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# FUNCTIONAL ANALYSIS AND THEORY OF SUMMABILITY

Matemaatika- ja mehaanikaalaseid töid

TARTU 1991

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# CHARACTERIZATION OF MATRIX TRANSFORMATIONS OF SUMMABILITY FIELDS

Ants Aasma

Let $c$ and $bv$ denote respeotively the space of convergent sequences and the space of absolutely convergent sequences, let

$$c^o = \left\{ x = (x_k) \mid \lim_k x_k = 0 \right\}$$

and

$$bv^o = \left\{ x = (x_k) \mid x \in bv \text{ and } \lim_k x_k = 0 \right\}.$$

Furthermore, let $A = (\alpha_{nk})$ be a reversible matrix over $\mathbb{C}$, i.e. the system

$$z_n = \sum_k \alpha_{nk} x_k \tag{1}$$

(shortly $z_n = A_n x$) has unique solution for every convergent sequence $(z_n)$ and $B = (\beta_{nk})$ be a matrix over $\mathbb{C}$. Moreover, let

$$s_A = \left\{ x = (x_k) \mid A_n x \text{ exists for each } n \in \mathbb{N} \right\},$$

$$c_A = \left\{ x = (x_k) \mid x \in s_A \text{ and } (A_n x) \in c \right\},$$

$(c_A, c_B)$ (respectively $(bv_A, c_B)$ or $(bv_A, bv_B)$) be the set of matrices $M = (m_{nk})$ over $\mathbb{C}$ for which the transformation

$$y_n = \sum_k m_{nk} x_k \tag{2}$$

maps $c_A$ into $c_B$ (respectively $bv_A$ into $c_B$ or $bv_A$ into $bv_B$) and let

$$b_M = \left\{ x = (x_k) \mid x \in s_M \text{ and } \sum_{k=1}^{\infty} m_{nk} x_k = O(1) \right\}.$$

Necessary and sufficient conditions in order that $M$ would belong to $(c_A, c_B)$, $(bv_A, c_B)$ or $(bv_A, bv_B)$ for a triangular matrix $B$ are given in [1, 3, 4-6]. The aim of the present paper is to find sufficient conditions in order that

$M$ would belong to $(c_A, c_B)$, $(bv_A, c_B)$ or $(bv_A, bv_B)$ for an infinite matrix $B$.

Further we shall need the following auxiliary results.

**LEMMA 1** ([2], p. 257 - 258). *Let $M = (m_{nk})$ be a matrix over $\mathbb{C}$. In order that the series*

$$\sum_k \Big( \sum_n t_n m_{nk} \Big) x_k \tag{3}$$

*converges for every absolutely convergent series $\sum_n t_n$ it is necessary and sufficient that $(x_k) \in b_M$ and $m_{nk} = O_k(1)$.*

*At this, if the series* (3) *is convergent for each convergent series $\sum_n t_n$, then*

$$\sum_k \Big( \sum_n t_n m_{nk} \Big) x_k = \sum_n t_n M_n x.$$

**LEMMA 2** ([7], p. 12 - 17 and 30 - 34). *Let $\mathfrak{A} = (a_{nk})$ be a sequence-to-sequence transformation. In order that*[1] *$\mathfrak{A} \in (c^o, c)$ (respectively $\mathfrak{A} \in (bv^o, c)$) it is necessary and sufficient that*

*1) there exist finite limits $\lim_n a_{nk} = a_k$,*

*2) $\sum_k |a_{nk}| = O(1)$ (respectively $\sum_{m=0}^{k} a_{nm} = O(1)$ ).*

*At this, $\lim_n \mathfrak{A}_n x = \sum_k a_k x_k$ for each $x = (x_k) \in c^o$ (respectively for each $x = (x_k) \in bv^o$).*

**LEMMA 3** ([7], p. 37). *Let $\mathfrak{A} = (a_{nk})$ be a sequence-to-sequence transformation. for $\mathfrak{A} \in (bv^o, bv)$ it is necessary and sufficient that*

$$\sum_n |r_{nk} - r_{n-1,k}| = O(1)$$

*where $r_{-1,k} \equiv 0$ and*

$$r_{nk} = \sum_{l=k}^{\infty} a_{nl}.$$

1. Let $(\eta_n)$ and $(\eta_{nk})$ for fixed $k$ be solutions of the system (1) in the case when $z_n = \delta_{nn}$ and $z_n = \delta_{nk}$

---

[1]Here and onwards $(\gamma, v)$ denotes the set of such matrixes, which transform the space of sequences $\gamma$ into the space of sequences $v$.

respectively (here $\delta_{nk} = 1$ if $n = k$ and $\delta_{nk} = 0$ if $n \neq k$). Moreover, let

$$B_{nl} = \sum_k \beta_{nk} m_{kl}, \qquad B' = (B_{nl}),$$

$$\gamma^n_{sk} = \sum_{l=0}^{s} B_{nl}\eta_{lk}, \qquad M^n_{sk} = \sum_{l=0}^{s} m_{nl}\eta_{lk}$$

for each $k, l, n, s \in \mathbb{N}$. At the same time we use these notations in the case when all series above are convergent.

It is easy to see that the transformation (2) exists for each $x \in c_A$ ($x \in bv_A$) if and only if the numbers $m_{nk}$ for fixed $n \in \mathbb{N}$ are convergence factors for $c_A$ (respectively for $bv_A$). Therefore, by Theorems 5 and 6 from [8] we have

**LEMMA 4.** *Let* $A = (\alpha_{nk})$ *be a reversible matrix and* $M = (m_{nk})$ *be a matrix over* $\mathbb{C}$. *For the existence of the transformation* (2) *for each* $(x_k) \in c_A$ *(respectively* $(x_k) \in bv_A$) *it is necessary and sufficient that*

*1) there exist finite limits* $\lim_s M^n_{sk} = M_{nk}$ *and series* $\sum_l m_{nl}\eta_l$ *are convergent,*

*2)* $\sum_k |M^n_{sk}| = O_n(1)$ *(respectively* $\sum_{k=0}^{l} M^n_{sk} = O_n(1)$).

It is easy to see that the equality

$$\sum_k \beta_{nk} y_k = \sum_k B_{nk} x_k \tag{4}$$

is true for a triangular matrix B if the transformation (2) exists. But in the case when B is not a triangular matrix it is not always so. Next we shall find the conditions for B and M in order that the equality (4) would be valid. By Lemma 1 we have

**LEMMA 5.** *Let* $A = (\alpha_{nk})$, $B = (\beta_{nk})$ *and* $M = (m_{nk})$ *be matrices over* $\mathbb{C}$. *If* $\sum_k |\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$ *then the equality* (4) *holds for each* $x \in c_A$ *(respectively* $x \in bv_A$) *if and only if* $m_{nk} = O_k(1)$ *and* $c_A \subseteq b_M$ *(respectively* $bv_A \subseteq b_M$).

**LEMMA 6.** *Let* $A = (\alpha_{nk})$ *be a reversible matrix and* $M = (m_{nk})$ *be a matrix over* $\mathbb{C}$. *In order that* $c_A \subseteq b_M$ *it is necessary and sufficient that condition 1) of Lemma 4 and conditions*

$$1)\ \sum_{k=0}^{\infty} m_{nk}\eta_k = O(1)$$

*and*

$$2)\ \sum_k |M^n_{\infty k}| = O(1)$$

*are fulfilled.*

*Proof. Necessity.* Let $c_A \subseteq b_M$. Then the transformation (2) exists for each $x \in c_A$ by the definition of $b_M$ and from $(\eta_n) \in c_A$ follows that $(\eta_n) \in b_M$. Therefore conditions 1) of Lemma 4 and 1) of Lemma 6 are fulfilled.

It is known (cf.[8], p.197) that the elements $x_k$ of the sequence $x = (x_k) \in c_A$ (for a reversible matrix A) may be represented in the form

$$x_k = Z\eta_k + \sum_l \eta_{kl}(z_l - Z) \tag{5}$$

where $z_l = A_l x$, $Z = \lim_l z_l$ and $\sum_l |\eta_{kl}| < \infty$. Now, it is easy to see that the series $\sum_l \eta_{kl}(z_l - Z)$ are convergent. Therefore the equality

$$\sum_{k=0}^{\infty} m_{nk}x_k = Z\sum_{k=0}^{\infty} m_{nk}\eta_k + \sum_k M^n_{\infty k}(z_k - Z) \tag{6}$$

holds for each $x = (x_k) \in c_A$. Hence

$$\sum_k M^n_{\infty k}(z_k - Z) = O(1)$$

for each $(z_k - Z) \in c^o$ by condition 1) because A is reversible. As $M^n_\infty$, defined by $M^n_\infty(x) = \sum_k M^n_{\infty k}x_k$ for each $x = (x_k) \in c^o$, are continuous linear functionals on $c^o$ by the principle of uniform boundedness we obtain that the sequence of norms of functionals $M^n_\infty$ is uniformly bounded. Consequently (cf.[9],p. 260), condition 2) holds.

*Sufficiency.* Let condition 1) of Lemma 4 and conditions 1) and 2) be fulfilled. Then condition 2) of Lemma 4 is also fulfilled. Hence (2) exists for each $(x_k) \in c_A$ by Lemma 4.

As the equality (6) holds for each $(x_k) \in c_A$, $(\eta_n) \in b_M$ by 1) and

$$|M^n_\infty(z_k - Z)| \le \sum_k |M^n_{\infty k}||z_k - Z| = O(1)$$

by 2) (as $(z_k - Z) \in c^o$), we have $c_A \subseteq b_M$.

**LEMMA 7.** *Let* $A = (\alpha_{nk})$ *be a reversible matrix and* $M = (m_{nk})$ *be a matrix over* $\mathbb{C}$. *For* $bv_A \subseteq b_M$ *it is necessary* *and sufficient that*

$$\sum_{k=0}^{l} M^n_{sk} = O(1)$$

*and conditions 1) of Lemma 4 and 1) of Lemma 6 are fulfilled.*

*Proof.* As $A_n\eta = \delta_{nn}$ (where $\eta = (\eta_n)$) and $\sum_k |\delta_{nn} - \delta_{n-1,n-1}| = 1$ ($\delta_{-1,-1} = 0$) we have $\eta \in bv_A$. Therefore the proof of Lemma 7 is similar to the proof of Lemma 6.

**THEOREM 1.** *Let* $A = (\alpha_{nk})$ *be a reversible matrix,* $B = (\beta_{nk})$ *and* $M = (m_{nk})$ *be matrices over* $\mathbb{C}$. *If* $\sum_k |\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$, $m_{nk} = O_k(1)$ *and condition 1) of Lemma 4 and conditions 1) and 2) of Lemma 6 are fulfilled then there exist finite limits* $\lim_s \gamma^n_{sk} = \gamma_{nk}$. *Moreover, if in addition*

*1) there exist finite limits* $\lim_n \gamma_{nk} = \gamma_k$,

*2) there exists finite limit* $\lim_n \sum_k B_{nk}\eta_k$,

*3)* $\sum_k |\gamma_{nk}| = O(1)$

*then* $M \in (c_A, c_B)$.

*Proof.* The equality (4) is true for each $x \in c_A$ by Lemmas 5 and 6. Consequently, it is sufficient to show that $c_A \subseteq c_B$. The elements $x_k$ of the sequence $x = (x_k) \in c_A$ are of the form (5) because A is reversible. Hence the equality (compare with (6))

$$\sum_{k=0}^{s} B_{nk}x_k = Z \sum_{k=0}^{s} B_{nk}\eta_k + \sum_k \gamma^n_{sk}(z_k - Z), \tag{7}$$

where $z_k = A_k x$ and $Z = \lim_k z_k$, holds for each $x \in c_A$. As

$$\sum_r |\beta_{nr}| \sum_{l=0}^{s} |m_{rl}\eta_{lk}| < \infty \tag{8}$$

for each $k, n, s \in \mathbb{N}$ and

$$\sum_r |\beta_{nr} M^r_{sk}| = O_n(1) \tag{9}$$

by condition 2) of Lemma 6 we have

$$\gamma^n_{sk} = \sum_r \beta_{nr} M^r_{sk} \tag{10}$$

and there exist finite limits $\lim_s \gamma^n_{sk} = \gamma_{nk}$ by condition 1) of Lemma 4. Moreover, the condition $\sum_k |\gamma^n_{sk}| = O_n(1)$ is fulfilled because

$$\sum_r |\beta_{nr}| \sum_k |M^r_{sk}| = O_n(1)$$

by condition 2) of Lemma 6. Therefore from (7) we obtain (by Lemma 2 and condition 2)) that the equality

$$\sum_k B_{nk} x_k = Z \sum_k B_{nk} \eta_k + \sum_k \gamma_{nk}(z_k - Z) \tag{11}$$

holds for each $x = (x_k) \in c_A$. At this, conditions 1) and 3) imply the existence of the finite limit $\lim_n \sum_k \gamma_{nk}(z_k - Z)$ for each $x \in c_A$ by Lemma 2 (since $(z_k - Z) \in c^0$ for each $x \in c_A$). Consequently $c_A \subseteq c_B$, by 2). This means that $M \in (c_A, c_B)$.

**THEOREM** 2. *Let* $A = (\alpha_{nk})$ *be a reversible matrix,* $B = (\beta_{nk})$ *and* $M = (m_{nk})$ *be matrices over* $\mathbb{C}$. *If* $\sum_k |\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$, $m_{nk} = O_k(1)$ *and condition 1) of Lemma 4, condition 1) of Lemma 6, condition of Lemma 7 and conditions 1) and 2) of Theorem 1 are fulfilled then there exist finite limits* $\lim_s \gamma^n_{sk} = \gamma_{nk}$. *Moreover, if in addition*

$$\sum_{l=0}^{k} \gamma_{nl} = O(1)$$

*then* $M \in (bv_A, c_B)$.

*Proof*. The equality (4) holds for each $x \in bv_A$ by Lemmas 5 and 7. Consequently, it is sufficient to show that $bv_A \subseteq c_B$. It is easy to see that the equality (7) is true for each $x \in bv_A$ since $bv_A \subseteq c_A$. As (8) is true (since $\sum_k |\beta_{nk}| < \infty$ for each $n \in \mathbb{N}$) and

$$|M^n_{sk}| = \left| \sum_{l=0}^{k} M^n_{sl} - \sum_{l=0}^{k-1} M^n_{sl} \right| = O(1)$$

by Lemma 7 the conditious (9) and (10) are valid. Hence there exist finite limits $\lim_s \gamma^n_{sk} = \gamma_{nk}$ by condition 1) of Lemma 4. Moreover, as

$$\sum_r |\beta_{nr}| \sum_{l=0}^{k} \sum_{i=0}^{s} |m_{ri}\eta_{il}| < \infty$$

for each $k, n, s, \in \mathbb{N}$ we have

$$\sum_{l=0}^{k} \gamma^n_{sl} = \sum_r \beta_{nr} \sum_{l=0}^{k} M^r_{sl}.$$

Thus the condition $\sum_{l=0}^{k} \gamma^n_{sl} = O_n(1)$ holds by Lemma 7. Therefore (7) implies the equality (11) for each $x \in bv_A$ by

Lemma 2 and condition 2) of Theorem 1. Hence, from condition 1) of Theorem 1 and from the condition of Theorem we obtain by Lemma 2 that there exists the finite limit $\lim_n \sum \gamma^n_{nk}(z_k - Z)$ for each $(x_k) \in bv_A$. For that reason $bv_A \subseteq c_B$, by condition 2) of Theorem 1. Consequently $M \in (bv_A, c_B)$.

**REMARK 1.** If $B = (\beta_{nk})$ is a matrix which has the property $\sum_k |\beta_{nk}| = O(1)$ then condition 3) of Theorem 1 and the condition of Theorem 2 are redundant.

By Lemma 3 we have

**THEOREM 3.** *Let* $A = (\alpha_{nk})$ *be a reversible matrix,* $B = (\beta_{nk})$ *and* $M = (m_{nk})$ *be matrices over* $\mathbb{C}$. *If* $\sum_k |\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$, $m_{nk} = O_k(1)$ *and condition 1) of Lemma 4, condition 1) of Lemma 6 and the condition of Lemma 7 are fulfilled then there exist finite limits* $\lim_s \gamma^n_{sk} = \gamma_{nk}$. *Moreover, if in addition*

*1)* $\sum_n |\sum_l (B_{nl} - B_{n-1,l})\eta_l| < \infty$,

*2)* $\sum |\rho_{nk} - \rho_{n-1,k}| = O(1)$,

*where* $\rho_{-1,k} \equiv 0$ *and*

$$\rho_{nk} = \sum_{l=k}^{\infty} \gamma_{nl},$$

*then* $M \in (bv_A, bv_B)$.

2. Let $(p_n)$ be a sequence of non-zero complex numbers, $P_n = p_0 + \ldots + p_n \neq 0$ for each $n \in \mathbb{N}$, $P_{-1} = 0$ and $(R, p_n) \doteq (\alpha_{nk})$ be the series-to-sequence Riesz method generated by $(p_n)$, i.e.

$$\alpha_{nk} = \begin{cases} 1 - P_{k-1}/P_n & \text{if } k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

We note that $(R, p_n)$ is a normal method. Therefore $(R, p_n)$ has the inverse matrix $(R, p_n)^{-1} = (\eta_{nk})$ where (cf. [7], p.116)

$$
\eta_{nk} = \begin{cases} P_k/p_k & \text{if } n = k, \\ -P_k(1/p_k + 1/p_{k+1}) & \text{if } n = k+1, \\ P_k/p_{k+1} & \text{if } n = k+2, \\ 0 & \text{if } n < k \text{ or } n > k+2. \end{cases} \tag{12}
$$

Now we shall give some results for the case $A = (R, p_n)$

**THEOREM 4.** *Let* $(R, p_n)$ *be a conservative method,* $B = (\beta_{nk})$ *and* $M = (m_{nk})$. *Moreover, let* $\sum_k |\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$. *If*[2]

*1)* $m_{n0} = O(p_0 P_0^{-1})$,

*2)* $\sum_k |P_k \Delta \frac{\Delta m_{nk}}{p_k}| = O(1)$,

*3) there exist finite limits* $\lim_n B_{nk} = B_k$,

*4)* $\sum_k |P_k \Delta \frac{\Delta B_{nk}}{p_k}| = O(1)$

*then* $M \in (c_{(R,p_n)}, c_B)$.

*Proof.* It is sufficient to show that all the assumptions and conditions of Theorem 1 are fulfilled. As $(R, p_n)$ is a conservative method there exists a number $M > 0$ such that

$$R_n = \sum_{k=0}^{n} |p_k| < M|P_n|, \quad n \in \mathbb{N}.$$

(cf. [7], Theorem 17.1). Hence

$$|P_n/p_n| > \frac{1}{M} + R_{n-1}/M|p_n| > \frac{1}{M}, \quad n \in \mathbb{N}.$$

Consequently, from condition 1) we obtain

$$m_{n0} = O(1). \tag{13}$$

For that reason $m_{nk} = O_k(1)$ and the condition 1) of Lemma 6 is fulfilled (since $\eta_n = \delta_{n0}$ (cf.[7],p. 58)). Moreover, in that case by (12) we have

---

[2] Here and onwards $\Delta x_n = x_n - x_{n+1}$.

$$M^n_{sk} = \begin{cases} M_{nk} & \text{if } k < s-1, \\ M_{n,s-1} - P_{s-1}m_{n,s+1}/p_s & \text{if } k = s-1, \\ P_s m_{ns}/p_s & \text{if } k = s, \\ 0 & \text{if } k > s \end{cases}$$

where

$$M_{nk} = P_k\Delta\frac{\Delta m_{nk}}{p_k}$$

and

$$\gamma_{nk} = P_k\Delta\frac{\Delta B_{nk}}{p_k}$$

Now it is easy to see that conditions 1) - 3) of Theorem 1 (by conditions 3) and 4)) and condition 1) of Lemma 4 are fulfilled. As

$$\sum_{k=0}^{s-1} P_k\Delta\frac{\Delta m_{nk}}{p_k} = \sum_{l=0}^{s-1} p_l\left(\sum_{k=l}^{s-1}\Delta\frac{\Delta m_{nk}}{p_k}\right) = \sum_{l=0}^{s-1}\Delta m_{nl} - \frac{\Delta m_{ns}}{p_s}\sum_{l=0}^{s-1}p_l =$$

$$= m_{no} - m_{n,s+1} - \frac{P_s}{p_s}\Delta m_{ns}$$

we have

$$\frac{P_s}{p_s}\Delta m_{ns} = m_{no} - m_{n,s+1} - \sum_{k=0}^{s-1}P_k\Delta\frac{\Delta m_{nk}}{p_k}. \tag{14}$$

Therefore

$$P_s\Delta m_{ns}/p_s = O(1) \tag{15}$$

by conditions 2) and (13). Onwards, it is easy to see that

$$\frac{P_{s-1}}{p_s}m_{n,s+1} = \frac{P_s}{p_s}m_{ns} - \frac{P_s}{p_s}\Delta m_{ns} - m_{n,s+1}. \tag{16}$$

Consequently

$$\sum_k |M^n_{sk}| \le \sum_{k=0}^{s-1}|M_{nk}| + \left|\frac{P_{s-1}}{p_s}m_{n,s+1}\right| + \left|\frac{P_s}{p_s}m_{ns}\right| = O(1)$$

by conditions 1) - 2), (13) and (15). So the condition 2) of Lemma 6 is fulfilled too and $M \in (c_{(R,p_n)}, c_B)$ by Theorem 1.

**THEOREM** 5. *Let* $(R,p_n)$ *be a absolute convergence preserving method,* $B = (\beta_{nk})$ *and* $M = (m_{nk})$ *be matrices over* $\mathbb{C}$. *Moreover, let* $\sum_k|\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$ *and conditions 1)* *and 3) of Theorem 4 be fulfilled. If*

*1)* $P_s \Delta m_{ns} = O(p_s)$,

*2)* $B_{nk} = O(1)$,

*3)* $P_s \Delta B_{ns} = O(p_s)$

*then* $M \in (bv_{(R,p_n)}, c_B)$.

*Proof.* It is sufficient to show that all the assumptions and conditions of Theorem 2 are fulfilled. It is clear that condition 1) of Lemma 4 is satisfied. We see that conditions 1) and 2) of Theorem 1 are also satisfied by condition 3) of Theorem 4 because $\eta_n = \delta_{no}$. As $(R,p_n)$ preserves absolute convergence, we have (cf.[7], Theorem 17.2)

$$P_{k-1} \sum_{n=k}^{\infty} \left|\frac{p_n}{P_n P_{n-1}}\right| = O(1).$$

Hence there exists a number $M > 0$ such that $|p_k/P_k| < M$ and so $|P_k/p_k| > 1/M$ independently of $k$. Consequently, from condition 1) of Theorem 4 we conclude that (13) holds. Therefore $m_{nk} = O_k(1)$ and condition 1) of Lemma 6 is fulfilled.

As equalities (14) and (16) hold then the condition of Lemma 7 is fulfilled by conditions (13) and 1). It is easy to see that the condition of Theorem 2 is also fulfilled. Indeed, the equality

$$\sum_{k=0}^{s-1} P_k \Delta \frac{\Delta B_{nk}}{p_k} = B_{no} - B_{n,s+1} - \frac{P_s}{p_s} \Delta B_{ns} \tag{17}$$

is true (compare with (14)). Consequently, by conditions 2) and 3) condition 1) of Theorem 2 is fulfilled. This implies that $M \in (bv_{(R,p_n)}, c_B)$ by Theorem 2.

**REMARK** 2. If $B = (\beta_{nk})$ is a matrix such that $\sum_k |\beta_{nk}| = O(1)$ then condition 4) of Theorem 4 and conditions 2) and 3) of Theorem 5 are redundant.

**THEOREM** 6. *Let* $(R,p_n)$ *be an absolute convegence preserving method,* $B = (\beta_{nk})$ *and* $M = (m_{nk})$ *be matrices over* $\mathbb{C}$. *Moreover, let* $\sum_k |\beta_{nk}| < \infty$ *for each* $n \in \mathbb{N}$ *and the condition*

*1) of Theorem 4 and condition 1) of Theorem 5 be fulfilled. If*

*1)* $\sum_n |B_{nk} - B_{n-1,k}| = O(1)$,

*2)* $P_k \sum_n |\Delta(B_{nk} - B_{n-1,k})| = O(p_k)$,

*3) there exist finite limits* $\lim_k d_{nk} = d_n$, *where*

$$d_{nk} = B_{n,k+1} + \frac{P_k}{p_k}\Delta B_{nk},$$

*then* $M \in (bv_{(R,p_n)}, bv_B)$.

*Proof*. We shall show that all the assumptions and conditions of Theorem 3 are fulfilled. In the proof of Theorem 5 it has been shown that $m_{nk} = O_k(1)$ and condition 1) of Lemma 4, condition 1) of Lemma 6 and the condition of Lemma 7 are fulfilled. It is easy to see that condition 1) of Theorem 3 is fulfilled by condition 1) because $\eta_n = \delta_{no}$. Condition 2) of Theorem 3 is fulfilled too. Indeed, by (17) the equality

$$\sum_{l=o}^{k-1} \varkappa_{nl} = B_{no} - d_{nk}$$

is true. Therefore we have

$$\rho_{nk} = \sum_l \varkappa_{nl} - \sum_{l=o}^{k-1} \varkappa_{nl} = \lim_k (B_{no} - d_{nk}) -$$

$$- (B_{no} - d_{nk}) = d_{nk} - d_n$$

by condition 3). Onwards, by conditions 1) and 2) we obtain

$$\sum_n |d_{nk} - d_{n-1,k}| = O(1)$$

from which it follows by condition 3) that $\sum_n |d_n - d_{n-1}| < \infty$. For that reason

$$\sum_n |\rho_{nk} - \rho_{n-1,k}| \le \sum_n |d_{nk} - d_{n-1,k}| + \sum_n |d_n - d_{n-1}| = O(1).$$

So condition 2) of Theorem 3 is fulfilled. Consequently, $M \in (bv_{(R,p_n)}, bv_B)$ by Theorem 3.

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Summeeruvusväljade maatriksteisenduste iseloomustus

Ants Aasma

Resümee

Olgu $A = (\alpha_{nk})$-kompleksarvuline reversiivne maatriks, s.t. süsteemil (1) on ühene lahend iga koonduva jada $(z_n)$ korral. Olgu $B = (\beta_{nk})$ ja $M = (m_{nk})$ kompleksarvulised maatriksid, $c_A$ maatriksi A summeeruvusväli ja $bv_A$ maatriksi A absoluutse summeeruvuse väli. Artiklis antakse piisavad tingimused selleks, et maatriksteisendus (2) teostaks kujutused $c_A \to c_B$, $bv_A \to c_B$ ja $bv_A \to bv_B$.

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**ON COMMUTATIVE LOCALLY M-CONVEX ALGEBRAS**

**Jorma Arhippainen**

Let A be a commutative locally m-convex topological algebra over the complex numbers. We also assume that A has a unit element which will be denoted by e. Let $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ be a family of seminorms which defines the topology in A. We assume that this topology is a Hausdorff topology, in other words, if $p_\lambda(x) = 0$ for all $\lambda \in \Lambda$ then $x = 0$. Furthermore we assume that $p_\lambda(e) = 1$ for all $\lambda \in \Lambda$. General properties of locally m-convex algebras can be found in [2,3,13] or in [15].

If $\lambda \in \Lambda$ then the following notations will be used:

$N_\lambda = \{x \in A \mid p_\lambda(x) = 0\}$,

$\underset{\sim}{A}_\lambda = A/N_\lambda$ is the quotient algebra of A by $N_\lambda$,

$A_\lambda$ is the completion of $\underset{\sim}{A}_\lambda$.

Obviously $\underset{\sim}{A}_\lambda$ is a normed algebra with the norm defined by $\dot{p}_\lambda(x + N_\lambda) = p_\lambda(x)$ for each $x + N_\lambda \in \underset{\sim}{A}_\lambda$. Furthermore, we shall denote by $\Delta(A)$ the set of all nontrivial continuous homomorphisms from A into $\mathbb{C}$. The set $\Delta(A)$ will be provided by the relative $\sigma(A',A)$-topology. Then $\Delta(A)$ is a completely regular space. As it is generally known this is the weakest topology for which each function $\hat{x} : \Delta(A) \to \mathbb{C}$ defined by the equation $\hat{x}(\tau) = \tau(x)$ for each $\tau \in \Delta(A)$ is continuous whenever $x \in A$. The mapping $\mathcal{G} : x \to \hat{x}$ will be called the *Gelfand mapping* and $\Delta(A)$ the *carrier space of* A. Let $\hat{A} = \{\hat{x} \mid x \in A\}$. Then $\hat{A} \subseteq C(\Delta(A))$. For any set S we shall denote by cl(S) the closure of S. If I is an ideal of A then

$$h(I) = \{\tau \in \Delta(A) \mid \hat{x}(\tau) = 0 \text{ for each } x \in I\}$$

is the *hull* of I. The *kernel* k(E) of a subset E of $\Delta(A)$ is defined by

$$k(E) = \{x \in A \mid \hat{x}(\tau) = 0 \text{ for each } \tau \in E\}$$

and for the empty set we define $k(\emptyset) = A$.

1. **Auxiliary results.** If $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ is a family of seminorms which generates the topology in $A$ we will denote this topology by $T(\mathcal{P})$ and the corresponding topological algebra by $(A, T(\mathcal{P}))$. If $(A, T(\mathcal{P}_1))$ and $(B, T(\mathcal{P}_2))$ are two locally m-convex algebras with corresponding families of seminorms $\mathcal{P}_i = \{p_\lambda \mid \lambda \in \Lambda_i\}$ $(i = 1, 2)$ then a mapping $S : (A, T(\mathcal{P}_1)) \to (B, T(\mathcal{P}_2))$ will be called *semi-isometric mapping* if there is a bijection $\varphi$ from $\Lambda_1$ onto $\Lambda_2$ such that

$$p_{\varphi(\lambda)}(Sx) = p_\lambda(x) \text{ for all } x \in A \text{ and } \lambda \in \Lambda_1. \qquad (1.1)$$

Semi-isometric mappings have, for example, the following properties:

**LEMMA 1.1.** *Let* $(A, T(\mathcal{P}_1))$ *and* $(B, T(\mathcal{P}_2))$ *be two locally m-convex algebras and let* $S : (A, T(\mathcal{P}_1)) \to (B, T(\mathcal{P}_2))$ *be a semi-isometric algebra-homomorphism. Then*

(a) $S$ *is continuous,*

(b) $S$ *is a bijection from* $A$ *onto* $S(A) \subset B$,

(c) $S^{-1}$ *from* $(S(A), T(\mathcal{P}_2))$ *onto* $(A, T(\mathcal{P}_1))$ *is semi-isometric,*

(d) $S^{-1}$ is *continuous.*

The proof can be carried out by exactly the same fashion as for an isometric map between two normed algebras.

We shall say that two locally m-convex algebras are *semi-isometrically isomorphic* if there is a semi-isometrical isomorphism between these two algebras.

**LEMMA 1.2.** *If* $(A, T(\mathcal{P}_1))$ *and* $(B, T(\mathcal{P}_2))$ *are two commutative semi-isometrically isomorphic locally m-convex algebras then the carrier spaces* $\Delta(A)$ *and* $\Delta(B)$ *are homeomorphic.*

*Proof.* Let $S$ be a semi-isometric linear isomorphism from $(A, T(\mathcal{P}_1))$ onto $(B, T(\mathcal{P}_2))$. For each $\tau \in \Delta(B)$ we denote by $\omega_\tau$ the $\mathbb{C}$-homomorphism defined by $\omega_\tau(x) = \tau(Sx)$ for each

$x \in A$. Then it is easy to see that the mapping $\tau \to \omega_\tau$ is a homeomorphism from $\Delta(B)$ onto $\Delta(A)$.

Next we shall consider some properties of the carrier space $\Delta(A)$. One of the most fundamental results dealing with this subject is

**LEMMA 1.3.** *Let* $(A, T(\mathcal{P}))$ *be a commutative locally m-convex algebra where* $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$. *Then*

$$\Delta(A) = \bigcup \{K_\lambda \mid \lambda \in \Lambda\}$$

*where* $K_\lambda = \Delta(A) \cap \overline{V}_\lambda^o$ *is compact for each* $\lambda \in \Lambda$ *and* $\overline{V}_\lambda^o$ *is the polar of* $\overline{V}_\lambda = \{x \in A \mid p_\lambda(x) \leq 1\}$.

