DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

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CONVERGENCE AND SUMMABILITY WITH SPEED OF FUNCTIONAL SERIES

NATALIA SAEALLE

TARTU UNIVERSITY
PRESS

Faculty of Mathematics and Computer Science, University of Tartu, Tartu, Estonia

Dissertation is accepted for the commencement of the degree of Doctor of Philosophy (PhD) in mathematics on September 16, 2005 by the Council of the Faculty of Mathematics and Computer Science, University of Tartu.

Supervisors:

PhD Heino Türnpu,
University of Tartu,
Tartu, Estonia

PhD Prof. Toivo Leiger,
University of Tartu,
Tartu, Estonia

Opponents:

PhD Prof. Igor Brui,
Baranovichi University,
Baranovichi, Belarus

PhD Prof. Andi Kivinukk,
Tallinn Pedagogical University,
Tallinn, Estonia

The public defence will take place on October 28, 2005.

Publication of this dissertation is granted by the Institute of Pure Mathematics of the University of Tartu (research project TMTPM0085) and the Estonian Science Foundation grants GMTPM3991, GMTPM5376, and DMTPM1632.

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LIST OF PAPERS CONTAINED IN THE THESIS

This thesis comprises the papers I, II, III, and IV. They will be presented, respectively, as Chapters I, II, III, and IV of the thesis.

- I N. Saealle and H. Türnpu, Riesz summability with speed of orthogonal series, *Acta et Commentationes Universitatis Tartuensis de Mathematica* **5** (2001), 3-14.
- II N. Saealle and H. Türnpu, Summability of orthogonal series with speed, *Analysis Mathematica (Szeged)* **31** (2005), 63-73.
- III N. Saealle and H. Türnpu, Convergence and λ-boundedness of functional series with respect to multiplicative systems, *Proceedings of the Estonian Academy of Sciences. Physics. Mathematics* **53** (2004), No. 1, 13-25.
- IV N. Saealle, Uniform convergence and A^{λ}-boundedness of series with respect to product systems, *Acta et Commentationes Universitatis Tartuensis de Mathematica* (to appear).

ACKNOWLEDGEMENTS

It is difficult to overstate my gratitude to my first supervisor, Heino Türnpu. He provided encouragement, sound advice, good teaching, lots of good ideas, and continuous support over the last ten years.

I am very grateful to my second supervisor, Toivo Leiger, who has supported me throughout my thesis-writing period with his patience. Without him, this thesis would not have been completed or written.

Many thanks to all my friends and colleagues at the University of Tartu for scientific and technical assistance, good company and positive attitude.

My special thanks to my parents and husband for their support and encouragement during all those years.

INTRODUCTION

The theory of orthogonal series is a classical area of mathematical analysis dealing with functional series

$$\sum_{k=0}^{\infty} \xi_k \varphi_k(t), \tag{1}$$

where $\varphi = \{\varphi_k\}$ is an orthogonal system, and with functions representable by these series. Due to applications in physics, the most well-known and best studied part of this theory is the trigonometric Fourier' series theory, which investigates series (1), where system $\varphi = \{1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots\}$. Besides the trigonometric system, the modern theory of orthogonal series considers also other certain systems, e.g. the systems of Rademacher, Walsh, Haar, etc.

One of the basic problems of the aforenamed theory is convergence and summability of series (1). Let A be a summability method. In order to investigate the summability of series (1) by the method A, instead of partial sums $\sum_{k=0}^{n} \xi_k \varphi_k(t)$ $(n = 0, 1, \ldots)$ we consider so-called A-means sequence $(\sigma_n(t))$. It is said that series (1) is A-summable if (σ_n) is convergent. The interest in summability methods is that they provide a way to understand series, which are not convergent. An A-means sequence may have better convergence properties than the sequence of partial sums. For example, by well-known Fejer's theorem, the arithmetic means of the Fourier' series of every continuous 2π-periodic function converge uniformly; but, on the other hand, there exist continuous 2π-periodic functions whose Fourier series diverge at infinitely many points.

Often it is important to estimate the speed of a convergence process. In 1969, Kangro laid down foundations of the theory of λ-convergence (i.e., convergence with speed) of sequences in a context of topological sequence spaces. This theory allows to apply methods of functional analysis in investigation of convergence speed. Let λ be a speed, i.e. a monotonically increasing sequence of positive numbers. It is said that series (1) is λ-convergent if the sequence $(\lambda_n \sum_{k=n+1}^{\infty} \xi_k \varphi_k(t))_n$ converges. If the A-means of series converge with a speed λ, then it is said that series is A^λ-summable.

In the thesis, maximal λ-convergence and P^λ-summability almost everywhere (a.e.) of series (1) are investigated, where P is a Riesz method of weighted means. For $p \geq 1$, series (1) is said to be p-maximal convergent a.e. on $[a,b]$ if it is convergent a.e. and

$$\int_a^b \sup_n \left| \sum_{k=0}^{n} \xi_k \varphi_k(t) \right|^p dt < \infty.$$

In this case, in addition to convergence a.e. of series (1), we get convergence in $L^p_{[a,b]}$ spaces. The notion of maximal convergence can be extended in natural way to convergence and summability with speed. This makes it possible to investigate A^λ-summability of series (1) in $L^p_{[a,b]}$ spaces.

There are several reasons to study Riesz methods in the current context. Firstly, in the case of orthogonal series, this method is universal in the sense that for every $(\xi_k) \in \ell^2$ there exists a Riesz method of weighted means $P = P((\xi_k))$ such that series (1) is P-summable a.e. The second important reason for using Riesz methods is that P is technically simpler to use.

Moreover, for any regular Riesz method P, we have at our disposal a useful characterization (due to Kangro) of the speeds λ such that every λ-convergent sequence is P^λ-summable.

In the thesis, the special attention is paid to series (1), where system $\{\varphi_k\}$ is a product system generated by arbitrary system $\{f_k\}$, i.e. $\varphi_0(t) = 1$ and $\varphi_k = f_{k_0+1} f_{k_1+1} \cdots f_{k_n+1}$, where $k = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_n}$ $(k_0 < k_1 < \ldots < k_n)$ is the dyadic representation of k. The most known product system is the Walsh system $\{w_k\}$ generated by the Rademacher system. The Walsh system is similar to the trigonometric system but simpler. By means of product systems one convergence problem can be reduced to another with better properties. In particular, some of the more difficult aspects of the trigonometric theory are easier to understand in the simpler Walsh case first.

The main aim of this thesis is to show that many results from the orthogonal series theory may be extended to the case of convergence and summability with speed. The starting points of this study are some classical works of Alexits [1], Kaczmarz [9-11], Kangro [6-8], and Tandori [13] and recent past papers of Mòricz [12], Schipp [19-21], and Türnpu [22-25]. Research methods of classical and functional analysis are used.

The main contributions of the present thesis to the theory of functional series are as follows.

1) Some classes of summability methods for which the boundedness of the corresponding Lebesgue function implies the λ-summability of series (1) for all $(\xi_k) \in \ell^2_\lambda := \{(\xi_k) : \sum_{k=0}^\infty \lambda_k^2 \xi_k^2 < \infty\}$ are described.

2) If $\{\varphi_k\}$ is the product system generated by a system $\{f_k\}$, connections between properties of $\{f_k\}$ (p-weak multiplicativity) and convergence properties of the series (1) (p-maximal λ-convergence) are fixed.

3) Let $\{\varphi_k\}$ be a product system and $\{w_k\}$ the Walsh system. For functions u from the classes $C_{[0,1]}$ and $L^p_{[0,1]}$ it is showed that some problems concerning the λ-convergence or λ-summability of the series $\sum_{k=0}^{\infty} < u, w_k > \varphi_k$ can be reduced to the corresponding problems for the Walsh-Fourier series $\sum_{k=0}^{\infty} < u, w_k > w_k$.

In Chapter I we consider a specific problem. Applying a general complicated theorem due to Türnpu in the case of Riesz method, we find when the boundedness of according Lebesgue functions implies the maximal P^λ-summability a.e. of series (1) for all sequences (ξ_k) from the Banach space ℓ^2_λ. In Chapter II, the same problem in the case of any regular λ^2-conservative summability method A is investigated. For this purpose, we compare A with a (suitably constructed) Riesz method $P(A)$ using our results about λ-inclusion of summability methods in the class of series (1).

In Chapters III and IV we consider series (1), where $\{\varphi_k\}$ is any product system (not necessarily orthogonal). Chapter III examines p-maximal λ-convergence and p-maximal λ-boundedness of series (1), where a system $\{\varphi_k\}$ satisfies some additional conditions. For example, it is proved that if $\sum_{k=0}^{\infty} \left| \int_a^b \varphi_k(t)dt \right| < \infty$ (i.e. the system $\{f_k\}$ is weakly multiplicative), then series (1) is 2-maximally λ-convergent a.e. on $[a, b]$ for every $(\xi_k) \in \ell^2_\lambda$. Our main attention is concentrated to the series

$$\sum_{k=0}^{\infty} < u, w_k > \varphi_k(t), \tag{2}$$

where $< u, w_k > := \int_0^1 u(t)w_k(t)dt$ are the Walsh-Fourier coefficients of any integrable function u. In the case of $u \in L^p_{[0,1]}$ $(1 < p < \infty)$, the problem of 1-maximal λ-boundedness of this series might be reduced to examination of p-maximal λ-boundedness of the Walsh-Fourier series $\sum_{k=0}^{\infty} < u, w_k > w_k(t)$.

Chapter IV investigates uniform convergence of series (2), uniform A-summability, uniform A^λ-boundedness and uniform regular A^λ-summability, where u is a continuous function on $[0, 1]$. It is shown that each of the above mentioned properties is equivalent to the same property of the corresponding Walsh-Fourier series.

SUMMARY

1 Preliminaries: summability methods and convergence with speed

1.1 Summability methods

We consider *series-to-sequence summability methods* $A = (\alpha_{nk})$ given by the matrix transformation [1]

$$\eta_n := \sum_{k=0}^{n} \alpha_{nk} u_k \quad (n \in \mathbf{N}),$$

where (α_{nk}) is a triangular matrix (i.e. $\alpha_{nk} = 0$ for $k > n$). The corresponding *sequence-to-sequence method* $A = (a_{nk})$ is defined by the transformation

$$\eta'_n := \sum_{k=0}^{n} a_{nk} \zeta_k \quad (n \in \mathbf{N})$$

with

$$a_{nk} := \alpha_{nk} - \alpha_{n,k+1}, \text{ or, equivalently, } \quad \alpha_{nk} = \sum_{\nu=k}^{n} a_{n\nu} \quad (n, k \in \mathbf{N}).$$

A series $\sum_{k=0}^{\infty} u_k$ (a sequence (ζ_k)) is called *A-summable* if the limit $\lim_n \eta_n$ ($\lim_n \eta'_n$) exists.

A series-to-sequence method A is said to be *regular* if

$$\lim_n \eta_n = \sum_{k=0}^{\infty} u_k$$

for every convergent series $\sum_{k=0}^{\infty} u_k$. It is well known (see, for example, [3, Theorem 1.3]) that A is regular if and only if

$$\lim_n \alpha_{nk} = 1 \ (k \in \mathbf{N}) \quad \text{and} \quad \sum_{k=0}^{\infty} |\alpha_{nk} - \alpha_{n,k+1}| = O(1).$$

In the thesis, the main attention has been paid to two certain summability methods: the Cesàro method and the Riesz method.

[1] We will denote the set of non-negative integers by $\mathbf{N}$.

1) The sequence-to-sequence *Cesàro method* (or *the method of arithmetic means*) $(C,1) = (a_{nk})$ is given by the matrix A with

$$a_{nk} = \begin{cases} \dfrac{1}{n+1} & \text{if} \quad k \le n, \\ 0 & \text{if} \quad k > n, \end{cases}$$

and the series-to-sequence method $(C,1) = (\alpha_{nk})$ is then given by

$$\alpha_{nk} = \begin{cases} 1 - \dfrac{k}{n+1} & \text{if} \quad k \le n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

2) The sequence-to-sequence *Riesz method* (or *the method of weighted means*) $P = (R, p_n) = (\alpha_{nk})$ is defined by the matrix A with

$$a_{nk} = \begin{cases} \dfrac{p_k}{P_n} & \text{if} \quad k \le n, \\ 0 & \text{if} \quad k > n, \end{cases}$$

where

$$P_n := \sum_{k=0}^{n} p_k \nearrow \infty$$

and (p_k) is a sequence of positive numbers. The series-to-sequence Riesz method is given by

$$\alpha_{nk} = \begin{cases} 1 - \dfrac{P_{k-1}}{P_n} & \text{if} \quad k \le n, \\ 0 & \text{if} \quad k > n, \end{cases}$$

where $P_{-1} := 0$.

Note that P is a regular method (cf. [3, Theorem 17.1]) and $(C,1)$ is a special case of it.

1.2. Convergence and summability with speed

Let $\lambda = (\lambda_k)$ be a scalar sequence such that $0 < \lambda_k \nearrow \infty$. By Kangro [7],[8], a sequence $z = (\zeta_k)$ is said to be

(a) λ-*convergent* (or *convergent with the speed* λ) if the limit $\lim_k \zeta_k =: \zeta$ exists and the sequence $(\lambda_k(\zeta_k - \zeta))$ is convergent;

(b) *regularly* λ-*convergent* if $\lim_k \lambda_k(\zeta_k - \zeta) = 0$;

(c) λ-*bounded* if the sequence $(\lambda_k(\zeta_k - \zeta))$ is bounded.

The set of all λ-convergent sequences is denoted by c^λ.

If a sequence z is summable by a sequence-to-sequence summability method A, then it is called A^λ*-summable*, provided that the limit

$$\lim_n \lambda_n \left(\sum_{k=0}^n \alpha_{nk}\zeta_k - \lim_m \eta'_m \right)$$

exists. *Regular* A^λ*-summability* and A^λ*-boundedness* are defined analogously.

A summability method A is said to be λ*-conservative* (or λ*-convergence preserving*) if the sequence (η'_n) is λ-convergent for any $z \in c^\lambda$. Note (see [6]) that a regular method A is λ-conservative if and only if

$$\lambda_n \sum_{k=0}^n \frac{|a_{nk}|}{\lambda_k} = O(1).$$

For example, a Riesz method P is λ-conservative if and only if

$$\frac{\lambda_n}{P_n} \sum_{k=0}^n \frac{p_k}{\lambda_k} = O(1).$$

2 Summability with speed of orthogonal series by Riesz methods

2.1 Lebesgue functions

We will consider convergence and summability almost everywhere (shortly, a.e.) on $[a,b]$ of the series

$$\sum_{k=0}^\infty \xi_k \varphi_k(t), \tag{1}$$

where $x = (\xi_k) \in \ell^2$ and $\varphi = \{\varphi_k\}$ is an orthogonal system of functions defined on $[a,b]$. Basic facts from the theory of orthogonal series can be found e.g. in [1] or [11]. The most familiar examples of orthogonal systems are following.

(a) The *trigonometric system*

$$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \dots\}$$

is orthogonal on $[0, 2\pi]$.

(b) Let r_0 be the function defined on $[0, 1)$ by

$$r_0(t) := \begin{cases} 1 & \text{if} \quad t \in [0, \frac{1}{2}), \\ -1 & \text{if} \quad t \in [\frac{1}{2}, 1) \end{cases}$$

and extended to the set of real numbers by periodicity of period 1. The *Rademacher system* $r := \{r_n(t)\}$ is defined by

$$r_n(t) := r_0(2^n t) \qquad (t \in [0, 1], n \in \mathbf{N}). \tag{2}$$

(c) The *Walsh*(*-Paley*) system $w = \{w_n(t)\}$ is defined by:

$$w_0(t) := 1 \text{ and } w_n(t) := r_{n_0+1}(t) r_{n_1+1}(t) \ldots r_{n_k+1}(t) \quad (t \in [0, 1]), \tag{3}$$

where $n = 2^{n_0} + 2^{n_1} + \ldots + 2^{n_k}$ $(n_0 < n_1 < \ldots < n_k)$ is the dyadic representation of n.

(d) The *Haar system* $h = \{h_n(t)\}$ is defined as follows. Set $h_0 := 1$. For $n, k \in \mathbf{N}$ with $0 \leq k < 2^n$ define h_n on [0,1] by

$$h_{2^n+k}(t) := \begin{cases} 2^{n/2} & \text{if} \quad t \in [2^{-(n+1)} 2k; 2^{-(n+1)}(2k+1)), \\ -2^{n/2} & \text{if} \quad t \in [2^{-(n+1)}(2k+1); 2^{-(n+1)}(2k+2)), \\ 0 & \text{otherwise.} \end{cases}$$

In problems dealing with the convergence of series with respect to orthogonal system φ a major role is played by the *Lebesgue functions*

$$L_n(\varphi, t) := \int_a^b \left| \sum_{k=0}^n \varphi_k(t) \varphi_k(\tau) \right| d\tau \quad (n \in \mathbf{N}).$$

A classical result of Kaczmarz [9] states that series (1) converges a.e. on $[a, b]$ for each $(\xi_k) \in \ell^2$ if $L_n(\varphi, t) = O(1)$ on $[a, b]$. Türnpu [23] showed that the last condition can be replaced by

$$L_n(\varphi, t) = O_t(1) \quad (t \in [a, b]).$$

For a summability method $A = (\alpha_{nk})$ with $\lim_n \alpha_{nk} = 1$ $(k \in \mathbf{N})$ the Lebesgue functions are defined by

$$L_n(A, \varphi, t) := \int_a^b \left| \sum_{k=0}^n \alpha_{nk} \varphi_k(t) \varphi_k(\tau) \right| d\tau \quad (n \in \mathbf{N}).$$

By Kaczmarz [9], [10], series (1) is $(C, 1)$-summable a.e. on $[a, b]$ for each $(\xi_k) \in \ell^2$ if $L_n(A, \varphi, t) = O(1)$ on $[a, b]$. On the other hand, Móricz and Tandori [13] showed that there exist a regular triangular summability method A

and an orthogonal system φ such that series (1) is not A-summable a.e. on $[a, b]$ for some $(\xi_k) \in \ell^2$. Móricz [12] and Türnpu [23] found certain classes of regular methods A for which the conditions $L_n(A, \varphi, t) = O(1)$ on $[a, b]$ and $L_n(A, \varphi, t) = O_t(1)$ $(t \in [a, b])$, respectively, imply the A-summability a.e. on $[a, b]$ of series (1) for each $(\xi_k) \in \ell^2$.

