EXPONUNTUR FORMULAE ANALYTICAE
QUIBUS
PERTURBATIO MOTUS GYRATORII TERRAE DETERMINATUR.

DISSERTATIO INAUGURALIS

QUAM

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haec dissertatio ea conditione, ut simulac typis excusa fuerit, quinque exemplaria collegio, cui librorum exploratio mandata est, tradantur.

Dorpati d. 22. Nov. 1832.

Dr. Blum, *Decanus*.

PRAEFATIO.

Secundum legem gravitationis Newtoni ingenio detectam sol et luna in terrae sphaeroides agentes situm ejus axis rotationis in spatio immutant, *eaque phaenomena procreant, quae astronomis* observatoribus nomine n u t a t i o n i s e t p r a e c e s s i o n i s a e q u i n o c t i o r u m notantur. Quae phaenomena in hac dissertatione analytice et quam brevissime exponere animus mihi erat.

Restat ut lectores benevolos moneam in hoc opusculo:

1. Omnia differentialia sumta esse respectu temporis t, quare loco $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$ .... denotationibus dx, dy, dz ... me usum esse.

2. Omnia systemata coordinatarum ita esse disposita, ut si in plano x, y steteris in angulo positivorum semiaxium semiaxi positivo z innixus, axis x dexter sit, axis vero y sinister.

3. Directionem lineae rectae in spatio determinatam esse per cosinus trium angulorum inter eam ipsam et quemcunque trium semiaxium positivorum systematis coordinatarum, quos cosinus clarissimo Professore **Bartels** auctore *determinantes* vocavimus et litteris ξ, η, ζ designavimus.

§. 1.

Formulae generales motus gyratorii.

1. **S**int x, y, z coordinatae rectangulares elementi dm corporis solidi, quod circa initium coordinatarum volvitur et in cujus omnia elementa vires X, Y, Z axibus coordinatarum parallelae agunt; tum notae formulae motus gyratorii erunt:

$$\left.\begin{aligned}\Sigma(yZ-zY)dm &= \Sigma(yd^2z-zd^2y)dm\\ \Sigma(zX-xZ)dm &= \Sigma(zd^2x-zd^2z)dm\\ \Sigma(xY-yX)dm &= \Sigma(xd^2y-yd^2x)dm\end{aligned}\right\}\quad \ldots \text{(I)}$$

ubi signum summationis Σ ad elementum dm refertur et per totam massam corporis solidi extendi debet.

2. Transformemus has formulas in alias. Quare sumamus novum systema coordinatarum x', y', z', quarum initium in eodem puncto est, quarumque situs definitur quantitatibus ξ, η, ζ quoad axem x'; ξ', η', ζ' quoad axem y' et ξ'', η'', ζ'' quoad axem z'; habebimus uti ex theoria mutationis coordinatarum constat:

$$\begin{aligned}x' &= x\xi + y\eta + z\zeta\\ y' &= x\xi' + y\eta' + z\zeta'\\ z' &= x\xi'' + y\eta'' + z\zeta''.\end{aligned}$$

Axibus τῶν x, y, z immobilibus in corpore positis, coordinatae elementi dm ad illos relatae erunt constantes movente se corpore, cum coordinatae x', y', z' pariter atque quantitates ξ, η, ζ, $\xi' \ldots \zeta''$ sint variabiles. Quare differentialia valorum praecedentium praebebunt:

$$dx' = xd\xi + yd\eta + zd\zeta$$
$$dy' = xd\xi' + yd\eta' + zd\zeta'$$
$$dz' = xd\xi'' + yd\eta'' + zd\zeta''.$$

Prima harum aequationum multiplicata per ξ, secunda per ξ', et tertia per ξ'' et summatis productis erit: *)

$$\xi dx' + \xi' dy' + \xi'' dz' = y(\xi d\eta + \xi' d\eta' + \xi'' d\eta'') + z(\xi d\zeta + \xi' d\zeta' + \xi'' d\zeta'')$$

Simili modo: $\eta dx' + \eta' dy' + \eta'' dz' = z(\eta d\zeta + \eta' d\zeta' + \eta'' d\zeta'') + x(\eta d\xi + \eta' d\xi' + \eta'' d\xi'')$

$$\zeta dx' + \zeta' dy' + \zeta'' dz' = x(\zeta d\xi + \zeta' d\xi' + \zeta'' d\xi'') + y(\zeta d\eta + \zeta' d\eta' + \zeta'' d\eta'')$$

aut posito brevitatis gratia:

$$\eta d\zeta + \eta' d\zeta' + \eta'' d\zeta' = p$$
$$\zeta d\xi + \zeta' d\xi' + \zeta'' d\xi'' = q$$
$$\xi d\eta + \xi' d\eta' + \xi'' d\eta'' = r$$

erit: $\xi dx' + \xi' dy' + \xi'' dz' = yr - zq$

$$\eta dx' + \eta' dy' + \eta'' dz' = zp - xr$$
$$\zeta dx' + \zeta dy' + \zeta'' dz' = xq - yp \text{ **)}.$$

Ex his aequationibus post differentiationem obtinebimus pro viribus acceleratricibus valores sequentes:

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*) Ope aequationum conditionis $\xi^2 + \xi'^2 + \xi''^2 = 1$, $\eta^2 + \eta'^2 + \eta''^2 = 1$, $\zeta^2 + \zeta'^2 + \zeta''^2 = 1$; $\eta\zeta + \eta'\zeta' + \eta''\zeta'' = 0$, $\zeta\xi + \zeta'\xi' + \zeta''\xi'' = 0$, $\xi\eta + \xi'\eta' + \xi''\eta'' = 0$, erit $\xi d\xi + \xi' d\xi' + \xi'' d\xi'' = 0$, $\eta d\eta + \eta' d\eta' + \eta'' d\eta'' = 0$, $\zeta d\zeta + \zeta' d\zeta' + \zeta'' d\zeta'' = 0$; $\eta d\zeta + \eta' d\zeta' + \eta'' d\zeta'' = -(\zeta d\eta + \zeta d\eta' + \zeta' d\eta'') \ldots$