*Proof*. See [2], p.227, or [5], p.28.

**LEMMA 1.4.** *Let* $(A, T(\mathcal{P}))$ *be as in Lemma 1.2. Then* $K_\lambda = h(N_\lambda)$ *for all* $\lambda \in \Lambda$.

*Proof*. By [15] we have $\Delta(A_\lambda) = \{\tau_\lambda \mid \tau \in K_\lambda\}$ where $\tau_\lambda$ is the mapping from $A_\lambda$ into $\mathbb{C}$ defined by $\tau_\lambda(x + N_\lambda) = \tau(x)$ for each $x + N_\lambda \in A_\lambda$. On the other hand, we have $h(N_\lambda) = \{\tau \mid \tau_\lambda \in \Delta(A_\lambda)\}$ by [13], Theorem 4.1. So we can see that $K_\lambda = h(N_\lambda)$ for all $\lambda \in \Lambda$.

**REMARK.** We shall always assume that the family $\mathcal{P} = \{p_\lambda \mid \lambda \in \Lambda\}$ is saturated, in other words, if $p_1, p_2 \in \mathcal{P}$ then $p_M = \max\{p_1, p_2\} \in \mathcal{P}$ where $p_M$ is the seminorm of $A$ defined by $p_M(x) = \max\{p_1(x), p_2(x)\}$ for each $x \in A$.

Let $\overline{V}_M = \{x \in A \mid p_M(x) \leq 1\}$ and $K_M = \Delta(A) \cap \overline{V}_M^o$. Then we have

**LEMMA 1.5.** $K_M = K_1 \cup K_2$ *where* $K_i = \Delta(A) \cap \overline{V}_i^o$ with $i = 1,2$.

*Proof*. Let $N_M = \{x \in A \mid p_M(x) = 0\}$. Then it is easy to see that $N_M = N_1 \cap N_2$ where $N_i = \{x \in A \mid p_i(x) = 0\}$ with $i = 1,2$. Thus,

$$K_M = h(N_M) = h(N_1 \cap N_2) = h(N_1) \cup h(N_2) = K_1 \cup K_2.$$

**COROLLARY 1.1.** *The family* $\mathcal{K}(\Lambda) = \{K_\lambda | \lambda \in \Lambda\}$ *is closed under a finite union.*

**2. On function algebras.** Let X be a completely regular space and let C(X) be the algebra of all continuous complex-valued functions defined on X. As it is generally known C(X) can be equipped by several kind of topologies, usually by the so-called *compact-open topology* which is defined by the family $\mathcal{P}(X) = \{p_K | K \in \mathcal{K}(X)\}$ of seminorms where

$$p_K(x) = \sup_{t \in K} |x(t)|$$

for each $x \in C(X)$ and $\mathcal{K}(X)$ is the set of all compact subsets of X.

Let $\mathcal{K}_0 \subset \mathcal{K}(X)$ be a family of compact subsets of X with properties

$$\bigcup \{K | K \in \mathcal{K}_0\} = X \tag{2.1}$$

and

$$\text{if } K_1, K_2 \in \mathcal{K}_0 \text{ then } K_1 \cup K_2 \in \mathcal{K}_0. \tag{2.2}$$

If $\mathcal{P}_0 = \{p_K | K \in \mathcal{K}_0\}$ then we shall denote by $T(\mathcal{P}_0)$ the topology in C(X) generated by the family $\mathcal{P}_0$. The properties of topological algebra $(C(X), T(\mathcal{P}_0))$ were considered in [16]. Obviously $T(\mathcal{P}_0)$ equals the compact-open topology if for each $K \in \mathcal{K}(X)$ there is $K_1 \in \mathcal{K}_0$ such that $K \subseteq K_1$. The properties of C(X) with the compact-open topology can be found, for example, in [7] or in [18]. In the following we shall give some results concerning the algebra $(C(X), T(\mathcal{P}_0))$.

**LEMMA 2.1.** *Let X be a completely regular space. Then*

(a) $\Delta(C(X), T(\mathcal{P}_0)) = \{\tau_t | t \in X\}$ where $\tau_t(g) = g(t)$ for each $g \in C(X)$,

(b) *if* I *is a closed ideal of* $(C(X), T(\mathcal{P}_0))$ *then* $k(h(I)) = I$,

(c) *if* $p_{K_1}, p_{K_2} \in \mathcal{P}_0$ *and* $p_M = \max\{p_{K_1}, p_{K_2}\}$ *then* $p_M = p_{K_1 \cup K_2}$.

*Proof.* For the proof of part (a) see [15], Example 7.6. Part (b) can be proved in a similar fashion as the corresponding result for the compact-open topology (cf.[13], p.333). Part (c) is obvious.

It is well known that $C(X)$ with the compact-open topology is complete if and only if $X$ is the so-called $k_R$-space. The space $X$ is called a $k_R$-*space* if from $g|K \in C(K)$ for each $K \in \mathcal{K}(X)$ follows that $g \in C(X)$ (cf., for example, [2] or [6]). We shall call $X$ a $k(\mathcal{K}_0)_R$-*space* if the condition $g|K \in C(K)$ for each $K \in \mathcal{K}_0$ implies that $g \in C(X)$.

**THEOREM 2.1.** *Let $X$ be a completely regular space. Then $(C(X), T(\mathcal{P}_0))$ is complete if and only if $X$ is a $k(\mathcal{K}_0)_R$-space.*

*Proof.* Suppose that $X$ is a $k(\mathcal{K}_0)_R$-space. Now it is easy to see that $(\mathbb{C}^X, T(\mathcal{P}_0))$ is complete when $\mathbb{C}^X$ is the algebra of all $\mathbb{C}$-valued functions defined on $X$. Obviously $(C(X), T(\mathcal{P}_0))$ is a closed subalgebra of $(\mathbb{C}^X, T(\mathcal{P}_0))$ from which it follows that $(C(X), T(\mathcal{P}_0))$ is complete. Suppose now that $(C(X), T(\mathcal{P}_0))$ is complete and let $g \in \mathbb{C}^X$ be a function for which $g|K \in C(K)$ for all $K \in \mathcal{K}_0$. Since each $K \in \mathcal{K}_0$ is compact and $X$ is completely regular there is an extension $G_K \in C(X)$ of $g$ for all $K$ by Tietzes extension theorem (see [5], Theorem 5.1). The family $\mathcal{K}_0$ is partially ordered by set inclusion. It is also directed because $\mathcal{K}_0$ has property (2.2). So $\{G_K\}_{K \in \mathcal{K}_0}$ is a net in $C(X)$. Obviously we have $\lim_K G_K = g$. Since $(C(X), T(\mathcal{P}_0))$ is complete we can see that $g \in C(X)$. Thus $X$ is a $k(\mathcal{K}_0)_R$-space.

**3. On functional representation of a locally m-convex algebra.** It was earlier noted that $\hat{A} \subseteq C(\Delta(A))$ when $A$ is a commutative locally m-convex algebra. If we now define a topology in $C(\Delta(A))$ by the family $\mathcal{P}(\Lambda) = \{p_{K_\lambda} \mid \lambda \in \Lambda\}$ of seminorms where

$$p_{K_\lambda}(g) = \sup_{\tau \in K_\lambda} |g(\tau)|$$

for each $g \in C(\Delta(A))$ and denote this topology by $T(\Lambda)$ then $(C(\Delta(A)), T(\Lambda))$ is a topological algebra of the same kind as was considered in Chapter 2. Namely, the family $\mathcal{K}(\Lambda)$ has by Lemmas 1.3 and 1.5 the following properties:

$$\bigcup \{K_\lambda \mid \lambda \in \Lambda\} = \Delta(A) \tag{3.1}$$

and

$$\text{if } K_1, K_2 \in \mathcal{K}(\Lambda) \text{ then } K_1 \cup K_2 \in \mathcal{K}(\Lambda). \tag{3.2}$$

Since $\hat{A}$ is a subalgebra of $C(\Delta(A))$ we can also equip it with the topology $T(\Lambda)$. So we can next consider a locally m-convex algebra $(\hat{A},T(\Lambda))$.

**LEMMA 3.1.** *Let* $(A,T(\mathcal{P}))$ *be a commutative locally m-convex algebra. Then the Gelfand mapping from* $(A,T(\mathcal{P}))$ *onto* $(\hat{A},T(\Lambda))$ *is continuous.*

*Proof*. Since $|\tau(x)| \le p_\lambda(x)$ for all $\tau \in K_\lambda$, $x \in A$ and $\lambda \in \Lambda$ we have $p_{K_\lambda}(\hat{x}) = \sup_{\tau\in K_\lambda} |\hat{x}(\tau)| \le p_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$ from which the result follows.

If each seminorm $p_\lambda \in \mathcal{P}$ has the property

$$p_\lambda(x^2) = p_\lambda(x)^2$$

for each $x \in A$ we shall call $(A,T(\mathcal{P}))$ a ***square algebra***. For square algebras we have

**THEOREM 3.1.** *Let* $(A,T(\mathcal{P}))$ *be a commutative locally m-convex algebra. Then* $(A,T(\mathcal{P}))$ *is a square algebra if and only if*

$$p_\lambda(x) = p_{K_\lambda}(\hat{x}) \text{ for all } x \in A \text{ and } \lambda \in \Lambda. \tag{3.3}$$

*Moreover, if* $(A,T(\mathcal{P}))$ *is a square algebra then the Gelfand mapping is a semi-isomorphism from* $(A,T(\mathcal{P}))$ *onto* $(\hat{A},T(\Lambda))$.

*Proof*. If $(A,T(\mathcal{P}))$ is a square algebra then

$$\dot{p}_\lambda((x + N_\lambda)^2) = p_\lambda(x^2) = p_\lambda(x)^2 = (\dot{p}(x + N_\lambda))^2$$

for all $\lambda \in \Lambda$ and we can see that $A_\lambda$ and therefore also $\tilde{A}_\lambda$ are normed square algebras. In the proof of Lemma 1.4 we saw that $\Delta(A_\lambda) = \{\tau_\lambda | \tau \in K_\lambda\}$ for all $\lambda \in \Lambda$ where $\tau_\lambda(x + N_\lambda) = \tau(x)$. Now the mapping $\tau \to \tau_\lambda$ is a homeomorphism from $K_\lambda$ onto $\Delta(A_\lambda)$. As $\Delta(A_\lambda)$ and $\Delta(\tilde{A}_\lambda)$ are homeomorphic by Corollary 2.1 of [13] then the elements of $\Delta(\tilde{A}_\lambda)$ will be denoted also by $\tau_\lambda$. Now by using Theorem 5.1.2 of [9] we have

$$\dot{p}_\lambda(x + N_\lambda) = \sup_{\tau_\lambda\in\Delta(\tilde{A}_\lambda)} |(x + N_\lambda)^\wedge(\tau_\lambda)|.$$

But $(x + N_\lambda)^\wedge(\tau_\lambda) = \hat{x}(\tau)$ for all $\tau \in K_\lambda$. So

$$p_\lambda(x) = \dot{p}_\lambda(x + N_\lambda) = \sup_{\tau\in K_\lambda} |\hat{x}(\tau)| = p_{K_\lambda}(\hat{x}).$$

Suppose now that the equality (3.3) holds. Then

$$p_\lambda(x^2) = p_{K_\lambda}(\hat{x}^2) = p_{K_\lambda}(\hat{x})^2 = p_\lambda(x)^2$$

for all $\lambda \in \Lambda$ and $x \in A$ which shows that $(A,T(\mathcal{P}))$ is a square algebra.

If now (3.3) is valid then for each seminorm $p_\lambda$ there is a unique seminorm $q_\lambda = p_{K_\lambda}$ such that $p_\lambda(x) = q_\lambda(\hat{x})$ for each $x \in A$. So, if $(A,T(\mathcal{P}))$ is a square algebra then the Gelfand mapping is a semi-isometric isomorphism from $(A,T(\mathcal{P}))$ onto $(\hat{A},T(\Lambda))$.

**COROLLARY 3.1.** *Let* $(A,T(\mathcal{P}))$ *be a square algebra. Then* $k(h(N_\lambda)) = N_\lambda$ *for all* $\lambda \in \Lambda$.

*Proof.* We only have to show that $k(h(N_\lambda)) \subseteq N_\lambda$. Let $x \in k(h(N_\lambda))$ be arbitrary. Then $\hat{x}(\tau) = 0$ for all $\tau \in h(N_\lambda) = K_\lambda$. Therefore

$$p_\lambda(x) = p_{K_\lambda}(\hat{x}) = \sup_{\tau \in K_\lambda} |\hat{x}(\tau)| = 0.$$

So we can see that $x \in N_\lambda$ which completes the proof.

**REMARK.** Some properties of $(\hat{A},T(\Lambda))$ have been studied in [9].

**4. On algebras with involution.** Let $(A,T(\mathcal{P}))$ be a locally m-convex algebra and $x \to x^*$ be an involution in $A$. We say that $(A,T(\mathcal{P}))$ is a *star algebra* if

$$p_\lambda(xx^*) = p_\lambda(x)^2$$

for each $x \in A$ and $\lambda \in \Lambda$. It is easy to see that a star algebra is a square algebra (cf. [2], p.222). Moreover, if $(A,T(\mathcal{P}))$ is a complete star algebra then each $A_\lambda$ is complete (cf.[4] or [18], p.179). So, for a complete star algebra, each factor algebra $A_\lambda$ is a $B^*$-algebra.

Next we shall consider the functional representation of algebra $(A,T(\mathcal{P}))$. This subject has been studied, for example, in the following papers [1,2,6,9,12,13,15,16,19].

**THEOREM 4.1.** *Let* $(A,T(\mathcal{P}))$ *be a commutative complete star algebra. Then the Gelfand mapping is semi-isometric isomorphism from* $(A,T(\mathcal{P}))$ *onto* $(C(\Delta(A)),T(\Lambda))$.

*Proof.* By Theorem 3.1. the Gelfand mapping is a semi-isometric isomorphism from $(A,T(\mathcal{P}))$ onto $(\hat{A},T(\Lambda))$. So it suffices to prove that $\hat{A} = C(\Delta(A))$. This result has been proved in [9] or in [15] by using the so-called projective limits. For the properties of projective limits see, for example, [8] or [15]. We shall however use here a more direct method.

Let $g \in C(\Delta(A))$ be arbitrary. Since $A_\lambda$ is a commutative $B^*$-algebra for all $\lambda \in \Lambda$ we have $\hat{A}_\lambda = C(\Delta(A_\lambda))$ for each $\lambda \in \Lambda$ by the Gelfand-Naimark Theorem (cf.,for example,[10], p.277, or [17], p. 230-232).

Let now $\lambda \in \Lambda$ and $\tau \in K_\lambda$. Then as above there exists $\tau_\lambda \in \Delta(A_\lambda)$ such that $\tau(x) = \tau_\lambda(x + N_\lambda)$ for each $x \in A$. Moreover, let $\varkappa_\lambda(x) = x + N_\lambda$ for each $x \in A$. Then $\tau = \tau_\lambda(\varkappa_\lambda) = \varkappa_\lambda^*(\tau_\lambda)$. As $g \circ \varkappa_\lambda^* \in C(\Delta(A_\lambda))$ then there exists an element $x_\lambda \in A$ such that $g \circ \varkappa_\lambda^* = (x_\lambda + N_\lambda)\hat{}$. Consequently, for each $\lambda \in \Lambda$ and $\tau \in K_\lambda$ we have

$$g(\tau) = g \circ \varkappa_\lambda^*(\tau_\lambda) = (x_\lambda + N_\lambda)\hat{}(\tau_\lambda) = \tau(x_\lambda) = \hat{x}_\lambda(\tau).$$

The index set $\Lambda$ can be partially ordered by setting $\lambda_1 \le \lambda_2 \leftrightarrow r_{K_{\lambda_1}}(f) \le r_{K_{\lambda_2}}(f)$ for all $f \in C(\Delta(A))$. Then $\Lambda$ is a directed set by (3.2). So $\{\hat{x}_\lambda\}_{\lambda\in\Lambda}$ is a net in $(\hat{A},T(\Lambda))$. Let now $\mu$ be a fixed element in $\Lambda$. It is easy to see that $r_{K_\mu}(\hat{x}_\lambda - g) = 0$ for all $\lambda \ge \mu$. Therefore $\lim_\lambda \hat{x}_\lambda = g$.

Because $(A,T(\mathcal{P}))$ and $(\hat{A},T(\Lambda))$ are semi-isometrically isomorphic by Theorem 3.1 they can be identified as topological algebras and therefore from the completeness of $(A,T(\mathcal{P}))$ it follows that $(\hat{A},T(\Lambda))$ is also complete. So $(\hat{A},T(\Lambda))$ is a closed subalgebra of $(C(\Delta(A)),T(\Lambda))$ and since $g$ is the limit of a net in $(\hat{A},T(\Lambda))$ we can see that $g \in \hat{A}$ which completes the proof.

**REMARK.** Since

$$(x + N_\lambda)\hat{}(\tau_\Lambda) = \overline{(x + N_\lambda)^*(\tau_\lambda)}$$

for all $\lambda \in \Lambda$ we can see that

$$(x^*)\hat{}(\tau) = \overline{\hat{x}(\tau)}$$

for all $x \in A$ and $\tau \in \Delta(A)$ where the bar denotes the complex conjugation. Thus, the Gelfand mapping is also a $^*$-isomorphism.

*Proof.* This result can be shown by similar fashion as for commutative $B^*$-algebras.

**COROLLARY 4.3.** *Let* $(A, T(\mathcal{P}))$ *be as in Corollary 4.2. Let* I *be a closed ideal of* A *and let* $\Lambda_0$ *be as in Corollary 4.1. Then*

$$I = \bigcap \{I + N_\lambda \mid \lambda \in \Lambda_0\}$$

*Proof.* We only have to note that $h(I + N_\lambda) = h(I) \cap h(N_\lambda) = E_\lambda$ for all $\lambda \in \Lambda$ from which the result follows, since

$$I + N_\lambda = k(h(I + N_\lambda)) = k(E_\lambda) = I_\lambda$$

for all $\lambda \in \Lambda$.

**COROLLARY 4.4.** *Let* $(A, T(\mathcal{P}))$ *be as in Corollary 4.1. Then* $A_\lambda$ *is complete for each* $\lambda \in \Lambda$.

*Proof.* It is easy to see that by Theorem 3.1 the mapping $x + N_\lambda \to \hat{x}|h(N_\lambda)$ where $x + N_\lambda \in A_\lambda$ is an isometric isomorphism from $A_\lambda$ onto $C(h(N_\lambda))$ from which the result follows.

In Chapter 2 we gave the necessary and sufficient condition for algebra $(C(X, T(\mathcal{P}_0))$ to be complete. By using Theorem 2.1 we obtain now the following result:

**COROLLARY 4.5.** *A commutative star algebra* $(A, T(\mathcal{P}))$ *for which* $\hat{A} = C(\Delta(A))$ *is complete if and only if the carrier space* $\Delta(A)$ *is a* $k(\mathcal{K}(A))_R$*-space.*

**5. On quotient algebras.** Let $(A, T(\mathcal{P}))$ be a commutative locally m-convex algebra and let $I \subset A$ be a closed ideal. Then the quotient algebra $A/I$ will also be a locally m-convex algebra if we define the topology in $A/I$ by the family $\dot{\mathcal{P}}$ of seminorms where $\dot{\mathcal{P}} = \{\dot{p}_\lambda \mid \lambda \in \Lambda\}$ and

$$\dot{p}_\lambda(x + I) = \inf_{y \in I} p_\lambda(x + y)$$

for all $x + I \in A/I$ and $\lambda \in \Lambda$. We shall denote this topology by $T(\dot{\mathcal{P}})$. Furthermore, we shall denote

$$\dot{N}_\lambda = \{x + I \mid \dot{p}_\lambda(x + I) = 0\},$$

**THEOREM 4.2.** *Let* $(A, T(\mathcal{P}))$ *be a commutative star algebra for which* $\hat{A} = C(\Delta(A))$. *Then* $k(h(I)) = I$ *for all closed ideals of* $A$.

*Proof.* Let $I$ be an arbitrary closed ideal of $A$. To prove the theorem it suffices to show that $k(h(I)) \subseteq I$. So let $x \in k(h(I))$. Then $\hat{x}(\tau) = 0$ for all $\tau \in h(I)$. If we now define the mapping $\tau \to \omega_\tau$ for each $\tau \in \Delta(A)$, where $\omega_\tau$ is defined by $\omega_\tau(\hat{x}) = \tau(x)$ for each $x \in A$, then by Theorem 3.1 and Lemma 1.2 this mapping is a homeomorphism from $\Delta(A)$ onto $\Delta(\hat{A}) = \Delta(C(\Delta(A)))$. Therefore we have

$$h(\hat{I}) = \{\omega_\tau | \tau \in h(I)\}$$

where $\hat{I} = \mathcal{G}(I)$ is a closed ideal in $C(\Delta(A))$. From the condition $\omega_\tau(\hat{x}) = \tau(x) = \hat{x}(\tau) = 0$ for each $\tau \in h(I)$, it follows that $\hat{x} \in k(h(\hat{I}))$. As $k(h(\hat{I})) = \hat{I}$ by Corollary 8.3.1 in [10] then $\hat{x} \in \hat{I}$, so $x \in I$ whereas $\mathcal{G}$ is an one-to-one mapping (cf. [2], p. 263).

**COROLLARY 4.1.** *Let* $I$ *be a closed ideal of a commutative star algebra* $(A, T(\mathcal{P}))$ *for which* $\hat{A} = C(\Delta(A))$. *Furthermore, let* $E_\lambda = h(I) \cap K_\lambda$ *for each* $\lambda \in \Lambda$ *and let*

$$\Lambda_0 = \{\lambda \in \Lambda | E_\lambda \neq \emptyset\}.$$

*If we define* $I_\lambda = k(E_\lambda)$ *then*

$$I = \cap \{I_\lambda | \lambda \in \Lambda_0\}.$$

*Proof.* We have

$$I = k(h(I)) = k(\cup_{\lambda \in \Lambda_0} E_\lambda) = \cap_{\lambda \in \Lambda_0} k(E_\lambda) = \cap_{\lambda \in \Lambda_0} I_\lambda .$$

We shall say that a locally m-convex algebra $(A, T(\mathcal{P}))$ is *normal* if the elements of $\hat{A}$ separate any two disjoint closed subsets $F_1$ and $F_2$ of the carrier space $\Delta(A)$. It is easy to see that for a normal locally m-convex algebra the carrier space is a normal topological space. For normal star algebra we have

**COROLLARY 4.2.** *Let* $(A, T(\mathcal{P}))$ *be a commutative normal algebra for which* $\hat{A} = C(\Delta(A))$ *and let* $I_1$ *and* $I_2$ *be two closed ideals in* $A$. *Then* $I_1 \oplus I_2$ *is also a closed ideal of* $A$ *or* $I_1 \oplus I_2 = A$.

$$\bar{\dot{V}}_\lambda = \{x + I \mid \dot{p}_\lambda(x + I) \le 1\},$$

$$\dot{K}_\lambda = \Delta(A/I) \cap \bar{\dot{V}}^o_\lambda .$$

If we define a mapping $\tau_\omega$ for $\omega \in \Delta(A/I)$ as following

$$\tau_\omega(x) = \omega(x + I)$$

for each $x \in A$ then by [13] (Theorem 4.1, p. 339) the mapping $\omega \to \tau_\omega$ is a homeomorphism from $\Delta(A/I)$ onto $h(I)$.

**THEOREM 5.1.** *Let $(A, T(\mathcal{P}))$ be a commutative locally m-convex algebra and let $I$ be a closed ideal of $A$. Then*

$$\{\tau_\omega \mid \omega \in h(\dot{N}_\lambda)\} = h(I) \cap K_\lambda$$

*for all $\lambda \in \Lambda$.*

*Proof.* Let $\lambda \in \Lambda$ be fixed and $\omega \in h(\dot{N}_\lambda)$ be arbitrary. Then we have $\omega(x + I) = 0$ for all $x + I \in \dot{N}_\lambda$. Now

$$\tau_\omega(u + v) = \omega(u + v + I) = \omega(v + I)$$

for each $u \in I$, $v \in N_\lambda$. But $|\omega(v + I)| \le \dot{p}_\lambda(v + I) \le p_\lambda(v) = 0$. Thus, $\omega(v + I) = 0$ and we have $\tau_\omega(u + v) = 0$ for all $u \in I$, $v \in N_\lambda$ from which it follows that $\tau_\omega \in h(I + N_\lambda)$. But $h(I + N_\lambda) = h(I) \cap h(N_\lambda) = h(I) \cap K_\lambda$.

Let now $\tau \in h(I) \cap K_\lambda$ be arbitrary. Since $h(I) = \{\tau_\omega \mid \omega \in \Delta(A/I)\}$ we can see that there is $\omega \in \Delta(A/I)$ such that $\tau = \tau_\omega$. It now suffices to prove that $\omega(x + I) = 0$ for all $x + I \in \dot{N}_\lambda$. Let $x + I \in \dot{N}_\lambda$. Then for each $\varepsilon > 0$ there is $y_0 \in I$ such that $p_\lambda(x + y_0) < \varepsilon$. So

$$|\omega(x + I)| = |\tau_\omega(x)| = |\tau(x)| = |\tau(x + y_0)| \le p_\lambda(x + y_0) < \varepsilon$$

which completes the proof.

**COROLLARY 5.1.** *Let $A$ and $I$ be as in Theorem 5.1. Then the mapping $\omega \to \tau_\omega$ is a homeomorphism from $h(\dot{N}_\lambda) = \dot{K}_\lambda$ onto $h(I) \cap K_\lambda$ for each $\lambda \in \Lambda$.*

Next we shall consider the functional representation of the quotient algebra $A/I$. Let $I$ be a closed ideal of algebra $(A, T(\mathcal{P}))$. Then the following notations will be used: $E_\lambda = h(I) \cap K_\lambda$ for each $\lambda \in \Lambda$, $\Lambda_0 = \{\lambda \in \Lambda \mid E_\lambda \ne \emptyset\}$ and $T(\dot{\mathcal{P}}_0)$ is the topology in $A/I$ generated by the family of seminorms $\dot{\mathcal{P}}_0 = \{\dot{p}_\lambda \mid \lambda \in \Lambda_0\}$. Furthermore, we shall denote by $T(\Lambda_0)$ the topology in $C(h(I))$ generated by $\{p_{E_\lambda} \mid \lambda \in \Lambda_0\}$.

**THEOREM 5.2.** *If* $(A, T(\mathcal{P}))$ *be a commutative normal star algebra for which* $\hat{A} = C(\Delta(A))$ *and* $I$ *be a closed ideal of* $A$ *then* $T(\dot{\mathcal{P}}_0)$ *is a locally m-convex topology in* $A/I$ *and the mapping*

$$x + I \to \hat{x}|\ h(I) \qquad (5.1)$$

*is a semi-isometric isomorphism from* $(A/I, T(\dot{\mathcal{P}}_0))$ *onto* $(C(h(I)), T(\Lambda_0))$ *such that*

$$\dot{p}_\lambda(x + I) = p_{E_\lambda}(\hat{x}) \text{ for all } x + I \in A/I \text{ and } \lambda \in \Lambda_0. \qquad (5.2)$$

*Furthermore, we have*

$$\dot{p}_\lambda((x + I)(x + I)^*) = \dot{p}_\lambda(x + I)^2 \text{ for all } x + I \in A/I \text{ and}$$

$$\lambda \in \Lambda_0. \qquad (5.3)$$

*Proof*. Obviously $T(\dot{\mathcal{P}}_0)$ has all the properties of a locally m-convex topology. We only have to prove that from the condition $\dot{p}_\lambda(x + I) = 0$ for all $\lambda \in \Lambda_0$ it follows that $x + I = 0$ or equivalently that $x \in I$. We shall show this after we have proved (5.2).

It is easy to see that the mapping $x + I \to \hat{x}|\ h(I)$ is a linear homomorphism from $A/I$ into $C(h(I))$. If $\hat{x}|\ h(I) = 0$, then $\hat{x}(\tau) = 0$ for all $\tau \in h(I)$. So $x \in k(h(I)) = I$ by Theorem 4.2 from which it follows that $x + I = 0$ and we have shown that the mapping defined in (5.1) is an injection. To prove the surjectivity let $g \in C(h(I))$ be arbitrary. Since $h(I)$ is a closed subset of normal space $\Delta(A)$ there is a function $G \in C(\Delta(A))$ by Tietzes extension theorem such that $G|\ h(I) = g$. Now $\hat{A} = C(\Delta(A))$. So there is $x \in A$ for which $\hat{x} = G$. Therefore

$$(x + I)\hat{}\,(\omega) = \hat{x}(\tau_\omega) = g(\tau_\omega)$$

for each $\omega \in \Delta(A/I)$ since $h(I) = \{\tau_\omega |\ \omega \in \Delta(A/I)\}$. This completes the proof of surjectivity.

Next we shall prove (5.2). If $x \in A$ and $y \in I$ then $p_\lambda(x + y) = p_{K_\lambda}(\hat{x} + \hat{y}) \geq p_{E_\lambda}(\hat{x} + \hat{y}) = p_{E_\lambda}(\hat{x})$ ($(\hat{y}(\tau) = 0$ for all $\tau \in E_\lambda$ since $E_\lambda \subset h(I))$. Thus,

$$\dot{p}_\lambda(x + I) = \inf_{y \in I} p_\lambda(x + y) \geq p_{E_\lambda}(\hat{x}).$$

Let now $\varepsilon > 0$, $x \in A$ and

$$U_\lambda = \{\tau \in \Delta(A)|\ |\hat{x}(\tau) - \hat{x}(\tau')| < \varepsilon \text{ for some } \tau' \in E_\lambda\ \}.$$

Then $U_\lambda$ is an open subset of $\Delta(A)$ and $E_\lambda \subset U$. Now for any

$\tau \in U_\lambda \cap K_\lambda$ there is $\tau' \in E_\lambda$ such that $|\hat{x}(\tau)| < |\hat{x}(\tau')| + \varepsilon$. As $\Delta(A)$ is a regular space and $h(I)$ is a closed subset of $\Delta(A)$ there is an open subset $V$ of $\Delta(A)$ such that $h(I) \subset V$. It is easy to see that $W = (X \setminus (U_\lambda \cup V)) \cup (K_\lambda \setminus U_\lambda)$ is a closed subset of $\Delta(A)$ and $W \cap h(I)$ is empty. Therefore by Urysohns lemma there is $y \in A$ for which $\hat{y}(\tau) = 1$ for each $\tau \in h(I)$ and $\hat{y}(\tau) = 0$ for each $\tau \in W$. Then $(xy)\hat{}\,(\tau) = \hat{x}(\tau)$ for each $\tau \in h(I)$ and therefore $x - xy \in k(h(I)) = I$ for which $x + I = xy + I$. So

$$\dot{p}_\lambda(x + I) = \dot{p}_\lambda(xy + I) \le p_\lambda(xy) = p_{K_\lambda}(\hat{x}\hat{y}) = p_{U_\lambda \cap K_\lambda}(\hat{x}\hat{y}) =$$

$$= \sup_{\tau \in U_\lambda \cap K_\lambda} |\hat{x}(\tau)| \le \sup_{\tau \in E_\lambda} |\hat{x}(\tau)| + \varepsilon = p_{E_\lambda}(\hat{x}) + \varepsilon$$

by Theorem 3.1. Thus $\dot{p}_\lambda(x + I) \le p_{\hat{E}_\lambda}(x)$ which completes the proof of (5.2).

Suppose that $\dot{p}_\lambda(x + I) = 0$ for all $\lambda \in \Lambda_0$. Then $\hat{x}(\tau) = 0$ for all $\tau \in E_\lambda$ where $\lambda \in \Lambda_0$. Because $\bigcup_{\lambda \in \Lambda_0} E_\lambda = h(I)$, we can see that $\hat{x}(\tau) = 0$ for all $\tau \in h(I)$ whence $x \in I$ which shows that $T(\dot{\mathcal{P}}_0)$ is a Hausdorff topology. The result (5.3) follows from (5.2).

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Kommutatiivsetest lokaalselt m-kumeratest algebratest
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Resümee

Olgu A kommutatiivne lokaalselt m-kumer ühikuga $\mathbb{C}$-algebra. Käesolevas töös uuritakse algebra A funktsionaalseid esitusi, Gelfandi teisenduse omadusi, kinniste ideaalide kirjeldusi ning faktoralgebra A/I funktsionaalseid esitusi kinnise ideaali I korral.