2.2 Summability with speed of orthogonal series

Let $A = (\alpha_{nk})$ be a triangular series-to-sequence summability method. Series (1) is said to be A^λ*-summable a.e. on* $[a, b]$ if the limits

$$\lim_n \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) =: f_x^A(t) \text{ and } \lim_n \lambda_n \left(\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x^A(t) \right)$$

exist a.e. on $[a, b]$. If, in addition,

$$\int_a^b \sup_n \lambda_n \left| \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x^A(t) \right| dt < \infty,$$

then series (1) is called *maximally* A^λ*-summable a.e.*

λ-convergence and λ-summability of orthogonal series was investigated by Kangro and Türnpu (see, for example, [6-8], [22-25]). In [25] Türnpu considered the A^λ-summability a.e. of series (1) with

$$(\xi_k) \in \ell^2_\lambda := \left\{ (\xi_k) | \sum_{k=0}^{\infty} \lambda_k^2 \xi_k^2 < \infty \right\}$$

and proved the following theorem.

Theorem 1 (cf. [25]). *Let A be λ^2-conservative and let*

$$\lim_n \alpha_{nk} = 1 \quad (k \in \mathbf{N}).$$

Series (1) is A^λ-summable a.e. on $[a, b]$ for all $x \in \ell^2_\lambda$ if and only if the following conditions hold:

1° *series (1) is A-summable a.e. on $[a, b]$ for every $x \in \ell^2_\lambda$;*

2° *for each $\varepsilon > 0$ there exist a measurable subset $T_\varepsilon \subset [a, b]$ satisfying $\operatorname{mes} T_\varepsilon > b - a - \varepsilon$ and a constant $M_\varepsilon > 0$ such that, for all measurable decompositions*

$$\mathcal{N}_m := \Big\{ \mathcal{N}_{mn} : n = 0, 1, \ldots, m; \mathcal{N}_{mk} \cap \mathcal{N}_{mn} = \emptyset \text{ if } k \neq n; \bigcup_{n=0}^{m} \mathcal{N}_{mn} \subset [a, b] \Big\}, \quad (4)$$

one has

$$A_m(\varepsilon) = \left| \int_{T_\varepsilon} \int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \sum_{\nu=0}^{m} \varphi_\nu(t)\varphi_\nu(\tau) D^m_{np\nu} dt d\tau \right| \le M_\varepsilon,$$

where χ_{mn} *is the characteristic function of* $\mathcal{N}_{mn}$ *and*

$$D^m_{np\nu} = \begin{cases} (\alpha_{m\nu} - \alpha_{n\nu})(\alpha_{m\nu} - \alpha_{p\nu})\lambda_n\lambda_p/\lambda_\nu^2, & \text{if} \quad 0 \le v \le n < p < m, \\ \alpha_{m\nu}(\alpha_{m\nu} - \alpha_{p\nu})\lambda_n\lambda_p/\lambda_\nu^2, & \text{if} \quad n < \nu \le p < m, \\ \alpha_{m\nu}^2 \lambda_n\lambda_p/\lambda_\nu^2, & \text{if} \quad n < p < \nu \le m. \end{cases}$$

This theorem is our starting point for investigation of maximally P^λ-summability a.e. of orthogonal series (1).

In **Chapter I** ([15]), by means of Theorem 1 we find the relationship between sequences (p_k) and (λ_k), which guarantees the maximal P^λ-summability a.e. on $[a,b]$ of series (1) for each $(\xi_k) \in \ell^2_\lambda$, provided that the Lebesgue functions $L_k(P,\varphi,t)$ of P satisfy the condition

$$\int_a^b \sup_k L_k(P,\varphi,t)dt < \infty.$$

The main result of Chapter I is the following theorem.

Theorem 2 (cf. [15]). *Let the method* P *be* λ^2*-conservative, i.e.*

$$\frac{\lambda_n^2}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} = O(1). \tag{5}$$

If

$$p_n = O(P_{n-1}), \tag{6}$$

$$\frac{\lambda_n^2}{P_{n-1}} \downarrow 0, \quad \frac{1}{p_n}\left(\frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n+1}^2}\right) \downarrow 0, \tag{7}$$

and

$$\int_a^b \sup_k L_k(P,\varphi,t)dt < \infty,$$

then series (1) *is maximally* P^λ*-summable a.e. on* $[a,b]$ *for each* $(\xi_k) \in \ell^2_\lambda$.

Example. Let $\lambda_k = (k+1)^\alpha$, $\alpha > 0$. Then, as an example of a Riesz method satisfying $(5) - (7)$, we may consider the method P with $p_k = (k+1)^\beta$, where $\beta > 2\alpha - 1$.

For example, if $0 < \alpha < 1/2$, we can put $\beta = 0$, i.e. $P = (R,1) = (C,1)$.

Remark. In [6], Kangro proved that if the sequence $\left(\frac{1}{\lambda_k}\right)$ is a sequence of summability factors of type (A, A^λ) (i.e. the series $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} u_k$ is A^λ-summable for each A-summable series $\sum_{k=0}^{\infty} u_k$), then the A-summability a.e. on $[a, b]$ of series (1), where $(\xi_k) \in \ell^2$, implies the A^λ-summability of the series $\sum_{k=0}^{\infty} \frac{\xi_k}{\lambda_k} \varphi_k(t)$ a.e. on $[a, b]$. From [3, Theorem 29.3], it follows, that $\left(\frac{1}{\lambda_k}\right)$ is a sequence of summability factors (P, P^λ), provided that conditions (5), (6), and (7) hold. Therefore the P-summability a.e. of series (1) for each $(\xi_k) \in \ell^2$ implies the P^λ-summability a.e. of the series $\sum_{k=0}^{\infty} \zeta_k \varphi_k(t)$ for each $(\zeta_k) \in \ell^2_\lambda$.

Note that the above argument does not imply the maximal P^λ-summability a.e. of the series $\sum_{k=0}^{\infty} \zeta_k \varphi_k(t)$ for each $(\zeta_k) \in \ell^2_\lambda$.

2.3 λ-inclusion of summability methods in the class of orthogonal series

Let A and B be two summability methods. If for series (1) from the B-summability a.e. follows the A-summability a.e. for every $(\xi_k) \in \ell^2$, then we say that A *includes* B *in the class of series* (1) (shortly $B \subset A$). Inclusion of summability methods (in the class of orthogonal series) is well investigated. A review of inclusion results may be found in [27].

The following result by Türnpu is the starting point in **Chapter II** ([17]), where we discuss the λ-inclusion $A^\lambda \supset B^\lambda$ in the class of series (1).

Theorem 3 (cf. [24]). *Let* $A = (\alpha_{nk})$ *and* $B = (\beta_{nk})$ *be regular triangular summability methods. If*

$$\sum_{k=\nu}^{\infty} \sup_{n \geq k} |a_{nk}| \left(\beta_{k\nu} - 1\right)^2 = O(1),$$

then $A \supset B$ *in the class of series* (1) .

Theorem 4 (cf. [24]). *Let* $A = (\alpha_{nk})$ *and* $B = (\beta_{nk})$ *be regular triangular summability methods and* $\sigma = (\sigma_k)$ *be a sequence such that* $0 < \sigma_k \downarrow 0$. *If* $\sup_{n \geq k} |a_{nk}| = O(\sigma_k)$, *then* $A \supset P(\sigma)$ *in the class of series* (1), *where* $P(\sigma)$ *is the Riesz method defined by the sequence* (P_k) *with*

$$P_k = \exp\left(\sum_{\nu=0}^{k} \sigma_\nu\right).$$

Our interest is focused to the inclusion with respect to maximal λ-summability of summability methods. The main result of Chapter II is the following

Theorem 5 (cf. [17]). *Let $A = (\alpha_{nk})$ and $B = (\beta_{nk})$ be regular λ^2-conservative methods and let*

$$\lambda_n^{-2} \sum_{k=n}^{m} (\beta_{kn} - 1)^2 \sup_{l \geq k} \lambda_l^2 |a_{lk}| = O(1) \ \text{ and } \ \sum_{k=n}^{m} (\beta_{kn} - 1)^2 \sup_{l \geq k} |a_{lk}| = O(1).$$

If the orthogonal series (1) *is B^λ-summable (maximally B^λ-summable) a.e. on $[a, b]$ for every $x \in \ell^2_\lambda$, then the orthogonal series* (1) *is also A^λ-summable (maximally A^λ-summable) a.e. on $[a, b]$ for every $x \in \ell^2_\lambda$.*

The proof of this Theorem is based on Banach-Steinhaus theorem and two lemmas on measurable functions due to Türnpu [22].

For some summability methods A, Theorem 5 enables us to reduce the problem of the maximal λ-summability of orthogonal series to the well-studied Riesz methods $P = P(A)$. We have proved the following theorem.

Theorem 6 (cf. [17]). *Let A be a regular λ^2-conservative method, where*

$$a_k := \sup_{n \geq k} |a_{nk}| \searrow 0 \ \text{ and } \ \lambda_n^2 |a_{nk}| \searrow 0 \quad (n \to \infty, \ k \in \mathbf{N}).$$

Let $P(A)$ be the Riesz method with

$$P_k = \exp\left(\sum_{\nu=0}^{k} a_\nu\right).$$

If conditions (5), (7), *and*

$$L_n(P(A), t) = O(1)$$

hold, then the orthogonal series (1) *is maximally A^λ-summable a.e. on $[a, b]$ for every $x \in \ell^2_\lambda$.*

Consider some examples of Riesz method P defined by regular summability methods A.

Example 1. Let $A = (C, 2)$ be the Cesàro method of order 2, where

$$a_{nk} = \begin{cases} \dfrac{2(n-k+1)}{(n+1)(n+2)} & \text{if} \quad k \leq n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

Then

$$a_k = \frac{1}{2k+1} \searrow 0,$$

and for the Riesz method $P((C,2))$ we have

$$P_k = \exp\left(\sum_{\nu=0}^{k} \frac{1}{2\nu+1}\right) > \sqrt{2k+3}$$

and

$$p_k = P_k - P_{k-1} = P_{k-1}\left(\exp\left(\frac{1}{2k+1}\right) - 1\right) < e.$$

Let $\lambda_n = (n+1)^\alpha$ $(0 < \alpha < 1/4)$. In this case, by asymptotic formula

$$\sum_{k=0}^{n} \frac{1}{(k+1)^\gamma} \sim \frac{1}{1-\gamma}(n+1)^{1-\gamma} \quad (0 < \gamma < 1),$$

the methods $(C,2)$ and $P((C,2))$ are λ^2-conservative. Conditions (7) are also fulfilled:

$$\frac{\lambda_n^2}{P_{n-1}} < \frac{\lambda_n^2}{\sqrt{2n+1}} \downarrow 0 \text{ and } \frac{1}{p_n}\left(\frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n+1}^2}\right) < \sqrt{2n+1}\left(\frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n+1}^2}\right) \downarrow 0.$$

Example 2. Consider the regular discontinuous Riesz method $(R^*,1,2)$ with

$$\alpha_{nk} = \begin{cases} \left(1 - \dfrac{k}{n+1}\right)^2 & \text{if} \quad k \le n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

For this method we have

$$a_{nk} = \alpha_{nk} - \alpha_{nk+1} = \frac{n-2k}{(n+1)^2} \quad \text{and} \quad a_k = \frac{1}{4(2k+1)}.$$

As in the previous example, we get that the method $P((R^*,1,2))$ is λ^2-conservative. Conditions (7) are fulfilled if $\lambda_n = (n+1)^\alpha$, $(0 < \alpha < 1/16)$, for instance.

3 Convergence and summability with speed of series with respect to product systems

3.1 Product systems

Let $\{f_k\}$ be a system of integrable functions on $[a,b]$ (orthogonality is not essential) such that $|f_k(t)| \le 1$ a.e. on $[a,b]$ $(k \in \mathbf{N})$. The system $\{g_n\}$ defined by

$$g_0(t) := 1 \quad \text{and} \quad g_n(t) := f_{n_0+1}(t) f_{n_1+1}(t) \dots f_{n_k+1}(t) \qquad (t \in [a,b]),$$

where $n = 2^{n_0} + 2^{n_1} + \ldots + 2^{n_k}$ $(n_0 < n_1 < \ldots < n_k)$ is the dyadic representation of n, is called the *product system of* $\{f_k\}$. For example, the Walsh system (3) is the product system of Rademacher system (2).

The system $\{f_k\}$ is said to be *weakly multiplicative* if the product system g satisfies the condition

$$\sum_{n=0}^{\infty} \left| \int_a^b g_n(t)dt \right| < \infty$$

(cf. [21, p.292]). If

$$\int_0^1 \left| \sum_{n=0}^{2^m-1} \left(\int_a^b g_n(\tau)d\tau \right) w_n(t) \right|^p dt = O(1),$$

then $\{f_k\}$ is called *p-weakly multiplicative* $(p \ge 1)$ (cf. [21, p. 330]). In particular, the system $\{f_k\}$ with

$$\sum_{n=0}^{\infty} \left(\int_a^b g_n(t)dt \right)^2 < \infty$$

is 2-weakly multiplicative. If $\{f_k\}$ is weakly multiplicative then it is p-weakly multiplicative for every p.

Note that

(a) the Rademacher system (2), systems $\{\sin 2^k t\}$ and $\{\cos 2^k t\}$ are weakly multiplicative;

(b) the system $\{\frac{1}{k}\}$ is not weakly multiplicative, but p-weakly multiplicative $(p \ge 1)$;

(c) the systems $\{1\}$, $\{\cos kx\}$, and $\{\sin kx\}$ are not p-weakly multiplicative for every $p \ge 1$.

In [20] it is proved that the series

$$\sum_{k=0}^{\infty} \xi_k f_k(t)$$

converges a.e. on $[a, b]$ for all rearrangements of $\{\xi_k f_k\}$ if $(\xi_k) \in \ell^2$ and $\{f_k\}$ is p-weakly multiplicative for a number p with $1 < p < \infty$.

The series

$$\sum_{k=0}^{\infty} \xi_k g_k(t) \tag{8}$$

is called *p-maximally convergent a.e. on* $[a, b]$ if it is convergent a.e. on $[a, b]$ and

$$\int_a^b \sup_n \left| \sum_{k=0}^{n} \xi_k g_k(t) \right|^p dt < \infty.$$

The following two theorems are starting points for investigations in **Chapter III** ([16]).

Theorem 7 (cf. [20]). *Series* (8) *is* 1*-maximally convergent a.e. on* $[a, b]$ *if* $(\xi_k) \in \ell^2$ *and* $\{g_k\}$ *is the product system of a p-weakly multiplicative system for* $2 \leq p < \infty$.

Theorem 8 (cf. [19], [21, p. 292]). *Series* (8) *is* 2*-maximally convergent a.e. on* $[a, b]$ *if* $(\xi_k) \in \ell^2$ *and* $\{g_k\}$ *is the product system of a weakly multiplicative system.*

3.2 Convergence and λ-boundedness of series with respect to multiplicative systems

In Chapter III we study p-maximal convergence (p-maximal boundedness) a.e. of series (8) in the sense of convergence with speed. If series (8) is λ-convergent (λ-bounded) a.e. on $[a, b]$ and

$$\int_a^b \sup_n \lambda_n^p \left| \sum_{k=n+1}^{\infty} \xi_k g_k(t) \right|^p dt < \infty,$$

then it is said to be *p-maximally λ-convergent* (*p-maximally λ-bounded*) a.e. on $[a, b]$.

The proof of the following statements is based on classical inequalities and Banach-Steinhaus theorem.

Theorem 9 (cf. [16]).

(a) *If* $(\xi_k) \in \ell^2_\lambda$ *and* $\{g_k\}$ *is the product system of a weakly multiplicative system, then series* (8) *is* 2*-maximally* λ*-convergent a.e. on* $[a, b]$.

(b) *If* $(\xi_k) \in \ell^2_\lambda$ *and* $\{g_k\}$ *is the product system of a* 2*-weakly multiplicative system, then series* (8) *is* 1*-maximally* λ*-convergent a.e. on* $[a, b]$.

Beside the trigonometric and Haar systems, the Walsh system is one of the most widely used complete orthonormal systems of functions in the theory of functions of a real variable. It is very similar to the trigonometric system, differing from the latter by its greater simplicity.

The *Walsh-Fourier coefficients of an integrable function* u are the numbers

$$< u, w_n > := \int_0^1 u(t) w_n(t) dt \quad (n \in \mathbf{N}),$$

and the *Walsh-Fourier series of* u is the series

$$\sum_{n=0}^{\infty} < u, w_n > w_n(t). \tag{9}$$

Properties of this series are well studied. Mention some of them. For example, it was shown in [26], in which Walsh introduced the system bearing his name, that for every point $\tau_0 \in [0, 1]$ there exists a continuous function u, whose Walsh-Fourier series diverges at that point. On the other hand, Walsh remarked, that

$$\lim_n \sum_{\nu=0}^{2^n-1} < u, w_\nu > w_\nu(\tau) = u(\tau) \text{ uniformly on } [0, 1]$$

for every $u \in C_{[0,1]}$. The Walsh-Fourier series is uniformly $(C, 1)$-summable for every $u \in C_{[0,1]}$ (cf. [4]).

The first treatise on the convergence of Walsh-Fourier series in space $L^p_{[0,1]} (1 < p < \infty)$ was Paley's paper [14]. In it he proved that for every function $u \in L^p_{[0,1]} (1 < p < \infty)$ the Walsh-Fourier series converges in the metric of $L^p_{[0,1]}$. The Walsh-Fourier series of $u \in L^1_{[0,1]}$ may diverge everywhere, but it is $(C, 1)$-summable in the metric of $L^1_{[0,1]}$ (cf. [2]). For more details about Walsh-Fourier series see e.g. [5], [21].

In Chapter III we study convergence and summability a.e. of series

$$\sum_{k=0}^{\infty} < u, w_k > g_k(t), \tag{10}$$

where $u \in L^p_{[a,b]}$, and $\{g_k\}$ is the product system of a system $\{f_k\}$ of integrable functions. By means of the Banach-Steinhaus theorem (in context of sublinear operators), the following theorems are proved.