**) Quarum aequationum quamlibet si nihilo aequalem ponamus, inveniemus sic dictas determinantes axis rotationis: $\frac{p}{\sqrt{p^2+q^2+r^2}}$, $\frac{q}{\sqrt{p^2+q^2+r^2}}$, $\frac{r}{\sqrt{p^2+q^2+r^2}}$.

$$d^2x' = \xi(ydr - zdq) + \eta(zdp - xdr) + \zeta(xdq - ydp) + d\xi(yr - zq)$$
$$+ d\eta(zp - xr) + d\zeta(xq - yp)$$
$$d^2y' = \xi'(ydr - zdq) + \eta'(zdp - xdr) + \zeta'(xdq - ydp) + d\xi'(yr - zq)$$
$$+ d\eta'(zp - xr) + d\zeta(xq - yp)$$
$$d^2z' = \xi''(ydr - zdq) + \eta''(zdp - xdr) + \zeta''(xdq - ydp) + d\xi''(yr - zq)$$
$$+ d\eta''(zp - xr) + d\zeta''(xq - yp).$$

Quae vires ut solvantur paralleliter axibus τῶν x, y, z, multiplicemus eas respective per ξ, ξ', ξ''; η, η', η'': ζ, ζ', ζ'' et summemus. Quo facto obtinebimus valores, quos in formulis (I) per d^2x d^2y d^2z designavimus:

$$d^2x = ydr - zdq + (zp - xr)r - (xq - yr)q$$
$$d^2y = zdp - xdr + (xq - yp)p - (yr - zq)r$$
$$d^2z = xdq - ydp + (yr - zq)q - (zp - xr)p.$$

Substituamus hos valores in aequationibus fundamentalibus (I) et sint axes coordinatarum x, y, z sic dictae principales, in quo casu

$$\Sigma yz\,dm = 0$$
$$\Sigma zx\,dm = 0$$
$$\Sigma xy\,dm = 0$$

eruemus
$$\Sigma(Yz - Zy)dm = dp.\Sigma(y^2 + z^2)dm - qr.\Sigma(y^2 - z^2)dm$$
$$\Sigma(Zx - Xz)dm = dq.\Sigma(z^2 + x^2)dm - rp.\Sigma(z^2 - x^2)dm$$
$$\Sigma(Xy - Yx)dm = dr.\Sigma(x^2 + y^2)dm - pq.\Sigma(x^2 - y^2)dm.$$

Aut designatis quantitatibus $\Sigma(y^2 + z^2)dm$, $\Sigma(z^2 + x^2)dm$, $\Sigma(x^2 + y^2)dm$ quae sunt momenta inertiae ratione axium coordinatarum x, y, z habita per A, B, C:

$$\Sigma(Yz - Zy)dm = Adp + qr(B - C)$$
$$\Sigma(Zx - Xz)dm = Bdq + rp(C - A)$$
$$\Sigma(Xy - Yx)dm = Cdr + pq(A - B).$$

Sint brevitatis gratia:

$$\Sigma(Yz - Zy)dm = S,\ \Sigma(Zx - Xz)dm = S',\ \Sigma(Xy - Yx)dm = S''.$$

Ex aequationibus praecedentibus obtinebimus:

$$\left.\begin{aligned} dp &= \frac{S - qr(B-C)}{A} \\ dq &= \frac{S' - rp(C-A)}{B} \\ dr &= \frac{S'' - pq(A-B)}{C} \end{aligned}\right\} \ldots\ldots (II)$$

Tales sunt aequationes differentiales, quarum ope determinantur quantitates p, q, r, notis momentis inertiae A, B, C et valoribus S, S', S''.

§. 2.

Evolutio virium perturbantium.

3. Sit initium coordinatarum in centro gravitatis corporis m, quod in motu suo gyratorio per attractionem corporis M perturbatur. Sint porro $x_{,}$ $y_{,}$ $z_{,}$ coordinatae centri gravitatis corporis M relatae ad axes τῶν x, y, z, ejus distantia ab initio $R = \sqrt{x_1^2 + y_1^2 + z_1^2}$ et ab elemento dm, cujus coordinatae sunt x, y, z ..... $R' = \sqrt{(x_{,} - x)^2 + (y_{,} - y)^2 + (z_{,} - z)^2}$.

Secundum legem attractionis vis, quae in dm agit, erit $\frac{M}{R'^2}$, atque dissoluta paralleliter axibus coordinatarum praebebit $\frac{M(x_{,} - x)}{R'^3}$, $\frac{M(y_{,} - y)}{R'^3}$, $\frac{M(z_{,} - z)}{R'^3}$. Ut vero obtineatur relativus elementi dm circa centrum gravitatis corporis m motus, has vires deminuamus oportet quantitatibus $\frac{Mx_{,}}{R^3}$, $\frac{My_{,}}{R^3}$, $\frac{Mz_{,}}{R^3}$; erit ergo:

$$X = \frac{M(x_{,} - x)}{R'^3} - \frac{Mx_{,}}{R^3}$$

$$Y = \frac{M(y_{,} - y)}{R'^3} - \frac{My_{,}}{R^3}$$

$$Z = \frac{M(z_{,} - z)}{R'^3} - \frac{Mz_{,}}{R^3}.$$

His valoribus substitutis in formulis $S=\Sigma(Yz-Zy)dm$, $S'=\Sigma(Zx-Xz)dm$, $S''=\Sigma(Xy-Yx)dm$, eruemus pro S, S', S'' expressiones sequentes:

$$S = M\Sigma(\frac{1}{R'^3}-\frac{1}{R^3})(y_{,}z-z_{,}y)dm$$

$$S' = M\Sigma(\frac{1}{R'^3}-\frac{1}{R^3})(z_{,}x-x_{,}z)dm$$

$$S'' = M\Sigma(\frac{1}{R'^3}-\frac{1}{R^3})(x_{,}y-y_{,}x)dm.$$

In evolvenda functione $\frac{1}{R'^3}=((x_{,}-x)^2+(y_{,}-y)^2+(z_{,}-z)^2)^{-\frac{3}{2}}$, quadrata x^2, y^2, z^2 negligi possunt ratione habita ad quantitates $x_{,}$, $y_{,}$, $z_{,}$. Hoc facto erit

$$\frac{1}{R'^3}=\frac{1}{R^3}+\frac{3(xx_{,}+yy_{,}+zz_{,})}{R^5}.$$

Igitur
$$S = \frac{3M}{R^5}\Sigma(xx_{,}+yy_{,}+zz_{,})(y_{,}z-z_{,}y)dm$$

$$S' = \frac{3M}{R^5}\Sigma(xx_{,}+yy_{,}+zz_{,})(z_{,}x-x_{,}y)dm$$

$$S'' = \frac{3M}{R^5}\Sigma(xx_{,}+yy_{,}+zz_{,})(x_{,}y-y_{,}x)dm.$$

Quia secundum peculiarem naturam axium principalium $\Sigma yz\,dm=0$, $\Sigma zx\,dm=0$, $\Sigma xy\,dm=0$, nostrae formulae fient:

$$S = \frac{3M}{R^5}\Sigma(z^2-y^2)y_{,}z_{,}dm = \frac{3M}{R^5}(B-C)y_{,}z_{,}$$

$$S' = \frac{3M}{R^5}\Sigma(x^2-z^2)z_{,}x_{,}dm = \frac{3M}{R^5}(C-A)z_{,}x_{,}$$

$$S'' = \frac{3M}{R^5}\Sigma(y^2-x^2)x_{,}y_{,}dm = \frac{3M}{R^5}(A-B)x_{,}y_{,}.$$

5. Substitutis his valoribus in aequationibus (II) habebimus:

$$\left.\begin{aligned} dp &= \frac{B-C}{A}(\frac{3M}{R^5}y_{,}z_{,}-qr) \\ dq &= \frac{C-A}{B}(\frac{3M}{R^5}z_{,}x_{,}-rp) \\ dr &= \frac{A-B}{C}(\frac{3M}{R^5}x_{,}y_{,}-pq) \end{aligned}\right\}\ldots\ldots\text{(III)}$$

6. Videmus nunc quomodo ope quantitatum p, q, r determinari possint nutatio axis terrae et praecessio aequinoctiorum. In applicatione formularum praecedentium ad motum gyratorium terrae, corpus m ipsam terram designabit. Figura vero ejus considerari potest tanquam ellipsoides per revolutionem circa axem minorem ortum. Coincidat nunc planum coordinatarum x, y cum plano aequatoris, planum vero coordinatarum x', y' cum plano eclipticae; sit porro ω angulus inter haec plana, ϱ ascensio recta axis x, et λ longitudo axis x', nostrae ξ, η, ζ, ... ζ'' fient:

$$\begin{aligned}
\xi &= \cos\varrho \cos\lambda + \sin\varrho \sin\lambda \cos\omega,\\
\eta &= -\sin\varrho \cos\lambda + \cos\varrho \sin\lambda \cos\omega,\\
\zeta &= \sin\lambda \sin\omega,\\
\xi' &= -\cos\varrho \sin\lambda + \sin\varrho \cos\lambda \cos\omega,\\
\eta' &= \sin\varrho \sin\lambda + \cos\varrho \cos\lambda \cos\omega,\\
\zeta' &= \cos\lambda \sin\omega,\\
\xi'' &= -\sin\varrho \sin\omega,\\
\eta'' &= -\cos\varrho \sin\omega,\\
\zeta'' &= \cos\omega.
\end{aligned}$$

Differentiatis his valoribus obtinebimus:

$$\begin{aligned}
d\xi &= d\varrho(-\sin\varrho \cos\lambda + \cos\varrho \cos\lambda \cos\omega) + d\lambda(-\cos\varrho \sin\lambda + \sin\varrho \cos\lambda \cos\omega)\\
&\qquad - d\omega \sin\varrho \sin\lambda \sin\omega\\
d\eta &= d\varrho(-\cos\varrho \cos\lambda - \sin\varrho \sin\lambda \cos\omega) + d\lambda(\sin\varrho \sin\lambda + \cos\varrho \cos\lambda \cos\omega)\\
&\qquad - d\omega \cos\varrho \sin\lambda \sin\omega\\
d\zeta &= d\lambda \cos\lambda \sin\omega + d\omega \sin\lambda \cos\omega\\
d\xi' &= d\varrho(\sin\varrho \sin\lambda + \cos\varrho \cos\lambda \cos\omega) + d\lambda(-\cos\varrho \cos\lambda - \sin\varrho \sin\lambda \cos\omega)\\
&\qquad - d\omega \sin\varrho \sin\lambda \sin\omega\\
d\eta' &= d\varrho(\cos\varrho \sin\lambda - \sin\varrho \cos\lambda \cos\omega) + d\lambda(\sin\varrho \cos\lambda - \cos\varrho \sin\lambda \cos\omega)\\
&\qquad - d\omega \cos\varrho \cos\lambda \sin\omega\\
d\zeta' &= -d\lambda \sin\lambda \sin\omega + d\omega \cos\lambda \cos\omega,
\end{aligned}$$

$$d\xi'' = -\, d\varrho \cos\varrho \sin\omega - d\omega \sin\varrho \cos\omega$$

$$d\eta'' = d\varrho \sin\varrho \sin\omega - d\omega \cos\varrho \cos\omega$$

$$d\zeta'' = -\, d\omega \sin\omega.$$

Substitutis his quantitatibus in:

$$p = \eta d\zeta + \eta' d\zeta' + \eta'' d\zeta''$$

$$q = \zeta d\xi + \zeta' d\xi' + \zeta'' d\xi''$$

$$r = \xi d\eta + \xi' d\eta' + \xi'' d\eta''.$$

Post omnes reductiones eruemus:

$$p = -\, d\lambda \sin\varrho \sin\omega + d\omega \cos\varrho$$

$$q = -\, d\lambda \cos\varrho \sin\omega - d\omega \sin\varrho$$

$$r = d\lambda \cos\omega - d\varrho.$$

Unde

$$d\omega = p \cos\varrho - q \sin\varrho$$

$$d\lambda \sin\omega = -\, p \sin\varrho - q \cos\varrho$$

$$d\varrho = d\lambda \cos\omega - r.$$

Prima harum aequationum praebebit immutationem obliquitatis eclipti-ces, aut sic dictam *nutationem* axis terrae, secunda immutationem longitudinis axis fixae x 'aut sic dictam *praecessionem aequinoctiorum*, tertia tandem immutationem ascensionis rectae axis x, aut motum gyratorium terrae.

7. Ad nostras aequationes (III) denuo regreasi, primum introducamus in producta $y_, z_,$, $z_, x_,$, $x_, y_,$, loco $x_,$, $y_,$, $z_,$ coordinatas ejusdem corporis $x'_,$, $y'_,$, $z'_,$ ad Systema axium x', y', z' relatas, quia plani eclipticès respectu habito corporum coelestium situs definiri solet. In hoc casu:

$$x_, = x'_, \xi + y'_, \xi' + z'_, \xi''$$

$$y_, = x'_, \eta + y'_, \eta' + z'_, \eta''$$

$$z_, = x'_, \zeta + y'_, \zeta + z'_, \zeta'.$$

Ergo habebimus:

$$y_1 z_1 = x_1'^2\eta\zeta + y_1'^2\eta'\zeta' + z_1'^2\eta''\zeta'' + y_1'z_1'(\eta'\zeta''+\zeta'\eta'') + z_1'x_1'(\eta''\zeta+\zeta''\eta) + x_1'y_1'(\eta\zeta'+\zeta\eta')$$

$$z_1 x_1 = x_1'^2\zeta\xi + y_1'^2\zeta'\xi' + z_1'^2\zeta''\xi'' + y_1'z_1'(\zeta'\xi''+\xi'\zeta'') + z_1'x_1'(\zeta''\xi+\xi''\zeta) + x_1'y_1'(\zeta\xi'+\xi\zeta')$$

$$x_1 y_1 = x_1'^2\xi\eta + y_1'^2\xi'\eta' + z_1'^2\xi''\eta'' + y_1'z_1'(\xi'\eta''+\eta'\xi'') + z_1'x_1'(\xi''\eta+\eta''\xi) + x_1'y_1'(\xi\eta'+\eta\xi').$$

Substitutis hic loco ξ, η, ζ eorum valoribus praecedentibus et posito simplicitatis gratia $\lambda = 0$, post omnes reductiones nanciscimur:

$$y_1 z_1 = ((y_1'^2 - z_1'^2)\sin\omega.\cos\omega + y_1'z_1'\cos 2\omega)\cos\varrho - (x_1'y_1'\sin\omega + z_1'x_1'\cos\omega)\sin\varrho$$

$$z_1 x_1 = ((y_1'^2 - z_1'^2)\sin\omega.\cos\omega + y_1'z_1'\cos 2\omega)\sin\varrho + (x_1'y_1'\sin\omega + z_1'x_1'\cos\omega)\cos\varrho$$

$$x_1 y_1 = \tfrac{1}{2}(y_1'^2\cos^2\omega + z_1'^2\sin^2\omega - 2y_1'z_1'\sin\omega\cos\omega - x_1'^2)\sin 2\varrho - (z_1'x_1'\sin\omega - x_1'y_1'\cos\omega)\cos 2\varrho.$$

Posito brevitatis gratia:

$$(y_1'^2 - z_1'^2)\sin\omega.\cos\omega + y_1'z_1'\cos 2\omega = \psi$$

$$x_1'y_1'\sin\omega + z_1'x_1'\cos\omega = \psi'.$$

Aequationes (III) hanc formam induent:

$$\left.\begin{aligned} dp &= \frac{B-C}{A}\left(\frac{3M}{R^5}(\psi\cos\varrho - \psi'\sin\varrho) - qr\right) \\ dq &= \frac{C-A}{B}\left(\frac{3M}{R^5}(\psi\sin\varrho + \psi'\cos\varrho) - rp\right) \\ dr &= \frac{A-B}{C}\left(\frac{3M}{R^5}x_1y_1 - pq\right). \end{aligned}\right\} \quad \ldots,\ (\mathrm{IV})$$

§. 3.

Ultima evolutio formularum praecedentium in earum applicatione ad motum gyratorium terrae.

8. Prima statim se nobis offert in applicatione formularum praecedentium definitio momentorum inertiae A, B, C, id quod facillimum est, quia

rata habita sumtione nostra de figura terrae omnis sectio ad axem z perpendicularis erit circulus, omnis sectio vero per axem ipsum transiens ellipsis. Sit itaque $2a$ diameter aequatoris, $2b$ axis rotationis, x, y, z coordinatae elementi dm, u linea ab hoc elemento ad axem rotationis perpendiculariter ducta, φ angulus inter hanc lineam et planum coordinatarum y, z, . . habebimus $y = u\cos\varphi$, $x = u\sin\varphi$, $dm = u\,du\,d\varphi\,dz$

$$\Sigma x^2 dm = \iiint u^3 \sin^2\varphi\, du\, d\varphi\, dz$$
$$\Sigma y^2 dm = \iiint u^3 \cos^2\varphi\, du\, d\varphi\, dz$$
$$\Sigma z^2 dm = \iiint u^2 z\, du\, d\varphi\, dz.$$