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# Product and direct sum of $L_\varphi$–K(X)–spaces and related K(X)–spaces

Johann Boos and Toivo Leiger*

**1.** Let $(X,\tau)$ be a locally convex space. With $X'$ and $X^\#$ we denote the topological dual of $(X,\tau)$ and the algebraic dual of $X$, respectively. A subset $S$ of $X$ is called *sequentially $\tau$-closed* if $x_k \in S$ $(k \in \mathbb{N})$ and $x_k \longrightarrow a\,(\tau)$ implies $a \in S$ and *sequentially $\tau$-dense*, if for each $a \in X$ there exists a sequence $(x_k)$ in $S$ with $x_k \longrightarrow a\,(\tau)$.
If $S$ is any subspace of $X$ then $\overset{\sqcup}{S}$ denotes the smallest sequentially $\tau$-closed subspace of $X$ containing $S$. Obviously, $\overset{\sqcup}{S}$ is the intersection of all sequentially $\tau$-closed subspaces of $X$ including $S$ (see [3]).

**2.** As usually, $\omega(X)$ and $\varphi(X)$ denotes the *set of all sequences $x = (x_k)$ in $X$* and the *set of all finite sequences in $X$*, respectively. A subspace of $\omega(X)$ is called *sequence space (over $X$)*.
A locally convex sequence space $(E,\tau_E)$ over $X$ is a *K(X)-space* if the coordinate mappings

$$\pi_k : (E,\tau_E) \longrightarrow (X,\tau)\,,\ x = (x_i) \longrightarrow x_k \qquad (k \in \mathbb{N})$$

are continuous. In case of a sequence space $E$ (over $X$) the *$\beta$-dual* is defined by

$$E^\beta := \left\{ (\psi_k) \in \omega(X') \;\middle|\; \sum_k \psi_k(x_k) \text{ converges for each } x = (x_k) \in E \right\}.$$

Each $(\psi_k) \in E^\beta$ defines a linear functional

$$\psi : E \longrightarrow \mathbb{K}\,,\ x = (x_k) \longrightarrow \sum_k \psi_k(x_k)\,;$$

therefore in case of $\varphi(X) \subset E$, this representation of $\psi$ is uniquely determined and we may identify $E^\beta$ as a subspace of $E^\#$. If $E$ is a K(X)–space containing $\varphi(X)$ then

$$\varphi(X') \subset E^\beta \cap E' \subset E' \quad \text{and} \quad E' = \overline{\varphi(X')}^{\sigma(E',E)} \quad \text{thus} \quad E' = \overline{E^\beta \cap E'}^{\sigma(E',E)}.$$

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DEFINITION (see [1]). Let $E$ be a $K(X)$-space containing $\varphi(X)$. $E$ has ***$\beta$-sequentially dense dual*** if $E^{\beta} \cap E'$ is sequentially $\sigma(E', E)$-dense in $E'$.
$E$ has a ***$\varphi$-sequentially dense dual*** if $\varphi(X')$ is sequentially $\sigma(E', E)$-dense in $E'$.
$E$ is called ***$L_{\varphi}$-space***, if $E' \cap \overline{\varphi(X')}^{\sigma(E^{\#},E)} = E'$.
Obviously, any subspace of an $L_{\varphi}$-K(X)-space $(E, \tau)$ containing $\varphi(X)$ is an $L_{\varphi}$-K(X)-space, and $E$ is also an $L_{\varphi}$-space if we replace $\tau$ by any weaker K(X)-topology.

**3.** Let $X_{\alpha}$ $(\alpha \in \mathcal{A})$ be linear spaces. For the product $X := \prod\limits_{\alpha} X_{\alpha}$ one defines for each $\alpha \in \mathcal{A}$ the following linear operators:

$$pr_{\alpha} : X \longrightarrow X_{\alpha}, \; x := (x^{\delta}) := (x^{\delta})_{\delta \in \mathcal{A}} \longrightarrow x^{\alpha},$$

$$g_{\alpha} : X_{\alpha} \longrightarrow X, \; a \longrightarrow x = (x^{\delta}) \;\text{ with }\; x^{\alpha} = a \;\text{ and }\; x^{\delta} = 0 \;\text{ for }\; \delta \neq \alpha.$$

Furthermore, we put $X_0 := \bigoplus\limits_{\alpha} X_{\alpha}$ where $\bigoplus$ marks the direct sum.

For each $\alpha \in \mathcal{A}$ let $\langle X_{\alpha}, Y_{\alpha} \rangle$ be a (total) duality. Then we may identify $Y_{\alpha}$ and $X_{\alpha}$ as a subspace of $X_{\alpha}^{\#}$ and $Y_{\alpha}^{\#}$, respectively. In a natural way $X$ and $Y_0 := \bigoplus\limits_{\alpha} Y_{\alpha}$ are a dual pair $\langle X, Y_0 \rangle$ with the bilinear mapping $\langle\, ,\, \rangle$ defined by

$$\langle x, y \rangle := \sum_{\alpha \in \mathcal{A}_y} \langle x^{\alpha}, y^{\alpha} \rangle$$

whereby on the right $\langle\, ,\, \rangle$ denotes the bilinear mapping of the duality $\langle X_{\alpha}, Y_{\alpha} \rangle$ and $\mathcal{A}_y$ is a finite subset of $\mathcal{A}$ such that $y^{\alpha} = 0$ for each $\alpha \in \mathcal{A} \setminus \mathcal{A}_y$. On the base of the statements

$$\langle pr_{\alpha} x, y^{\alpha} \rangle = \langle x, g_{\alpha} y^{\alpha} \rangle \quad\text{and}\quad \langle a, pr_{\alpha} y \rangle = \langle g_{\alpha} a, y \rangle$$

we get for each $\alpha \in \mathcal{A}$ the continuity of the projections

$$pr_{\alpha} : \big(X, \sigma(X, Y_0)\big) \longrightarrow \big(X_{\alpha}, \sigma(X_{\alpha}, Y_{\alpha})\big),$$

$$pr_{\alpha} : \big(Y_0, \sigma(Y_0, X)\big) \longrightarrow \big(Y_{\alpha}, \sigma(Y_{\alpha}, X_{\alpha})\big),$$

and of the injections

$$g_{\alpha} : \big(X_{\alpha}, \sigma(X_{\alpha}, Y_{\alpha})\big) \longrightarrow \big(X, \sigma(X, Y_0)\big),$$

$$g_{\alpha} : \big(Y_{\alpha}, \sigma(Y_{\alpha}, X_{\alpha})\big) \longrightarrow \big(Y_0, \sigma(Y_0, X)\big).$$

Proving, for example, the continuity of the operator listed at last, we assume $x \in X$ and that $\big(a^{(\gamma)}\big)_{\gamma \in \Gamma}$ is a net in $(Y_{\alpha}, \sigma(Y_{\alpha}, X_{\alpha}))$ converging to $a \in Y_{\alpha}$. Thus we get

$$\langle x, g_{\alpha} a^{(\gamma)} \rangle = \langle pr_{\alpha} x, a^{(\gamma)} \rangle \xrightarrow{\Gamma} \langle pr_{\alpha} x, a \rangle = \langle x, g_{\alpha} a \rangle$$

and therefore $g_\alpha a^{(\gamma)} \xrightarrow{\Gamma} g_\alpha a \ (\sigma(Y_0, X))$ which proves the weak continuity of the injection $g_\alpha : Y_\alpha \longrightarrow Y_0$.

On account of the continuity of the operators listed above, the spaces $\big(X_\alpha, \sigma(X_\alpha, Y_\alpha)\big)$ and $\big(g_\alpha(X_\alpha), \sigma(X, Y_0)|_{g_\alpha(X_\alpha)}\big)$ and the spaces $\big(Y_\alpha, \sigma(Y_\alpha, X_\alpha)\big)$ and $\big(g_\alpha(Y_\alpha), \sigma(Y_0, X)|_{g_\alpha(Y_\alpha)}\big)$ are in each case algebraicly and topologically isomorphic.

**PROPOSITION 1.** *The statement*

$$\overset{\sqcup}{\bigoplus_\alpha N_\alpha}^{\sigma(Y_0,X)} = \bigoplus_\alpha \overset{\sqcup}{N_\alpha}^{\sigma(Y_\alpha,X_\alpha)}$$

*holds for all subspaces $N_\alpha$ of $Y_\alpha$ $(\alpha \in \mathcal{A})$.*

*Proof.* First of all, we prove that $\bigoplus_\alpha \overset{\sqcup}{N_\alpha}$ is sequentially $\sigma(Y_0, X)$–closed.

If $y^{(n)} \longrightarrow y\,(\sigma(Y_0, X)))$ for $y^{(n)} \in \bigoplus_\alpha \overset{\sqcup}{N_\alpha}$ and $y = (y^\alpha) \in Y_0$ then we obtain

$$pr_\alpha y^{(n)} \in \overset{\sqcup}{N_\alpha}, \quad pr_\alpha y^{(n)} \overset{n\to\infty}{\longrightarrow} y^\alpha \ (\sigma(Y_\alpha, X_\alpha)),$$

implying $y^\alpha \in \overset{\sqcup}{N_\alpha}$ $(\alpha \in \mathcal{A})$. Consequently, $y \in \left(\prod_\alpha \overset{\sqcup}{N_\alpha}\right) \cap Y_0 = \bigoplus_\alpha \overset{\sqcup}{N_\alpha}$ after which the sequential $\sigma(Y_0, X)$–closedness of $\bigoplus_\alpha \overset{\sqcup}{N_\alpha}$ is verified. By that and $\bigoplus_\alpha N_\alpha \subset \bigoplus_\alpha \overset{\sqcup}{N_\alpha}$ we get $\overset{\sqcup}{\bigoplus_\alpha N_\alpha} \subset \bigoplus_\alpha \overset{\sqcup}{N_\alpha}$.

To prove the converse inclusion we remark first of all that

$$\overset{\sqcup}{g_\alpha(N_\alpha)}^{\sigma(Y_0,X)} = g_\alpha\left(\overset{\sqcup}{N_\alpha}^{\sigma(Y_\alpha,X_\alpha)}\right) \qquad (*)$$

is valid for each $\alpha \in \mathcal{A}$. Namely, $g_\alpha\left(\overset{\sqcup}{N_\alpha}\right)$ is sequentially $\sigma(Y_0, X)$–closed in $Y_0$ because of the weak continuity of the projection $pr_\alpha : Y_0 \longrightarrow Y_\alpha$; therefore we have $\overset{\sqcup}{g_\alpha(N_\alpha)} \subset g_\alpha\left(\overset{\sqcup}{N_\alpha}\right) \subset g_\alpha(Y_\alpha)$. Since $g_\alpha : (Y_\alpha, \sigma(Y_\alpha, X_\alpha)) \longrightarrow (g_\alpha(Y_\alpha), \sigma(Y_0, X)|_{g_\alpha(Y_\alpha)})$ is an algebraic and topological isomorphism, we have proved $(*)$.

Now, let $y = (y^\alpha) \in \bigoplus_\alpha \overset{\sqcup}{N_\alpha}$. Then we may represent $y$ by $y = \sum_{\alpha \in \mathcal{A}_y} g_\alpha y^\alpha$ whereby $\mathcal{A}_y$ is a suitable finite subset of $\mathcal{A}$. On account of $(*)$ we get

$$g_\alpha y^\alpha \in g_\alpha\left(\overset{\sqcup}{N_\alpha}\right) = \overset{\sqcup}{g_\alpha(N_\alpha)} \subset \overset{\sqcup}{\bigoplus_\alpha N_\alpha}.$$

Since $\overline{\bigoplus_\alpha N_\alpha}$ is a subspace of $\bigoplus_\alpha Y_\alpha$ we obtain $y \in \overline{\bigoplus_\alpha N_\alpha}$ with which the inclusion $\overline{\bigoplus_\alpha N_\alpha} \supset \bigoplus_\alpha \overline{N_\alpha}$ is verified. □

In the following proposition we are dealing with the case $\mathcal{A} := \mathbb{N}$.

PROPOSITION 2. *If* $M_i$ $(i \in \mathbb{N})$ *are subspaces of* $X_i$ *then*

$$\overline{\prod_i M_i}^{\sigma(X,Y_0)} = \prod_i \overline{M_i}^{\sigma(X_i,Y_i)} .$$

*Proof.* The inclusion '$\subset$' may be proved analoguously to the corresponding inclusion in Proposition 1. For a prove of the converse inclusion, first of all, we check the statement

$$\bigoplus_i \overline{M_i} \text{ is sequentially } \sigma(X,Y_0)\text{–dense in } \prod_i \overline{M_i} . \qquad (**)$$

Let $x = (x^i) \in \prod_i \overline{M_i}$. Then $x^{[m]} := (x^1, \ldots, x^m, 0, \ldots)$ $(m \in \mathbb{N})$ are members of the subspace $\bigoplus_i \overline{M_i}$. For any given $y \in Y_0$ there exists an $n \in \mathbb{N}$ such that $y^i = 0$ for $i > n$. Therefore,

$$\langle x^{[m]}, y\rangle = \sum_{i=1}^{\min\{n,m\}} \langle x^i, y^i\rangle \overset{m\to\infty}{\longrightarrow} \sum_{i=1}^{n} \langle x^i, y^i\rangle = \langle x, y\rangle$$

and consequently $x^{[m]} \overset{m\to\infty}{\longrightarrow} x \ \big(\sigma(X,Y_0)\big)$. By it, $(**)$ is proved.

With Proposition 1 we get

$$\bigoplus_i \overline{M_i}^{\sigma(X_i,Y_i)} = \overline{\bigoplus_i M_i}^{\sigma\left(\bigoplus_i X_i, \prod_i Y_i\right)} \subset \overline{\bigoplus_i M_i}^{\sigma(X,Y_0)} \subset \overline{\prod_i M_i}^{\sigma(X,Y_0)} .$$

By it and $(**)$ we get the desired inclusion $\overline{\prod_i M_i} \supset \prod_i \overline{M_i}$. □

**4.** Let $X_\alpha$ $(\alpha \in \mathcal{A})$ be locally convex spaces. We endow the product space $X := \prod_\alpha X_\alpha$ with the product topology, that is, with the coarsest topology such that all projections $pr_\alpha : X \longrightarrow X_\alpha$ $(\alpha \in \mathcal{A})$ are continuous. It is well–known that $X' \cong \bigoplus_\alpha X_\alpha{}'$, that means, $X'$ is algebraicly isomorphic to the direct sum $\bigoplus_\alpha X_\alpha{}'$.

Now, for each $\alpha \in \mathcal{A}$ let $F_\alpha$ be a $K(X_\alpha)$–space containing $\varphi(X_\alpha)$. We put

$$F^* := \prod_\alpha F_\alpha = \left\{\chi^* = (\chi^\alpha)_\alpha = \left((x_k^\alpha)_{k\in\mathbb{N}}\right)_{\alpha\in\mathcal{A}} \;\middle|\; \chi^\alpha := (x_k^\alpha)_k \in F_\alpha\,,\ \alpha\in\mathcal{A}\right\}$$

and endow $F^*$ with the product topology, too. Furthermore, we put $\chi_k := (x_k^\alpha)_\alpha$ for $\chi^* \in F^*$ and

$$F := \left\{\chi := (\chi_k) \;\middle|\; \chi^\alpha \in F_\alpha\,,\ \alpha\in\mathcal{A}\right\}.$$

Obviously, $F$ together with

$$\chi + v := \left((x_k^\alpha)_\alpha + (y_k^\alpha)_\alpha\right)_k, \quad \lambda\chi := \left(\lambda(x_k^\alpha)_\alpha\right)_k \qquad (\chi, v \in F\,,\ \lambda\in\mathbb{K})$$

is a vector space being algebraicly isomorphic to $F^*$. Namely, the mapping

$$J : F \longrightarrow F^*\,,\ \chi = \left((x_k^\alpha)_\alpha\right)_k \longrightarrow \chi^* = \left((x_k^\alpha)_k\right)_\alpha.$$

is a linear bijection. On the linear space $F$ we define a locally convex topology $\tau_F$ by the neighbourhood basis $\left\{J^{-1}(U_\beta) \;\middle|\; \beta\in\mathcal{B}\right\}$ whereby $\left\{U_\beta \;\middle|\; \beta\in\mathcal{B}\right\}$ is a neighbourhood basis of the product topology of $F^*$. Thus, with these topologies $F$ and $F^*$ are topologically isomorphic, too. Among other things we get by it

$$f\in F' \iff \exists\, f^*\in F^{*\prime} : f = f^*\circ J$$

and, since $F^{*\prime} \cong \bigoplus_\alpha F_\alpha{}'$,

$$f\in F' \iff f(\chi) = f^*(\chi^*) = \sum_{\alpha\in\mathcal{A}_f} f_\alpha(\chi^\alpha) \quad \text{for each } \chi\in F$$

where $f_\alpha$ is a continuous linear functional on $F_\alpha$ $(a\in\mathcal{A})$ and $\mathcal{A}_f$ is a suitable finite subset of $\mathcal{A}$ such that $f_\alpha = 0$ $(\alpha\in\mathcal{A}\setminus\mathcal{A}_f)$.
Obviously, $F$ is a sequence space over $X = \prod_\alpha X_\alpha$ containing $\varphi(X)$. We are going to prove that $(F, \tau_F)$ is a K(X)–space. The coordinate mappings

$$\pi_k : F \longrightarrow \prod_\alpha X_\alpha\,,\ \chi = (\chi_i) \longrightarrow \chi_k = (x_k^\alpha)_\alpha \qquad (k\in\mathbb{N})$$

may be represented by $\pi_k = \pi_k^*\circ J$ whereby

$$\pi_k^* : F^* \longrightarrow \prod_\alpha X_\alpha\,,\ \chi^* \longrightarrow \chi_k \qquad (k\in\mathbb{N}).$$

For each $k\in\mathbb{N}$ the mapping $\pi_k^*$, therefore $\pi_k$, is continuous since for each $\alpha\in\mathcal{A}$ the projections $pr_\alpha : F^* \longrightarrow F_\alpha$ and the coordinate mapping $\pi_k^\alpha : F_\alpha \longrightarrow X_\alpha$ are continuous and since in the product space $X$ convergence of nets is equivalent to coordinatewise convergence of nets. □

Now, we consider the $\beta$–Dual

$$F^\beta := \left\{(f_k)\in\omega(X') \;\middle|\; \sum_k f_k(\chi_k) \ \text{ converges for all } \ \chi\in F\right\}$$

of $F$. Each $f_k = (f_k^\alpha)_\alpha \in X' \cong \bigoplus_\alpha X_\alpha{}'$ may be represented by

$$f_k(\chi_k) = \sum_{\alpha \in \mathcal{A}_k} f_k^\alpha(x_k^\alpha)$$

where $f_k^\alpha \in X_\alpha'$ and $\mathcal{A}_k$ is a suitable finite subset of $\mathcal{A}$ with $f_k^\alpha = 0$ $(\alpha \in \mathcal{A} \setminus \mathcal{A}_k)$. Therefore, $(f_k) \in \omega(X')$ is a member of $F^\beta$ if and only if the series $\sum_k \sum_{\alpha \in \mathcal{A}_k} f_k^\alpha(x_k^\alpha)$ converges for all $\chi = \big((x_k^\alpha)_\alpha\big)_k \in F$. Each $(f_k) \in F^\beta$ defines a linear functional

$$\Phi : F \longrightarrow \mathrm{IK},\ \chi \longrightarrow \sum_k f_k(\chi_k),$$

thus we have got the inclusion $\varphi(X') \subset F^\beta \cap F'$. Immediately, we may state

$$\varphi(X') = \bigoplus_\alpha \varphi(X_\alpha{}'). \tag{1}$$

PROPOSITION 3. $\quad F^{\,\beta} \cap F' = \bigoplus_\alpha (F_\alpha{}^\beta \cap F_\alpha{}')$.

*Proof.* If $\Phi \in F^\beta \cap F'$ then we have on one hand

$$\Phi(\chi) = \sum_k \sum_{\alpha \in \mathcal{A}_k} f_k^\alpha(x_k^\alpha) \qquad \text{for each } \chi \in F$$

with $\big((f_k^\alpha)_\alpha\big)_k \in F^\beta$ and a suitably chosen finite subset $\mathcal{A}_k$ of $\mathcal{A}$, and on the other hand

$$\Phi(\chi) = \sum_{\alpha \in \mathcal{A}_\Phi} \psi_\alpha(\chi^\alpha) \qquad \text{for each } \chi \in F \tag{$***$}$$

where $\psi_\alpha \in F_\alpha{}'$ $(\alpha \in \mathcal{A}_\Phi)$ and $\mathcal{A}_\Phi$ is a suitable finite subset of $\mathcal{A}$ with $\psi_\alpha = 0$ $(\alpha \in \mathcal{A} \setminus \mathcal{A}_\Phi)$. For a fixed $\alpha \in \mathcal{A}$ and $\boldsymbol{a} \in F_\alpha$ we consider the sequence $\zeta = (\zeta_k) \in F$ such that $\zeta^* = (\zeta^\delta)_{\delta \in \mathcal{A}} \in F^*$ satisfies $\zeta^\alpha = \boldsymbol{a} \in F_\alpha$ and $\zeta^\delta = 0$ for $\delta \neq \alpha$.
From the representations of $\Phi$ we get

$$\Phi(\zeta) = \psi_\alpha(\boldsymbol{a}) = \sum_k f_k^\alpha(a_k) \qquad (\boldsymbol{a} := (a_k) \in F_\alpha),$$

that is $\psi_\alpha \in F_\alpha{}^\beta$. Using $(***)$ we obtain $\Phi \in \bigoplus_\alpha (F_\alpha{}^\beta \cap F_\alpha{}')$.

Conversely, let $\Phi \in \bigoplus_\alpha (F_\alpha{}^\beta \cap F_\alpha{}')$. Then $\Phi \in F'$ and

$$\Phi(\chi) = \sum_{\alpha \in \mathcal{A}_\Phi} f^\alpha(\chi^\alpha) \quad \text{for each } \chi \in F$$

with $f^\alpha \in F_\alpha{}^\beta \cap F_\alpha{}'$ and suitable finite subset $\mathcal{A}_\Phi$ of $\mathcal{A}$. Since each $f^\alpha \in F_\alpha{}^\beta$ may be represented by

$$f^\alpha(\chi^\alpha) = \sum_k f_k^\alpha(x_k^\alpha) \quad \text{for each } \chi^\alpha \in F_\alpha$$

with $f_k^\alpha \in X_\alpha{}'$ $(k \in \mathbb{N})$ we obtain

$$\Phi(\chi) = \sum_{\alpha \in \mathcal{A}_\Phi} \sum_k f_k^\alpha(x_k^\alpha) = \sum_k \sum_{\alpha \in \mathcal{A}_\Phi} f_k^\alpha(x_k^\alpha) \quad \text{for each } x \in F,$$

thus $\Phi \in F^\beta$. Therefore, we have proved $\Phi \in F^\beta \cap F'$. □

THEOREM 4.

(a) $F$ has $\beta$-sequentially dense dual if and only if this is true for $F_\alpha$ $(\alpha \in \mathcal{A})$.

(b) $F$ has $\varphi$-sequentially dense dual if and only if this coincides for $F_\alpha$ $(\alpha \in \mathcal{A})$.

(c) $F$ is an $L_\varphi$-space if and only if $F_\alpha$ is an $L_\varphi$-space for each $\alpha \in \mathcal{A}$.

*Proof.* (a) First of all, we assume that $F$ has $\beta$-sequentially dense dual, that is, $F^\beta \cap F'$ is sequentially $\sigma(F', F)$-dense in $F'$. Let $\alpha \in \mathcal{A}$ be fixed and let $\psi_\alpha \in F_\alpha{}'$, then $g_\alpha \psi_\alpha \in \bigoplus_\alpha F_\alpha{}'$; thus we may choose a sequence $(\Phi^{(n)})$ in $F^\beta \cap F'$ such that $\Phi^{(n)} \longrightarrow g_\alpha \psi_\alpha \ (\sigma(F', F))$. On account of the continuity of the projection $pr_\alpha : \bigoplus_\alpha F_\alpha{}' \longrightarrow F_\alpha{}'$ we get $pr_\alpha \Phi^{(n)} \longrightarrow \psi_\alpha \ (\sigma(F_\alpha{}', F_\alpha))$ and therefore – since $pr_\alpha \Phi^{(n)} \in F_\alpha{}^\beta \cap F_\alpha{}'$ (see Proposition 3) – that $F_\alpha{}^\beta \cap F_\alpha{}'$ is sequentially $\sigma(F_\alpha{}', F_\alpha)$-dense in $F_\alpha{}'$. Thus $F_\alpha$ has $\beta$-sequentially dense dual.

Conversely, for each $\alpha \in \mathcal{A}$ let $F_\alpha{}^\beta \cap F_\alpha{}'$ be sequentially $\sigma(F_\alpha{}', F_\alpha)$-dense in $F_\alpha{}'$. Furthermore, let $f \in F'$, that is

$$f(\chi) = \sum_{\alpha \in \mathcal{A}_f} f_\alpha(\chi^\alpha) \qquad \text{for each } \chi \in F$$

with suitably chosen $f_\alpha \in F_\alpha{}'$ and finite set $\mathcal{A}_f$. For each $\alpha \in \mathcal{A}_f$ we may choose a sequence $\left(\psi_\alpha^{(n)}\right)$ in $F_\alpha{}^\beta \cap F_\alpha{}'$ such that $\psi_\alpha^{(n)} \overset{n\to\infty}{\longrightarrow} f_\alpha \ (\sigma(F_\alpha{}', F_\alpha))$. Furthermore, we define $(\Phi^{(n)})$ by

$$\Phi^{(n)} := \sum_{\alpha \in \mathcal{A}_f} g_\alpha \psi_\alpha^{(n)} \in \bigoplus_\alpha (F_\alpha{}^\beta \cap F_\alpha{}').$$

Obviously, $\Phi^{(n)} \longrightarrow f \ (\sigma(F', F))$. This proves $F^\beta \cap F'$ to be sequentially $\sigma(F', F)$-dense in $F'$.

(b) The proof of (b) is based on (1) and runs up quite similar as the proof of (a).

(c) We assume $F$ to be an $L_\varphi$-space and we fix $\alpha \in \mathcal{A}$.
Let $N_\alpha$ be a subspace of $F_\alpha{}^\#$ containing $\varphi(X_\alpha{}')$ and being sequentially $\sigma(F_\alpha{}^\#, F_\alpha)$-closed in $F_\alpha{}^\#$, and let $N := \bigoplus_{\delta \in \mathcal{A}} N_\delta$ where $N_\delta := \overline{\varphi(X_\alpha{}')}^{\sigma(F_\alpha{}^\#, F_\alpha)}$ for $\delta \neq \alpha$.
From Proposition 1 we obtain



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$$\overline{N}^{\sigma\left(\bigoplus_\alpha F_\alpha{}^\#,F\right)} = \overline{\bigoplus_\delta N_\delta}^{\sigma\left(\bigoplus_\alpha F_\alpha{}^\#,F\right)} = \bigoplus_\delta \overline{N_\delta}^{\sigma(F_\alpha{}^\#,F)} = N\,;$$

therefore, $N$ is sequentially $\sigma\left(\bigoplus_\alpha F_\alpha{}^\#, F\right)$–closed and because of $\bigoplus_\alpha F_\alpha{}^\# \subset F^\#$ also sequentially $\sigma(F^\#,F)$–closed. Since $F$ is an $L_\varphi$–space we get from $N \supset \bigoplus_\alpha \varphi(X_\alpha{}')$ the inclusion $N \supset F'$ which implies $N_\alpha \supset F_\alpha{}'$. Thus, we have proved the inclusion $\overline{\varphi(X_\alpha{}')}^{\sigma(F_\alpha{}^\#,F_\alpha)} \supset F_\alpha{}'$, that is, $F_\alpha$ is an $L_\varphi$–space.

Conversely, for each $\alpha \in \mathcal{A}$ let $F_\alpha$ be an $L\varphi$–space. With Proposition 1 we get

$$\overline{\bigoplus_\alpha \varphi(X_\alpha{}')}^{\sigma\left(\bigoplus_\alpha F_\alpha{}^\#,F\right)} = \bigoplus_\alpha \overline{\varphi(X_\alpha{}')}^{\sigma(F_\alpha{}^\#,F_\alpha)}$$

and on account of $\overline{\varphi(X_\alpha{}')}^{\sigma(F_\alpha{}^\#,F_\alpha)} \supset F_\alpha{}' \quad (\alpha \in \mathcal{A})$ we obtain

$$F' = \bigoplus_\alpha F_\alpha{}' \subset \bigoplus_\alpha \overline{\varphi(X_\alpha{}')} = \overline{\bigoplus_\alpha \varphi(X_\alpha{}')}^{\sigma\left(\bigoplus_\alpha F_\alpha{}^\#,F\right)}$$

implying (see (1))

$$F' \subset \overline{\bigoplus_\alpha \varphi(X_\alpha{}')}^{\sigma(F^\#,F)} = \overline{\varphi(X')}^{\sigma(F^\#,F)}$$

Thus $F$ is an $L_\varphi$–space.

**5.** For each $\alpha \in X_\alpha$ let $X_\alpha$ be a locally convex space.
We endow the direct sum $X_0 := \bigoplus_\alpha X_\alpha$ with the sum topology, that is the strongest locally convex topology such that each injection $g_\alpha : X_\alpha \longrightarrow X_0$ is continuous. If $\{U_\beta^\alpha \mid \beta \in \mathcal{B}\}$ denotes a neighbourhood basis of $X_\alpha$ for $\alpha \in \mathcal{A}$, then the absolutely convex hulls

$$\Gamma_\alpha\, g_\alpha(U_\beta^\alpha) := \left\{ \sum_\alpha \lambda_\alpha g_\alpha a^\alpha \;\Big|\; \sum_\alpha |\lambda_\alpha| \le 1,\ a^\alpha \in U_\beta^\alpha \right\}$$

of $\bigcup_{\alpha\in\mathcal{A}} g_\alpha(U_\beta^\alpha)$ for $\beta \in \mathcal{B}$ form a zero neighbourhood basis of the sum topology (see [2], 18.5(1)). As everybody knows, $X_0{}^\#$ and $\prod_\alpha X_\alpha{}^\#$ as well as $X_0{}'$ and $\prod_\alpha X_\alpha{}'$ are algebraicly isomorphic.

Now, let $F_\alpha$ be K($X_\alpha$)–spaces containing $\varphi(X_\alpha)$. We put

$$F_0^* := \bigoplus_\alpha F_\alpha = \left\{ \chi^* \in F^* \;\Big|\; \chi^\alpha \ne 0 \quad \text{for (at most) finite many} \quad \alpha \in \mathcal{A} \right\}$$

and

$$F_0 := J^{-1}(F_0^*)$$

where $J : F \longrightarrow F^*$ is defined as in section 4. Consequently, $F_0$ is a subspace of $F$, and $\chi := \left((x_k^\alpha)_\alpha\right)_k \in F$ is a member of $F_0$, if and only if there exists a finite subset $\mathcal{A}_\chi$ of $\mathcal{A}$ such that $x_k^\alpha = 0$ for each $k \in \mathrm{IN}$ and $\alpha \in \mathcal{A} \setminus \mathcal{A}_\chi$. A simple proof shows $\varphi(X_0) \subset F_0 \subset \omega(X_0)$. We endow $F_0^*$ with the sum topology and define a locally convex topology on $F_0$ by the zero neighbourhood basis $\{J^{-1}(U_\beta) \mid \beta \in \mathcal{B}\}$ whereby $\left\{U_\beta \mid \beta \in \mathcal{B}\right\}$ is a zero neighbourhood basis of the sum topology on $F_0^*$. Therefore, $F_0$ and $F_0^*$ are algebraicly and topologically isomorphic, which implies

$$\begin{aligned} f \in F_0{}' \quad &\Longleftrightarrow \quad \exists\, f^* \in F_0^{*\prime} : f = f^* \circ J \\ &\Longleftrightarrow \quad f(\chi) = \sum_{\alpha \in \mathcal{A}_\chi} f_\alpha(\chi^\alpha) \qquad (\chi \in F_0) \end{aligned}$$

where $f_\alpha \in F_\alpha{}'$.

PROPOSITION 5. $F_0$ is a $K(X_0)$-space.