Theorem 10 (cf. [16]). *Let $1 < p, q < \infty$ be conjugate exponents $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ and let u be a function in $L^p_{[0,1]}$. If g is the product system of a q-weakly multiplicative system, then series* (10) *is 1-maximally convergent a.e. on* $[a, b]$.

Theorem 11 (cf. [16]). *If $\{g_k\}$ is the product system of a weakly multiplicative system, then series* (10) *with $u \in L^p_{[0,1]}$ $(1 < p < \infty)$ is p-maximally convergent a.e. on* $[a, b]$.

Let $u \in L^p_{[a,b]}$. With help of Cesáro means

$$h_n(t) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) < u, g_k > w_k(t) \quad (n \in \mathbf{N}),$$

of the series $\sum_{k=0}^{\infty} < u, g_k > w_k(t)$, by use of an estimate due to Balashov and Rubinstein [2] we obtain that in context of Theorem 11

$$< u, g_k > := \int_a^b u(t) g_k(t) dt = < h, w_k > \quad (k \in \mathbf{N}),$$

where $h := \lim_n h_n$ in $L^p_{[0,1]}$. Then we have

Theorem 12 (cf. [16]). *If $\{g_k\}$ is a product system of a weakly multiplicative system, then the series*

$$\sum_{k=0}^{\infty} < u, g_k > g_k(t),$$

is p-maximally convergent a.e. on $[a, b]$ for every $u \in L^p_{[a,b]}$.

Let $\{g_k\}$ be the product system of a weakly multiplicative system. From Theorem 10 it follows that for every $u \in L^p_{[0,1]}$ $(1 < p < \infty)$ series (10) is p-maximally convergent a.e. on $[a, b]$ (and in $L^p_{[a,b]}$) to some function $v \in L^p_{[0,1]}$. We prove that the problem of the p-maximally λ-boundedness of series (10) may be reduced to the same problem for Walsh-Fourier series.

Theorem 13 (cf. [16]). *Let $\{g_k\}$ be the product system of a weakly multiplicative system and let $u \in L^p_{[0,1]}$. If series* (9) *is p-maximally λ-bounded a.e. on $[0, 1]$, then series* (10) *is p-maximally λ-bounded a.e. on $[a, b]$ for the same u.*

Theorem 14 (cf. [16]). *Let $\{g_k\}$ be the product system of a q-weakly multiplicative system and let $u \in L^p_{[0,1]}$ where $\frac{1}{p} + \frac{1}{q} = 1$. If series* (9) *is p-maximally λ-bounded a.e. on $[0, 1]$ for u, then series* (10) *is 1-maximally λ-bounded a.e. on $[a, b]$ for the same u.*

Proofs of these results are essentially based on orhogonality of the Walsh system and classical inequalities.

3.3 Uniform convergence and λ-boundedness of series with respect to product systems

Let g be the product system of a system of measurable functions f_k ($k \in \mathbf{N}$) satisfying

$$f_0(t) = 1 \quad \text{and} \quad |f_k(t)| \leq 1 \text{ on } [a, b].$$

For a function $u \in C_{[0,1]}$, relationships between uniform convergence properties of series (10) and of the Walsh-Fourier series (9) are investigated in **Chapter IV** ([18]).

Theorem 15 (cf. [18]). *Let $u \in C_{[0,1]}$ and let A be a regular triangular summability method.*

(a) *Series* (10) *is convergent (A-summable, A^λ-bounded, regularly A^λ-summable) uniformly on* $[a, b]$*, if series* (9) *is convergent (A-summable, A^λ-bounded, regularly A^λ-summable) uniformly on* $[0, 1]$.

(b) *If $\{g_k\}$ is an orthogonal system, then series* (10) *is convergent (A-summable, A^λ-bounded, regularly A^λ-summable) uniformly on* $[a, b]$*, if and only if series* (9) *is convergent (A-summable, A^λ-bounded, regularly A^λ-summable) uniformly on* $[0, 1]$.

The proof is based on good convergence properties for 2^nth partial sums of series with respect to product systems, which essentially follows from Banach-Steinhaus theorem.

As a consequence of this theorem we have, that uniform convergence of Walsh-Fourier series of a continuous function u implies convergence of series of Walsh-Fourier coefficients of the same u.

An additional consequence of Theorem 15 is the following

Corollary 16 (cf. [18]). *Series* (10) *is uniformly* $(C, 1)$*-summable on* $[a, b]$ *for every* $u \in C_{[0,1]}$.

FUNKTSIONAALRIDADE KIIRUSEGA KOONDUVUS JA SUMMEERUVUS

KOKKUVÕTE

Vaatleme funktsionaalridu kujul

$$\sum_{k=0}^{\infty} \xi_k \varphi_k(t). \tag{1}$$

Kui λ on mingi kiirus, s.o. monotoonselt kasvav positiivsete reaalarvude jada, siis koonduva rea (1) λ-koonduvus (ehk koonduvus kiirusega λ) tähendab jada $\left(\lambda_n \sum_{k=n+1}^{\infty} \xi_k \varphi_k(t)\right)_n$ koonduvust. Olgu $A = (\alpha_{nk})$ rida-jada summeerimismenetlus. Öeldakse, et rida (1) on A^{λ}-summeeruv (ehk A-summeeruv kiirusega λ), kui see on A-summeeruv ja jada $\left(\lambda_n \sum_{k=n+1}^{\infty} \alpha_{nk} \xi_k \varphi_k(t)\right)$ koondub.

Doktoritöö põhieesmärgiks on näidata, et funktsionaalridade teooria paljud tuntud tulemused on laiendatavad kiirusega koonduvuse ja kiirusega summeeruvuse juhule. Töö lähtepunktideks on mitmed Alexitsi [1], Kaczmarzi [9-11], Kangro [6-8] ja Tandori [13] klassikalised tööd ning Mòriczi [12], Schippi [19-21] ja Türnpu [22-25] artiklid lähemast minevikust. Väidete tõestamisel on töös kasutatud nii klassikalise analüüsi kui ka funktsionaalanalüüsi uurimismeetodeid.

Käesoleva doktoritöö olulisemad tulemused on järgmised.

1) On kirjeldatud selliste summeerimismenetluste klasse, mille puhul vastava Lebesgue'i funktsiooni tõkestatus garanteerib rea (1) maksimaalse λ-summeeruvuse kõigi $(\xi_k) \in \ell^2_\lambda := \{(\xi_k) : \sum_{k=0}^{\infty} \lambda_k^2 \xi_k^2 < \infty\}$ korral

2) Eeldusel, et $\{\varphi_k\}$ on mingi süsteemi $\{f_k\}$ korrutissüsteem, on leitud seosed süsteemi $\{f_k\}$ omaduste (p-nõrk multiplikatiivsus) ja rea (1) koonduvusomaduste (p-maksimaalne λ-koonduvus) vahel.

3) Kui $\{\varphi_k\}$ on mingi süsteemi korrutissüsteem ja $\{w_k\}$ on Walshi süsteem, siis funktsioonide u puhul klassidest $C_{[0,1]}$ ning $L^p_{[0,1]}$ on näidatud, et rea $\sum_{k=0}^{\infty} < u, w_k > \varphi_k$ mitmed kiirusega koonduvuse probleemid saab taandada Walsh-Fourier' rea $\sum_{k=0}^{\infty} < u, w_k > w_k$ vastavatele probleemidele.

Töö koosneb viiest osast: kokkuvõte ja peatükid I, II, III ja IV, mis kujutavad endast teaduslikke artikleid (artiklite loetelu vt. lk. 7).

Töö **I peatükis** lahendatakse konkreetne ülesanne: lähtudes Türnpu poolt tõestatud üldisest ja keerulisest teoreemist (vt. [25]), leitakse need seosed Rieszi kaalutud keskmiste menetluse P ja kiiruse λ vahel, mille puhul vastavate Lebesgue'i funktsioonide tõkestatusest järeldub rea (1) maksimaalne P^λ-summeeruvus peaaegu kõikjal (p.k.) iga jada (ξ_k) korral Banachi ruumist ℓ^2_λ.

Töö **II peatükis** lahendatakse sama ülesanne teatavate regulaarsete λ^2-konservatiivsete menetluste $A = (a_{nk})$ jaoks, võrreldes neid Rieszi menetlusega $P(A)$, mis on konstrueeritud järgmiselt:

$$P_k = \exp\left(\sum_{\nu=0}^{k} \sup_{n\geq\nu} |a_{n\nu}|\right) \qquad (k = 0, 1, 2, \ldots).$$

See arutelu baseerub eelnevas peatükis saadud tulemustel ja summeerimismenetluste sisalduvusest kiirusega summeeruvuse mõttes ortogonaalridade klassis.

On mitu põhjust, miks uurimiseks on valitud just Rieszi menetlused. Ühelt poolt on nende menetluste klass ortogonaalridade puhul universaalne selles mõttes, et iga jada $(\xi_k) \in \ell^2$ jaoks leidub selline kaalutud keskmiste menetlus $P = P((\xi_k))$, mis summeerib vastava rea (1). Teine oluline põhjus on, et P on tehniliselt lihtsalt käsitletav ja tema puhul on Kangro poolt efektiivselt lahendatud kiiruste säilitamise probleem (vt. [6]–[8]), s.o. küsimus sellest, milliste kiiruste λ puhul teisendab menetlus kõik λ-koonduvad jadad λ-koonduvateks jadadeks.

Töö III ja IV peatükis vaadeldakse rida (1), kus $\{\varphi_k\}$ on mingi teise süsteemi $\{f_k\}$ korrutissüsteem, s.t.

$$\varphi_0(t) = 1 \qquad \text{ja} \qquad \varphi_k(t) = f_{k_0+1}(t) f_{k_1+1}(t) \ldots f_{k_n+1}(t),$$

kus $k = 2^{k_0} + 2^{k_1} + \ldots 2^{k_n}$ $(k_0 < k_1 \ldots < k_n)$. **III peatükk** uurib ridade (1) p-maksimaalset λ-koonduvust ja p-maksimaalset λ-tõkestatust korrutissüsteemi $\{\varphi_k\}$ korral. Seejuures nimetatakse rida (1) p-maksimaalselt λ-koonduvaks (p-maksimaalselt λ-tõkestatuks), kui ta on λ-koonduv (λ-tõkestatud) p.k. lõigus $[a, b]$ ja

$$\int_a^b \sup_n \lambda_n^p \left| \sum_{k=n+1}^{\infty} \xi_k \varphi_k(t) \right|^p dt < \infty,$$

kus $1 < p < \infty$. Muuhulgas selles peatükis tõestatakse, et kui

$$\sum_{k=0}^{\infty} \left| \int_a^b \varphi_k(t) dt \right| < \infty$$

(sel juhul öeldakse, et lähtesüsteem $\{f_k\}$ on nõrgalt multiplikatiivne), siis rida (1) on 2-maksimaalselt λ-koonduv p.k. lõigus $[a,b]$ iga $(\xi_k) \in \ell^2_\lambda$ korral. Erilise tähelepanu all on read

$$\sum_{k=0}^{\infty} < u, w_k > \varphi_k(t), \tag{2}$$

kus $< u, w_k > := \int_0^1 u(t)w_k(t)dt$ on mingi integreeruva funktsiooni u Fourier' kordajad Walshi süsteemi $\{w_k\}$ suhtes. Osutub, et selle rea 1-maksimaalse λ-tõkestatuse probleemi saab $u \in L^p_{[0,1]}$ puhul $(1 < p < \infty)$ (sobivatel eeldustel süsteemi $\{\varphi_k\}$ suhtes) taandada funktsiooni u Walsh-Fourier rea $\sum_{k=0}^{\infty} < u, w_k > w_k(t)$ p-maksimaalsele λ-tõkestatusele.

IV peatükk uurib ridade (2) ühtlast koonduvust, ühtlast A-summeeruvust, ühtlast A^λ-tõkestatust ja ühtlast regulaarset A^λ-summeeruvust, kus u on lõigus $[0,1]$ pidev funktsioon. Näidatakse, et ortogonaalse süsteemi $\{\varphi_k\}$ puhul on iga nimetud omadus samaväärne funktsiooni u Walsh-Fourier' rea sama omadusega.

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PUBLICATIONS

CHAPTER I

Acta et Commentationes Universitatis Tartuensis de Mathematica **5** (2001), 3-14

Riesz summability with speed of orthogonal series

Natalia Saealle and Heino Türnpu

Abstract. Sufficient conditions for summability with speed of orthogonal series are found.

1. Main result

Let $\varphi = \{\varphi_k\}$ be a system of orthogonal functions on $[a, b]$, and let $\lambda = (\lambda_k)$ be a sequence with $0 < \lambda_k \nearrow \infty$. We will consider the series of the form

$$\sum \xi_k \varphi_k(t),$$

where $x = (\xi_k) \in \ell^2_\lambda$, i.e. $\sum \xi_k^2 \lambda_k^2 < \infty$.

We will use the following definitions from [1].

Let $A = (a_{nk})$ be a triangular summability method and let $z = (\zeta_k) \in c$ with $\lim \zeta_k = \zeta$.

The sequence z is said to be *convergent with speed* λ or λ-*convergent*, if the limit

$$\lim_n \lambda_n(\zeta_n - \zeta)$$

exists. The set of all λ-convergent sequences is denoted by c^λ.

The sequence z is said to be A-*summable with speed* λ or A^λ-*summable*, if $y = (\eta_n) \in c^\lambda$, where

$$\eta_n = \sum_{k=0}^{n} a_{nk} \zeta_k.$$

The summability method A is said to be λ-*convergence preserving* if every element of the set c^λ is A^λ-summable.

2000 *Mathematics Subject Classification.* 40C05, 42C15.

Key words and phrases. Orthogonal series, summability with speed, Riesz summability, maximal summability, Lebesgue functions.

Supported in part by Estonian Science Foundation Grant 3991.

The series $\sum \xi_k\varphi_k(t)$ is said to be *A^λ-summable almost everywhere (a.e.) on* $[a,b]$ if it is A-summable a.e. on $[a,b]$, i.e. the limit

$$\lim_n \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) = f_x(t) \tag{1}$$

exists a.e. on $[a,b]$, and the limit

$$\lim_n \beta_n(A,x,t) \tag{2}$$

exists a.e. on $[a,b]$, where

$$\beta_n(A,x,t) = \lambda_n \left(\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x(t) \right)$$

and

$$\alpha_{nk} = \sum_{\nu=k}^{n} a_{n\nu}.$$

The series $\sum \xi_k\varphi_k(t)$ is said to be *maximally A^λ-summable* if the limits (1) and (2) exist and

$$\int_a^b \sup_n |\beta_n(A,x,t)|dt < \infty.$$

The starting point of this paper is the following theorem.

Theorem 1 (see [7]). *Let A be λ^2-convergence preserving and let*

$$\lim_n \alpha_{nk} = 1 \quad \textit{for all} \quad k \in \mathbf{N}.$$

The series $\sum \xi_k\varphi_k(t)$ is A^λ-summable a.e. on $[a,b]$ for all $x \in \ell^2_\lambda$ if and only if the following conditions hold:

1° $\sum \xi_k\varphi_k(t)$ *is A-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$;*

2° *For each $\varepsilon > 0$ there exist a measurable subset $T_\varepsilon \subset [a,b]$ satisfying* $\text{mes}T_\varepsilon > b-a-\varepsilon$ *and a constant $M_\varepsilon > 0$ such that, for all measurable decompositions*

$$\mathcal{N}_m := \Big\{ \mathcal{N}_{mn} : n = 0,1,\ldots,m; \mathcal{N}_{mk} \cap \mathcal{N}_{mn} = \emptyset \text{ if } k \neq n; \bigcup_{n=0}^{m} \mathcal{N}_{mn} \subset [a,b] \Big\}, \tag{3}$$

one has

$$A_m(\varepsilon) = \left| \int_{T_\varepsilon} \int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \sum_{\nu=0}^{m} \varphi_\nu(t)\varphi_\nu(\tau) D_{np\nu}^m dt d\tau \right| \le M_\varepsilon,$$

where $\chi_{mn} = \chi_{\mathcal{N}_{mn}}$ *and*

$$D_{np\nu}^m = \begin{cases} (\alpha_{m\nu} - \alpha_{n\nu})(\alpha_{m\nu} - \alpha_{p\nu}) \dfrac{\lambda_n \lambda_p}{\lambda_\nu^2}, & \text{if } 0 \le v \le n < p < m, \\ \alpha_{m\nu}(\alpha_{m\nu} - \alpha_{p\nu}) \dfrac{\lambda_n \lambda_p}{\lambda_\nu^2}, & \text{if } n < \nu \le p < m, \\ \alpha_{m\nu}^2 \dfrac{\lambda_n \lambda_p}{\lambda_\nu^2}, & \text{if } n < p < \nu \le m. \end{cases}$$

In the present paper we will mainly consider the case, when A is the Riesz summability method P, i.e.

$$a_{nk} = \begin{cases} \dfrac{p_k}{P_n}, & k \le n, \\ 0, & k > n, \end{cases}$$

where $p_k > 0$ and $P_n = \sum_{k=0}^n p_k \nearrow \infty$.

Note that the Riesz summability method P is λ-convergence preserving if and only if (see [2])

$$\frac{\lambda_n}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k} = O(1).$$

If P is λ-convergence preserving, then clearly

$$\frac{\lambda_n}{P_n} = O(1) \frac{\lambda_k}{P_k} \quad \text{for } k \le n \ , \quad k, n \in \mathbf{N}. \tag{4}$$

Hence, if the method P is λ^2-convergence preserving, i.e.

$$\frac{\lambda_n^2}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} = O(1), \tag{5}$$

then

$$\frac{\lambda_n^2}{P_n} = O(1) \frac{\lambda_k^2}{P_k} \quad \text{for } k \le n \ , \quad k, n \in \mathbf{N}. \tag{6}$$

Since by the Cauchy inequality

$$\frac{\lambda_n}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k} \le \left(\frac{\lambda_n^2}{P_n^2} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} \sum_{k=0}^{n} p_k \right)^{1/2} = \left(\frac{\lambda_n^2}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} \right)^{1/2},$$

we have that *if P is λ^2-convergence preserving, then P is also λ-convergence preserving.*

The main objective of this paper is to prove the following theorem.