In his expressionibus integralia extendi debent quoad quantitatem φ a $\varphi = 0$ usque ad $\varphi = 360° = 2\pi$, quoad quant. u ab $u = 0$ usque ad u aequale quantitati ex aequatione $a^2b^2 = a^2z^2 + b^2u^2$ determinandae, quoad z tandem a $z = 0$ usque ad z aequale axi minori b ellipsoidis. Ita progredientes obtinebimus:

$$\Sigma(y^2 + z^2)dm = A = \tfrac{4}{15}\pi\, a^2b(a^2 + b^2)$$
$$\Sigma(z^2 + x^2)dm = B = \tfrac{4}{15}\pi\, a^2b(a^2 + b^2) = A$$
$$\Sigma(x^2 + y^2)dm = C = \tfrac{8}{15}\pi\, a^4 b$$
$$2\iiint u\, du\, d\varphi\, dz = m = \tfrac{4}{3}\pi\, a^2 b.$$

Aut $$A = B = \tfrac{1}{5} m(a^2 + b^2)$$
$$C = \tfrac{2}{5} m a^2.$$

9. Ex his valoribus momentorum A, B, C sequitur, omnes axes qui in plano aequatoris positi sunt, esse principales et momentum C respectu axis rotationis omnium maximum. Quare aequationes (IV) ient:

$$dp = -\frac{C-A}{A}\left(\frac{3M}{R^5}(\psi\cos\varrho - \psi'\sin\varrho) - qr\right)$$

$$dq = \frac{C-A}{A}\left(\frac{3M}{R^5}(\psi\sin\varrho + \psi'\cos\varrho) - rp\right)$$

$dr = 0$, ergo $r = const. = h$.

10. Inter omnia corpora coelestia sol et luna sola sunt, quorum attractio in terram admodum influit, quare in his solis duobus corporibus operam nostram collocemus. Designata itaque longitudine utriusque corporis per l et latitudine per β habebimus:

pro sole $x'_1 = R\cos l,\;\; y'_1 = R\sin l,\;\; z'_1 = 0$

et pro luna $x'_1 = R\cos l,\;\; y'_1 = R\sin l,\;\; z'_1 = R\sin\beta$

quia angulus β semper ita parvus est, ut liceat radium vectorem lunae sumere loco ejus projectionis in plano ecliptices.

In evolvendis functionibus ψ et ψ', denotantes brevitatis gratia $\cos\omega$ per c et $\cos 2\omega$ per c', habebimus neglecto quadrato $\sin^2\beta$:

$$\psi = R^2\sin l(c\sin\omega.\sin l + c'\sin\beta)$$
$$\psi' = R^2\cos l(\sin\omega\,.\sin l + c\sin\beta).$$

Posito $\varrho = \varrho' - 90^\circ$ et denotando sinum inclinationis plani orbitae lunae ad planum ecliptices per i et longitudinem nodi ascendentis per Ω, erit $\sin\beta = i\sin(l-\Omega)$

$$\psi\cos\varrho = R^2\sin l(c\sin\omega.\sin l + c'i\sin(l-\Omega))\sin\varrho'$$
$$-\psi'\sin\varrho = R^2\cos l(\sin\omega\,.\sin l + ci\sin(l-\Omega))\cos\varrho'$$
$$\psi\sin\varrho = -R^2\sin l(c\sin\omega.\sin l + c'i\sin(l-\Omega))\cos\varrho'$$
$$\psi'\cos\varrho = R^2\cos l(\sin\omega\,.\sin l + ci\sin(l-\Omega))\sin\varrho'.$$

Hae expressiones facile transformantur in sequentes:

$$\psi\cos\varrho = \tfrac{1}{2}R^2[c\sin\omega(1-\cos 2l) + c'i(\cos\Omega - \cos(2l-\Omega))]\sin\varrho'$$
$$-\psi'\sin\varrho = \tfrac{1}{2}R^2[\sin\omega.\sin 2l + ci(\sin(2l-\Omega) - \sin\Omega)]\cos\varrho'$$
$$\psi\sin\varrho = -\tfrac{1}{2}R^2[c\sin\omega(1-\cos 2l) + c'i(\cos\Omega - \cos(2l-\Omega))]\cos\varrho'$$
$$\psi'\cos\varrho = \tfrac{1}{2}R^2[\sin\omega.\sin 2l + ci(\sin(2l-\Omega) - \sin\Omega)]\sin\varrho'.$$

Post quasdam reductiones neglectis quantitatibus, quae pendent $a\,2l-\Omega+\varrho'$ et $a\,2l-\Omega-\varrho'$ (: quia inaequalitates, quarum argumenta $2l-\Omega+\varrho'$ et $2l-\Omega-\varrho'$ sunt, non superant 0,004″ :), nanciscimur:

$$\psi \cos\varrho = \tfrac{3}{4}R^2[c \sin\omega(2\sin\varrho' - \sin(2l+\varrho') + \sin(2l-\varrho')) + c'i(\sin(\varrho'+\Omega) + \sin(\varrho'-\Omega))]$$
$$-\psi' \sin\varrho = \tfrac{3}{4}R^2[\sin\omega(\sin(2l+\varrho') + \sin(2l-\varrho')) + ci(\sin(\varrho'+\Omega) - \sin(\varrho'-\Omega))]$$
$$\psi \sin\varrho = -\tfrac{3}{4}R^2(c\sin\omega[2\cos\varrho' - \cos(2l+\varrho') - \cos(2l-\varrho')) + c'i(\cos(\varrho'+\Omega) + \cos(\varrho'-\Omega))]$$
$$\psi' \cos\varrho = \tfrac{3}{4}R^2(\sin\omega[\cos(2l-\varrho') - \cos(2l+\varrho')) + ci(\cos(\varrho'-\Omega) - \cos(\varrho'+\Omega))].$$