*Proof.* The coordinate mappings $\pi_k : F_0 \longrightarrow X_0$ may be represented by $\pi_k = \pi_k^* \circ J$ whereby

$$\pi_k^* : F_0^* \longrightarrow X_0\,,\ \chi^* \longrightarrow \chi_k \qquad (k \in \mathrm{IN}).$$

We have to prove the continuity of $\pi_k^*$ $(k \in \mathrm{IN})$.
For that we fix $k \in \mathrm{IN}$ and assume $U$ to be a zero neighbourhood in $X_0$ with respect to the sum topology which has the form $U := \Gamma_\alpha g_\alpha(U_\alpha)$, that is

$$U = \left\{ \sum_\alpha \lambda_\alpha g_\alpha x_k^\alpha \;\middle|\; \sum_\alpha |\lambda_\alpha| \leq 1\,,\ x_k^\alpha \in U_\alpha \right\},$$

where $U_\alpha$ is a zero neighbourhood in $X_\alpha$. Furthermore, we put $V := \Gamma_\alpha g_\alpha(V_\alpha)$, where $V_\alpha := \prod_i U_\alpha^i$; $U_\alpha^k := U_\alpha$ and $U_\alpha^i = X_\alpha$ for all $i \neq k$. Since $F_\alpha$ is a $K(X_\alpha)$-space $V_\alpha$ is a zero neighbourhood in $F_\alpha$. Thus, $V$ is a zero neighbourhood in $F^*$. Therefore, the statement of Proposition 5 is proved if $\pi_k^*(V) \subset U$ holds.
Each $\chi^* \in V$ may be represented by $\chi^* = \sum_\alpha \lambda_\alpha g_\alpha \chi^\alpha$ with $\sum_\alpha |\lambda_\alpha| \leq 1$ and $\chi^\alpha = (x_i^\alpha)_i \in V_\alpha$. Consequently, we get $x_i^\alpha \in U_\alpha^i$, thus $x_k^\alpha \in U_\alpha$. This implies

$$\pi_k^* \chi^* = \sum_\alpha \lambda_\alpha g_\alpha x_k^\alpha \in U$$

for each $\chi^* \in V$. □

For the following examinations, we consider the $\beta$-dual

$$F_0{}^\beta := \left\{ (f_k) \in \omega(X_0') \mid \sum_k f_k(\chi_k) \text{ converges for each } \chi \in F_0 \right\}$$

of $F_0$ where $f_k \in X_0{}'$ has a representation of the type.

$$f_k(\chi_k) = \sum_{\alpha \in \mathcal{A}_\chi} f_k^\alpha(x_k^\alpha) \qquad (\chi = (\chi_k) \in F_0,\, k \in \mathbb{N})$$

with $f_k^\alpha \in X_\alpha{}'$ and a suitable finite subset $\mathcal{A}_\chi$ of $\mathcal{A}$. Obviously, we have

$$\varphi(X_0{}') \subset F_0{}^\beta$$

and each $(f_k) \in F_0{}^\beta$ defines a linear operator

$$\Phi : F_0 \longrightarrow \mathbb{K},\, \chi \longrightarrow \sum_k f_k(\chi_k).$$

As one may check immediately, we have

$$\varphi(X_0{}') = \prod_\alpha \varphi(X_\alpha{}'). \tag{2}$$

PROPOSITION 6.

(a) $F_0{}^\beta = \prod_\alpha F_\alpha{}^\beta.$

(b) $F_0{}^\beta \cap F_0{}' = \prod_\alpha (F_\alpha{}^\beta \cap F_\alpha{}').$

*Proof.* (a) Because of

$$\sum_k f_k(\chi_k) = \sum_{\alpha \in \mathcal{A}_\chi} \sum_k f_k^\alpha(x_k^\alpha) \qquad (\chi = (\chi_k) \in F_0)$$

we get $\Phi := (f_k) \in F_0{}^\beta$ if and only if the series $\sum_k f_k^\alpha(x_k^\alpha)$ converges for each $\chi^\alpha = (x_k^\alpha) \in F_\alpha$ and $\alpha \in \mathcal{A}$. Since thereby $\Phi$ may be represented by

$$\Phi(\chi) = \sum_{\alpha \in \mathcal{A}_\chi} f^\alpha(\chi^\alpha) \qquad (\chi^* = (\chi^\alpha) \in F_0^*, \text{ that is } \chi \in F_0)$$

with

$$f^\alpha(\chi^\alpha) := \sum_k f_k^\alpha(x_k^\alpha)$$

we obtain

$$\Phi \in F_0{}^\beta \iff \Phi \in \prod_\alpha F_\alpha{}^\beta.$$

The statement (b) follows from (a) and the identity $F_0{}' \cong \prod_\alpha F_\alpha{}'$ :

$$F_0{}^\beta \cap F_0{}' = \left(\prod_\alpha F_\alpha{}^\beta\right) \cap \left(\prod_\alpha F_\alpha{}'\right) = \prod_\alpha (F_\alpha{}^\beta \cap F_\alpha{}'). \qquad \square$$

THEOREM 7.

**(a)** $F_0$ has $\beta$-sequentially dense dual if and only if this is true for $F_\alpha$ $(\alpha \in \mathcal{A})$.

**(b)** $F_0$ has $\varphi$-sequentially dense dual if and only if this coincides for $F_\alpha$ $(\alpha \in \mathcal{A})$.

**(c)** In case of $\mathcal{A} = \mathbb{N}$ the space $F_0$ is an $L_\varphi$-space if and only if $F_\alpha$ is an $L_\varphi$-space for each $\alpha \in \mathbb{N}$.

*Proof.* **(a)** First of all, we assume that $F_0$ has $\beta$-sequential dense dual, that is, ${F_0}^\beta \cap {F_0}'$ is sequentially $\sigma({F_0}', F_0)$-dense in ${F_0}'$. For a fixed $\alpha \in \mathcal{A}$ let $\psi_\alpha \in {F_\alpha}'$. Then $g_\alpha \psi_\alpha \in {F_0}'$ and therefore there exists a sequence $(\Phi^{(n)})$ in ${F_0}^\beta \cap {F_0}'$ such that $\Phi^{(n)} \overset{n\to\infty}{\longrightarrow} g_\alpha\psi_\alpha$ $(\sigma({F_0}', F_0))$. On account of the weak continuity of the projection $pr_\alpha : \prod_\alpha {F_\alpha}' \longrightarrow {F_\alpha}'$ we get $pr_\alpha \Phi^{(n)} \overset{n\to\infty}{\longrightarrow} \psi_\alpha$ $(\sigma({F_\alpha}', F_\alpha))$ and since $pr_\alpha\Phi^{(n)} \in {F_\alpha}^\beta \cap {F_\alpha}'$ according to Proposition 6(b), we have established that ${F_\alpha}^\beta \cap {F_\alpha}'$ is sequentially $\sigma({F_\alpha}', F_\alpha)$-dense in ${F_\alpha}'$. Thus, $F_\alpha$ has $\beta$-sequential dense dual.

Conversely, for each $\alpha \in \mathcal{A}$ let ${F_\alpha}^\beta \cap {F_\alpha}'$ be sequential $\sigma({F_\alpha}', F_\alpha)$-dense in ${F_\alpha}'$ and let $f \in {F_0}'$, that is

$$f(x) = \sum_{\alpha \in \mathcal{A}_x} f_\alpha(x^\alpha) \qquad (x^* = (x^\alpha) \in F_0^*, \text{ that is } x \in F_0)$$

with $f_\alpha \in {F_\alpha}'$. For each $\alpha \in \mathcal{A}$ we choose $(\psi_\alpha^{(n)})$ in ${F_\alpha}^\beta \cap {F_\alpha}'$ such that $\psi_\alpha^{(n)} \overset{n\to\infty}{\longrightarrow} f_\alpha$ $(\sigma({F_\alpha}', F_\alpha))$. Then for each $x \in F_0$ we obtain

$$\Phi^{(n)}(x) := \sum_{\alpha \in \mathcal{A}_x} \psi_\alpha^{(n)}(x^\alpha) \overset{n\to\infty}{\longrightarrow} \sum_{\alpha \in \mathcal{A}_x} f_\alpha(x^\alpha) = f(x).$$

Because of $\Phi^{(n)} = (\psi_\alpha^{(n)})_\alpha \in {F_0}^\beta \cap {F_0}' = \prod_\alpha ({F_\alpha}^\beta \cap {F_\alpha}')$ (see Proposition 6(b)) we have established that ${F_0}^\beta \cap {F_0}'$ is sequentially $\sigma({F_0}', F_0)$-dense in ${F_0}'$.

**(b)** The proof of statement (b) runs up quite similar to the proof of (a), and therefore we omit it.

**(c)** First of all , we assume that $F_0$ is a $L_\varphi$-space and we fix a $k \in \mathbb{N}$. Furthermore let $N_k$ be a sequentially $\sigma({F_k}^\#, F_k)$-closed subspace of ${F_k}^\#$ containing $\varphi({X_k}')$. Furthermore, we put $N_i := \overline{\varphi({X_i}')}^{\sigma({F_i}^\#, F_i)}$ for $i \neq k$ $(i \in \mathbb{N})$ and define $N := \prod_i N_i$. According to Proposition 2 we get

$$\overline{N}^{\sigma({F_0}^\#, F_0)} = \prod_i \overline{N_i}^{\sigma({F_i}^\#, F_i)} = \prod_i N_i = N;$$

thus $N$ is sequentially $\sigma(F_0{}^{\#}, F_0)$-closed. Since $F_0$ is a $L_\varphi$-space we obtain $N \supset F_0{}'$ from the inclusion $N \supset \prod_i \varphi(X_i{}')$. By that and (2),

$$N_k = \underline{pr_k(N)} \supset pr_k(F_0{}') = F_k{}',$$

therefore $\varphi(X_k{}') \supset F_k{}'$. This proves $F_k$ to be an $L_\varphi$-space ($k \in \mathbb{N}$).

Conversely, for each $i \in \mathbb{N}$ let $F_i$ be an $L_\varphi$-space. According to Proposition 2 we get

$$\overline{\prod_i \varphi(X_i{}')}^{\,\sigma(F_0{}^{\#}, F_0)} = \prod_i \overline{\varphi(X_i{}')}^{\,\sigma(F_\alpha{}^{\#}, F_\alpha)} \supset \prod_i F_i{}' = F_0{}'.$$

Thus $F_0$ is an $L_\varphi$-space. □

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## $L_\varphi$–K(X)–ruumide ning nendega sarnaste K(X)–ruumide otsekorrutis ja otsesumma

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Resümee

Käesolevas artiklis vaadeldakse lokaalselt kumeraid jadaruume $F$, mille elementideks on mingi lokaalselt kumera ruumi $X$ elementide jadad $x = (x_k)$. Sellist ruumi nimetatakse K(X)–ruumiks, kui koordinaatoperaatorid $\pi_k : F \to X$, $x \to x_k$ $(k \in \mathbb{N})$ on pidevad. Sel juhul sisaldab ruumi $F$ (topoloogiline) kaasruum $F'$ ruumi $X$ kaasruumi $X'$ kõigi lõplike jadade hulga $\varphi(X')$. Olgu $\underline{\varphi(X')}$ ruumi $F$ algebralise kaasruumi $F^{\#}$ kõigi jadaliselt $\sigma(F^{\#}, F)$-kinniste, hulka $\varphi(X')$ sisaldavate alamruumide ühisosa. K(X)–ruumi $F$ nimetatakse $L_\varphi$–ruumiks, kui $\underline{\varphi(X')} \cap F' = F'$. Tõestatakse (teoreemid 4 ja 7), et suvalise arvu $L_\varphi$–ruumide otsekorrutis ning loenduva arvu $L_\varphi$–ruumide otsesumma on $L_\varphi$–ruumid. Analoogilised väited kehtivad ka selliste K(X)–ruumide $F$ korral, kus vastavalt $\varphi(X')$ ja $\beta$-kaasruum $F^\beta$ on jadaliselt $\sigma(F', F)$-tihedad.

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# THE STATISTICAL CONVERGENCE IN BANACH SPACES

Enno Kolk

**Introduction.** The notion of statistical convergence was introduced by Fast [3] and has been investigated in the papers [1,5,7,10,11]. Following Freedman and Sember [4] the author [7], taking in Fast's definition of statistically convergent sequence and in Fridy's definition of statistically Cauchy sequence an arbitrary non-negative regular matrix A instead of Cesaro matrix $C_1$, introduced the notions of A-statistically convergent and A-statistically Cauchy sequences in normed spaces. Independently Maddox [10] introduced the statistical convergence in locally convex spaces.

In Section 2 of this paper it is proved that in a Banach space X the sets of A-statistically convergent and A-statistically Cauchy sequences coincide. It is also shown that a sequence $(x_k)$, $x_k \in X$, converges A-statistically to $x_o \in X$ if and only if there exists an infinite index set $\{k_i\}$ with the A-density 1 such that the subsequence $(x_{k_i})$ converges to $x_o$. These results were presented in [7] and they generalize corresponding results of Fridy [5] and Šalat [11] about number sequences (in the case $A = C_1$).

In Section 3 the relations between A-statistical convergence and strong A-summability defined by a sequence of moduli are studied. Our results extend some results of Connor ([1], Theorem 2.1, Corollary 2.2) and of Maddox ([10], Theorems 1 and 2).

**1. Notation and preliminaries.** Let X be a Banach space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. By s(X), m(X) and c(X) we denote the vector *spaces* of all X-valued sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$, of all bounded sequences in X and of all convergent sequences in X, respectively. In the case $X = \mathbb{K}$



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we write s, m and c instead of s(X), m(X) and c(X).

Let $\lambda(X)$ and $\mu(X)$ be two subsets of s(X) and $A = (a_{nk})$ an infinite matrix with $a_{nk} \in \mathbb{K}$. If for each $x = (x_k) \in \lambda(X)$ the series

$$A_n x = \sum_k a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N})$$

converge in X and the sequence

$$Ax = (A_n x)$$

belongs to $\mu(X)$ then we say that A maps $\lambda(X)$ into $\mu(X)$ and write $A: \lambda(X) \to \mu(X)$.

A matrix A (or matrix map A) is called regular on s(X) if $A: c(X) \to c(X)$ and $\lim_n A_n x = \lim_k x_k$ in X for all $x \in c(X)$. It is known that a matrix A is regular on s(X) if and only if it is regular on s. The well-known Silvermann--Toepliz's theorem asserts that $A = (a_{nk})$ is regular on s if and only if (see [2], Theorem 4.1, II)

(R1) $\lim_n a_{nk} = 0 \quad (k \in \mathbb{N})$,

(R2) $\lim_n \sum_k a_{nk} = 1$,

(R3) $\sup_n \sum_k |a_{nk}| < \infty$.

A matrix A is called uniformly regular if it satisfies the conditions (R2), (R3) and

(R4) $\lim_n \sup_k |a_{nk}| = 0$.

We denote by $\mathcal{T}$ and $\mathcal{UT}$, respectively, the sets of all regular matrices and all uniformly regular matrices. We use also the notation

$$\mathcal{T}^+ = \{A \in \mathcal{T}: a_{nk} \geq 0\},$$

$$\mathcal{UT}^+ = \mathcal{UT} \cap \mathcal{T}^+.$$

For example, the Cesaro matrix $C_1$, defined by

$$a_{nk} = \begin{cases} 1/n & \text{if } k \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

is uniformly regular and non-negative, so $C_1 \in \mathcal{UT}^+$.

A sequence $x = (x_k) \in s(X)$ is said to be strongly A-summable with index $p > 0$ to $x_0 \in X$ if (cf. [9])

$$\lim_n \sum_k a_{nk} \| x_k - x_0 \|^p = 0.$$

The set of all strongly A-summable sequences in X is denoted by $w_A^p(X)$. We write $w_A^p = w_A^p(\mathbb{K})$, $w^p(X) = w_{C_1}^p(X)$ and $w^p = w^p(\mathbb{K})$.

We recall that the modulus $f$ is a function $f\colon [0,\infty) \to [0,\infty)$ such that (see [8])

(i) $f(t) = 0$ if and only if $t = 0$;

(ii) $f(t + u) \le f(t) + f(u)$ for all $t \ge 0$, $u \ge 0$;

(iii) $f$ is increasing;

(iv) $f$ is continuous from the right at 0.

Maddox [8,10] used the modulus $f$ to construct the sequence space

$$w^p(f) = \{x \in s\colon \lim_n n^{-1}\sum_k [f(|x_k - x_0|)]^p = 0 \text{ for some } x_0\}.$$

In [6] this idea was generalized by taking in place of one modulus a sequence of moduli $F = (f_k)$. Here we consider together with $w_A^p(X)$ a more general space

$$w_A^p(F,X) = \{x \in s(X)\colon \lim_n \sum_k a_{nk}[f_k(\|x_k - x_0\|)]^p = 0 \text{ for some } x_0 \in X\}.$$

In the case where $x \in w_A^p(F,X)$ we write $w_A^p(F)\text{-}\lim x_k = x_0$. We write also $w_A^p(f,X)$ instead of $w_A^p(F,X)$ in the case where $f_k = f$ $(k \in \mathbb{N})$. Thus $w^p(f) = w_{C_1}^p(f,\mathbb{K})$ and $w^p = w^p(f)$ for $f(t) = t$.

In the following we mean by an index set a finite or infinite subset $\{k_i\}$ of $\mathbb{N}$ with $k_i < k_{i+1}$. Thus an infinite index set $\{k_i\}$ is precisely the sequence $(k_i)$ of indices. The set of all $k \in K$ with $k \le n$ is denoted by $K(\le n)$.

Let $K = \{k_i\}$ be an index set and let $\phi^K$ be the characteristic sequence of $K$, i.e. $\phi^K = (\phi_j^K)$, where

$$\phi_j^K = \begin{cases} 1 & \text{if } j = k_i \quad (i \in \mathbb{N}), \\ 0 & \text{otherwise}. \end{cases}$$

If $\phi^K$ is $C_1$-summable then the limit

$$\lim_n n^{-1}\sum_{j=1}^n \phi_j^K$$

is called the asymptotic density of $K$ and is denoted by $\delta(K)$.

For a non-negative regular matrix $A$, following Freedman and Sember [4], an index set $K = \{k_i\}$ will be said to have A-density

$$\delta_A(K) = \lim_n A_n\phi_j^K$$

when $A\phi^K \in c$. Thus

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk} = \lim_n \sum_i a_{n,k_i}.$$

Brudno (see [2], p. 155-156) proved the following result: for every $A \in \mathcal{T}^+$ there exists a normal matrix $A' \in \mathcal{T}^+$ such that $c_{A'} \cap m = c_A \cap m$ and

$$\lim_n A'_n x = \lim_n A_n x \qquad (x \in c_A \cap m),$$

where $c_A = \{x \in s\colon Ax \in c\}$ (a matrix $A = (a_{nk})$ is called, normal, if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$). By Brudno's theorem we may assume that the matrix A in the definition of A-density is normal.

In [3] the definition of statistical convergence was given: a sequence $x = (x_k) \in s$ is said to be statistically convergent to a number $x_0$ if $\delta(K_\varepsilon) = 0$ for every $\varepsilon > 0$, where

$$K_\varepsilon = \{k\colon |x_k - x_0| \geq \varepsilon\}.$$

The notion of statistically Cauchy sequence was introduced by Fridy [5]: a sequence $x = (x_k) \in s$ is said to be statistically Cauchy if for every $\varepsilon > 0$ there exists an index $n(\varepsilon)$ such that $\delta(K_{n(\varepsilon)}) = 0$, where

$$K_{n(\varepsilon)} = \{k\colon |x_k - x_{n(\varepsilon)}| \geq \varepsilon\}.$$

If we take here A-density instead of asymptotic density then we arrive to the following definitions [7]. A sequence $x = (x_k) \in s(X)$ is said to be A-statistically convergent to $x_0$, briefly $st_A(X)$-$\lim x_k = x_0$, if $\delta_A(L_\varepsilon) = 0$ for every $\varepsilon > 0$, where

$$L_\varepsilon = \{k\colon \| x_k - x_0 \| \geq \varepsilon\}.$$

A sequence $x = (x_k) \in s(X)$ is said to be A-statistically Cauchy if for every $\varepsilon > 0$ there exists an index $n(\varepsilon)$ such that $\delta_A(L_{n(\varepsilon)}) = 0$, where

$$L_{n(\varepsilon)} = \{k\colon |x_k - x_{n(\varepsilon)}| \geq \varepsilon\}.$$

By the symbol $st_A(X)$ we denote the set of all A-statistically convergent sequences in X and by $st^0(X)$ the set of all A-statistically null sequences in X.

It should be noted that A-statistical convergence is defined only for $A \in \mathcal{T}^+$. It is clear that $c(X) \subset st_A(X)$. A theorem of Agnew (see [2], Theorem 8.5, III) shows that for all $A \in \mathcal{U}\mathcal{T}^+$ the inclusion $c(X) \subset st_A(X)$ is strict.

**2. A-statistically convergent and A-statistically Cauchy sequences in Banach space.** Modifying the arguments of Šalat [11], Lemma 1.1, we prove a useful lemma.

**LEMMA 2.1** *Let* $(y_{k,j})$ *be a double sequence in a Banach space* X. *The following two statements are equivalent:*

(i) *for every* $\varepsilon > 0$ *there exists an index* $n(\varepsilon)$ *such that*

$$\delta_A(\{k: \|y_{k,n(\varepsilon)}\| < \varepsilon\}) = 1;$$

(ii) *there is an infinite index set* $K = \{k_i\}$ *such that* $\delta_A(K) = 1$ *and for every* $\varepsilon > 0$ *there exist indices* $l(\varepsilon)$ *and* $k_0 = k_0(\varepsilon)$ *such that*

$$\|y_{k,l(\varepsilon)}\| < \varepsilon \qquad (k \in K,\ k \geq k_0).$$

*Proof*. If (ii) holds then

$$K_0 = \{k \in K: k \geq k_0\} \subset \{k: \|y_{k,l(\varepsilon)}\| < \varepsilon\}$$

and by $\delta_A(K_0) = 1$ we have

$$\delta_A(\{k: \|y_{k,l(\varepsilon)}\| < \varepsilon\}) = 1.$$

Thus (ii) implies (i) with $n(\varepsilon) = l(\varepsilon)$.

Next suppose that (i) is true. Then $\delta_A(K_m) = 1$ $(m \in \mathbb{N})$ where

$$K_m = \{k: \|y_{k,n(1/m)}\| < \frac{1}{m}\}.$$

If we define

$$S_j = \bigcap_{m=1}^{j} K_m \qquad (j \in \mathbb{N})$$

then $S_1 \supset S_2 \supset \ldots \supset S_j \supset \ldots$ and $\delta_A(S_j) = 1$ $(j \in \mathbb{N})$, i.e.

$$\lim_n A_n(S_j) = 1, \qquad (2.1)$$

where

$$A_n(S_j) = \sum_{k \in S_j} a_{nk}.$$

Let us choose an arbitrary number $v_1 \in S_1$ with $v_1 > 1$. In view of (2.1) there exists a number $v_2 \in S_2$ such that $v_2 > v_1$ and

$$A_n(S_2) > \frac{1}{2} \qquad (n \geq v_2).$$

Further, again by (2.1), there exists a number $v_3 \in S_3$ with $v_3 > v_2$ and

$$A_n(S_3) > \frac{2}{3} \qquad (n > v_3)$$

a.s.o. Thus we can construct by induction a sequence $(v_j)$ of indices such that $v_j \in S_j$, $v_j < v_{j+1}$ and

$$A_n(S_j) > \frac{j-1}{j} \qquad (n \geq v_j,\ j \in \mathbb{N}). \tag{2.2}$$

Now we define

$$K = \bigcup_{j=0}^{\infty} (S_j \cap [v_j, v_{j+1})),$$

where $v_0 = 1$ and $S_0 = \mathbb{N}$. Then for $v_j \leq n \leq v_{j+1}$ we have $K(\leq n) \supset S_j(\leq n)$. So by (2.2) we get

$$A_n(K) = \sum_{k \in K} a_{nk} > \sum_{k \in S_j} a_{nk} = A_n(S_j) > \frac{j-1}{j},$$

from which it follows that $\delta_A(K) = 1$.

Let $\varepsilon > 0$ and choose a number $j_0$ with $1/j_0 \leq \varepsilon$. If $k_0$ is the least element in $S_{j_0} \cap [v_{j_0}, v_{j_0+1})$ then by

$$S_j \subset S_{j_0} \subset K_{j_0} \qquad (j \geq j_0)$$

we have

$$\| y_{k,n(1/j_0)} \| < \frac{1}{j_0} \leq \varepsilon \qquad (k \in K,\ k \geq k_0).$$

Thus (ii) holds with $l(\varepsilon) = n(1/j_0)$. The lemma is proved.

Let $x = (x_k)$ be a sequence in a Banach space $X$. For $y_{k,j} = x_k - x_j$ the statement (i) of Lemma 2.1 means that $x$ is a A-statistically Cauchy sequence. At that time the equivalent statement (ii) means that $x$ contains a Cauchy subsequence $(x_{k_j})$ with $\delta_A(\{k_j\}) = 1$. By completeness of $X$ subsequence $(x_{k_j})$ must converge to an element $x_0 \in X$. The same meaning has (ii) for $y_{k,j} = x_k - x_0$. But (i) states in this case that $x$ is A-statistically convergent to $x_0$. Hence we have proved the following results.

**THEOREM 2.2.** *In a Banach space $X$ the sets of A-statistically convergent and A-statistically Cauchy sequences coincides.*

**THEOREM 2.3.** *The sequence $x = (x_k)$ converges A-statistically to $x_0$ in a Banach space $X$ if and only if* *there exists an infinite index set* $K = \{k_j\}$ *with* $\delta_A(K) = 1$ *such that the subsequence* $(x_{k_j})$ *converges to* $x_o$.

**3. A-statistical convergence and strong A-summability.** In this section we investigate the relations between A-statistical convergence and strong A-summability defined by a sequence $F = (f_k)$ of moduli.

**THEOREM 3.1.** *Let* X *be a Banach space and* $F = (f_k)$ *a sequence of moduli. Then*

$$w_A^p(F)\text{-}\lim x_k = x_o \Rightarrow st_A\text{-}\lim x_k = x_o \qquad (p > 0,\ A \in \mathcal{T}^+) \qquad (3.1)$$

*if and only if*

$$(F1) \qquad \inf_k f_k(t) > 0 \qquad (t > 0).$$

*Proof.* If (F1) holds then there exists a number $s_o > 0$ such that

$$f_k(t) \geq s_o \qquad (t > 0,\ k \in \mathbb{N}).$$

Let $\varepsilon > 0$. If $w_A^p(F)\text{-}\lim x_k = x_o$ and $L_\varepsilon = \{k\colon \|x_k - x_o\| \geq \varepsilon\}$ then

$$\sigma_n = \sum_k a_{nk}[f_k(\|x_k - x_o\|)]^p \geq s_o^p \sum_{k \in L_\varepsilon} a_{nk},$$

whence

$$\sum_{k \in L_\varepsilon} a_{nk} \leq s_o^{-p}\sigma_n \qquad (n \in \mathbb{N}).$$

So by $\lim_n \sigma_n = 0$ we get

$$\delta_A(L_\varepsilon) = \sum_{k \in L_\varepsilon} a_{nk} = 0.$$

Thus $st_A\text{-}\lim x_k = x_o$ and the sufficiency of (F1) is proved.

To prove necessity we suppose that (3.1) holds but (F1) fails. Then there exist a number $t_o > 0$ and an infinite index set $K = \{k_i\}$ such that $k_{i+1} > k_i + 1$ and

$$\lim_i f_{k_i}(t_o) = 0. \qquad (3.2)$$

For an arbitrary sequence of indices $l_n$ $(n \in \mathbb{N})$ with $k_n < l_n < k_{n+1}$ we consider the infinite matrix $B = (b_{nk})$, where

$$b_{nk} = \begin{cases} 1/2 & \text{if } k = k_n \text{ or } k = l_n, \\ 0 & \text{otherwise.} \end{cases}$$

It is difficult to see that $B \in \mathcal{T}^+$ and

$$\delta_B(K) = 1/2. \tag{3.3}$$

Now we define $y = (y_k)$ by $y_k = t_o z$ for $k = k_i$ and $y_k = \theta$ otherwise, where $z \in X$, $\|z\| = 1$. Then $\|y_{k_i}\| = t_o$, and so by (3.2) we get $\lim_k f_k(\|y_k\|) = 0$, whence $\lim_k [f_k(\|y_k\|)]^p = 0$. By regularity of B it follows that

$$\lim_n \sum_k b_{nk}[f_k(\|y_k^p\|)] = 0,$$

i.e. $w_B^p(F)\text{-}\lim y_k = \theta$. At that time for any $\varepsilon$ with $0 < \varepsilon \le t_o$ we have

$$L_\varepsilon = \{k\colon \|y_k\| \ge \varepsilon\} = K.$$

From (3.3) it follows that $\delta_B(L_\varepsilon) = \frac{1}{2} \ne 0$, which implies $st_A\text{-}\lim y_k \ne \theta$, contrary to (3.1). Thus (F1) must hold and the proof is complete.

If $f_k = f$ ($k \in \mathbb{N}$) for a modulus f then (F1) is automatically fulfilled. Thus we get

**COROLLARY 3.2.** *Let f be a modulus. Then*

$$w_A^p(f)\text{-}\lim x_k = x_o \Rightarrow st_A\text{-}\lim x_k = x_o \quad (p > 0,\ A \in \mathcal{T}^+)$$

*in a Banach space X.*

Maddox ([10], Theorem 1) proved Corollary 3.2 for a locally convex space X in the case p = 1 and $A = C_1$. Connor ([1], Theorem 2.1) examined the case where $X = \mathbb{K}$, $A = C_1$ and $f(t) = t$.

**THEOREM 3.3.** *The implication*

$$st_A\text{-}\lim x_k = x_o \Rightarrow w_A^p(f)\text{-}\lim x_k = x_o \quad (p > 0,\ A \in \mathcal{UT}^+) \tag{3.4}$$

*holds in a Banach space X if and only if*

(F2) $\lim_{t \to 0+} \sup_k f_k(t) = 0$,

(F3) $\sup_t \sup_k f_k(t) < \infty$.

*Proof*. First we prove the necessity of (F2). By a theorem of Steinhaus (see [2], Theorem 4.4, III) for every

$A \in \mathcal{T}^+$ there exists a sequence $s = (s_k)$ of 0's and 1's which is not A-summable, i.e. the sequence $As = (A_n s)$ is not convergent. If we suppose that (F2) fails then there exist a number $\varepsilon > 0$, a positive null sequence $(t_i)$ and an index set $\{k(i)\}$ such that

$$f_{k(i)}(t_i) \geq (\varepsilon_0)^{1/p} \qquad (i \in \mathbb{N}). \tag{3.5}$$

For an element $z \in X$ with $\|z\| = 1$ and for the index set $\{i_j\} = \{i: s_i = 1\}$ we consider the sequence $x = (x_k)$, where

$$x_k = \begin{cases} t_{i_j} z & \text{if } k = k(i_j), \\ \theta & \text{otherwise.} \end{cases}$$

Then $\lim_k x_k = \theta$, and so $st_A\text{-}\lim x_k = \theta$. By (3.4) it follows that

$$\sigma_n = \sum_k a_{nk}[f_k(\|x_k\|)]^p \to 0 \qquad (n \to \infty). \tag{3.6}$$

But according to (3.5) we have

$$\sigma_n = \sum_j a_{n,k(i_j)}[f_{k(i_j)}(t_{i_j})]^p \geq \varepsilon_0 \sum_k a_{nk}s_k = \varepsilon_0 A_n s.$$

Since $(A_n s)$ is not convergent, this implies $\lim_n \sigma_n \neq \theta$, contrary to (3.6). Thus (3.4) implies (F2).