Theorem 2. *Let condition* (5) *hold, and let*

$$\frac{\lambda_n^2}{P_{n-1}} \downarrow 0\ , \quad p_n = O(P_{n-1}), \tag{7}$$

$$\frac{1}{p_n} \Delta \frac{1}{\lambda_n^2} \downarrow 0, \tag{8}$$

where

$$\Delta \frac{1}{\lambda_n^2} = \frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n+1}^2}.$$

If

$$\int_a^b \sup_k L_k(P, t) dt < \infty, \tag{9}$$

where

$$L_k(P, t) = \int_a^b \left| \sum_{\nu=0}^{k} \left(1 - \frac{P_{\nu-1}}{P_k}\right) \varphi_\nu(t)\varphi_\nu(\tau) \right| d\tau,$$

with $P_{-1} = 0$, are the Lebesgue functions of the method P, then the series $\sum \xi_k \varphi_k(t)$ is maximally P^λ-summable a.e. on $[a, b]$ for every $x \in \ell^2_\lambda$.

Let us remark that, in 1969, G. Kangro proved the following result.

Theorem 3 (cf. [2])**.** *If $(1/\lambda_k)$ is a sequence of summability factors of type (A, A^λ), i.e. the series*

$$\sum \frac{1}{\lambda_k} \zeta_k$$

is A^λ-summable for every A-summable series $\sum \zeta_k$, then the A-summability a.e. on $[a, b]$ of the series $\sum \xi_k^0 \varphi_k(t)$, where $x_0 \in \ell^2$, implies the A^λ-summability of the series $\sum \frac{\xi_k^0}{\lambda_k} \varphi_k(t)$ a.e. on $[a, b]$.

If conditions (5), (7) and (8) are fulfilled, then from Theorem 29.3 of [1], it follows that $\left(\frac{1}{\lambda_k}\right)$ is a sequence of summability factors of type (P, P^λ). Therefore we have that if conditions (5), (7) and (8) hold, then the P-summability a.e. of the series $\sum \xi_k \varphi_k(t)$ for every $x \in \ell^2$ implies the P^λ-summability of the series $\sum \xi_k \varphi_k(t)$ for every $x \in \ell^2_\lambda$. Note that the above argument does not imply the maximal P^λ-summability of the series $\sum \xi_k \varphi_k(t)$ for every $x \in \ell^2_\lambda$.

2. Main Lemma

The proof of Theorem 2 is based on the following lemma.

Lemma 4. *If conditions* (5), (7) *and* (8) *hold, then for each* $\varepsilon > 0$ *there exists a measurable subset* $T_\varepsilon \subset [a,b]$ *satisfying* $\text{mes}T_\varepsilon > b-a-\varepsilon$ *such that for all decompositions* (3) *one has*

$$A_m(\varepsilon) = O(1)\int_{T_\varepsilon} \sup_{k\le m} L_k(P,t)dt. \tag{10}$$

Proof. Denote

$$R_j(t,\tau) = \sum_{\nu=0}^{j} \alpha_{j\nu}\varphi_\nu(t)\varphi_\nu(\tau),$$

where

$$\alpha_{j\nu} = 1 - \frac{P_{\nu-1}}{P_j}.$$

Then

$$\varphi_\nu(t)\varphi_\nu(\tau) = \sum_{k=0}^{\nu} \eta_{\nu k} R_k(t,\tau),$$

where $(\eta_{nk}) = P^{-1}$ is the inquotation matrix of P.

From [1] (see p. 193) it follows that

$$\sum_{\nu=k}^{m} \eta_{\nu k} D^m_{np\nu} = P_k \Delta \frac{\Delta D^m_{npk}}{p_k},$$

and therefore

$$\begin{aligned} A_m(\varepsilon) &= \left| \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{n=0}^{m-2}\chi_{mn}(t) \sum_{p=n+1}^{m-1}\chi_{mp}(\tau) \sum_{\nu=0}^{m}\sum_{k=0}^{\nu} \eta_{\nu k} R_k(t,\tau) D^m_{np\nu}dtd\tau \right| \\ &= \left| \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{n=0}^{m-2}\chi_{mn}(t) \sum_{p=n+1}^{m-1}\chi_{mp}(\tau) \sum_{k=0}^{m} R_k(t,\tau) P_k \Delta \frac{\Delta D^m_{np\nu}}{p_k} dtd\tau \right| \\ &= \Bigg| \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{n=0}^{m-2}\chi_{mn}(t) \sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\lambda_n\lambda_p \Bigg[\sum_{k=0}^{m} R_k(t,\tau)\Big(\Delta^1_k(n,p,m) \\ &\quad + \Delta^2_k(p,m) + \Delta^3_k(p,m)\Big)\Bigg] dtd\tau \Bigg|, \end{aligned}$$

where

$$\Delta_k^1(n,p,m)=\begin{cases} P_k\Delta\dfrac{\Delta\Big[(\alpha_{mk}-\alpha_{nk})(\alpha_{mk}-\alpha_{pk})\frac{1}{\lambda_k^2}\Big]}{p_k} & \text{if } 0\le k<n,\\ \dfrac{P_n\alpha_{nn}(\alpha_{mn}-\alpha_{pn})}{\lambda_n^2 p_n} & \text{if } k=n,\\ 0 & \text{if } k>n,\end{cases}$$

$$\Delta_k^2(p,m)=\begin{cases} P_k\Delta\dfrac{\Delta\Big[\alpha_{mk}(\alpha_{mk}-\alpha_{pk})\frac{1}{\lambda_k^2}\Big]}{p_k} & \text{if } n\le k<p,\\ \dfrac{P_p\alpha_{mp}\alpha_{pp}}{\lambda_p^2 p_p} & \text{if } k=p,\\ 0 & \text{if } k<n, k>p,\end{cases}$$

and

$$\Delta_k^3(p,m)=\begin{cases} P_k\Delta\dfrac{\Delta\Big[\alpha_{mk}^2\frac{1}{\lambda_k^2}\Big]}{p_k} & \text{if } p\le k<m-1,\\ P_{m-1}\Big(\dfrac{\alpha_{m,m-1}^2}{\lambda_{m-1}^2p_{m-1}}-\dfrac{\alpha_{mm}^2}{\lambda_m^2p_{m-1}}-\dfrac{\alpha_{mm}^2}{\lambda_m^2p_m}\Big) & \text{if } k=m-1,\\ P_m\dfrac{\alpha_{mm}^2}{\lambda_m^2p_m} & \text{if } k=m,\\ 0 & \text{if } k<p.\end{cases}$$

Observe, that

$$\Big|\Delta_k^1(n,p,m)\Big|\le\frac{P_k}{P_nP_p}\left|\Delta\frac{\Delta\frac{P_{k-1}^2}{\lambda_k^2}}{p_k}\right| \quad \text{for} 0<k<n,$$

by (4)

$$\Delta_n^1(n,p,m)=O(1)\frac{1}{\lambda_n^2P_p}=O(1)\frac{1}{\lambda_n\lambda_pP_n},$$

and

$$\Delta_p^2(p,m)=O(1)\frac{1}{\lambda_p^2}.$$

Denote

$$M_n=\frac{P_n}{p_n}\lambda_n^2\Delta\frac{1}{\lambda_n^2}.$$

From (7) it follows that $M_n\le 1$ for all $n\in\mathbf{N}$. Therefore

$$\Delta_{m-1}^3(p,m)=\frac{P_{m-1}}{p_{m-1}}\Delta\frac{\alpha_{m,m-1}^2}{\lambda_{m-1}^2}-\frac{p_m}{\lambda_m^2P_m^2}=O(1)M_{m-1}\frac{1}{\lambda_{m-1}^2}+O(1)\frac{1}{\lambda_m^2},$$

and

$$\Delta_m^3(p,m) = O(1)\frac{1}{\lambda_m^2}.$$

Thus

$$\begin{aligned} A_m(\varepsilon) \quad &\leq \quad \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \frac{\lambda_n^2}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left| P_k \Delta \frac{\Delta \frac{P_{k-1}^2}{\lambda_k^2}}{p_k} \right| |R_k(t,\tau)|\, dt d\tau \\ &+ \quad \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{p=1}^{m-1} \chi_{mp}(\tau)\lambda_p^2 \left[\sum_{k=0}^{p-1} \left|\Delta_k^2(p,m)\right| |R_k(t,\tau)| \right. \\ &\qquad \left. + \sum_{k=p}^{m-2} \left|\Delta_k^3(p,m)\right| |R_k(t,\tau)| \right] dt d\tau \\ &+ \quad O(1) \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \Big[|R_n(t,\tau)| + |R_p(t,\tau)| \\ &\qquad + |R_{m-1}(t,\tau)| + |R_m(t,\tau)| \Big] dt d\tau. \end{aligned}$$

Now we have

$$\begin{aligned} A_m(\varepsilon) \leq &\int_{T_\varepsilon} \sup_{k<m} L_k(P,t) dt \sup_{n<m} \frac{\lambda_n^2}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left| P_k \Delta \frac{\Delta \frac{P_{k-1}^2}{\lambda_k^2}}{p_k} \right| \\ &+ \int_{T_\varepsilon} \sup_{k<m} L_k(P,\tau) d\tau \sup_{p<m} \lambda_p^2 \sum_{k=0}^{p-1} \left|\Delta_k^2(p,m)\right| \\ &+ \int_{T_\varepsilon} \sup_{k<m} L_k(P,\tau) d\tau \sup_{p} \lambda_p^2 \sum_{k=p}^{m-2} \left|\Delta_k^3(p,m)\right| \\ &+ O(1) \int_{T_\varepsilon} \sup_{k<m} L_k(P,t) dt. \end{aligned}$$

Therefore, in order to prove (10), it is sufficient to show that

$$V_{npm}^i = O(1), \quad i = 1, 2, 3,$$

where

$$V_{npm}^1 = \frac{\lambda_n^2}{P_n P_{n-1}} \sum_{k=0}^{n-1} P_k \left| \Delta \frac{\Delta \frac{P_{k-1}^2}{\lambda_k^2}}{p_k} \right|,$$

$$V_{npm}^2 = \lambda_p^2 \sum_{k=0}^{p-1} \left|\Delta_k^2(p,m)\right|, \; V_{npm}^3 = \lambda_p^2 \sum_{k=p}^{m-2} \left|\Delta_k^3(p,m)\right|.$$

By [4] (see p. 220) we have that, for any sequence $(a_n) \subset \mathbf{R}$,

$$P_k \Delta \frac{\Delta \frac{a_k}{\lambda_k^2}}{p_k} = \left(\Delta \frac{1}{\lambda_k^2} + \Delta \frac{1}{\lambda_{k+1}^2} \right) P_k \frac{\Delta a_k}{p_k} \tag{11}$$
$$+ \frac{1}{\lambda_{k+2}^2} P_k \Delta \frac{\Delta a_k}{p_k} + P_k a_{k+1} \Delta \left(\frac{1}{p_k} \Delta \frac{1}{\lambda_k^2} \right).$$

Consider the case when $i = 1$; then, by (11), we have

$$V_{npm}^1 \leq \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \left| \left(\Delta \frac{1}{\lambda_k^2} + \Delta \frac{1}{\lambda_{k+1}^2} \right) \frac{\Delta P_{k-1}^2}{p_k} + \frac{1}{\lambda_{k+2}^2} \Delta \frac{\Delta P_{k-1}^2}{p_k} + P_k^2 \Delta \Big(\frac{1}{p_k} \Delta \frac{1}{\lambda_k^2} \Big) \right|.$$

Since by (7)

$$\sum_{k=0}^{n-1} \left| P_k^2 \Delta \left(\frac{1}{p_k} \Delta \frac{1}{\lambda_k^2} \right) \right|$$
$$= P_0^2 \frac{1}{p_0} \Delta \frac{1}{\lambda_0^2} - \sum_{k=0}^{n-1} \Delta P_k^2 \frac{1}{p_{k+1}} \Delta \frac{1}{\lambda_{k+1}^2} - \frac{P_{n-1}^2}{p_n} \Delta \frac{1}{\lambda_n^2},$$

we have

$$V_{npm}^1 \leq \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \left(\left(\Delta \frac{1}{\lambda_k^2} + \Delta \frac{1}{\lambda_{k+1}^2} \right) (P_{k-1} + P_k) + \frac{1}{\lambda_{k+2}^2} (p_k + p_{k+1}) \right)$$
$$+ \frac{\lambda_n^2}{P_n} O(1) + \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \left| \Delta P_k^2 \frac{1}{p_{k+1}} \Delta \frac{1}{\lambda_{k+1}^2} \right| + \frac{\lambda_n^2}{P_n^2} \frac{P_{n-1}^2}{p_n} \Delta \frac{1}{\lambda_n^2}.$$

Now, by (5) and (6), we have

$$V_{npm}^1 \leq 2 \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \left(\frac{p_k}{\lambda_k^2} M_k + \frac{p_{k+1}}{\lambda_{k+1}^2} M_{k+1} + p_{k+1} \Delta \frac{1}{\lambda_{k+1}^2} \right)$$
$$+ \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \left(\frac{p_k}{\lambda_k^2} + \frac{p_{k+1}}{\lambda_{k+1}^2} \right) + O(1)$$
$$+ 2 \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \frac{p_{k+1}}{\lambda_{k+1}^2} M_{k+1} + \frac{\lambda_n^2}{P_n^2} \frac{P_n}{\lambda_n^2} M_n + O(1)$$
$$= O(1) \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} + O(1) \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \frac{p_{k+1}}{\lambda_{k+1}^2} + O(1)$$
$$= O(1).$$

Analogously we have

$$V_{npm}^2 \le \frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left|P_k\Delta\frac{\Delta\frac{\alpha_{mk}P_{k-1}}{\lambda_k^2}}{p_k}\right|$$

$$\le \frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left|\left(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\right)P_k\frac{\Delta(\alpha_{mk}P_{k-1})}{p_k}+\frac{1}{\lambda_{k+2}^2}P_k\Delta\frac{\Delta(\alpha_{mk}P_{k-1})}{p_k}\right|$$

$$+\frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left|\alpha_{mk}P_k^2\Delta\left(\frac{1}{p_k}\Delta\frac{1}{\lambda_k^2}\right)\right|$$

$$= O(1)\frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left[\left(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\right)P_k+\frac{P_k}{\lambda_{k+2}^2}\left|\Delta\left(\frac{P_k}{P_m}-\left(1-\frac{P_{k-1}}{P_m}\right)\right)\right|\right]$$

$$+\frac{\lambda_p^2}{P_p}O(1)+\frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\Delta(\alpha_{mk}P_k^2)\frac{1}{p_{k+1}}\Delta\frac{1}{\lambda_{k+1}^2}+\alpha_{m,p-1}\frac{\lambda_p^2}{P_p}\frac{P_p}{\lambda_p^2}M_p = O(1).$$

Finally, for $i=3$, we have

$$V_{npm}^3$$

$$\le \lambda_p^2\sum_{k=p}^{m-2}\left|\left(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\right)P_k\frac{\Delta\alpha_{mk}^2}{p_k}\right.$$

$$\left.+\frac{1}{\lambda_{k+2}^2}P_k\Delta\frac{\Delta\alpha_{mk}^2}{p_k}+P_k\alpha_{mk+1}^2\Delta\left(\frac{1}{p_k}\Delta\frac{1}{\lambda_k^2}\right)\right|$$

$$= O(1)\lambda_p^2\sum_{k=p}^{m-2}\left(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\right)+\frac{\lambda_p^2}{P_m}\sum_{k=p}^{m-2}\left|\frac{P_k}{\lambda_{k+2}^2}\Delta(\alpha_{mk}+\alpha_{m,k+1})\right|$$

$$+\alpha_{mp+1}\lambda_p^2\frac{P_p}{p_p}\Delta\frac{1}{\lambda_p^2}+O(1)\lambda_p^2\sum_{k=p}^{m-2}\Delta\frac{1}{\lambda_{k+1}^2}=O(1).$$

The proof is complete.

3. Proof of Theorem 2

In the proof of Theorem 2, we will make use of the following

Lemma 5 (see [5], pp. 142-144)**.** *Let (f_n) be a sequence of integrable functions on $[a,b]$. Then for each measurable subset $T\subset[a,b]$ and for each $m\in\mathbf{N}$ one has*

$$\int_T \sup_{n\le m}|f_n(t)|dt \le 2\sup_{\mathcal{N}_m}\left|\int_T\sum_{n=0}^{m}\chi_{mn}(t)f_n(t)dt\right|,$$

where $\mathcal{N}_m$ ranges over all decompositions defined by (3).

Proof of Theorem 2. By [6] (see p. 201) the condition

$$L_n(P,t) = O_t(1) \quad \text{a.e. on } [a,b]$$

implies that the series $\sum \xi_k \varphi_k(t)$ is P-summable a.e. on $[a,b]$ for every $x \in \ell^2$.

From Theorem 1 and Lemma 4 it follows that the series $\sum \xi_k \varphi_k(t)$ is P^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$.

To show the maximal P^λ–summability we prove that

$$\int_{T_\varepsilon} \sup_{n \le m} |\beta_n(A,x,t)| dt = O(\|x\|_{l^2_\lambda}) + \sup_{\mathcal{N}_m} \{A_m(\varepsilon)\}^{1/2}, \tag{12}$$

where $T_\varepsilon \subset [a,b]$ is a measurable subset with $\text{mes} T_\varepsilon > b-a-\varepsilon$ and $\mathcal{N}_m$ ranges over all decompositions defined by (3).