Ergo:

$$\psi\cos\varrho - \psi'\sin\varrho = \tfrac{3}{4}R^2[((c+c')\sin(\varrho'-\Omega) - (c-c')\sin(\varrho'+\Omega))i + \sin\omega(2c\sin\varrho'$$
$$+(1-c)\sin(2l+\varrho') + (1+c)\sin(2l-\varrho')] = \tfrac{3}{4}R^2.V$$
$$\psi\sin\varrho + \psi'\cos\varrho = -\tfrac{3}{4}R^2[((c+c')\cos(\varrho'-\Omega) - (c-c')\cos(\varrho'+\Omega))i + \sin\omega(2c\cos\varrho'$$
$$-(1+c)\cos(2l+\varrho') + (1-c)\cos(2l+\varrho')] = -\tfrac{3}{4}R^2.V'.$$

Fient igitur aequationes differentiales (IV), posito $\frac{C-A}{A} = \varepsilon$, $\frac{M}{R^3} = n^2$

$$dp = \varepsilon(dq - \tfrac{3}{4}n^2 V)$$
$$dh = -\varepsilon(ph + \tfrac{3}{4}n^2 V').$$

11. Ad integrandas has aequationes differentiemus illas, consideratis ε, h et n^2 tanquam constantibus:

$$d^2p = \varepsilon(hdq - \tfrac{3}{4}n^2 dV)$$
$$d^2q = \varepsilon(hdp + \tfrac{3}{4}n^2 dV')$$

substitutis loco dp, dq eorum valoribus praecedentibus:

$$d^2p + (h\varepsilon)^2 p = -\tfrac{3}{4}n^2\varepsilon(dV + h\varepsilon V')$$
$$d^2q + (h\varepsilon)^2 q = -\tfrac{3}{4}n^2\varepsilon(dV' + h\varepsilon V).$$

Aut posito $h\varepsilon = \alpha$, $dV + h\varepsilon V' = W$, $dV' - h\varepsilon V = W'$

$$d^2p + \alpha^2 p = -\tfrac{3}{4}n^2\varepsilon W$$
$$d^2q + \alpha^2 q = -\tfrac{3}{4}n^2\varepsilon W'.$$

12. In evolutione functionum W, W' motum medium corporum coelestium loco eorum motus veri introducere possumus. Denotato itaque per

n motu progressivo solis aut lunae, per $-\sigma n$ motu retrogrado nodorum orbitae lunae et per h motu gyratorio terrae, id est posito $dl = n$, $d\varrho = h$ et $d\Omega = -\sigma n$ erit:

$$dV = ((c+c')(h+\sigma n)\cos(\varrho'-\Omega) - (c-c')(h-\sigma n)\cos(\varrho'+\Omega))i$$
$$+ (2ch\cos\varrho' + (1-c)(2n+h)\cos(2l+\varrho') + (1+c)(2n-h)\cos(2l-\varrho'))\sin\omega$$
$$dV' = -((c+c')(h+\sigma n)\sin(\varrho'-\Omega) - (c-c')(h-\sigma n)\sin(\varrho'+\Omega))i$$
$$-(2ch\sin\varrho' - (1+c)(2n-h)\sin(2l-\varrho') + (1-c)(2n+h)\sin(2l+\varrho'))\sin\omega.$$

Unde:

$$W = dV + h\epsilon V' = ((c+c')(h(1+\epsilon)+\sigma n)\cos(\varrho'-\Omega) - c - c')(h(1+\epsilon)-\sigma n)\cos(\varrho'+\Omega))i$$
$$+\sin\omega(2ch(1+\epsilon)\cos\varrho' + (1-c)(2n+h(1+\epsilon)\cos(2l+\varrho') + (1+c)(2n-h(1+\epsilon))\cos(2l-\varrho'))$$
$$W' = dV' - h\epsilon V = -((c+c')(h(1+\epsilon)+\sigma n)\sin(\varrho'-\Omega) - (c-c')(h(1+\epsilon)-\sigma n)\sin(\varrho'+\Omega))i$$
$$-\sin\omega((2ch(1+\epsilon)\sin\varrho' - (1+\epsilon)(2n-h)1+\epsilon)(\sin(2l-\varrho') + (1-c)(2n+h(1+\epsilon))\sin(2l+\varrho')).$$

13. Quia $p = \sin\alpha t$ et $p = \cos\alpha t$ satisfaciunt aequationi $d^2p + \alpha^2 p = 0$, integrale completum aequationis $d^2p + \alpha^2 p = -\frac{3}{4} n^2 \epsilon W$ hanc formam induere debet:

$$p = P\sin\alpha t + P'\cos\alpha t$$

unde $\qquad dp = \alpha P\cos\alpha t - \alpha P'\sin\alpha t + dP\sin\alpha t + dP'\cos\alpha t$

propter quantitates indeterminatas P et P' possumus statuere:

$$dP\sin\alpha t + dP'\cos\alpha t = 0 \,\ldots\ldots\ldots\, (\mathrm{a})$$

Ergo differentiale secundi ordinis quant. p erit:

$$d^2p = \alpha\, dP\cos\alpha t - \alpha dP'\sin\alpha t - \alpha^2(P\sin\alpha t + P'\cos\alpha t).$$

Ex hac aequatione nanciscimur:

$$\alpha d P\cos\alpha t - \alpha dP'\sin\alpha t = d^2p + \alpha^2 p = -\tfrac{3}{4} n^2\epsilon W.$$