The necessity of (F3) we also prove by contradiction. Let $A \in \mathcal{U}\mathcal{T}^+$. We may assume that A is normal, and so $\lim_n a_{nn} = 0$. If (F3) is true then, using also a theorem of Agnew (see [2], Theorem 8.5, III), we find an index set $K = \{k_i\}$ with $\delta_A(K) = 0$ and numbers $0 < t_1 < \ldots < t_i < t_{i+1} < \ldots$ such that

$$f_{k_i}(t_i) \geq (1/a_{k_i,k_i})^{1/p} \qquad (i \in \mathbb{N}). \tag{3.7}$$

By Theorem 2.3 the sequence $x = (x_k)$ with $x_k = t_i z$ for $k = k_i$ and $x_k = \theta$ otherwise, where $z \in X$, $\|z\| = 1$, converges A-statistically to $\theta$. So by assumption (3.4) we have $\lim_n \sum_k a_{nk}[f_k(\|x_k\|)]^p = 0$, which implies

$$\lim_n a_{nn}[f_k(\|x_k\|)]^p = 0. \tag{3.8}$$

But in view of (3.7) we get

$$a_{k_i,k_i}[f_{k_i}(\|x_{k_i}\|)]^p = a_{k_i,k_i}[f_{k_i}(t_i)]^p \geq 1 \qquad (i \in \mathbb{N}),$$

contrary to (3.8). Thus the necessity of (F3) is also proved

Let $st_A\text{-}\lim x_k = x_0$ and choose $\varepsilon > 0$. We split the sum $\sigma_n = \sum_k a_{nk} [f_k(\|x_k - x_0\|)]^p$ into two sums $\Sigma_1$ and $\Sigma_2$ over $L_\varepsilon = \{k: \|x_k - x_0\| \geq \varepsilon\}$ and $\{k: \|x_k - x_0\| < \varepsilon\}$, respectively. Since by (F3) there exists a constant $M > 0$ such that $f_k(t) \leq M$ $(k \in \mathbb{N},\ t > 0)$, we find

$$\Sigma_1 \leq M^p \sum_{k \in L_\varepsilon} a_{nk}.$$

Further, if we write $h(t) = \sup_k f_k(t)$, then by the increase of $f_k$ we have

$$\Sigma_2 \leq h(\varepsilon) \sum_k a_{nk}.$$

Consequently, by $\delta_A(L_\varepsilon) = 0$ and (F2) we get $\lim_n \sigma_n \leq h(\varepsilon)$. By (F2) it follows that $\lim_n \sigma_n = 0$, i.e. $w_A^p(F)\text{-}\lim x_k = x_0$. The theorem is proved.

From Theorems 3.1 and 3.3 we deduce the following result.

**COROLLARY 3.4.** *Let* X *be a Banach space and* $F = (f_k)$ *a sequence of moduli. Then*

$$st_A(X) = w_A^p(F, X) \qquad (p > 0,\ A \in \mathcal{U}\mathcal{T}^+)$$

*if and only if the conditions* (F1), (F2), *and* (F3) *are satisfied.*

In the case $f_k = f$ $(k \in \mathbb{N})$ the conditions (F1) and (F2) hold. Thus we get

**Corollary 3.5.** *Let* f *be a modulus. Then* $st_A(X) = w_A^p(f, X)$ $(p > 0,\ A \in \mathcal{U}\mathcal{T}^+)$ *in a Banach space* X *if and only if* f *is bounded.*

For $p = 1$ and $A = C_1$ Corollary 3.5 is contained in Theorem 2 of Maddox [10].

The next theorem characterizes the relation between A-statistical convergence and strong A-summability for bounded sequences.

**THEOREM 3.6.** *The implication*

$$st_A\text{-}\lim x_k = x_o \;\Rightarrow\; w_A^p(F)\text{-}\lim x_k = x_o \quad (p > 0,\ A \in \mathcal{T}^+)$$

*holds in* $m(X)$ *if and only if* (F2) *is satisfied.*

*Proof.* The necessity of (F2) is already proved in Theorem 3.3.

Assume that (F2) holds. Then

$$h(t) = \sup_k f_k(t) < \infty \quad (t > 0). \tag{3.9}$$

If $st_A\text{-}\lim x_k = x_o$ in $X$ and $\|x_k\| \le M$, then

$$f_k(\|x_k - x_o\|) \le f_k(M + \|x_o\|) \le h(M + \|x_o\|) < \infty,$$

and $w_A^p(F)\text{-}\lim x_k = x_o$ follows from the proof of necessity in Theorem 3.3 with $h(M + \|x_o\|)$ instead of $M$. The proof is complete.

Using also Theorem 3.1 we get

**COROLLARY 3.7.** *Let* $X$ *be a Banach space and* $F = (f_k)$ *a sequence of moduli. Then*

$$st_A(X) \cap m(X) = w_A^p(F,X) \cap m(X) \quad (p > 0,\ A \in \mathcal{T}^+)$$

*if and only if* (F1) *and* (F2) *are satisfied.*

In the case $f_k = f$ $(k \in \mathbb{N})$ from Corollary 3.7 we deduce

**COROLLARY 3.8.** *For any modulus* $f$ *we have*

$$st_A \cap m(X) = w_A^p(f,X) \cap m(X) \quad (p > 0,\ A \in \mathcal{T}^+)$$

*in a Banach space* $X$.

Connor ([1], Corollary 2.2) proved this result in the case where $X = \mathbb{K}$, $A = C_1$ and $f(t) = t$.

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Statistiline koonduvus Banachi ruumides

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Resümee

Artikli esimeses osas tõestatakse autori poolt konverentsi teesides [7] sõnastatud teoreemid A-statistilise koonduvuse kohta. Teises osas uuritakse A-statistilise koonduvuse ning tugeva A-summeeruvuse vahekorda.

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**T-DUAL SPACES WITH RATE AND T-SECTIONALLY SUMMABLE SPACES WITH RATE IN THE CASE OF DOUBLE SEQUENCES**

**Ivar Lepasson**

1. **Introduction.** G. Kangro introduced the notions of space $c^{\lambda}_{T}$ of simple sequences $\lambda$-convergent by the method T [5] and space $m^{\lambda}_{T}$ of simple sequences $\lambda$-bounded by the method T [4]. S. Baron [1] has studied $\lambda$-boundedness in the case of double sequences. Starting with it we introduce the notion of spaces which are called $\gamma_{T^{\lambda}}$ - and $\beta_{T^{\lambda}}$ -dual. Also we introduce the notions of $\lambda$-T-sectional boundedness ($T^{\lambda}B$) and $\lambda$-T-sectional boundedly convergence ($T^{\lambda}K$). In the present work we investigate the connections between these notions. In the case of simple sequences the analogous result has been proved by M. Buntinas [2] and in the case of simple sequences with rate by the author [6].

2. **Definitions.** Let $T = (t_{mnkl})$ be a triangular infinite matrix where $t_{mnkl} = 0$ when $m < k$ or $n < l$, let E be a Hausdorff locally convex double sequence space (l. c. d. s. s.) and $x = (x_{kl})$ a double sequence of real (or complex) numbers. We say that $E = [E; p_{mn}]$ is an FK-space if there is a finite number or a denumerable set of quasinorms $p_{mn}(x)$ with following properties:

1° when $p_{mn}(x) = 0$ then always $x = \theta$,

2° E is perfect,

3° when $x^{(r,s)} \to x$ in E then always $x^{(r,s)}_{kl} \to x_{kl}$ $(r,s \to \infty)$.

Let $\lambda = (\lambda_{mn})$ be a double sequence of positive numbers, monotonically increasing both by the indices m and n. In this case $\lambda$ is called a rate. A double sequence x is said to be

$\lambda$-*bounded* if

$$\beta_{mn} = \lambda_{mn}(x_{kl} - \lim_{k,l} x_{kl}) = O(1),$$

$\lambda$-*convergent* if there exists

$$\lim_{m,n} \beta_{mn}$$

and $\lambda$-*boundedly convergent* if $x$ is $\lambda$-bounded and $\lambda$-convergent. In this article the convergence means the convergence in Pringsheim's sense.

We define:

$t^{mn} = (t_{mnkl})$ are double sequences;

$x \cdot y = (x_{kl} \cdot y_{kl})$; $y_{mn} = \sum_{k,l} t_{mnkl} x_{kl}$;

$y = (y_{mn})$; $\lim_{m,n} y_{mn} = \eta$; $\gamma_{mn} = \lambda_{mn}(y_{mn} - \eta)$;

$$y^{mn} = \sum_{k,l=0}^{m,n} t_{mnkl} x_{kl} e^{kl}$$

where $e^{mn} = (\delta^{mn}_{kl})$ and $\delta^{mn}_{kl} = \begin{cases} 1, & \text{if } k = m \text{ and } l = n, \\ 0, & \text{otherwise;} \end{cases}$

$c^{\lambda}_{T} = \{ x\colon \exists \lim_{m,n} \gamma_{mn} \}$;

$m^{\lambda}_{T} = \{ x\colon \gamma_{mn} = O(1) \}$;

$bc^{\lambda}_{T} = \{ x\colon x \in c^{\lambda}_{T} \text{ and } x \in m^{\lambda}_{T} \}$;

$E^{\beta_{T^{\lambda}}} = \{ x\colon \forall z \in E,\ x \cdot z \in bc^{\lambda}_{T} \}$;

$E^{\gamma_{T^{\lambda}}} = \{ x\colon \forall z \in E,\ x \cdot z \in m^{\lambda}_{T} \}$;

$E_{T^{\lambda}B} = \{ x\colon (t^{mn}x) \text{ is } \lambda\text{-bounded in } E \}$;

$E_{FT^{\lambda}K} = \{ x\colon \forall f \in E',\ f(t^{mn}x)$ is a $\lambda$-boundedly convergent double sequence $\}$;

$E_{T^{\lambda}K} = \{ x : \lambda_{mn}(y^{mn} - y) \text{ is boundedly convergent in } E \}$;

$E^{\Phi} = \{ x\colon \exists f \in E',\ x_{kl} = f(e^{kl}),\ k,l \in \mathbb{Z}^{+} \}$;

$E_{AD}$ is the closure of the span of the sequences $e^{mn}$;

A double sequence $x$ in $E$ is said to have the property $T^{\lambda}B$ (T-sectional boundedness with the rate $\lambda$) if $x \in E_{T^{\lambda}B}$, $T^{\lambda}K$ (T-sectional bounded convergence with the rate $\lambda$) if

$x \in E_{T^\lambda K}$ and the property $FT^\lambda K$ (functional T-sectional bounded convergence with the rate $\lambda$) if $x \in E_{FT^\lambda K}$. If $E \subset E_{T^\lambda B}$ ($E = E_{T^\lambda K}$, $E \subset E_{FT^\lambda K}$, $E = E_{AD}$) then $E$ is called $T^\lambda B$-space ($T^\lambda K$-space, $FT^\lambda K$-space, AD-space). $E$ is called $\gamma_{T^\lambda}$-space ($\beta_{T^\lambda}$-space) if $E = E^{\gamma_{T^\lambda}}$ ($E = E^{\beta_{T^\lambda}\beta_{T^\lambda}}$). Each double sequence space $E$ considered here will be assumed to contain all unit double sequences $e^{kl}$.

## 3. $\gamma_{T^\lambda}$ - and $\beta_{T^\lambda}$ - duality.

**THEOREM 3.1.** *For each* $E$, $E_{T^\lambda B} = (E^\Phi)^{\gamma_{T^\lambda}}$.

*Proof.* A subset of a l. c. d. s. s. is bounded if and only if it is weakly bounded. Hence, $x \in E_{T^\lambda B}$ if and only if

$$\sup_{m,n} |f\{\lambda_{mn}(t^{mn}x - \lim_{m,n} t^{mn}x)\}| =$$

$$= \sup_{m,n} |\lambda_{mn}\{f(t^{mn}x) - \lim_{m,n} f(t^{mn}x)\}| < \infty \text{ , } f \in E' \leftrightarrow$$

$$\leftrightarrow \sup_{m,n} |\lambda_{mn}(\sum_{k,l} t_{mnkl}x_{kl}y_{kl} - \lim_{m,n} \sum_{k,l} t_{mnkl}x_{kl}y_{kl})| < \infty,$$

$\forall\, y \in E^\Phi$.

**COROLLARY 3.2.** *For each* $E$, $E_{T^\lambda B}$ *is a* $\gamma_{T^\lambda}$*-space.*

**COROLLARY 3.3.** *For each* $E$

$$E \subset E_{T^\lambda B} \leftrightarrow E^\Phi \subset E^{\gamma_{T^\lambda}} \leftrightarrow E^{\gamma_{T^\lambda}\gamma_{T^\lambda}} \subset E_{T^\lambda B}$$

*Proof.* If $E \subset E_{T^\lambda B}$ then, for every $x \in E$ and $y \in E^\Phi$, $x \cdot y \in m^\lambda_T$. Hence $E^\Phi \subset E^{\gamma_{T^\lambda}}$. If $E^\Phi \subset E^{\gamma_{T^\lambda}}$ then

$E^{\gamma_{T^\lambda}\gamma_{T^\lambda}} \subset (E^\Phi)^{\gamma_{T^\lambda}} = E_{T^\lambda B}$. While $E \subset E^{\gamma_{T^\lambda}\gamma_{T^\lambda}}$ then $E \subset E_{T^\lambda B}$.

**THEOREM 3.4.** *For each* $E$, $E_{FT^\lambda K} = (E^\Phi)^{\beta_{T^\lambda}}$.

*Proof*. We have $x \in E_{FT^\lambda K}$ if and only if $\forall\, y \in E^\Phi$, $\exists \lim_{m,n} \lambda_{mn}\left(\sum_{k,l} t_{mnkl}x_{kl}y_{kl} - \lim_{m,n} \sum_{k,l} t_{mnkl}x_{kl}y_{kl}\right) \leftrightarrow$ for every $y \in E^\Phi$, $x \cdot y \in bc^\lambda_T$.

**COROLLARY 3.5.** *For each* $E$, $E_{FT^\lambda K} = E^{\beta_{T^\lambda}\beta_{T^\lambda}}$.

**COROLLARY 3.6.** *For each* $E$,

$$E \subset E_{FT^\lambda K} \leftrightarrow E^\Phi \subset E^{\beta_{T^\lambda}} \leftrightarrow E^{\beta_{T^\lambda}\beta_{T^\lambda}} \subset E_{FT^\lambda K}.$$

*Proof*. Similar to Corollary 3.3.

**THEOREM 3.7.** *Suppose that is an FK-space and there exist* $\lim_{m,n} t_{mnkl} \neq 0$. *Then*

(a) $E^{\gamma_{T^\lambda}} \subset E^\Phi$;

(b) $E_{T^\lambda B} \subset E^{\gamma_{T^\lambda}\gamma_{T^\lambda}}$ *and* $E_{FT^\lambda K} \subset E^{\beta_{T^\lambda}\beta_{T^\lambda}}$;

(c) $E \subset E_{T^\lambda B} \leftrightarrow E^\Phi = E^{\gamma_{T^\lambda}} \leftrightarrow E^{\gamma_{T^\lambda}\gamma_{T^\lambda}} = E_{T^\lambda B}$;

(d) $E \subset E_{FT^\lambda K} \leftrightarrow E^\Phi = E^{\beta_{T^\lambda}} \leftrightarrow E^{\gamma_{T^\lambda}\gamma_{T^\lambda}} = E_{T^\lambda B}$;

(e) $E \subset E_{FT^\lambda K} \leftrightarrow E^\Phi = E^{\beta_{T^\lambda}} = E^{\gamma_{T^\lambda}} \leftrightarrow E \subset E_{T^\lambda B}$ *and* $E^{\beta_{T^\lambda}} = E^{\gamma_{T^\lambda}}$.

*Proof*. (a)We have $E^{\gamma_{T^\lambda}} \subset E^{\gamma_T} = \{\, x\colon \forall\, y \in E,\ x \cdot y \in m_T\}$, thus (analogically with the case of simple sequences [2]) $E^{\gamma_{T^\lambda}} \subset E^{\gamma_T} \subset E^{\Phi}$.

(b) is a corollary from (a).

(c) The equivalences follow from Corollary 3.3, (a) and (b).

(d) The equivalences follow from Corollary 3.6., (a) and (b).

(e) If $E \subset E_{T^\lambda B}$ and $E^{\beta_{T^\lambda}} = E^{\gamma_{T^\lambda}}$ then, by (c), $E^{\Phi} = E^{\gamma_{T^\lambda}}$. Thus $E^{\beta_{T^\lambda}} = E^{\gamma_{T^\lambda}} = E^{\Phi}$. If $E^{\beta_{T^\lambda}} = E^{\Phi}$ then, by (d) $E \subset E_{FT^\lambda K}$ and $E^{\beta_{T^\lambda}} = E^{\Phi} = E^{\gamma_{T^\lambda}} \leftrightarrow E \subset E_{T^\lambda B}$ and $E^{\beta_{T^\lambda}} = E^{\gamma_{T^\lambda}}$.

For example, if E and F are sequence spaces, the multiplier space $(E \to F)$ is the space of all sequences x such that, for all y in E, $x \cdot y \in F$. The multiplier spaces $(E \to c^{\lambda}_{T})$ and $(E \to m^{\lambda}_{T})$ are the $\beta_{T^\lambda}$- and $\gamma_{T^\lambda}$-duals of E, respectively. By theorem, if

$$\lambda = \frac{1}{(m+1)^{-1}+(n+1)^{-1}}$$

and $T = C^{11}$ then $m^{\lambda}_{T}$ is a $T^{\lambda}B$-space (see [1]) and

$$(m^{\lambda}_{T})^{\Phi} = (m^{\lambda}_{T} \to m^{\lambda}_{T}).$$

**DEFINITION**. Let p be a continuous seminorm on l. c. d. s. s. E. Then $p^{\lambda}_{T}$ is a seminorm on $E_{T^\lambda B}$ defined (analogously to the case of $\lambda$-bounded simple sequences [4]) by

$$p^{\lambda}_{T}(x) = \sup_{m,n} \{ p[\lambda_{mn}( t^{mn}x - \lim_{m,n} t^{mn}x)];\ p(\lim_{m,n} t^{mn}x) \}.$$

**THEOREM 3.8.** *Suppose* E *is a* BK - *space and there exist* $\lim_{m,n} t_{mnkl} \neq 0$. *Then* $E^{\Phi}$ *is a* BK - *space and*

$$(E^{\Phi})_{T^\lambda B} = (E_{AD})^{\gamma_{T^\lambda}} \subset E^{\Phi}.$$

*Proof.* If E is a BK-space, then $E_{AD}$ is a BK-space with the norm of E. $(E_{AD})'$ can be identified with $E^{\Phi}$ and hence $E^{\Phi}$

is a BK-space (analogously to the case of simple sequences [2], Proposition 1 or [7]). Let $f_{t^{mn}z} \in E'$ be defined by

$$f_{t^{mn}z}(x) = \sum_{k,l} t_{mnkl} z_{kl} x_{kl},$$

then

$$z \in (E_{AD})^{\gamma_{T^\lambda}} \Leftrightarrow \sup_{m,n} |\lambda_{mn}(f_{t^{mn}z}(x) - \lim_{m,n} f_{t^{mn}z}(x)| =$$
$$= \sup_{m,n}|\lambda_{mn}(\sum_{k,l} t_{mnkl} z_{kl} x_{kl} - \lim_{m,n} \sum_{k,l} t_{mnkl} z_{kl} x_{kl})| < \infty \qquad \forall x \in E_{AD}.$$

Due to the uniform boundedness this is equivalent to the condition that $(t^{mn}z)$ is $\lambda$-bounded in $E^{\Phi}$. Thus

$$(E^{\Phi})_{T^\lambda B} = (E_{AD})^{\gamma_{T^\lambda}} \subset (E_{AD})^{\Phi} = E^{\Phi}.$$

**THEOREM 3.9.** *Let* $\lim_{m,n} t_{mnkl} = 1$. *Suppose* $E$ *is a BK-space and* $E = E_{AD}$. *Then* $E = E_{T^\lambda K}$ *iff* $E^{\Phi} \subset (E^{\Phi})_{T^\lambda B}$.

*Proof.* If $E = E_{AD}$, then $E = E_{T^\lambda K}$ if $E_{T^\lambda K} = E = E_{AD}$. *By Theorem 3.7, (c), and Theorem 3.8* $E = E_{AD}$ *and* $E \subset E_{T^\lambda B}$ *if and only if*

$$(E^{\Phi})_{T^\lambda B} = E^{\gamma_{\lambda_T}} = E^{\Phi}.$$

**THEOREM 3.10.** *If there exists a FK-space which is also a* $T^\lambda B$ *- space, then* $l \subset bc^\lambda_T$.

*Proof.* If for every $e^{kl} \in E_{T^\lambda B}$ then

$$\sup_{k,l} \sup_{k,l} |\lambda_{mn}(t_{mnkl} - \lim_{m,n} t_{mnkl})| < \infty,$$

then $\sup_{k,l} p^\lambda_T(t^{mn}) < \infty$. Thus $l \subset bc^\lambda_T$. (3.1, [3])

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Kiirusega T-duaalsed ruumid ja kiirusega lõike T-summeeruvad ruumid kahekordsete jadade korral

Ivar Lepasson

Resümee

Käesolevas artiklis on sisse toodud $\gamma_{T}^{\lambda}$- ja $\beta_{T}^{\lambda}$-duaalsete ruumide mõiste. Samuti on sisse toodud λ- T-lõike

tõkestatuse ($T^\lambda B$), $\lambda$- T- lõike tõkestatult koonduvuse ($T^\lambda K$) ja $\lambda$- T- lõike tõkestatult koonduvate funktsionaalide ($FT^\lambda K$) mõisted. Töös uuritakse nendevahelisi seoseid.

ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS
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**ON CORES OF SEMICONTINUOUS SEQUENTIAL SUMMABILITY METHODS**

**Leiki Loone**

This paper in an Englich version of the paper [5] which is unreadable by reason of unsatisfactory printing.

Let $A_m = (a_{mnk})$ $(m = 0,1,\ldots)$ be matrices, where $a_{mnk} \in \mathbb{R}$. A sequence of real numbers $x = (\xi_k)$ is called $\alpha$-summmable to $a$ if

$$\lim_m \sum_k a_{mnk}\xi_k = a \qquad \text{uniformly in } n \tag{1}$$

(see [1]). Let $\mathbb{Q}$ be the set of all operators $q : \mathbb{N} \to \mathbb{N}$ and let $B_q = (a_{mq(m)k})$. It means that the set

$$\{B_q : q \in \mathbb{Q}\}$$

is the family of all possible matrices which can be produced by selecting the first row of the matrix from the rows of the matrix $A_1$, the second row from the rows of the matrix $A_2$ etc. The following theorem is due to Petersen (see [1]).

**THEOREM 1.** *A sequence* $x = (\xi_k)$ *is* $\alpha$*-summable to* $a$ *iff for each* $q \in \mathbb{Q}$ *it is* $B_q$*-summable to* $a$.

Let $m$ be the set of bounded sequences with the norm

$$\|x\| = \sup_k |\xi_k|.$$

Let $K^o$ be the set of all linear continuous functionals on $m$ satisfying the following conditions:

$1^o$ $\langle e_k,f\rangle = 0$ $\forall k = 0,1,\ldots,$

$2^o$ $\langle e,f\rangle = 1,$

$3^o$ $\|f\| = 1,$

where $e_k = (\underbrace{0,\ldots,0}_{k},1,0,\ldots)$ and $e = (1,1,\ldots,1,\ldots)$. This set $K^o$ determines the Knopp's core in $m$. This means that for an arbitrary bounded sequence $x$ the set

$$K^o(x) = \{\langle x,f\rangle : f \in K^o\}$$

is the Knopp's core for x (see[3]). It is well-known that a sequence x converges to a number $a$ iff its core $K^o(x)$ is a singleton which contains only $a$ (see [2, Ch.6]).

Let ${}^tA$ be the conjugate matrix to a matrix A. The set

$$ {}^tA(K^o) = \{{}^tAf : f \in K^o\} $$

determines A-summability in the sence of sequence x being A-summable to $a$ iff the set

$$ \{\langle x,f\rangle : f \in {}^tA(K^o)\} $$

is a singleton which contains only $a$. The concept of the core $K_\alpha(x)$ was introduced in [4], based on Theorem 1. This core is the set

$$ K_\alpha(x) = \{\langle x,f\rangle : f \in K_\alpha\}, $$

where

$$ K_\alpha = \text{clco} \bigcup \{{}^tB_q(K^o) : q \in \mathbb{Q}\}. \tag{2} $$

Here "clco" denotes the close and convex hull of the set. This core determines $\alpha$-summability in $m$ in the sence that a sequence x is $\alpha$-summable to $a$ iff $K_\alpha(x)$ is a singleton which contains $a$ (see [4]).

Let $\mathcal{U}_-(\tau_0)$ be an arbitrary fixed left-hand neighbourhood of a number $\tau_0 \in \mathbb{R}$. Suppose that for every $\tau \in \mathcal{U}_-(\tau_0)$ there is a matrix $A(\tau) = (a_{nk}(\tau))$ such that

$$ \sup_n \sum_k |a_{nk}(\tau)| < \infty \quad \forall \tau \in \mathcal{U}_-(\tau_0). $$

**DEFINITION 1.** *It is said that a sequence* $x = (\xi_k)$ *is summable by a semicontinuous sequential summability method* $(A(\tau))$ *(for short: "$\alpha(\tau)$-summable") to a number* $a$ *if*

$$ \lim_{\tau \to \tau_0 -} \sum_k a_{nk}(\tau)\xi_k = a $$

*uniformly in* n.

The set of all $\alpha(\tau)$-summable sequences is denoted by $c_{\alpha(\tau)}$. Semicontinuous sequential summability method $(A(\tau))$ is called regular if every convergent sequence is $\alpha(\tau)$-summable to the previous limit.

In special case of

$$ a_{nk}(\tau) = a_k(\tau) \quad \forall n \in \mathbb{N} $$

the $\alpha(\tau)$-summability method $(A(\tau))$ turns into ordinal semi-continuous summability method $(a_k(\tau))$.

Let $\mathcal{W}$ be the set of all sequences $(\tau_m) \subset \mathcal{U}_-(\tau_0)$ which are convergent to $\tau_0$. It means that

$$\mathcal{W} = \{w = (\tau_m) : \tau_m \to \tau_0, \ \tau_m \in \mathcal{U}_-(\tau_0) \quad \forall m \in \mathbb{N}\}.$$

Let $w = (\tau_m)$ be an arbitrarily fixed element from $\mathcal{W}$ and let us define the $\alpha$-method $(A_m)$ where $a_{mnk} = a_{nk}(\tau_m)$. If a sequence x is $\alpha$-summable by this $\alpha$-method $(A_m)$ we say for short that it is w-summable. The set which defines the core for the w-summability is denoted by $K_w$.

**THEOREM 2.** *A sequence* $x = (\xi_k)$ *is* $\alpha(\tau)$*-summable to a number* $a$ *iff it is* w*-summable to* $a$ *for every* $w \in \mathcal{W}$.

*Proof.* It follows from the concept of limit given by Heine.

**COROLLARY 2.1.** *A sequence* $x = (\xi_k)$ *is* $\alpha(\tau)$*-summable to* $a$ *iff*

$$K_w(x) = \{a\} \quad \forall w \in \mathcal{W}.$$

Let us introduce now the concept of core for the semicontinuous sequential summability method. This concept is based on Theorem 2.

**DEFINITION 2.** *The core for the* $\alpha(\tau)$*-method* $(A(\tau))$ *is the core defined by the set*

$$K = \operatorname{clco} \bigcup \{K_w : w \in \mathcal{W}\}, \tag{3}$$

**THEOREM 3.** *The set* $c_{\alpha(\tau)}$ *coincides with the set of all sequences* x *for which the core* $K(x)$ *is a singleton.*

*Proof.* If $K(x) = \{a\}$, it follows from (3) that $K_w(x) = \{a\}$ for each w from $\mathcal{W}$. Hence, by Corollary 2.1, x is $\alpha(\tau)$-summable.

Suppose now that x is $\alpha(\tau)$-summable to $a$ and $\delta \in K(x)$. It means that in the set

$$\bigcup \{K_w : w \in \mathcal{W}\}$$

there exist sequences $(g_n)$ and $(h_n)$ such that

$$\lim_n \langle x, \lambda_n g_n + (1-\lambda_n)h_n \rangle = \delta$$

for some $(\lambda_n)$ with $0 \le \lambda_n \le 1$ for all $n \in \mathbb{N}$. As $\langle x, g_n \rangle = a$ and $\langle x, h_n \rangle = a$ for every $n$, we have $\delta = a$. It means that $K(x) = \{a\}$ which gives us the desired result.

**COROLLARY 3.1.** *If* $x \in c_{\alpha(\tau)}$ *and* (1) *holds then* $K(x) = \{a\}$.

**COROLLARY 3.2.** *The core which determines a semicontinuous summability method* $A = (a_k(\tau))$ *is the core defined by the set*

$$K_A = \text{clco} \bigcup \{{}^tB_w(K^o) : w \in \mathcal{W}\}, \tag{4}$$

*where* $B_w = (b_{mk})$ *and* $b_{mk} = a_k(\tau_m)$.

*Proof.* For an arbitrary $w = (\tau_m) \subset \mathcal{W}$ the $\alpha$-method $(A(\tau_m))$ is the matrix method $B_w = (b_{mk})$, where $b_{mk} = a_k(\tau_m)$. Hence, the set K given by the formula (3) has the form (4) and now Corollary 3.1 completes the proof.

**THEOREM 4.** *An* $\alpha(\tau)$*-method* $A = (a_{nk}(\tau))$ *is regular iff*

$1^o$ $\lim\limits_{\tau \to \tau_o -} \sup\limits_n |a_{nk}(\tau)| = 0 \quad \forall k = 0,1,\ldots,$

$2^o$ $\lim\limits_{\tau \to \tau_o -} \sum\limits_k a_{nk}(\tau) = 1$ *uniformly in* $n$,

$3^o$ $\sup\limits_n \sum\limits_k |a_{nk}(\tau)| < M$ *for every* $\tau \in \mathcal{U}\_(\tau_0)$.

*Proof* is entailed by Theorem 2 if one applies to necessary and sufficient conditions for the regularity of the $\alpha$-method (see [4]).

**THEOREM 5.** *The inclusion*

$$K(x) \subset K^o(x) \quad \forall x \in m \tag{5}$$

*holds iff*

$1^o$ $\alpha(\tau)$*-method is regular,* (6)

$2^o$ $\lim\limits_{\tau \to \tau_o -} \sup\limits_n \sum\limits_k |a_{nk}(\tau)| = 1.$ (7)

*Proof.* The inclusion (5) is equivalent to the inclusion $K \subset K^{o}$.

Necessity. Let $K \subset K^{o}$. Then $K(x) = K^{o}(x)$ for every $x \in c$. It means that the $\alpha(\tau)$-method is regular. It follows from the inclusion $K \subset K^{o}$ that $K_w \subset K^{o}$ for every $w \in \mathcal{W}$. Consequently,

$$\lim_m \sup_n \sum_k |a_{nk}(\tau_m)| = 1 \quad \forall\ (\tau_m) \subset \mathcal{W}$$

(see [4]). The condition (7) follows now from the concept of limit given by Heine.

Sufficiency. If $\alpha(\tau)$-method is regular and (7) holds then for every $(\tau_m) \in \mathcal{W}$ $\alpha$-method $(A(\tau_m))$ is core-regular (see [4]). Then

$$K_w \subset K^{o} \quad \forall\ w \in \mathcal{W}.$$

The set $K^{o}$ is closed and convex, therefore

$$\text{clco} \bigcup \{K_w : w \in \mathcal{W}\} \subset K^{o},$$

it means that $K \subset K^{o}$.

**COROLLARY 5.1.** *For a semicontinuous matrix method* $A = (a_k(\tau))$ *the inclusion*

$$K_A(x) \subset K^{o}(x) \quad \forall\ x \in m \tag{8}$$

*holds iff*

$1^{o}$ *method* A *is regular,*

$2^{o}$ $\lim_{\tau \to \tau_o} \sum_k |a_k(\tau)| = 1.$

Let $L(x)$ be the set of Banach limits of a sequence $x$. This set is the core of almost convergency of $x$ (see [1;4]).

**THEOREM 6.** *The inclusion*

$$K(x) \subset L(x) \quad \forall\ x \in m \tag{9}$$

*holds iff*

$1^{o}$ $\alpha(\tau)$*-method is regular,*

$2^{o}$ $\lim_{\tau \to \tau_o -} \sup_n \sum_k |a_{nk}(\tau)| = 1,$

$3^{o}$ $\lim_{\tau \to \tau_o -} \sup_n \sum_k |a_{nk}(\tau) - a_{nk+1}(\tau)| = 0.$

*Proof* is analoguous to the proof of Theorem 5. In the case we need the necessary and sufficient conditions for the inclusion $K_w \subset L$ (see [4]) instead of $K_w \subset K^{o}$ in Theorem 5.