If condition (12) holds, then from (9) and (10) it follows that the series $\sum \xi_k \varphi_k(t)$ is maximally P^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$. We now prove (12). By Lemma 5

$$\int_{T_\varepsilon} \sup_{n \le m} |\beta_n(A,x,t)| dt = O(1) \sup_{\mathcal{N}_m} \left| \int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t) \beta_n(A,x,t) dt \right|.$$

Denote

$$\bar{\alpha}_{pk} = \alpha_{pk} - \alpha_{p-1,k},$$

then

$$\beta_n(A,x,t) = \lambda_n \sum_{p=n+1}^{\infty} \sum_{k=0}^{p} \bar{\alpha}_{pk} \xi_k \varphi_k(t) = B_{mn}(x,t) + C_{mn}(x,t),$$

where

$$B_{mn}(x,t) = \lambda_n \sum_{p=n+1}^{m} \sum_{k=0}^{p} \bar{\alpha}_{pk} \xi_k \varphi_k(t),$$

$$C_{mn}(x,t) = \lambda_n \sum_{p=m+1}^{\infty} \sum_{k=0}^{p} \bar{\alpha}_{pk} \xi_k \varphi_k(t).$$

Therefore

$$\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t) \beta_n(A,x,t) dt = O(1) \int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t) B_{mn}(x,t) dt$$
$$+ O(1) \int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t) C_{mn}(x,t) dt.$$

By orthogonality of φ we have

$$\begin{aligned}
&\left|\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)C_{mn}(x,t)dt\right| \\
&\le \int_a^b \sum_{n=0}^{m}\chi_{mn}(t)\lambda_n\left|\sum_{k=0}^{m}\alpha_{mk}\xi_k\varphi_k(t)-f_x(t)\right|dt \\
&\le\sqrt{b-a}\left[\sup_{k\le m}\frac{\lambda_m|\alpha_{mk}-1|}{\lambda_k}\left(\sum_{k=0}^{m}\xi_k^2\lambda_k^2\right)^{1/2}+\left(\sum_{k=m+1}^{\infty}\xi_k^2\lambda_k^2\right)^{1/2}\right].
\end{aligned}$$

If A is λ-convergence preserving, then by [3] (see Lemma 3)

$$\lambda_m|\alpha_{mk}-1| = O(\lambda_k)\quad (k\le m) \tag{13}$$

and therefore

$$\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)C_{mn}(x,t)dt = O(\|x\|_{l^2_\lambda}).$$

Denoting

$$A_p^m(t)=\sum_{n=0}^{p-1}\chi_{mn}(t)\lambda_n,$$

we have

$$\begin{aligned}
\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)B_{mn}(x,t)dt &= \sum_{k=0}^{m}\xi_k\int_{T_\varepsilon}\sum_{p=k}^{m}\bar{\alpha}_{pk}\varphi_k(t)A_p^m(t)dt \\
&\quad+\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)\lambda_n(\alpha_{m0}-\alpha_{n0})\xi_0\varphi_0(t)dt.
\end{aligned}$$

Now by (13)

$$\begin{aligned}
&\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)B_{mn}(x,t)dt \\
&=\sum_{k=0}^{m}\xi_k\int_{T_\varepsilon}\sum_{p=k}^{m}\bar{\alpha}_{pk}\varphi_k(t)A_p^m(t)dt+O(\|x\|_{\ell^2_\lambda}).
\end{aligned}$$

Using the principle of uniform boundedness we get

$$\begin{aligned}
&\sum_{k=0}^{m}\xi_\nu\int_{T_\varepsilon}\sum_{p=k}^{m}\bar{\alpha}_{pk}\varphi_k(t)A_p^m(t)dt \\
&=O(1)\left(\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{k=0}^{m}\frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2}\sum_{p=k}^{m}\bar{\alpha}_{pk}A_p^m(t)\sum_{\nu=k}^{m}\bar{\alpha}_{\nu k}A_\nu^m(\tau)dtd\tau\right)^{1/2}\|x\|_{\ell^2_\lambda}.
\end{aligned}$$

Finally

$$\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{k=0}^{m}\frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2}\sum_{p=k}^{m}\bar{\alpha}_{pk}A_p^m(t)\sum_{\nu=k}^{m}\bar{\alpha}_{\nu k}A_\nu^m(\tau)dtd\tau = A_m(\varepsilon)+E_m,$$

where

$$E_m = \int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)\chi_{mn}(\tau)\sum_{k=0}^{m}\varphi_k(t)\varphi_k(\tau)\left[\frac{\lambda_n}{\lambda_k}(\alpha_{mk}-\alpha_{nk})\right]^2 dtd\tau.$$

By (13) and Bessel's inequality we get

$$\begin{aligned}
E_m &= \sum_{n=0}^{m-1}\sum_{k=0}^{m}\left[\frac{\lambda_n}{\lambda_k}(\alpha_{mk}-\alpha_{nk})\right]^2\left(\int_{T_\varepsilon}\varphi_k(t)\chi_{mn}(t)dt\right)^2\\
&= O(1)\sum_{n=0}^{m}\sum_{\nu=0}^{\infty}\left(\int_{T_\varepsilon}\varphi_\nu(t)\chi_{mn}(t)dt\right)^2\\
&= O(1)\sum_{n=0}^{m}\int_{T_\varepsilon}\chi_{mn}^2(t)dt\\
&= O(1)\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)dt\\
&= O(1).
\end{aligned}$$

Therefore condition (12) holds.

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CHAPTER II

Analysis Mathematica (Szeged) **31** (2005), 63-73

Summability of orthogonal series with speed

Natalia Saealle and Heino Türnpu

Abstract. Some sufficient conditions are found for summability of orthogonal series with speed.

1. Introduction

Let $\varphi = \{\varphi_k\}$ be a system of integrable (in special case: orthonormal) functions on an interval

$[a, b]$, and let $\lambda = (\lambda_k)$ be a sequence of real numbers such that $0 < \lambda_k \nearrow \infty$. We will consider the series of the form

$$\sum_{k=0}^{\infty} \xi_k \varphi_k(t), \tag{1}$$

where $x = (\xi_k) \in \ell^2$, or $x = (\xi_k) \in \ell^2_\lambda$, that is, $\sum_{k=0}^{\infty} \xi_k^2 \lambda_k^2 < \infty$.

In this paper, we use the following basic definitions and facts.

The sequence $(\zeta_k) \in c$ is said to be λ-*convergent* (see [2, p. 251]) if the limit

$$\lim_n \lambda_n(\zeta_n - \zeta) \quad \text{exists, where } \lim_n \zeta_n =: \zeta.$$

The set of all λ-convergent sequences is denoted by c^λ. The series (1) is said to be *convergent with speed* λ or λ-*convergent almost everywhere* (a.e.) *on* $[a, b]$ if the limits

$$\lim_n \sum_{k=0}^{n} \xi_k \varphi_k(t) =: f_x(t)$$

and

$$\lim_n \lambda_n \left(\sum_{k=0}^{n} \xi_k \varphi_k(t) - f_x(t) \right)$$

exist a.e. on $[a, b]$.

Throughout this paper, we assume that $A = (\alpha_{nk})$ is a triangular summability method and denote

$$a_{nk} := \alpha_{n,k} - \alpha_{n,k+1}.$$

In particular, we will study the Riesz summability method P with

$$\alpha_{nk} = 1 - \frac{P_{k-1}}{P_n}, \qquad \text{or} \qquad a_{nk} = \frac{p_k}{P_n} \qquad (k \le n \quad n, k \in \mathbf{N}),$$

where

$$P_{-1} = 0,\ P_n := \sum_{k=0}^{n} p_k$$

and (p_k) is a sequence of real numbers. We assume, that $p_k \ge 0$ and $P_n \nearrow \infty$. In this case the Riesz method is regular.

The sequence $(\zeta_k) \in c$ is said to be A*-summable with speed* λ or A^λ*-summable* if $(\eta_n) \in c^\lambda$, where

$$\eta_n := \sum_{k=0}^{n} a_{nk}\zeta_k.$$

The method A is said to be λ*-convergence preserving* if every element of the set c^λ is A^λ-summable.

If A is a regular summability method, then (see [6]) A is λ-convergence preserving if and only if

$$\lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k} = O(1).$$

In the present paper, we assume that the regular method A is λ^2-convergence preserving, where $\lambda^2 = (\lambda_n^2)$. Since by the Cauchy-Bunyakovsky inequality, we have

$$\lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k} \le \lambda_n \left(\sum_{k=0}^{n} |a_{nk}| \right)^{1/2} \left(\sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k^2} \right)^{1/2} = O(1) \left(\lambda_n^2 \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k^2} \right)^{1/2},$$

this means that if A is λ^2-convergence preserving, then A is also λ-convergence preserving.

Series (1) is said to be A^λ*-summable a.e. on* $[a, b]$ (see [2, p. 252]) if it is A-summable a.e. on $[a, b]$ (that is, if the limit

$$\lim_n \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) =: f_x^A(t)$$

exists a.e. on $[a,b]$), and the limit

$$\lim_n \beta_n(A,x,t),$$

also exists a.e. on $[a,b]$, where

$$\beta_n(A,x,t) := \lambda_n \left(\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x^A(t) \right).$$

Series (1) is said to be *maximally A^λ-summable* if it is A^λ-summable and

$$\int_a^b \sup_n \lambda_n \left| \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x^A(t) \right| dt < \infty.$$

If φ is an orthonormal system and A is regular then by the Fisher-Riesz theorem, we have $f_x^A(t) = f_x(t)$, where $f_x(t)$ is the sum of the orthogonal series (1) in $L^2_{[a,b]}$. The functions

$$L_n^\varphi(A,t) := \int_a^b \left| \sum_{k=0}^{n} \alpha_{nk}\varphi_k(t)\varphi_k(\tau) \right| d\tau$$

are called the *Lebesgue functions of the method A associated with φ.*

First, let φ be an orthonormal system on $[a,b]$ and let $A = C^1$ be the Cesàro method, that is,

$$\alpha_{nk} = 1 - \frac{k-1}{n}.$$

In this case, Kaczmarz proved (see [4], [5]) that if the Lebesgue functions of C^1 are bounded on $[a,b]$, then series (1) is C^1-summable a.e. on $[a,b]$ for every $x \in \ell^2$. On the other hand, if $\alpha_{nk} \equiv 1$ for $k \le n$, then it is proved in [4], [5] that from the boundedness of Lebesgue functions on $[a,b]$ it follows that series (1) converges a.e. on $[a,b]$ for every $x \in \ell^2$.

It has been proved by Alexits and Sharma in [1] that the result of Kaczmarz is true if the φ_k are integrable (not necessarily orthogonal) functions on $[a,b]$.

Now, Móricz and Tandori (see [7]) proved that there exist a triangular regular summability method $A^0 = (\alpha_{nk}^0)$, a sequence $x_0 = (\xi_k^0) \in \ell^2$ and a system $\varphi_0 = (\varphi_k^0)$ orthonormal on $[a,b]$ such that the Lebesgue functions $L_n^{\varphi^0}(A^0,t)$ are bounded on $[a,b]$, but the series $\sum_{k=0}^\infty \xi_k^0\varphi_k^0(t)$ is not A^0-summable a.e. on $[a,b]$.

Móricz [8] and Türnpu [11] found certain classes of regular summability methods A for which the condition

$$L_n^\varphi(A,t) = O(1) \qquad \Big(\text{or (see [11])} \;\; L_n^\varphi(A,t) = O_t(1) \Big)$$

implies that series (1) is A-summable a.e. on $[a,b]$ for every $x \in \ell^2$.

For example, for the case of the Riesz method P from the boundedness of Lebesgue functions a.e. on $[a,b]$ it follows that the series (1) is P-summable a.e. on $[a,b]$ for every $x \in \ell^2$.

On the other hand, necessary and sufficient conditions for A-summability of series (1) a.e. on $[a,b]$ for all $x \in \ell^2$ are founded in [12] as follows. It is proved that from the conditions

$$\lim_n \alpha_{nk} = 1$$

and

$$\int_a^b \sup_{p \geq n} \left| \sum_{k=0}^{n} \alpha_{nk}\alpha_{pk}\varphi_k(t)\varphi_k(\tau) \right| d\tau = O_t(1)$$

a.e. on $[a,b]$ it follows that the series (1) is A-summable a.e. on $[a,b]$ for all $x \in \ell^2$.

Necessary and sufficient conditions for A^λ-summability of series (1) a.e. on $[a,b]$ for all $x \in \ell^2_\lambda$

are found in [14]. We proved there that if A is λ^2-convergence preserving, $\lim\limits_n \alpha_{nk} = 1$ and the series (1) is A-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$, then the condition

$$\int_a^b \sup_{p \geq n} \left| \sum_{\nu=0}^{m-1} \varphi_\nu(t)\varphi_\nu(\tau) D^m_{np\nu} \right| d\tau = O_t(1)$$

a.e. on $[a,b]$, where

$$D^m_{np\nu} := \begin{cases} (\alpha_{m\nu} - \alpha_{n\nu})(\alpha_{m\nu} - \alpha_{p\nu})\lambda_n\lambda_p/\lambda_\nu^2 & \text{if } 0 \leq \nu \leq n < p < m, \\ \alpha_{m\nu}(\alpha_{m\nu} - \alpha_{p\nu})\lambda_n\lambda_p/\lambda_\nu^2 & \text{if } n < \nu \leq p < m, \\ \alpha_{m\nu}^2\lambda_n\lambda_p/\lambda_\nu^2 & \text{if } n < p < \nu \leq m, \end{cases}$$

implies that series (1) is A^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$.

Since the form of the above condition is very complicated, in [9] we considered the case $A = P$, the Riesz summability method.

Theorem A (see [9]). *Let*

$$\frac{\lambda_n^2}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} = O(1), \tag{2}$$

$$\frac{\lambda_n^2}{P_{n-1}} \searrow 0\ , \quad p_n = O(P_{n-1}), \tag{3}$$

and

$$\frac{1}{p_n}\left(\frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n+1}^2}\right) \searrow 0. \tag{4}$$

If

$$\int_a^b \sup_k L_k^\varphi(P,t)dt < \infty, \tag{5}$$

then the orthogonal series (1) *is maximally* P^λ*-summable a.e. on* $[a,b]$ *for every* $x \in \ell_\lambda^2$.

2. Two new theorems

The main aim of this paper is to prove the following theorems.

Theorem B. *Let* A *be a regular* λ^2*-convergence preserving method, where*

$$a_k := \sup_{n\geq k} |a_{nk}| \searrow 0 \quad \text{and} \quad \lambda_n^2 |a_{nk}| \searrow 0 \quad (n \to \infty,\ k \in \mathbf{N}). \tag{6}$$

and let $P = P(A)$ *be the Riesz summability method with*

$$P_k = \exp\left(\sum_{\nu=0}^k a_\nu\right).$$

If (2), (3), (4) *and the condition*

$$L_n(P(A),t) = O(1)$$

hold, then the orthogonal series (1) *is maximally* A^λ*-summable a.e. on* $[a,b]$ *for every* $x \in \ell_\lambda^2$.

In the proof of Theorem B, we will use the following

Theorem C. *Let* $A = (\alpha_{nk})$ *and* $B = (\beta_{nk})$ *be regular* λ^2*-convergence preserving methods, and let*

$$\lambda_n^{-2} \sum_{k=n}^m (\beta_{kn}-1)^2 \sup_{l\geq k} \lambda_l^2 |a_{lk}| = O(1) \text{ and } \sum_{k=n}^m (\beta_{kn}-1)^2 \sup_{l\geq k} |a_{lk}| = O(1). \tag{7}$$

If the orthogonal series (1) *is* B^λ*-summable (maximally* B^λ*-summable) a.e. on* $[a,b]$ *for every* $x \in \ell_\lambda^2$*, then the orthogonal series* (1) *is also* A^λ*-summable (maximally* A^λ*-summable) a.e. on* $[a,b]$ *for every* $x \in \ell_\lambda^2$.

3. Proofs of Theorems B and C

We need the following lemmas.

Lemma 1 (see [10]). *Let f be a measurable function on $[a,b]$. Then*

$$|f(t)| < \infty \quad a.e.\ on \quad [a,b]$$

if and only if for each $\varepsilon > 0$ there exists a measurable subset $T_\varepsilon \subset [a,b]$ such that mes $T_\varepsilon > b - a - \varepsilon$ *and*

$$\int_{T_\varepsilon} |f(t)|dt < \infty.$$

Lemma 2 (see [10]). *Let (f_n) be a sequence of integrable functions on $[a,b]$. Then*

$$\sup_n |f_n(t)| < \infty \ \text{ a.e. on } [a,b]$$

if and only if for each $\varepsilon > 0$, there exist a measurable subset $T_\varepsilon \subset [a,b]$ with mes $T_\varepsilon > b - a - \varepsilon$ *and a constant $M_\varepsilon > 0$ such that for all measurable decompositions*

$$\mathcal{N}_m := \Big\{\mathcal{N}_{mn} : n = 0,1,\dots,m; \mathcal{N}_{mk} \cap \mathcal{N}_{mn} = \emptyset \ \textit{if} \ k \neq n; \\ \bigcup_{n=0}^{m} \mathcal{N}_{mn} \subset [a,b]\Big\}, \qquad (8)$$

one has

$$B_m^\varepsilon := \left|\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t) f_n(t) dt\right| \leq M_\varepsilon, \ \textit{where} \ \chi_{mn} := \chi_{\mathcal{N}_{mn}}. \qquad (9)$$

Remark. In [10] we have actually proved that under the conditions of Lemma 2, for each measurable subset $T \subseteq [a,b]$ and for each $m \in \mathbf{N}$ one has

$$\int_T \max_{n \leq m} |f_n(t)| dt \leq 2 \sup_{\mathcal{N}_m} \left|\int_T \sum_{n=0}^{m} \chi_{mn}(t) f_n(t) dt\right|.$$

Since the space ℓ^2_λ endowed with the norm

$$\| x \|_{\ell^2_\lambda} = \left(\sum_{k=0}^{\infty} \xi_k^2 \lambda_k^2\right)^{1/2}$$

is a Banach space and the set $\{e_i = (\delta_{ki})_{k=0}^{\infty} : i \in \mathbf{N}\}$, where δ_{ki} is the Kronecker symbol , forms a total set in ℓ^2_λ (that is, the linear combinations of e_i are everywhere dense in ℓ^2_λ), we can use the Banach theorem.

Lemma 3 (see [3], p. 361)**.** *Let* $(D_n : n \in \mathbf{N})$ *be continuous linear operators from* ℓ^2_λ *to the Frechet space* $M_{[a,b]}$ *of all functions totally measurable on* $[a,b]$*. Suppose that the following conditions hold:*

1° $\sup_n |D_n(x,t)| < \infty$ *a.e. on* $[a,b]$ *for every* $x \in \ell^2_\lambda$*;*

2° *the limit* $\lim_n D_n(e_i,t)$ *exists a.e. on* $[a,b]$ *for every* $i \in \mathbf{N}$*.*

Then the limit $\lim_n D_n(x,t)$ *exists a.e. on* $[a,b]$ *for all* $x \in \ell^2_\lambda$*.*

Proof of Theorem C. Let the second equality in (7) hold. By [13, Corollary], if the orthogonal series (1) is B-summable a.e. on $[a,b]$ for $x^0 \in \ell^2$, then it is A-summable a.e. on $[a,b]$ for the same x_0. In Theorem C we assume that series (1) is B^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda \subset \ell^2$, therefore it is B-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$. So, by [13, Corollary 1], series (1) is A-summable a.e. on $[a,b]$ to some function f_x for every $x \in \ell^2_\lambda$. Furthermore, the operator $p = f_x(t)$ defined by

$$p : \ell^2_\lambda \to M_{[a,b]}, \qquad x \mapsto f_x$$

is continuous and linear.