Unde ope formulae (a) odtinebimus:

$$dP = -\tfrac{3}{4}\frac{n^2\epsilon}{\alpha} W co$$

$$dP' = \quad \tfrac{3}{4}\frac{n^2\epsilon}{\alpha} W\sin\alpha t$$

Integrando eruemus:

$$P = C - \tfrac{3}{4}\frac{n^2\varepsilon}{\alpha}\int W \cos\alpha t$$

$$P' = C' - \tfrac{3}{4}\frac{n^2\varepsilon}{\alpha}\int W \sin\alpha t.$$

Ergo tandem:

$$p = C\sin\alpha t + C'\cos\alpha t - \tfrac{3}{4}\frac{n^2\varepsilon}{\alpha}\sin\alpha t\int W\cos\alpha t + \tfrac{3}{4}\frac{n^2\varepsilon}{\alpha}\cos\alpha t\int W\sin\alpha t.$$

14. Ad integrandas functiones $W\cos\alpha t$ et $W\sin\alpha t$, observamus quemcunque terminum functionem W constituentem habere formam $\cos mt$, secundum vero formulas generales integrationis est:

$$\int\cos mt.\cos\alpha t = \tfrac{1}{2}\int(\cos(m+\alpha)t + \cos(m-\alpha)t) = \tfrac{1}{2}\left(\frac{\sin(m+\alpha)t}{m+\alpha} + \frac{\sin(m-\alpha)t}{m-\alpha}\right)$$

$$\int\cos mt.\sin\alpha t = \tfrac{1}{2}\int(\sin(m+\alpha)t - \sin(m-\alpha)t) = -\tfrac{1}{2}\left(\frac{\cos(m+\alpha)t}{m+\alpha} - \frac{\cos(m-\alpha)t}{m-\alpha}\right)$$

multiplicata prima harum aequationum per $-\frac{\sin\alpha t}{\alpha}$ et secunda per $\frac{\cos\alpha t}{\alpha}$ et summatis productis, habebimus:

$$-\frac{\sin\alpha t}{\alpha}\int\cos mt.\cos\alpha t + \frac{\cos\alpha t}{\alpha}\int\cos mt.\sin\alpha t = \frac{\cos mt}{m^2-\alpha^2}.$$

Ergo quicunque terminus functionis W introducit in expressionem quantitatis p terminum, cuius est forma $\frac{\cos mt}{m^2-\alpha^2}$. Omnibus itaque reductionibus factis eruemus:

$$p = C\sin\alpha t + C'\cos\alpha t + \tfrac{3}{4}n^2\varepsilon\left(\frac{(c+c')\cos(\varrho'-\Omega)}{\sigma n + h(1-\varepsilon)} + \frac{(c-c')\cos(\varrho'+\Omega)}{(\sigma n - h(1-\varepsilon)}\right)i$$

$$+ \tfrac{3}{4}n^2\varepsilon\left(\frac{2c\cos\varrho'}{h(1-\varepsilon)} + \frac{(1-c)\cos(2l+\varrho')}{2n+h(1-\varepsilon)} + \frac{(1-c)\cos(2l-\varrho')}{2n-h(1-\varepsilon)}\right)\sin\omega$$

Simili modo procedentes nanciscimur:

$$q = C_1\sin\alpha t + C_1'\cos\alpha t - \tfrac{3}{4}n^2\varepsilon\left(\frac{(c+c')\sin(\varrho'+\Omega)}{\sigma n + h(1-\varepsilon)} + \frac{(c-c')\sin(\varrho'+\Omega)}{\sigma n - h(1-\varepsilon)}\right)i$$

$$- \tfrac{3}{4}n^2\varepsilon\left(\frac{2c\sin\varrho'}{h(1-\varepsilon)} + \frac{(1-c)\sin(2l+\varrho')}{2n+h(1-\varepsilon)} - \frac{(1-c)\sin(2l-\varrho')}{2n-h(1-\varepsilon)}\right)\sin\omega.$$

Quia in his expressionibus quantitates constantes $C \sin \alpha t + C' \cos \alpha t$ et $C_{\prime} \sin \alpha t + C'_{\prime} \cos \alpha t$ a viribus perturbantibus non pendent, eas facili negotio praeterire possumus.

15. Substituamus nunc quantitates p, q, r in aequationibus:

$$d\omega = p \cos \varrho - q \sin \varrho = p \sin \varrho' + q \cos \varrho'$$

$$d\lambda \sin \omega = - p \sin \varrho - q \cos \varrho = p \cos \varrho' - q \sin \varrho'$$

$$d\varrho = d\lambda \cos \omega - r$$

et statuamus $1 - \varepsilon = 1$ propter tenuitatem quantitatis $\varepsilon = \frac{C - A}{A} = \frac{1 - (\frac{b}{a})^2}{1 + (\frac{b}{a})^2}$,

erit: $d\omega = \frac{3}{4} n^2 \varepsilon i \sin \Omega \left(\frac{c + c'}{\sigma n + h} - \frac{c - c'}{\sigma n - h} \right) + \frac{3}{4} n^2 \varepsilon \left(\frac{1 + c}{2n - h} - \frac{1 - c}{2n + h} \right) \sin 2l \sin \omega$

$$d\lambda = \frac{3}{4} n^2 \varepsilon i \frac{\cos \Omega}{\sin \omega} \left(\frac{c + c'}{\sigma n + h} + \frac{c - c'}{\sigma n - h} \right) + \frac{3}{4} n^2 \varepsilon \left(\frac{1 - c}{2n + h} + \frac{1 + c}{2n - h} \right) \cos 2l + \frac{3}{2} n^2 \varepsilon \frac{c}{h}$$

$$d\varrho = d\lambda \cos \omega - h.$$

16. Integrando habebimus:

$$\omega = \Omega + \frac{3}{4} n^2 \varepsilon i \left(\frac{c + c'}{\sigma n + h} - \frac{c - c'}{2n - h} \right) \cos \Omega - \frac{3}{8} n^2 \varepsilon \left(\frac{1 + c}{2n - h} - \frac{1 - c}{2n + h} \right) \cos 2l \sin \omega$$

$$\lambda = \Lambda - \frac{3}{4} n^2 \frac{\varepsilon i}{\sin \omega} \left(\frac{c + c'}{\sigma n + h} + \frac{c - c'}{\sigma n - h} \right) \sin \Omega + \frac{3}{8} n^2 \varepsilon \left(\frac{1 - c}{2n + h} + \frac{1 + c}{2n - h} \right) \sin 2l + \frac{3}{2} n^2 \varepsilon \frac{c}{h} l$$

$$\varrho = P + \lambda \cos \omega - ht$$

ubi Ω, Λ, P sunt quantitates arbitrariae constantes per integrationem introductae.