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Poolpidevad jadamenetlused ja nende tuumad

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Resümee

Antud töös defineeritakse poolpideva jadalise summeeruvusmenetluse ja sellega määratud tuuma mõiste ( vt. definitsioonid 1 ja 2).

Selle tuuma järgi koonduvate jadade hulk ühtib $\alpha(\tau)$-summeeruvate jadade hulgaga (vt. teoreem 3 ja järeldus 3.1.).

Kasutades teoreemi 2 ja tulemusi artiklist [4] antakse tarvilikud ja piisavad tingimused $\alpha(\tau)$-menetluse regulaarsuseks (teoreem 4). On antud ka tarvilikud ja piisavad tingimused sisalduvusteks (5), (8) ja (9).

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## INCLUSION BETWEEN THE CORES CONCERNING WEIGHTED MEANS AND POWER SERIES

Leiki Loone

Suppose throughout that $(p_k)$ is a sequence of real numbers with $p_k > 0$ for all $k = 0,1,2,\ldots$ .Let

$$p(\tau) = \sum_k p_k \tau^k < \infty \quad \forall\, \tau \in (0,1) \tag{1}$$

and let

$$\lim_m P_m = \infty, \tag{2}$$

where

$$P_m = \sum_{k=0}^{m} p_k .$$

Let $x = (\xi_k)$ be a sequence of real numbers. The weighted mean summability method $(R,p_k)$ and the power series method $(J,p_k)$ are defined as follows.

We say that $x = (\xi_k)$ is $(R,p_k)$-summable to a number $a$ if

$$\lim_m \frac{1}{P_m} \sum_{k=0}^{m} p_k \xi_k = a.$$

The set of all $(R,p_k)$-summable sequences is denoted by $c_R$, and the set of all $(R,p_k)$-bounded sequences is denoted by $m_R$. It means that

$$m_R = \{x = (\xi_k) : \sup_m \left|\frac{1}{P_m} \sum_{k=0}^{m} p_k \xi_k\right| < \infty\}.$$

We say that $x = (\xi_k)$ is $(J,p_k)$-summable to $a$ if the series

$$\sum_k p_k \tau^k \xi_k \tag{3}$$

is convergent for every $\tau \in (0,1)$ and

$$\lim_{\tau \to 1-} \frac{1}{p(\tau)} \sum_k p_k \tau^k \xi_k = a.$$

The set of all $(J,p_k)$-summable sequences is denoted by $c_J$.

The set of all $x = (\xi_k)$ for which the series (3) is convergent for every $\tau \in (0,1)$ is denoted by $c_J$.

Let $K^o(x)$ be the Knopp's core of the sequence $x = (\xi_k)$ and let $K_R(x)$ be the Knopp's core of the sequence $y = (\eta_m)$ where

$$\eta_m = \frac{1}{P_m} \sum_{k=0}^{m} p_k \xi_k .$$

Let the set $\mathcal{W}$ be defined as follows

$$\mathcal{W} = \{w = (\tau_m) : \tau_m \to 1,\ \tau_m \in (0,1),\ m = 0,1,\dots\}.$$

For an arbitrary $w = (\tau_m) \in \mathcal{W}$ the Knopp's core of the sequence $y^* = (\eta_m^*)$, where

$$\eta_m^* = \frac{1}{p(\tau_m)} \sum_{k=0}^{\infty} p_k \tau_m^k \xi_k ,$$

is denoted by $K_w(x)$.

**DEFINITION 1.** *The core $K_J(x)$ of a sequence $x = (\xi_k) \in c_J$ is the set*

$$\operatorname{clco} \bigcup \{K_w(x) : w \in \mathcal{W}\}.$$

It is obvious that $x \in c_J$ iff $K_J(x)$ is a singleton.

The core concerning a semicontinuous summability method was defined in [3] only for the bounded sequences $x \in m$. Definition 1 gives the concept of the core for the method $(J, p_n)$ in the space $c_J$. It is obvious that $m \subset c_J$. The next theorem shows the relations between those conceptions.

**THEOREM 1.** *For every $x \in m$ the core $K_J(x)$ is identical to the core defined by the set*

$$K = \operatorname{clco} \bigcup \{{}^tB_w(K^o) : w \in \mathcal{W}\},$$

*where*

$$B_w = (b_{mk}) \quad \textit{and} \quad b_{mk} = p_k \tau_m^k / p(\tau_m).$$

*Proof* is entailed by Corollary 3.2 from paper [3] if one applies the Definition 1 in this paper.

It is well-known that $c_R \subset c_J$ (see [2]). This result can be strenghtened as follows.

**THEOREM 2.** *The inclusion*

$$K_J(x) \subset K_R(x) \tag{4}$$

*holds for all* $x \in c_J \cap m_R$.

*Proof.* Let $A = (a_{mk})$ be the Riesz matrix i.e.

$$a_{mk} = \begin{cases} p_k/P_m & \text{if } k \le m, \\ 0 & \text{if } k > m. \end{cases}$$

The inverse matrix $A^{-1} = (\alpha_{mk})$ to Riesz matrix is as follows

$$\alpha_{mk} = \begin{cases} P_m/p_m & \text{if } k = m, \\ -P_{n-1}/p_n & \text{if } k = m-1, \\ 0 & \text{if } k < m-1 \text{ or } k > m \end{cases}$$

(see [4]). The inclusion (4) is equivalent to the following

$$K_J(x) \subset K^o(Ax) \quad \forall\ x \in c_J \cap m_R$$

and this is identical to the inclusion

$$K_J(A^{-1}x) \subset K^o(x) \quad \forall\ x,\ \ A^{-1}x \in c_J \cap m_R. \tag{5}$$

To prove that (5) holds we have to show according to Definition 1 that for all $w \in W$ the inclusion

$$K_w(A^{-1}x) \subset K^o(x) \quad \forall\ x,\ \ A^{-1}x \in c_J \cap m_R. \tag{6}$$

holds. Let $w$ be an arbitrarily fixed element from $W$. Let $C$ be the matrix method $C = (c_{mk})$, where $c_{mk} = p_k\tau_m^k/p(\tau_m)$, and let $G = (g_{mk})$ be such that $G = CA^{-1}$. It means that

$$g_{mk} = \sum_{\nu} c_{m\nu}\alpha_{\nu k} = \sum_{\nu} \frac{p_\nu\tau_m^\nu}{p(\tau_m)}\alpha_{\nu k} = \frac{p_k\tau_m^k P_k}{p(\tau_m)p_k} - \frac{p_{k+1}\tau_m^{k+1}P_k}{p_{k+1}p(\tau_m)} =$$

$$= \frac{P_k}{p(\tau_m)}\tau_m^k(1-\tau_m).$$

It is obvious that the inclusion (6) holds if for all $w = (\tau_m)$ the method $G$ is core regular in $m$ i.e.

$$K^o(Gx) \subset K^o(x) \quad \forall\ x \in m. \tag{7}$$

The necessary and sufficient conditions for (7) are as follows (see [5])

$1^\circ \quad \lim_{m\to\infty} g_{mk} = 0 \quad \forall\, k = 0,1,\dots,$ (8)

$2^\circ \quad \lim_{m\to\infty} \sum_k g_{mk} = 1,$ (9)

$3^\circ \quad \lim_{m\to\infty} \sum_k |g_{mk}| = 1.$ (10)

Since $(\tau_m) \in (0,1)$ and $p(\tau_m) \to \infty$ as $m \to \infty$, the equality (8) holds for any $k = 0,1,\dots$ .As $|g_{mk}| = g_{mk}$ the conditions (9) and (10) coincide.

$$\lim_m \sum_k g_{mk} = \lim_m \frac{1-\tau_m}{p(\tau_m)} \sum_k P_k \tau_m^k.$$

We have chosen $(p_k)$ such that (1) holds, therefore

$$\sum_k P_k \tau_m^k = \frac{1}{1-\tau_m} \sum_k p_k \tau_m^k = \frac{p(\tau_m)}{1-\tau_m}$$

and consequently (9) is valid. It means that for an arbitrary w method G is core regular in $m$ and due to it (4) holds for all $x \in c_J \cap m_R$.

**COROLLARY 2.1.** *If*

$$\lim_{\tau\to 1} \frac{p(\tau^2)}{p(\tau)} = 1 \tag{11}$$

*then*

$$K_J(x) = K_R(x) \quad \forall\, x \in c_J \cap m_R. \tag{12}$$

*Proof.* It is known that if (11) holds then

$$K_J(x) = K_R(x) \quad \forall\, x \in c_J$$

(see [1]). As $c_R \subset c_J$, it follows that $c_J = c_R$. Therefore, for all $w \in \mathcal{W}$ the method G in the proof of Theorem 2 is equivalent to the convergency. As $g_{mk} \ge 0$ for all $m,k = 0,1,\dots$ this equivalence gives us

$$K^\circ(Gx) = K^\circ(x) \quad \forall\, x \in m,$$

(see [4],p.125) and due to it (12) holds.

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Rieszi menetluse ja astmerea abil defineeritud poolpideva menetluse poolt määratud tuumade sisalduvus

Leiki Loone

Resümee

Olgu $(p_k)$ positiivsete reaalarvude jada. Olgu

$$p(\tau) = \sum_k p_k \tau^k < \infty \quad \forall\, \tau \in (0,1)$$

ja olgu $\lim_m P_m = \infty$,

kusjuures

$$P_m = \sum_{k=0}^{m} p_k .$$

Me ütleme, et arvjada $x = (\xi_k)$ on $(R,p_k)$-summeeruv artvuks $a$ kui

$$\lim_m \frac{1}{P_m} \sum_{k=0}^{m} p_k \xi_k = a.$$

Olgu $m_R$ kõikide $(R,p_k)$-tõkestatud jadade hulk, s.t.

$$m_R = \{x = (\xi_k) : \sup_m \left|\frac{1}{P_m} \sum_{k=0}^{m} p_k \xi_k\right| < \infty\}.$$

Hulka, mis koosneb kõikidest jadadest $x = (\xi_k)$, mille korral rida

$$\sum_k p_k \tau^k \xi_k$$

on koonduv iga $\tau \in (0,1)$ korral, tähistame sümboliga $\diamond_J$.

Me ütleme, et jada $x \in (\xi_k)$ on $(J,p_k)$-summeeruv arvuks $a$, kui ta kuulub hulka $\diamond_J$, ja kui

$$\lim_{\tau \to 1-} \frac{1}{p(\tau)} \sum_k p_k \tau^k \xi_k = a.$$

Antud töös defineeritakse tuuma mõiste poolpideva menetluse $(J,p_n)$ jaoks ruumis $\diamond_J$ (vt. definitsioon 1). Tõestatakse sisalduvus

$$K_J(x) \subset K_R(x) \quad \forall x \in m_R \cap \diamond_J$$

ja sellest järelduv võrdus (12) tingimusel (11). Nendes seostes on $K_J(x)$ definitsioonis 1 antud tuum ja $K_R(x)$ on Rieszi menetlusega määratud tuum.

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**A DESCRIPTION OF MEASURE SPACES WITH LIFTINGS**

**Aleksander Monakov-Rogozkin**

Since Fremlin [1] has constructed an example of a measure-complete locally determined Maharam measure space which is not decomposable (or equivalently has no lifting), the question arises how to describe all measure-complete decomposable spaces and their liftings. A solution of this problem is presented in Section 6 of this paper where a description of all measure-complete decomposable spaces with a given Stone space is obtained. Finally an example of a Maharam measure space with lifting which is neither measure-complete nor locally determined is given.

**1. Notations and definitions.** Throughout this paper we use the terminology of Fremlin's book [2], * is the end of the proof. For a measure space $\mathcal{T} = (T, \Sigma, \mu)$ we put $\mathfrak{N}(\mu) = \{E \in \Sigma: \mu E = 0\}$, $\Sigma^f = \{E \in \Sigma: \mu E < \infty\}$. Let $\mathfrak{B}(\mu) = \Sigma/\mu$ be the associated measure algebra and $\pi_\mu: \Sigma \to \mathfrak{B}(\mu)$ the canonical homomorphism. We write also $\tilde{E}_\mu$ instead of $\pi_\mu(E)$ and $E \sim G \pmod{\mu}$ if $\tilde{E}_\mu = \tilde{G}_\mu$, $E, G \in \Sigma$. If there is no confusion, we use the notations $\pi$, $\tilde{E}$ and $E \sim G$.

A measure space $\mathcal{T}$ is *a Maharam measure space* if the following conditions are satisfied

1) $\mathcal{T}$ is a *semi-finite measure space*, i.e. for every $E \in \Sigma$ with $\mu E > 0$ there is a set $B \in \Sigma$ such that $B \subset E$ and $0 < \mu B < \infty$;

2) $\mathfrak{B}(\mu)$ is a Dedekind complete Boolean algebra.

Let us note that in general we do not assume for a measure space to be complete. A semi-finite measure space is *locally determined* if $E \in \Sigma$ whenever $E \subset T$ and $E \cap T \in \Sigma$ for every $F \in \Sigma^f$. We say that a measure space $\mathcal{T}$ is *decomposable (strictly localisable)* if there is a partition

$\langle T_i \rangle_{i \in I}$ of T into sets of finite measure such that

$$\Sigma = \{E : E \subset T,\ E \cap T_i \in \Sigma \text{ for each } i \in I\},$$

$$\mu E = \sum_{i \in I} \mu(E \cap T_i) \text{ for } E \in \Sigma.$$

Every decomposable space is Maharam and locally determined [2]. A complete locally determined measure space is decomposable iff there is a *lifting* of this space [3], i.e. a Boolean homomorphism $\rho : \mathfrak{B}(\mu) \to \Sigma$ such that $\rho(\tilde{T}) = T$, $\rho(\tilde{\emptyset}) = \emptyset$ and $\pi_\mu \circ \rho = \mathrm{id}_{\mathfrak{B}(\mu)}$.

We say that two measure spaces $\mathcal{T}_1$ and $\mathcal{T}_2$ are *measure-isomorphic* iff there is a measure-preserving isomorphism between their measure algebras.

Let $\mathcal{T} = (T,\Sigma,\mu)$ be a measure space and X a subset of T. Put

$$\Sigma_X = \{E \cap X : E \in \Sigma\},$$

$$\mu_X F = \inf\{\mu E : E \in \Sigma,\ E \supset F\},\ F \in \Sigma_X.$$

Then $\mathcal{X} = (X,\Sigma_X,\mu_X)$ is a *subspace* of a measure space $\mathcal{T}$ (cf.[1]).

Let $\mathcal{T} = (T,\Sigma,\mu)$ be a measure space and $\varphi : Y \to T$ a map from a nonempty set Y into T. Put $X = \varphi(Y)$,

$$\Xi = \varphi^{-1}(\Sigma_X) = \{\varphi^{-1}(E) : E \in \Sigma_X\},$$

$$\eta(\varphi^{-1}(E)) = \mu_X E, \quad E \in \Sigma_X.$$

Then $\mathcal{Y} = (Y,\Xi,\eta)$ is a measure space which will be called a *preimage* of the space $\mathcal{T}$ (under the map $\varphi$). It is clear that $\mathcal{Y}$ is also a preimage of the subspace $\mathcal{X} = (X,\Sigma_X,\mu_X)$ of $\mathcal{T}$ under the surjective map $\varphi : Y \to X$.

**2. The Stone space of Maharam measure space.** Let $\mathcal{T} = (T,\Sigma,\mu)$ be a Maharam measure space and Q the Stone space of the Boolean algebra $\mathfrak{B}(\mu)$, i.e. an extremally disconnected Hausdorff topological space with a compact topology $\varkappa$ such that the algebra $\mathcal{A}$ of all open-closed subsets of Q is isomorphic to $\mathfrak{B}(\mu)$. The isomorphism $\tau : \mathfrak{B}(\mu) \to \mathcal{A}$ transfers the measure $\mu$ onto $\mathcal{A}$. Let $\mathcal{M}_Q$ be the collection of all subsets of the first category of Q and

$$\Omega = \{A \triangle M : A \in \mathcal{A},\ M \in \mathcal{M}_Q\},$$

where $A \triangle M$ denotes the symmetrical difference of the sets A and M. Then $\Omega$ is the $\sigma$-algebra of subsets of Q generated by

$\mathscr{A}$ and $\mathscr{M}_Q$ (see e.g. [4]) and there exists a unique extension of the measure $\nu = \tau(\mu)$ from $\mathscr{A}$ onto $\Omega$. The space Q is "hyperstonean", in particular, every set from $\mathscr{M}_Q$ is nowhere dense in Q. Now the topological measure space $Q = (Q, \Omega, \nu, \varkappa)$ is the *Stone space* of the Maharam measure space $\mathscr{T}$. It is clear that $\mathscr{T}$ is measure-isomorphic to its Stone space.

Let $\mathfrak{M} = \mathfrak{M}(Q)$ be the class of all Maharam measure spaces that have the same Stone space Q. The class $\mathfrak{M}$ always contains a measure-complete decomposable spaces, e.g. the space $(Q, \Omega, \nu)$. For every $G \in \Omega$ there is a unique $A \in \mathscr{A}$ such that $A \sim G \pmod{\nu}$ and the equality $\sigma_0(\tilde{G}) = A$ defines the unique *strong lifting* $\sigma_0$ *of the space* Q. In the paper [1] D. H. Fremlin has constructed an example of a measure-complete locally determined Maharam space which is not decomposable and thus has no liftings. Therefore the question arises how to describe all measure-complete decomposable spaces of a given class $\mathfrak{M} = \mathfrak{M}(Q)$. We shall see that this problem is connected with the construction of all pairs $(\mathscr{T}, \rho)$ where $\mathscr{T} \in \mathfrak{M}$ and $\rho$ is a lifting of $\mathscr{T}$.

The notations of this section will be essentially used below.

Given a space $\mathscr{T} \in \mathfrak{M}$ one can obtain some new spaces belonging to the same class $\mathfrak{M}$. The assertions of the next proposition are well known.

**PROPOSITION 1.** *Let* $\mathscr{T} = (T, \Sigma, \mu)$ *be a Maharam measure space and* $(Q, \Omega, \nu, \varkappa)$ *its Stone space.*

(a) *If the set* $X \subset T$ *is thick in the space* $\mathscr{T}$ *then the corresponding subspace* $\mathscr{X} = (X, \Sigma_X, \mu_X)$ *is measure-isomorphic to* $\mathscr{T}$.

(b) *The set* $X \subset Q$ *is thick in the space* $(Q, \Omega, \nu)$ *iff it is dense in the topological space* $(Q, \varkappa)$.

Let $\mathscr{T} = (T, \Sigma, \mu)$ and $\mathscr{Y} = (Y, \Xi, \lambda)$ be two measure-isomorphic spaces. We say that *a measure-preserving isomorphism* $u : \mathscr{B}(\mu) \to \mathscr{B}(\lambda)$ is *generated by the map* $\varphi : Y \to T$ if

$$u(\tilde{E}_\mu) = \widetilde{[\varphi^{-1}(E)]_\lambda} \qquad (1)$$

for every $E \in \Sigma$ (cf.[5]). Let us note that the formula (1) defines a measure-preserving isomorphism between $\mathfrak{B}(\mu)$ and $\mathfrak{B}(\lambda)$ iff the following conditions are satisfied:

C.1) $\varphi^{-1}(E) \in \Xi$ for every $E \in \Sigma$;

C.2) $\lambda(\varphi^{-1}(E)) = \mu E$ for every $E \in \Sigma$;

C.3) for every $G \in \Xi$ there is a set $E \in \Sigma$ such that $\varphi^{-1}(E) \sim G \pmod{\lambda}$.

The next proposition follows immediately from the definition of a preimage of a measure space.

**PROPOSITION 2.** *Let $\mathcal{Y} = (Y,\Xi,\eta)$ be the preimage of a space $\mathcal{T} = (T,\Sigma,\mu)$ under the map $\varphi : Y \to T$ such that the set $\varphi(Y)$ is thick in the space $\mathcal{T}$. Then the map $u : \mathfrak{B}(\mu) \to \mathfrak{B}(\eta)$ defined by the equalities*

$$u(\tilde{E}_\mu) = [\widetilde{\varphi^{-1}(E)}]_\eta, \quad E \in \Sigma, \tag{2}$$

*is a measure-preserving isomorphism of $\mathfrak{B}(\mu)$ onto $\mathfrak{B}(\eta)$. If $\mathcal{T}$ is decomposable then so is $\mathcal{Y}$.*

**3. Pointwize maps connected with liftings.** The example in Section 7 below shows that there are Maharam measure spaces with liftings which complete or even locally determined. Therefore we shall consider some constructions connected with liftings of Maharam spaces.

Let $\rho$ be a lifting of a Maharam space $\mathcal{T} = (T,\Sigma,\mu)$ and $Q = (Q,\Omega,\nu,\varkappa)$ the Stone space of $\mathcal{T}$. For each $t \in T$ the set $\mathcal{N}_{\rho,t} = \{\tilde{E} \in \mathfrak{B}(\mu) : t \notin \rho(\tilde{E})\}$ is a maximal ideal of the Boolean algebra $\mathfrak{B}(\mu)$ and we shall identify it with the point $z \in Q$. Thus we write $z = \mathcal{N}_{\rho,t}$ and we obtain a map $\varphi_\rho : T \to Q$ where $\varphi_\rho(t) = \mathcal{N}_{\rho,t} = = z$. In the following propositions we use the notations of the previous section.

**PROPOSITION 3.** *Let $\rho$ be a lifting of a Maharam measure space $\mathcal{T} = (T,\Sigma,\mu)$. Then the corresponding map $\varphi_\rho : T \to Q$ has the following properties:*

(a) *the set $\varphi_\rho(T)$ is dense in $Q$;*

(b) $\rho(\tilde{E}) = \varphi_\rho^{-1}(\tau(\tilde{E}))$ *for every* $E \in \Sigma$;

(c) $\varphi_\rho^{-1}(U) \in \Sigma$ *for every* $U \in \mathcal{A}$;

(d) *if* $M \subset Q$ *is such that the inner measure* $\mu_*(\varphi_\rho^{-1}(M)) = 0$, *then* $M \in \mathcal{M}_Q$;

(e) *for every* $M \in \mathcal{M}_Q$ *the set* $\varphi_\rho^{-1}(M)$ *is locally negligible in the space* $\mathcal{T}$.

**NOTE.** (a) is essentially due to [3], (b)-(e) are taken from [6].

**PROPOSITION 4.** *Let* $\rho$ *be a lifting of measure-complete decomposable space* $\mathcal{T} = (T,\Sigma,\mu)$ *and* $Q = (Q,\Omega,\nu,\varkappa)$ *the Stone space of* $\mathcal{T}$. *Put* $X = \varphi_\rho(T)$. *Then*

(a) *for* $M \in \Omega$ *we have* $\nu M = 0$ *iff* $\mu(\varphi_\rho^{-1}(M)) = 0$;

(b) $\varphi_\rho^{-1}(G) \in \Sigma$ *for every* $G \in \Omega$;

(c) *the map* $\varphi_\rho$: $T \to Q$ *generates a measure-preserving isomorphisms* $v$: $\mathcal{B}(\nu) \to \mathcal{B}(\mu)$ *and* $g$: $\mathcal{B}(\nu_X) \to \mathcal{B}(\mu)$ *whereas the map* $\sigma_0 = \tau \circ v$ *is the strong lifting of the space* $Q$.

**NOTE.** This result is taken from [3] and [6]. For the other properties of the map $\varphi_\rho$ and for the proof of the following theorem see also [6].

**THEOREM 1.** *Let* $\mathcal{T} = (T,\Sigma,\mu)$ *be a Maharam measure space,* $(Q,\Omega,\nu,\varkappa)$ *its Stone space and* $\varphi : T \to Q$ *a map with the following properties*:

(i) $\varphi^{-1}(G) \in \Sigma$ for every $G \in \Omega$;

(ii) *if* $E \in \Sigma$ *and* $\mu E > 0$, *then* $E \cap \varphi^{-1}(\tau(\tilde{E}_\mu)) \neq \emptyset$.

*Then the equalities*

$$\rho(\tilde{E}_\mu) = \varphi^{-1}(\tau(\tilde{E}_\mu)),\ E \in \Sigma, \tag{3}$$

*define the lifting* $\rho$ *of the space* $\mathcal{T}$ *whereas* $\varphi = \varphi_\rho$.

This theorem shows that any lifting $\rho$ of the space $\mathcal{T}$ is uniquely determined by the pointwise map $\varphi : T \to Q$.

**4. Extremal extensions of measure space.** We shall say that a measure space $\mathcal{T} = (T,\Sigma,\mu)$ is an *extremal extension* of a space $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ if the measure $\mu$ extends $\mu_0$ and for every $E \in \Sigma$ there is a set $E_0 \in \Sigma_0$ such that $E \sim E_0 \pmod{\mu}$.

**NOTE**. It seems to be more natural to use the term "equivalent extension", but under the conditions of the previous definition the measure $\mu$ represents a well known extremal extension of measure $\mu_0$ (cf. e.g. [7], [8]).

The following proposition is an easy consequence of the above definition.

**PROPOSITION 5**. *Let $\mathcal{T}$ be an extremal extension of $\mathcal{T}_0$. Then*

(a) *the spaces $\mathcal{T}$ and $\mathcal{T}_0$ are measure-isomorphic;*

(b) *$\mathcal{T}$ is a semi-finite measure space iff $\mathcal{T}_0$ is;*

(c) *$\mathcal{T}$ is a Maharam iff $\mathcal{T}_0$ is.*

**NOTE**. In (a) the isomorphism $w : \mathfrak{B}(\mu_0) \to \mathfrak{B}(\mu)$ is generated by the identity map $e : T \to T$.

For any measure space $\mathcal{T} = (T,\Sigma,\mu)$ its completion $\overline{\mathcal{T}} = (T,\overline{\Sigma},\overline{\mu})$ is obviously an extremal extension of $\mathcal{T}$. A partial case of an extremal extension of a Maharam measure space $\mathcal{T}$ is a *locally determined version* $\mathcal{T}^o = (T,\Sigma^o,\mu^o)$ of $\mathcal{T}$ where

$$\Sigma^o = \{E \subset T : E \cap F \in \Sigma \text{ for every } F \in \Sigma^f\},$$

$$\mu^o E = \sup\{\mu(E \cap F) : F \in \Sigma^f\},\ E \in \Sigma^o$$

(see [1]). The same property has a *complete locally determined version* $\mathcal{T}' = (T,\Sigma',\mu')$ of a Maharam space $\mathcal{T}$ (see [1]) which may be defined as $(\overline{\mathcal{T}})^o$ or $\overline{(\mathcal{T}^o)}$.

Now we shall describe how to construct an extremal extension of a measure space. For a family $\mathcal{G}$ of subsets of $T$ we denote by $sa\mathcal{G}$ the $\sigma$-algebra of subsets of $T$ generated by $\mathcal{G}$.

**PROPOSITION 6**. *Let $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ be a measure space and $\mathfrak{R}$ a family of subsets of $T$ with the following properties:*

(j) *$\mathfrak{N}$ is closed under at most countable unions;*

(jj) *every element of $\mathfrak{N}$ is of inner measure zero.*

*Set $\Sigma = \mathrm{sa}(\Sigma_0 \cup \mathfrak{N})$. Then the measure $\mu_0$ can be uniquely extended to the measure $\mu$ on $\Sigma$ and the space $\mathcal{T} = (T,\Sigma,\mu)$ is an extremal extension of $\mathcal{T}_0$.*

*Conversely, if a space $\mathcal{T} = (T,\Sigma,\mu)$ is an extremal extension of a space $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$, then $\mathcal{T}$ may be obtained from $\mathcal{T}_0$ and from a suitable family $\mathfrak{N}$ as described above.*

*Proof.* The first assertion is well known (see e.g. [9]). To prove the converse, denote $\mathfrak{N} = \mathfrak{N}(\mu)$. Then $\mathfrak{N}$ satisfies (j) and (jj). Clearly $\mathrm{sa}(\Sigma_0 \cup \mathfrak{N}) \subset \Sigma$. For every $E \in \Sigma$ there is a set $E_0 \in \Sigma_0$ such that $E \sim E_0$ (mod $\mu$). Now we have $E = E_0 \triangle N$, where $N \in \mathfrak{N}(\mu) = \mathfrak{N}$. Thus $E \in \mathrm{sa}(\Sigma_0 \cup \mathfrak{N})$ so that $\Sigma = \mathrm{sa}(\Sigma_0 \cup \mathfrak{N})$. *

**REMARK.** If $\mathfrak{N}$ is a $\sigma$-ideal of subsets of T, then (jj) means that $\mathfrak{N} \cap \Sigma_0 \subset \mathfrak{N}(\mu_0)$. We have $\mathrm{sa}(\Sigma_0 \cup \mathfrak{N}) = \{A \triangle N : A \in \Sigma_0,\ N \in \mathfrak{N}\}$ and the extension $\mu$ of $\mu_0$ is defined by $\mu(A \triangle N) = \mu_0 A$. In this case the measure space $\mathcal{T} = (T,\Sigma,\mu)$ is complete if one of the following sufficient conditions aresatisfied: (a) $\mathcal{T}_0$ is measure-complete; (b) $\mathfrak{N}(\mu_0) \subset \mathfrak{N}$ i.e. $\mathfrak{N} \cap \Sigma_0 = \mathfrak{N}(\mu_0)$. In this case $\mathfrak{N} = \mathfrak{N}(\mu)$.

**THEOREM 2.** *Let $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ be a measure space and $\mathfrak{N}$ a $\sigma$-ideal of subsets of T such that $\mathfrak{N} \cap \Sigma_0 = \mathfrak{N}(\mu_0)$. Then the space $\mathcal{T} = (T,\Sigma,\mu)$, where $\Sigma = \mathrm{sa}(\Sigma_0 \cup \mathfrak{N})$ and $\mu(A \triangle N) = \mu_0 A$ for $A \in \Sigma_0$, $N \in \mathfrak{N}$, is a measure-complete extremal extension of the space $\mathcal{T}_0$ whereas $\mathfrak{N} = \mathfrak{N}(\mu)$. Conversely, if $\mathcal{T} = (T,\Sigma,\mu)$ is a measure-complete extremal extension of a measure-space $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$, then it coincides with the extremal extension of $\mathcal{T}_0$ constructed by $\Sigma_0$ and by the $\sigma$-ideal $\mathfrak{N} = \mathfrak{N}(\mu)$.*

*Proof.* From Proposition 6 we obtain that the space $\mathcal{T} = (T,\Sigma,\mu)$ is an extremal extension of $\mathcal{T}_0$. The equality $\mathfrak{N} = \mathfrak{N}(\mu)$ and thus the completeness of $\mathcal{T}$ are both obvious.

Conversely, let $\mathcal{T}$ be a measure-complete extremal extension of $\mathcal{T}_0$. Put $\mathfrak{N} = \mathfrak{N}(\mu)$. Then $\mathfrak{N}$ is a $\sigma$-ideal of subsets of $T$, $\mathfrak{N}(\mu_0) \subset \mathfrak{N}(\mu) = \mathfrak{N}$, $\Sigma_0 \subset \Sigma$, the measure $\mu$ extends $\mu_0$ and for each $E \in \Sigma$ there is a set $A \in \Sigma_0$ such that $\mu(A \Delta E) = 0$. Hence $\mu(A \setminus E) = \mu(E \setminus A) = 0$. Set $A \setminus E = N_1 \in \mathfrak{N}$, $E \setminus A = N_2 \in \mathfrak{N}$ and $N = N_1 \cup N_2 \in \mathfrak{N}$. Then $E = A \Delta N$. From this we conclude that

$$\Sigma \subset \{A \Delta N : A \in \Sigma_0,\ N \in \mathfrak{N}\} = sa(\Sigma_0 \cup \mathfrak{N}).$$

Since the converse inclusion is obvious, we have $\Sigma = sa(\Sigma_0 \cup \mathfrak{N})$. It is clear that $\mathfrak{N} \cap \Sigma_0 = \mathfrak{N}(\mu_0)$ and $\mu(A \Delta N) = \mu_0 A$ for every $A \in \Sigma_0$, $N \in \mathfrak{N}$. *

5. **Constructing new spaces with liftings.** Now we shall show how to construct new measure spaces with liftings using a given space $\mathcal{T}$ with a lifting $\rho$.