Let

$$D_n(x,t) = \lambda_n \left(\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x(t) \right).$$

The operator $D_n(x,t)$ from ℓ^2_λ into $M_{[a,b]}$ is continuous and linear. We will use Lemma 3 and show that conditions 1° and 2° are fulfilled.

By Lemma 2, for condition 1° it is sufficient to show that inequality (9) with $f_n = D_n$ holds for every decompositions (8), that is, for every $\varepsilon > 0$ and fixed $x \in \ell^2_\lambda$ there exists a measurable subset $T_\varepsilon = T_\varepsilon(x) \subset [a,b]$ with mes $T_\varepsilon(x) > b - a - \varepsilon$ and a constant $M_\varepsilon = M_\varepsilon(x) > 0$ such that for all decomposition (8) one has

$$B^\varepsilon_m = \left| \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t) D_n(x,t)dt \right| \le M_\varepsilon(x).$$

By Abel's transformation, we obtain

$$\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) = \sum_{k=0}^{n} a_{nk} \sum_{\nu=0}^{k} \xi_\nu\varphi_\nu(t),$$

and by using the Cauchy-Bunyakovsky inequality, we have

$$
\begin{aligned}
B_m^\varepsilon &= \left| \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \left(\sum_{k=0}^{n} a_{nk} \sum_{\nu=0}^{k} \xi_\nu \varphi_\nu(t) - f_x(t) \right) dt \right| \\
&\le \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \left| \sum_{k=0}^{n} a_{nk} \sum_{\nu=0}^{k} \beta_{k\nu}\xi_\nu \varphi_\nu(t) - f_x(t) \right| dt \\
&\quad + \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \left| \sum_{k=0}^{n} a_{nk} \sum_{\nu=0}^{k} (\beta_{k\nu}-1)\xi_\nu \varphi_\nu(t) \right| dt \\
&\le \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \left| \sum_{k=0}^{n} a_{nk} \left(\sum_{\nu=0}^{k} \beta_{k\nu}\xi_\nu \varphi_\nu(t) - f_x(t) \right) \right| dt \\
&\quad + \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \left| \sum_{k=0}^{n} a_{nk} - 1 \right| |f_x(t)|\, dt \\
&\quad + \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \left\{ \sum_{k=0}^{n} |a_{nk}| \right\}^{\frac{1}{2}} \times \\
&\qquad\qquad \times \left\{ \sum_{k=0}^{n} |a_{nk}| \left(\sum_{\nu=0}^{k} (\beta_{k\nu}-1)\xi_\nu \varphi_\nu(t) \right)^2 \right\}^{\frac{1}{2}} dt.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
B_m^\varepsilon &= \int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n \sum_{k=0}^{n} \frac{|a_{nk}|}{\lambda_k}\lambda_k \left| \sum_{\nu=0}^{k} \beta_{k\nu}\xi_\nu \varphi_\nu(t) - f_x(t) \right| dt \\
&+ O(1) \sup_n \lambda_n \left| \sum_{k=0}^{n} a_{nk} - 1 \right| \left(\int_a^b (f_x(t))^2 dt \right)^{1/2} \\
&+ O(1) \left(\int_{T_\varepsilon(x)} \sum_{n=0}^{m} \chi_{mn}(t)\lambda_n^2 \sum_{k=0}^{n} |a_{nk}| \left(\sum_{\nu=0}^{k} (\beta_{k\nu}-1)\xi_\nu \varphi_\nu(t) \right)^2 dt \right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $(1,1,1,\ldots) \in c^\lambda$ and A is λ-convergence preserving, we have

$$
\sup_n \lambda_n \left| \sum_{k=0}^{n} a_{nk} - 1 \right| = O(1).
$$

So, we find that

$$
\begin{aligned}
B_m^\varepsilon &= O(1) \int_{T_\varepsilon(x)} \sup_k \lambda_k \left| \sum_{\nu=0}^{k} \beta_{k\nu}\xi_\nu \varphi_\nu(t) - f_x(t) \right| dt + O(1)\|f_x(t)\|_{L^2} \\
&\quad + O(1) \left(\sum_{\nu=0}^{m} \lambda_\nu^2 \xi_\nu^2 \lambda_\nu^{-2} \sum_{k=\nu}^{m} (\beta_{k\nu}-1)^2 \sup_{l\ge k} \lambda_l^2 |a_{lk}| \right)^{1/2}.
\end{aligned}
$$

By (7), we get

$$B_m^\varepsilon = O(1)\int_{T_\varepsilon(x)} \sup_k \lambda_k \left|\sum_{\nu=0}^{k} \beta_{k\nu}\xi_\nu\varphi_\nu(t) - f_x(t)\right| dt$$
$$+O(1)\|f_x\|_{L^2} + O(1)||x||_{\ell^2_\lambda}.$$

Since series (1) is B^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$, by Lemma 1, we have that for every $\varepsilon > 0$ a nd every fixed $x \in \ell^2_\lambda$ there exist a measurable set $E_\varepsilon(x) \subset [a,b]$ with mes$E_\varepsilon(x) > b - a - \varepsilon$ and a constant $N_\varepsilon(x) > 0$ such that one has

$$\int_{E_\varepsilon(x)} \sup_k \lambda_k \left|\sum_{\nu=0}^{k} \beta_{k\nu}\xi_\nu\varphi_\nu(t) - f_x(t)\right| dt = N_\varepsilon(x).$$

Thus there exist a measurable subset $T_\varepsilon(x) = E_\varepsilon(x)$ and the constant $M_\varepsilon(x)$ such that

$$M_\varepsilon(x) = O(1)N_\varepsilon(x) + O(1)||f_x||_{L^2} + O(1)||x||_{\ell^2_\lambda}.$$

Therefore, we have

$$B_m^\varepsilon \le M_\varepsilon(x),$$

which means that condition 1° of Lemma 3 holds.

Let δ_{ki} be the Kronecker symbol. Since the series $\sum_{k=0}^\infty \delta_{ki}$ is λ-convergent, A is regular and λ-convergence preserving, the limit

$$\lim_n \lambda_n\Big(\sum_{k=0}^{n} \alpha_{nk}\delta_{ki}\varphi_k(t) - \varphi_i(t)\Big) = \lim_n \lambda_n(\alpha_{ni} - 1)\varphi_i(t)$$

exists a.e. on $[a,b]$, that is, the limit

$$\lim_n D_n(e_i, t)$$

exists a.e. on $[a,b]$ for every $i \in \mathbf{N}$, which means that condition 2° of Lemma 3 also holds. From Lemma 3 it follows that if the series (1) is B^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$, then series (1) is also A^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$.

Assume that the series (1) be maximally B^λ-summable a.e. on $[a,b]$ for all $x \in \ell^2_\lambda$, then

$$\int_a^b \sup_n \lambda_n \left|\sum_{k=0}^{n} \beta_{nk}\xi_k\varphi_k(t) - f_x(t)\right| dt = O_x(1).$$

Now, by the above Remark we have

$$\begin{aligned}
&\int_a^b \max_{n\le m} \lambda_n \left|\sum_{k=0}^n \alpha_{nk}\xi_k\varphi_k(t) - f_x(t)\right| dt \\
&\le 2\sup_{\mathcal{N}_m}\left|\int_a^b \sum_{n=0}^m \chi_{mn}(t)\lambda_n \left|\sum_{k=0}^n \alpha_{nk}\xi_k\varphi_k(t) - f_x(t)\right| dt\right| \\
&= O(1)\int_a^b \sup_k \lambda_k \left|\sum_{\nu=0}^k \beta_{k\nu}\xi_\nu\varphi_\nu(t) - f_x(t)\right| dt + O(1)||f_x||_{L^2} + O(1) \\
&= M_\varepsilon(x),
\end{aligned}$$

that is, the series (1) is maximally A^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$. The proof of the theorem is now complete.

Proof of Theorem B. From (6) it follows that

$$\sup_{n\ge k} \lambda_n^2 |a_{nk}| \le \lambda_k^2 |a_{kk}| \le \lambda_k^2 a_k. \tag{10}$$

If $P = P(A)$, then

$$p_k = P_{k-1}(e^{a_k} - 1),$$

that is,

$$\frac{p_k}{P_{k-1}} \ge a_k. \tag{11}$$

We will show that condition (7) in Theorem C is satisfied with $P = P(A)$ in place of B. Using (3), (10) and (11) gives

$$\begin{aligned}
\lambda_n^{-2}\sum_{k=n}^m (1-\beta_{kn}^2)\sup_{m\ge k}\lambda_m^2|a_{mk}| &\le \lambda_n^{-2}\sum_{k=n}^m \frac{P_{n-1}^2}{P_k^2}\lambda_k^2\frac{p_k}{P_{k-1}} \\
&\le \frac{\lambda_n^2 P_{n-1}^2}{\lambda_n^2 P_{n-1}}\sum_{k=n}^m \frac{p_k}{P_k P_{k-1}} \\
&= P_{n-1}\left(\frac{1}{P_{n-1}} - \frac{1}{P_m}\right) = O(1).
\end{aligned}$$

From Theorem C it follows that series (1) is maximally A^λ-summable a.e. on $[a,b]$ if series (1) is maximally $P(A)^\lambda$-summable a.e. on $[a,b]$. By inequality (5), the boundedness of the Lebesgue functions, and conditions (2), (3) and (4), Theorem A gives that series (1) is maximally $P(A)^\lambda$-summable a.e. on $[a,b]$ for every $x \in \ell_2^\lambda$.

Consequently, series (1) is maximally A^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$. The proof is complete.

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CHAPTER III

Proceedings of the Estonian Academy of Sciences.
Physics. Mathematics **53** (2004), No. 1, 13-25

Convergence and λ-boundedness of functional series with respect to multiplicative systems

N. Saealle and H. Türnpu

Abstract. The series $\sum c_k g_k(t)$, where $\{g_k\}$ is a product system defined by a multiplicative system, is studied. Some sufficient conditions for p-maximal convergence with speed of this series are found. Also the series $\sum < f, w_k > g_k(t)$ with $f \in L^p_{[0,1]}$, and $\{w_k\}$ being a Walsh system is considered. It is proved that this series converges almost everywhere for various product systems. In the last section the λ-boundedness of this series is discussed.

1. INTRODUCTION

Let $f = \{f_k\}_{k=0}^{\infty}$ be a system of integrable functions on $[a, b]$ satisfying

$$|f_k(t)| \leq 1 \quad \text{a.e. on } [a, b].$$

The *product system* $\{g_n\}$ *of* $\{f_k\}$ is then given by

$$g_0(t) = 1 \quad \text{and} \quad g_n(t) = f_{n_0+1}(t) f_{n_1+1}(t) \dots f_{n_k+1}(t) \qquad (t \in [a, b]),$$

where $n = 2^{n_0} + 2^{n_1} + \ldots + 2^{n_k}$ $(n_0 < n_1 < \ldots < n_k)$ is the dyadic representation of n. If $\{g_n\}$ is orthogonal, then $\{f_k\}$ is called *orthogonal multiplicative.* If

$$\int_a^b g_n(t)dt = 0 \quad \text{for } n = 1, 2 \ldots ,$$

then it is said that $\{f_k\}$ is *strongly multiplicative system* (see [1]). For example, the Rademacher system is orthogonal multiplicative and the Walsh

Key words and phrases. Multiplicative systems, Walsh functions, convergence with speed, convergence almost everywhere, λ-boundedness.

system $\{w_n\}_{n=0}^{\infty}$ is their product system. If

$$\sum_{n=0}^{\infty}\left|\int_a^b g_n(t)dt\right| < \infty,$$

then the system $\{f_k\}$ is called *weakly multiplicative* (see [5], p.292). If

$$\int_0^1\left|\sum_{n=0}^{2^m-1}\left(\int_a^b g_n(\tau)d\tau\right)w_n(t)\right|^p dt = O(1),$$

then $\{f_k\}$ is called *p-weakly multiplicative* $(1 \le p \le \infty)$ (see [5], p.330). Particularly, the system $\{f_k\}$ with

$$\sum_{n=0}^{\infty}\left(\int_a^b g_n(t)dt\right)^2 < \infty$$

is 2-weakly multiplicative (see [8]).

Clearly, every orthogonal multiplicative system, strongly multiplicative system and weakly multiplicative system is p-weakly multiplicative system.

We first consider series

$$\sum_{k=0}^{\infty} c_k f_k(t) \tag{1}$$

and

$$\sum_{k=0}^{\infty} c_k g_k(t). \tag{2}$$

Notice that if the series (2) converges a.e. on $[a,b]$ for all $(c_k) \in \ell^2$, then the same statement is true for the series (1).

In [7] it is proved that the series (1) converges a.e. on [a,b] for all rearrangements of $\{c_k f_k\}$ if $(c_k) \in \ell^2$ and $\{f_k\}$ is p-weakly multiplicative system for a number p with $1 < p < \infty$.

The series (2) is called *p-maximally convergent a.e. on* $[a,b]$ if it is convergent a.e. on $[a,b]$ and

$$\int_a^b \sup_n\left|\sum_{k=0}^{n} c_k g_k(t)\right|^p dt < \infty.$$

Theorem A ([7]). *A series* (2) *is* 1*-maximally convergent a.e. on* $[a,b]$ *if* $(c_k) \in \ell^2$ *and* $\{g_k\}$ *is the product system of a* p*-weakly multiplicative system for* $2 \le p < \infty$.

On the other hand Schipp in [6] proved

Theorem B ([6]). *A series* (2) *is* 2*-maximally convergent a.e. on* $[a,b]$ *if* $(c_k) \in \ell^2$ *and* $\{g_k\}$ *is the product system of a weakly multiplicative system.*

In this paper we study p-maximally convergence a.e. of the series

$$\sum_{k=0}^{\infty} c_k g_k(t)$$

in the sense of the convergence with speed. Let $\lambda = (\lambda_k)$ be a sequence such that $0 < \lambda_k \nearrow \infty$. The series (2), which is convergent a.e. on $[a,b]$, is called

1)λ*-convergent* (or *convergent with speed* λ) *a.e. on* $[a,b]$ if the limit

$$\lim_n \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t)$$

exists a.e. on $[a,b]$;

2) λ*-bounded a.e. on* $[a,b]$ if

$$\sup_n \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| < \infty \quad \text{a.e. on } [a,b].$$

Clearly, that the λ-convergence implies the λ-boundedness.

Definition 1. *If a series* (2) *is* λ *-convergent a.e. on* $[a,b]$ *and*

$$\int_a^b \sup_n \lambda_n^p \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right|^p dt < \infty, \tag{3}$$

then it is said that the series (2) *is* p*-maximally* λ*-convergent a.e. on* $[a,b]$.

Definition 2. *If the series* (2) *is* λ*-bounded and* (3) *is valid, then it is said that the series* (2) *is* p*-maximally* λ*-bounded.*

In Section 2 we will characterize p-maximally λ-convergence a.e. of the series (2) for $p = 1$ and $p = 2$. For this, we consider the sequence space

$$\ell_\lambda^2 := \{c = (c_k) | \sum_{k=0}^{\infty} \lambda_k^2 c_k^2 < \infty\}.$$

Obviously, ℓ_λ^2 endowed with the norm

$$\| c \| = \left(\sum_{k=0}^{\infty} c_k^2 \lambda_k^2 \right)^{1/2}$$

is a Banach space and the sequences $e_i := (\delta_{ki})_{k=0}^{\infty}$ $(i = 0, 1, \dots)$ form a total set in $(\ell^2_\lambda, \| \ \|)$ (cf. [4], p. 138).

In Section 3 we will consider the series (2) where

$$c_k =< f, w_k >:= \int_0^1 f(t) w_k(t) dt \qquad (f \in L^p_{[0,1]})$$

or

$$c_k =< f, g_k >:= \int_a^b f(t) g_k(t) dt \qquad (f \in L^p_{[a,b]})$$

and we have found some sufficient conditions for p-maximal convergence a.e. $(1 \le p < \infty)$ of these series.

In Section 4 we will characterize p-maximal λ-boundedness a.e. of the series $\sum_{k=0}^{\infty} < f, g_k > g_k(t)$, where $f \in L^p_{[a,b]}$.

2. p-MAXIMAL λ-CONVERGENCE

We will prove the following theorem.

Theorem 1. *If* $(c_k) \in \ell^2_\lambda$ *and* $\{g_k\}$ *is the product system of a weakly multiplicative system, then the series* (2) *is* 2*-maximally* λ*-convergent a.e. on* $[a, b]$*.*

To prove Theorem 1 we need the following corollary of the Banach-Steinhaus theorem.