17. Determinemus nunc coefficientes numericos, posito $\omega = 23° 27' 50''$:

$$\log. \sin \omega = 9{,}6000696 \;\ldots\; \sin \omega = 0{,}398171$$

$$\log. \cos \omega = 9{,}9625167 \;\ldots\; \cos \omega = 0{,}9173112 = c$$

$$\cos 2\omega = 0{,}6829196 = c'$$

$1 + c = 1{,}9173112$	$\frac{c}{\sin \omega} = 2{,}303812$	$\frac{c + c'}{\sin \omega} = 4{,}018954$
$1 - c = 0{,}0826888$		
$c + c' = 1{,}6002308$	$\frac{c'}{\sin \omega} = 1{,}715142$	$\frac{c - c'}{\sin \omega} = 0{,}588670$
$c - c' = 0{,}2343916$		
	$c \sin \omega = 0{,}365247$	$(1 + c) \sin \omega = 0{,}763418$
		$(1 - c) \sin \omega = 0{,}032924$

$$\omega = \Omega + \tfrac{3}{4}n^2\varepsilon i\left(\frac{1,6000308}{in+h} - \frac{0,2343916}{in-h}\right)\cos\Omega - \tfrac{3}{8}n^2\varepsilon\left(\frac{0,763418}{2n-h} - \frac{0,032924}{2n+h}\right)\cos 2l$$

$$\lambda = \Lambda - \tfrac{3}{4}n^2\varepsilon i\left(\frac{4,018954}{in+h} + \frac{0,588670}{in-h}\right)\sin\Omega + \tfrac{3}{8}n^2\varepsilon\left(\frac{1,9173112}{2n-h} + \frac{0,0826888}{2n+h}\right)\sin 2l$$

$$+ \tfrac{3}{2}\varepsilon n^2 \frac{c}{h} t$$

pro luna $n = 47435'',03$, pro sole $n' = 3548'',33$

$i = 0,00402185$, $h = 360°\,59'\,8'' = 1299548'',5$ $in = 190'',776$

$in + h = 1299639'',276$ $2n + h = 1394416'',56$ $2n' + h = 1306645'',16$

$h - in = 1299357,294$ $h - 2n = 1204678,44$ $h - 2n' = 1292451'',84$

$$\omega = \Omega + \tfrac{3}{4}n^2 i\left(\frac{1,6002308}{1299739,276} + \frac{0,2343916}{1299357,224}\right)\cos\Omega$$

$$+ \tfrac{3}{8}\varepsilon n\left(\frac{0,763418}{1204678'',44} + \frac{0,032924}{1394418'',56}\right)\cos 2\text{☾}$$

$$+ \tfrac{3}{8}\varepsilon n'\left(\frac{0,763418}{1292451,84} + \frac{0,032924}{1306645'',16}\right)\cos 2\odot$$

$$\lambda = \Lambda - \tfrac{3}{4}\varepsilon n^2 i\left(\frac{4,018954}{1299739,276} - \frac{0,588670}{1299357,224}\right)\sin\Omega$$

$$- \tfrac{3}{8}n\varepsilon\left(\frac{1,9173112}{1204678,44} - \frac{0,0826888}{1394418,56}\right)\sin 2\text{☾}$$

$$- \tfrac{3}{8}n'\varepsilon\left(\frac{1,9173112}{1292451,84} - \frac{0,0826888}{1306645,16}\right)\sin 2\odot$$

$$+ \tfrac{3}{2}\frac{n^2 c}{h}t,$$

$$+ \tfrac{3}{2}\frac{n'^2 c}{h}t.$$

$$\omega = \Omega + 2975'',96\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)\cos\Omega + 30'',92\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)\cos 2\text{☾} + 169'',033\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)\cos 2\odot$$

$$\lambda = \Lambda + (30'',543 + 13'',33)\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)t - 5563'',81\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)\sin\Omega$$

$$- 72'',076\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)\sin 2\text{☾} - 389'',785\left(\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)\sin 2\odot.$$

Secundum Bradlium nutatio proportionalis est quantitati $9'',63 \cos\Omega$, unde

$$\frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2} = 0{,}003236.$$

Itaque habebimus tandem:

$$\omega = \Omega + 9'',63 \cos\Omega + 0'',547 \cos 2\odot + 0'',1 \cos 2\text{☾}$$

$$\lambda = \Lambda + 0'',1417\, t - 18'' \sin\Omega - 1'',26 \sin 2\odot - 0'',233 \sin 2\text{☾}.$$

Hae sunt formulae, quarum ope determinantur praecessio aequinoctiorum et nutatio axis terrae, ex quibusque omnes leges horum phaenomenorum deduci possunt, quod ex libris de Astronomia tractantibus perspici potest.

THESES.

1. Statica a Dynamica separata scholasticam rationem redolet, quia omnes leges motus ex legibus aequilibrii deduci possunt.
2. Geometrica analytica majore cum utilitate, quam trigonometria sphaerica ad calculos astronomicos adhiberi potest.
3. Methodus analytica in Mathesi pura praeferenda est Methodo syntheticae; haec tamen non prorsus negligenda est.
4. Quantitates infinitae et imaginariae quamquam absurdae, tamen in Mathesi ad novas veritates inveniendas adhiberi possunt.
5. Corpore solido in medio fluido se movente lex resistentiae proportionalis quadrato celeritatis a priori demonstrari potest.