**PROPOSITION 7.** *Let X be a thick subset of a measure space $\mathcal{T} = (T,\Sigma,\mu)$. If $\mathcal{T}$ has a lifting, then so has the space $\mathcal{X} = (X,\Sigma_X,\mu_X)$.*

*Proof*. Let $\rho$ be a lifting of $\mathcal{T}$. Applying to Proposition 1 (or Proposition 2) we denote by $h : \mathfrak{B}(\mu_X) \to \mathfrak{B}(\mu)$ the measure-preserving isomorphism (induced by inclusion $X \subset T$). Let $\pi^* : \Sigma_X \to \mathfrak{B}(\mu_X)$ be the canonical homomorphism. Put $A^* = \pi^*(A)$ and

$$\rho'(A^*) = \rho(h(A^*)) \cap X, \quad A \in \Sigma_X.$$

It is easy to check that $\rho' : \mathfrak{B}(\mu_X) \to \Sigma_X$ is a lifting of the space $\mathcal{X}$. *

**PROPOSITION 8.** *Let $\mathcal{Y} = (Y,\Xi,\eta)$ be the preimage of a measure space $\mathcal{T} = (T,\Sigma,\mu)$ under the map $\varphi : Y \to T$ such that $\varphi(Y)$ is a thick subset of T. Let also $u : \mathfrak{B}(\mu) \to \mathfrak{B}(\eta)$ be the measure-preserving isomorphism generated by $\varphi$, and $\rho$ a lifting of the space $\mathcal{T}$. Then the equalities*

$$\sigma(\tilde{G}_\eta) = \varphi^{-1}(\rho(u^{-1}(\tilde{G}_\eta))), \quad G \in \Xi, \tag{4}$$

*define a lifting of the space $\mathcal{Y}$.*

*Proof*. It is clear that $\sigma(\tilde{Y}_\eta) = Y$, $\sigma(\tilde{\emptyset}_\eta) = \emptyset$ and $\sigma$ is a Boolean homomorphism. Therefore we must prove only that $\pi_\eta \circ \sigma = \mathrm{id}_{\mathcal{B}(\eta)}$, in other words that $\sigma(\tilde{G}_\eta) \sim G \pmod{\eta}$ for every $G \in \Xi$. Take a set $E \in \Sigma$ such that $G = \varphi^{-1}(E)$. By Proposition 2, we have

$$\tilde{G}_\eta = [\widetilde{\varphi^{-1}(E)}]_\eta = u(\tilde{E}_\mu),$$

whence $\tilde{E}_\mu = u^{-1}(\tilde{G}_\eta)$ and

$$\sigma(\tilde{G}_\eta) \,\Delta\, G = \sigma(\tilde{G}_\eta) \,\Delta\, \varphi^{-1}(E) = \varphi^{-1}(E \,\Delta\, \rho(u^{-1}(\tilde{G}_\eta))) =$$
$$= \varphi^{-1}(E \,\Delta\, \rho(\tilde{E}_\mu)).$$

As $E \,\Delta\, \rho(\tilde{E}_\mu) \sim \emptyset \pmod{\mu}$, we conclude that $\sigma(\tilde{G}_\eta) \sim G \pmod{\eta}$.*

**PROPOSITION 9.** *Let $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ be a measure space, $\mathcal{T} = (T,\Sigma,\mu)$ its extremal extension and $w : \mathcal{B}(\mu_0) \to \mathcal{B}(\mu)$ the measure-preserving isomorphism generated by the identity map $e : T \to T$. Then for every lifting $\rho_0$ of the space $\mathcal{T}_0$ there exists a unique lifting $\rho$ of the space $\mathcal{T}$ such that*

$$\rho(\tilde{E}_\mu) = \rho_0(w^{-1}(\tilde{E}_\mu))$$

*for every $E \in \Sigma$. Moreover, if $\tau : \mathcal{B}(\mu) \to \mathcal{A}$ and $\tau_0 : B(\mu_0) \to \mathcal{A}$ are the canonical maps then $w = \tau^{-1} \circ \tau_0$.*

The proof is easy. Let us remark that for any lifting $\rho$ of the space $\mathcal{T}$ one can consider the map $\rho' : \Sigma \to \Sigma$ defined as follows

$$\rho'(E) = \rho(\tilde{E}_\mu), \ E \in \Sigma.$$

The map $\rho'$ is called a *lifting* of the $\sigma$-algebra $\Sigma$ (cf. [3]). Thus, Proposition 9 may be formulated as follows. Let $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ be a measure space and $\mathcal{T} = (T,\Sigma,\mu)$ its extremal extension. Then every lifting $\rho_0'$ of $\Sigma_0$ can be uniquely extended to the lifting $\rho'$ of $\Sigma$.

**6. A description of Maharam measure spaces with liftings.** Now we shall prove the main theorems.

**PROPOSITION 10.** *Let a measure space $\mathcal{T} = (T,\Sigma,\mu)$ be an extremal extension of the preimage of some dense subspace*

*$\mathcal{X} = (X,\Omega_X,\nu_X)$ of hyperstonean space $Q = (Q,\Omega,\nu,\varkappa)$ under some surjective map $\varphi : T \to X$. Then there is a lifting $\rho$ of the space $\mathcal{T}$ such that $\varphi = \varphi_\rho$.*

*Proof*. Denote by $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ the preimage of the space $\mathcal{X}$ under the map $\varphi : T \to X$. It is clear that $\mathcal{T}_0$ is also the preimage of the space $Q$. By Propositions 1, 2 and 5, the spaces $\mathcal{T}$, $\mathcal{T}_0$, $\mathcal{X}$ and $Q$ are measure-isomorphic. Consider the canonical maps $\pi : \Sigma \to \mathfrak{B}(\mu)$, $\pi_0 : \Sigma_0 \to \mathfrak{B}(\mu_0)$ and the canonical isomorphisms $\tau : \mathfrak{B}(\mu) \to \mathcal{A}$, $\tau_0 : \mathfrak{B}(\mu_0) \to \mathcal{A}$. Let $\sigma_0 : \mathfrak{B}(\nu) \to \mathcal{A}$ be the strong lifting of the space $Q$. Put $\tilde{E} = \pi(E), A^* = \pi_0(A)$ for $E \in \Sigma$, $A \in \Sigma_0$. Let $u : \mathfrak{B}(\nu) \to \mathfrak{B}(\mu_0)$ be the measure-preserving isomorphism generated by $\varphi$.

$$\rho_0(E^*) = \varphi^{-1}(\sigma_0(u^{-1}(E^*))),\ E \in \Sigma_0,$$

define a lifting of the space $\mathcal{T}_0$. From the assertion (c) of Proposition 4 we obtain $\sigma_0 = \tau_0 \circ u$, so that $\sigma_0 \circ u^{-1} = \tau_0$ and

$$\rho_0\ (E^*) = \varphi^{-1}(\tau_0(E^*))$$

for every $E \in \Sigma_0$.

Let $w : \mathfrak{B}(\mu_0) \to \mathfrak{B}(\mu)$ be the measure-preserving isomorphism generated by the identity map $e : T \to T$. By Proposition 9, there is a unique lifting $\rho$ of the space $\mathcal{T}$ such that

$$\rho(\tilde{E}) = \rho_0(w^{-1}(\tilde{E}))$$

for every $E \in \Sigma$. Since $w = \tau^{-1} \circ \tau_0$ we have $\tau_0 \circ w^{-1} = \tau$ and

$$\rho(\tilde{E}) = \rho_0(w^{-1}(\tilde{E})) = \varphi^{-1}(\tau_0(w^{-1}(\tilde{E}))) = \varphi^{-1}(\tau(\tilde{E})). \qquad (5)$$

From (5) and the assertion (b) of Proposition 3 it follows that $\varphi^{-1}(U) = \varphi_\rho^{-1}(U)$ for every open-closed set $U \in \mathcal{A}$. The Stone space $Q$ is a Hausdorff topological space and $\mathcal{A}$ is a base of its topology. Therefore $\varphi = \varphi_\rho$.*

**THEOREM 3.** *Let $\mathcal{T} = (T,\Sigma,\mu)$ be a measure-complete locally determined Maharam measure space. The following assertions are equivalent:*

*(1) $\mathcal{T}$ is decomposable.*

*(2) There is a lifting of the space $\mathcal{T}$.*

*(3) The space $\mathcal{T}$ is an extremal extension of the preimage of some dense subspace $\mathcal{X} = (X,\Omega_X,\nu_X)$ of the Stone*

*space* $Q = (Q,\Omega,\nu,*)$ *of* $\mathcal{T}$ *under some surjective map* $\varphi : T \to X$.

*Proof*. (1) $\Leftrightarrow$ (2) is well known (see e.g. [3]). (3) $\Rightarrow$ (2) follows from Proposition 10.

Check that (2) $\Rightarrow$ (3). Let $\rho$ be a lifting of the space $\mathcal{T}$ and $\varphi_\rho$: $T \to Q$ the corresponding map. Put $X = \varphi_\rho(T)$ and $\mathcal{X} = (X,\Omega_X,\nu_X)$. By Propositions 1 and 3, the set $X$ is dense in $Q$. Let $\mathcal{T}_0,\Sigma_0,\mu_0)$ be the preimage of the space $\mathcal{X}$ under the map $\varphi_\rho$. Applying to Proposition 4 we obtain that $\varphi_\rho^{-1}(G \cap X) = \varphi_\rho^{-1}(G) \in \Sigma$ for every $G \in \Omega$ and therefore $\varphi_\rho^{-1}(G) \in \Sigma$ for every $G \in \Omega_X$ so that $\Sigma_0 \subset \Sigma$. Clearly $\mathcal{T}_0$ is also the preimage of $Q$ under the map $\varphi_\rho$.

Consider the canonical homomorphism $\pi : \Sigma \to \mathfrak{B}(\mu)$ and put $\tilde{E} = \pi(E)$, for $E \in \Sigma$. By the assertion (b) of Proposition 3 we have $\rho(\mathfrak{B}(\mu)) \subset \Sigma_0$. Hence for each $E \in \Sigma$ there is a set $E_0 \in \Sigma_0$ such that $E \sim E_0$ (mod $\mu$), namely we can take $E_0 = \rho(\tilde{E})$. Now we have only to prove that the measure $\mu$ extends $\mu_0$.

In fact, if $E, F \in \Sigma_0$ and $E \sim F$ (mod $\mu$), then $\mu(E \,\Delta\, F) = 0$. As $E \,\Delta\, F \in \Sigma_0$ there is a set $M \in \Omega$ such that $E \,\Delta\, F = \varphi_\rho^{-1}(M)$. By the assertion (a) of Proposition 4 we have $\nu M = 0$. This yields that $\mu_0(E \,\Delta\, F) = \nu M = 0$. Since $E \sim \rho(\tilde{E})$ (mod $\mu$) for every $E \in \Sigma_0$, it follows that $E \sim (\tilde{E})$ (mod $\mu_0$), whence $\mu_0 E = \mu_0(\rho(\tilde{E}))$.

Finally for every $E \in \Sigma_0$ we have by Propositions 3 and 4 $\mu\tilde{E} = \mu(\rho(\tilde{E})) = \nu_X(\varphi_\rho(\rho(\tilde{E}))) = \mu_0(\varphi_\rho^{-1}(\varphi_\rho(\rho(\tilde{E})))) = \mu_0(\rho(\tilde{E})) = \mu_0 E$.
Thus, the measure $\mu$ extends $\mu_0$.*

Since the completeness of a measure is not used in the proof of the implication (2) $\to$ (1) in Theorem 3, we have also the following theorem.

**THEOREM 4**. *A locally determined Maharam measure space which has a lifting, is decomposable.*

**THEOREM 5**. *Let* $\mathcal{T} = (T, \Sigma, \mu)$ *be a Maharam measure space and* $Q = (Q,\Omega,\nu,*)$ *its Stone space. The following assertions are equivalent:*

(1) $\mathcal{T}$ *is measure-complete and decomposable.*

(2) $\mathcal{T}$ *coincides with the complete locally determined*

*version of an extremal extension of the preimage of the Stone space* $Q$ *under such map* $\varphi : T \to Q$ *that* $\varphi(T)$ *is a dense subset in* $Q$.

*Proof.* (1) $\Rightarrow$ (2) follows from Theorem 3. The implication (2) $\Rightarrow$ (1) follows from Propositions 1, 2, 8, 9, Theorem 4 and the definition of complete locally determined version.*

From Theorem 5 it follows that every measure-complete decomposable space may be obtained in four steps at most: 1) take a suitable dense subset $X$ of the corresponding Stone space, 2) take a preimage of the subspace $\mathcal{X} = (X,\Omega_X,\nu_X)$ under some surjective map $\varphi : T \to X$, 3) take a suitable extremal extension of the previous space, 4) take the complete locally determined version of the last space.

In the paper [10] V. L. Levin has stated the question whether there always exists a *separating lifting* of the measure-complete decomposable space $\mathcal{T} = (T,\Sigma,\mu)$, i.e. such lifting $\rho$ that for every $t_1,t_2 \in T$ ($t_1 \neq t_2$) there is a set $E \in \Sigma$ with $t_1 \in \rho(E)$ and $t_2 \notin \rho(E)$. Clearly $\rho$ is a separating lifting iff the map $\varphi_\rho : T \to Q$ is injective. Now we can easily see that the answer is in general "no". Let $\mathcal{T}$ be the preimage of a hyperstonean space $(Q,\Omega,\nu,\varkappa)$ under a surjective map $\varphi : T \to Q$. Then $\mathcal{T}$ is decomposable, but if card $T >$ card $Q$, then there is no separating lifting of the space $\mathcal{T}$. The completion $\overline{\mathcal{T}}$ of $\mathcal{T}$ is obviously a measure--complete decomposable space which has no separating liftings.

Finally we shall describe all Maharam measure spaces $\mathcal{T} = (T,\Sigma,\mu)$ with a given Stone space $(Q,\Omega,\nu,\varkappa)$ that have a lifting. Let $\mathcal{D}$ be the $\sigma$-algebra of Baire sets in $Q$ and $\nu_0$ the restriction of $\nu$ to $\mathcal{D}$. Then the space $(Q,\mathcal{D},\nu_0)$ has obviously the natural strong lifting $\sigma : \mathcal{B}(\nu_0) \to \mathcal{A}$ and $(Q,\Omega,\nu)$ is an extremal extension of $(Q,\mathcal{D},\nu_0)$.

**THEOREM 6.** *Let* $\mathcal{T} = (T,\Sigma,\mu)$ *be a Maharam measure space,* $Q = (Q,\Omega,\nu,\varkappa)$ *its Stone space and* $(Q,\mathcal{D},\nu_0)$ *the corresponding measure space with Baire sets as measurable ones. Then* $\mathcal{T}$ *has a lifting iff it is an extremal extension of the preimage of* *the space* $(Q,\mathcal{D},\nu_0)$ *under such map* $\varphi : T \to Q$ *that* $\varphi(T)$ *is a dense subset of* $Q$.

*Proof*. Let $\rho$ be a lifting of $\mathcal{T}$ and $\mathcal{T}_0 = (T,\Sigma_0,\mu_0)$ the preimage of $(Q,\mathcal{D},\nu_0)$ under the map $\varphi_\rho : T \to Q$. Then $\varphi_\rho^{-1}(G) \in \Sigma$ for every $G \in \mathcal{D}$ so that $\Sigma_0 \subset \Sigma$. It is clear that for every $E \in \Sigma$ there is a set $\tilde{E}_0 \in \Sigma_0$ such that $E \sim E_0$ (mod $\mu$) (one can take $E_0 = \rho(E\mu)$).

Now we shall prove that $\mu$ extends $\mu_0$. If $E \in \Sigma_0$ and $\mu E = 0$ then $\mu_0 E = 0$. In fact, $E = \varphi_\rho^{-1}(A)$ for some $A \in \mathcal{D} \subset \Omega$ and if we assume that $\mu_0 E \neq 0$, then $\nu A \neq 0$. In this case we have Int $A \neq \emptyset$ and there is a set $\tilde{G} \in \mathcal{A}$, $G \neq \emptyset$, such that $G \subset A$. Then $E \supset \varphi_\rho^{-1}(\tilde{G}) = \rho(\tilde{H})$ for some $H \in \Sigma$ whereas $\rho(\tilde{H}) \neq \emptyset$, whence $\mu\rho(H) > 0$ and $\mu E > 0$. Now we have $E \sim \rho(\tilde{E})$ (mod $\mu$) for every $E \in \Sigma_0$, i.e. $E \,\Delta\, \rho(\tilde{E}) \sim \emptyset$ (mod $\mu$). From this it follows that $E \,\Delta\, \rho(\tilde{E}) \sim \emptyset$ (mod $\mu_0$), i.e. $\mu_0 E = \mu_0\rho(\tilde{E})$. Finally for every $E \in \Sigma_0$ we obtain
$\mu E = \mu\rho(\tilde{E}) = \nu(\tau(\tilde{E})) = \nu_0\varphi(\rho(\tilde{E})) = \mu_0\varphi_\rho^{-1}(\varphi_\rho(\rho(\tilde{E}))) = \mu_0\rho(\tilde{E}) =$
$= \mu_0 E$.

The converse assertion follows immediately from Propositions 1, 8 and 9.*

Let us remark that from Theorem 6 it follows that the spaces $(X,\mathcal{D}_X,\nu_{0_X})$, where $X$ is dense subset in $Q$, are the "poorest" spaces with liftings that have the given Stone space $(Q,\Omega,\nu,\varkappa)$.

7. **Example**. Let $Q = (Q,\Omega,\nu,\varkappa)$ be a hyprestonean extremally disconnected compact space with a sufficiently positive semi-finite measure $\nu$ and let us assume that $\nu$ is not $\sigma$-finite. Then there is a disjoint family $\langle Q_i\rangle_{i\in I}$ of open- -closed subsets of $Q$ with $\nu Q_i < \infty$, $i \in I$, such that

$$\nu(Q \setminus \bigcup_{i\in I} Q_i) = 0$$

(cf. [4]). Let $T_1$ and $T_2$ be two disjoint dense subsets of $Q$ such that $T_1 \cup T_2 = Q$; note that $T_1, T_2 \notin \Omega$. Denote by $\mathfrak{N}$ the collection of all subsets of $T_2$ that have nonempty intersection with at most countable family of sets $Q_i$. Then $\mathfrak{N}$ is a $\sigma$-ideal of subsets of $T_2$ and $Q$. Consider the space $\mathcal{T} = (Q,\Sigma,\mu)$ where $\Sigma = sa(\Omega \cup \mathfrak{N})$ and $\mu(E \,\Delta\, N) = \nu E$, $E \in \Omega$, $N \in \mathfrak{N}$, constructed as in Proposition 6 (see also the corresponding remark). The space $\mathcal{T}$ is an extremal extension of the space Q and by Proposition 9, there is a lifting of the space $\mathcal{T}$. It is clear that $\mathcal{T}$ is a measure-complete Maharam space.

Now we show that $\mathcal{T}$ is not locally determined. We shall verify that the set $T_2$ is locally negligible (whence it is locally measurable), otherwise it is easy to see that $T_2 \notin \mathfrak{N}$ and $T_2 \notin \Sigma$.

Let $F \in \Sigma^f$, then $F = E \triangle N$ where $E \in \Omega^f$ and $N \in \mathfrak{N}$. By the definition of the $\sigma$-algebra $\Omega$ (see Section 2) we have also $E = A \triangle M$ where $A \in \mathcal{A}$, $\nu A < \infty$ and $M \in \mathfrak{N}(\nu) = \mathcal{N}_Q$. As the space Q is decomposable, there exists an at most countable set $J \subset I$ and a set $K \in \mathfrak{N}(\nu)$ such that

$$A = \bigcup_{i \in J} (A \cap Q_i) \cup K.$$

We have

$$T_2 \cap F = T_2 \cap ((A \triangle M) \triangle N) = (T_2 \cap A) \triangle (T_2 \cap (M \triangle N)).$$

Put $L = T_2 \cap (M \triangle N)$. As $M \triangle N \in \mathfrak{N}(\mu)$ and $\mathcal{T}$ is measure-complete, we deduce that $L \in \mathfrak{N}(\mu)$. Now we have $T_2 \cap F = (T_2 \cap A) \triangle L$ and

$$T_2 \cap A = T_2 \cap [\bigcup_{i \in J} (A \cap Q_i) \cup K] =$$

$$= [\bigcup_{i \in J} A \cap (T_2 \cap Q_i)] \cup (T_2 \cap K).$$

Again because $\mathcal{T}$ is measure-complete we have $T_2 \cap K \in \mathfrak{N}(\mu)$. As $T_2 \cap Q_i \in \mathfrak{N} \subset \mathfrak{N}(\mu)$ for each $i \in I$ and $\mathfrak{N}$ is a $\sigma$-ideal, we obtain that $T_2 \cap A \in \mathfrak{N}(\mu)$. Therefore $T_2 \cap F \in \mathfrak{N}(\mu)$, consequently, $T_2 \cap F \in \Sigma$. Thus $T_2 \cap F \in \Sigma$ for every $F \in \Sigma^f$, but $T_2 \notin \Sigma$, which means that the space $\mathcal{T}$ is not locally determined.

If we take the preimage $\mathcal{Y} = (Y, \Xi, \eta)$ of the space $\mathcal{T}$ under a surjective but not injective map $\varphi : Y \to Q$, we obtain a Maharam measure space which has a lifting but is neither measure-complete nor locally determined.

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Liftingut omavate mõõduga ruumide kirjeldus
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Resümee

Artiklis [1] on D. H. Fremlin konstrueerinud näite sellisest täieliku mõõduga lokaliseeruvast (artikli [10]

mõttes) Maharami ruumist, mis pole rangelt lokaliseeruv ning seepärast ei oma liftingut. Käesolevas artiklis uuritakse liftinguga ruumide konstrueerimist ja antakse näiteks kõigi täieliku mõõduga rangelt lokaliseeruvate ruumide kirjeldus, millel on üks ja seesama Stone'i ruum. On toodud näide niisugusest Maharami ruumist, mis omab liftingut, kuid pole täielik mõõdu järgi ega lokaliseeruv.

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**REMARKS ON THE DUAL OF THE SPACE OF CONTINUOUS LINEAR OPERATORS**

**Eve Oja**

1. Let $L(E,F)$ and $K(E,F)$ be the Banach spaces of continuous linear and compact operators from a Banach space $E$ into a Banach space $F$. In [5] we obtained three decomposition theorems for the dual $L(E,F)^*$ into the direct sum of $K(E,F)^\perp$ and a subspace $K(E,F)^\#$ isometrically isomorphic to $K(E,F)^*$ in the case where $E$ or $F$ are closed subspaces in Banach spaces having certain variants of metric compact approximation property (MCAP in short).

The present note is an appendix to [5]. We shall prove a result announced in [5] in a slightly more general form. This permits us to deduce the quotient spaces variants of the composition theorems of [5]. This permits us also to show that the property "$K(X,X)$ is an M-ideal in $L(X,X)$" is preserved by passing to closed subspaces of $X$ which are as well quotients of $X$ and which have the MCAP. This answers partially a question mentioned in [7].

2. Let $X$ be a Banach space. Let $I_X$ denote the identity on $X$. Let $j : E \to X$ denote the canonical injection for a closed subspace $E$ of $X$, and let $q : X \to E$ denote the canonical surjection for a quotient $E$ of $X$. The following result was announced in [5] for the subspace case.

**PROPOSITION.** *Let $X^*$ or $X^{**}$ have the Radon-Nikodym property. Suppose that there is a net $(A_\alpha)$ in $K(X,X)$ such that*

$$\lim_\alpha A_\alpha^* x^* = x^* \quad \textit{for all } x^* \in X^*. \tag{1}$$

*Let $E$ be a closed subspace or a quotient of $X$. If $E$ has the*

*MCAP and the unique extension property (UEP in short), then there are a net* $(P_m)_{m\in\Delta}$ *of convex combinations of* $(A_\alpha)$ *and a net* $(S_m)_{m\in\Delta}$ *in the unit ball of* $K(E,E)$ *such that*

$$\lim_{m\in\Delta} \|P_m j - jS_m\| = 0 \text{ or } \lim_{m\in\Delta} \|qP_m - S_m q\| = 0. \tag{2}$$

*Proof.* Let $(T_\beta)$ be a net in the unit ball of $K(E,E)$ such that $\lim_\beta T_\beta e = e$ for all $e \in E$. The $w^*$-compacteness of the unit ball of $L(E^{**},E^{**}) = (E^* \hat{\otimes} E^{**})^*$ and the UEP permit us to conclude that $T_\beta^* \to I_{E^*}$ in the $w^*$-topology (cf. [2], Theorem 2.2). Let $B_{(\alpha,\beta)} = A_\alpha j - jT_\beta$ in the subspace case or $C_{(\alpha,\beta)} = qA_\alpha - T_\beta q$ in the quotient case, where $\{(\alpha,\beta)\}$ is directed by the product ordering. Consider $X^* \otimes E^{**}$ and $E^* \otimes X^{**}$ as vector subspaces of $K(E,X)^*$ and $K(X,E)^*$ respectively. These subspaces are norm-dense since $X^*$ and $E^*$ (or $E^{**}$ and $X^{**}$) have the Radon-Nikodym property (cf. e.g. [2], p.674). Therefore it is clear that $B_{(\alpha,\beta)} \to 0$ weakly in $K(E,X)$ for the subspace case, and $C_{(\alpha,\beta)} \to 0$ weakly in $K(X,E)$ for the quotient case. The convex combinations of these nets which converge to zero in norm (cf. [1], p.40) will give us the nets satisfying (2) (note that $\Delta = \{(\alpha,\beta)\} \times \mathbb{N}$ ).

3. Using Proposition, one can prove in the similar fashion the quotient spaces variants of Theorems 1 and 2 of [5]. For formulate these results, one must simply replace the word "closed subspace" by " quotient" in Theorems 1 and 2 of [5]. So we do not reformulate these results here and we mention only some of their corollaries.

**COROLLARY 1.** *Let* $E$ *be a Banach space and* $F$ *a quotient or a closed subspace of* $l_p(\Gamma)$ *or* $d(w,p)$, $1< p < \infty$. *If* $F$ *has the CAP then* $K(E,F)$ *is an HB-subspace.*

**COROLLARY 2.** *Let* $F$ *be a Banach space and* $E$ *a quotient or a closed subspace of* $l_p(\Gamma)$ *or* $d(w,p)^*$, $1< p < \infty$. *If* $E$ *has the CAP then* $K(E,F)$ *is an HB-subspace.*

The following result is a generalization of Theorem 3 from [5] and Proposition 1.2 from [6]. For $A_\alpha \in L(X,X)$ (where $\alpha$ is an index) we put $A^\alpha = I_X - A_\alpha$.

**THEOREM 1.** *Let* X *and* Y *be two Banach spaces such that* $X^*$ *or* $X^{**}$ *and* $Y^*$ *or* $Y^{**}$ *have the Radon-Nikodym property. Suppose that there are two nets* $(A_\alpha)$ *and* $(B_\beta)$ *in the unit balls of* $K(X,X)$ *and* $K(Y,Y)$ *respectively, satisfying the conditions (1) and*

$$B_\beta y \to y \quad \text{for all} \quad y \in Y, \tag{3}$$

$$B_\beta^* y^* \to y^* \quad \text{for all} \quad y^* \in Y^*. \tag{4}$$

*Suppose moreover that there are* $\lambda > 0$ *and two functions* $N_1$ *and* $N_2$ *on* $[0,\infty) \times [0,\infty)$ *such that* $N_2$ *is convex,* $N_2(a,b) \le N_2(c,d) \le N_1(c,d/\lambda)$ *for* $a \le c$, $b \le d$, *and that for all* $\varepsilon > 0$ *there are* $\alpha_0$ *and* $\beta_0$ *such that for all* $\alpha > \alpha_0$ *and* $\beta > \beta_0$ *we have*

$$N_1(\| A_\alpha x \|, \| A^\alpha x \|) \le (1+\varepsilon) \| x \|,$$

$$\| B_\beta y + B^\beta z \| \le (1+\varepsilon) N_2(\| y \|, \| z \|)$$

*for all* $x \in X$ *and* $y, z \in Y$. *Let* E *and* F *be quotients or closed subspaces of* X *and* Y *respectively. If* E *and* F *have the MCAP and the UEP then* $K(E,F)$ *has the property SU and* $\| g \| + \lambda \| h \| \le \| f \|$ *for* $f = g + h \in L(E,F)^*$, $g \in K(E,F)^\#$, $h \in K(E,F)^\perp$.

*Proof.* By Proposition, there are the nets $(P_m)$, $(S_m)$ and $(Q_n)$, $(T_n)$ corresponding to $(A_\alpha)$ and $(B_\beta)$ respectively. In view of the nets variant of Theorem 2 in [11], we have to show that E with $(S_m)$ and F with $(T_n)$ satisfy (1) and (3), (4) respectively, which is obvious, and that

$$\overline{\lim_m}\ \overline{\lim_n} \| T_n S S_m + \lambda T^n T S^m \| \le 1 \tag{5}$$

for all S and T in the unit ball of $L(E,F)$.

Let us first consider the case where F is a subspace. Then we have

$$\overline{\lim_n} \| T_n S S_m + \lambda T^n T S^m \| \le$$

$$\leq \overline{\lim_{n}} \sup_{\|e\|<1} \|Q_n SS_m e + \lambda Q^n TS^m e\| \leq$$

$$\leq \overline{\lim_{\beta}} \sup_{\|e\|<1} \|B_\beta SS_m e + \lambda B^\beta TS^m e\| .$$

Let $\varepsilon$ be any positive number. Then for all $\beta > \beta_0$, m and $e \in E$, $\| e \| < 1$, we have

$$\|B_\beta SS_m e + \lambda B^\beta TS^m e\| \leq (1 + \varepsilon) N_2 (\|S_m e\|, \lambda \|S^m e\|).$$

Denote $\varepsilon_m = \|P_m j - jS_m\|$ if E is a subspace and $\varepsilon_m = \|qP_m - S_m q\|$ if E is a quotient (we may suppose that $\varepsilon_m < 1$). Then for some $x \in X$, $\| x \| \leq 1$, and all m

$$N_2 (\|S_m e\|, \lambda \|S^m e\|) \leq N_2(\varepsilon_m + \|P_m x\|, \lambda\varepsilon_m + \lambda \|P^m x\| =$$

$$= N_2(\varepsilon_m(1+\|P_m x\|) + (1-\varepsilon_m)\|P_m x\|, \varepsilon_m(\lambda+\lambda\|P^m x\|)+(1-\varepsilon_m)\lambda\|P^m x\|) \leq$$

$$\leq \varepsilon_m N_2(1+ \|P_m x\|, \lambda+\lambda\|P^m x\|) + (1- \varepsilon_m)N_2(\|P_m x\|, \lambda\|P^m x\|) \leq$$

$$\leq \varepsilon_m N_2(3, 4\lambda) + (1- \varepsilon_m)N_2(\|P_m x\|, \lambda\|P^m x\|).$$

And since

$$\overline{\lim_{m}} \sup_{\|x\|\leq 1} N_2(\|P_m x\|, \lambda\|P^m x\|) \leq$$

$$\leq \overline{\lim_{\alpha}} \sup_{\|x\|\leq 1} N_2(\|A_\alpha x\|, \lambda\|A^\alpha x\|),$$

we are done.

Let us finally suppose that F is a quotient. Let $\varepsilon < 1$ be any positive number. We have

$$\overline{\lim_{n}} \|T_n SS_m + \lambda T^n TS^m\| \leq$$

$$\leq \overline{\lim_{n}} \sup \|Q_n y + \lambda Q^n z\| \leq$$

$$\leq \overline{\lim_{\beta}} \sup \|B_\beta y + \lambda B^\beta z\| \leq$$

$$\leq (1 + \varepsilon) \sup N_2(\|y\|, \lambda\|z\|),$$

where the supremum is taken over all those y and z in Y that the canonical surjection sends to $SS_m e$ and $TS^m e$ respectively for some $e \in E$, $\|e\| < 1$, so that the conditions $\|y\| \leq \|SS_m e\| + \varepsilon$ and $\|z\| \leq \|TS^m e\| + \varepsilon$ are fulfilled. Since

$$N_2(\|y\|, \lambda\|z\|) \leq N_2(\varepsilon + \|S_m e\|, \lambda\varepsilon + \lambda\|S^m e\|),$$

we can conclude as above (supposing that $\varepsilon_m + \varepsilon < 1$).