Lemma ([3], p. 361)**.** *Let* D_n $(n = 0, 1, \dots)$ *be continuous sublinear operators from a Banach space* X *to the Frechet space* $M_{[a,b]}$ *of all functions totally measurable on* $[a, b]$*. Suppose that the following conditions hold:*

1° $\sup_n |D_n(x, t)| < \infty$ *a.e. on* $[a, b]$ *for every* $x \in X$*,*

2° *the limit* $\lim_n D_n(\bar{x}, t)$ *exists a.e. on* $[a, b]$ *for every* $\bar{x}$ *from a total set in* X*.*

Then the limit $\lim_n D_n(x, t)$ *exists a.e. on* $[a, b]$ *for all* $x \in X$*.*

Proof of Theorem 1. Let $\{g_k\}$ be the product system of a weakly multiplicative system. Because

$$K_m(t, u) := \sum_{j=0}^{2^m - 1} g_j(t) w_j(u) \ge 0 \qquad (t \in [a, b],\ u \in [0, 1],\ m = 0, 1, \dots)$$

(see [5], p. 293) and the Walsh system is orthogonal, by the Cauchy-Schwartz inequality we get

$$\left\{\int_a^b \left(\sum_{k=0}^m c_k g_k(t)\right)^2 dt\right\}^{1/2} = \left\{\int_a^b \left(\int_0^1 \sum_{k=0}^m c_k w_k(\tau) K_m(t,\tau)d\tau\right)^2 dt\right\}^{1/2}$$

$$\leq \left\{\int_a^b \left(\int_0^1 \left(\sum_{k=0}^m c_k w_k(\tau)\right)^2 K_m(t,\tau)d\tau\right)\left(\int_0^1 K_m(t,u)du\right) dt\right\}^{1/2}$$

$$= \left\{\int_0^1 \left(\sum_{k=0}^m c_k w_k(\tau)\right)^2 \left(\int_a^b K_m(t,\tau)dt\right) d\tau\right\}^{1/2}$$

$$\leq \left\{\sum_{k=0}^m c_k^2\right\}^{1/2} \left\{\sum_{\nu=0}^{2^m-1} \left|\int_a^b g_\nu(t)dt\right|\right\}^{1/2} = O(1)\left\{\sum_{k=0}^m c_k^2\right\}^{1/2}.$$

Thus the sequence (A_m) of the continuous linear operators

$$A_m : \ell^2_\lambda \to L^2_{[a,b]}, \qquad (c_k) \mapsto \sum_{k=0}^m c_k g_k(t)$$

is pointwise bounded. Since

$$\lim_m \| A_m(e_k) \| = \lim_m \left\{\int_a^b \left(\sum_{k=0}^m \delta_{ki} g_k(t)\right)^2 dt\right\}^{1/2} = \left\{\int_a^b g_i^2(t)dt\right\}^{1/2}$$

for each $k = 0, 1, \dots$, then by the Banach-Steinhaus theorem we have that (A_m) is pointwise convergent to a linear operator

$$A : \ell^2_\lambda \to L^2_{[a,b]}, \qquad (c_k) \mapsto \sum_{k=0}^\infty c_k g_k(t)$$

which is continuous. Consequently,

$$\lim_m \left\{\int_a^b \left(\sum_{k=m+1}^\infty c_k g_k(t)\right)^2\right\}^{1/2} = 0 \quad \text{for each } (c_k) \in \ell^2_\lambda.$$

Therefore

$$\left\{\int_a^b \left(\sum_{k=m+1}^\infty c_k g_k(t)\right)^2\right\}^{1/2} = O(1)\left\{\sum_{k=m+1}^\infty c_k^2\right\}^{1/2} \qquad \left((c_k) \in \ell^2_\lambda\right)$$

and using the Minkowski inequality we have

$$\left\{\int_a^b \max_{n\le m} \lambda_n^2 \left(\sum_{k=n+1}^{\infty} c_k g_k(t)\right)^2 dt\right\}^{1/2}$$

$$\le \left\{\int_a^b \max_{n\le m} \lambda_n^2 \left(\sum_{k=n+1}^{m} c_k g_k(t)\right)^2 dt\right\}^{1/2}$$

$$+\left\{\int_a^b \lambda_m^2 \left(\sum_{k=m+1}^{\infty} c_k g_k(t)\right)^2 dt\right\}^{1/2}$$

$$\le \left\{\int_a^b \max_{n\le m} \lambda_n^2 \left(\sum_{k=n+1}^{m} c_k g_k(t)\right)^2 dt\right\}^{1/2} + O(1)\left\{\sum_{k=m+1}^{\infty} c_k^2\lambda_k^2\right\}^{1/2}.$$

By Abel's transformation in view of

$$\sum_{k=n+1}^{m} a_k u_k = \sum_{k=n+1}^{m-1}(a_k - a_{k+1})\sum_{\nu=0}^{k} u_\nu - a_{n+1}\sum_{k=0}^{n} u_k + a_m \sum_{k=0}^{m} u_k \qquad (4)$$

we obtain

$$\left\{\int_a^b \max_{n\le m} \lambda_n^2 \left(\sum_{k=n+1}^{m} c_k g_k(t)\right)^2 dt\right\}^{1/2}$$

$$\le O(1)\left\{\int_a^b \max_{k\le m}\left(\sum_{\nu=0}^{k} c_\nu\lambda_\nu g_\nu(t)\right)^2 dt\right\}^{1/2} \max_{n\le m}\lambda_n \sum_{k=n+1}^{m-1}\left(\frac{1}{\lambda_k}-\frac{1}{\lambda_{k+1}}\right)$$

$$+\left\{\int_a^b \max_{n\le m}\left(\sum_{\nu=0}^{n} c_\nu\lambda_\nu g_\nu(t)\right)^2 dt\right\}^{1/2} + \left\{\int_a^b \left(\sum_{\nu=0}^{m} c_\nu\lambda_\nu g_\nu(t)\right)^2 dt\right\}^{1/2}$$

$$= O(1)\left\{\int_a^b \max_{k\le m}\left(\sum_{\nu=0}^{k} c_\nu\lambda_\nu g_\nu(t)\right)^2 dt\right\}^{1/2}.$$

Then by Theorem B

$$\left\{\int_a^b \max_{n\le m} \lambda_n^2 \left(\sum_{k=n+1}^{\infty} c_k g_k(t)\right)^2 dt\right\}^{1/2} = O(1) \qquad \left((c_k)\in \ell_\lambda^2\right) \qquad (5)$$

which gives

$$\sup_n \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| < \infty \quad \text{a.e. on } [a,b] \text{ for each } (c_k) \in \ell^2_\lambda. \tag{6}$$

Therefore the linear operators

$$D_n : \ell^2_\lambda \longrightarrow M_{[a,b]}, \quad (c_k) \mapsto \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t) \quad (n = 0, 1, \ldots)$$

are continuous and the statements 1° (cf. (6)) and 2° from Lemma are fulfilled. By Lemma, the limit

$$\lim_n \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t)$$

exists a.e. on $[a,b]$ for every $(c_k) \in \ell^2_\lambda$. Hence the series (2) is λ-convergent a.e. on $[a,b]$ and by (5) it is 2-maximally λ-convergent. The proof of the theorem is now complete.

Analogously, if $\{g_k\}$ is product system of a 2-weakly multiplicative system, then by orthogonality of the Walsh system we have

$$\begin{aligned}
\int_a^b \left| \sum_{k=0}^{m} c_k g_k(t) \right| dt &\le \int_0^1 \left| \sum_{k=0}^{m} c_k w_k(\tau) \right| \left(\int_a^b K_m(t,\tau) dt \right) d\tau \\
&\le \left\{ \int_0^1 \left(\sum_{k=0}^{m} c_k w_k(\tau) \right)^2 d\tau \right\}^{1/2} \left\{ \int_0^1 \left(\int_a^b K_m(t,\tau) \right)^2 d\tau \right\}^{1/2} \\
&= \left\{ \sum_{k=0}^{m} c_k^2 \right\}^{1/2} \left\{ \sum_{\nu=0}^{2^m-1} \left(\int_a^b g_\nu(t) dt \right)^2 \right\}^{1/2} = O(1) \left\{ \sum_{k=0}^{m} c_k^2 \right\}^{1/2}.
\end{aligned}$$

Applying the Banach-Steinhaus theorem we get that for every $c \in \ell^2_\lambda$

$$\int_a^b \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| dt = O(1) \left\{ \sum_{k=n+1}^{\infty} c_k^2 \right\}^{1/2}.$$

By Abel's transformation (4) and Theorem A we obtain

$$\int_a^b \max_{n\le m} \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| dt$$

$$= O(1) \int_a^b \max_{n\le m} \left| \sum_{k=0}^{n} c_k \lambda_k g_k(t) \right| dt + \int_a^b \lambda_m \left| \sum_{k=m+1}^{\infty} c_k g_k(t) \right| dt$$

$$= O(1) \int_a^b \max_{n\le m} \left| \sum_{k=0}^{n} c_k \lambda_k g_k(t) \right| dt + O(1) \parallel c \parallel_{l^2_\lambda} = O(1) \parallel c \parallel_{l^2_\lambda} .$$

Using Lemma we get the following result.

Theorem 2. *If* $(c_k) \in \ell^2_\lambda$ *and* $\{g_k\}$ *is the product system of a* 2*-weakly multiplicative system, then the series* (2) *is* 1*-maximally* λ*-convergent a.e. on* $[a, b]$.

3. p-MAXIMAL CONVERGENCE OF THE SERIES $\sum < f, w_k > g_k(t)$ AND $\sum < f, g_k > g_k(t)$

We will prove the following theorem.

Theorem 3. *Let* $1 < p, q < \infty$ *be conjugate exponents* $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ *and let* f *be a function in* $L^p_{[0,1]}$. *If* $\{g_k\}$ *is the product system of a* q*-weakly multiplicative system, then the series*

$$\sum_{k=0}^{\infty} < f, w_k > g_k(t) \tag{7}$$

is 1-maximally convergent a.e. on $[a, b]$.

Proof. On the one hand,

$$\int_a^b \max_{n\le m} \left| \sum_{k=0}^{n} < f, w_k > g_k(t) \right| dt$$

$$= \int_a^b \max_{n\le m} \left| \int_0^1 \sum_{k=0}^{n} < f, w_k > w_k(\tau) K_m(t,\tau) d\tau \right| dt$$

$$\le \int_0^1 \max_{n\le m} \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right| \left| \int_a^b K_m(t,\tau) dt \right| d\tau.$$

On the other hand from [5], p.103 it follows that

$$\sup_n \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right| \in L^p_{[0,1]}. \tag{8}$$

Therefore by the Hölder inequality

$$\begin{aligned}\int_a^b \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > g_k(t) \right| dt &= O(1) \left\{ \int_0^1 \left| \int_a^b K_m(t,\tau) dt \right|^q d\tau \right\}^{1/q} \\ &= O(1).\end{aligned}$$

The assertion now follows from Lemma.

Since by the Hölder inequality

$$\begin{aligned}&\left\{ \int_a^b \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > g_k(t) \right|^p dt \right\}^{1/p} \\ &= \left\{ \int_a^b \max_{n \le m} \left| \int_0^1 \sum_{k=0}^{n} < f, w_k > w_k(\tau) K_m(t,\tau) d\tau \right|^p dt \right\}^{1/p} \\ &\le \left\{ \int_a^b \int_0^1 \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right|^p K_m(t,\tau) d\tau \times \right. \\ &\qquad\qquad \left. \times \left[\int_0^1 K_m(t,\tau) d\tau \right]^{p/q} dt \right\}^{1/p} \\ &= \left\{ \int_0^1 \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right|^p \left(\int_a^b K_m(t,\tau) dt \right) d\tau \right\}^{1/p} \\ &= O(1) \left\{ \int_0^1 \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right|^p d\tau \right\}^{1/p} \left\{ \sum_{\nu=0}^{\infty} \left| \int_a^b g_\nu(t) dt \right| \right\}^{1/p},\end{aligned}$$

then by (8) we get

$$\left\{ \int_a^b \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > g_k(t) \right|^p dt \right\}^{1/p} = O(1).$$

Now Lemma leads to the following theorem.

Theorem 4. *If $\{g_k\}$ is the product system of a weakly multiplicative system, then the series* (7) *with $f \in L^p_{[0,1]}$ $(1 < p < \infty)$ is p-maximally convergent a.e. on $[a,b]$.*

Set

$$h_n(t) := \sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right) < f, g_k > w_k(t),$$

where $f \in L^p_{[a,b]}$ and $\{g_k\}$ is the product system of a weakly multiplicative system. We will prove that $h_n \in L^p_{[0,1]}$. Indeed, since (see [2])

$$\operatorname{vrai}\sup_n \int_0^1 \left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right) w_k(\tau)w_k(t)\right| d\tau = O(1),$$

using the Hölder inequality we get

$$\begin{aligned}
&\left\{\int_0^1 |h_n(t)|^p\, dt\right\}^{1/p} \\
&= \left\{\int_0^1 \left|\int_0^1 \sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right) w_k(t)w_k(\tau) \sum_{\nu=0}^{2^n-1} w_\nu(\tau) < f, g_\nu > d\tau\right|^p dt\right\}^{\frac{1}{p}} \\
&\le \left\{\int_0^1 \int_0^1 \left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right) w_k(t)w_k(\tau)\right| \left|\sum_{\nu=0}^{2^n-1} w_\nu(\tau) < f, g_\nu >\right|^p d\tau \right. \\
&\qquad\qquad \left. \times \left[\int_0^1 \left|\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right) w_k(t)w_k(\tau)\right| d\tau\right]^{p/q} dt\right\}^{1/p} \\
&= O(1)\left\{\int_0^1\int_0^1 \left|\sum_{k=0}^{n}(1-\frac{k}{n+1})w_k(t)w_k(\tau)\right| dt \left|\sum_{\nu=0}^{2^n-1} w_\nu(\tau) < f, g_\nu >\right|^p d\tau\right\}^{1/p} \\
&= O(1)\left\{\int_0^1 \left|\int_a^b f(u)K_n(u,\tau)du\right|^p d\tau\right\}^{1/p}
\end{aligned}$$

and using the Hölder inequality once again, we have

$$\begin{aligned}
&\left\{\int_0^1 |h_n(t)|^p\, dt\right\}^{1/p} \\
&\quad = O(1)\left\{\int_0^1 \int_a^b |f(u)|^p K_n(u,\tau)du \left[\int_a^b K_n(u,\tau)du\right]^{p/q} d\tau\right\}^{1/p} \\
&\quad = O(1)\left\{\int_a^b |f(u)|^p \int_0^1 K_n(u,\tau)d\tau du\right\}^{1/p} \left\{\sum_{\nu=0}^{2^n-1}\left|\int_a^b g_\nu(u)du\right|\right\}^{1/q} \\
&\quad = O(1)\left\{\int_a^b |f(u)|^p\, du\right\}^{1/p}.
\end{aligned}$$

Therefore $h(t) := \lim_n h_n(t) \in L^p_{[0,1]}$ and $< f, g_\nu >$ are the Walsh-Fourier coefficients of h for every $k = 0, 1, 2, \ldots$:

$$
\begin{aligned}
< h, w_\nu > &= \int_0^1 w_\nu(t) \lim_n \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) < f, g_k > w_k(t)dt \\
&= \lim_n \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) < f, g_k > \int_0^1 w_k(t) w_\nu(t) dt \\
&= \lim_n \left(1 - \frac{\nu}{n+1}\right) < f, g_\nu > = < f, g_\nu > .
\end{aligned}
$$

This yields the following result.

Theorem 5. *If $\{g_k\}$ is a product system of a weakly multiplicative system, then the series*

$$\sum_{k=0}^{\infty} < f, g_k > g_k(t),$$

where $f \in L^p_{[a,b]}$, is p-maximally convergent a.e. on $[a, b]$.

4. p-MAXIMAL λ-BOUNDEDNESS

Let $\{g_k\}$ be the product system of a weakly multiplicative system. From Theorem 4 it follows that the series (7) is for every $f \in L^p_{[a,b]}$ $(1 < p < \infty)$ p-maximally convergent a.e. on $[a, b]$ (and in $L^p_{[a,b]}$) to some function $g \in L^p_{[a,b]}$.

We will prove the following theorem.

Theorem 6. *Let $\{g_k\}$ be the product system of a weakly multiplicative system and let $f \in L^p_{[0,1]}$. If the series*

$$\sum_{k=0}^{\infty} < f, w_k > w_k(t) \tag{9}$$

is p-maximally λ-bounded a.e. on $[0, 1]$, then the series (7) for the same f is p-maximally λ-bounded a.e. on $[a, b]$.

Proof. Let (s_m) be a sequence of natural numbers. Because the Walsh system is orthogonal, by the Minkowski inequality we obtain

$$C_m := \left\{ \int_a^b \max_{n\le m} \lambda_n^p \left| \sum_{k=0}^{n} < f, w_k > g_k(t) - g(t) \right|^p dt \right\}^{1/p}$$

$$\le \left\{ \int_a^b \max_{n\le m} \lambda_n^p \left[\int_0^1 \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \right| K_{s_m}(t,\tau) d\tau \right]^p dt \right\}^{1/p}$$

$$+ \left\{ \int_a^b \max_{n\le m} \lambda_n^p \left| \int_0^1 f(\tau) K_{s_m}(t,\tau) d\tau - g(t) \right|^p dt \right\}^{1/p}.$$

By the Hölder inequality it follows that

$$C_m \le \left\{ \int_a^b \max_{n\le m} \lambda_n^p \int_0^1 \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \right|^p K_{s_m}(t,\tau) d\tau \times \right.$$

$$\left. \times \left[\int_0^1 K_{s_m}(t,\tau) d\tau \right]^{\frac{p}{q}} dt \right\}^{\frac{1}{p}}$$

$$+ \left\{ \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t) \right|^p dt \right\}^{1/p}.$$

Thus

$$C_m$$

$$\le \left\{ \int_0^1 \max_{n\le m} \lambda_n^p \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \right|^p \left(\int_a^b K_{s_m}(t,\tau) dt \right) d\tau \right\}^{1/p}$$

$$+ \left\{ \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t) \right|^p dt \right\}^{1/p}$$

$$= O(1) \left\{ \int_0^1 \max_{n\le m} \lambda_n^p \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \right|^p d\tau \right\}^{1/p}$$

$$+ \left\{ \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t) \right|^p dt \right\}^{1/p}.$$

From Theorem 4 it follows that there exists a subsequence (s_m) of natural numbers such that

$$\lim_m \int_a^b \lambda_m^p \left| \sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t) \right|^p dt = 0.$$

Therefore we have

$$C_m = O(1)\left\{\int_0^1 \max_{n\le m} \lambda_n^p \left|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\right|^p d\tau\right\}^{1/p} + O(1).$$

The proof is complete.

Using Theorem 3 we can prove the following theorem.

