theorem 1.16), W is dense in S_E (note that we apply the theorem to E, which is finite-dimensional and hence separable).

Let us first show the following equality:

$$\|y_1 - y_2\| = \sup\{x^*(y_1 - y_2) : x \in W\}.$$

Let $y = y_1 - y_2$. From (3) it follows that for each $x \in W$ we have $x^*(y) \leq \|y\|$. Therefore, we have $\sup\{x^*(y) : x \in W\} \leq \|y\|$. Now, we need to show that $\|y\| \leq \sup\{x^*(y) : x \in W\}$, for which it suffices to show that for each $\varepsilon > 0$ there exists $x \in W$ such that $|x^*(y) - \|y\|| < \varepsilon$. Since $\|y\| = x^*(\|y\|x)$, the expression $|x^*(y) - \|y\||$ takes the form $|x^*(y) - x^*(\|y\|x)|$. Using (2), the latter can be estimated from above by $\|y - \|y\|x\|$, which can be rewritten as $\|y\|\|\frac{y}{\|y\|} - x\|$. Therefore, it suffices to show that for each $\varepsilon > 0$ there exists $x \in W$ such that $\|y\|\|\frac{y}{\|y\|} - x\| < \varepsilon$ (or, equivalently, $\|\frac{y}{\|y\|} - x\| < \frac{\varepsilon}{\|y\|}$). But this is a direct consequence of W being dense in S_E. Indeed, since W is dense in S_E and $\frac{y}{\|y\|} \in S_E$, there should exist $x \in W$ with $\|\frac{y}{\|y\|} - x\| < \frac{\varepsilon}{\|y\|}$.

Now we have the equality $\|y_1 - y_2\| = \sup\{x^*(y_1 - y_2) : x \in W\}$. Since W is the set of smooth points of S_E, Corollary 1.15 implies that x^* is linear on E for $x \in W$, hence the supremum $\sup\{x^*(y_1 - y_2) : x \in W\}$ can be rewritten as $\sup\{x^*(y_1) - x^*(y_2) : x \in W\}$. Furthermore, the linearity implies that for $x \in W$ we have $x^*(-y_1) = -x^*(y_1)$ and $x^*(-y_2) = -x^*(y_2)$, from which we obtain equalities $x^*(y_1) = (F(x))^*(F(y_1))$ and $x^*(y_2) = (F(x))^*(F(y_2))$ using the previous lemma. The expression $\sup\{x^*(y_1) - x^*(y_2) : x \in W\}$ is therefore the same as

$$\sup\{(F(x))^*(F(y_1)) - (F(x))^*(F(y_2)) : x \in W\}.$$

According to (2), the latter can be estimated from above by $\|F(y_1) - F(y_2)\|$. To sum up, we have obtained the following chain of equalities and inequalities:

$$\begin{aligned}\|y_1 - y_2\| &= \sup\{x^*(y_1 - y_2) : x \in W\} \\ &= \sup\{x^*(y_1) - x^*(y_2) : x \in W\} \\ &= \sup\{(F(x))^*(F(y_1)) - (F(x))^*(F(y_2)) : x \in W\} \\ &\leq \|F(y_1) - F(y_2)\|.\end{aligned}$$

This shows that $\|y_1 - y_2\| \leq \|F(y_1) - F(y_2)\|$, as was needed. $\square$

Since for a strictly convex space the conditions (i), (ii), and (iii) from Theorem 2.3 are satisfied, the last lemma immediately implies the main theorem.

Theorem 2.6 ([1, Theorem 2.6]). *The unit ball of a strictly convex Banach space is a plastic metric space.*

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