**REMARK.** If $X = Y$ in Theorem 1 then the hypotheses of the Radon-Nikodym property for $X^*$ or $X^{**}$ and of the UEP for $E$ and $F$ are superfluous. For, one can show similarly to Theorem 5 in [10] that $K(X,X)$ has the property SU in $L(X,X)$. But then $X$ is Hahn-Banach smooth [4]. Therefore $X^*$ has the Radon-Nikodym property [8] and all quotients and closed subspaces of $X$ have the UEP [2].

Applying Theorem 1 with $N_1(a,b) = (a^p+b^p)^{1/p}$ and $N_2(a,b) = (a^q+b^q)^{1/q}$ yields

**COROLLARY 3.** *Let $E$ be a quotient or a closed subspace of $l_p(\Gamma_1)$ or $d(v,p')^*$, and let $F$ be a quotient or a closed subspace of $l_q(\Gamma_2)$ or $d(w,q)$, where $1 < p \leq q < \infty$ and $1/p + 1/p' = 1$. If $E$ and $F$ have CAP then $K(E,F)$ is an $M$-ideal in $L(E,F)$.*

**4.** It was shown in [3] that a Banach space $X$ with $K(X) = K(X,X)$ being an M-ideal in $L(X) = L(X,X)$ necessarily must enjoy the MCAP. It is clear from Corollary 3 that for quotients and closed subspaces of $l_p(\Gamma)$, $1 < p < \infty$, the MCAP already ensures this property. In [7], there was mentioned the question whether the property "$K(X)$ is an M-ideal in $L(X)$" is preserved by passing to quotients and closed subspaces of $X$ having the MCAP. By the following result, we give a partial affirmative answer to this question.

**THEOREM 2.** *Let $K(X)$ be an M-ideal in $L(X)$. Let $E$ be a quotient and $F$ a closed subspace of $X$. If $E$ and $F$ have the MCAP then $K(E,F)$ is an M-ideal in $L(E,F)$. In particular, if $E$ is a quotient as well as a closed subspace of $X$, and $E$ has the MCAP, then $K(E)$ is an M-ideal in $L(E)$.*

*Proof.* Since $K(X)$ is an M-ideal in $L(X)$, there is a net $(B_\beta)$ in the unit ball of $K(X)$ satisfying the conditions (3), (4) and

$$\overline{\lim_{\beta}} \| B_\beta A + B^\beta B \| \le 1$$

for all $A$ and $B$ in the unit ball of $L(X)$ (cf. [9]). By Proposition, there are the nets $(P_m)$, $(S_m)$ for $E$ and $(Q_n)$, $(T_n)$ for $F$ corresponding to $(B_\beta)$. As in the proof of Theorem 1, it is sufficient to prove (5) (with $\lambda=1$, for this time).

Let $q : X \to E$ and $j : F \to X$ be the canonical surjection and injection. Let $\varepsilon$ be any positive number. Since

$$\overline{\lim_{m}} \|S^m q\| = \overline{\lim_{m}} \|qP^m\| \le \overline{\lim_{m}} \|P^m\| \le \overline{\lim_{\beta}} \|B^\beta\| \le 1,$$

there is an $m_0$ such that $\|S^m q\| \le 1 + \varepsilon$ for $m > m_0$. We have for $m > m_0$

$$\begin{aligned} &\overline{\lim_{n}} \|T_n SS_m + T^n TS^m\| \le \\ &\le \overline{\lim_{n}} \|Q_n jSS_m + Q^n jTS^m\| \le \\ &\le \overline{\lim_{\beta}} \|B_\beta jSS_m + B^\beta jTS^m\| = \\ &= \overline{\lim_{\beta}} \|B_\beta jSS_m q + B^\beta jTS^m q\| \le 1+\varepsilon, \end{aligned}$$

and thus we are done.

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Märkmeid pidevate lineaarsete operaatorite ruumi kaasruumi kohta

Eve Oja

Resümee

Tõestatakse üks autori artiklis [5] sõnastatud tulemus mõnevõrra üldisemal kujul. See lubab tuletada artikli [5] lahutusteoreemide jaoks faktorruumide variandid. See lubab ka osaliselt vastata ühele artiklis [7] püstitatud küsimusele järgmisel kujul.

**TEOREEM 2.** Moodustagu kompaktsete operaatorite alamruum K(X,X) M-ideaali kõigi Banachi ruumis X tegutsevate pidevate lineaarsete operaatorite ruumis L(X,X). Olgu E ruumi X faktorruum ja F ruumi X kinnine alamruum. Kui ruumidel E ja F on meetriline kompaktne approksimatsiooniomadus, siis K(E,F) on M-ideaal ruumis L(E,F).

ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS
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**SUMMABILITY FACTORS FOR STRONG SUMMABILITY**

**Virge Soomer**

Let $A = (a_{nk})$, $a_{nk} \geq 0$, be an infinite matrix and let $p = (p_k)$ be a sequence of positive numbers. A sequence $x = (x_k)$ is called strongly $A$-summable (to $l$) with exponent $p$ if

$$\lim_n \sum_k a_{nk} |x_k - l|^{p_k} = 0.$$

Let $\alpha = (A_i)$ be a sequence of matrices $A_i = (a_{nik})$, $a_{nik} \geq 0$. A sequence $x = (x_k)$ is called strongly $\alpha$-summable with exponent $p$ if

$$\lim_n \sum_k a_{nik} |x_k - l|^{p_k} = 0$$

uniformly in $i$. The sets of strongly $A$-summable, strongly $A$-summable to zero, strongly $\alpha$-summable and strongly $\alpha$-summable to zero sequences are denoted respectively by $[c_A]^p$, $[c_A]_o^p$, $[c_\alpha]^p$ and $[c_\alpha]_o^p$.

**REMARK.** If $\alpha = (A)$, $A = (a_{nk})$, then $[c_\alpha]^p = [c_A]^p$.

The purpose of this paper is to characterize the sequences $\varepsilon = (\varepsilon_k)$ which have the following properties:

$$x = (x_k) \in [c_\alpha]^p \quad \text{implies} \quad \varepsilon x = (\varepsilon_k x_k) \in [c_\alpha]^q \tag{1}$$

or

$$x = (x_k) \in [c_\alpha]_o^p \quad \text{implies} \quad \varepsilon x = (\varepsilon_k x_k) \in [c_\alpha]_o^q. \tag{2}$$

We call a sequence $\varepsilon = (\varepsilon_k)$ satisfying (1), resp. (2), a summability factor (notation $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$, resp. $(\varepsilon_k) \in ([c_\alpha]_o^p, [c_\alpha]_o^q)$).

For $p = (p_k)$, $q = (q_k)$, $0 < q_k < p_k$ we have the following two theorems.

**THEOREM 1.** *Suppose that* $0 < q_k < p_k \leq M < \infty$, $r =$

$= \sup(q_k/p_k) < 1$ *and* $\lambda = \inf(q_k/p_k) > 0$. *If the conditions*

$$\sup_{n,i} \sum_k a_{nik} |\varepsilon_k|^{\frac{q_k}{1-r}} < \infty \tag{3}$$

*and*

$$\sup_{n,i} \sum_k a_{nik} |\varepsilon_k|^{\frac{q_k}{1-\lambda}} < \infty \tag{4}$$

*are fulfilled then* $(\varepsilon_k) \in ([c_\alpha]_o^p, [c_\alpha]_o^q)$.

*Proof*. Put $|x_k|^{p_k} = w_k$ and $\lambda_k = q_k/p_k$. Let $x = (x_k) \in [c_\alpha]_o^p$, then

$$\lim_n \sum_k a_{nik} |x_k|^{p_k} = \lim_n \sum_k a_{nik} w_k = 0 \tag{5}$$

uniformly in i. Define

$$u_k = \begin{cases} w_k & , w_k \geq 1, \\ 0 & , w_k < 1, \end{cases}$$

and

$$v_k = \begin{cases} w_k & , w_k < 1, \\ 0 & , w_k \geq 1. \end{cases}$$

Then $w_k = u_k + v_k$, $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_k^{\lambda_k} \leq u_k^r$ , $v_k^{\lambda_k} \leq v_k^\lambda$ ,and by Hölder's inequality we obtain

$$\sum_k a_{nik} |\varepsilon_k x_k|^{q_k} = \sum_k a_{nik} |\varepsilon_k|^{q_k} w_k^{\lambda_k} =$$

$$= \sum_k a_{nik} |\varepsilon_k|^{q_k} u_k^{\lambda_k} + \sum_k a_{nik} |\varepsilon_k|^{q_k} v_k^{\lambda_k} \leq$$

$$\leq \sum_k (a_{nik} u_k)^r a_{nik}^{1-r} |\varepsilon_k|^{q_k} + \sum_k (a_{nik} v_k)^\lambda a_{nik}^{1-\lambda} |\varepsilon_k|^{q_k} \leq$$

$$\leq \Big[ \sum_k a_{nik} u_k \Big]^r \Big[ \sum_k a_{nik} |\varepsilon_k|^{\frac{q_k}{1-r}} \Big]^{1-r} +$$

$$+ \Big[ \sum_k a_{nik} v_k \Big]^\lambda \Big[ \sum_k a_{nik} |\varepsilon_k|^{\frac{q_k}{1-\lambda}} \Big]^{1-\lambda} \leq$$

$$\le \Big(\sum_k a_{nik} w_k\Big)^r \Big(\sum_k a_{nik} |\varepsilon_k|^{\overline{\frac{q_k}{1-r}}}\Big)^{1-r} +$$

$$+ \Big(\sum_k a_{nik} w_k\Big)^\lambda \Big(\sum_k a_{nik} |\varepsilon_k|^{\overline{\frac{q_k}{1-\lambda}}}\Big)^{1-\lambda} .$$

Now it follows by (5) that conditions (3) and (4) are sufficient for $(\varepsilon_k) \in ([c_\alpha]_0^p, [c_\alpha]_0^q)$. This completes the proof.

Let $e = (1,1,1,\dots)$, then $e \in [c_\alpha]^p$, $\varepsilon e = \varepsilon$, and consequently from $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$ it follows that $(\varepsilon_k) \in [c_\alpha]^q$, i.e. $([c_\alpha]^p, [c_\alpha]^q) \subset [c_\alpha]^q$.

Let $x = (x_k) \in [c_\alpha]^p$ and $\varepsilon = (\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q) \subset$ $\subset [c_\alpha]^q$. Then there exist some numbers $l$ and $\eta$ such that

$$\lim_n \sum_k a_{nik} |x_k - l|^{p_k} = 0$$

uniformly in i and

$$\lim_n \sum_k a_{nik} |\varepsilon_k - \eta|^{q_k} = 0 \text{ uniformly in } i. \tag{6}$$

For $H = \sup_k q_k$ and $K = \max(1, 2^{H-1})$ we have (see [2])

$$|a_k + b_k| \le K(|a_k|^{q_k} + |b_k|^{q_k}). \tag{7}$$

It follows from (7) that

$$\sum_k a_{nik} |\varepsilon_k x_k - \eta l|^{q_k} = \sum_k a_{nik} |\varepsilon_k x_k - \varepsilon_k l + \varepsilon_k l - \eta l|^{q_k} \le$$

$$\le K\Big(\sum_k a_{nik} |\varepsilon_k (x_k - l)|^{q_k} + \sup_k |l|^{q_k} \sum_k a_{nik} |\varepsilon_k - \eta|^{q_k}\Big).$$

Since $(x_k - l) \in [c_\alpha]_0^p$ it follows by Theorem 1 and by the condition (6) that conditions (3) and (4) are sufficient for $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$. Thus we have proved

**THEOREM 2.** *Suppose that* $0 < q_k < p_k \le M < \infty$, $r = \sup(q_k/p_k) < 1$ *and* $\lambda = \inf(q_k/p_k) > 0$. *Then conditions* (3) *and* (4) *are sufficient for* $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$.

**REMARK.** Using Hölder's inequality it is easy to show that in case $\sup_{n,i} \sum_k a_{nik} < \infty$ the condition (4) follows from

(3). In this case every bounded sequence $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$.

For $0 < p_k \le q_k$ we have the following result:

**THEOREM 3.** *Suppose that* $0 < r \le p_k \le q_k \le M < \infty$. *If*

$$\varepsilon_k = O(1)\, a_k^{(q_k - p_k)/q_k p_k}, \tag{8}$$

*where* $a_k = \sup_{n,i} a_{nik}$, *then* $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$.

*Proof.* Let $x = (x_k) \in [c_\alpha]^p$. Then there exists $l$ and $K_1$ such that $a_k\, |x_k - l|^{p_k} < K_1$ for all $k$. Since $0 < r \le p_k$, there exists a number $K_0$ so that

$$a_k^{1/p_k}\, |x_k - l| < K_1^{1/p_k} < K_0. \tag{9}$$

We obtained that for every $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$ condition (6) is fulfilled and so it follows from (6), (7), (8) and (9) that

$$\sum_k a_{nik} |\varepsilon_k x_k - \eta l|^{q_k} \le$$

$$\le K \left( \sum_k a_{nik} |\varepsilon_k|^{q_k}\, |x_k - l|^{q_k} + \sup_k |l|^{q_k} \sum_k a_{nik} |\varepsilon_k - \eta|^{q_k} \right) =$$

$$= K \sum_k a_{nik} |x_k - l|^{p_k}\, |\varepsilon_k|^{q_k}\, |x_k - l|^{q_k - p_k} + o(1) =$$

$$= O(1) \sum_k a_{nik} |x_k - l|^{p_k}\, (a_k^{1/p_k} |x_k - l|)^{q_k - p_k} + o(1) =$$

$$= O(1) \sum_k a_{nik} |x_k - l|^{p_k} + o(1) = o(1).$$

Hence (8) implies that $(\varepsilon_k) \in ([c_\alpha]^p, [c_\alpha]^q)$. The proof is completed.

**REMARK.** Let $\alpha = (A)$, $A = (a_{nk})$. Then $[c_\alpha]^p = [c_A]^p$ and in case $p_k = \text{const}$, $q_k = \text{const}$, (3) (for $q_k < p_k$) and (8) (for $q_k \ge p_k$) are necessary and sufficient for $(\varepsilon_k) \in ([c_A]^p, [c_A]^q)$ (see [1]).

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Summeeruvustegurid tugeva summeeruvuse korral
Virge Soomer
Resümee


Olgu $\alpha = (A_i)$ - maatriksite $A_i = (a_{nik})$ jada, kusjuures $a_{nik} \geq 0$, ja olgu $p = (p_k)$ positiivsete arvude jada. Jada $x = (x_k)$ nimetatakse tugevalt $\alpha$-summeeruvaks arvuks 1, kui

$$\lim_n \sum_k a_{nik} |x_k - 1|^{p_k} = 0$$

ühtlaselt $i$ suhtes.

Tähistame kõigi tugevalt $\alpha$-summeeruvate jadade hulga sümboliga $[c_\alpha]^p$. Käesolevas artiklis on leitud piisavad tingimused, et arvud $\varepsilon_k$ oleksid $([c_\alpha]^p, [c_\alpha]^q)$-tüüpi summeeruvustegurid, s.t. selleks, et sisalduvusest $(x_k) \in [c_\alpha]^p$ järelduks sisalduvus $(\varepsilon_k x_k) \in [c_\alpha]^q$.

**TEOREEM 2.** Olgu $0 < q_k < p_k \leq M < \infty$, $r = \sup(q_k/p_k) < 1$ ja $\lambda = \inf(q_k/p_k) > 0$. Siis selleks, et jada $(\varepsilon_k)$ oleks $([c_\alpha]^p, [c_\alpha]^q)$-tüüpi summeeruvustegur, on piisav, et oleksid

täidetud tingimused (3) ja (4).

**TEOREEM 3.** Olgu $0 < r \le p_k \le q_k \le M < \infty$. Siis on tingimus (8) piisav selleks, et jada $(\varepsilon_k)$ oleks $([c_\alpha]^p, [c_\alpha]^q)$-tüüpi summeeruvustegur.

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**WEYL FACTORS FOR SUMMABILITY WITH SPEED OF ORTHOGONAL SERIES**

**Heino Türnpu**

1. Let $\varphi = \{\varphi_k\}$ be an orthogonal system in $e = [a,b]$. We consider the orthogonal series

$$\Sigma\, \xi_k \varphi_k(t) \tag{1}$$

where $x = (\xi_k) \in l^2$.

Let $A$ and $B$ be regular[1] summability methods, given by triangular matrices $(\alpha_{nk})$ and $(\beta_{nk})$, respectively.

The series (1) is called $A$-summable almost everywhere (a.e.) in e, if the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) \tag{2}$$

exists a.e. in e.

Let $\lambda = (\lambda_k)$ be a sequence of real numbers with $0 < \lambda_k \nearrow \infty$.

We say that the series (1) is $A_0$-summable a.e. in e with speed $\lambda$ or, in short, $A_0^\lambda$-summable a.e. in e, if the limit (2) exists a.e. in e and

$$\lim_{n\to\infty} \lambda_n \left[\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f(t)\right] = 0, \tag{3}$$

a.e. in e where f is the sum of the series (1).

We say that $(\omega_k)$ with $0 < \omega_k \nearrow \infty$ is a sequence of Weyl factors for the $A_0^\lambda$-summability a.e. in e if the condition

$$\Sigma\, \xi_k^2\, \omega_k^2 < \infty$$

implies the $A_0^\lambda$-summability a.e. in e of the series (1).

In the case where $A$ is the Riesz summability method $P$ with

[1] We use definitions from [3].

$$\alpha_{nk} = 1 - \frac{P_{k-1}}{P_n}$$

where $0 < P_k \nearrow \infty$ is a sequence of real numbers, the Weyl factors are well known (see e.g.[1,2,6])..

In [2] the following Theorem A was proved.

**THEOREM A.** *Let* $\omega_k = \lambda_k \ln\ln P_k$ *where*

$$\frac{\lambda_k}{P_k} \searrow 0.$$

*Then* $(\omega_k)$ *is a sequence of Weyl factors for* $|P_o^{\lambda}$*-summability a.e. in* $e$ *of the series (1).*

In this paper we shall generalize Theorem A.

The series

$$\Sigma\, \eta_k \tag{4}$$

where $\eta_k$ are real numbers is called $\lambda$-convergent if the series (4) is convergent and, furthermore, the limit

$$\lim_{n\to\infty} \lambda_n \sum_{k=n+1}^{\infty} \eta_k.$$

exists.

We say that the summability method $A$ is *$\lambda$-convergence preserving* if for every $\lambda$-convergent series (4) the limit

$$\lim_{n\to\infty} \lambda_n \left( \sum_{k=0}^{n} \alpha_{nk}\eta_k - y\right)$$

exists where $y$ is the sum of the series (4).

Let $A$ be a summability method for which

$$a_i := \sup_{n\geq i} |a_{ni}| = o(1) \tag{5}$$

where $a_{ni} = \alpha_{ni} - \alpha_{n,i+1}$.

We prove the following

**THEOREM.** *Let* $A$ *be a* $\lambda^2$*-convergence preserving summability method for which* (5) *is fulfilled and*

$$\lambda_n^2 \Big/ \exp \sum_{i=0}^{n} a_i \searrow 0. \tag{6}$$

*Then the sequence* $\omega_k = \lambda_k \ln \sum_{i=0}^{k} a_i$ *is a sequence of Weyl* *factors for the* $A_0^\lambda$*-summability* *a.e. in* e *of the series* (1).

2. To prove Theorem we consider the Riesz summability method $P$ where[2]

$$P_k = \exp \sum_{i=0}^{k} a_i$$

From (5) it evidently follows that there exist constants $l>0$ and $L>0$ so that

$$l a_k \le \frac{P_k}{P_{k-1}} \le L a_k . \tag{7}$$

From Theorem A it follows that if the condition (6) holds and

$$\sum \xi_k^2 \lambda_k^2 \ln^2 \sum_{i=0}^{k} a_i < \infty \tag{8}$$

then

$$\lambda_n \left| \sum_{k=1}^{n} \left(1 - \frac{P_{k-1}}{P_n}\right) \xi_k \varphi_k(t) - f(t) \right| = o_t(1) \tag{9}$$

a.e. in e.

Let $\mathfrak{M}$ denote a decomposition of e, i.e.

$$\mathfrak{M} = \{ \mathfrak{M}_{mn},\ n=0,1,\dots,m\},$$

$$\bigcup_{n=0}^{m} \mathfrak{M}_{mn} = e$$

and

$$\mathfrak{M}_{mk} \cap \mathfrak{M}_{mn} = \emptyset$$

if $k \neq n$.

We use the following results.

**LEMMA 1** (see [7]). *Let* $(f_n)$ *be a sequence of integrable functions in* e. *Then*

$$f_n(t) = O_t(1)$$

*a.e. in* e *iff for each* $\varepsilon>0$ *there exists a measurable subset*

---

[2] If the summability method $P$ is $\lambda^2$-convergence preserving then the condition (6) holds (see [5], p.140)

$T_\varepsilon \subset e$ *where mes* $T_\varepsilon > b-a-\varepsilon$ *and a constant* $M_\varepsilon > 0$ *so that the inequality*

$$| \int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t) f_n(t) dt | \le M_\varepsilon$$

*holds uniformly for all the decompositions* $\mathfrak{M}$ *of* $e$ *where*

$$\chi_{mn} = \chi_{\mathfrak{M}_{mn}}$$

**LEMMA 2** (see [5]). *If a regular triangular summability method A is* $\lambda^2$*-convergence preserving then the conditions*

$$1^\circ \ \exists \lim_{n\to\infty} \lambda_n^2 \Big( \sum_{k=0}^{n} a_{nk} - 1 \Big)$$

*and*

$$2^\circ \ \lambda_n^2 \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k^2} = O(1)$$

*hold.*

**LEMMA 3** (see [4,p.361]). *Let* $D_n$ $(n \in \mathbb{N})$ *be continuous homogeneous operators from a Banach space* $X$ *into the space* $M_e$ *of all functions measurable in* $e$ *for which the inequality*

$$|D_m(x_1+x_2,t)| \le |D_m(x_1,t)| + |D_m(x_2,t)|$$

*holds. If*

$$1^\circ \quad D_n(x,t) = O_t(1)$$

*a.e. in* $e$ *for any* $x \in X$ *and*

$$2^\circ \quad \lim_{n\to\infty} D_n(x',t) = 0$$

*a.e. in* $e$ *for any* $x'$ *from a total set in* $X$ *then*

$$\lim_{n\to\infty} D_n(x,t) = 0$$

*a.e. in* $e$ *for each* $x \in X$.

**Lemma 4**. *Let* A *be a triangular regular* $\lambda^2$*-convergence preserving summability method for which* (5) *and* (6) *hold. Then*

$$A_n(x,t) = O_t(1), \tag{10}$$

*where*

$$A_n(x,t) = \lambda_n \sum_{k=0}^{n} a_{nk} \sum_{i=0}^{k} \frac{P_{i-1}}{P_k} \xi_i \varphi_i(t) \tag{11}$$

*and*

$$P_k = \exp \sum_{i=0}^{k} a_i ,$$

*holds a.e. in* e *for each* $x = (\xi_k) \in l^2$ *with*

$$\sum \xi_k^2 \lambda_k^2 < \infty. \tag{12}$$

*Proof*. From Lemma 1 it follows that (10) holds for fixed x iff for each $\varepsilon>0$ there exists a measurable subset $T_\varepsilon \subset e$ where $\operatorname{mes}T_\varepsilon > b-a-\varepsilon$ and a constant $M_\varepsilon>0$ so that the inequality

$$B_m^\varepsilon(x) \le M_\varepsilon ,$$

where

$$B_m^\varepsilon(x) = \left| \int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t)\, A_n(x,t)dt \right| ,$$

holds uniformly for all the decompositions $\mathfrak{M}$ of e.

Using Cauchy-Bunyakovsky inequality we get

$$B_m^\varepsilon(x) \le O(1)\left\{\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t)\, c_n \sum_{k=0}^{n} \lambda_k^2 |a_{nk}| \left( \sum_{i=0}^{k} \frac{P_{i-1}}{P_k} \xi_i \varphi_i(t) \right)^2 dt \right\}^{1/2}$$

where

$$c_n = \lambda_n^2 \sum_{l=0}^{n} \frac{|a_{nl}|}{\lambda_l^2} .$$

From the condition $2^\circ$ of Lemma 2 and (5) it follows that

$$B_m^\varepsilon(x) \le O(1)\left\{\int_{T_\varepsilon} \sum_{k=0}^{m} \lambda_k^2 \sum_{n=k}^{m} |a_{nk}| \chi_{mn}(t) \left[ \sum_{i=0}^{k} \frac{P_{i-1}}{P_k} \xi_i \varphi_i(t) \right]^2 dt \right\}^{1/2} \le$$

$$\le O(1)\left\{ \int_{T_\varepsilon} \sum_{k=0}^{m} \lambda_k^2 a_k \left[ \sum_{i=0}^{k} \frac{P_{i-1}}{P_k} \xi_i \varphi_i(t) \right]^2 dt \right\}^{1/2} .$$

From Bessel's inequality we get

$$B_m^\varepsilon(x) \le O(1)\, N_\varepsilon \left\{ \sum_{k=0}^{n} \lambda_k^2 a_k \sum_{i=0}^{k} \frac{P_{i-1}^2}{P_k^2} \xi_i^2 \right\}^{1/2} \le$$

$$\le O(1)\, N_\varepsilon \left\{ \sum_{i=0}^{m} \xi_i^2 P_{i-1}^2 \sum_{k=i}^{m} \frac{a_k \lambda_k^2}{P_k^2} \right\}^{1/2} .$$

Since from (6) and (7) it follows that

$$\sum_{k=\iota}^{m} \frac{a_k\lambda_k^2}{P_k^2} \le \frac{\lambda_\iota^2}{1P_\iota} \sum_{k=\iota}^{m} \frac{p_k}{P_k^2} \le \frac{\lambda_\iota^2}{1P_\iota} \sum_{k=\iota}^{m} \Big( \frac{1}{P_{k-1}} - \frac{1}{P_k} \Big) =$$

$$= \frac{\lambda_\iota^2}{1P_\iota} \Big( \frac{1}{P_{\iota-1}} - \frac{1}{P_m} \Big),$$

we finally have by using (12) that

$$B_m^{\varepsilon}(x) = O(1)\ N_\varepsilon \Big\{ \sum_{\iota=0}^{m} \xi_\iota^2\lambda_\iota^2 \Big( \frac{P_{\iota-1}^2}{P_\iota P_{\iota-1}} + \frac{P_{\iota-1}^2}{P_\iota P_m} \Big)\Big\}^{1/2} \le$$

$$\le O(1)\ N_\varepsilon \Big\{ \sum_{\iota=0}^{m} \xi_\iota^2\lambda_\iota^2 \Big\}^{1/2} = O(1)N_\varepsilon = M_\varepsilon .$$

The proof of Lemma 4 is complete.

3. *Proof of Theorem.* We have

$$\lambda_n \Big| \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f(t)\Big| = \lambda_n \Big| \sum_{k=0}^{n} a_{nk} \sum_{\iota=0}^{k} \xi_\iota\varphi_\iota(t) - f(t)\Big| \le$$

$$\le A_n(x,t) + B_n(x,t) + C_n(x,t) \tag{13}$$

where $A_n(x,t)$ is defined by formula (11),

$$B_n(x,t) = \lambda_n \Big| \sum_{k=0}^{n} a_{nk} - 1\Big| |f(t)|$$

and

$$C_n(t) = \lambda_n \Big| \sum_{k=0}^{n} a_{nk} \Big( \sum_{\iota=0}^{k} \Big[1 - \frac{P_{\iota-1}}{P_k}\Big]\xi_\iota\varphi_\iota(t) - f(t)\Big]\Big| .$$

From the condition $1^{\circ}$ of Lemma 2 it follows that

$$B_n(x,t) = O_\iota(1)$$

holds a.e. in e.

As the inequality (8) holds, the condition (12) is fulfilled and from Lemma 4 we get that

$$A_n(x,t) = O_\iota(1)$$

a.e. in e.

Since the summability method $A$ is regular and $\lambda^2$-convergence preserving and

$$\lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k} \le \left\{ \sum_{k=0}^{n} |a_{nk}| \right\}^{1/2} \left\{ \lambda_n^2 \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k^2} \right\}^{1/2},$$

it follows from the condition $2^o$ of Lemma 2 that

$$\lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k} = O(1).$$

But since

$$|C_n(x,t)| \le \lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k} \lambda_k | \sum_{i=0}^{k} \left(1 - \frac{P_{i-1}}{P_k}\right) \xi_i \varphi_i(t) - f(t)|,$$

we finally get using the inequality (9) that

$$|C_n(x,t)| \le O_t(1) \lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k} = O_t(1)$$

a.e. in $e$ for each $x$ for which the inequality (8) holds.

Now by using the inequality (13) we have that

$$F_n(x,t) = O_t(1) \tag{14}$$

a.e. in $e$ for each $x$ for which the inequality (8) holds, where

$$F_n(x,t) = \lambda_n | \sum_{k=0}^{n} \alpha_{nk} \xi_k \varphi_k(t) - f(t)|. \tag{15}$$

It is evident that for each $n \in \mathbb{N}$ the equality (15) defines a bounded linear operator $F_n$ from $l_\omega^2$ into $M_e$ where $l_\omega^2$ is the Banach space of all sequences for which the inequality (8) holds.

The set $\{e_j\}$ is a total set in $l_\omega^2$ where $e_j = (\delta_{kj})$ and $\delta_{kj}$ is the Kronecker's symbol and

$$\lim_{n\to\infty} F_n(e_i, t) = \lim_{n\to\infty} \lambda_n \left( \sum_{k=0}^{n} \alpha_{nk} \delta_{ki} \varphi_i(t) - \varphi_i(t) \right) =$$

$$= \lim_{n\to\infty} \lambda_n \varphi_i(t)(\alpha_{ni} - 1).$$

Since the method $A$ is $\lambda^2$-convergence preserving and the series $\sum_k \delta_{ki}$ is $\lambda^2$-convergent we have

$$\lim_{n\to\infty} \lambda_n \varphi_i(t)(\alpha_{ni} - 1) = \lim_{n\to\infty} \lambda_n \left( \sum_{k=0}^{n} \alpha_{nk} \delta_{ki} - 1 \right) = 0$$

and therefore

$$\lim_{n\to\infty} F_n(e_i, t) = 0$$

a.e. in $e$, i.e. the condition $2^o$ of the Lemma 4 holds. By using Lemma 4 we get from inequality (14) that

$$\lim_{n\to\infty} F_n(x,t) = 0$$

a.e. in $e$ for all $x$ for which the inequality (8) is fulfilled, i.e.

$$\omega_k = \lambda_k \ln \sum_{i=0}^{k} a_i$$

is a sequence of Weyl factors for the $A_0^\lambda$-summability a.e. in $e$.

The proof of our Theorem is complete.

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Weyli tegurid ortogonaalridade
kiiruvusega summeeruvuseks

Heino Türnpu

Resümee

Olgu $\varphi = \{\varphi_k\}$ lõigus $e = [a,b]$ defineeritud ortonormaalne süsteem.

Olgu $A$ regulaarne kolmnurkne summeerimismenetlus, mis on antud maatriksiga $(\alpha_{nk})$.

Öeldakse, et rida

$$\sum \xi_k \varphi_k(t)$$

on $A_o^\lambda$-summeeruv p.k. lõigul e, kui eksisteerivad piirväärtused

$$\lim_{n\to\infty} \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t)$$

ja

$$\lim_{n\to\infty} \lambda_n \left[ \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f(t)\right] = 0.$$

Öeldakse, et jada $(\omega_k)$, kus $0 < \omega_k \nearrow \infty$, on Weyli tegurite jada $A_o^\lambda$-summeeruvuseks p.k. lõigul e, kui tingimus

$$\sum \xi_k^2 \, \omega_k^2 < \infty ,$$

garanteerib rea $\sum \xi_k\varphi_k(t)$ $A_o^\lambda$-summeeruvuse p.k. lõigul e.

Me tõestame, et kehtib järgmine

**TEOREEM.** Olgu A niisugune $\lambda^2$-koonduvust säilitav summeerimismenetlus, mille korral on täidetud tingimused (5) ja (6). Siis jada $(\omega_k)$, kus

$$\omega_k = \lambda_k \ln \sum_{i=0}^{k} a_i ,$$

on Weyli tegurite jada $A_0^\lambda$-summeeruvuseks p.k. lõigul e.

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