Theorem 7. *Let $\{g_k\}$ be the product system of a q-weakly multiplicative system and let $f \in L^p_{[0,1]}$ where $\frac{1}{p}+\frac{1}{q}=1$. If the series (9) is p-maximally λ-bounded a.e. on $[0,1]$ for f, then the series (7) is 1-maximally λ-bounded a.e. on $[a,b]$ for the same f.*

Proof. Let (s_m) be a sequence of natural numbers. As in proof of Theorem 6, we obtain

$$\begin{aligned}
D_m &:= \int_a^b \max_{n\le m} \lambda_n \left|\sum_{k=0}^{n} < f, w_k > g_k(t) - g(t)\right| dt \\
&\le \int_a^b \max_{n\le m} \lambda_n \left|\int_0^1 \left(\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\right) K_{s_m}(t,\tau) d\tau\right| dt \\
&\le \int_0^1 \max_{n\le m} \lambda_n \left|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\right| \left|\int_a^b K_{s_m}(t,\tau)dt\right| d\tau \\
&\qquad + \int_a^b \lambda_m \left|\sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t)\right| dt.
\end{aligned}$$

By Theorem 3, the series (7) is 1-maximally λ-convergent a.e. on $[a,b]$. Therefore the series (7) converges in $L^1_{[a,b]}$ as well. So, there exists a sequence of natural numbers s_m such that

$$\lim_m \int_a^b \lambda_m \left|\sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t)\right| dt = 0.$$

Therefore by the Hölder inequality we have

$$\begin{aligned}
D_m \le &\left\{\int_0^1 \max_{n\le m} \lambda_n^p \left|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\right|^p d\tau\right\}^{\frac{1}{p}} \\
&\times \left\{\int_0^1 \left|\int_a^b K_{s_m}(t,\tau)dt\right|^q d\tau\right\}^{\frac{1}{q}} + O(1)
\end{aligned}$$

and the proof is complete by the hypotheses of theorem.

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Multiplikatiivsete süsteemidega määratud funktsionaalridade koonduvus ja λ-tõkestatus

N.Saealle ja H. Türnpu

Artiklis on käsitletud rida $\sum c_k g_k(t)$, kus süsteem $\{g_k\}$ on mingi multiplikatiivse süsteemi korrutissüsteem, ja leitud piisavaid tingimusi selle rea p-maksimaalse kiirusega koonduvuse jaoks. On vaadeldud ka rida $\sum < f, w_k > g_k(t)$, kus $f \in L^p_{[0,1]}$ ja $\{w_k\}$ on Walshi süsteem, ning tõestatud, et see rida koondub peaaegu kõikjal erinevate korrutissüsteemide korral. Töö viimases osas on uuritud selle rea λ-tõkestatust peaaegu kõikjal.

CHAPTER IV

Acta et Commentationes
Universitatis Tartuensis de Mathematica (to appear)

Uniform convergence and A^λ-boundedness of series with respect to product systems

Natalia Saealle

Abstract. Let $\{g_k\}$ be an orthogonal product system. For a continuous function u it is proved that the series $\sum_k < u, w_k > g_k(t)$ with the Walsh-Fourier coefficients $< u, w_k >$ is convergent (A-summable, A^λ-bounded, regularly A^λ-summable) uniformly if and only if the Walsh-Fourier series $\sum_k < u, w_k > w_k(t)$ has the same property.

1. Introduction and statement of the results

Let $\{f_k\}_{k=0}^\infty$ be a system of measurable functions such that

$$f_0(t) = 1 \text{ and } |f_k(t)| \le 1 \text{ on } [a, b].$$

The *product system* $\{g_n\}_{n=0}^\infty$ *of* $\{f_k\}$ is given by

$$g_0(t) = 1 \quad \text{and} \quad g_n(t) = f_{n_0+1}(t) f_{n_1+1}(t) \dots f_{n_k+1}(t) \qquad (t \in [a, b]),$$

where $n = 2^{n_0} + 2^{n_1} + \dots + 2^{n_k} (n_0 < n_1 < \dots < n_k)$ is the dyadic representation of n. For example, the product system of Rademacher system is the Walsh system $\{w_n\}_{n=0}^\infty$ (in the Paley enumeration), which is complete and orthonormal (see e.g. [1], pp. 12, 60).

In this paper we consider the series

$$\sum_{k=0}^{\infty} < u, w_k > g_k(t), \tag{1}$$

2000 *Mathematics Subject Classification.* Primary 42C10, 40A30; secondary 40C05, 40G05.

Key words and phrases. Walsh-Fourier series, product systems, uniform convergence, matrix summability methods, boundedness with speed.

Supported in part by Estonian Science Foundation Grant 5376.

where

$$< u, w_k >:= \int_0^1 u(\tau) w_k(\tau) d\tau \qquad (k = 0, 1, \ldots)$$

are the Walsh-Fourier coefficients of u. In our previous work [4], we considered p-maximal λ-boundedness of series (1) in the case of functions $u \in L^p_{[0,1]}$ $(1 < p < \infty)$. Now, we suppose, that u is continuous on $[0, 1]$ and study the uniform convergence, the uniform A-summability, the uniform A^λ-boundedness, and the uniform regular A^λ-summability of series (1).

Let $\lambda = (\lambda_k)$ be a sequence of real numbers such that $0 < \lambda_k \nearrow \infty$ and let $A = (\alpha_{nk})$ be a triangular regular summability method. For a function $u \in C_{[0,1]}$ we put

$$b_n(A, t) := \lambda_n \left(\sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) - \lim_m \sum_{k=0}^{m} \alpha_{mk} < u, w_k > g_k(t) \right).$$

A series (1) uniformly A-summable on $[a, b]$ is called

1) *uniformly A^λ-bounded on* $[a, b]$, if

$$\sup_n |b_n(A, t)| = O(1) \text{ uniformly in } t \in [a, b];$$

2) *uniformly regularly A^λ-summable on* $[a, b]$, if

$$\lim_n b_n(A, t) = 0 \text{ uniformly in } t \in [a, b]$$

(cf. [2], [3]).

If series (1) is uniformly Σ^λ-bounded (uniformly regularly Σ^λ-summable), where $\Sigma = (\sigma_{nk})$ is the triangular matrix with $\sigma_{nk} = 1$ $(k = 0, 1, \ldots;$ $n = 0, 1, \ldots)$, then it is called uniformly λ-bounded (uniformly regularly λ-convergent).

We will consider the relationship between the convergence properties of series (1) and of the Walsh-Fourier series

$$\sum_{k=0}^{\infty} < u, w_k > w_k(\tau). \tag{2}$$

The series (2) are well studied. For example, it was shown in [6] that for every point $\tau_0 \in [0, 1]$ there is a continuous function u, whose Walsh-Fourier series diverges at that point. On the other hand, Walsh remarked, that

$$\lim_n \sum_{\nu=0}^{2^n - 1} < u, w_\nu > w_\nu(\tau) = u(\tau) \text{ uniformly in } \tau \in [0, 1] \tag{3}$$

for every $u \in C_{[0,1]}$.

We prove the following

Theorem 1. *Let* $u \in C_{[0,1]}$ *and let* A *be a regular triangular summability method.*

(a) *Series* (1) *is convergent (*A*-summable,* A^λ*-bounded, regularly* A^λ*-summable) uniformly on* $[a,b]$, *if series* (2) *is convergent (*A*-summable,* A^λ*-bounded, regularly* A^λ*-summable) uniformly on* $[0,1]$.

(b) *If* $\{g_k\}$ *is an orthogonal system, then series* (1) *is convergent (*A*-summable,* A^λ*-bounded, regularly* A^λ*-summable) uniformly on* $[a,b]$, *if and only if series* (2) *is convergent (*A*-summable,* A^λ*-bounded, regularly* A^λ*-summable) uniformly on* $[0,1]$.

Let A be the summability method of arithmetic means, i.e. $A = C^1 = (\gamma_{nk})$, where

$$\gamma_{nk} := \begin{cases} 1 - \dfrac{k}{n+1}, & \text{if} \quad k \le n, \\ \hfill 0, & \text{if} \quad k > n. \end{cases}$$

This method is regular. It is well known, that the Walsh-Fourier series is uniformly C^1-summable for every $u \in C_{[0,1]}$ (see [5], p. 265, or [1], p.103). An immediate consequence of Theorem 1 is the following

Corollary 2. *Series* (1) *is uniformly* C^1*-summable on* $[a,b]$ *for every* $u \in C_{[0,1]}$.

2. Proof of Theorem 1

We need the following

Lemma 3. *Let* (m_n) *be an increasing sequence of natural numbers. Then the subsequence of partial sums*

$$\sum_{k=0}^{2^{m_n}-1} < u, w_k > g_k(t)$$

converges uniformly on $[a,b]$ *for every* $u \in C_{[0,1]}$.

Remark 4. Let

$$v_u(t) := \lim_n \sum_{k=0}^{2^{m_n}-1} < u, w_k > g_k(t).$$

From Lemma 3 it follows that for every speed (λ_p) there exists a subsequence (s_p) of (m_n) such that

$$\lim_p \lambda_p \left| \sum_{k=0}^{2^{s_p}-1} < u, w_k > g_k(t) - v_u(t) \right| = 0 \text{ uniformly on } [a,b]. \qquad (4)$$

Indeed, by Lemma 3, for every $\varepsilon = 1/p^{\lambda_p}$ $(p = 1, 2, \ldots)$ there exists $N = N(p)$ such that

$$\lambda_p \left| \sum_{\nu=0}^{2^{s_p}-1} < u, w_k > g_k(t) - v_u(t) \right| < \frac{\lambda_p}{p^{\lambda_p}} \qquad (t \in [a,b])$$

for all $p > N$. The right side of this inequality converges to zero, hence (4) holds.

Proof of Lemma 3. First, note that the kernel

$$K_n(t,\tau) := \sum_{k=0}^{2^n-1} g_k(t) w_k(\tau)$$

is non-negative for every $t \in [a,b]$ and $\tau \in [0,1]$, therefore

$$\int_0^1 |K_n(t,\tau)|\, d\tau = 1$$

(cf. [3], p. 233). We consider the sequence of continuous linear operators

$$B_n : C[0,1] \longrightarrow L^\infty_{[a,b]} \quad (n = 0, 1, \ldots)$$

defined by

$$B_n(u,t) = \sum_{k=0}^{2^{m_n-1}} < u, w_k > g_k(t).$$

On the one hand, we have

$$\begin{aligned} |B_n(u,t)| &= \left| \int_0^1 u(\tau) \sum_{k=0}^{2^{m_n-1}} w_k(\tau) g_k(t) d\tau \right| \\ &\leq \|u\|_{C_{[0,1]}} \int_0^1 |K_{m_n}(t,\tau)|\, d\tau = \|u\|_{C_{[0,1]}} \ (t \in [a,b],\ n = 0, 1, \ldots), \end{aligned}$$

thus the sequence (B_n) is uniformly bounded.

On the other hand, we have

$$B_n(w_i,t) = \sum_{k=0}^{2^{m_n}-1} < w_i, w_k > g_k(t) = g_i(t) \quad (t \in [a,b],\ n = 0, 1, 2, \ldots)$$

for $2^{m_n} \geq i+1$. Therefore $(B_n(P,t))$ is uniformly convergent on $[0,1]$ for every $P \in \mathcal{P}$, where $\mathcal{P}$ is the collection of finite linear combinations of Walsh functions. It is known that $\mathcal{P}$ is dense in $C_{[0,1]}$ (cf. [1], p. 63).

The assertion of Lemma follows from the Banach-Steinhaus theorem.

Proof of Theorem 1. (a) Let $\{g_n\}$ be a product system. By the orthogonality of the Walsh system, we have

$$g_k(t) = \int_0^1 w_k(\tau) \sum_{\nu=0}^{2^{m_n}-1} w_\nu(\tau) g_\nu(t) d\tau.$$

Consequently,

$$\begin{aligned}
&\lambda_n \left| \sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) - v_u(t) \right| \\
&= \lambda_n \left| \int_0^1 \sum_{k=0}^{n} \alpha_{nk} < u, w_k > w_k(\tau) \sum_{\nu=0}^{2^{m_n}-1} w_\nu(\tau) g_\nu(t) d\tau - v_u(t) \right| \\
&= \lambda_n \Bigg| \int_0^1 \left(\sum_{k=0}^{n} \alpha_{nk} < u, w_k > w_k(\tau) - u(\tau) + u(\tau) \right) \times \\
&\qquad\qquad \times \sum_{\nu=0}^{2^{m_n}-1} w_\nu(\tau) g_\nu(t) d\tau - v_u(t) \Bigg| \\
&= \lambda_n \Bigg| \int_0^1 u(\tau) \sum_{\nu=0}^{2^{m_n}-1} w_\nu(\tau) g_\nu(t) d\tau - v_u(t) \\
&\quad + \int_0^1 \left(\sum_{k=0}^{n} \alpha_{nk} < u, w_k > w_k(\tau) - u(\tau) \right) \sum_{\nu=0}^{2^{m_n}-1} w_\nu(\tau) g_\nu(t) d\tau \Bigg|.
\end{aligned}$$

Since

$$\sum_{\nu=0}^{2^{m_n}-1} g_\nu(t) \int_0^1 w_\nu(\tau) d\tau = g_0(t) = 1,$$

then the inequality

$$\begin{aligned}
&\lambda_n \left| \sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) - v_u(t) \right| \\
&\leq \lambda_n \left| \sum_{\nu=0}^{2^{m_n}-1} < u, w_\nu > g_\nu(t) - v_u(t) \right| \qquad (5) \\
&\qquad + \lambda_n \max_{0 \leq \tau \leq 1} \left| \sum_{k=0}^{n} \alpha_{nk} < u, w_k > w_k(\tau) - u(\tau) \right|
\end{aligned}$$

holds for every regular triangular matrix $A = (\alpha_{nk})$ and speed (λ_n). If series (2) is uniformly A^λ-bounded or uniformly regularly A^λ-summable, then, by (4) and (5), series (1) enjoys the same property. To prove that the uniform convergence of series (2) on $[0,1]$ implies the uniform convergence of series (1) on $[a,b]$, we use (4) and (5), where $\lambda_n = 1$ $(n = 0, 1, \ldots)$ and $A = \Sigma$. Similarly, using Lemma 3 and (5) we can prove the same statement concerning the A-summability.

(b) Suppose that the product system $\{g_n\}$ is orthogonal. Then

$$\lambda_n \left| \sum_{k=0}^{n} \alpha_{nk} < u, w_k > w_k(\tau) - u(\tau) \right|$$

$$= \lambda_n \left| \int_a^b \left(\sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) - v_u(t) + v_u(t) \right) \times \right.$$

$$\left. \times \sum_{\nu=0}^{2^{m_n}-1} g_\nu(t) w_\nu(\tau) dt - u(\tau) \right|$$

$$= \lambda_n \left| \sum_{\nu=0}^{2^{m_n}-1} < v_u, g_\nu > w_\nu(\tau) - u(\tau) \right.$$

$$\left. + \int_a^b \left(\sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) - v_u(t) \right) \sum_{\nu=0}^{2^{m_n}-1} g_\nu(t) w_\nu(\tau) dt \right|.$$

By the non-negativity of K_{m_n}, we have

$$\int_a^b |K_{m_n}(t,\tau)|\, dt = (b-a) w_0(\tau) = b - a.$$

On the other hand, by the orthogonality of $\{g_k\}$,

$$< v_u, g_\nu > = \int_a^b g_\nu(t) \lim_n \sum_{k=0}^{2^{m_n}-1} < u, w_k > g_k(t) dt = < u, w_\nu > .$$

Therefore, the inequality

$$\lambda_n \left| \sum_{k=0}^{n} \alpha_{nk} < u, w_k > w_k(\tau) - u(\tau) \right|$$

$$\leq \lambda_n \left| \sum_{\nu=0}^{2^{m_n}-1} < u, w_\nu > w_\nu(\tau) - u(\tau) \right| \tag{6}$$

$$+ (b-a) \lambda_n \max_{a \leq t \leq b} \left| \sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) - v_u(t) \right|$$

holds. Moreover, by Lemma 3, from the A-summability of (1) it follows that

$$\lim_{n} \sum_{k=0}^{n} \alpha_{nk} < u, w_k > g_k(t) = v_u(t) \text{ uniformly on } [a, b]. \tag{7}$$

Now, to prove that series (2) converges uniformly on $[a, b]$, if series (1) converges uniformly on $[0, 1]$, we use (3), (7), and inequality (6), where $\lambda_n = 1$ $(n = 0, 1, \ldots)$, $A = \Sigma$. The converse assertion follows from part (a) of this theorem. Similarly we can prove the statements concerning the A-summability. The assertion concerning the A^λ-boundedness and regular A^λ-summability follows from (3), (6), (7), and part (a) of this theorem. The proof is complete.

Acknowledgements. This article is a part of my Ph.D. thesis, written under the direction of Heino Türnpu. I am very grateful to his valuable help. I thank Toivo Leiger and the referee for helpful suggestions which have improved the representation of this work.

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CURRICULUM VITAE

Natalia Saealle

Citizenship:	Estonian Republic
Born:	March 23, 1976, Kiviõli, Estonia
Martial status:	married
Address:	Institute of Pure Mathematics University of Tartu 2 Liivi Str. 50409 Tartu, Estonia
Telephone:	+372 53966767
E-mail:	natalia@ut.ee

Education

1993-1997	Faculty of Mathematics, University of Tartu *baccalaureus scientiarum* in mathematics
1997-1999	MSc student at Institute of Pure Mathematics, University of Tartu *magister scientiarum* in Mathematics
1999-2004	PhD student at Institute of Pure Mathematics, University of Tartu

Professional employment

2001-2003	0.5 time Researcher, Institute of Pure Mathematics, University of Tartu
Since 2003	Assistant, Institute of Pure Mathematics, University of Tartu

CURRICULUM VITAE

Natalia Saealle

Kodakondsus:	Eesti Vabariik
Sünniaeg ja -koht:	23. märts 1976, Kiviõli, Eesti
Perekonnaseis:	abielus
Aadress:	Puhta Matemaatika Instituut Liivi 2-615 50409 Tartu, Eesti
Telefon:	+372 53966767
E-mail:	natalia@ut.ee

Haridus

1993-1997	Tartu Ülikool, Matemaatikateaduskond, bakalaureuseõpe *baccalaureus scientiarum* matemaatika erialal
1997-1999	Tartu Ülikool, Matemaatikateaduskond, magistriõpe *magister scientiarum* matemaatika erialal
1999-2004	Tartu Ülikool, Matemaatika-informaatikateaduskond, Puhta Matemaatika Instituut, doktoriõpe

Erialane teenistuskäik

2001-2003	TÜ Puhta Matemaatika Instituut, teadur (0,5 kohta)
Alates 2003	TÜ Puhta Matemaatika Instituut, assistent

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